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Bi-directional evolutionary topology optimization of continuum structures with one or multiple materials

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Abstract There are several well-established techniques for the generation of solid-void optimal topologies such as solid isotropic material with penalization (SIMP) method and evolutionary structural optimization (ESO) and its later version bi-directional ESO (BESO) methods. Utilizing the material interpolation scheme, a new BESO method with a penalization parameter is developed in this paper. A number of examples are presented to demonstrate the capabilities of the proposed method for achieving convergent optimal solutions for structures with one or multiple materials. The results show that the optimal designs from the present BESO method are independent on the degree of penalization. The resulted optimal topologies and values of the objective function compare well with those of SIMP method.

Keywords Bi-directional evolutionary structural optimization (BESO) · Material interpolation scheme · Optimal design

1 Introduction

Topology optimization for continuum structures has attracted considerable attention in the last three decades and many methods have been developed based on the finite element analysis [1–10]. Two of the commonly used methods are the solid isotropic material with penalization (SIMP) method [2–6] and the evolutionary structural optimization (ESO) method [7–10]. In SIMP method one defines at each element a density of material that varies continuously between 0 and 1. The elastic properties for intermediate densities are expressed in terms of the density, for example, using a power-law interpolation which leads to nearly 0/1 designs throughout optimization [11]. ESO method is based on an empirical concept that a structure evolves towards an optimum by slowly removing inefficient material [7,8]. Bi-directional evolutionary optimization (BESO) is an extension of that idea which allows for efficient material to be added to the structure at the same time as the inefficient one to be removed [12–14]. It is noted that the sequential linear programming (SLP)-based approximate optimization method followed by the Simplex algorithm is equivalent to ESO/BESO if the strain energy criterion is utilized [15].

Topology optimization problems for continuous structures are well known to be ill-posed problems, if one seeks an optimal distribution of solid and void materials without restriction. This shows up through numerical instabilities such as mesh-dependency [2,16,17]. In topology optimization using SIMP method, some restrictions on the density design field variation are normally required in order to ensure existence of solutions. A variety of restriction methods have been developed, for example, by perimeter control [18], gradient constraint [19] or blurring filters [20,21]. SIMP method incorporated with these restrictions leads to convergent, mesh-independent and nearly solid-void designs when a large penalty exponent (normally $p \geq 3$) is used.

ESO/BESO methods solve an optimization problem using discrete design variables (0 or 1). Several previous papers have dealt with the numerical instability problems of ESO/BESO methods. For example, to prevent checkerboarding, a smoothing algorithm using the surrounding elements' reference factors has been proposed by Li et al. [22].

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However, this smoothing procedure depends on the assigned mesh and fails to overcome the mesh-dependency problem. To overcome mesh-dependency, an additional restriction by perimeter control [23] or filter [24] has been introduced into BESO method more recently.

As has been pointed out by Zhou and Rozvany [25], removing (hard-killing) elements may cause non-optimal solution when it is used to solve a statically indeterminate structure with a highly inefficient support with a coarse mesh. The breakage of such boundary support could completely change the original optimization problem. Therefore BESO leads to a non-optimal solution if it fails to detect this change. To overcome this problem, Edwards et al. [26] and Huang and Xie [27] have proposed in their parallel but independent studies a fine mesh so that the breakage of the boundary is avoided and a member with a small size is allowed to remain in the optimal design.

In BESO method, the optimal topology is determined according to the relative ranking of sensitivity numbers. The sensitivity numbers for solid elements can be easily estimated by the approximate variation of objective function due to removing these individual elements. The sensitivity numbers for void elements is hardly determined because these void elements are not included in the finite element analysis. Therefore, elements were directly added to the regions of the structure neighboring those elements with higher sensitivity numbers [12,14]. Yang et al. [13] calculated the sensitivity numbers of void elements through the linear extrapolation of the displacement field. However, these BESO methods failed to obtain convergent solutions due to their separate criteria for material removal and addition. Recently, a new BESO algorithm has been developed [24] where sensitivity numbers for void elements are set to be zero initially, then modified by the filter scheme. Using a consistent removal and addition criterion the BESO algorithm leads to a convergent and mesh-independent solution and the resulted topology is very similar to the optimal topology of SIMP method.

Similarities in the resulted topologies may well indicate some intrinsic relationships between BESO sensitivity number and SIMP model. Therefore, we propose a new BESO method utilizing the material interpolation scheme with penalization in this paper. The sensitivity number with a penalty parameter is derived from the sensitivity analysis. It indicates that sensitivity numbers for solid elements and void elements can be set to be elemental strain energy and zero respectively when the penalty exponent tends to be infinity. However, the strain energy criterion is invalid for the topology optimization problems of structures involving two or more materials. In the end, a number of examples are presented to show effect of the penalty exponent and demonstrate the effectiveness of the present BESO method to solve topology optimization problems of structures with one or more materials.

