HW4

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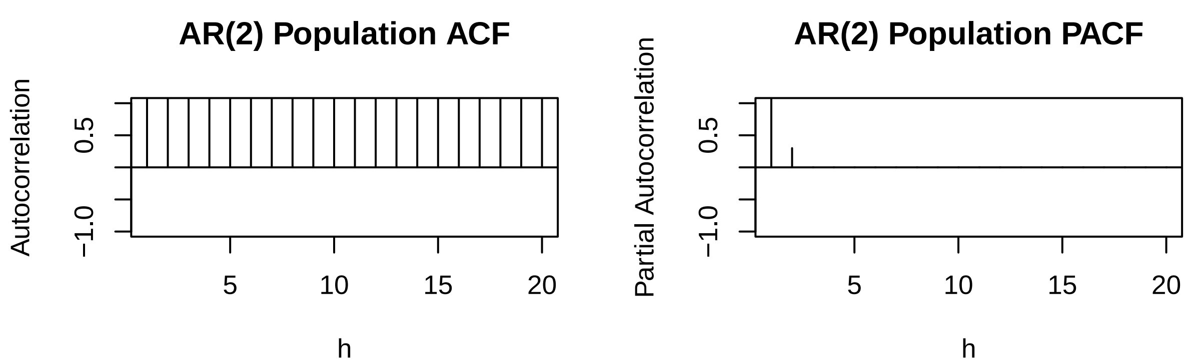
2023-03-15

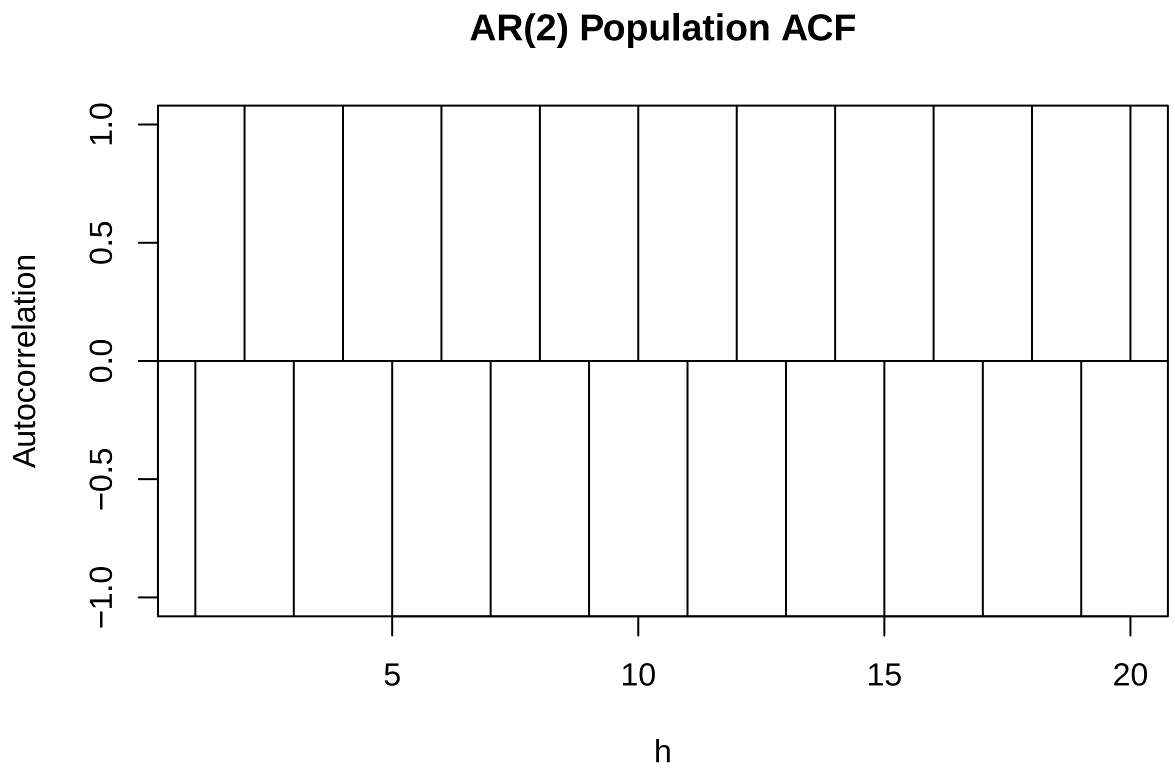
#1 ##(i)

```
par(mfrow=c(2,2))
y = ARMAacf(ar = c(1.1,0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popul
abline(h = 0)
y = ARMAacf(ar = c(1.1, 0.3), lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h",
ylab = "Partial Autocorrelation", main = "AR(2) Population PACF")
abline(h = 0)
```

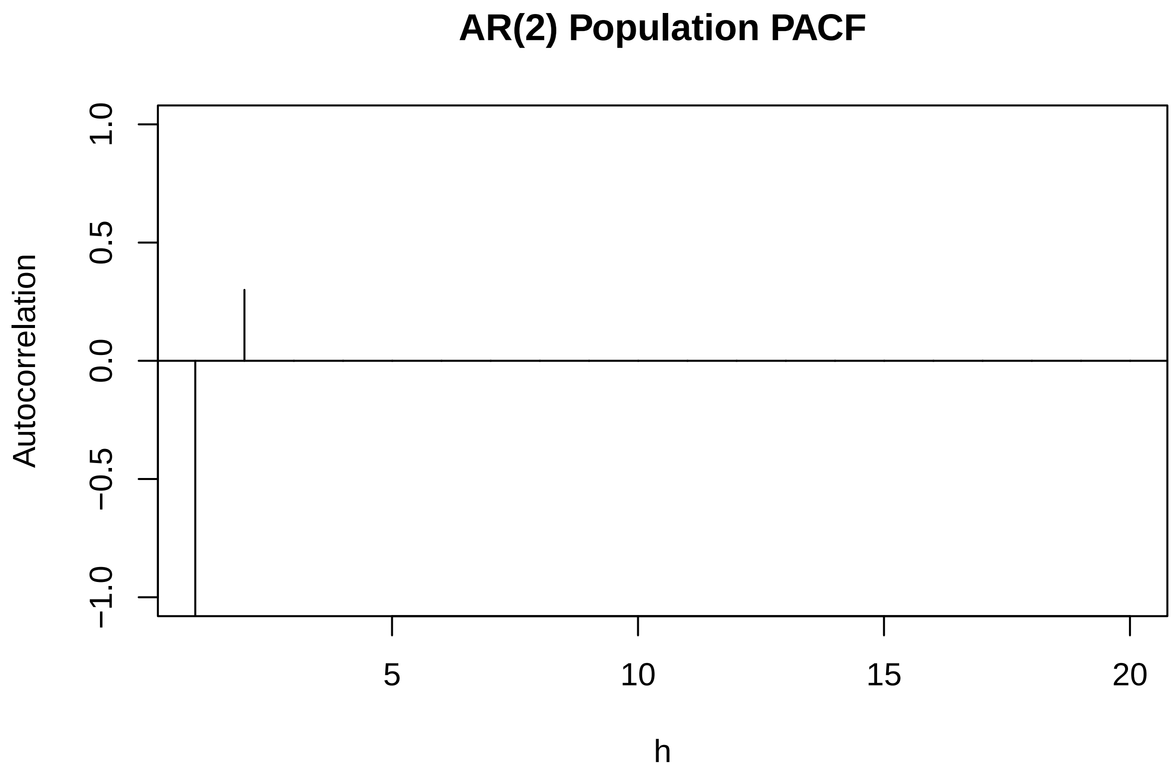


##(ii)

```
y = ARMAacf(ar = c(-1.1,0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
```

