A Proof of A Special Case of Generalized Cyclotomic Expansion

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Abstract

Habiro[H] proved that colored Jones polynomial has a special algebraic property called cyclotomic expansion. Later on Chen-Liu-Zhu[CLZ] conjectured that a generalized cyclotomic expansion is true for (reduced) colored $SU(n)$-invariant. Formulas of (reduced) colored $SU(n)$-invariants for torus knots can be easily deduced from the formulas of corresponding colored HOMFLY-PT invariants obtained by Lin-Zheng[LZ]. Based on these previous results, in this paper we prove that for any torus knot $T(2, 2k+1)$, its (reduced) colored $SU(3)$-invariant has certain generalized cyclotomic expansion predicted by Chen-Liu-Zhu[CLZ].

1 INTRODUCTION

The mathematical studies of knots begins with Guass and in the 1860s many people became interested in knot because of Lord Kelvin's theory that atoms were knots in the ether. In the early 20th century mathematicians like Max Dehn, J. W. Alexander study knot from the point of view of the knot group and homology theory. The Alexander polynomial discovered by Alexander is the first knot polynomial. In the late 1970s, William Thurston introduced hyperbolic geometry into the study of knots.

In the 1980s Vaughan Jones[J] discovered Jones polynomial and in 1989 a deep connection between Jones polynomial and quantum field theory was revealed by Witten[W]. Now we know that Jones polynomial is an example of quantum invariants. Briefly speaking quantum invariants are invariants obtained from quantum group. Quantum group first appeared in physics and was then formalized by Vladimir Drinfeld and Michio Jimbo as a particular class of Hopf algebra. Several mathematicians including Reshetikhin and Turaev[RT1][RT2] developed a systematic way of producing quantum invariants from representations of quantum groups. For each colored tangle diagram, we can slice it into pieces. These pieces corresponds to morphisms of representations and we can get a map by tensoring and compositing them according to some rules. The quantum trace of that resulting map is a quantum invariant of the closure of that tangle. Such an invariant always depends on framing but we can easily get an unframed invariant from it. The framed invariant can be used to construct invariants of 3-manifolds which is an important breakthrough in low dimensional topology (cf. [RT2]).

Quantum invariants becomes very important in the study of knots since its birth. A nice algebraic property of colored Jones polynomial called cyclotomic expansion was proved by Habiro in [H]:

Theorem 1. *For any knot $\mathcal{K}$, their exist $H_k(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of $N (N > 0)$, such that*

$$J_N^{SU(2)}(\mathcal{K}; q) = \sum_{k=0}^{N} C_{N+1,k}^{(2)} H_k^{(2)}(\mathcal{K}). \tag{1.1}$$

Note that $H_k^{(2)}(\mathcal{K})$ has nothing to do with N. If we know the formulas of $J_N^{SU(2)}(\mathcal{K}; q)$ and $H_0^{(2)}(\mathcal{K}), H_1^{(2)}(\mathcal{K}), ..., H_{N-1}^{(2)}(\mathcal{K})$, we can obtain the formula of $H_N^{(2)}(\mathcal{K})$. Chen-Liu-Zhu[CLPZ] proposed the following conjecture generalizing Habiro's cyclotomic expansion.

Conjecture 1. *For any knot $\mathcal{K}$, there exist $H_k^{(n)}(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of $N (N > 0)$, such that*

$$J_N^{SU(n)}(\mathcal{K}; q) = \sum_{k=0}^{N} C_{N+1,k}^{(n)} H_k^{(n)}(\mathcal{K}), \tag{1.2}$$

where $C_{N+1,k}^{(n)} = \{N-(k-1)\}\{N-(k-2)\} \cdots \{N-1\}\{N\}\{N+n\}\{N+n+1\} \cdots \{N+n+(k-1)\}$, for $k = 1, ..., N$, and $C_{N+1,0}^{(n)} = 1$. In particular, $J_0^{SU(n)}(\mathcal{K}; q) = H_0^{(n)}(\mathcal{K}) = 1$.

In this paper we prove a special case of that conjecture, i.e.,

Theorem 2. *For each torus knot $T(2, 2k+1)$, there exist $H_1^{(3)}(T(2, 2k+1)), H_2^{(3)}(T(2, 2k+1)) \in \mathbb{Z}[q, q^{-1}]$, such that*

$$J_2^{SU(3)}(T(2, 2k+1); q) = 1 + \{2\}\{5\} H_1^{(3)}(T(2, 2k+1)) + \{1\}\{2\}\{5\}\{6\} H_2^{(3)}(T(2, 2k+1)). \tag{1.3}$$

In section 2 we introduce some basic concepts of links. We also introduce some functions of partitions for they appear in the formula of (reduced) colored $SU(n)$-invariant for torus knots $T(2, 2k+1)$. In section 3 we describe the general procedure of producing quantum invariants. In section 4 we introduce homfly skein theory and introduce the (reduced) colored $SU(n)$-invariant. This skein theory is developed by Morton's school in Liverpool and it provides some tools for concrete computation of (reduced) colored $SU(n)$-invariant. In section 5 we give the proof of theorem 2. We use induction method to prove our main result.

ACKNOWLEDGEMENTS I would like to thank my supervisors Giovanni Felder and Qingtao Chen for their help on this master thesis.

2 PRELIMINARY

2.1 LINK AND RIBBON

Definition 1. *A link $\mathcal{L}$ is the image of several non-intersecting smooth embeddings of S^1 into R^3.Its connected components are called link components and denoted by $\mathcal{L}_i$. A link with only one component is called a knot. A link is oriented if each component is given an orientation.*

For example, the torus knot $T(m, n)$ with m, n coprime is given by the parametrization

$$\begin{aligned} x &= rcos(m\phi) \\ y &= rsin(m\phi) \\ z &= -sin(n\phi) \end{aligned}$$

where $r = cos(n\phi) + 2$ and $0 \leq \phi \leq 2\pi$. Its orientation can be induced by the interval $[0, 2\pi]$.

We usually represent a link by its diagram. A link diagram is a nice projection of a link on the R^2 plane such that

(1) the tangent lines to the link at all points are projected onto lines on the plane;

(2) no more than two distinct points at the link are projected on one and the same point of the plane;

(3) the set of crossing points is finite and at each crossing point the projections of the two tangents do not coincide.

The following situations are forbidden: when points of three distinct branches of the link project on one and the same point or when the projections of two branches are tangent.