2 Problem statements and material interpolation scheme

2.1 Problem statements

To maximizing the stiffness of a structure, the total strain energy of the structure or mean compliance (which is half of compliance) is minimized for a given volume of the material. The topology optimization problem can be stated as

$$\begin{aligned} &\text{minimize}: C = \tfrac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} \\ &\text{Subject to}: V^* - \sum_{i=1}^{N} V_i x_i = 0 \\ &x_i = x_{\min} \text{ or } 1 \end{aligned} \tag{1}$$

where $\mathbf{K}$ and $\mathbf{u}$ are the global stiffness matrix of the structure and displacement vectors. V_i is the volume of an individual element and V^* the prescribed total structural volume. The binary design variable x_i denotes the density of ith element. To avoid the singularity of the stiffness matrix, a small value of $x_{\min}$, for example, 0.001 is used to denote the void elements.

Sometimes, we may seek the optimal distribution of multiple materials and their elasticity modulus are $E_1, E_2, \ldots, E_n$ (where $E_1 > E_2 \cdots > E_n$). The corresponding optimization problem can be stated as

$$\begin{aligned} &\text{minimize}: C = \tfrac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} \\ &\text{Subject to}: V_j^* - \sum_{i=1}^{N} V_i x_{ij} - \sum_{i=1}^{j-1} V_i^* = 0 \\ &x_{ij} = x_{\min} \text{ or } 1 (j = 1,\ 2,\ \ldots,\ n-1) \end{aligned} \tag{2}$$

where V_j^* denotes the prescribed volume of jth material, the design variables x_{ij} denote the density of ith element for jth material and

$$x_{ij} = \begin{cases} 1 & \text{for } E \geq E_j \\ x_{\min} & \text{for } E \leq E_{j+1} \end{cases} \tag{3}$$

2.2 Material interpolation scheme

In Bendsøe and Sigmund [28], the material interpolation schemes were compared to various bounds for effective material properties in composite, for example, the Hashin–Shtrikman bounds. In the solid-void design case, Young's modulus of intermediate material can be interpolated as the function of the element density variable as

$$E(x_i) = E_1 x_i^p \tag{4}$$

where E_1 denotes the Young's modulus for solid material, p is the penalty exponent.

In the two non-zero materials case, the Young's modulus of two materials are E_1 and E_2 and $E_1 > E_2 \neq 0$. The material interpolation scheme can be expressed as

$$E(x_i) = x_i^p E_1 + (1 - x_i^p) E_2 \tag{5}$$

If the optimal design contains n types of materials and corresponding elasticity modulus are $E_1 > E_2 > \cdots > E_n$. It is straightforward to interpolate the material properties between two neighboring phases, for example, E_j and E_{j+1} as

$$E(x_{ij}) = x_{ij}^p E_j + (1 - x_{ij}^p) E_{j+1} 3 \tag{6}$$

where the subscript $j = 1, \ldots, n - 1$.

3 Sensitivity numbers and modified sensitivity numbers

3.1 Sensitivity numbers

The gradient of the objective function with respect to individual element density can be easily derived by the adjoint method [2]. For solid-void designs, the sensitivity of ith element can be expressed by

$$\frac{\partial C}{\partial x_i} = -\frac{p x_i^{p-1}}{2} \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_i \mathbf{u}_\mathbf{i} \tag{7}$$

where $\mathbf{K}_i$ are the elemental stiffness matrix.

In the evolutionary structural optimization method, a structure can be optimized using discrete design variables. That is to say that only two bound materials are allowed in the design. Also, the sensitivity number used in the evolutionary structural optimization method denotes the relative ranking of the sensitivity of an individual element. Therefore, it can be defined by

$$\alpha_i = -\frac{1}{p}\frac{\partial C}{\partial x_i} = \begin{cases} \frac{1}{2} \mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i & \text{when } x_i = 1 \\ \frac{x_{\min}^{p-1}}{2} \mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i & \text{when } x_i = x_{\min} \end{cases} \tag{8}$$

It can be seen that the above sensitivity numbers of "void" elements depends on the selection of the penalty exponent. When the penalty exponent tends to infinity, the above sensitivity number can be simply expressed as

$$\alpha_i = \begin{cases} \frac{1}{2} \mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i & \text{when } x_i = 1 \\ 0 & \text{when } x_i = x_{\min} \end{cases} \tag{9}$$

The above equation indicates that the sensitivity numbers for solid elements and "void" elements are equal to the elemental strain energy and zero respectively. It is consistent with the sensitivity numbers used in the original BESO method [24]. According to the material interpolation scheme Eq. (4), the stiffness of "void" elements also becomes zero as p tends to infinity. Therefore these "void" elements can be totally removed from the design domain except that the removal causes the singularity of global stiffness matrix or any change of boundary conditions [22,23].

