ECON 7020
Philip Shaw
Problem Set 4
Due date: April 8, 2022

Problem 1. Suppose we have an economy with $j = 1, ..., n$ risky assets A^j_{t+i+1} all with their own rate of return r^j_{t+i+1}. The dynamic budget constraint is given by $\sum_{j=1}^n A^j_{t+i+1} = \sum_{j=1}^n (1 + r^j_{t+i})A^j_{t+i} + y_{t+1} - c_{t+i}$. Assume utility is defined by $u(c_t)$ and satisfies the assumptions required for risk aversion.

a. Formulate the Bellman equation carefully pointing out the state(s) and controls.

b. Prove that the first order condition for each asset j is given by:

$$u'(c_t) = \frac{1}{1+\rho} E_t[(1 + r^j_{t+1})u'(c_{t+1})] \tag{1}$$

c. Re-write the Euler equation in terms of the stochastic discount factor, pointing out what the stochastic discount factor represents.

d. Derive the fundamental result of the C-CAPM model using the fact that : $E_t[X_{t+1}Y_{t+1}] = E_t[X_{t+1}]E_t[Y_{t+1}] + cov_t(X_{t+1}, Y_{t+1})$. The fundamental result of the C-CAPM model shows the relationship between the expected return on the risky asset in excess of the safe asset.

e. Now assume utility takes the following functional form $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$. Find the expression for the stochastic discount factor and explain how the excess return depends on the covariance between the stochastic discount factor and a risky asset's return.

Problems from McCandless and Wallace:
Chapter 1 Exercises:
1.1-1.4, 1.11

Problem 1. Suppose we have an economy with $j = 1, ..., n$ risky assets A^j_{t+i+1} all with their own rate of return r^j_{t+i+1}. The dynamic budget constraint is given by $\sum_{j=1}^n A^j_{t+i+1} = \sum_{j=1}^n (1 + r^j_{t+i})A^j_{t+i} + y_{t+1} - c_{t+i}$. Assume utility is defined by $u(c_t)$ and satisfies the assumptions required for risk aversion.

a. Formulate the Bellman equation carefully pointing out the state(s) and controls.

set up Bellman Eqn:

$$V(A, y) = \max_c \{ u(c) + \beta V(A', y') \}$$

$$= \max_c \{ u(c) + \beta V[(1+r)A + y - c,\ y'] \}$$

state var: A, y

control var: c.

b. Prove that the first order condition for each asset j is given by:

$$u'(c_t) = \frac{1}{1+\rho} E_t[(1 + r^j_{t+1})u'(c_{t+1})] \quad (1)$$

FOC w.r.t c in Bellman:

$$u'(c) = \beta V_1(A', y')$$

Apply Envelop to A:

$$V_1(A, y) = \beta V_1(A', y') \cdot (1+r)$$

$$\therefore V_1(A', y') = \frac{V_1(A, y)}{\beta(1+r)}$$

$$\therefore u'(c) = \cancel{\beta} \frac{V_1(A, y)}{\cancel{\beta}(1+r)}$$

$$\therefore (1+r)u'(c) = V_1(A, y)$$

$$\therefore (1+r')u'(c') = V_1(A', y')$$

$$\therefore u'(c) = \beta(1+r')u'(c')$$

substitute $'$ w/ $t+1$ and add time t, add Expectation given uncertainty in the future.

$$\Rightarrow u'(c_t) = \beta E_t[(1+r^j_{t+1})u'(c_{t+1})]$$

where r^j_{t+1} is the return rate for specific asset j

Let $\beta = \frac{1}{1+\rho} = \text{const}$.

$$\Rightarrow u'(c_t) = \frac{1}{1+\rho} E_t[(1+r^j_{t+1})u'(c_{t+1})]$$

1

c. Re-write the Euler equation in terms of the stochastic discount factor, pointing out what the stochastic discount factor represents.