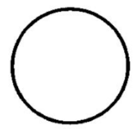

Figure 2.1: the diagram of unknot

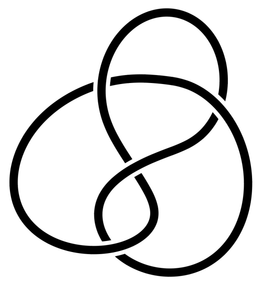

Figure 2.2: the diagram of figure eight knot

A "link diagram in the annulus" is a projection of a link on a given annulus. We will need this concept in the latter part of this paper.

Definition 2. *Two links $\mathcal{L}_1, \mathcal{L}_2$ are isotopic if there exist homeomorphisms $f_t : R^3 \to R^3$, $t \in [0,1]$ such that*

(1)$F : R^3 \times [0,1] \to R^3$ given by $(x,t) \to f_t(x)$ is differentiable;

(2)f_0 is identity and $f_1(\mathcal{L}_1) = \mathcal{L}_2$.

Analogously we can define plane isotopy. Its precise definition is not very important. It tells us that changing a link diagram in the following way does not change the link isotopic class it represents.

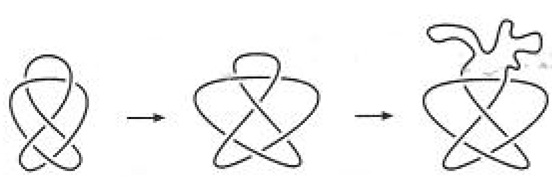

Figure 2.3: plane isotopy

It is good to know how to tell if two links are isotopic from their diagrams and we have the following theorem.

Theorem 3. *Two link diagrams corresponds to isotopic links if and only if one can be obtained from another by finite Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$ and plane isotopies.* $\square$

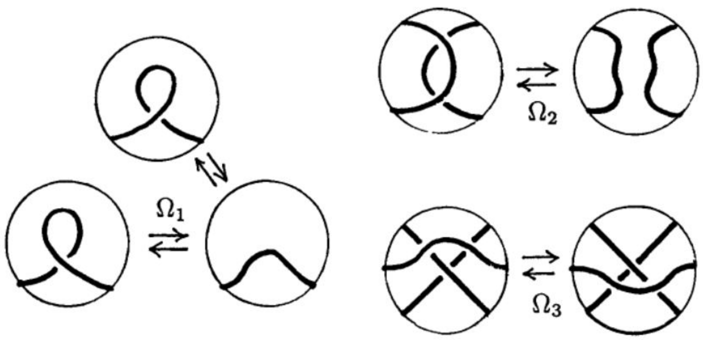

Figure 2.4: Reidemeister moves

For a link diagram of two oriented knots $\mathcal{K}_1, \mathcal{K}_2$, classify its crossings into four types as shown in the figure and count them. The linking number $lk(\mathcal{K}_1, \mathcal{K}_2)$ is defined as $\frac{n_1+n_2-n_3-n_4}{2}$.

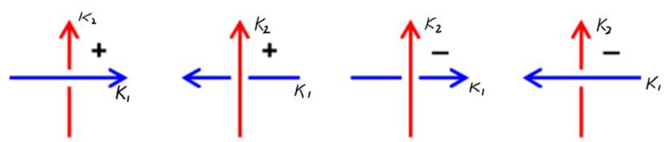

Figure 2.5: n_1, n_2, n_3, n_4

The writhe $w(\mathcal{L})$ is a property of an oriented link diagram $\mathcal{L}$. It is the total number of positive crossings minus the total number of negative crossings.

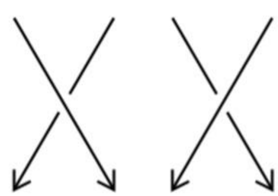

Figure 2.6: negative crossing and positive crossing

Definition 3. *A ribbon is the image of several non-intersecting smooth embeddings of annulus $S^1 \times [0, 1]$ in R^3.*

Like links, we care about isotopic classes of ribbons. Technically speaking, a ribbon is a framed link, i.e., a link with some extra information called framing. Denote one edge of ribbon component $\mathcal{R}_i$ as $\mathcal{L}_i$. The linking number of two edges of $\mathcal{R}_i$ is a integer which is called the framing of $\mathcal{L}_i$. (When computing the linking number, we adopt the convention that the both edges of $\mathcal{R}_i$ have parallel orientation.) In other words, each isotopic class of ribbons corresponds to a isotopic class of links with each component assigned a integer.

By the belt trick, a twist of ribbon can be put on a plane after isotopy so that we get one way to present a ribbon by a plane graph. First take one edge of each ribbon component to get a link diagram. Then count the linking number of two edges for each component of link diagram (imagine drawing it by a big chalk). Finally compare the difference of framing and the linking numbers, and add curls on the graph to fix it. We may view this final link diagram as a very good projection of ribbon in the space. Each link diagram can present a ribbon graph if we adopt the convention that the writhe numbers are the framings. In other words, we cannot eliminate twists of a ribbon graph by stretching it.

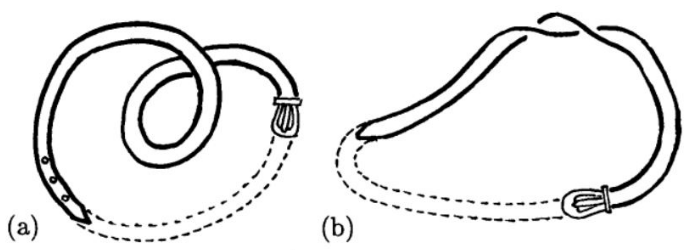

Figure 2.7: belt trick

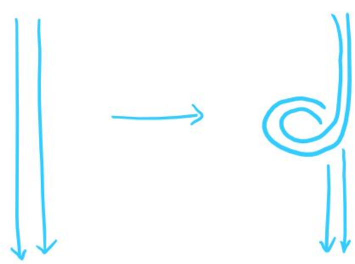

Figure 2.8: add a positive curl to a ribbon graph amounts to add one framing

Theorem 4. *Two ribbon graph corresponds to isotopic links with the same framings if and only if one can be obtained from another by finite Reidemeister moves* $\Omega_1', \Omega_2, \Omega_3$ *and plane isotopies. (See [PS, theorem 19.7].)* $\square$

Figure 2.9: the double Reidemeister move Ω_1'

For a given ribbon, the images of $S^1 \times \{1/2\}$ is called the core of that ribbon. Giving orientation to core we get a oriented ribbon. The concept of linking number and writhe can be naturally generalized to ribbon graph as well.

