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Math for Machine Learning

Linear algebra - Week 2

Solving systems of equations
Matrix row reduction
Row operations that preserve singularity
Row-reduced echelon form
Row echelon form
Rank of a matrix

W2 Lesson 1

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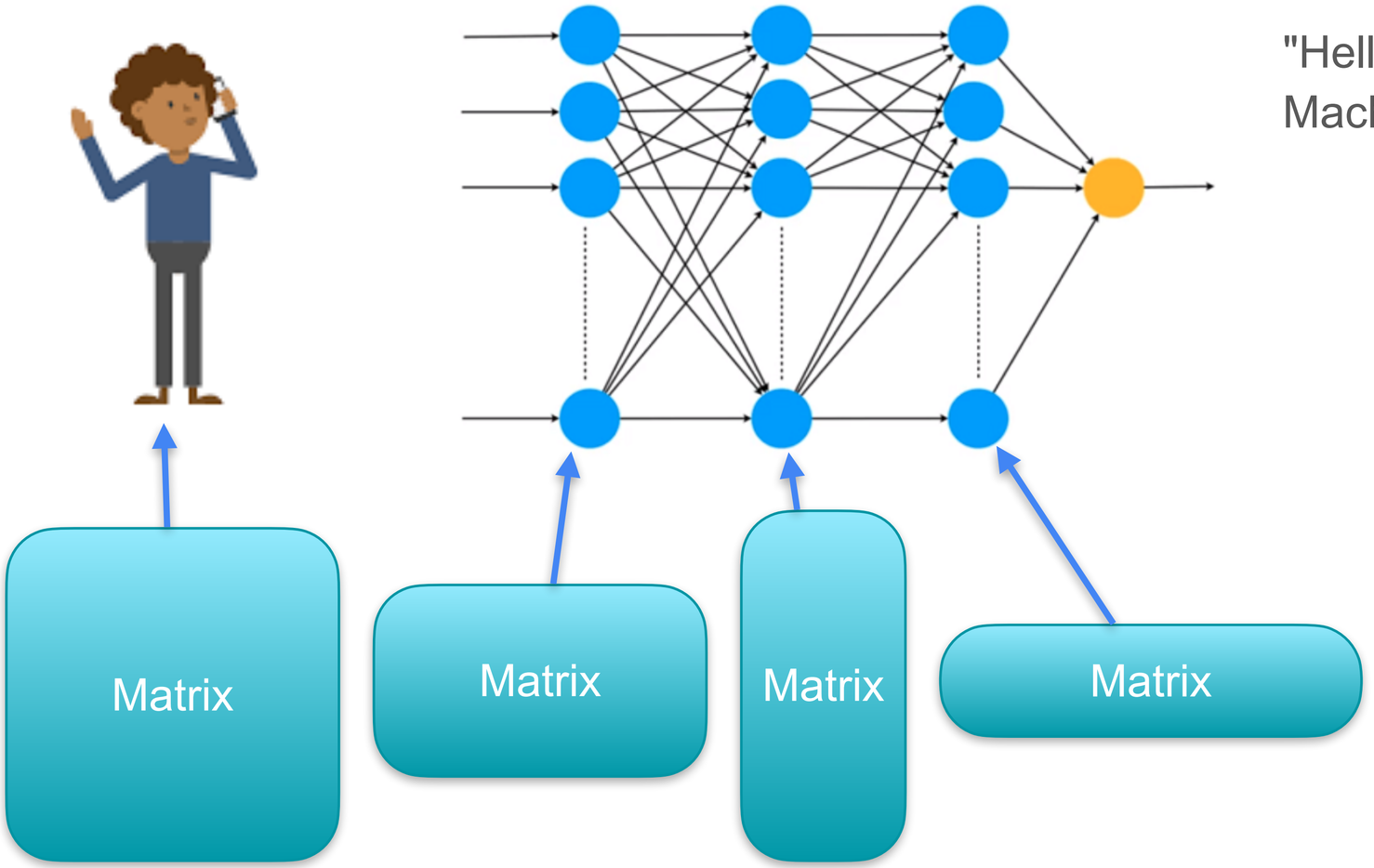
Solving System of Linear Equations

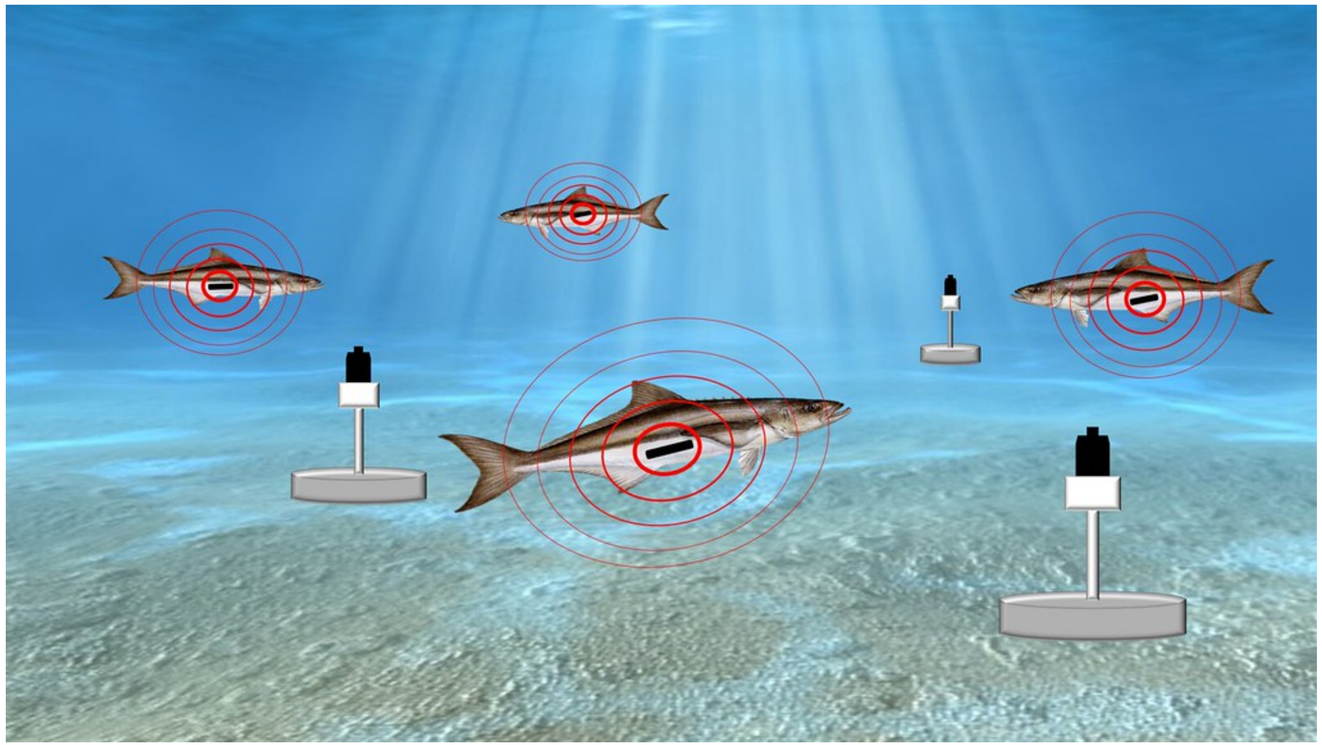
Machine learning motivation

Neural networks - Matrix operations



"Hello! Welcome to Math for Machine Learning!"