Rearrage items in eqn ①

$$1 = E_t\left[(1+r^j_{t+1})\boxed{\frac{u'(C_{t+1})}{u'(C_t)}\frac{1}{1+\rho}}\right]$$

as stochastic discount factor
Denote it as M_{t+1}

$$= E_t\left[(1+r^j_{t+1})M_{t+1}\right] \quad (2)$$

d. Derive the fundamental result of the C-CAPM model using the fact that : $E_t[X_{t+1}Y_{t+1}] = E_t[X_{t+1}]E_t[Y_{t+1}] + cov_t(X_{t+1}, Y_{t+1})$. The fundamental result of the C-CAPM model shows the relationship between the expected return on the risky asset in excess of the safe asset.

Using the formula $E[X_{t+1}Y_{t+1}] = E_t(X_{t+1})E_t(Y_{t+1}) + cov_t(X_{t+1}, Y_{t+1})$

to apply into (2)

$$1 = E_t[1+r^j_{t+1}]E_t[M_{t+1}] + cov_t(1+r^j_{t+1}, M_{t+1})$$

$$\therefore E_t(1+r^j_{t+1}) = \frac{1-cov(1+r^j_{t+1}, M_{t+1})}{E_t(M_{t+1})} \quad (3)$$

if we have risk free asset w/ return rate r^0_t?

$$\therefore 1+r^0_t = \frac{1-0}{E_t(M_{t+1})} = \frac{1}{E_t(M_{t+1})} \quad (4)$$

$\therefore$ plug (4) into (3)

$$\therefore E_t(1+r^j_{t+1}) = 1+r^0_t\left(1-cov(1+r^j_{t+1}, M_{t+1})\right)$$

$$\cancel{1+} E_t(r_{t+1}^j) = \cancel{1+} r_t^0 - (1+r_t^0)\,cov(1+r_{t+1}^j, M_{t+1})$$

$$\therefore \underbrace{E_t(r_{t+1}^j) - r_t^0}_{\text{excess return}} = -(1+r_t^0)\,cov(1+r_{t+1}^j, M_{t+1}) \quad (5)$$

for expected risk asset
relative to risk-free asset

e. Now assume utility takes the following functional form $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$. Find the expression for the stochastic discount factor and explain how the excess return depends on the covariance between the stochastic discount factor and a risky asset's return.

$$\because u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\therefore u'(c_t) = c_t^{-\gamma}. \quad (6)$$

put (6) into (2)

$$\therefore 1 = E_t\left[\frac{1}{1+\rho}(1+r_{t+1}^j)\frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}\right]$$

$$= E_t\left[\frac{1}{1+\rho}(1+r_{t+1}^j)\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \quad (7)$$

Take log of (7)

$$\therefore 0 = -\rho + \log E_t \left[(1 + r_{t+1}^j) \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right]$$

$$= -\rho + E_t [r_{t+1}^j - \gamma \log \Delta C_{t+1}] + \frac{1}{2} \Sigma_j \quad (8)$$

where Σ_j =

$$E_t \left\{ \left[(r_{t+1}^j - \gamma \log \Delta C_{t+1}) - E_t [r_{t+1}^j - \gamma \log \Delta C_{t+1}] \right]^2 \right\}$$

$$= \sigma_j^2 + \gamma^2 \sigma_c^2 - 2 \gamma \sigma_{jc} \quad (9)$$

plug (9) into (8)

$$0 = -\rho + E_t [r_{t+1}^j - \gamma \log \Delta C_{t+1}] + \frac{1}{2} (\sigma_j^2 + \gamma^2 \sigma_c^2 - 2 \gamma \sigma_{jc})$$

$$\Rightarrow E_t r_{t+1}^j = \rho + \gamma E_t \log \Delta C_{t+1} - \frac{1}{2} \sigma_j^2 - \frac{\gamma^2 \sigma_c^2}{2} + \gamma \sigma_{jc}$$

(10)