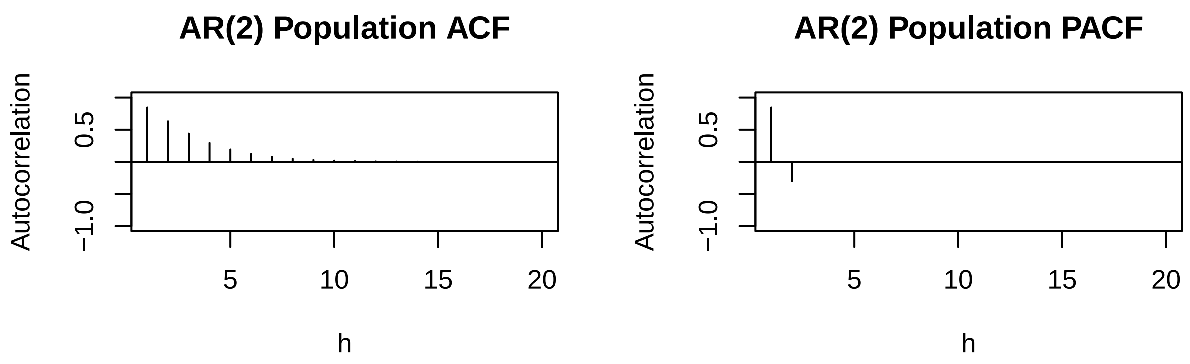



```
y = ARMAacf(ar = c(-1.1,0.3), lag.max = 20, pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial
Autocorrelation", main = "AR(2) Population PACF")
abline(h = 0)
```



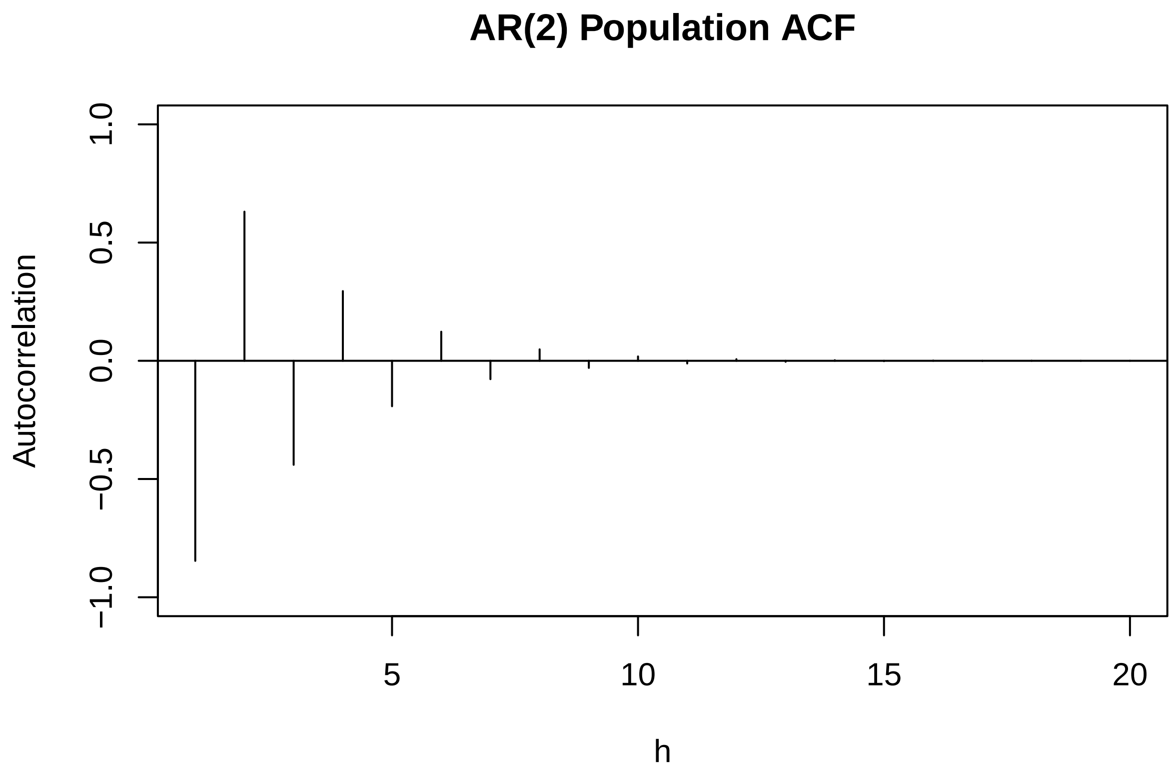
##(iii)

```
par(mfrow=c(2,2))
y = ARMAacf(ar = c(1.1,-0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
y = ARMAacf(ar = c(1.1,-0.3), lag.max = 20, pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial
Autocorrelation", main = "AR(2) Population PACF")
abline(h = 0)
```

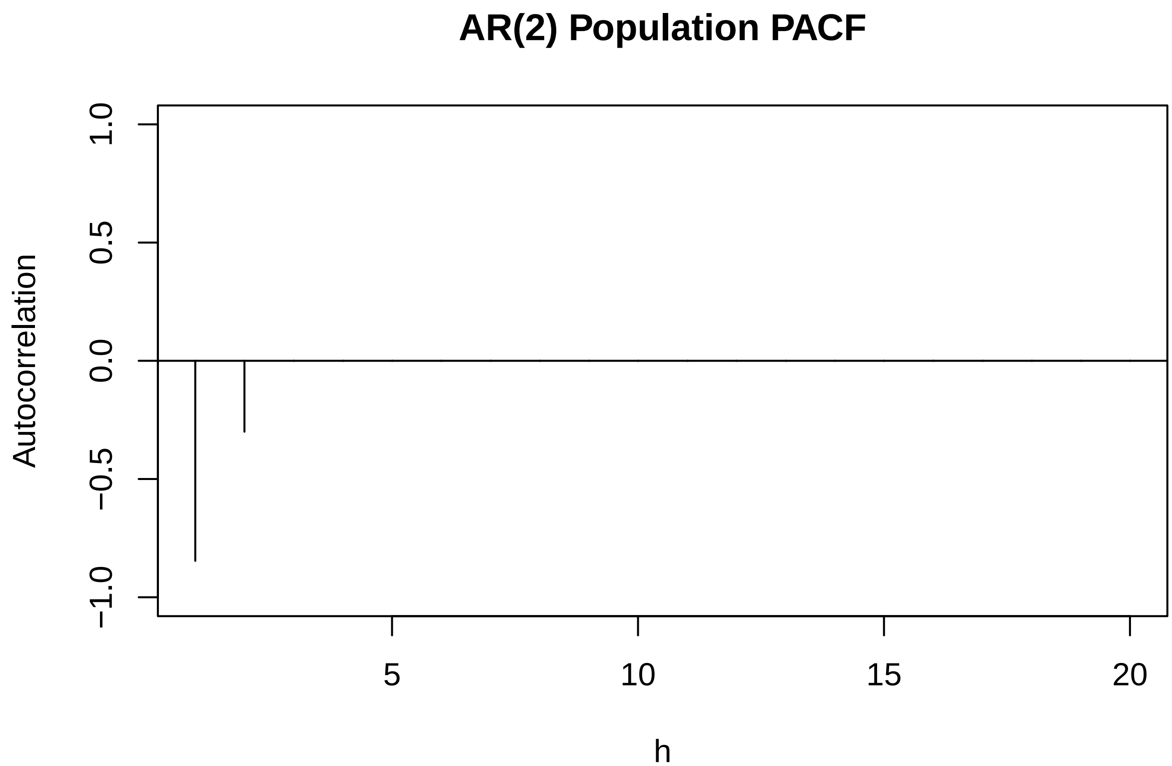


##(iv)

