"Ans To P Set 2, Part A"

Econ 6020: Macro Theory I

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II DGE Capital Accumulation, Part A:

Additional Problem 1:

We have, using the Cobb douglass Production function,

$$\boxed{\Delta K_{t+1} = A K_t^{\alpha} - C_t - \delta K_t \qquad (1)}$$

and

$$\boxed{\Delta C_{t+1} = -\frac{u'(C_t)}{u''(C_t)}\left[1 - \frac{1}{\beta\left[\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\right]}\right] \qquad (2)}$$

(a). Consider $\Delta\theta > 0$ from θ_0 to $\theta_1 > \theta_0$

(Since $\beta \equiv \frac{1}{1+\theta}$, an increase in θ is a decline in β.)

→ Note from (2) that $\Delta C_{t+1} = 0$ where

$$\beta\left[\alpha A K_S^{\alpha-1} + 1 - \delta\right] = 1$$

or using $\beta = \frac{1}{1+\theta}$

$$\alpha A K_S^{\alpha-1} = \theta + \delta$$

or

↘

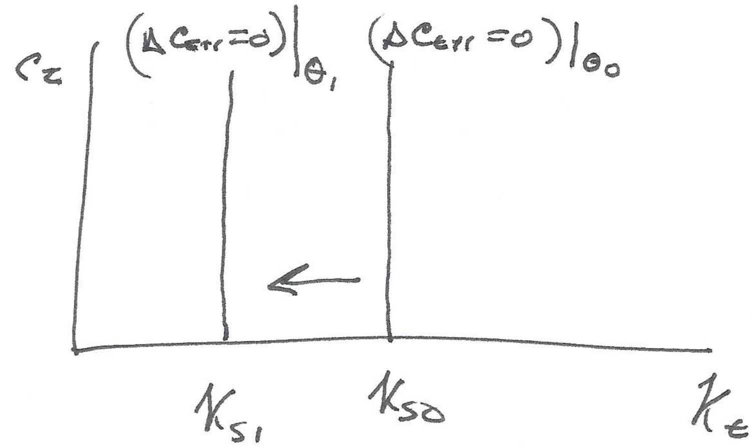
$$K_s^{\alpha-1} = \left[\frac{\theta+\delta}{\alpha A}\right] \quad \text{or}$$

$$K_s = \left[\frac{\theta+\delta}{\alpha A}\right]^{\frac{1}{\alpha-1}} \quad \text{or}$$

$$\boxed{K_s = \left[\frac{\alpha A}{\theta+\delta}\right]^{\frac{1}{1-\alpha}} \qquad (3)}$$

Based on (3) we see that $\Delta\theta > 0$ from θ_0 to $\theta_1 > \theta_0$ causes $\Delta K_s < 0$ from K_{s0} to $K_{s1} < K_{s0}$.