Similarly, if the design composed of two non-zero materials the sensitivity number can be found through sensitivity analysis [28] as

$$\alpha_i = -\frac{1}{p}\frac{\partial C}{\partial x_i} = \frac{1}{2} x_i^{p-1} (\mathbf{u}_i^T \mathbf{K}_i^1 \mathbf{u}_i - \mathbf{u}_i^T \mathbf{K}_i^2 \mathbf{u}_i) \tag{10}$$

where K_i^1 and K_i^2 denote the elemental stiffness matrix calculated with mechanical properties of material 1 and 2, respectively. If we assumed both materials with a same Poisson's ratio, the sensitivity number for two materials can be expressed explicitly as

$$\alpha_i = \begin{cases} \frac{1}{2}\left[1 - \frac{E_2}{E_1}\right] \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_i^1 \mathbf{u}_\mathbf{i} & \text{for material 1} \\ \frac{1}{2} \frac{x_{\min}^{p-1}(E_1 - E_2)}{x_{\min}^p E_1 + (1 - x_{\min}^p) E_2} \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_i^2 \mathbf{u}_\mathbf{i} & \text{for material 2} \end{cases} \tag{11}$$

For the extreme case that p tends to be infinity, the sensitivity numbers can also be simplified as

$$\alpha_i = \begin{cases} \frac{1}{2}\left[1 - \frac{E_2}{E_1}\right] \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_\mathbf{i} \mathbf{u}_\mathbf{i} & \text{for material 1} \\ 0 & \text{for material 2} \end{cases} \tag{12}$$

It can be seen that the ranking of the above sensitivity number totally differs from the elemental strain energy criterion used in the original BESO method.

If we seek optimal designs with n types of materials, the sensitivity numbers can be derived by sensitivity analysis with the design variables x_{ij} as

$$\alpha_{ij} = -\frac{1}{p}\frac{\partial C}{\partial x_{ij}} = \frac{1}{2} x_{ij}^{p-1} (\mathbf{u}_i^T \mathbf{K}_i^j \mathbf{u}_i - \mathbf{u}_i^T \mathbf{K}_i^{j+1} \mathbf{u}_i) \tag{13}$$

where K_i^j and K_i^{j+1} denote the elemental stiffness matrix calculated with E_j and E_{j+1}. It should be noted that the sensitivity number α_{ij} is defined in the whole design domain for later filter scheme although it is only used for adjusting the material between material j and $j + 1$. Therefore, the sensitivity number can be explicitly expressed as

$$\alpha_{ij} = \begin{cases} \frac{1}{2}\left[1 - \frac{E_{j+1}}{E_j}\right] \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_i^j \mathbf{u}_\mathbf{i} \\ \quad \text{for materials } 1, \ldots, j \\ \frac{1}{2} \frac{x_{\min}^{p-1}(E_j - E_{j+1})}{x_{\min}^p E_j + (1 - x_{\min}^p) E_{j+1}} \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_i^{j+1} \mathbf{u}_\mathbf{i} \\ \quad \text{for materials } j+1, \ldots, n \end{cases} \tag{14}$$

When p tends to infinity, the sensitivity number can be written as

$$\alpha_{ij} = \begin{cases} \frac{1}{2}\left[1 - \frac{E_{j+1}}{E_j}\right] \mathbf{u}_\mathbf{i}^\mathbf{T} \mathbf{K}_\mathbf{i} \mathbf{u}_\mathbf{i} \\ \quad \text{for materials } 1, \ldots, j \\ 0 \\ \quad \text{for materials } j+1, \ldots, n \end{cases} \tag{15}$$

It can be seen that there is $n - 1$ groups of sensitivity numbers in the system to adjust neighboring materials.

