ECON 7020
Philip Shaw
Problem Set 5
Due Date: April 22, 2022

Problems from McCandless and Wallace:

Chapter 2 Exercises:
2.1, 2.5

Problem 1. Take a simple two-period heterogeneous agent OLG model. Suppose we have agents that differ in their abilities a_i for $i = l, h$. Low ability agents are assigned the value $a_l = 1$ and high ability agents are assigned the value $a_h = 2$. The population of high ability agents is given by N^h and the population of low ability agents is given by N^l. Assume that both types of agents have the same utility function $u_t^h = c_t^h(t)c_t^h(t+1)$ and that the population of each type is constant over time. Furthermore assume that the ability level of each agents allows them to transform their endowments when young such that $w_t^h = [a_i\tilde{w}_t^h(t), \tilde{w}_t^h(t+1)]$ for $i = l, h$. Assume that each type of agent is assigned a pre-transformed endowment of $\tilde{w}_t^h = [1, 1]$.

a. Define a competitive equilibrium.

b. Assuming $N^h = 100$ and $N^l = 50$, solve the the competitive equilibrium. What is the equilibrium interest rate? What the the savings of the high ability and low ability agents? What are the consumption levels for each type of agent?

c. Describe in words the trading arrangements between the high and low ability agents. Do they make sense?

d. Now suppose we fix $N^l = 50$ but allow the population of high ability agents to remain unspecified at N^h. What is the limiting behavior of the interest rate as population of high ability agents becomes arbitrarily large or small? What is the limiting behavior of the individual savings functions? Explain your results intuitively. (Note: You should be able to say exactly what the limiting behavior is.)

Problem 2. Building on the model presented in the first problem, assume that generations transition over time in ability according to the following transition matrix:

$$P = \begin{bmatrix} p_{hh} & p_{hl} \\ p_{lh} & p_{ll} \end{bmatrix} \tag{1}$$

where p_{hh} gives the probability of high ability agents giving "birth" to high ability agents and p_{hl} is the probability of high ability agents giving "birth" to low ability agents. We can think of ability as following a Markov chain (a, P, π_0) where a is the ability type, P is a transition matrix, and π_0 is the initial distribution of each type of agent. Assume that $N^h + N^l = 1$ where the population of agents at time t is given by the proportion of each type of agent contained in $\pi_t = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$. Furthermore assume that $p_{ij} = .5$ for all i and j.

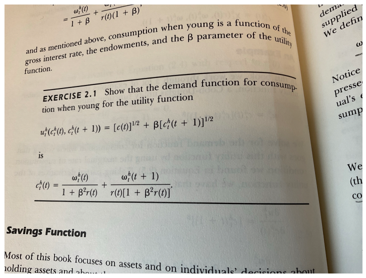
a. How does the stochastic nature of the model impact the decision of each individual?

b. What is the time t equilibrium interest rate?

c. What is the stationary distribution for each type of agent?

d. Define a stationary competitive equilibrium.

e. Solve for the stationary equilibrium interest rate. How does this interest rate compare to the one computed in part b? What are the individual savings for each type of consumer? What are the individual consumption levels for each type of consumer?



$$= \frac{\omega_t^h(t)}{1+\beta} + \frac{\omega_t^h(\ldots)}{r(t)(1+\beta)},$$

and as mentioned above, consumption when young is a function of the gross interest rate, the endowments, and the β parameter of the utility function.

EXERCISE 2.1 Show that the demand function for consumption when young for the utility function

$$u_t^h(c_t^h(t), c_t^h(t+1)) = [c(t)]^{1/2} + \beta[c_t^h(t+1)]^{1/2}$$

is

$$c_t^h(t) = \frac{\omega_t^h(t)}{1+\beta^2 r(t)} + \frac{\omega_t^h(t+1)}{r(t)[1+\beta^2 r(t)]}.$$