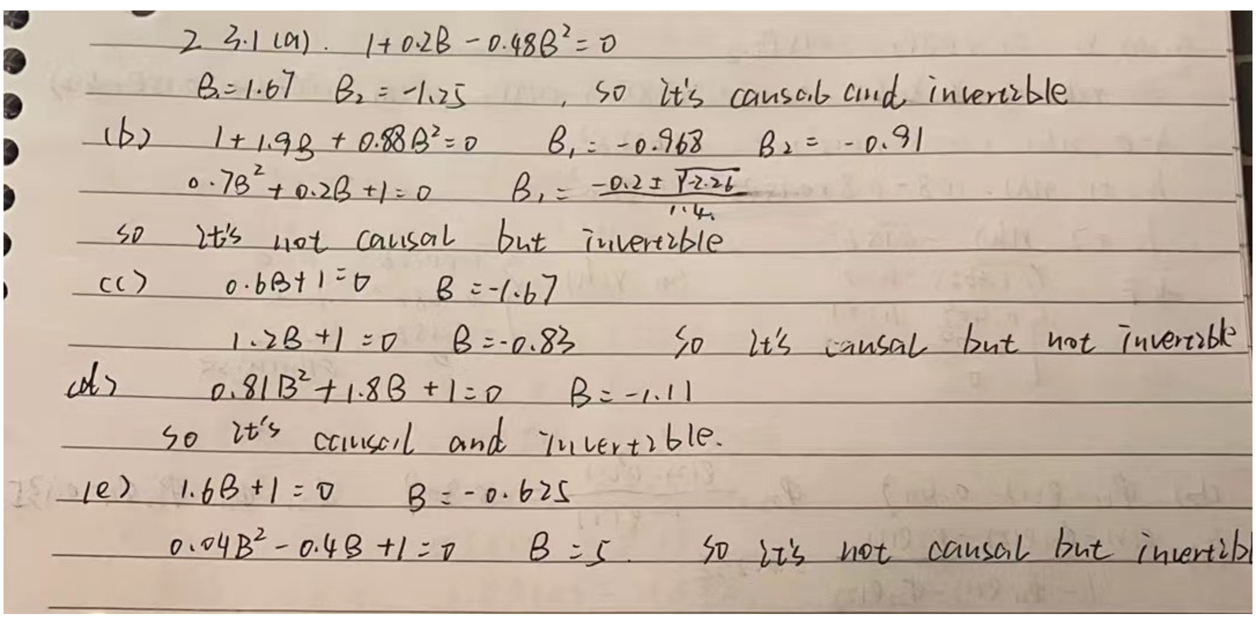
```
y = ARMAacf(ar = c(-1.1,-0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
```




```
y = ARMAacf(ar = c(-1.1,-0.3), lag.max = 20, pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial
Autocorrelation", main = "AR(2) Population PACF")
abline(h = 0)
```



#2 ##3.1



2. 3.1 (a). $1+0.2B-0.48B^2=0$

$B_1=1.67$ $B_2=-1.25$, so it's causal and invertible

(b) $1+1.9B+0.88B^2=0$ $B_1=-0.968$ $B_2=-0.91$

$0.7B^2+0.2B+1=0$ $B_1=\frac{-0.2\pm\sqrt{-2.26}}{1.4}$

so it's not causal but invertible

(c) $0.6B+1=0$ $B=-1.67$

$1.2B+1=0$ $B=-0.83$ so it's causal but not invertible

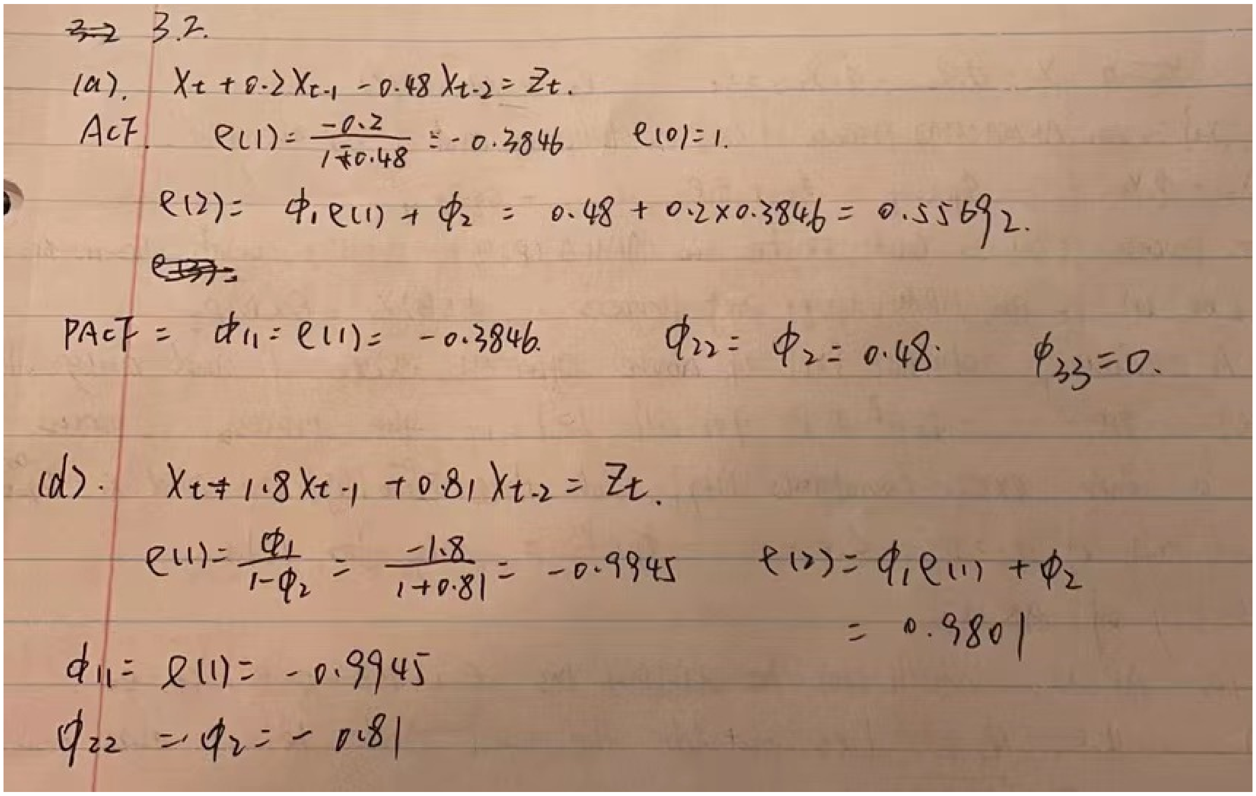
(d) $0.81B^2+1.8B+1=0$ $B=-1.11$

so it's causal and invertible.

(e) $1.6B+1=0$ $B=-0.625$

$0.04B^2-0.4B+1=0$ $B=5$. so it's not causal but invertibl

Figure 1: Problem 2 (3.1)



~~3.2~~ 3.2.

(a). $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$.

ACF $\rho(1) = \frac{-0.2}{1\mp 0.48} = -0.3846$ $\rho(0) = 1$.

$\rho(2) = \phi_1\rho(1) + \phi_2 = 0.48 + 0.2\times 0.3846 = 0.55692$.

~~ρ(3)=~~

PACF = $\phi_{11} = \rho(1) = -0.3846$. $\phi_{22} = \phi_2 = 0.48$. $\phi_{33} = 0$.

(d). $X_t \mp 1.8X_{t-1} + 0.81X_{t-2} = Z_t$.