If asset j is risk-free asset

$$r_{t+1}^0 = \rho + \gamma E_t \log \Delta C_{t+1} - \frac{\gamma^2 \sigma_c^2}{2}$$

$$\therefore E_t r_{t+1}^j = r_{t+1}^0 - \frac{1}{2} \sigma_j^2 + \gamma \sigma_{jc}$$

$$\therefore\ E_t r_{t+1}^{j} - r_{t+1}^{0} = -\frac{1}{2}\sigma_j^2 + r_{jc} \quad (11)$$

OLG Problem sets

The definition formalizes the idea that a given consumption allocation (independent of the mechanism by which it was chosen) can be achieved (and is therefore feasible) if the total consumption for every period is less than or equal to the total resources available in that period. Simply put, do we always have enough of the good around to allow people to have the particular consumption pattern we are studying?

The following four exercises should help familiarize you with some of the basic notions of our model. In all of them you are to show that a particular consumption allocation is feasible. These exercises also provide practice in the use of our notation. Try these exercises before going on. We work through Exercise 1.1 later, but try it first.

EXERCISE 1.1 Let $N(t) = N > 0$ and let $Y(t) = yN > 0$ for all t. Prove that if $0 < \alpha < 1$, then

$$c_t^h(t) = \alpha y, \qquad c_{t-1}^h(t) = (1-\alpha)y$$

for all h and $t \geq 1$ is feasible.

EXERCISE 1.2 Let $N(t+1)/N(t) = n$ and let $Y(t) = yN(t)$ for all t. Prove that if $0 < \alpha < 1$, then

$$c_t^h(t) = \alpha y, \qquad c_{t-1}^h(t) = n(1-\alpha)y$$

for all h and $t \geq 1$ is feasible.

EXERCISE 1.3 Let $N(t-1) = 2$ and $Y(t) = 2$ for all $t \geq 1$. Prove that the following allocation is feasible:

$$c_0^h(1) = 1/2 \quad \text{for} \quad h = 1, 2;$$

$$c_1^1(1) = 1/4, \qquad c_1^1(2) = 3/4;$$

$$c_1^2(1) = 3/4, \qquad c_1^2(2) = 1/4;$$

$$c_t^h(t) = c_t^h(t+1) = 1/2 \quad \text{for } h = 1, 2 \text{ and all } t \geq 2.$$

EXERCISE 1.4 Let $N(t-1) = 1$ and $Y(t) = 1$ for all $t \geq 1$. Show that the following allocation is feasible:

$$c_{t-1}(t) = (1/2) - (1/2)^{t+1},$$

$$c_t(t) = (1/2) + (1/2)^{t+1}, \quad \text{all } t \geq 1.$$

Exercise 1

$$C(t) = \sum_h^{N(t)} c_t^h(t) + c_{t-1}^h(t)$$

$$= \sum_h^{N(t)} \alpha y + (1-\alpha) y \qquad \text{if } \alpha \in (0,1)$$

$$= y \cdot N = Y(t)$$

$\therefore$ it's feasible.

EXERCISE 1.2 Let $N(t+1)/N(t) = n$ and let $Y(t) = yN(t)$ for all t. Prove that if $0 < \alpha < 1$, then

$$c_t^h(t) = \alpha y, \qquad c_{t-1}^h(t) = n(1-\alpha)y$$

for all h and $t \geq 1$ is feasible.

$$C(t) = \alpha y \cdot N(t) + \frac{N(t)}{\cancel{n}} \cdot \cancel{n}(1-\alpha)y$$

$$= \alpha N(t) y + (1-\alpha) N(t) y$$

$$= N(t) y = Y(t)$$

$\therefore$ it's feasible

EXERCISE 1.3 Let $N(t-1) = 2$ and $Y(t) = 2$ for all $t \geq 1$. Prove that the following allocation is feasible:

$c_0^h(1) = 1/2$ for $h = 1, 2$;

$c_1^1(1) = 1/4, \quad c_1^1(2) = 3/4$;

$c_1^2(1) = 3/4, \quad c_1^2(2) = 1/4$;

$c_t^h(t) = c_t^h(t+1) = 1/2$ for $h = 1, 2$ and all $t \geq 2$.