A more general concept is tangle diagram. Take a square $[0,1] \times [0,1]$ with n distinct points at $[0,1] \times \{0\}$ and m distinct points at $[0,1] \times \{1\}$. A (n,m)-tangle is a collection of oriented arcs and oriented simple closed curves that join these $n+m$ points ($n+m$ has to be even otherwise a tangle will not exist). Note that a $(0,0)$-tangle is a link (or ribbon if it is framed).

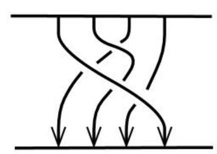

Figure 2.10: a (4, 4)-tangle

2.2 PARTITION

In this section we introduce some functions of partitions. They appear in the formulas of some link invariants.

Definition 4. *A partition $\lambda \vdash N$ is a finite sequence of integers $(\lambda_1 \lambda_2 \lambda_3 ...)$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots$ and $|\lambda| = \sum \lambda_i = N$. The length $l(\lambda)$ is the number of integers in λ.*

A partition $\lambda \vdash N$ may encoded by a Young diagram. For example, (421) corresponds to

One can transpose a Young diagram λ to get another diagram λ^t.

For a given partition λ, define $\kappa_\lambda = \sum_{j=1}^{l(\lambda)} \lambda_j(\lambda_j - 2j + 1)$ and $z_\lambda = \prod_{j=1}^{|\lambda|} j^{m_j(\lambda)} m_j(\lambda)!$ where $m_j(\lambda)$ is the times of j appearing in partition λ.

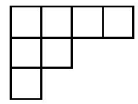

Figure 2.11: the young diagram of (421)

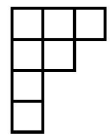

Figure 2.12: the young diagram of $(421)^t$

Definition 5. *The power sum symmetric function of many variables* $x = (x_1, x_2, x_3, ...)$ *is defined by* $p_n(x) = \sum_i x_i^n$. *Given a partition* λ, *define*

$$p_\lambda(x) = \prod_{j=1}^{l(\lambda)} p_{\lambda_j}(x). \tag{2.1}$$

Definition 6. *The Schur function is*

$$s_\lambda(x) = \sum_{|\mu|=|\lambda|} \frac{\chi_\lambda(C_\mu)}{z_\mu} p_\mu(x), \tag{2.2}$$

where χ_λ *is the character of the irreducible representation of the symmetric group* $S_{|\mu|}$ *corresponding to* λ. C_μ *denotes the conjugate class of symmetric group* $S_{|\mu|}$ *corresponding to partition* μ.

3 QUANTUM INVARIANT OF LINKS AND RIBBONS

In this section we describe a systematic way to produce a great many link invariants. From now on we will work with oriented links and oriented ribbons.

3.1 RIBBON HOPF ALGEBRA

A ribbon Hopf algebra $\mathcal{A}$ is both an algebra and a coalgebra over a commutative ring k, i.e. there are multiplication $\zeta : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, unit $\iota : k \to \mathcal{A}$, comultiplication

$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\varepsilon : \mathcal{A} \to k$ which satisfy

$$\begin{aligned}
&\zeta(id_{\mathcal{A}} \otimes \iota) = \zeta(\iota \otimes id_{\mathcal{A}}) = id_{\mathcal{A}} \\
&\zeta(\zeta \otimes id_{\mathcal{A}}) = \zeta(id_{\mathcal{A}} \otimes \zeta) \\
&(id_{\mathcal{A}} \otimes \varepsilon)\Delta = (\varepsilon \otimes id_{\mathcal{A}})\Delta = id_{\mathcal{A}} \\
&(\Delta \otimes id_{\mathcal{A}})\Delta = (id_{\mathcal{A}} \otimes \Delta)\Delta
\end{aligned}$$

Multiplication and unit are homomorphisms of coalgebras, and, equivalently, commultiplication and counit are homomorphisms of algebras. Furthermore we require the existence of a map called antipode $S : \mathcal{A} \to \mathcal{A}$ satisfying

$$\zeta(S \otimes id_{\mathcal{A}})\Delta = \iota\varepsilon = \zeta(id_{\mathcal{A}} \otimes S)\Delta.$$

We also require the existence of a universal R-matrix, i.e., an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$ and an invertible and central element $v \in \mathcal{A}$ such that

$$\begin{aligned}
&\Delta^{op}(x) = R\Delta(x)R^{-1} \text{ for all } x \in \mathcal{A}, \\
&(\Delta \otimes id_{\mathcal{A}})(R) = R_{13}(1 \otimes R), \\
&(id_{\mathcal{A}} \otimes \Delta)(R) = R_{13}(R \otimes 1), \\
&v^2 = uS(u) \\
&\Delta(v) = (R_{21}R)^{-1}(v \otimes v) \\
&\varepsilon(v) = 1 \\
&S(v) = v
\end{aligned}$$

where $R = \Sigma_i s_i \otimes t_i, u = \Sigma_i S(t_i)s_i, R_{13} = \Sigma_i s_i \otimes 1 \otimes t_i, R_{21} = \Sigma_i t_i \otimes s_i$, and $\Delta^{op} = \tau_{\mathcal{A},\mathcal{A}}\Delta$ where $\tau_{\mathcal{A},\mathcal{A}}\Delta$ is the flip of the components of $\mathcal{A} \otimes \mathcal{A}$.

The tensor product of any two $\mathcal{A}$-modules and the dual of any $\mathcal{A}$-modules are also equipped with $\mathcal{A}$-module structures. For $\mathcal{A}$-modules V and W, the $a(v \otimes w)$ can be defined by $\Delta(a)(v \otimes w)$ where $a \in \mathcal{A}, v \in V, w \in W$. The V^* becomes an $\mathcal{A}$-module by defining $\langle a\xi, v\rangle = \langle \xi, S(a)v\rangle$ where $\xi \in V^*$ and $\langle\ ,\ \rangle$ is the natural pairing.