Savings Function

Most of this book focuses on assets and on individuals' decisions about holding assets and about

$$\therefore\ r(t) = MRS = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial u}{\partial c_{t+1}}} = \frac{\frac{1}{2} c_t^h(t)^{-\frac{1}{2}}}{\beta \frac{1}{2}\, c_t^h(t+1)^{-\frac{1}{2}}} = \frac{\beta\, c_t^h(t+1)^{\frac{1}{2}}}{c_t^h(t)^{\frac{1}{2}}}$$

$$\therefore\ c_t^h(t+1) = r(t)[w_t^h(t) - c_t^h(t)] + w_t^h(t+1)$$

$$\therefore\ r(t) = \frac{\beta\left[r(t)[w_t^h(t) - c_t^h(t)] + w_t^h(t+1)\right]^{\frac{1}{2}}}{c_t^h(t)^{\frac{1}{2}}}$$

$$\therefore\ c_t^h(t)^{\frac{1}{2}} = \frac{\beta\left[r(t)[w_t^h(t) - c_t^h(t)] + w_t^h(t+1)\right]^{\frac{1}{2}}}{r(t)}$$

$$\therefore\ c_t^h(t) = \frac{\beta^2 r(t) \cdot w_t^h(t) - \beta^2 r(t) c_t^h(t) + \beta^2 w_t^h(t+1)}{r(t)^2}$$

$$\therefore\ \left[r(t)^2 + \beta^2 r(t)\right] c_t^h(t) = \beta^2 r(t) w_t^h(t) + \beta^2 w_t^h(t+1)$$

$$\therefore\ c_t^h(t) = \frac{w_t^h(t)}{1+\beta^2 r(t)} + \frac{w_t^h(t+1)}{r(t)[1+\beta^2 r(t)]}$$

$$[\omega_t^h(t), \omega_t^h(t+1)] = \begin{cases} \ldots, & h = 1, 2, \ldots, 60, \\ [1, 1], & h = 61, 62, \ldots, 100. \end{cases}$$

f. Same as a, except that for all $t \geq 0$,

$$[\omega_t^h(t), \omega_t^h(t+1)] = \begin{cases} [1, 1], & t = 1, 3, 5, \ldots, \\ [2, 1], & t = 2, 4, 6, \ldots. \end{cases}$$

EXERCISE 2.5 (Pareto optimality) Prove that the competitive equilibrium of each of the following economies of Exercise 2.4 is *not* Pareto optimal: 2.4a, 2.4c, 2.4d, 2.4e, 2.4f. (You may find it helpful to review the material on feasible, Pareto superior, and Pareto optimal allocations.)

In Exercise 2.4d when 50 of each generation (the odd) have an endowment of [1, 1] and 50 of each generation (the even) have an endow-

in 2.4a.
utility fct:

$$u_t^h = c_t^h(t)\left(c_t^h(t+1)\right)^\beta$$

(a) if $\beta = 1$ $\quad u_t^h = c_t^h(t)\, c_t^h(t+1)$

$$[w_t^h(t), w_t^h(t+1)] = [2, 1]$$

$$\therefore c_t^h(t) = \frac{w_t^h(t)}{1+\beta} + \frac{w_t^h(t+1)}{r(t)(1+\beta)}$$

$$s_t^h(t) = w_t^h(t) - c_t^h(t)$$

$$= w_t^h(t) - \frac{w_t^h(t)}{1+\beta} - \frac{w_t^h(t+1)}{r(t)(1+\beta)}$$

$$\boxed{= \frac{\beta}{1+\beta} w_t^h(t) - \frac{w_t^h(t+1)}{r(t)(1+\beta)}} \qquad \text{eqn (1)}$$

$\because \beta = 1$

$$= \frac{1}{2} w_t^h(t) - \frac{w_t^h(t+1)}{2r(t)}$$

$\because w_t^h(t) = 2, \quad w_t^h(t+1)$

$$= \frac{1}{2} \times 2 - \frac{1}{2r(t)} = 1 - \frac{1}{2r(t)}$$

$$\therefore S_t(t) = N(t) \cdot s_t^h(t)$$

$$= 100 - \frac{50}{r(t)} = 0$$

$$\therefore r(t) = \frac{1}{2}$$

$\therefore C_t^h(t) = \frac{2}{2} + \frac{1}{\frac{1}{2} \times 2}$

$= 2$

$C_t^h(t+1) = (W_t^h(t) - C_t^h(t))\frac{1}{2} + W_t^h(t+1)$

$= 1.$

$\therefore$ in CE: $[C_t^h(t), C_t^h(t+1)] = [2, 1]$

it's not pareto optimal, as

$[1, 2]$ is pareto superior.