This is a shift to the left of the $\Delta C_{t+1} = 0$ locus

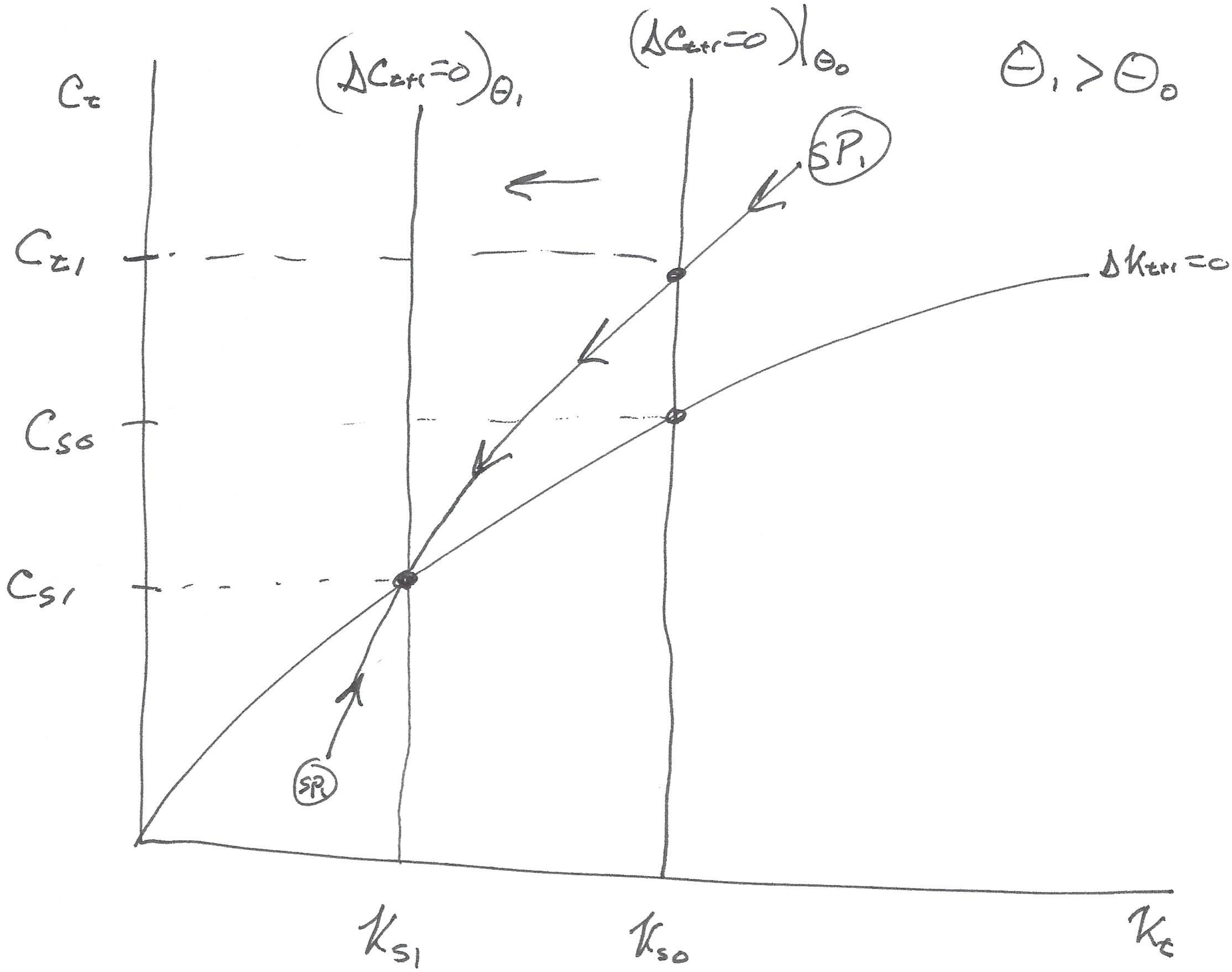


→ Next, note that $\Delta K_{t+1} = 0$ where

$$\boxed{C_t\Big|_{\Delta K_{t+1}=0} = A K_t^{\alpha} - \delta K_t \qquad (4)}$$

From (4) we see that $\Delta\Theta > 0$ has no effect on the $\Delta K_{t+1} = 0$ locus.

→ Assume we start from the original steady state with $\Theta = \Theta_0$. Call this steady state (K_{s0}, C_{s0})



→ As shown above $\Delta\Theta > 0$ to $\Theta_1 > \Theta_0$ causes the $\Delta C_{t+1} = 0$ curve to shift to the left. The new steady state is (K_{S1}, C_{S1}). The new saddle path is shown and labelled "SP₁".

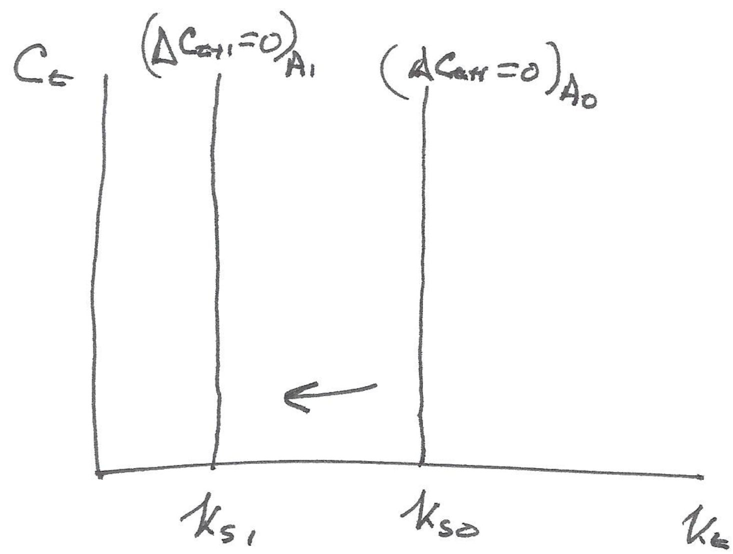
→ When Θ increases to Θ_1 there is an immediate increase in consumption to C_{Z1}. Since the capital stock does not change output stays the same: $y_{S0} = A K_{S0}^{\alpha}$. Thus, the increase in consumption requires a decline in investment. With ~~lower~~ lower investment the capital stock depreciates. (Investment at the steady state is at the "replacement rate" – replacing depreciating capital and nothing more. The increase in consumption requires that investment fall below the ~~stay~~ replacement rate and the capital stock begins to decline.)