3.2 RIBBON GRAPH INVARIANT

Given a ribbon Hopf algebra $\mathcal{A}$, we can construct an invariant for framed tangles and ribbon graphs. First we assign each component a $\mathcal{A}$-module (These modules are always called colors). Then we slice this link diagram into horizontal pieces and translate each pieces into a map according to the following rules:

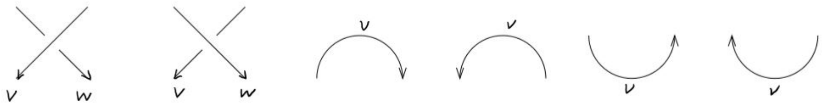

Figure 3.1: $\rho_{V,W}$, $\rho^{-1}_{W,V}$, F_1, F_1, F_3, F_4

$\rho_{V,W} : V \otimes W \to W \otimes V$ is first multiply the universal R-matrix then switch the factors of $V \bigotimes W$, that is to say, map $v \otimes w$ to $w \otimes v$ and extend this switch linearly;

$F_1 : V^* \otimes V \to k, F_1(g \otimes v) = g(v)$;

$F_2 : V \otimes V^* \to k, F_2(v \otimes g) = g(\mu v)$;

$F_3 : k \to V \otimes V^*, F_3(1) = \Sigma_m v_m \otimes v^m$;

$F_4 : k \to V^* \otimes V, F_4(1) = \Sigma_m v^m \otimes (\mu v_m)$, where $\{v_m\}$ is basis for V, and $\{v^m\}$ is the corresponding dual basis for V^*;

An arrow point downward colored by V is $id_V : V \to V$. An arrow point upward with color V is id_{V^*}. Juxtaposition of small pieces is translated to tensor product of morphisms.

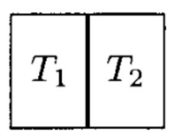

Figure 3.2: $T_1 \otimes T_2$

For any (n, m)-tangle diagram, read it from bottom to top and composite the corresponding morphisms by this order. This composition will be a morphism from a tensor product of n $\mathcal{A}$-modules to a tensor product of m $\mathcal{A}$-modules. For a (0,0)-tangle this morphism corresponds to a element in the ground ring k. It is well known that morphisms obtained in this way is an invariant of colored isotopic oriented ribbons.

One important example of ribbon Hopf algebra is the deformed universal enveloping algebra $U_h(sl(N))$. In this case the ground ring is a set of Laurent polynomials.

One may wonder why we have to work with framed diagram? After all, the above procedure can also be applied to unframed diagram. After some computations one may notice that a curl will not be translated to the identity map, and we cannot get a link invariant since the Reidemeister move Ω_3 eliminate curls.

However we can still obtain unframed polynomial from framed polynomial. Suppose

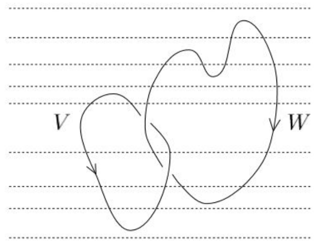

Figure 3.3: slice a graph and read it from bottom to top

we know that adding one framing on a link component $\mathcal{L}_i$ with color V_i amounts to multiply framed polynomial by θ_i, we can multiply the framed polynomial by $\theta_1^{-w(\mathcal{L}_1)} \cdots \theta_L^{-w(\mathcal{L}_L)}$. This new polynomial has nothing to do with framing. In the rest of this paper we mainly concern such family of unframed polynomials.

4 SKEIN THEORY

Most link polynomials can be defined by its skein relations. For example in most expository articles, the famous Jones polynomial are defined by skein relation $(t^{1/2} - t^{-1/2})V(L_0) = t^{-1}V(L_+) - tV(L_-)$, where t is a formal variable and L_0, L_+, L_- represent three link diagrams that only differ in a small disk.

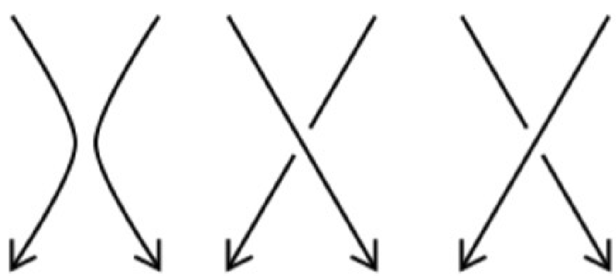

Figure 4.1: from left to right: L_0, L_-, L_+

Each link diagram can be simplified to the unknot by gradually applying this skein relation and finally we get its Jones polynomial.

Let Λ be $\mathbb{Z}[x^{\pm 1}, v^{\pm 1}, s^{\pm 1}, \delta]/\langle \delta(s - s^{-1}) = v^{-1} - v \rangle$. Consider the set of all Λ-linear combinations of ribbon graph in R^2. Quotient this space by plane isotopy and the following skein relations and denote the quotient space by $\mathcal{S}(R^2)$. Just like Jones polynomial can be obtained by skein relations, it is easy to show that all elements of $\mathcal{S}(R^2)$ belongs to Λ. For a framed link diagram D, the element $H(D)$ of Λ representing it is called the framed homfly polynomial of D. It is easy to show that $H(unknot) = \delta = \frac{v^{-1}-v}{s-s^{-1}}$.

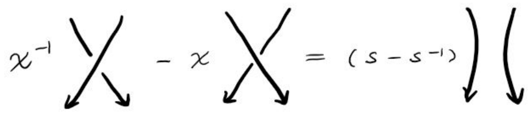

Figure 4.2: first homfly relation

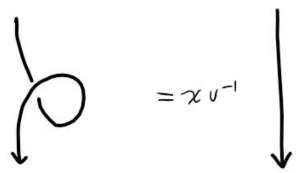

Figure 4.3: second homfly relation

Besides skein theory of links or ribbons, we can also consider skein theory of (n, n)-tangles. Like what we did in R^2, we consider the λ-module generated by all the (n, n)-tangles and quotient it by the same skein relations and plane isotopy. The resulting space is denoted by H_n. If we require all the orientations are from top to bottom then H_n has an algebra structure. The product T_1T_2 is given by stacking T_1 above T_2.

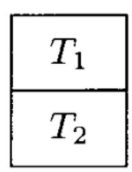

Figure 4.4: product in H_n

The juxtaposition of $T_1 \in H_n$ and $T_2 \in H_m$ is denoted by $D_1 \otimes D_2$ and belongs to H_{n+m}.

Each element π of symmetric group S_n determines a positive permutation braid $w_\pi \in H_n$ such that strings starting from the top points $i < j$ do not cross each other if $\pi(i) < \pi(j)$ and positively cross each other only once if $\pi(i) > \pi(j)$.

We can wire a tangle T into the annulus as indicated below. This element will be called the closure of the tangle T.