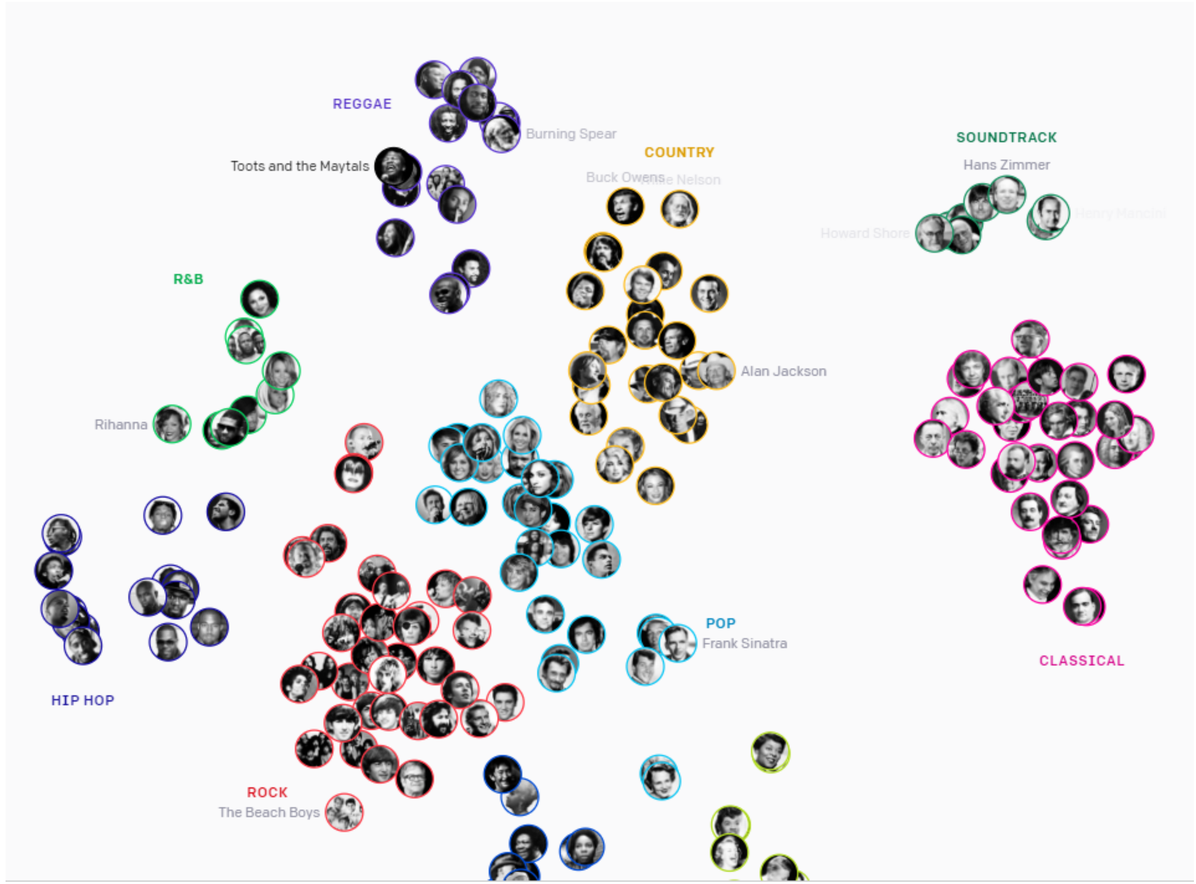
Neural networks - Sound recognition



Acoustic monitoring: Monitoring ecosystems through sounds

- Sound recognition: tracking species through sound to preserve bio-habitats.

Neural Networks - AI-generated music



Neural network generates music

- Automatic music generation: compressing music to discrete codes, then training the model on a specific genre to produce new music.

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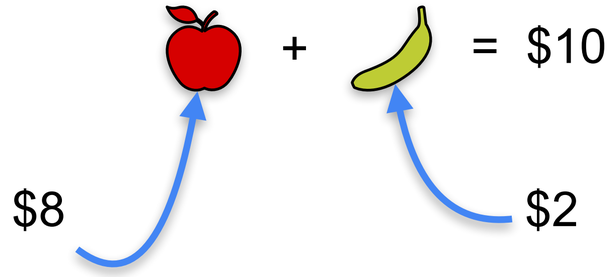
Solving System of Linear Equations

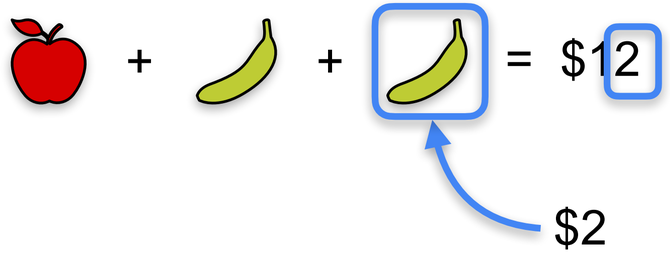
Solving non-singular system of linear equations

Solving systems of equations

System

- a + b = 10
- a + 2b = 12





Solving systems of equations

System

- a + b = 10

- a + 2b = 12

Some process

Manipulating equations

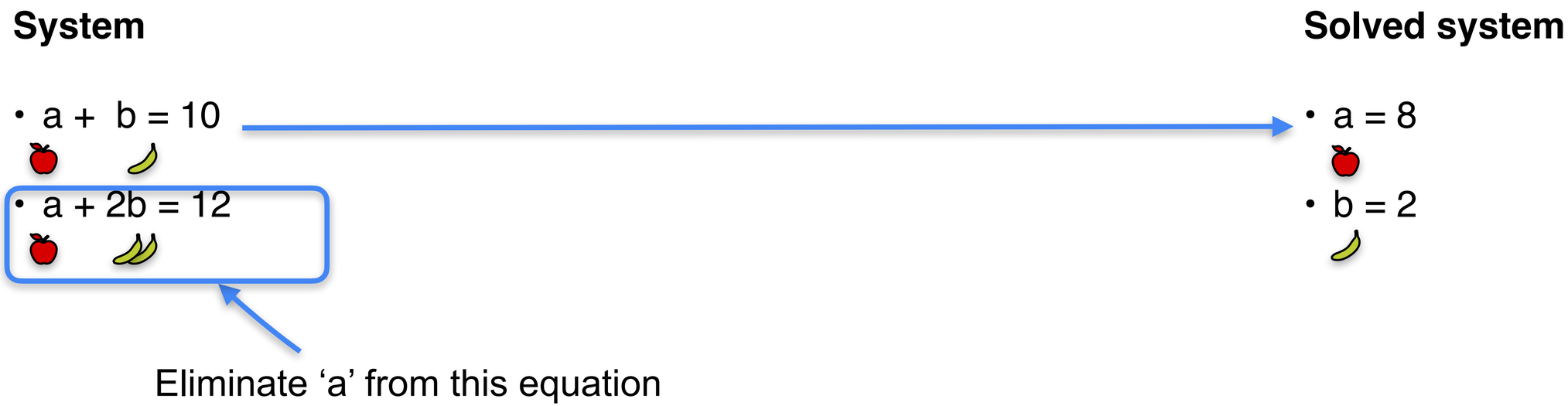
Swapping equations
Adding equations
Multiplying equations by a constant

Solved system

- a = 8

- b = 2

Solving systems of equations



Manipulating equations

Multiplying by a constant

$$a + b = 10$$

$$\times \quad 7$$

$$7a + 7b = 70$$

Adding two equations

$$a + b = 10$$

$$+ \quad 2a + 3b = 22$$

$$3a + 4b = 32$$

Systems of equations

System

- $5a + b = 17$
- $4a - 3b = 6$

Eliminate ‘a’ from this equation

Divide by coefficient of a

- $a + 0.2b = 3.4$
- $a - 0.75b = 1.5$

Subtract equation 1 from equation 2

$$a - 0.75b = 1.5$$

$$-\quad a + 0.2b = 3.4$$

$$0a - 0.95b = -1.9$$

$$-0.95b = -1.9$$

$$b = 2$$

Solved system

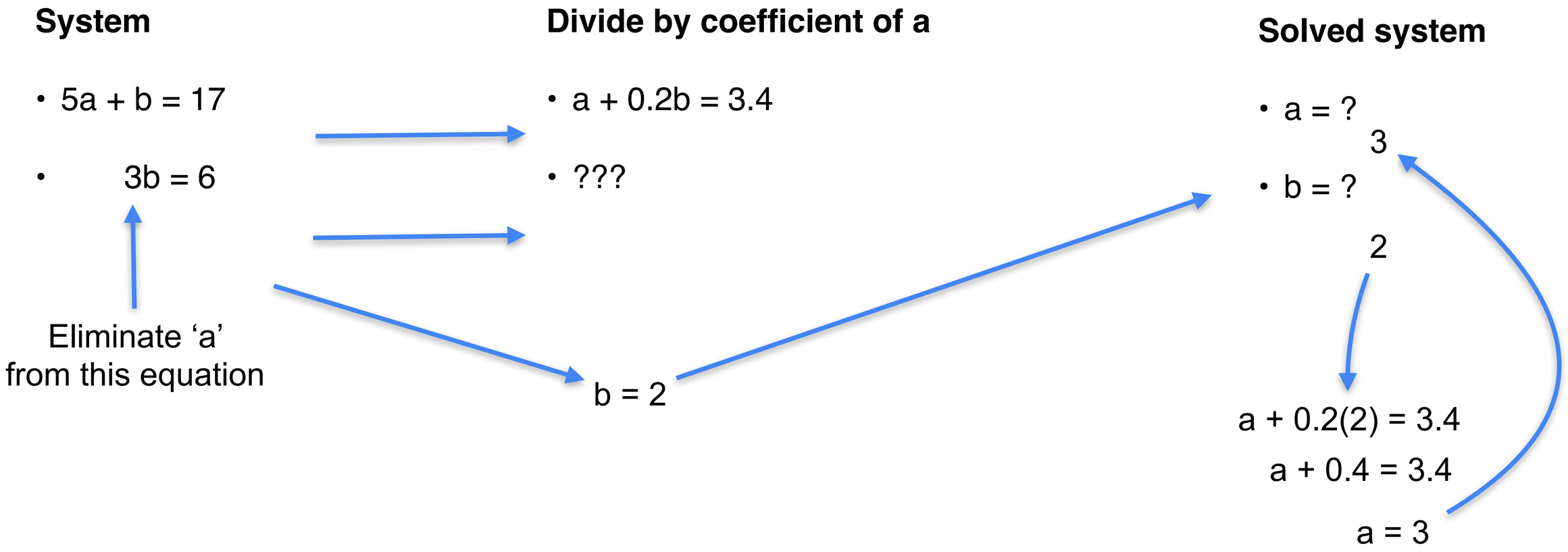
- $a = ?$ 3
- $b = ?$ 2

$$a + 0.2(2) = 3.4$$

$$a + 0.4 = 3.4$$

$$a = 3$$

What if one of the coefficients of a is zero?