3.2 Filter scheme and stabilization of evolutionary process

To avoid numerical instabilities such as checkerboard and mesh-dependency, the filter technique is applied to smooth the sensitivity number in the whole design domain. First, the nodal sensitivity numbers (α_i^n) which do not carry any physical meaning on their own are defined by averaging the sensitivity numbers of connected elements. Then, these nodal sensitivity numbers must be converted back into element before the topology can be determined. Here, a filter scheme is used to carry out this process. The defined filter functions are based on a length scale $r_{\min}$. The primary role of the scale parameter $r_{\min}$ in the filter scheme is to identify the nodes that influence the sensitivity of ith element. This can be visualized by drawing a circle of radius $r_{\min}$ centered at the centroid of ith element, thus generating the circular sub-domain Ω_i. Nodes located inside Ω_i contribute to the computation of the modified sensitivity number of ith element as

$$\hat{\alpha}_i = \frac{\sum_{j=1}^{M} w(r_{ij})\alpha_j^n}{\sum_{j=1}^{M} w(r_{ij})} \tag{16}$$

where M is the total number of nodes in the sub-domain Ω_i and r_{ij} denotes the distance between the center of the element i and node $j.w(r_{ij})$ is weight factor given as

$$w(r_{ij}) = \begin{cases} r_{\min} - r_{ij} & \text{for } r_{ij} < r_{\min} \\ 0 & \text{for } r_{ij} \geq r_{\min} \end{cases} \tag{17}$$

However, the BESO algorithm using above sensitivity numbers is hard to convergent. Computational experience has shown that averaging the sensitivity number with its historical information is an effective way to avoid this problem [24]. It can be simply expressed by

$$\tilde{\alpha}_i = \frac{1}{2}(\hat{\alpha}_{i,k} + \hat{\alpha}_{i,k-1}) \tag{18}$$

where k is the current iteration number. Then let $\hat{\alpha}_{i,k} = \tilde{\alpha}_i$, thus the modified sensitivity number considers the sensitivity information in the previous iterations. It should be noted that the above equation has almost no effect on the solution when it is convergent. For optimizing structures with multiple materials, all sensitivity numbers α_{ij} are modified by the above filter scheme and averaging with their corresponding historical information.

3.3 Convergence criterion

To stop the optimization process, the convergence criterion should be given after the objective volumes for all materials are achieved. The following convergence criterion is defined in the variation of the objective function

$$error = \frac{\left|\sum_{i=1}^{N} (C_{k-i+1} - C_{k-N-i+1})\right|}{\sum_{i=1}^{N} C_{k-i+1}} \leq \tau \tag{19}$$

where k is the current iteration number, τ is a allowable convergence error and N is integral number which are set to be 0.01% and 5 throughout this paper. It means a stable compliance at least in successive 10 iterations, thus the effect of Eq. (18) on the final solution can be nearly eliminated.

4 Bi-directional evolutionary structural optimization procedure

The evolutionary iteration procedure for solid-void and two non-zero material designs are same except for the calculation of sensitivity numbers. It can be outlined as follows:

Step 1: Discrete the whole design domain using a finite element mesh.

Step 2: Define the BESO parameters such as objective volume, V^*, evolutionary ratio ER and penalty exponent, p.

Step 3: Carry out a finite element analysis (FEA) which is conducted by ABAQUS in this paper.

Step 4: Calculate elemental the sensitivity numbers (Eq. (8) for solid-void designs and Eq. (11) for two non-zero material designs) and update sensitivity numbers by filtering (Eq. (16)) and averaging with historical information (Eq. (18)).

Step 5: Determine the target volume for the next design. When the current volume V_i is larger than the objective volume V^*, the target volume for the next design can be calculated by

$$V_{i+1} = V_i(1 - ER) \tag{20}$$

If the calculated volume for the next design is less than the objective volume V^*, the target volume for the next design V_{i+1} is set to be V^*.

Step 6: Reset the design variables of all elements. For solid elements, the elemental density is switched from 1 to $x_{\min}$ if the following criterion is satisfied.

$$\alpha_i \leq \alpha_{th} \tag{21a}$$

For "void" elements, the elemental density is switched from $x_{\min}$ to 1 if the following criterion is satisfied.

$$\alpha_i > \alpha_{th} \tag{21b}$$

Where α_{th} is the threshold of the sensitivity number which can be easily determined by the target material volume and ranking of the sensitivity number.

Step 7: Repeat 3–6 until the objective volume is achieved and the convergent criterion defined in Eq. (19) is satisfied.