$\rho(1) = \frac{\phi_1}{1-\phi_2} = \frac{-1.8}{1+0.81} = -0.9945$ $\rho(2) = \phi_1\rho(1) + \phi_2 = 0.9801$

$\phi_{11} = \rho(1) = -0.9945$

$\phi_{22} = \phi_2 = -0.81$

Figure 2: Problem 2 (3.2)

3.2(a)

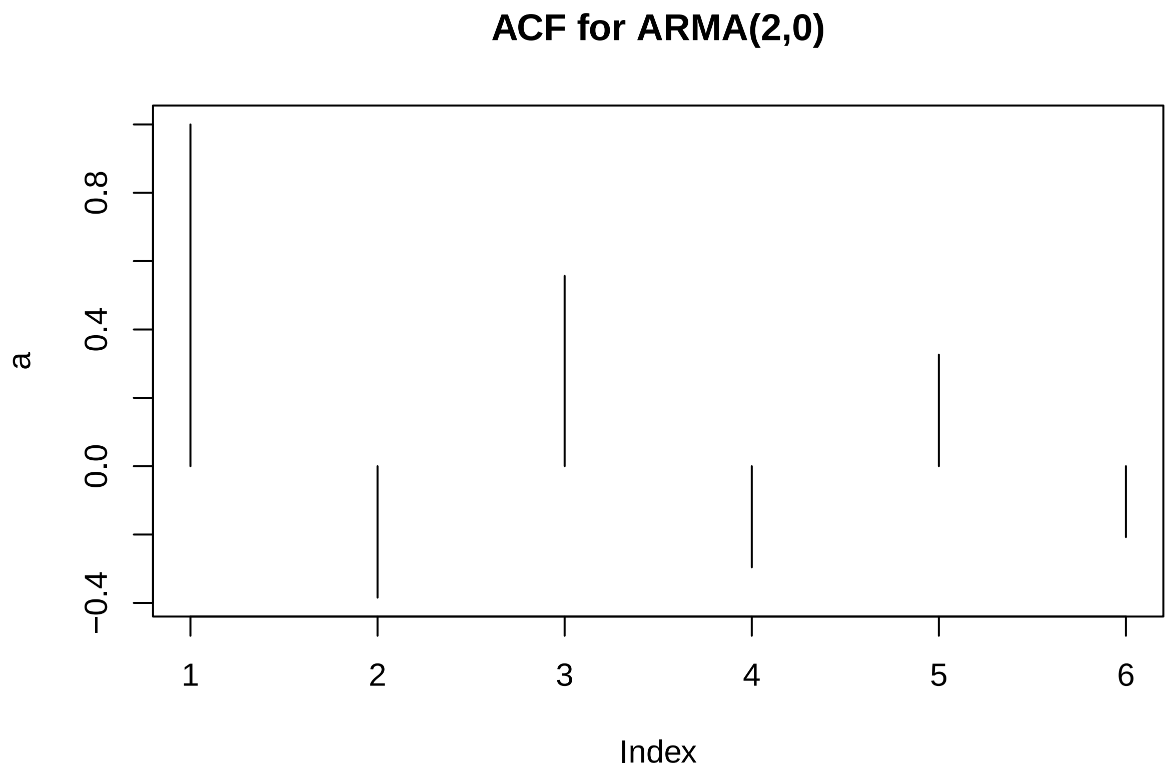
```
a <- ARMAacf(ar =c(-0.2,0.48), ma = 0, lag.max = 5)
b <- ARMAacf(ar =c(-0.2,0.48), ma = 0, lag.max = 5,pacf=T)
print(a)
```

```
##         0          1          2          3          4          5
##  1.0000000 -0.3846154  0.5569231 -0.2960000  0.3265231 -0.2073846
```

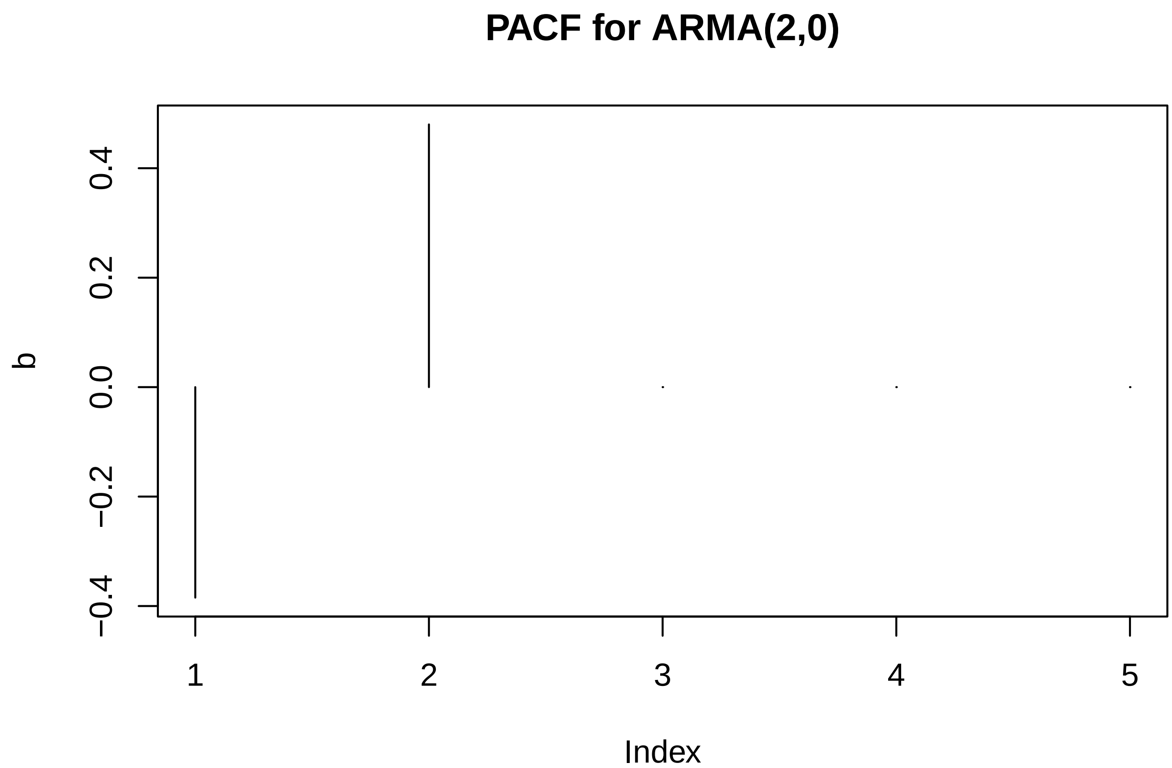
```
print(b)
```

```
## [1] -3.846154e-01  4.800000e-01  0.000000e+00  8.465242e-17  7.759805e-17
```

```
plot(a, type='h', main='ACF for ARMA(2,0)')
```




```
plot(b, type='h', main='PACF for ARMA(2,0)')
```



##3.2(d)

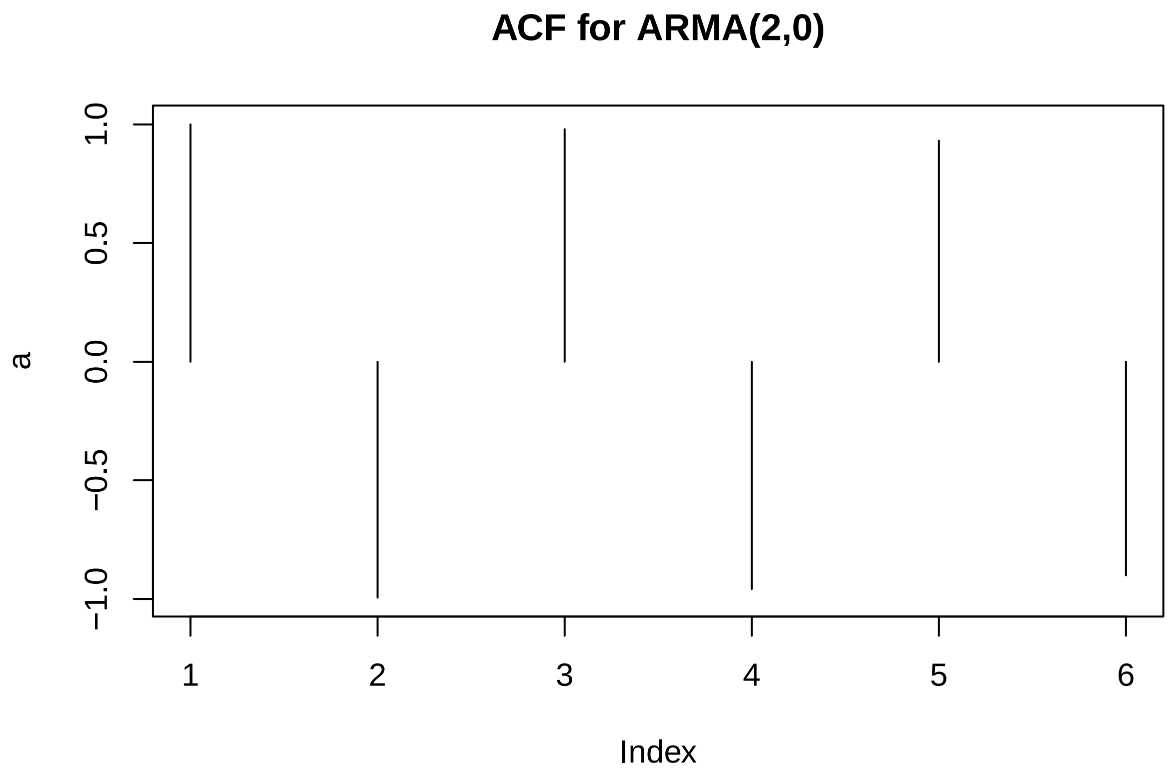
```
a <- ARMAacf(ar =c(-1.8,-0.81), ma = 0, lag.max = 5)
b <- ARMAacf(ar =c(-1.8,-0.81), ma = 0, lag.max = 5,pacf=T)
print(a)
```

```
##           0          1          2          3          4          5
##   1.0000000 -0.9944751  0.9800552 -0.9585746  0.9315895 -0.9004157
```