→ From (K_{S0}, C_{t1}) Capital, hence output, and consumption decline converging to their new ~~equi~~ steady state equilibrium values K_{S1}, C_{S1}, and $Y_{S1} = A K_{S1}^{\alpha}$.

b) Consider $\Delta A < 0$.

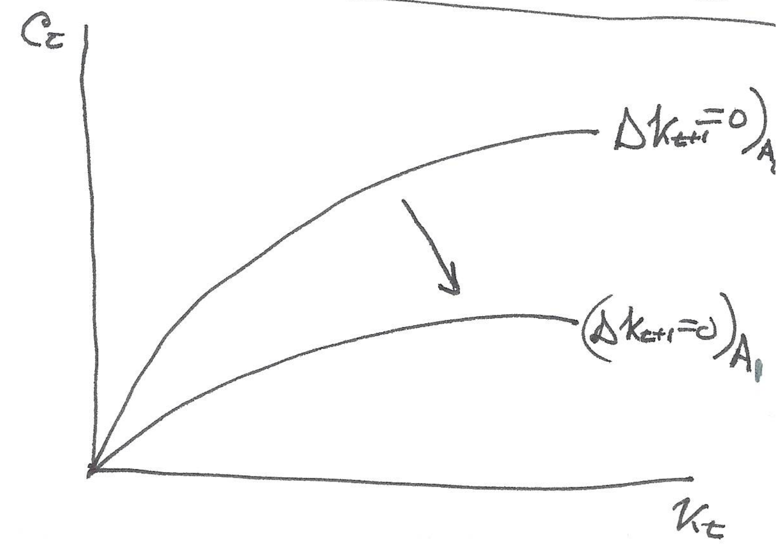
→ Note from (3) that $\Delta A < 0$ from, say, A_0 to $A_1 < A_0$, will cause a decline in K_S.

Thus $\Delta A < 0$ will shift the $\Delta C_{t+1} = 0$ locus to the left.



→ Note also from eqn (4) that a decline in A will cause the $\Delta K_{t+1} = 0$ locus to rotate down in a ~~counter~~ clockwise direction.