Quiz

- Solve the following system of equations

System

- $2a + 5b = 46$
- $8a + b = 32$

Solution

- Solve the following system of equations

System

- $2a + 5b = 46$
- $8a + b = 32$

Solution

- $a = 3$
- $b = 8$

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Solving System of Linear Equations

Solving singular system of linear equations

What if the system is singular (redundant)?

System

- a + b = 10
- 2a + 2b = 20

Eliminate ‘a’ from this equation

Divide by coefficient of a

- a + b = 10
- a + b = 10

Subtract equation 1 from equation 2

a + b = 10

\- a + b = 10

0 = 0

Degree of freedom x

Solved system

- a = ? x
- b = ? 10 - x

Solved system

- a + b = 10
- no other equation

What if the system is singular (contradictory)?

System

- a + b = 10
- 2a + 2b = 24

Eliminate ‘a’
from this equation

Divide by coefficient of a

- a + b = 10
- a + b = 12

Subtract equation 1 from equation 2

$$\begin{array}{rl} & a + b = 12 \\ - & a + b = 10 \\ \hline & 0 = 2 \end{array}$$

Contradiction!

Solved system

- a = ?
- b = ?

Quiz

- Solve the following system of equations

System

- $5a + b = 11$
- $10a + 2b = 22$

Solution

- Solve the following system of equations

System

- $5a + b = 11$
- $10a + 2b = 22$

Solution: If you look closely into the two equations in the system, you'll find that if equation 2 is divided by 2 you'll obtain equation 1.

Therefore, the system has infinitely many solutions.

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Solving System of Linear Equations

Solving system of equations with more variables

Elimination method

System

- a + b + 2c = **12**
- 3a - 3b - c = **3**
- 2a - b + 6c = **24**

Leave ‘a’ by itself

Divide each row by the coefficient of ‘a’

- a + b + 2c = **12**
- a - b - 1/3 c = **1**
- a - b/2 + 3c = **12**

Use the first equation to remove ‘a’ from the others

- a + b + 2c = **12**
- -2b - 7/3 c = **-11**
- - 3/2 b + c = **0**

Isolated ‘a’

Solve this new system of 2 equations

Elimination method

System

- a + b + 2c = **12**
- -2b - 7/3 c = **-11**
- - 3/2 b + c = **0**

Divide last two rows by the coefficient of b

- a + b + 2c = **12**
- b + 7/6 c = **11/2**
- b - 2/3 c = **0**

Use the second equation to remove ‘b’ from the third

- a + b + 2c = **12**
- b + 7/6 c = **11/2**
- -11/6 c = **-11/2**

Isolated ‘b’

c = 3

Elimination method

System

- a + b + 2c = **12**
- b + 7/6 c = **11/2**
- c = **3**

a + 2 + 6 = **12**
a = **4**

b + 7/2 = **11/2**
b = **2**

Replace c = 3 in the second equation, get b = 2

Replace c = 3 and b = 2 in the first equation, get a = 4

The solution is
a = 4
b = 2
c = 3

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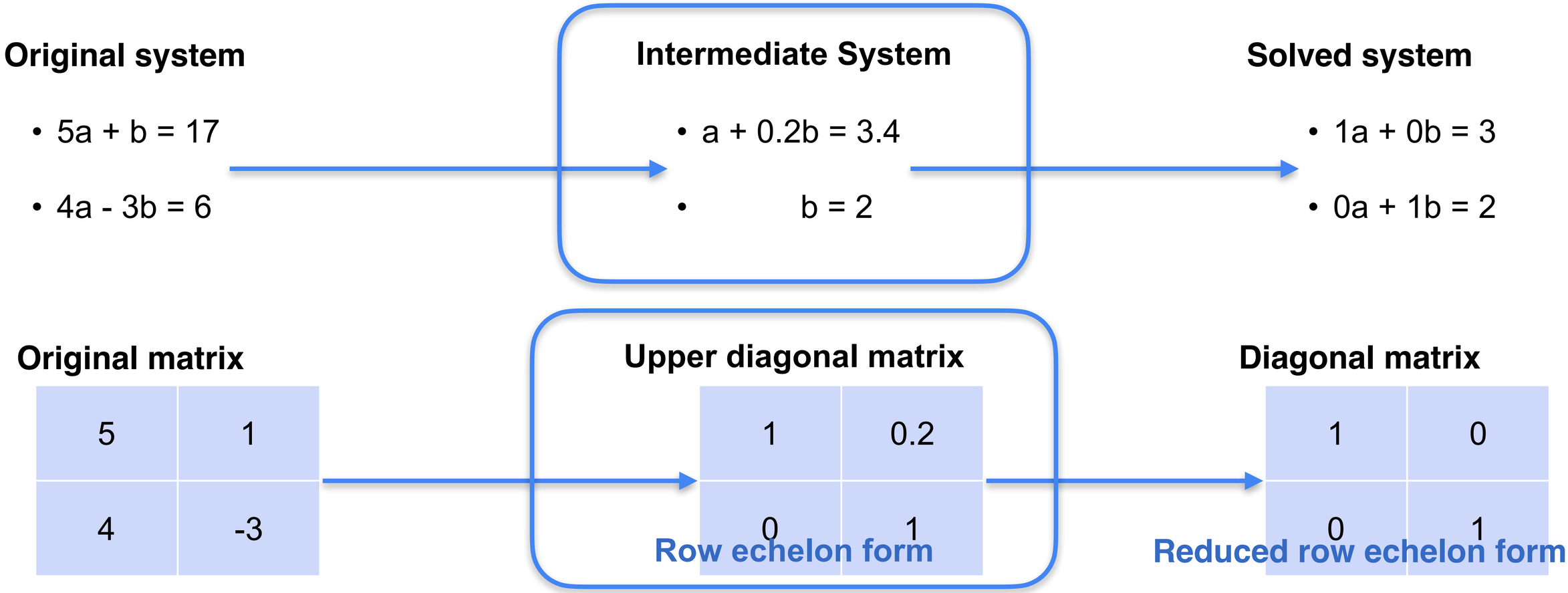
Solving System of Linear Equations

Matrix row reduction

Systems of equations to matrices



Systems of equations to matrices

Original system

- a + b = 10
- 2a + 2b = 20

Intermediate System

- a + b = 10
- 0a + 0b = 0

Original matrix

1	1
2	2

Upper diagonal matrix

1	1
0	0

Row echelon form

Systems of equations to matrices

Original system

- $5a + b = 11$
- $10a + 2b = 22$

Intermediate System

- $a + 0.2b = 2.2$
- $0a + 0b = 0$

Original matrix

5	1
10	2

Upper diagonal matrix

1	0.2
0	0

Row echelon form

Systems of equations to matrices

Original system

- 0a + 0b = 0
- 0a + 0b = 0

Intermediate System

- 0a + 0b = 0
- 0a + 0b = 0

Original matrix

0	0
0	0

Upper diagonal matrix

0	0
0	0

Row echelon form

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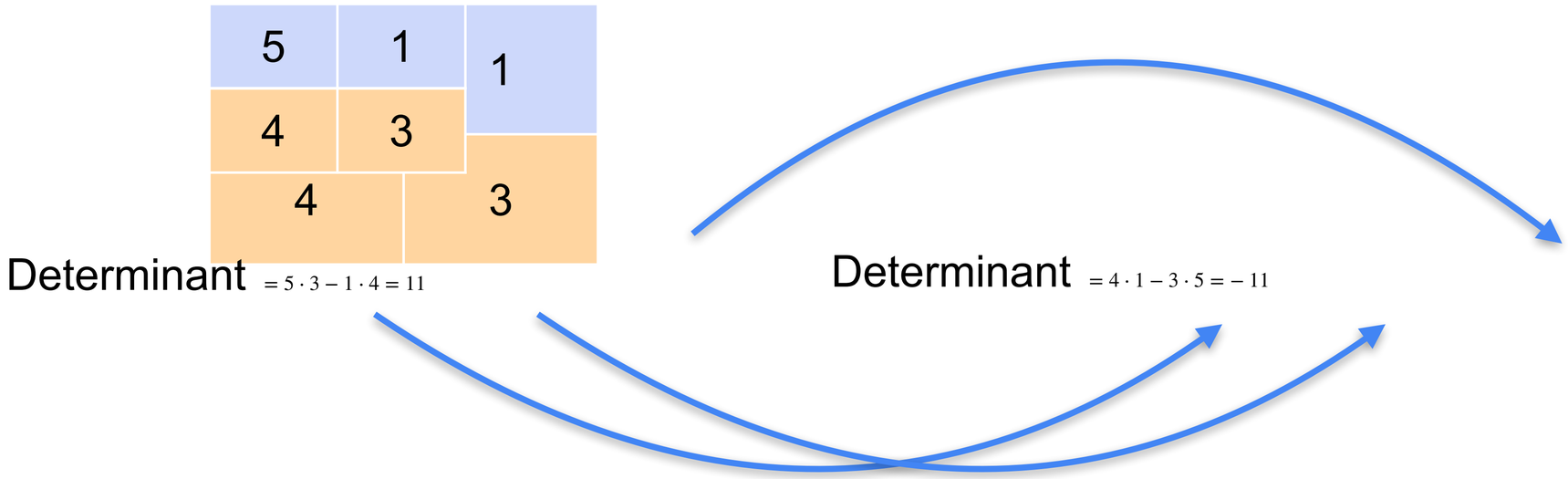
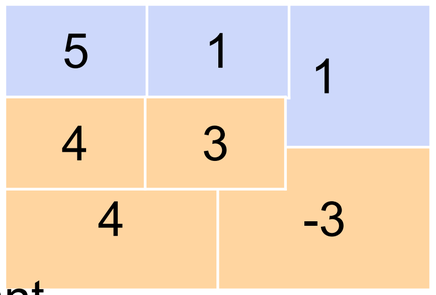
Solving System of Linear Equations