The basic BESO procedure for multiple material designs is similar to that for solid-void designs except that the above calculation should be applied for every group of sensitivity number α_{ij} according to Eq. (14). BESO starts from the full design with material 1 and triggers by evolutionary ratio ER which is defined as the proportion of volume reduction of material 1 relative to the total volume of material 1 in the current design. At the same time, the volume of material 2 gradually increases until the objective volume is achieved. Thereafter, the volume of material 2 keeps constant and the volume of material 3 gradually increases until the objective volume is achieved, and so on. Transition between materials 1 and 2 is carried out according to the target volume of material 1 and sensitivity numbers $\tilde{\alpha}_{i1}$. Similarly, transition between materials 2 and 3 is carried out according to the target volume of material 2 and sensitivity numbers $\tilde{\alpha}_{i2}$, and so on. The whole optimization procedure is stopped when objective volumes for all materials are achieved and the convergent criterion is satisfied.

5 Numerical examples

5.1 Examples for solid-void designs

The topology optimization problem of a cantilever beam is depicted in Fig. 1 where the 80 mm × 50 mm domain with a unit thickness is discretized by four-node 80 × 50 quadrilateral element mesh. The material properties are Young's modulus $E = 100$ GPa, Poisson's ratio $v = 0.3$. The filter radius is $r_{\min} = 3$ mm. The lower bound of the material density is set to be 0.001 which denotes void element in the following simulations. The objective is to design the stiffest structure by minimizing its mean compliance with 50% of material in the design domain. The evolutionary ratio is set to be $ER = 2\%$.

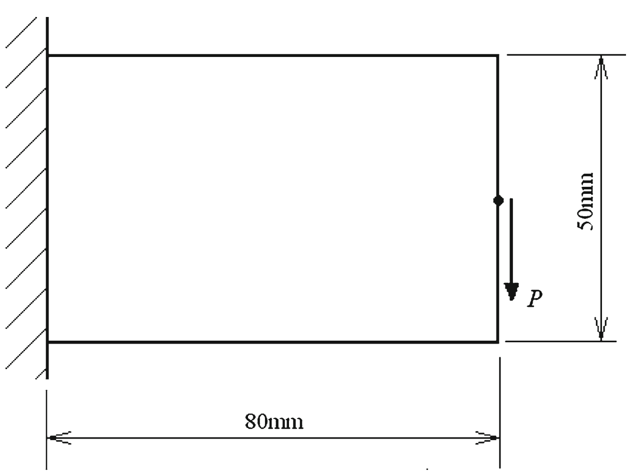

Fig. 1 Design domain for a cantilever beam

Figure 2 shows the evolutionary histories of the mean compliance, volume fraction and topologies for the proposed BESO method using $p = 3.0$. It indicates that the mean compliance increases initially as the ‘volume fraction decreases, then it is convergent to an almost constant value after the objective volume is achieved. The final structure is convergent to a stable topology after 65 iterations.

Figure 3a and b shows the BESO optimal designs for $p = 1.5$ and $p = 3.0$, respectively. The mean compliances of both designs are 1.865 Nmm. The results show the penalty exponent p has almost no effect on the optimal design. As a result, both optimal topologies and mean compliances are very close to the results of the "hard-killing" BESO method as shown in the reference [24]. Figure 3c shows a optimal topology

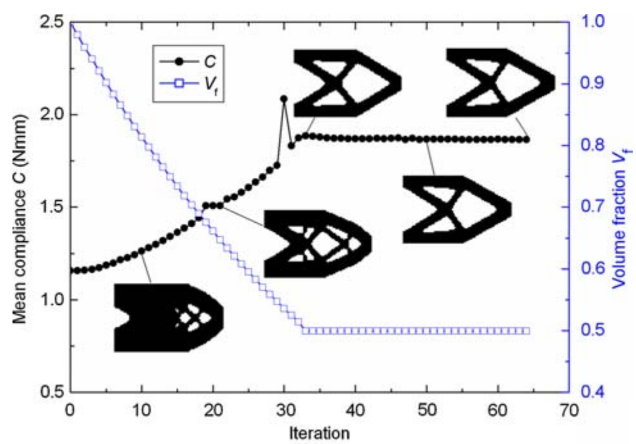

Fig. 2 Evolutionary histories of the mean compliance, volume fraction and topology when $p = 3.0$

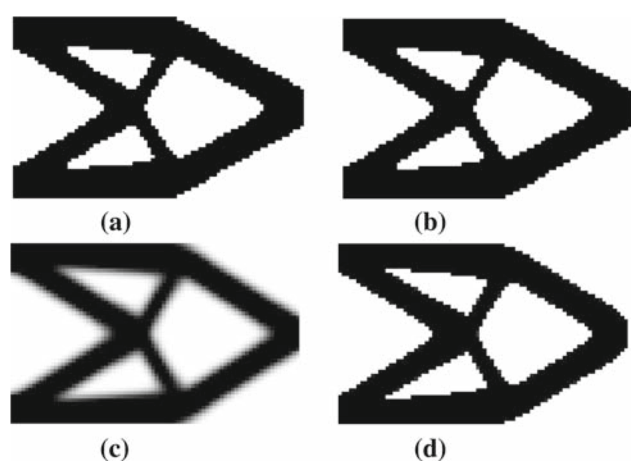

Fig. 3 Optimal designs **a** BESO with $p = 1.5$, **b** BESO with $p = 3.0$, **c** SIMP with $p = 3.0$ and **d** the continuation method