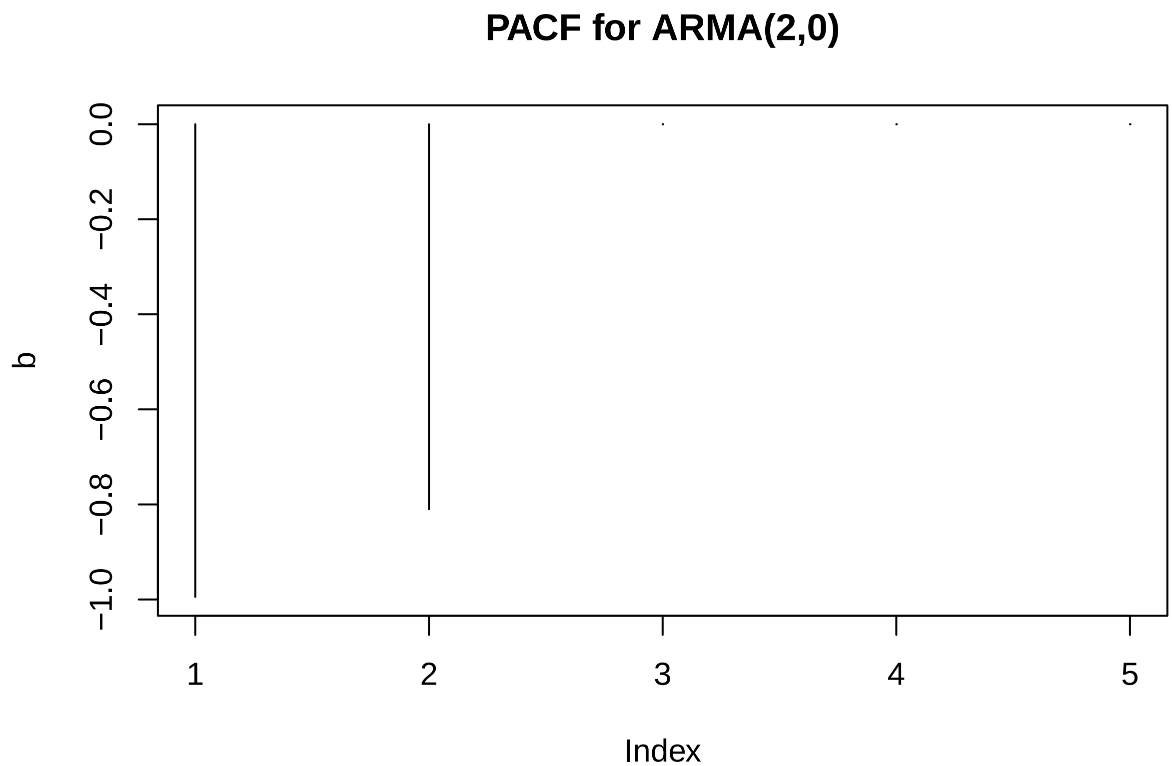
```
print(b)
```

```
## [1] -9.944751e-01 -8.100000e-01 -2.050813e-13 -8.796459e-14  3.370792e-14
```

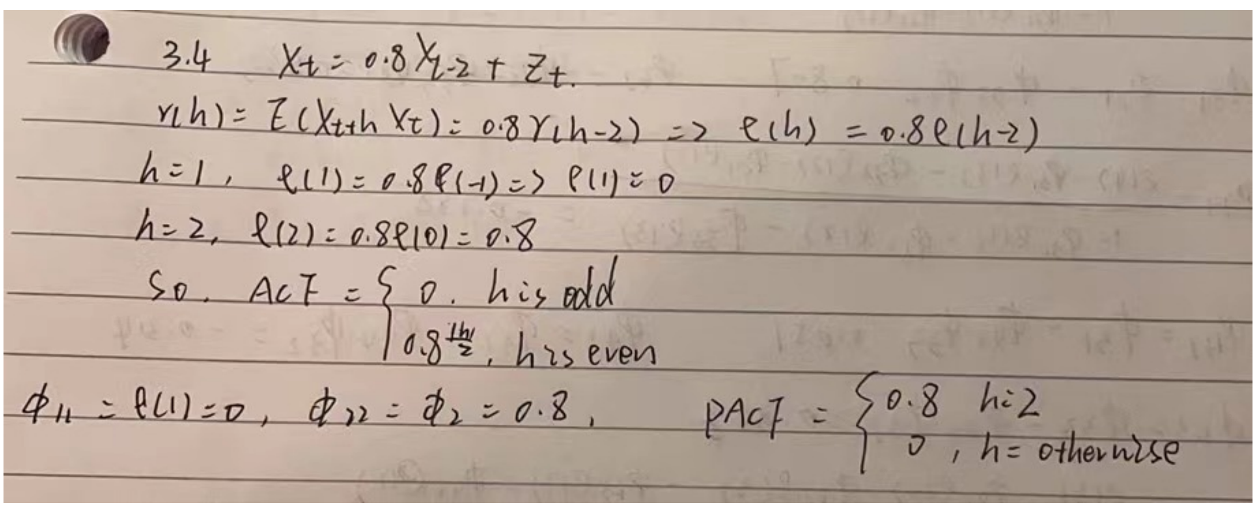
```
plot(a, type='h', main='ACF for ARMA(2,0)')
```




```
plot(b, type='h', main='PACF for ARMA(2,0)')
```



##3.4



3.4 $X_t = 0.8 X_{t-2} + Z_t$.

$\gamma(h) = E(X_{t+h} X_t) = 0.8\gamma(h-2) \Rightarrow \rho(h) = 0.8\rho(h-2)$

$h=1$, $\rho(1) = 0.8\rho(-1) \Rightarrow \rho(1) = 0$

$h=2$, $\rho(2) = 0.8\rho(0) = 0.8$

So. $ACF = \begin{cases} 0. & h \text{ is odd} \\ 0.8^{\frac{|h|}{2}}, & h \text{ is even} \end{cases}$

$\phi_{11} = \rho(1) = 0$, $\phi_{22} = \phi_2 = 0.8$, $PACF = \begin{cases} 0.8 & h=2 \\ 0, & h = \text{otherwise} \end{cases}$

Figure 3: Problem 2 (3.4)

#3(a)

#3(b)

```
ARMAtoMA(ar = c(1.1,-0.3), ma = 0, lag.max = 5)
```

```
## [1] 1.10000 0.91000 0.67100 0.46510 0.31031
```

3. ~~X_t~~ a. $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = e_t$ $\qquad e_t \sim WN(0, \sigma^2)$

$\{X_t\}$ is an ARMA(p,q) process if $\{X_t\}$ is stationary and if for every t.