Row operations that preserve singularity

Switching rows



Multiplying a row by a (non-zero) scalar



Determinant $= 5 \cdot 3 - 1 \cdot 4$

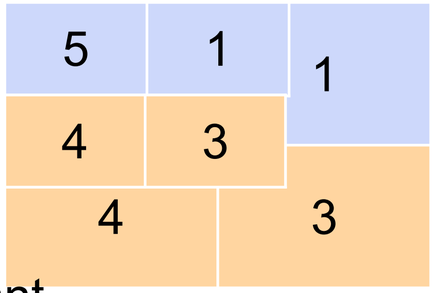
$= 11$

x 10 =

Determinant $= 5 \cdot (10 \cdot 3) - 1 \cdot (10 \cdot 4)$

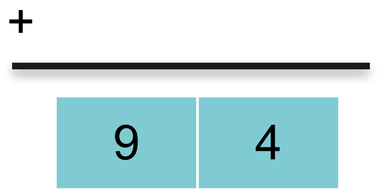
$= 10 \cdot 11$

Adding a row to another row



Determinant $= 5 \cdot 3 - 1 \cdot 4$

$= 11$



Determinant $= 9 \cdot 3 - 4 \cdot 4$

$= 11$

W2 Lesson 2

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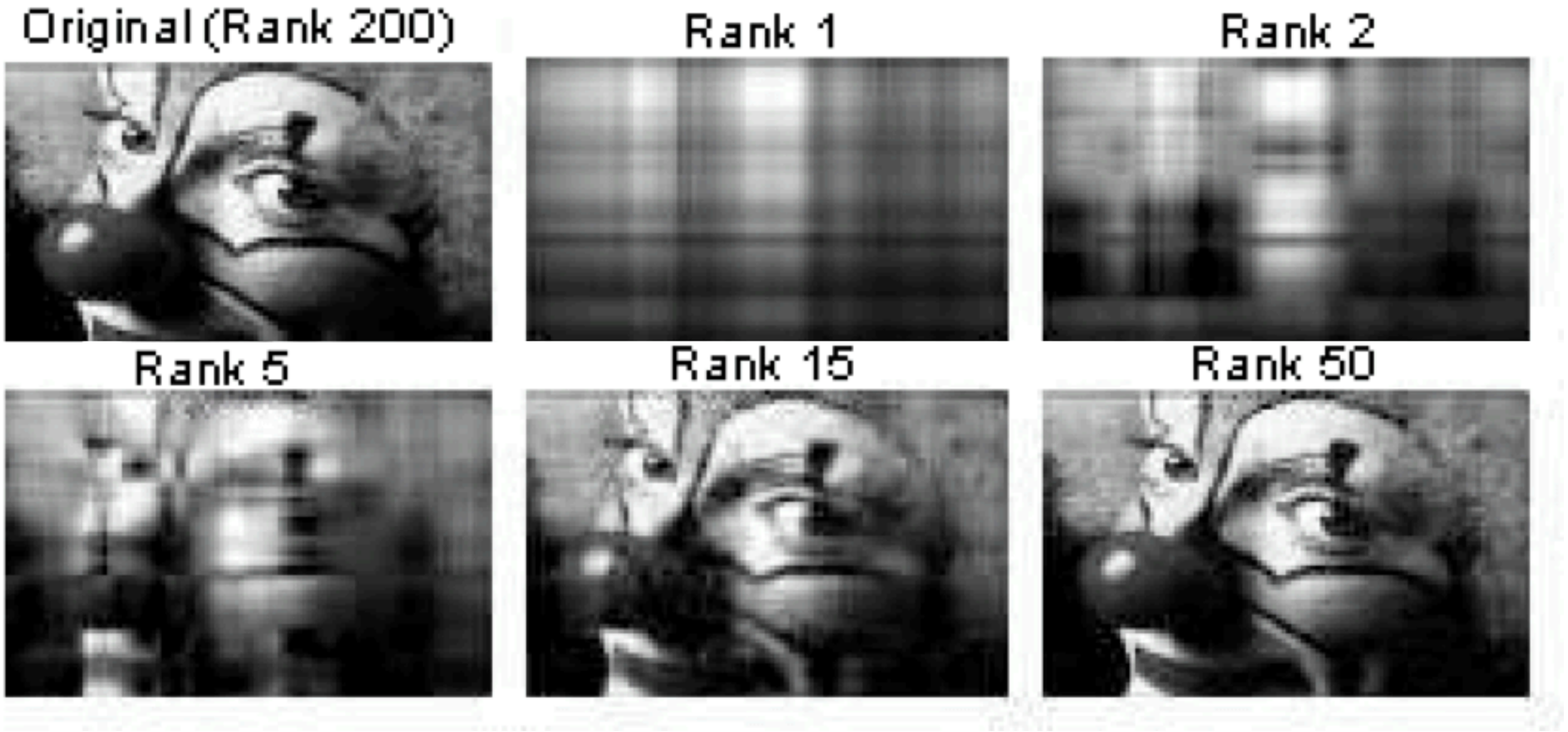
Solving System of Linear Equations