using SIMP method [29] with $p = 3.0$. Its topology is similar to those of above BESO designs except some "grey" elements. The mean compliance of the design is 2.05 Nmm which is about 10% higher than that of BESO designs due to those inefficient "grey" elements. To avoid premature convergence to local minima in SIMP method, the continuation method was proposed by gradually increasing penalty parameter from lower bound to upper bound [30,31]. For mesh-independence filter Sigmund [32] and Sigmund and Torquato [33] suggested starting with a large value of the filter size and gradually to decrease it, to end up with a 0–1 design. Figure 3d shows the final design via the continuation method which decreased the filter size from $r_{\min} = 3$ mm to 1 mm in increments 0.05 mm. The resulted optimal topology is almost identical to these BESO topologies. Its mean compliance is 1.856 Nmm that is just about 0.4% lower than that of BESO designs.

Next, we show a topology optimization problem for a beam structure. The design domain with 240 mm × 40 mm is supported by both ends and vertically loaded ($P = 10$N) in the middle of its lower edge as depicted in Fig. 4. Due to the symmetry, the computations are performed in only right-half of the domain with 120 × 40 four-node plane stress elements. The linear material is assumed with Young's modulus $E = 1$ GPa, Poisson's ratio $v = 0.3$. The final volume constraint is set to 30% of the design domain. The parameters used for all of following simulations are $x_{\min} = 0.001$, $r_{\min} = 3.0$ mm.

The BESO starts from the full design and gradually decreases the total volume according to evolutionary ratio, $ER = 2\%$. Figure 5a shows the optimal design for $p = 2.0$ after 102 iterations. Its mean compliance is 7.26 Nmm. Figure 5b and c show the optimal designs for SIMP method with $p = 3.0$ and the continuation method. Their corresponding mean compliances are 9.28 and 7.18 Nmm.

Both examples show that the proposed BESO method can obtain optimal topologies which are similar to SIMP method. The values of the objective function are lower than that of SIMP method and very close to that of the continuation method. It is also shown the penalty exponent almost has no effect on the obtained optimal designs using the present BESO method with the given meshes. Therefore, the penalty exponent can be set to be any value even infinity which is equivalent to the original BESO method for solid-void designs. The researches had shown that the optimization

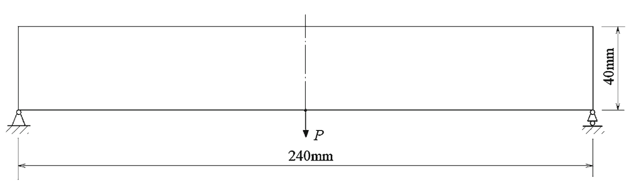

Fig. 4 Design domain and support conditions for a beam

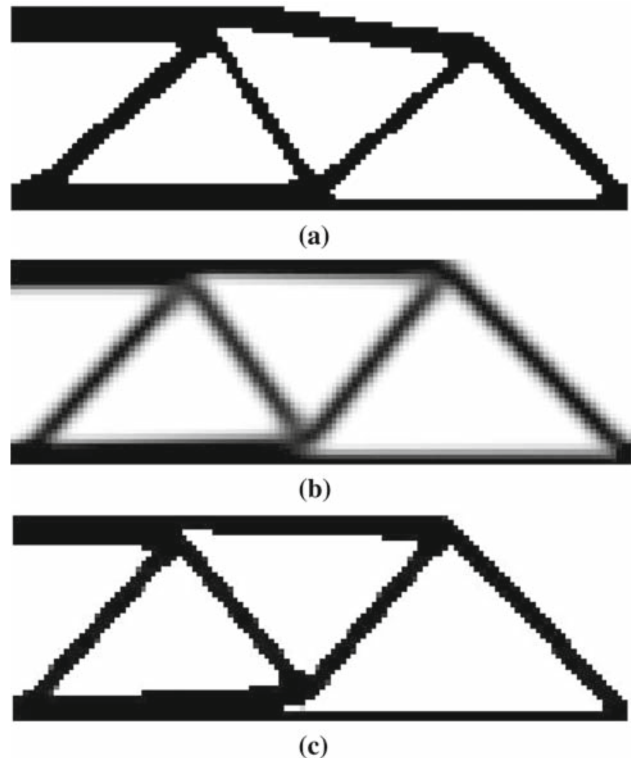

Fig. 5 Optimal designs **a** BESO with $p = 2.0$, **b** SIMP with $p = 3.0$ and **c** the continuation method

using the material extrapolation scheme resulted in nearly solid-void design if one chooses p sufficiently big ($p \geq 3$ is usually required) [28,30]. Numerical experience indicates that $p \geq 1.5$ is usually required for the present BESO method to obtain a convergent solution.