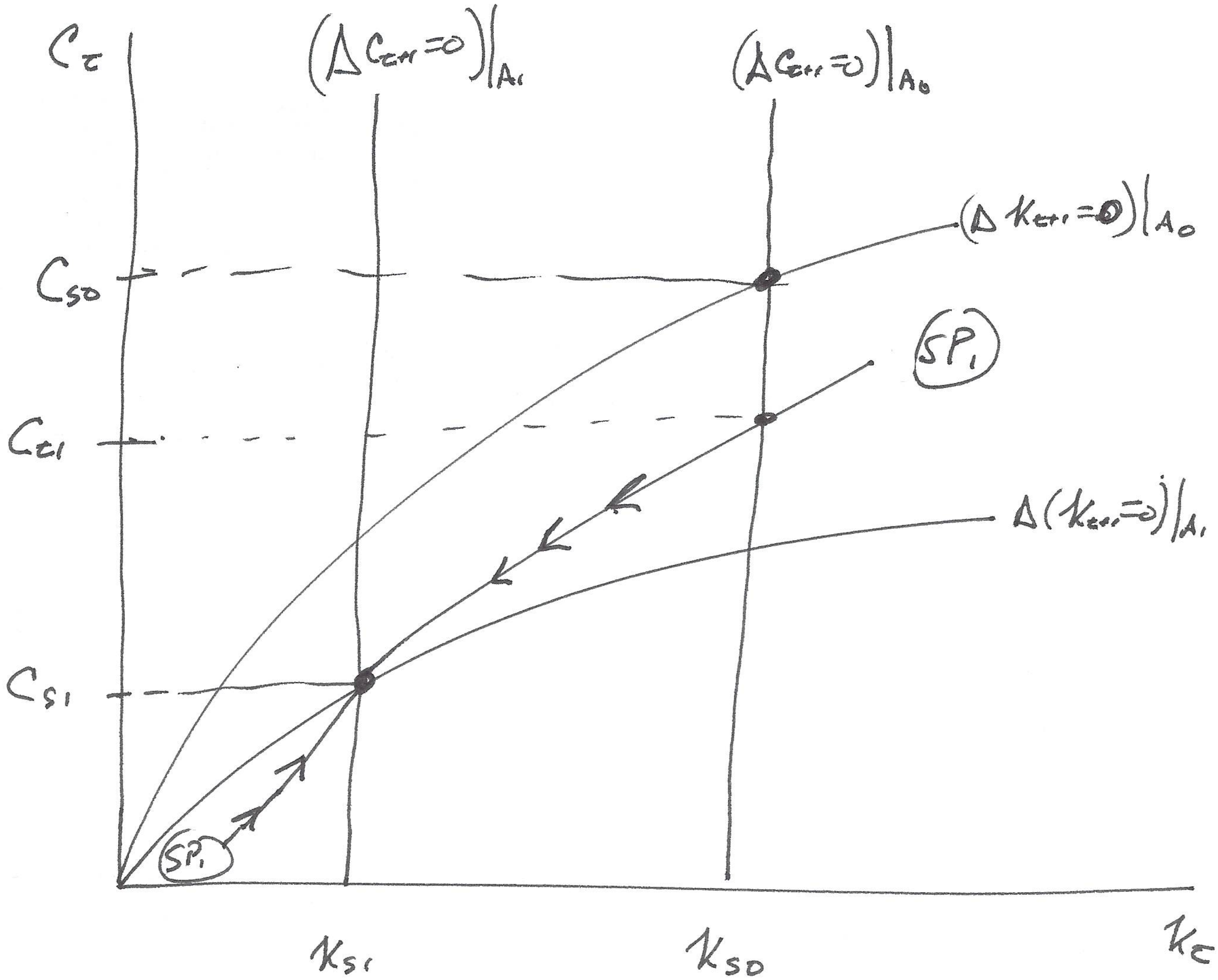
$\Delta A < 0$ will lower $C_t \big|_{\Delta K_{t+1}=0}$ at any given K_t. The locus will still pass through the origin. Thus it rotates down.



→ putting the two shifts together (Figure Next Page) we see that the new saddlepath will be SP_1.

→ Assume we start from C_{S0} K_{S0}. The decline in A to $A_1 < A_0$ causes optimal consumption to immediately drop to C_{t1}. Note also that $\Delta A < 0$ causes $\Delta Y_t < 0$ because $Y_t = A K_t^{\alpha}$.

→ Although consumption declines, output also declines. We can infer from the dynamics implied by the saddlepath that K_t will decline through time. This will cause further declines in Y_t. C_t will also decline through time.


C_t
$(\Delta C_{t+1}=0)|_{A_1}$
$(\Delta C_{t+1}=0)|_{A_0}$
$(\Delta K_{t+1}=0)|_{A_0}$
C_{s0}
SP_1
C_{t1}
$\Delta(K_{t+1}=0)|_{A_1}$
C_{s1}
SP_1
K_{s1}
K_{s0}
K_t

Furthermore, we can infer from declining K_t that investment at K_{s0} (after $\Delta A < 0$) is below the replacement rate.

→ K_t and C_t will decline converging to the new steady state equilibrium K_{s1} C_{s1}. Note that $\Delta A < 0$ causes $\Delta K_s < 0$, $\Delta C_s < 0$ and $\Delta Y_s < 0$.

(c). Consider $\Delta \delta > 0$.

Note from eqn (3) that $\Delta \delta > 0$ will reduce K_s. Note from eqn (4) that $\Delta \delta > 0$ will cause the $\Delta K_{t+1} = 0$ locus to rotate down in a clockwise direction. Thus the shift in the phase diagram is qualitatively the same as in the previous example ($\Delta A < 0$), but the economic interpretation would focus on the effect of higher depreciation of capital.

Additional Problem 2

The Planner's Problem is

$$\max \sum_{s=0}^{\infty} \beta^s u(c_{t+s}, l_{t+s}) \qquad (1)$$

S.T. $\quad l_t + n_t = 1 \qquad (2)$

and $\quad k_{t+1} = (1-\delta)k_t + F(k_t, n_t) - c_t \qquad (5)$

Note that (5) combines (4) and (3).

