Problem Set 1

Due Sep 12, 2024

Problem 1. (Dirac notation) Recall that the computational basis vectors are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

and the plus and minus states are

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Also define the following states

$$|y_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad\qquad |y_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Calculate the following expressions in *matrix notation*. **(20%, for each 10%)**

(a) $|-\rangle\langle y_+|$

(b) $\langle y_-|+\rangle$

Problem 2. (Pauli matrices) Recall that the Hadamard and the Pauli matrices are

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Calculate the eigenvalues and normalized eigenvectors for the Hadamard matrix and all three Pauli matrices.**(40%, for each 10%)**

Problem 3. (Tensor product) Calculate the following expressions, and write the results in *matrix notation*: **(30%, for each 10%)**

(a) $X \otimes Z$

(b) $(\langle -| \otimes \langle +|)(|1\rangle \otimes |0\rangle)$

(c) $(\langle y_-| \otimes \langle -|)(\sigma_y \otimes \sigma_y)(|+\rangle \otimes |y_-\rangle)$

Problem 4. (Unitary operator) Recall that a Unitary matrix U satisfy $U^\dagger = U^{-1}$. Prove that all eigenvalues of a Unitary matrix satisfy $\|\lambda\|^2 = 1$. (Hint: consider $\langle\psi| U^\dagger U |\psi\rangle$ and $(\langle\psi| U^\dagger)(U|\psi\rangle)$ and $U|\psi\rangle = \lambda|\psi\rangle$, where λ is an eigenvalue and $|\psi\rangle$ is an eigenvector of U)**(10%)**