5.2 Examples for design with two non-zero materials

Consider the topology optimization problem as shown in Fig. 4 using two non-zero materials. The topology optimizations were performed for the following two cases: (a) two materials with $E_1 = 1$ GPa and $E_2 = 0.1$ GPa, $p = 3$; (b) two materials with $E_1 = 1$ GPa and $E_2 = 0.2$ GPa, $p =$infinity. In both cases, the objective volume of material 1 is set to be 50% of the design domain. The parameters used for all of following simulations are $ER = 2\%$ and $r_{\min} = 3$ mm. $x_{\min} = 1.0 \times 10^{-5}$ is used when $p = 3$.

Figure 6 shows the evolutionary histories of mean compliance, topology and volume fraction of material 1 for case a. It demonstrates that the proposed BESO method leads to a convergent solution for objective function and topology after 57 iterations. Figure 7 shows the final optimal topologies for both cases. Obviously, the optimal topologies are totally different as different materials are used because the sensitivity numbers depend on the Young's modulus ratio of two materials.

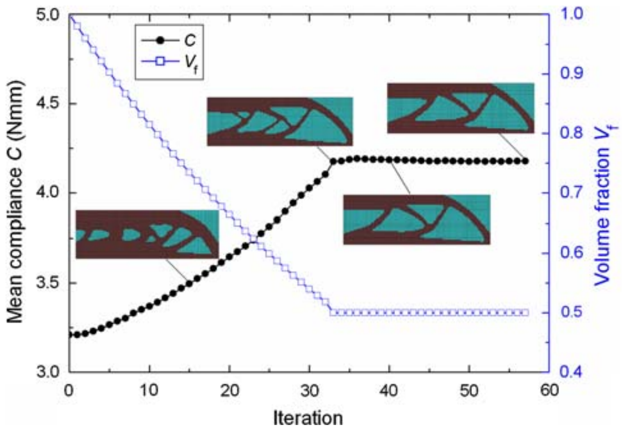

Fig. 6 Evolutionary histories of the mean compliance, volume fractions and topology for a two material structure ($E_1 = 1\,\mathrm{GPa}$, $E_2 = 0.1\,\mathrm{GPa}$) with $p = 3.0$

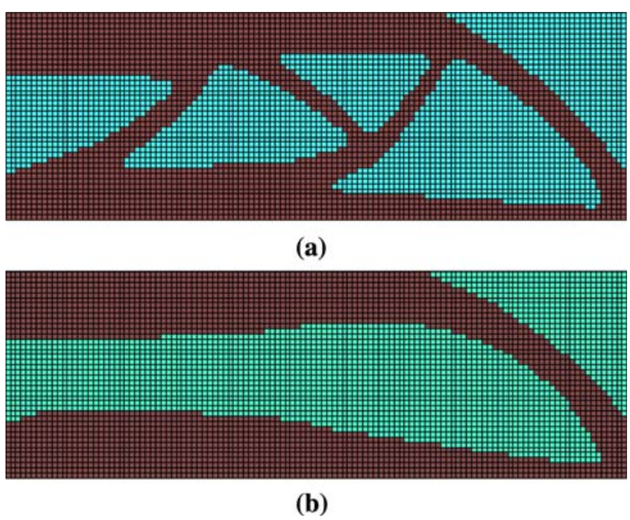

Fig. 7 BESO optimal designs for a two material structure **a** $E_1 = 1\,\mathrm{GPa}$, $E_2 = 0.1\,\mathrm{GPa}$; **b** $E_1 = 1\,\mathrm{GPa}$, $E_2 = 0.2\,\mathrm{GPa}$

5.3 Examples for designs with void and two materials

We consider the above optimization problem shown in Fig. 4 where the optimal design is composed with void and two materials with $E_1 = 1\,\mathrm{GPa}$, $E_2 = 0.1\,\mathrm{GPa}$. The Poisson's ratio is 0.3 for all material. The objective volume of materials 1 and 2 are set to be 15 and 25% of the whole design domain. The above BESO method for multiple materials is applied for p =infinity and void elements are totally removed from designs. The optimal design is shown in Fig. 8 and its mean compliance is 13.0 Nmm. The optimal topology shows a sandwich structure with a stiff skin (material 1) and soft truss core (material 2). Figure 9 shows the evolutionary histories of volume fractions for materials 1 and 2, topology and mean compliance. Both topology and objective function start to be convergent to stable solutions after 130 iterations.