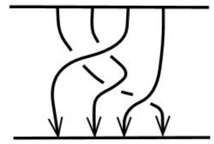

Figure 4.5: $w_{(431)}$ for $(431) \in S_4$

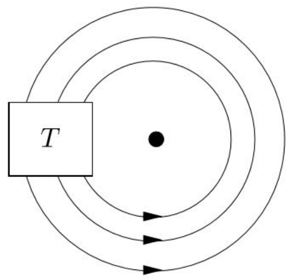

Figure 4.6: closure of T

Now we construct a special element $e_\lambda \in H_n$. Define

$$a_n = \sum_{S_n} (x^{-1}s)^{w(w_\pi)} w_\pi,$$
$$b_n = \sum_{S_n} (-x^{-1}s^{-1})^{w(w_\pi)} w_\pi,$$

and

$$E_\lambda(a) = a_{\lambda_1} \otimes a_{\lambda_2} \otimes \cdots a_{\lambda_{l(\lambda)}},$$
$$E_\lambda(b) = b_{\lambda_1} \otimes b_{\lambda_2} \otimes \cdots b_{\lambda_{l(\lambda)}},$$

for $\lambda \vdash n$.

We assign numbers $1, 2, ..., n$ to the cells of any Young diagram from left to right and top to bottom. The map $(i, j) \in \lambda$ to $(j, i) \in \lambda^t$ induces a permutation π_λ. The element e_λ is

$$E_\lambda(a) w_{\pi_\lambda} E_{\lambda^t}(b) w_{\pi_\lambda}^{-1}. \tag{4.1}$$

e_λ is a quasi-idempotent, i.e., there is a scalar $\alpha_\lambda \in \Lambda$ such that $e_\lambda e_\lambda = \alpha_\lambda e_\lambda$. It was shown that α_λ is a function of s.

The following theorem is known to experts.

Theorem 5. *(cf. [L]) Let $\mathcal{L}$ be a framed link whose L components are colored by $V_{A^1}, ...V_{A^L}$ (the numbers of cells of Young diagrams $A^1, ..., A^L$ may vary). Then the*

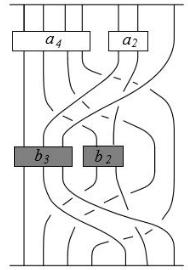

Figure 4.7: $e_{(421)}$

$U_h(sl(n))$-invariant of this link equals to the framed homfly polynomial of $\mathcal{L}$ with decorations $Q_{A^1}, ...Q_{A^L}$ on its components after the substitutions $x = e^{-h/n}$, $v = e^{-nh}$, $s = e^h$. We denote $\mathcal{L}$ with decorations $Q_{A^1}, ...Q_{A^L}$ as $\mathcal{L} \star \bigotimes_{i=1}^{L} Q_{A^i}$.

For a partition $A \vdash |A|$, Q_A is the closure of $\frac{1}{\alpha_\lambda} e_\lambda$.

The decoration of a diagram in the annulus to another diagram in R^2 is shown in the following figure.

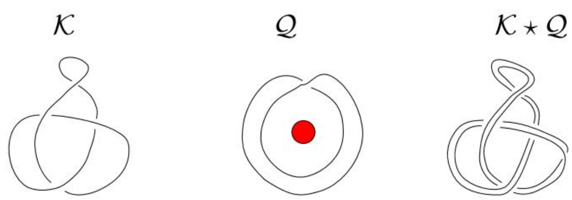

Figure 4.8: decoration

Aiston and Morton [AM] shown that, for each partition $\lambda \vdash |\lambda|$ with $l(\lambda) \leq n$, one has $\theta_{V_\lambda} = e^{h(\kappa_\lambda + |\lambda| n - \frac{|\lambda|^2}{n})} id_{V_\lambda}$.

Recall how to get an unframed polynomial from a framed one. The unframed $U_h(sl(n))$-invariant of a link is $e^{h(-\sum\limits_{i=1}^{L}(\kappa_{A^i} + |A^i| n - \frac{|A^i|^2}{n}) w(\mathcal{L}_i))} H(\mathcal{L} \star \otimes_{i=1}^{L} Q_{A^i})$.

Although $H(\mathcal{L} \star \bigotimes_{i=1}^{L} Q_{A^i})$ is a polynomial of x, s, v, after the substitution we do not know whether the above invariant is a polynomial of e^h. We have to drop the x term off. Consider the following skein relations. We denote the homfly polynomial obtained by this as H'.

Lemma 1. *Suppose $x = e^{-h/n}, s = e^h = q, e^{nh} = t = v^{-1}$, then $x^{-w(\mathcal{L})} H = H'$.*

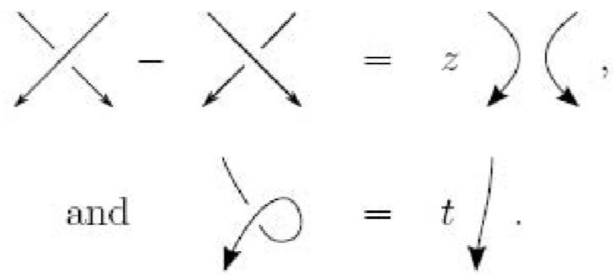

Figure 4.9: another homfly skein theory, where $z = q - q^{-1}$

Proof. It suffices to show $x^{-w(\mathcal{L})}H$ satisfies the skein relations of H'

$$\begin{aligned} x^{-1}H(L_+) - xH(L_-) &= (q - q^{-1})H(L_0) \\ x^{-1-w(L_0)}H(L_+) - x^{1-w(L_0)}H(L_-) &= (q - q^{-1})x^{-w(L_0)}H(L_0) \\ x^{-w(L_+)}H(L_+) - x^{-w(L_-)}H(L_-) &= (q - q^{-1})x^{-w(L_0)}H(L_0) \\ H(curl) &= xv^{-1}H(straight) \\ x^{-1-w(straight)}H(curl) &= tx^{-w(straight)}H(straight) \\ x^{-w(curl)}H(curl) &= tx^{-w(straight)}H(straight) \end{aligned}$$

□

To simplify the notation, we may just denote e^h by q and e^{nh} by t. One can show that for a colored link

$$q^{-\sum_{i=1}^{L}(\kappa_{A^i}+|A^i|n-\frac{|A^i|^2}{n})w(\mathcal{L}_i)} H(\mathcal{L}\star\otimes_{i=1}^{L}Q_{A^i}) = q^{-\sum_{i=1}^{L}(\kappa_{A^i}+|A^i|n-\frac{|A^i|^2}{n})w(\mathcal{L}_i)-\frac{w(\mathcal{L}^{|A^1|,\dots,|A^L|})}{n}} H'(\mathcal{L}\star\otimes_{i=1}^{L}Q'_{A^i}), \tag{4.2}$$

where $\mathcal{L}^{|A^1|,\dots,|A^L|}$ is $\mathcal{L}$ with $\mathcal{L}_i$ cabled by $|A^i|$ parallel lines and Q'_{A^i} of skein theory H' is constructed in the same way as Q_{A^i}.