Rank of a matrix

Compressing Images - Reducing rank



Systems of information

System 1

The dog is **black**
The cat is **orange**

Two sentences

Two pieces of information

Rank = 2

System 2

The dog is **black**
The dog is **black**

Two sentences

One piece of information

Rank = 1

System 3

The dog
The dog

Two sentences

Zero pieces of information

Rank = 0

Systems of equations

System 1

a + b = 0

a + 2b = 0

1	1
1	2

Rank = 2

Two equations

Two pieces of information

Rank = 2

System 2

a + b = 0

2a + 2b = 0

1	1
2	2

Rank = 1

Two equations

One piece of information

Rank = 1

System 3

0a + 0b = 0

0a + 0b = 0

0	0
0	0

Rank = 0

Two equations

Zero pieces of information

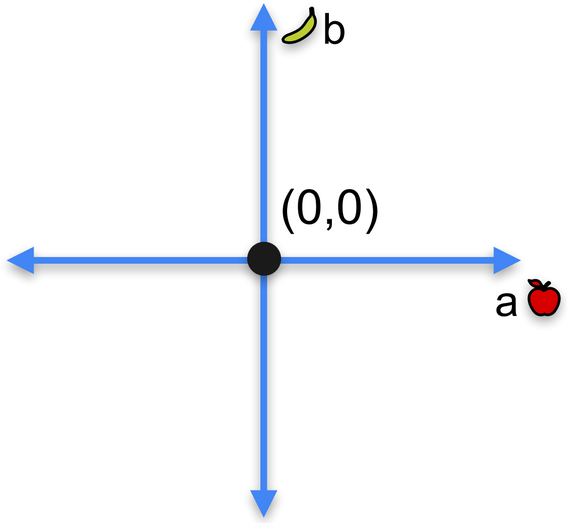
Rank = 0

Rank and solutions to the system

1	1
1	2

Rank = 2

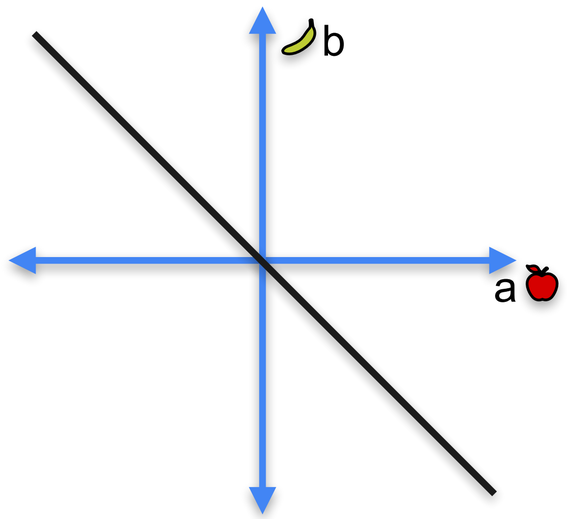
Dimension of solution space = 0



1	1
2	2

Rank = 1

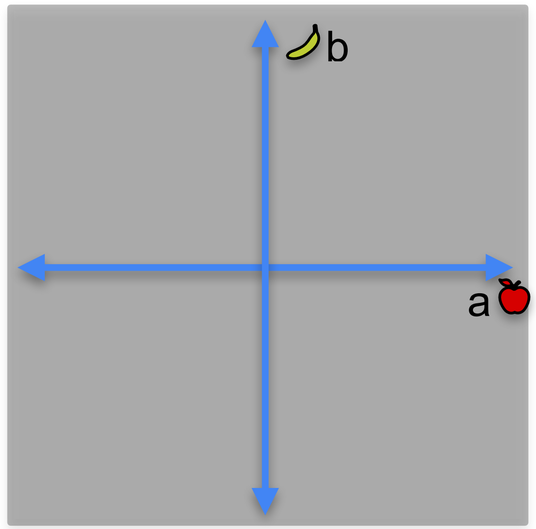
Dimension of solution space = 1



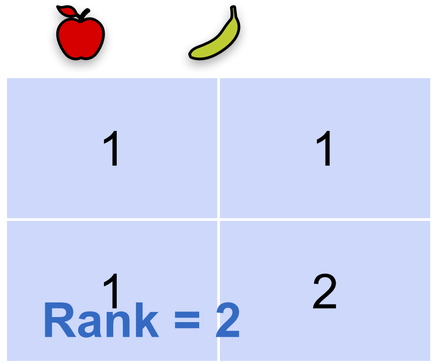
0	0
0	0

Rank = 0

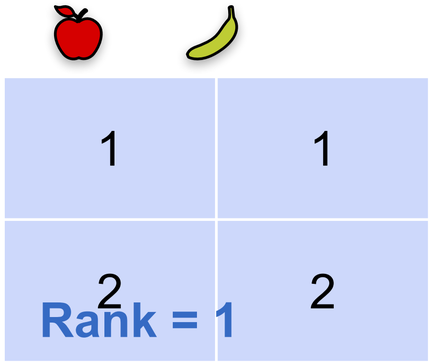
Dimension of solution space = 2



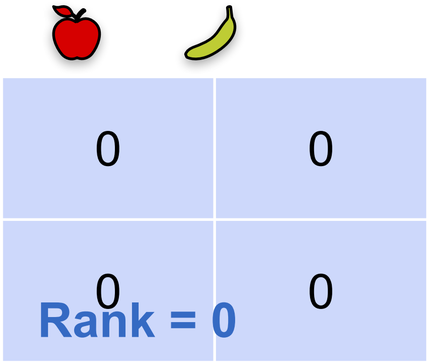
Rank of a matrix



Dimension of solution space = 0



Dimension of solution space = 1



Dimension of solution space = 2

Rank = 2 - (Dimension of solution space)

Rank and singularity

1	1
1	2

Rank = 2

Non-singular

1	1
2	2

Rank = 1

Singular

0	0
0	0

Rank = 0

Singular

Quiz: Rank of a matrix

Determine the rank of the following two matrices

Matrix 1

5	1
-1	3

Matrix 2

2	-1
-6	3

Solutions: Rank of a matrix

Determine the rank of the following two matrices

Matrix 1: Since the solution space had dimension 0, the rank is **2**.

5	1
-1	3

Matrix 2: Since the solution space had dimension 1, the rank is **1**.

2	-1
-6	3

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Solving System of Linear Equations

Rank of a matrix: General case

Rank for matrices

System 1

$a + b + c = 0$ ✓
$a + 2b + c = 0$ ✓
$a + b + 2c = 0$ ✓

3 Equations
3 Pieces of information

Rank 3

1	1	1
1	2	1
1	1	2

System 2

$a + b + c = 0$ ✓
$a + b + 2c = 0$ ✗
$a + b + 3c = 0$ ✓

3 Equations
2 Pieces of information

Rank 2

1	1	1
1	1	2
1	1	3

System 3

$a + b + c = 0$ ✓
$2a + 2b + 2c = 0$ ✗
$3a + 3b + 3c = 0$ ✗

3 Equations
1 Piece of information

Rank 1

1	1	1
2	2	2
3	3	3

System 4

$0a + 0b + 0c = 0$ ✗
$0a + 0b + 0c = 0$ ✗
$0a + 0b + 0c = 0$ ✗

3 Equations
0 Pieces of information

Rank 0

0	0	0
0	0	0
0	0	0

Question

- Is there an easier way to calculate the rank?
- Answer: Yes! As before, it is the number of ones in the diagonal of the reduced row echelon form of the matrix.

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Solving System of Linear Equations

Row echelon form

Row echelon form of a matrix

Original matrix

5	1
4	-3
5	1
10	2
0	0
0	0

Row echelon form

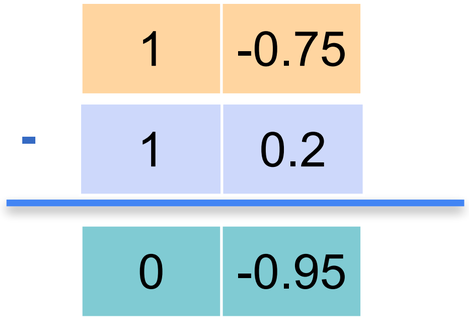
1	0.2
0	1
1	1
0	0
0	0
0	0

Row echelon form

Original matrix

Row echelon form

Divide each row by the leftmost coefficient



Divide the second row by the leftmost non-zero coefficient

Row echelon form for singular matrices

Original matrix

5	1
10	2

Divide each row by the leftmost coefficient

1	0.2
1	0.2

1	0.2
- 1	0.2
0	0

Row echelon form

1	0.2
0	0

Divide the second row by the leftmost non-zero coefficient

1	0.2
?	?

Row echelon form for singular matrices

Row echelon form

Original matrix

Divide each row by
the leftmost coefficient

Row echelon form, singularity, and rank

Non-singular matrix

5	1
4	-3

1	0.2
0	1

Rank 2

2 ones in the diagonal

Singular matrix

5	1
10	2

1	0.2
0	0

Rank 1

1 one in the diagonal

Singular matrix

0	0
0	0

0	0
0	0

Rank 0

0 ones in the diagonal

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Solving System of Linear Equations

Row echelon form: General case

Row echelon form

System

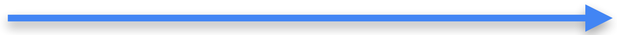
- $a + b + 2c = 12$
- $3a - 3b - c = 3$
- $2a - b + 6c = 24$

