Problem Set 2

Due Sep. 19, 2024

Problem 1 (Cascaded measurement)**.** Suppose $\{L_\ell\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{L_\ell\}$ followed by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{\ell m}\}$ with $N_{\ell m} \equiv M_m L_\ell$. **(10%)**

Problem 2 (Measurements)**.** Calculate the outcome *probabilities* and *corresponding post measurement states* of the following measurement settings. Please write clearly for which measurement you do and its corresponding outcome. **(60%, for each 15%)**

(a) Measure $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ against the measurement $I \otimes \sigma_z$.

(b) Measure $|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ against the orthonormal basis $\left\{ |v_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, |v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$.

(c) For the state $|\psi_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$, measure the observable $A = \begin{pmatrix} 2 & 0 & -i \\ 0 & 3 & 0 \\ i & 0 & 2 \end{pmatrix}$.

(d) Measure $|\psi_3\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with measurement $\left\{ M_0 = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}, M_1 = \frac{1}{10}\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \right\}$.

Problem 3 (Pauli measurements)**.** For any $\vec{v}$ that is a 3-dimensional unit vector, we can define an observable $\vec{v} \cdot \vec{\sigma} := v_1\sigma_x + v_2\sigma_y + v_3\sigma_z$

(a) Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_\pm = (\mathcal{I} \pm \vec{v} \cdot \vec{\sigma})/2$. **(10%)**

(b) The eigenvector of $\vec{v} \cdot \vec{\sigma}$ with respect to the eigenvalue $+1$ can be represented as $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ in the form of the polar coordinate system. Find the eigenvector with respect to -1 in the form of the polar coordinate system. Show the corresponding Cartesian coordinates, i.e., the value of v_1, v_2 and v_3, in terms of θ and ϕ. **(10%)**

(c) Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v} \cdot \vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained? **(10%)**