We give the proof of the above formula for the case $l(A^i) = 1, i = 1, ..., L$. The notation for the general case is too complicated to be written down. Now let $a_{A^i} =$

$\sum\limits_{S_{|A^i|}}(x^{-1}s)^{w(w_{\pi_i^a})}w_{\pi_i^a}$, and $b_{A^i}=\sum\limits_{S_{|A^i|}}(-x^{-1}s^{-1})^{w(w_{\pi_i^b})}w_{\pi_i^b}$, then

$$(\prod_{i=1}^{L}\frac{1}{\alpha_{A^i}})\sum_{\pi^a,\pi^b}(q^{\frac{1}{n}}q)^{\sum\limits_{i=1}^{L}w(w_{\pi_i^a})}(-q^{\frac{1}{n}}q^{-1})^{\sum\limits_{i=1}^{L}w(w_{\pi_i^b})}H(\mathcal{L}\star\otimes_{i=1}^{L}w_{\pi_i^a}w_{\pi_{A^i}}w_{\pi_i^b}w_{\pi_{A^i}}^{-1})$$
$$=(\prod_i\frac{1}{\alpha_{A^i}})\sum_{\pi^a,\pi^b}q^{\frac{1}{n}\sum\limits_i w(w_{\pi_i^a})+w(w_{\pi_i^b})}q^{-\frac{1}{n}(w(\mathcal{L}\star\otimes_i w_{\pi_i^a}w_{\pi_{A^i}}w_{\pi_i^b}w_{\pi_{A^i}}^{-1}))}$$
$$\sum_{\pi^a,\pi^b}q^{\sum\limits_i w(w_{\pi_i^a})}(-q^{-1})^{\sum\limits_i w(w_{\pi_i^b})}H'(\mathcal{L}\star\otimes_{i=1}^{L}w_{\pi_i^a}w_{\pi_{A^i}}w_{\pi_i^b}w_{\pi_{A^i}}^{-1})$$
$$=(\prod_i\frac{1}{\alpha_{A^i}})q^{-\frac{w(\mathcal{L}^{|A^1|,\ldots,|A^L|})}{n}}\sum_{\pi^a,\pi^b}q^{\sum\limits_i w(w_{\pi_i^a})}(-q^{-1})^{\sum\limits_i w(w_{\pi_i^b})}H'(\mathcal{L}\star\otimes_i w_{\pi_i^a}w_{\pi_{A^i}}w_{\pi_i^b}w_{\pi_{A^i}}^{-1})$$
$$=q^{-\frac{w(\mathcal{L}^{|A^1|,\ldots,|A^L|})}{n}}H'(\mathcal{L}\star\otimes_{i=1}^{L}Q'_{A^i}),$$

where $\pi^a=(\pi_1^a,\ldots,\pi_L^a)\in\prod\limits_i S_{|A_i|}$. So

$$q^{-\sum\limits_{i=1}^{L}(\kappa_{A^i}+|A^i|n-\frac{|A^i|^2}{n})w(\mathcal{L}_i)}H(\mathcal{L}\star\otimes_{i=1}^{L}Q_{A^i})=q^{-\sum\limits_{i=1}^{L}(\kappa_{A^i}+|A^i|n-\frac{|A^i|^2}{n})w(\mathcal{L}_i)-\frac{w(\mathcal{L}^{|A^1|,\ldots,|A^L|})}{n}}H'(\mathcal{L}\star\otimes_{i=1}^{L}Q'_{A^i}),\tag{4.3}$$

One notices that $\sum\limits_{i=1}^{L}|A^i|^2w(\mathcal{L}_i)-w(\mathcal{L}^{|A^1|,\ldots,|A^L|})=-2\sum\limits_{i<j}|A^i||A^j|lk(\mathcal{L}_i,\mathcal{L}_j)$. After dropping this factor we get the colored homfly polynomial,

$$W_{A^1,\ldots,A^L}(\mathcal{L};q,t)=q^{-\sum\limits_{i=1}^{L}\kappa_{A^i}w(\mathcal{L}_i)}t^{-\sum\limits_{i=1}^{L}|A^i|w(\mathcal{L}_i)}H'(\mathcal{L}\star\otimes_{i=1}^{L}Q'_{A^i};q,t)\in\mathbb{Z}[q,q^{-1}]\tag{4.4}$$

Definition 7. *(cf. [CLPZ].)For a link $\mathcal{L}$, the reduced $SU(n)$-invariant is*

$$J_N^{SU(n)}(\mathcal{L};q)=\frac{q^{-2lk(\mathcal{L})(N(N-1)+nN)}W_{(N)}(\mathcal{L};q,q^n)}{s_{(N)}(q,q^n)},\tag{4.5}$$

where $s_{(N)}(q,q^n)$ denote $s_{(N)}(q^{n-1},q^{n-3},\ldots,q^{-(n-1)})$ and $lk(\mathcal{L})=\sum\limits_{i<j}lk(\mathcal{L}_i,\mathcal{L}_j)$. For a knot $\mathcal{K}$,

$$J_N^{SU(n)}(\mathcal{K};q)=\frac{W_{(N)}(\mathcal{K};q,q^n)}{s_{(N)}(q,q^n)}.\tag{4.6}$$

5 STATEMENT OF RESULT

Throughout this section $\{n\}=q^n-q^{-n}$.

In [H] Habiro proved that

Theorem 6. *(=Theorem 1) For any knot $\mathcal{K}$, their exist $H_k(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of $N(N > 0)$, such that*

$$J_N^{SU(2)}(\mathcal{K}; q) = \sum_{k=0}^{N} C_{N+1,k}^{(2)} H_k^{(2)}(\mathcal{K}), \tag{5.1}$$

where $C_{N+1,k}^{(2)} = \{N-(k-1)\}\{N-(k-2)\}\cdots\{N-1\}\{N\}\{N+2\}\{N+2+1\}\cdots\{N+2+(k-1)\}$, for $k = 1, ..., N$, and $C_{N+1,0}^{(2)} = 1$. In particular, $J_0^{SU(2)}(\mathcal{K}; q) = H_0^{(2)}(\mathcal{K}) = 1$.