A. Lagrangian

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \Big\{ \beta^s u(c_{t+s}, l_{t+s}) + \mu_{t+s}\left[1 - l_{t+s} - n_{t+s}\right]$$
$$+ \lambda_{t+s}\left[(1-\delta)k_{t+s} + F(k_{t+s}, n_{t+s}) - c_{t+s} - k_{t+s+1}\right]\Big\} \qquad (6)$$

First-order conditions

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u_c(c_{t+s}, l_{t+s}) - \lambda_{t+s} = 0 \qquad (7)$$

$$\frac{\partial \mathcal{L}_t}{\partial l_{t+s}} = \beta^s u_l(c_{t+s}, l_{t+s}) - \mu_{t+s} = 0 \qquad (8)$$

$$\frac{\partial \mathcal{L}_t}{\partial n_{t+s}} = -\mu_{t+s} + \lambda_{t+s} F_n(k_{t+s}, n_{t+s}) = 0 \qquad (9)$$

$$\frac{\partial \mathcal{L}_t}{\partial k_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}\left[1 - \delta + F_k(k_{t+s+1}, n_{t+s+1})\right] = 0 \qquad (10)$$

$\dfrac{\partial \mathcal{L}_t}{\partial \mu_{t+s}} = 0$ gives (2) and $\dfrac{\partial \mathcal{L}_t}{\partial \lambda_{t+s}}$ gives (5)

B.) Together, (7) and (10) give

$$\boxed{U_c(c_{t+s}, \ell_{t+s}) = \beta U_c(c_{t+s+1}, \ell_{t+s+1})\left[F_K(K_{t+s+1}, n_{t+s+1}) + 1 - \delta\right] \quad (11)}$$

This is the <u>INTERTEMPORAL OPTIMALITY CONDITION</u>

Together (7), (8), and (9) give

$$-\beta^s U_\ell(c_{t+s}, \ell_{t+s}) + \beta^s U_c(c_{t+s}, \ell_{t+s}) F_n(K_{t+s}, n_{t+s}) = 0$$

or $$F_n(K_t, n_t) = \frac{U_\ell(c_t, \ell_t)}{U_c(c_t, \ell_t)} \quad (12)$$

This, eqn (12), is the <u>INTRATEMPORAL OPTIMALITY CONDITION</u>

C.) Compare (11) to the intertemporal Euler eqn in the model developed in class, which is

$$U'(c_t) = \beta U'(c_{t+1})\left[F_K(K_{t+1}) + 1 - \delta\right] \quad (13)$$

Both (11) AND (13) HAVE the <u>SAME form</u>, which is

$$MUC(t) = \beta\, MUC(t+1)\left[MPK(t+1) + 1 - \delta\right] \quad (14)$$

<u>The Difference</u> is that when VARIABLE LABOR/leisure is included, AS in EQN (11) The MUC CAN BE Affected by the choice of labor/leisure and the MPK can be affected by the LABOR input.

Wickens. Problem 2.4

The central planner's problem is to

$$\text{Maximize} \sum_{s=0}^{\infty} \beta^s U(C_{t+s}, l_{t+s}) \qquad (4.1)$$

~~Subject to~~

where $U(C_{t+s}, l_{t+s}) = \ln C_{t+s} + \phi \ln l_{t+s}$, $\qquad (4.2)$

$$\beta = \frac{1}{1+\theta}, \quad \theta > 0.$$

Maximization is subject to

$$y_t = C_t + I_t \qquad (4.3)$$

$$\Delta K_{t+1} = I_t - \delta K_t \qquad (4.4)$$

$$y_t = F(K_t, n_t) \qquad (4.5)$$

$$F(K_t, n_t) = A\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right]^{\frac{1}{1-\frac{1}{\gamma}}} \qquad (4.6)$$

$$\text{and } l_t + n_t = 1 \qquad (4.7)$$

(a) Derive the expressions from which the long-run equilibrium values of optimal consumption, labor, and capital can be obtained.

(b) Derive the long-run equilibrium real interest rate and real wage rate

(c) Comment on the implications for labor of having an elasticity of substitution between capital and labor different from unity. (That is, what are the implications of $\gamma \neq 1$).

(d) Derive the long-run equilibrium capital-labor ratio.