$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}$.

the process $\{X_t\}$ is said to be an ARMA(p,q) process with mean u. if $\{X_t - u\}$ is an ARMA(p,q) ~~prog~~ process, $\phi(B) X_t = \theta(B) e_t$.

A stationary solution $\{X_t\}$ of above equation exists if and only if $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$ for all $|z| = 1$. the process is causal. that is there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$ if and only if $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0$ only for $|z| > 1$.

causality of AR(2).

For AR(2), which can be written as $(1 - \phi_1 B - \phi_2 B^2) X_t = e_t$, $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ lies outside the unit circle $|z| = 1$. this can be written as $\left| \dfrac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2} \right| > 1$. the causitivity condition for AR(p) and ARMA(p,q) are same, where, $X_t = \dfrac{\theta(B)}{\phi(B)} e_t = \psi(B) e_t$.

that is $\phi(B)\psi(B) = \theta(B)$ in terms of polynomials

$(1 - \phi_1 z - \cdots - \phi_p z^p)(\psi_0 + \psi_1 z + \cdots) = 1 + \theta_1 z + \cdots + \theta_q z^q$

equalizing the coefficients of z^j, $j = 0, 1, \ldots$ we obtain $1 = \psi_0$

$\theta_1 = \psi_1 - \psi_0 \phi_1 \quad \theta_2 = \psi_2 - \psi_1 \phi_1 - \psi_0 \phi_2 \quad \theta_j = \psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k}$

Therefore $\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$

Figure 4: Problem 3 (a)

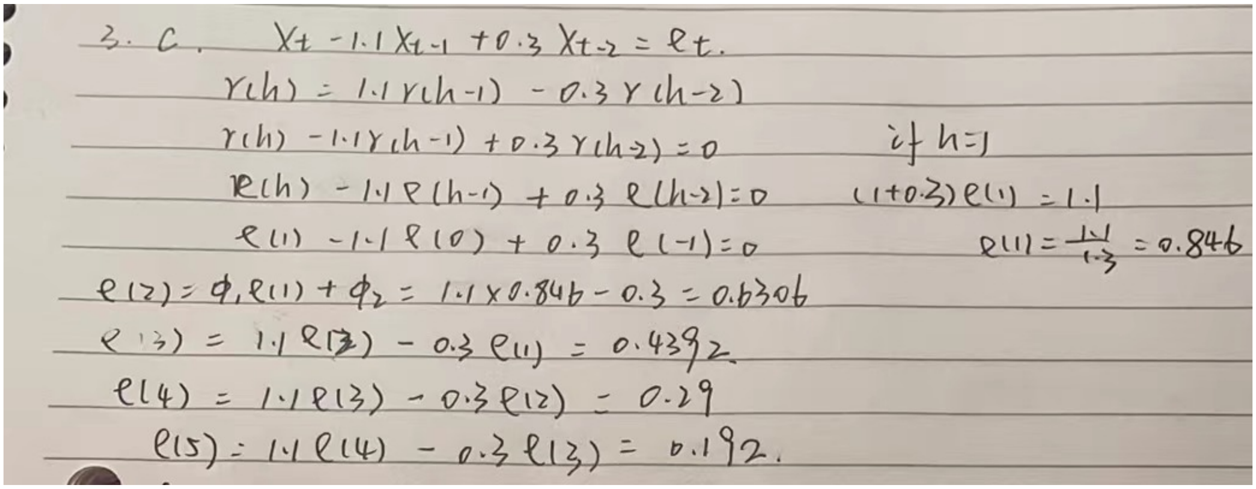
3 b. $\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2} \qquad \psi_1 = \phi_1 \psi_0 + \phi_2 \psi_{-1}$

$\psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 \qquad \psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1$

$\psi_4 = \phi_1 \psi_3 + \phi_2 \psi_4 \qquad \psi_5 = \phi_1 \psi_4 + \phi_2 \psi_3$

Figure 5: Problem 3(b)

#3(c)



3\. c. $X_t - 1.1X_{t-1} + 0.3X_{t-2} = e_t$.

$\gamma(h) = 1.1\gamma(h-1) - 0.3\gamma(h-2)$

$\gamma(h) - 1.1\gamma(h-1) + 0.3\gamma(h-2) = 0$ if $h=1$

$\rho(h) - 1.1\rho(h-1) + 0.3\rho(h-2) = 0$ $(1+0.3)\rho(1) = 1.1$

$\rho(1) - 1.1\rho(0) + 0.3\rho(-1) = 0$ $\rho(1) = \frac{1.1}{1.3} = 0.846$

$\rho(2) = \phi_1\rho(1) + \phi_2 = 1.1\times 0.846 - 0.3 = 0.6306$

$\rho(3) = 1.1\rho(2) - 0.3\rho(1) = 0.4392$

$\rho(4) = 1.1\rho(3) - 0.3\rho(2) = 0.29$

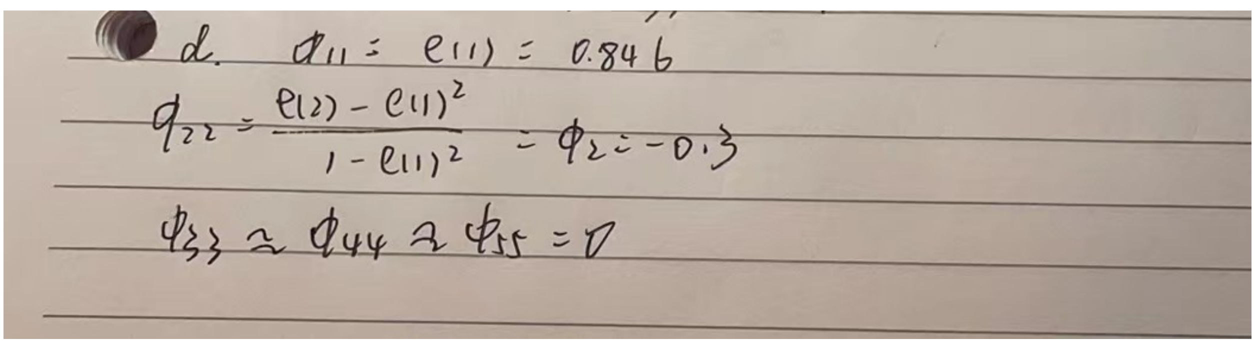
$\rho(5) = 1.1\rho(4) - 0.3\rho(3) = 0.192$.

Figure 6: Problem 3(c)

```
ARMAacf(ar =c(1.1,-0.3), ma = 0, lag.max = 5)
```

```
##         0         1         2         3         4         5
## 1.0000000 0.8461538 0.6307692 0.4400000 0.2947692 0.1922462
```

#3(d)



d. $\phi_{11} = \rho(1) = 0.846$

$\phi_{22} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = \phi_2 = -0.3$

$\phi_{33} = \phi_{44} = \phi_{55} = 0$

Figure 7: Problem 3 (d)

```
ARMAacf(ar =c(1.1,-0.3),ma=0, lag.max = 5,pacf=T)
```