The above result is called cyclotomic expansion. Note that $H_k^{(2)}(\mathcal{K})$ has nothing to do with N. If we know the formulas of $J_N^{SU(2)}(\mathcal{K}; q)$ and $H_0^{(2)}(\mathcal{K}), H_1^{(2)}(\mathcal{K}), ..., H_{N-1}^{(2)}(\mathcal{K})$, we can obtain the formula of $H_N^{(2)}(\mathcal{K})$.

After some numerical computation, Chen-Liu-Zhu[CLZ] proposed the following conjecture generalizing Habiro's cyclotomic expansion.

Conjecture 2. *(=Conjecture 1) For any knot $\mathcal{K}$, there exist $H_k^{(n)}(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of $N(N > 0)$, such that*

$$J_N^{SU(n)}(\mathcal{K}; q) = \sum_{k=0}^{N} C_{N+1,k}^{(n)} H_k^{(n)}(\mathcal{K}), \tag{5.2}$$

where $C_{N+1,k}^{(n)} = \{N-(k-1)\}\{N-(k-2)\}\cdots\{N-1\}\{N\}\{N+n\}\{N+n+1\}\cdots\{N+n+(k-1)\}$, for $k = 1, ..., N$, and $C_{N+1,0}^{(n)} = 1$. In particular, $J_0^{SU(n)}(\mathcal{K}; q) = H_0^{(n)}(\mathcal{K}) = 1$.

In this paper I prove that this conjecture is true for the colored $SU(3)$-invariant of torus knot $T(2, 2k+1)$ with $N = 2$, i.e.,

Theorem 7. *(=Theorem 2) For each torus knot $T(2, 2k+1)$, there exist $H_1^{(3)}(T(2, 2k+1)), H_2^{(3)}(T(2, 2k+1)) \in \mathbb{Z}[q, q^{-1}]$, such that*

$$J_2^{SU(3)}(T(2, 2k+1); q) = 1+\{2\}\{5\}H_1^{(3)}(T(2, 2k+1))+\{1\}\{2\}\{5\}\{6\}H_2^{(3)}(T(2, 2k+1)). \tag{5.3}$$

The fact that $H_1^{(3)}(\mathcal{K}) = \frac{J_1^{SU(3)}(\mathcal{K})-1}{\{1\}\{4\}}$ is Laurent polynomial is already known (for $T(2, 2k+1)$, this can be proved by direct computation), so it suffices to prove that

$$H_2^{(3)}(T(2, 2k+1)) = \frac{\frac{J_2^{SU(3)}(T(2,2k+1);q)-1}{\{2\}\{5\}} - H_1^{(3)}(T(2, 2k+1))}{\{1\}\{6\}} \tag{5.4}$$

is a Laurent polynomial for each k.

6 Proof

The tedious computations are in appendix.

Suppose a, n are all integers, then $\frac{q^{na}-1}{q^a-1} \in \mathbb{Z}[q, q^{-1}]$.

Suppose we manage to arrange the terms of a Laurent polynomial into pairs such that the exponents of each pair is different by a multiple of n, then we can apply the above lemma to say that this polynomial can be divided by $\{n\}$.

Recall the formula of Lin and Zheng (cf. [LZ, theorem 5.1], [Z, equation 5.8], [MM]) for torus knot $T(m, n)$,

$$W_A(T(m,n); q, q^l) = q^{-mn\kappa_A}(q^l)^{-n(m-1)|A|} \sum_{\mu \vdash m|A|} C_A^{\mu} q^{\frac{n}{m}\kappa_\mu} s_\mu(q, q^l), \tag{6.1}$$

where $C_A^{\mu} = \sum_B \frac{\chi_A(C_B)}{z_B}\chi_\mu(C_{mB})$.

Then we have

$$J_1^{SU(3)}(T(2, 2k+1); q) = \frac{\{2k+2\}q^{-6k} - \{2k\}q^{-6k-6}}{\{2\}}, \tag{6.2}$$

and

$$J_2^{SU(3)}(T(2,2k+1);q) = \frac{1}{1+q^4}(q^{-10-20k}(1+q^4-q^{4k}-q^{2+4k}-q^{4+4k}-q^{6+4k}-q^{8+4k} + q^{2+12k}+q^{6+12k}+q^{8+12k}+q^{10+12k}+q^{14+12k})), \tag{6.3}$$

then we have

$$\frac{J_2^{SU(3)}-1}{\{2\}} - H_1^{(3)}\{5\} = \frac{1-q^{8k}}{1-q^8}(q^{-16k}+q^{-8k}(1+q^2+q^4+q^6+q^8)) + (-q^{-8-20k}+q^{-6-16k}+q^{-6-12k}-q^{2-8k}-q^{2-4k}-q^{-4-8k}(1+q^2+q^4+q^6+q^8))\frac{1-q^{4k}}{1-q^4} \tag{6.4}$$

Denote $\frac{J_2^{SU(3)}-1}{\{2\}} - H_1^{(3)}\{5\}$ by f_k, we need to prove that f_k can be divided by $\{1\}\{5\}\{6\}$. When $k = 1$, $f_1 = q^{-14}\{6\}\{5\}\{3\}$.

Now we introduce $g_k = f_{k+1} - q^{-8}f_k$. It suffices to show g_k can be divided by $\{1\}\{5\}\{6\}$.

Now compute $f_{k+1} - q^{-8} f_k$. It equals to

$$(1 - q^{-20-20k})(1 + q^{-4} + q^{-8}) - (1 - q^{-12-12k})(1 + q^2 + q^4 + q^6 + q^8). \tag{6.5}$$

Fact 1. $(1 - q^{-12-12k})(1 + q^2 + q^4 + q^6 + q^8)$ *can be divided by* $\{5\}$.

Proof. Let's see the ones digits of the exponents of q. Both in the positive and negative exponentials, the exponential that has 0,2,4,6,8 as ones digits appears just once. For example, when $k = 1$, $(1 - q^{-24})(1 + q^2 + q^4 + q^6 + q^8)$ can be rewritten as

$$(1 - q^{-20}) + (q^2 - q^{-22}) + (q^4 - q^{-24}) + (q^6 - q^{-16}) + (q^8 - q^{-18}).$$

□

Fact 2. $(1 - q^{-20-20k})(1 + q^{-4} + q^{-8})$ *can be divided by* $\{6\}$.