$C(1) = c_0^h(1) \cdot 2 + c_1^1(1) + c_1^2(1)$

$= 1 + \frac{1}{4} + \frac{3}{4} = 2 \leq Y(1) = 2. \quad \therefore$ ✓

when $t = 2$,

$C(2) = c_1^1(2) + c_1^2(2) + c_2^1(2) + c_2^2(2)$

$= \frac{1}{4} + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} = 2 \leq Y(2) = 2$ ✓

when $t > 2$

$C(t) = 2 \cdot c_{t-1}^h(t) + 2 \cdot c_t^h(t)$

$= 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 2 \leq Y(t) = 2$ ✓

So this allocation is feasible at all time.

EXERCISE 1.4 Let $N(t-1) = 1$ and $Y(t) = 1$ for all $t \geq 1$. Show that the following allocation is feasible:

$c_{t-1}(t) = (1/2) - (1/2)^{t+1}$,

$c_t(t) = (1/2) + (1/2)^{t+1}$, all $t \geq 1$.

if the Num of each generation is const.

$C(t) = c_{t-1}(t) + c_t(t) = 1 = Y(t) = 1 \quad \therefore$ feasible.

if there is positive population growth rate.

$C(t) > Y(t) = 1 \quad \therefore$ not feasible

... Consider the setup and allocation of Exercise 1.3. Show that this allocation is not Pareto optimal if preferences are expressed by the utility function in Exercise 1.6.

EXERCISE 1.11 This problem is a generalization of the last exercise. Suppose that $u_t^h(c_t^h(t), c_t^h(t+1))$ is the utility function of person h in generation t, $t \geq 1$. Let the marginal rate of substitution (MRS) for this person be defined by

$$\frac{u_{t1}^h(c_t^h(t), c_t^h(t+1))}{u_{t2}^h(c_t^h(t), c_t^h(t+1))},$$

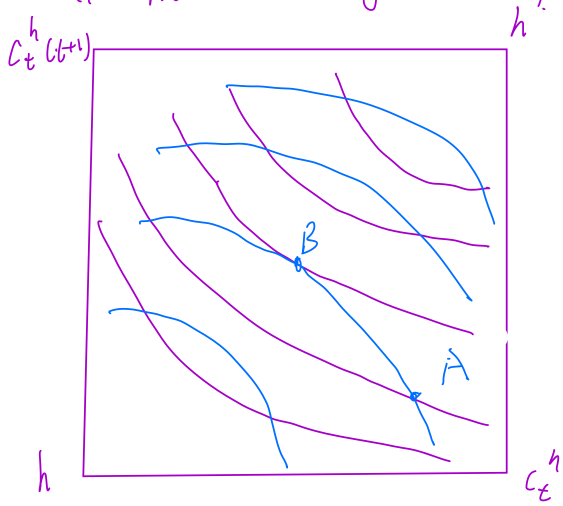
where $u_{tj}^h(c_t^h(t), c_t^h(t+1))$ is the partial derivative of the utility function with respect to its jth argument. Suppose h and h' are two members of generation t, for some $t \geq 1$. Prove the following. ① A feasible allocation that assigns positive first- and second-period consumptions to h and h' and ② implies different values for the MRS of h and h' is not Pareto optimal.

Exercise 1.7 shows us that efficiency is a necessary condition for Par-

Main Reference P26-P27 of the textbook:

∵ MRS can be unstood as the slope to the indifference curve to an agent.

∴ Draw an Edgeworth Box



From the graph, we can draw. there allocation is positive.

We prove by contradiction:

if B is not Pareto superior to A, insted A is.

Since In A, for h, if we move for A to B, h has more utility, as its indifference is upper than A.

But at B, there is no change on utility for h' as they are on the same indifference curve.

∴ B is pareto superior to A.

And A is not Pareto optimal.