System

- $a + b + 2c = 12$
- $-6b - 7c = -33$
- $6c = 18$

Matrix

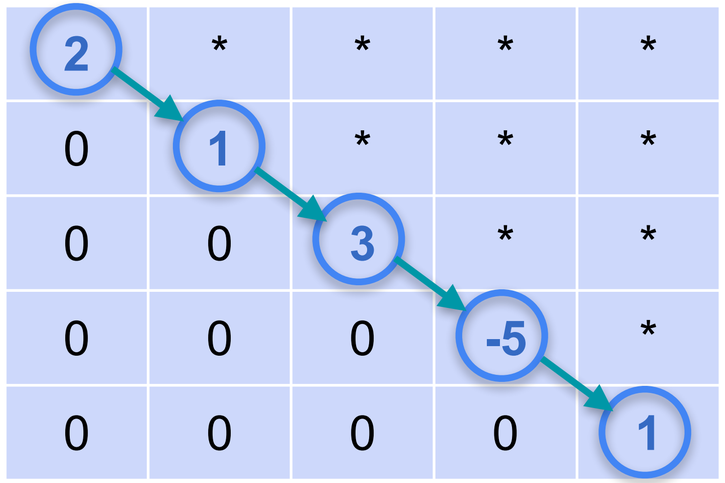
1	1	2
3	-3	-1
2	-1	6



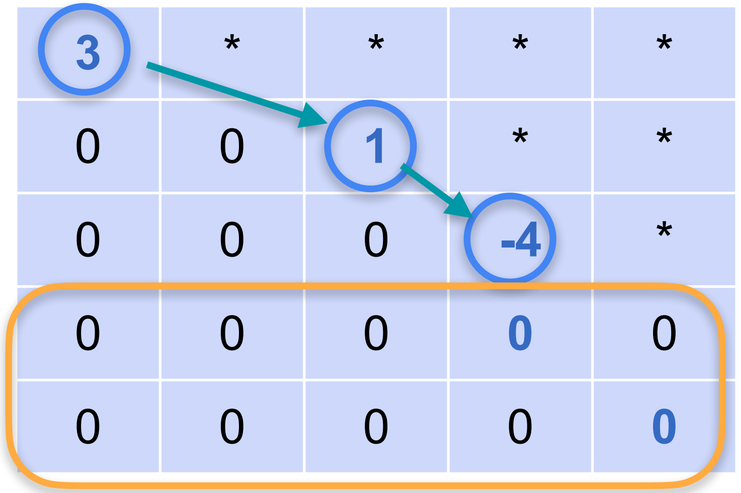
Row echelon form matrix

1	1	2
0	-6	7
0	0	6

Row echelon form



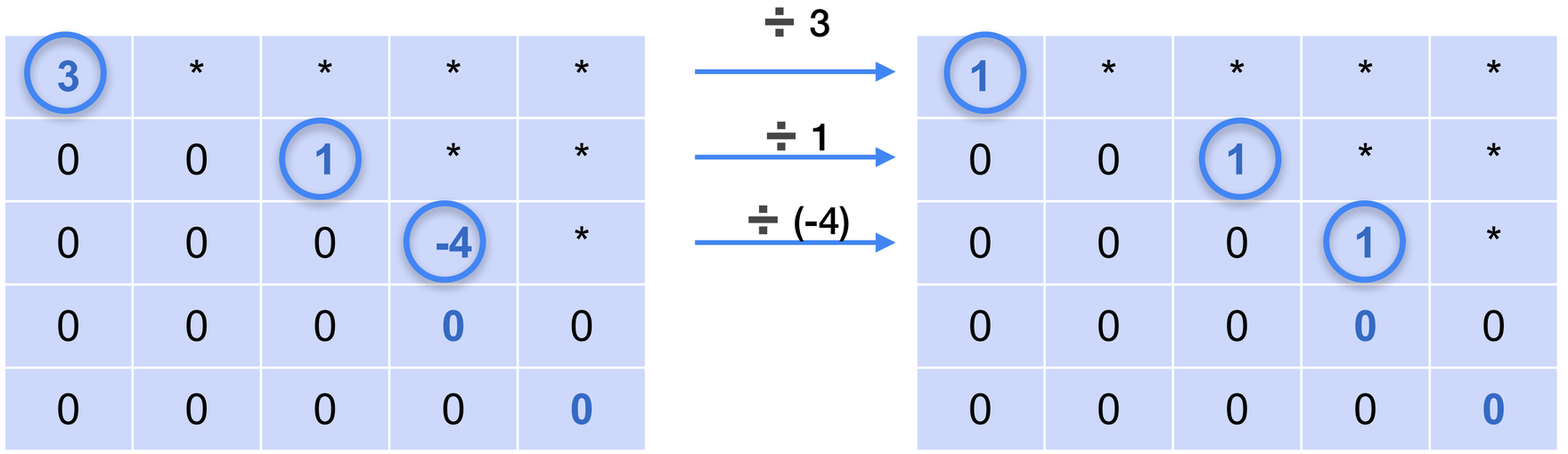
Rank 5



Rank 3

- Zero rows at the bottom
- Each row has a pivot (leftmost non-zero entry)
- Every pivot is to the right of the pivots on the rows above
- Rank of the matrix is the number of pivots

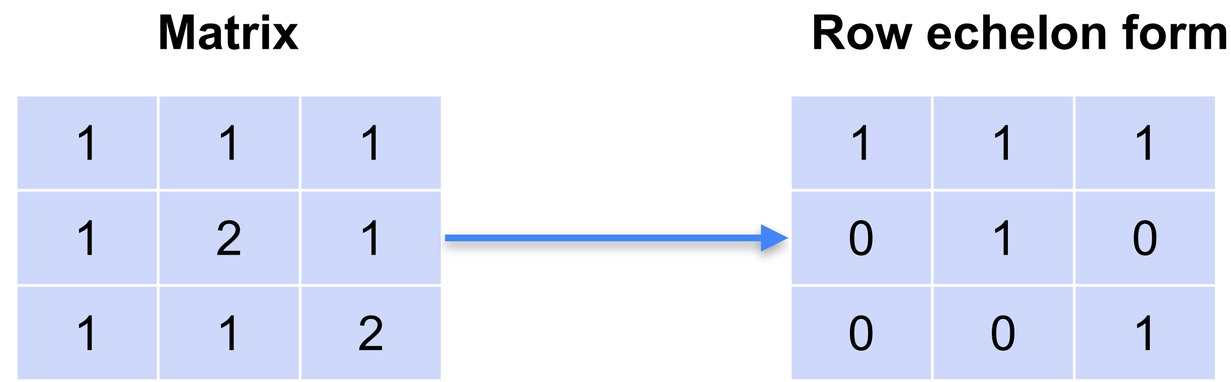
Row echelon form



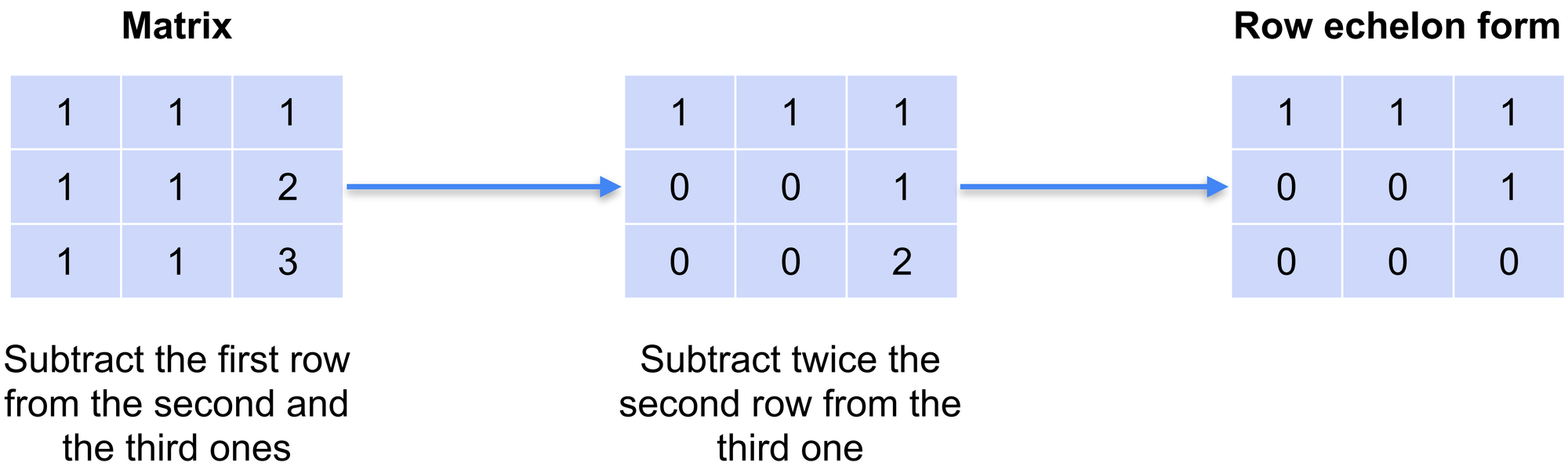
Note:

- In general, pivots different than 1 are allowed
- For this class, pivots are 1. This makes no mathematical difference.

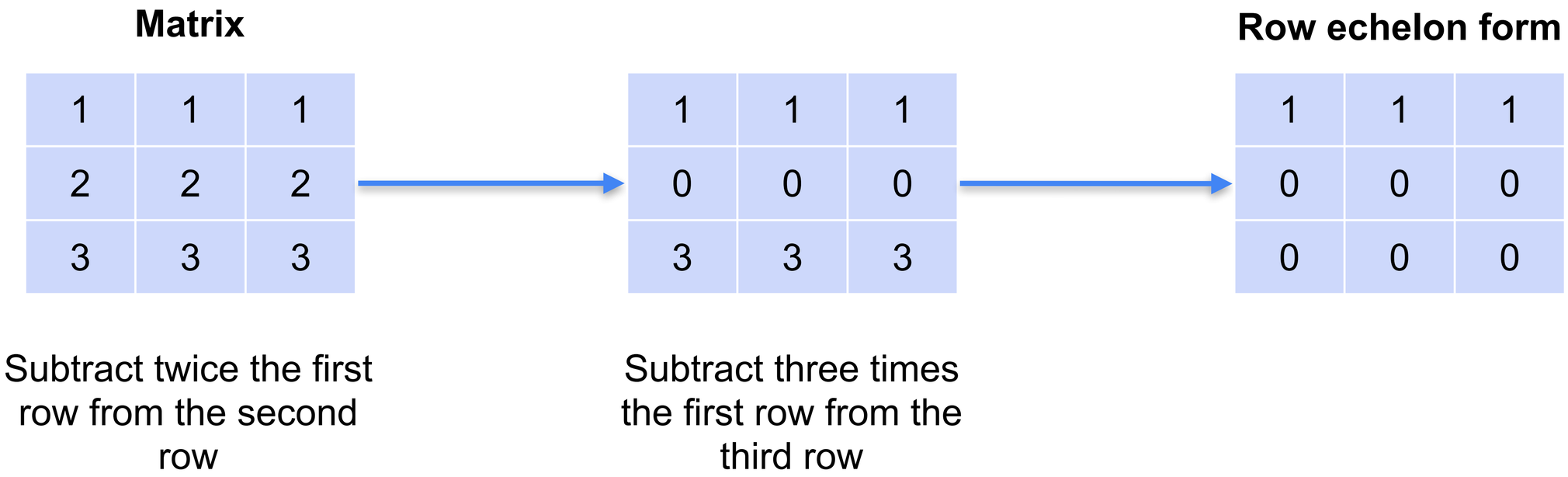
Another example



What if the matrix is singular?



What if the matrix is singular?