Proof. We can rearrange the terms for $k = 1, 2, 3, 4, 5, 6$ (see appendix) and apply the lemma in the beginning of this section. This is enough to show that such pairing exists for each positive k, since the pairing of terms for the case $k = l$ also works for the case $k = l + 6$. □

Fact 3. g_k *can be divided by* $\{1\}^3$.

Proof. Let $l = k + 1$, then

$$\frac{g_k}{1 - q^{-4-4k}} = 1 + q^{-4} + q^{-8} + q^{-8-16l} + q^{-4-16l} + q^{-16l} - q^{-6-12l} - q^{-2-12l} - q^{-6-8l} - q^{-2-8l} - q^{-6-4l} - q^{-2-4l}.$$

Rearrange the terms it equals to

$$(1 - q^{-2-8l})(1 - q^{-6-8l}) + (q^{-8} - q^{-6-4l} + q^{-4} - q^{-2-4l})(1 - q^{2-12l}),$$

which can be divided by $\{1\}^2$. □

Hence we have shown that if f_k can be divided by $\{1\}\{6\}\{5\}$, so is f_{k+1}.

7 APPENDIX

First compute $\frac{J_2^{SU(3)} - 1}{\{2\}}$,

$$J_2^{SU(3)}(T(2, 2k+1); q) = q^{-10-20k}(1 - q^{4k} - q^{4k+2} + q^{12k+2} + q^{10+12k} - q^{8+4k}\frac{1 - q^{8k}}{1 + q^4}), \tag{7.1}$$

$$
\begin{aligned}
\frac{J_2^{SU(3)}-1}{\{2\}} &= \frac{q^{-10-20k}(1-q^{4k}-q^{4k+2}+q^{12k+2}+q^{10+12k}-q^{8+4k}\frac{1-q^{8k}}{1+q^4})-1}{\{2\}} \\
&= -q^{-8-20k}\frac{1-q^{4k}-q^{4k+2}+q^{12k+2}+q^{10+12k}-q^{8+4k}\frac{1-q^{8k}}{1+q^4}-q^{10+20k}}{1-q^4} \\
&= -q^{-8-20k}((1+(-q^{4k+2}+q^{10+12k})(1+q^{4k}))\frac{1-q^{4k}}{1-q^4}-q^{8+4k}\frac{1-q^{8k}}{1-q^8}) \\
&= -q^{-8-20k}((1-q^{2+4k}-q^{2+8k}+q^{10+12k}+q^{10+16k})\frac{1-q^{4k}}{1-q^4}-q^{8+4k}\frac{1-q^{8k}}{1-q^8}).
\end{aligned}
\tag{7.2}
$$

Now compute $H_1^{(3)}(T(2,2k+1))\{5\}$,

$$
J_1^{SU(3)}-1=(q^{-4k}-q^{3(-2k-2)+2k-2})\frac{1-q^{-4k}}{1-q^{-4}}+q^{-8k}-1, \tag{7.3}
$$

$$
\frac{J_1^{SU(3)}-1}{\{4\}}=q^{-4-4k}\frac{1-q^{-4k}}{1-q^{-4}}-q^{-4}\frac{1-q^{-8k}}{1-q^{-8}}, \tag{7.4}
$$

$$
H_1^{(3)}(T(2,2k+1))\{5\}=\frac{(J_1^{SU(3)}-1)\{5\}}{\{4\}\{1\}}=(q^{-8k-4}\frac{1-q^{4k}}{1-q^4}-q^{-8k}\frac{1-q^{8k}}{1-q^8})(1+q^2+q^4+q^6+q^8). \tag{7.5}
$$

$$
\begin{aligned}
f_k=\frac{J_2^{SU(3)}-1}{\{2\}}-H_1^{(3)}\{5\} &= (q^{-16k}+q^{-8k}(1+q^2+q^4+q^6+q^8))\frac{1-q^{8k}}{1-q^8} \\
&+(-q^{-8-20k}+q^{-6-16k}+q^{-6-12k}-q^{2-8k}-q^{2-4k}-q^{-4-8k}(1+q^2+q^4+q^6+q^8))\frac{1-q^{4k}}{1-q^4}
\end{aligned}
\tag{7.6}
$$

Now we know that

$$
\begin{aligned}
f_{k+1} &= (q^{-16k-16}+q^{-8k-8}(1+q^2+q^4+q^6+q^8))(\frac{1-q^{8k}}{1-q^8}+q^{8k}) \\
&+(-q^{-20-8-20k}+q^{-16-6-16k}+q^{-12-6-12k}-q^{-8+2-8k}-q^{-4+2-4k} \\
&-q^{-8-4-8k}(1+q^2+q^4+q^6+q^8))(\frac{1-q^{4k}}{1-q^4}+q^{4k}),
\end{aligned}
\tag{7.7}
$$

and

$$
\begin{aligned}
&f_{k+1}-q^{-8}f_k=(q^{-16}-q^{-8})q^{-16k}\frac{1-q^{8k}}{1-q^8}\\
&+(-(q^{-20}-q^{-8})q^{-8-20k}+(q^{-16}-q^{-8})q^{-6-16k}+(q^{-12}-q^{-8})q^{-6-12k}-(q^{-4}-q^{-8})q^{2-4k})\frac{1-q^{4k}}{1-q^4}\\
&+(q^{-16k-16}+q^{-8k-8}(1+q^2+q^4+q^6+q^8))q^{8k}\\
&+(-q^{-20-8-20k}+q^{-16-6-16k}+q^{-12-6-12k}-q^{-8+2-8k}-q^{-4+2-4k}\\
&-q^{-8-4-8k}(1+q^2+q^4+q^6+q^8))q^{4k}.
\end{aligned}
\tag{7.8}
$$

Let see why $(1-q^{-20-20k})(1+q^{-4}+q^{-8})$ can be divided by 6. How to rearrange terms for $k=1,2,3,4,5,6$?

$$
\begin{aligned}
&k=1,(1-q^{-48})+(q^{-4}-q^{-40})+(q^{-8}-q^{-44});\\
&k=2,(1-q^{-60})+(q^{-4}-q^{-64})+(q^{-8}-q^{-68});\\
&k=3,(1-q^{-84})+(q^{-4}-q^{-88})+(q^{-8}-q^{-80});\\
&k=4,(1-q^{-108})+(q^{-4}-q^{-100})+(q^{-8}-q^{-104});\\
&k=5,(1-q^{-120})+(q^{-4}-q^{-124})+(q^{-8}-q^{-128});\\
&k=6,(1-q^{-144})+(q^{-4}-q^{-148})+(q^{-8}-q^{-140}).
\end{aligned}
$$

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