Solution (a) This problem fits the general form of the problem in chapter 2.6 but it adds the particular ~~specifications~~ specifications of the utility function, eqn (4.2) above, and the production function, eqn (4.6) above.

The Lagrangian is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \Big\{ \beta^s U(c_{t+s}, l_{t+s}) + \lambda_{t+s}\big[F(K_{t+s}, n_{t+s}) - c_{t+s} - K_{t+s+1} + (1-\delta)K_{t+s}\big] + \mu_{t+s}\big[1 - n_{t+s} - l_{t+s}\big]\Big\} \qquad (4.8)$$

The First-order Conditions Are

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s U_c(t+s) - \lambda_{t+s} = 0 \qquad (4.9a)$$

cf W(2.25)

$$\frac{\partial \mathcal{L}_t}{\partial l_{t+s}} = \beta^s U_l(t+s) - \mu_{t+s} = 0 \qquad (4.9b)$$

cf W(2.26)

$$\frac{\partial \mathcal{L}_t}{\partial n_{t+s}} = \lambda_{t+s} F_n(t+s) - \mu_{t+s} = 0 \qquad (4.9c)$$

cf W(2.27)

$$\frac{\partial \mathcal{L}_t}{\partial K_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}\left[F_K(t+s+1) + 1 - \delta\right] = 0 \qquad (4.9d)$$

cf W(2.28)

From (4.9a) we have that

$$\lambda_{t+s} = \beta^s U_c(t+s) \qquad (4.10)$$

Using (4.10) to eliminate λ_{t+s} and λ_{t+s+1} in (4.9d) gives

$$-\beta^s U_c(t+s) + \beta^{s+1} U_c(t+s+1)\left[F_K(t+s+1) + 1 - \delta\right] = 0$$

or, for $s=0$,

$$\beta \frac{u_c(t+1)}{u_c(t)}\left[F_K(t+1) + 1 - \delta\right] = 1 \qquad (4.11)$$

Eqn (4.11) is the intertemporal Euler Equation.

Note from (4.2) that

$$u_c(C_{t+s}) = \frac{1}{C_{t+s}} \qquad (4.12)$$

Also, from (4.6) we can obtain

$$F_K(K_t, M_t) = A\left(\frac{1}{1-\frac{1}{\gamma}}\right)\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha)M_t^{1-\frac{1}{\gamma}}\right]^{\left(\frac{1}{1-\frac{1}{\gamma}}\right)-1} \cdot \left(1-\frac{1}{\gamma}\right)\alpha K_t^{\frac{-1}{\gamma}}$$

or $$F_K(K_t, M_t) = \alpha A\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha)M_t^{1-\frac{1}{\gamma}}\right]^{\left(\frac{1}{1-\frac{1}{\gamma}}\right)-1} \cdot K_t^{\frac{-1}{\gamma}}$$

But note that

$$\frac{1}{1-\frac{1}{\gamma}} - 1 = \frac{1-1+\frac{1}{\gamma}}{1-\frac{1}{\gamma}} = \frac{\frac{1}{\gamma}}{1-\frac{1}{\gamma}} = \left(\frac{1}{1-\frac{1}{\gamma}}\right)\frac{1}{\gamma}$$

So we can write

$$F_K(K_t, M_t) = \alpha A^{1-\frac{1}{\gamma}} A^{\frac{1}{\gamma}}\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha)M_t^{1-\frac{1}{\gamma}}\right]^{\left(\frac{1}{1-\frac{1}{\gamma}}\right)\frac{1}{\gamma}} K_t^{\frac{-1}{\gamma}}$$

or $$F_K(K_t, M_t) = \alpha A^{1-\frac{1}{\gamma}} Y_t^{\frac{1}{\gamma}} K_t^{\frac{-1}{\gamma}}$$

or

$$F_K(K_t, M_t) = \alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y_t}{K_t}\right)^{\frac{1}{\gamma}} \qquad (4.13)$$

Using (4.12) and (4.13) in (4.11)

$$\beta \frac{C_t}{C_{t+1}} \left[\alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y_{t+1}}{K_{t+1}}\right)^{\frac{1}{\gamma}} + 1 - \delta\right] = 1 \qquad (4.14)$$