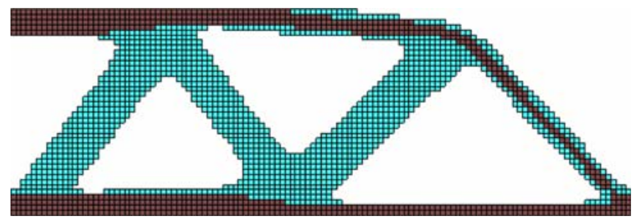

Fig. 8 BESO optimal design for a three-phase structure

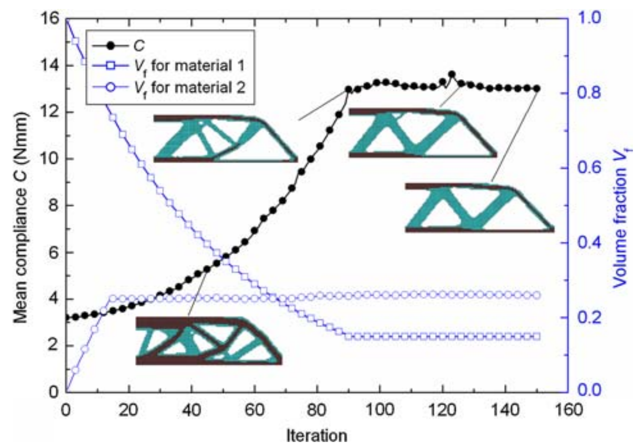

Fig. 9 Evolutionary histories of the mean compliance, volume fractions and topology for a three-phase structure

5.4 Discussion on the "hard-kill" BESO method

It is well known that the solid-void topology optimization problem for continuum structures without a size limitation lacks solution. In the BESO method with an infinite penalty exponent, any member in the structure which is thinner than one-element width will be immediately penalized to void elements. Thereafter, the structure evolves to an optimum which is composed with members whose width are more than or equal to one-element size. Therefore, it is also called as the "hard-kill" BESO method.

However, hard-killing elements may cause the failure of the BESO method when a prescribed boundary support is totally broken for a statically indeterminate structure such as a cantilever tie-beam in Zhou and Rozvany [25]. Therefore, Huang and Xie [27] suggested a simple way to detect the possible failure of the "hard-kill" BESO method by checking the boundary conditions of the design after each iteration. If a breakdown of boundary support occurs before a satisfactory solution is obtained, it may well indicate that the used mesh is too coarse for the BESO method. The scenario can be compared to the situation when the design loses its integrity (the breakage of the connection between external forces and boundary supports). Fortunately, the later case can be detected by finite element analysis automatically due to the singularity of the stiffness matrix. For both cases, the problem needs to be resolved with a finer mesh. The details may refer to the recent paper by Huang and Xie [27].

The limit of the mesh size for the BESO method depends on the optimization problem such as loading and support conditions and the volume constraint. It may not be possible to determine the appropriate mesh size beforehand except for a simple problem. Therefore, a disadvantage of the "hard-kill" BESO method is that the optimal solution, especially for small volume fraction, may be obtained after several trials on mesh refinements.

6 Conclusions

Using material interpolation schemes with penalization, this paper has developed a new and generalized BESO method. For solid-void design, it is shown that the original BESO is a special case of the present BESO method where the penalty exponent tends to infinity. Numerical examples show the resulted topologies are similar to those obtained by SIMP and continuation methods. The value of the objective function is much lower than that of SIMP method and very close to that of the continuation method. These examples demonstrate that the optimal designs of the present BESO method are independent of the selection of penalty exponent (normally $p \geq 1.5$) and therefore are the same as those of the original "hard-kill" BESO method for solid-void designs.

However, the sensitivity number in the present BESO method is totally different from the strain energy criterion in the original BESO method when the material interpolation scheme utilized for topology optimization problems of structures with two or more materials. Numerical examples show that the present BESO method also results in convergent solutions for this type of topology optimization problems and the optimal designs are independent of the degree of penalization.

The present BESO method can only be applied to the self-adjoint optimization problems with simple volume constraints. Further work on the more complex optimization problems needs to be conducted in the future.

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