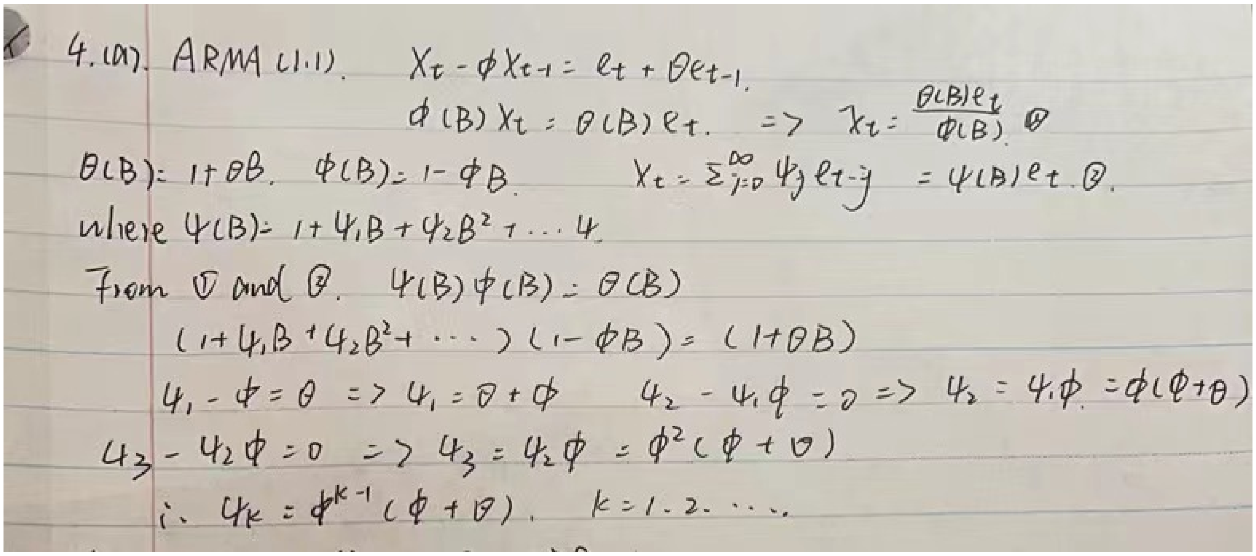
```
## [1]  8.461538e-01 -3.000000e-01  4.295506e-16 -2.250027e-16  5.077171e-17
```

#4

```
ARMAtoMA(ar =0.6, ma = -0.2, lag.max = 5)
```

```
## [1] 0.40000 0.24000 0.14400 0.08640 0.05184
```

#5(a)



4.(a). ARMA(1,1). $X_t - \phi X_{t-1} = e_t + \theta e_{t-1}$.

$\phi(B) X_t = \theta(B) e_t$. $\Rightarrow X_t = \frac{\theta(B) e_t}{\phi(B)}$ ①

$\theta(B) = 1 + \theta B$. $\phi(B) = 1 - \phi B$. $X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = \psi(B) e_t$ ②.

where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \dots \psi$

From ① and ②. $\psi(B)\phi(B) = \theta(B)$

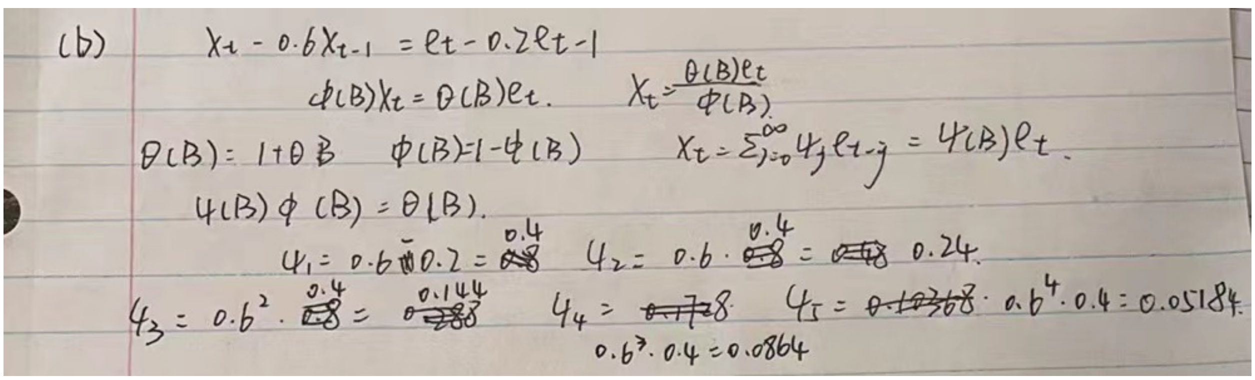
$(1 + \psi_1 B + \psi_2 B^2 + \cdots)(1 - \phi B) = (1 + \theta B)$

$\psi_1 - \phi = \theta \Rightarrow \psi_1 = \theta + \phi$ $\quad \psi_2 - \psi_1 \phi = 0 \Rightarrow \psi_2 = \psi_1 \phi = \phi(\phi + \theta)$

$\psi_3 - \psi_2 \phi = 0 \Rightarrow \psi_3 = \psi_2 \phi = \phi^2(\phi + \theta)$

$\therefore \psi_k = \phi^{k-1}(\phi + \theta)$, $k = 1, 2, \dots$

Figure 8: Problem 4 (a)



(b) $X_t - 0.6 X_{t-1} = e_t - 0.2 e_{t-1}$

$\phi(B) X_t = \theta(B) e_t$. $X_t = \frac{\theta(B) e_t}{\phi(B)}$

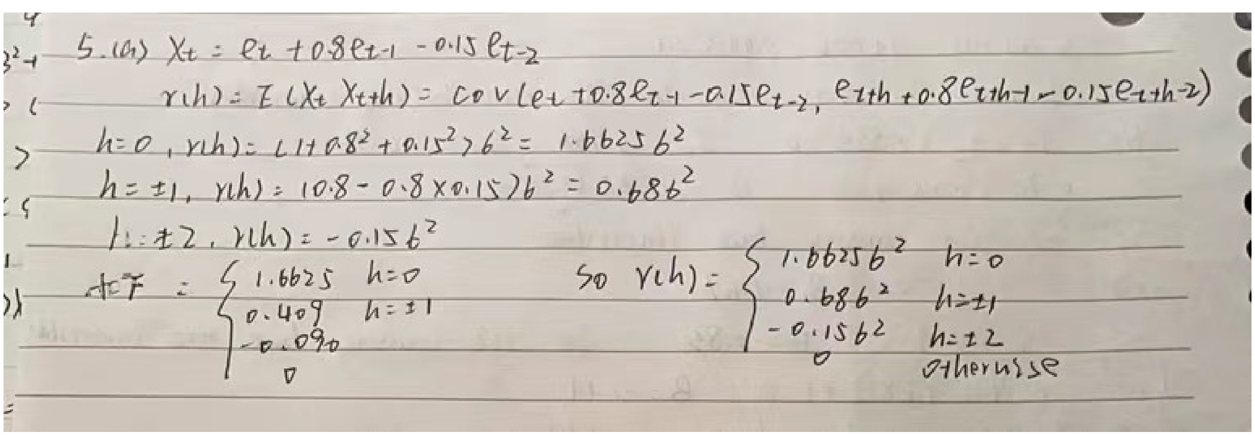
$\theta(B) = 1 + \theta B$ $\quad \phi(B) = 1 - \psi(B)$ $\quad X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = \psi(B) e_t$.

$\psi(B)\phi(B) = \theta(B)$.

$\psi_1 = 0.6 - 0.2 = \cancel{0.8}\ 0.4$ $\quad \psi_2 = 0.6 \cdot \cancel{0.8}\ 0.4 = \cancel{0.48}\ 0.24$.

$\psi_3 = 0.6^2 \cdot \cancel{0.8}\ 0.4 = \cancel{0.288}\ 0.144$ $\quad \psi_4 = \cancel{0.1728}\ 0.6^3 \cdot 0.4 = 0.0864$ $\quad \psi_5 = \cancel{0.10368} \cdot 0.6^4 \cdot 0.4 = 0.05184$.