Let variables with no subscript denote long-run (steady-state) values. Eqn (4.14) then gives

$$\beta \frac{C}{C}\left[\alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y}{K}\right)^{\frac{1}{\gamma}} + 1 - \delta\right] = 1 \quad \text{or}$$

$$\alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y}{K}\right)^{\frac{1}{\gamma}} = 1 + \theta - 1 + \delta = \theta + \delta \quad \text{or}$$

$$\left(\frac{Y}{K}\right)^{\frac{1}{\gamma}} = A^{\left(\frac{1}{\gamma} - 1\right)} \frac{\theta + \delta}{\alpha} \quad \text{or}$$

$$\boxed{\left(\frac{Y}{K}\right) = A^{1-\gamma} \left(\frac{\theta + \delta}{\alpha}\right)^{\gamma} \qquad (4.15)}$$

Eqn (4.15) gives the long-run (steady-state) value of (Y/K) but it does not seperately determine Y or K.

For c/l begin from (4.6) which gives

$$\mu_{t+s} = \beta^s U_l(t+s)$$

Since from (4.2) $U_l(t+s) = \dfrac{\varphi}{l_{t+s}}$ we have that

$$\mu_{t+s} = \beta^s \frac{\varphi}{l_{t+s}} \qquad (4.16)$$

Using (4.16) with (4.10) in (4.9c) and setting $s=0$ gives

$$\frac{1}{C_t} F_n(t) = \frac{\varphi}{l_t} \qquad (4.17)$$

From (4.6) and using a derivation that parallels the derivation of (4.13) we obtain that

$$F_n(t) = (1-\alpha) A^{1-\frac{1}{\gamma}} \left(\frac{Y_t}{M_t}\right)^{\frac{1}{\gamma}} \qquad (4.18)$$

Use (4.18) in (4.17) to get

$$\frac{\varphi}{l_t} = \frac{(1-\alpha) A^{1-\frac{1}{\gamma}} \left(\frac{y_t}{m_t}\right)^{\frac{1}{\gamma}}}{C_t} \qquad (4.19) \; [w(3)]$$

From this, and using $l = 1 - m$, gives that the long-run (steady-state) equilibrium values satisfy

$$\boxed{\frac{c}{l} = \frac{c}{1-m} = \left(\frac{1-\alpha}{\varphi}\right) A^{1-\frac{1}{\gamma}} \left(\frac{y}{m}\right)^{\frac{1}{\gamma}} \qquad (4.20)}$$

(b) The implied equilibrium real wage is equal to the marginal product of labor.

Thus

$$w_t = F_m(K_t, m_t) \qquad (4.21)$$

Using (4.18) and evaluating at the long-run (steady-state) values

$$\boxed{w = (1-\alpha) A^{1-\frac{1}{\gamma}} \left(y/m\right)^{\frac{1}{\gamma}} \qquad (4.22)}$$

The implied Equilibrium Real interest rate is equal to the marginal product of capital net of the depreciation rate. Thus,

$$r_t = F_K(K_t, M_t) - \delta \qquad (4.23)$$

Using this in (4.11) and evaluating at the steady state gives

$$\beta[r+1] = 1 \qquad \text{or} \qquad 1+r = 1+\theta$$

or

$$\boxed{r = \theta \qquad (4.24)}$$

(d) To find the long-run Equilibrium value of $\frac{K}{m}$ begin from (4.22) which can be written as

$$\left(\frac{y}{m}\right)^{\frac{1}{\gamma}} = \frac{w}{1-\alpha} A^{\frac{1}{\gamma}-1} \qquad \text{or}$$

$$\left(\frac{y}{m}\right) = \left(\frac{w}{1-\alpha}\right)^{\gamma} A^{1-\gamma} \qquad (4.25)$$

Note That

$$\frac{K}{m} = \left(\frac{y}{m}\right)\left(\frac{K}{y}\right)$$ and use (4.25) and (4.15]

To get

$$\frac{K}{m} = \left(\frac{w}{1-\alpha}\right)^{\gamma} A^{1-\gamma} A^{\gamma-1} \left(\frac{\theta+\delta}{\alpha}\right)^{-\gamma}$$

$$\frac{K}{m} = \left[\frac{\alpha w}{(1-\alpha)(\theta+\delta)}\right]^{\gamma}$$

or, since $r = \theta$, from eqn (4.24),

$$\boxed{\frac{K}{m} = \left[\frac{\alpha w}{(1-\alpha)(r+\delta)}\right]^{\gamma} \qquad (4.26)}$$