Rank for matrices

Matrix 1

1	1	1
1	2	1
1	1	2

Rank = 3

Matrix 2

1	1	1
1	1	2
1	1	3

Rank = 2

Matrix 3

1	1	1
2	2	2
3	3	3

Rank = 1

Matrix 4

0	0	0
0	0	0
0	0	0

Rank = 0

Row echelon forms

1	1	1
0	1	0
0	0	1

Number of pivots = 3

1	1	1
0	0	1
0	0	0

Number of pivots = 2

1	1	1
0	0	0
0	0	0

Number of pivots = 1

0	0	0
0	0	0
0	0	0

Number of pivots = 0

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Solving System of Linear Equations

Reduced row echelon form

Systems of equations to matrices

Original system

- $5a + b = 17$
- $4a - 3b = 6$

Intermediate System

- $a + 0.2b = 3.4$
- $b = 2$

Solved system

- $1a + 0b = 3$
- $0a + 1b = 2$

Original matrix

5	1
4	-3

Upper diagonal matrix

1	0.2
0	1

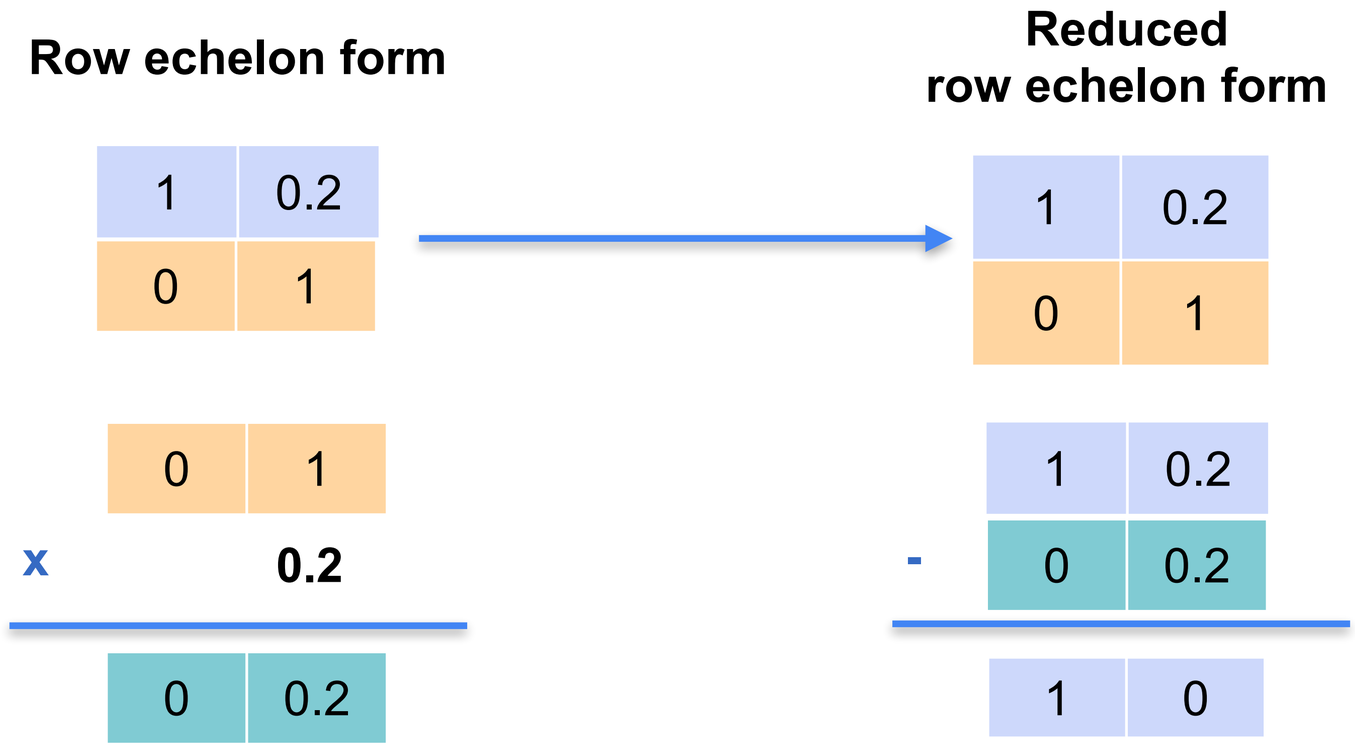
Row echelon form

Diagonal matrix

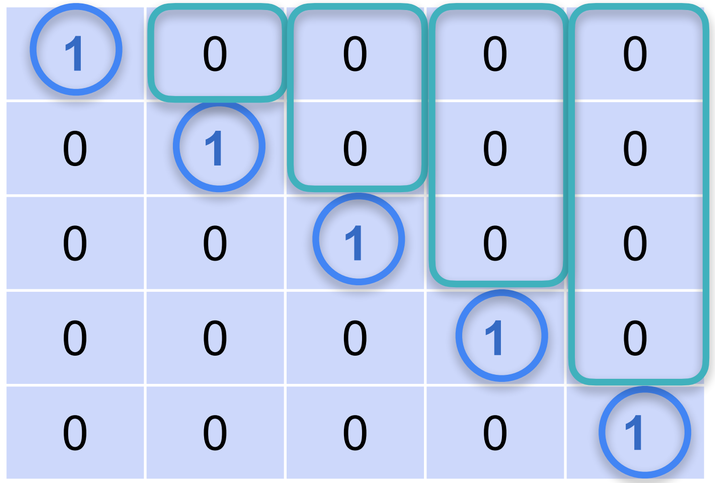
1	0
0	1

Reduced row echelon form

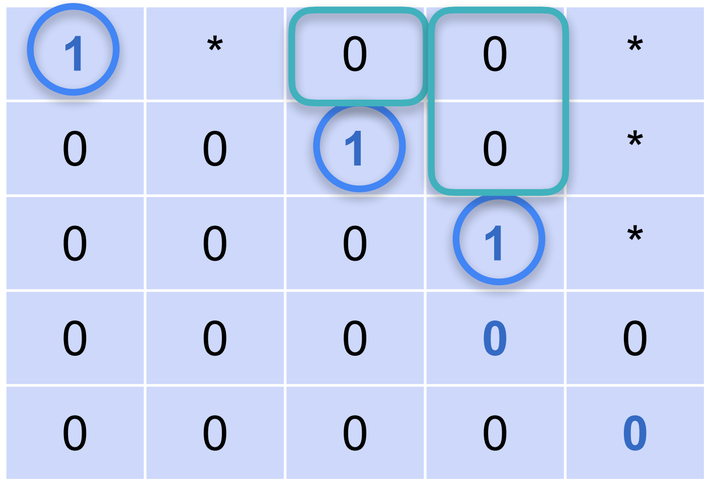
Reduced row echelon form



Reduced row echelon form



Rank 5



Rank 3

- Is in row echelon form
- Each pivot is a 1
- Any number above a pivot is 0
- Rank of the matrix is the number of pivots

Reduced row echelon form

Row echelon form

3	*	*	*	*
0	0	2	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

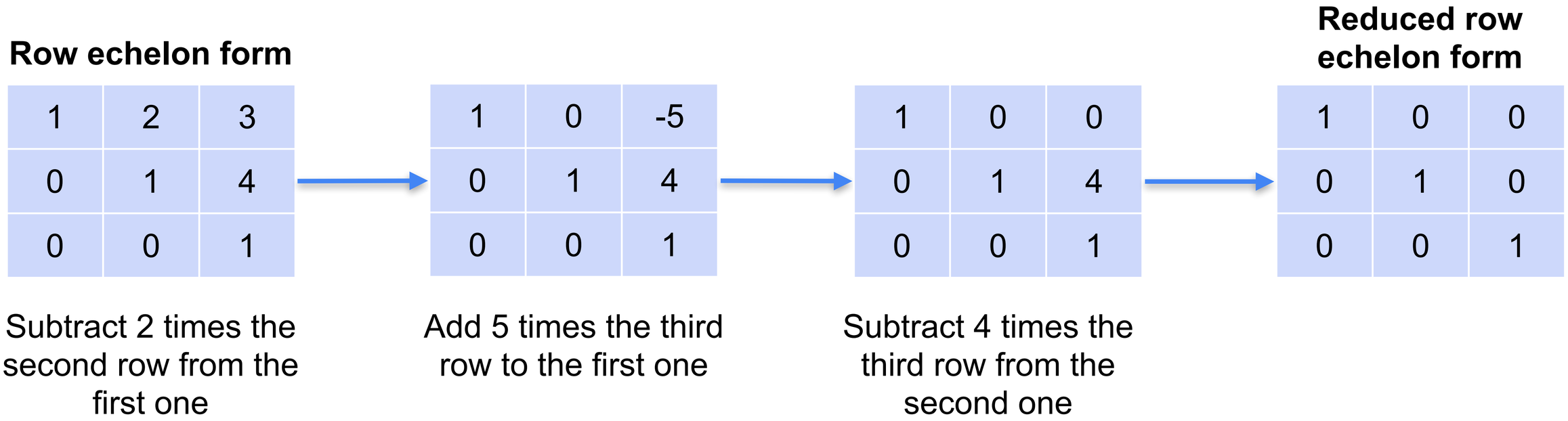
Divide each row by the value of the pivot

Reduced row echelon form

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Turn anything above a pivot to 0

Reduced row echelon form



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Solving System of Linear Equations

The Gaussian Elimination Algorithm

Augmented matrix

$$\begin{aligned} 2a - b + c &= 0 \\ 2a + 2b + 4c &= 0 \\ 4a + b &= 0 \end{aligned}$$

Augmented matrix

2	-1	1
2	2	4
4	1	0

Proceed with the elimination method

Augmented matrix

$$2a - b + c = 1$$

$$2a + 2b + 4c = -2$$

$$4a + b = -1$$

$$\begin{array}{c|cccc} & & & & \\ R_1 & 2 & -1 & 1 & 1 \\ R_2 & 2 & 2 & 4 & -2 \\ R_3 & 4 & 1 & 0 & -1 \end{array}$$