Figure 9: Problem 4 (b)



5.(a) $X_t = e_t + 0.8 e_{t-1} - 0.15 e_{t-2}$

$\gamma(h) = E(X_t X_{t+h}) = cov(e_t + 0.8 e_{t-1} - 0.15 e_{t-2},\ e_{t+h} + 0.8 e_{t+h-1} - 0.15 e_{t+h-2})$

$h = 0$, $\gamma(h) = (1 + 0.8^2 + 0.15^2)\sigma^2 = 1.6625\sigma^2$

$h = \pm 1$, $\gamma(h) = (0.8 - 0.8 \times 0.15)\sigma^2 = 0.68\sigma^2$

$h = \pm 2$, $\gamma(h) = -0.15\sigma^2$

[illegible] $= \begin{cases} 1.6625 & h = 0 \\ 0.409 & h = \pm 1 \\ -0.090 & \\ 0 & \end{cases}$

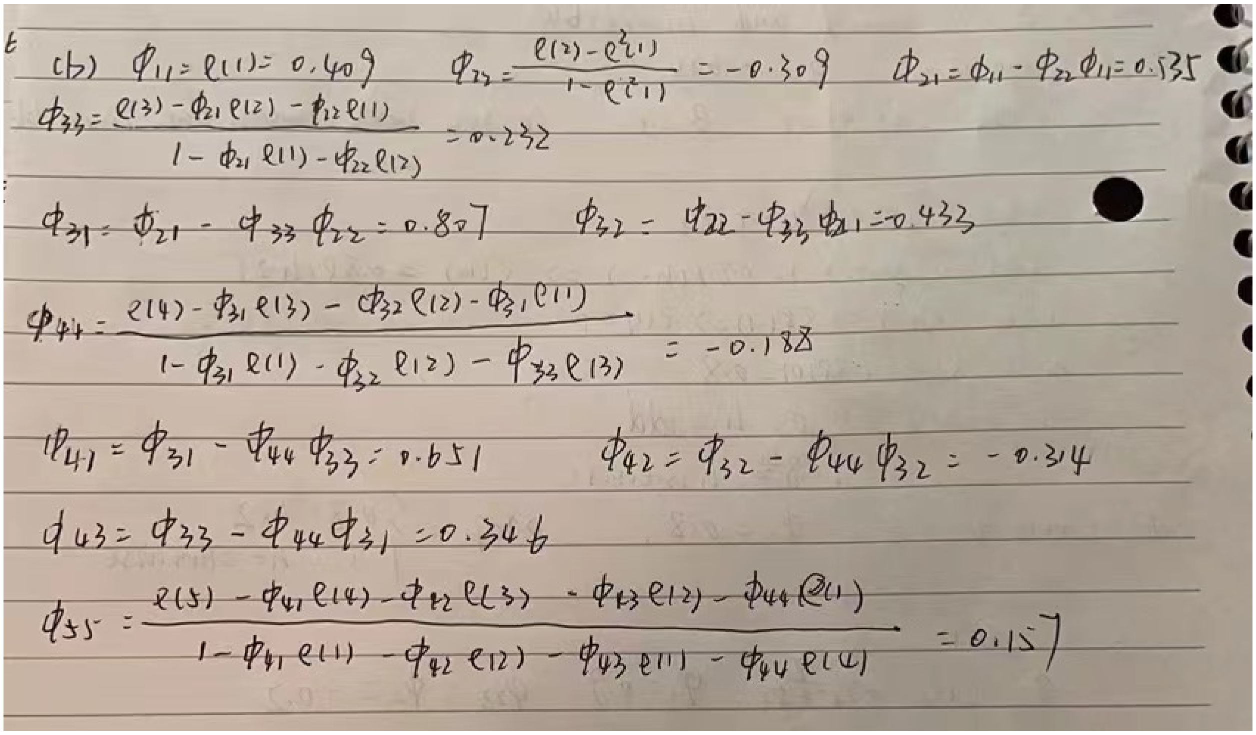
So $\gamma(h) = \begin{cases} 1.6625\sigma^2 & h = 0 \\ 0.68\sigma^2 & h = \pm 1 \\ -0.15\sigma^2 & h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$

Figure 10: Problem 5 (a)

```r
ARMAacf(ar =0, ma =c(0.8,-0.15), lag.max = 5)
```

```
##          0           1           2           3           4           5
##  1.00000000  0.40902256 -0.09022556  0.00000000  0.00000000  0.00000000
```

#5(b)



(b) $\phi_{11} = \rho(1) = 0.409$ $\phi_{22} = \frac{\rho(2)-\rho^2(1)}{1-\rho^2(1)} = -0.309$ $\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = 0.535$

$\phi_{33} = \frac{\rho(3) - \phi_{21}\rho(2) - \phi_{22}\rho(1)}{1 - \phi_{21}\rho(1) - \phi_{22}\rho(2)} = 0.232$

$\phi_{31} = \phi_{21} - \phi_{33}\phi_{22} = 0.807$ $\phi_{32} = \phi_{22} - \phi_{33}\phi_{21} = -0.433$

$\phi_{44} = \frac{\rho(4) - \phi_{31}\rho(3) - \phi_{32}\rho(2) - \phi_{31}\rho(1)}{1 - \phi_{31}\rho(1) - \phi_{32}\rho(2) - \phi_{33}\rho(3)} = -0.188$

$\phi_{41} = \phi_{31} - \phi_{44}\phi_{33} = 0.651$ $\phi_{42} = \phi_{32} - \phi_{44}\phi_{32} = -0.314$

$\phi_{43} = \phi_{33} - \phi_{44}\phi_{31} = 0.346$

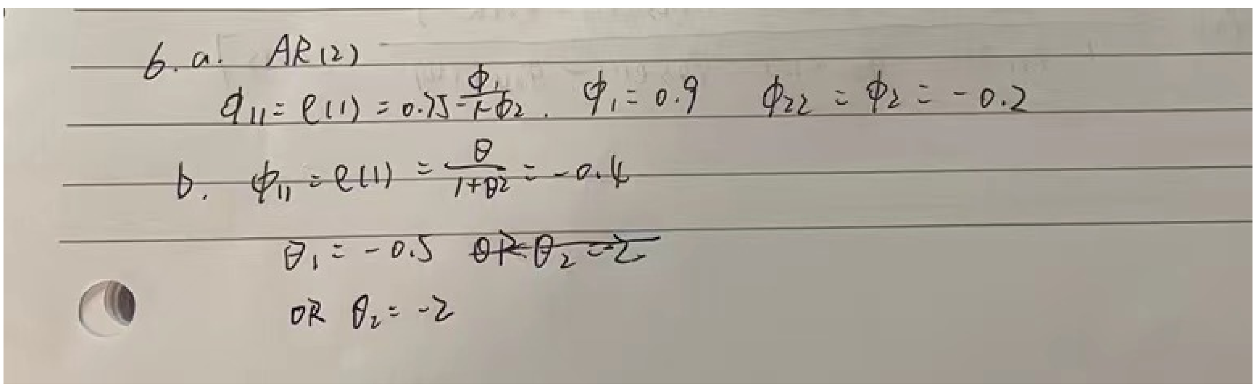
$\phi_{55} = \frac{\rho(5) - \phi_{41}\rho(4) - \phi_{42}\rho(3) - \phi_{43}\rho(2) - \phi_{44}\rho(1)}{1 - \phi_{41}\rho(1) - \phi_{42}\rho(2) - \phi_{43}\rho(1) - \phi_{44}\rho(4)} = 0.157$

Figure 11: Problem 5 (b)

```r
ARMAacf(ar =0, ma =c(0.8,-0.15), lag.max = 5,pacf=T)
```

```
## [1]  0.4090226 -0.3092649  0.2321388 -0.1881747  0.1574740
```

#6



6. a. AR(2)

$\phi_{11} = \rho(1) = 0.75 = \frac{\phi_1}{1-\phi_2}$. $\phi_1 = 0.9$ $\phi_{22} = \phi_2 = -0.2$

b. $\phi_{11} = \rho(1) = \frac{\theta}{1+\theta^2} = -0.4$

$\theta_1 = -0.5$ ~~OR~~ $\cancel{\theta_2 = -2}$

OR $\theta_2 = -2$

Figure 12: Problem 6