Pivoting

$$2a - b + c = 1$$
$$2a + 2b + 4c = -2$$
$$4a + b = -1$$

R_1	2	-1	1	1
R_2	2	2	4	-2
R_3	4	1	0	-1

$$R_1 \leftarrow \frac{1}{2} R_1$$

$$R_1 \leftarrow \frac{1}{2}$$

$=$

1	-1/2	1/2	1/2

Pivoting

$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}1$

$2a + 2b + 4c = -2$

$4a + b = -1$

R_1 | 1 | -1/2 | 1/2 | 1/2 |

R_2 | 2 | 2 | 4 | -2 |

R_3 | 4 | 1 | 0 | -1 |

$R_2 \leftarrow R_2 - 2R_1$

	2	2	4	-2
-2	1	-1/2	1/2	1/2
$R_2 \leftarrow$	0	3	3	-3

$R_3 \leftarrow R_3 - 4R_1$

	4	1	0	-1
-4	1	-1/2	1/2	1/2
$R_3 \leftarrow$	0	3	-2	2

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$3b + 3c = -3$$

$$3b - 2c = 2$$

R_1	1	-1/2	1/2	1/2
R_2	0	3	3	-3
R_3	0	3	-2	2

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$3b + 3c = -3$$

$$3b - 2c = 2$$

R_1	1	-1/2	1/2	1/2
R_2	0	3	3	-3
R_3	0	3	-2	2

$$R_2 \leftarrow \frac{1}{3}R_2$$

$$R_2 \leftarrow \frac{1}{3}$$

0	3	3	-3	=	0	1	1	-1

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$3b - 2c = 2$$

R_1	1	-1/2	1/2	1/2
R_2	0	1	1	-1
R_3	0	3	-2	2

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$3b - 2c = 2$$

R_1	1	-1/2	1/2	1/2
R_2	0	1	1	-1
R_3	0	3	-2	2

$$R_3 \leftarrow R_3 - 3R_2$$

	0	3	-2	2
-3	0	1	1	-1
$R_3 \leftarrow$	0	0	-5	5

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$-5c = 5$$

R_1 | 1 | -1/2 | 1/2 | 1/2 |

R_2 | 0 | 1 | 1 | -1 |

R_3 | 0 | 0 | -5 | 5 |

$$R_3 \leftarrow -\frac{1}{5}R_3$$

$$R_3 \leftarrow -\frac{1}{5}$$

0	0	-5	5	=	0	0	1	-1

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$c = -1$$

R_1	1	-1/2	1/2	1/2
R_2	0	1	1	-1
R_3	0	0	1	-1

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$c = -1$$

R_1	1	-1/2	1/2	1/2
R_2	0	1	1	-1
R_3	0	0	1	-1

Pivoting

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$c = -1$$

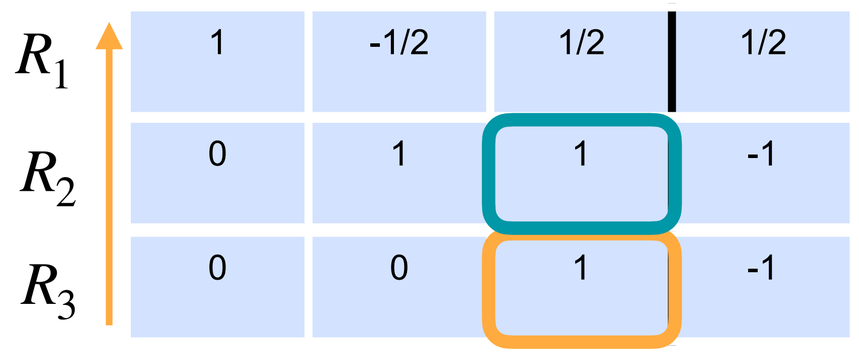
R_1	1	-1/2	1/2	1/2
R_2	0	1	1	-1
R_3	0	0	1	-1

Back substitution

$$a - \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}$$

$$b + c = -1$$

$$c = -1$$



$$R_2 \leftarrow R_2 - R_3$$

	0	1	1	-1
−	0	0	1	-1
$R_2 \leftarrow$	0	1	0	0

$$R_1 \leftarrow R_1 - \frac{1}{2}R_3$$

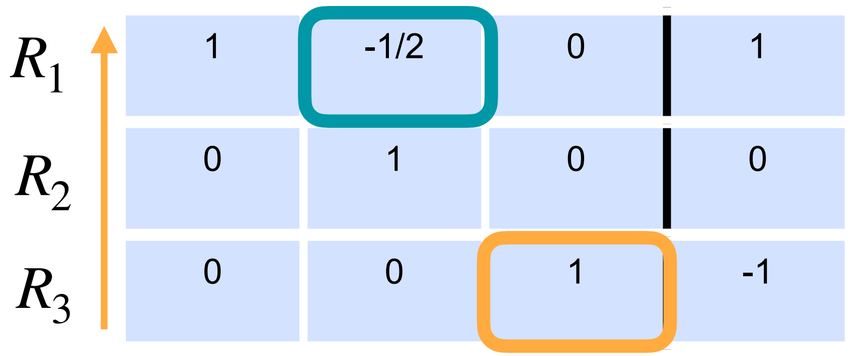
	0	0	1	-1
$-\frac{1}{2}$	1	-1/2	1/2	1/2
$R_1 \leftarrow$	1	-1/2	0	1

Back substitution

$$a - \frac{1}{2}b = 1$$

$$b = 0 \quad -1$$

$$c = -1$$



$$R_1 \leftarrow R_1 + \frac{1}{2}R_2$$

$$+\frac{1}{2}$$

$$R_1 \leftarrow$$

1	0	0	1

The result

a = 1

b = 0

c = -1

R_1 | 1 | 0 | 0 | 1

R_2 | 0 | 1 | 0 | 0

R_3 | 0 | 0 | 1 | -1

Solution to the system

Identity matrix

What if the system is singular?

1	2	-1	5
2	4	5	1
3	6	4	6

After row reduction…

1	2	-1	5
0	0	-7	9
0	0	0	0

There is no need to worry!

Checking if it has infinitely many or no solutions

1	2	-1	5
2	4	5	1
3	6	4	6

After row reduction…

Look at the column of constants

1	2	-1	5
0	0	-7	9
0	0	0	0

$0a + 0b + 0c = 0$

Infinitely many solutions

Checking if it has infinitely many or no solutions

Look at the column of constants

1	2	-1	5
2	4	5	1
3	6	4	10

After row reduction…

1	2	-1	5
0	0	-7	9
0	0	0	0

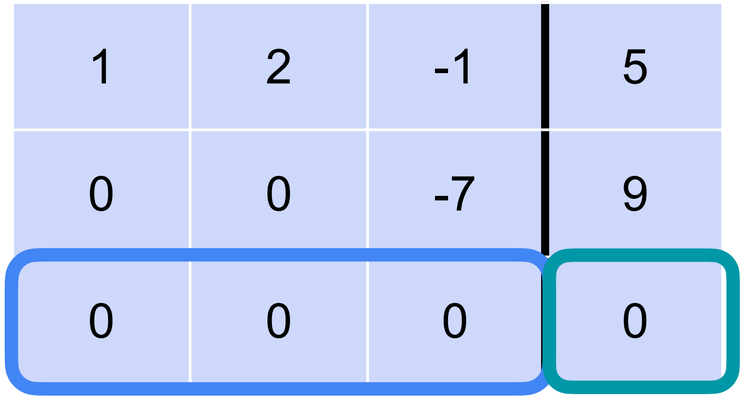
4

$0a + 0b + 0c = 4$

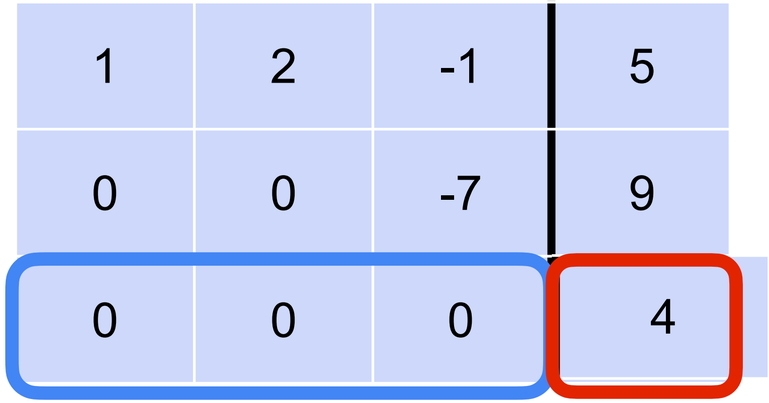
The system has no solutions

Checking if it has infinitely many or no solutions

- Row full of zeroes in row echelon form
- Constant in that row is zero
- **Infinitely many solutions**



- Row full of zeroes in row echelon form
- Constant in that row is not zero
- **No solutions**



Gaussian Elimination - Summary

1. Create the augmented matrix
2. Get the matrix into reduced row echelon form
3. Complete back substitution
4. Stop if you encounter a row of 0s

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Solving System of Linear Equations

Conclusion