(c) $U_t^h = [C_t^h(t)]^{\frac{1}{2}} + \beta [C_t^h(t+1)]^{\frac{1}{2}}$

$$\gamma(t) = \frac{\frac{\partial U_t^h}{\partial C_t^h(t)}}{\frac{\partial U_t^h}{\partial C_t^h(t+1)}} = \frac{\cancel{\frac{1}{2}} C_t^h(t)^{-\frac{1}{2}}}{\beta \cancel{\frac{1}{2}} C_t^h(t+1)^{-\frac{1}{2}}}$$

$$= \frac{1}{\beta}\left[\frac{C_t^h(t+1)^{\frac{1}{2}}}{C_t^h(t)^{\frac{1}{2}}}\right]$$

$C_t^h(t+1) = \gamma(t)[W_t^h(t) - C_t^h(t)] + W_t^h(t+1)$

$$\therefore\ r(t) = \frac{1}{\beta}\left[\frac{r(t)[W_t^h(t) - C_t^h(t)] + W_t^h(t+1)}{C_t^h(t)}\right]^{\frac{1}{2}}$$

$$\therefore\ \beta^2 r^2(t) = \frac{r(t)[W_t^h(t) - C_t^h(t)] + W_t^h(t+1)}{C_t^h(t)}$$

$$\therefore\ \beta^2 r^2(t)\, C_t^h(t) = -r(t) C_t^h(t) + r(t) W_t^h(t) + W_t^h(t+1)$$

$$\therefore\ r(t)[\beta^2 r(t) + 1]\, C_t^h(t) = r(t) W_t^h(t) + W_t^h(t+1)$$

$$\therefore\ C_t^h(t) = \frac{r(t) W_t^h(t) + W_t^h(t+1)}{r(t)[\beta^2 r(t) + 1]}$$

$$= \frac{W_t^h(t)}{\beta^2 r(t) + 1} + \frac{W_t^h(t+1)}{r(t)[\beta^2 r(t) + 1]}$$

$$\therefore\ S(t) = W(t) - C(t) = 0$$

$$\Rightarrow\ r = \frac{\sqrt{2}}{2}$$

$\therefore\ S^h(t) = 0$ $\qquad C_1^h(1) = 2$

$C_0^h(1) = 1$ $\qquad \therefore\ C_t^h = [2, 1]$

$2 > 1$

$\therefore\ C_t^h = [1, 2]$ is pareto superior

(d) following the same procedure w/ (a)(c)

let $S(t) = 0$, to get $r(t) = \frac{2}{3}$

$\therefore S_t^{odd}(t) = -\frac{1}{4}$ and $S_t^{even}(t) = \frac{1}{4}$

$C_0^{even}(1) = 1$ $\quad C_t^{even} = [\frac{7}{4}, \frac{7}{6}]$

$C_0^{odd}(1) = 1$ $\quad C_t^{odd} = [\frac{5}{4}, \frac{5}{6}]$

$\therefore C_t^{even} = [\frac{7}{6}, \frac{7}{4}]$ $\quad C_t^{odd} = [\frac{5}{6}, \frac{5}{4}]$ is pareto optimal.

for 2.4

(e): Same method to derive $r = \frac{5}{8}$, $S_t^h(t) = \frac{1}{5}$

for $h = 1, 2, \cdots 60$: $C_t^h = [\frac{9}{5}, \frac{9}{8}]$

for $h = 61, \cdots 100$ $\quad C_t^h = [\frac{13}{10}, \frac{13}{16}]$

$\Rightarrow \therefore$ for $h = 1, 2, \cdots 60$: $\quad C_t^h = [\frac{9}{8}, \frac{9}{5}]$

for $h = 61, \cdots 100$: $C_t^h = [\frac{13}{16}, \frac{13}{10}]$ is pareto superior.

(f) for

$t = 1, 3, \cdots$ $\quad C_t^h = [0.8, 1.25]$

$t = 2, 4, \cdots$ $\quad C_t^h = [1.25, 1.2]$

are pareto superior.

Problem 1. Take a simple two-period heterogeneous agent OLG model. Suppose we have agents that differ in their abilities a_i for $i = l, h$. Low ability agents are assigned the value $a_l = 1$ and high ability agents are assigned the value $a_h = 2$. The population of high ability agents is given by N^h and the population of low ability agents is given by N^l. Assume that both types of agents have the same utility function $u_t^h = c_t^h(t)c_t^h(t+1)$ and that the population of each type is constant over time. Furthermore assume that the ability level of each agents allows them to transform their endowments when young such that $w_t^h = [a_i\tilde{w}_t^h(t), \tilde{w}_t^h(t+1)]$ for $i = l, h$. Assume that each type of agent is assigned a pre-transformed endowment of $\tilde{w}_t^h = [1, 1]$.

$[a_l = 1, a_h = 2]$

$N = [N^l, N^h]$

a. Define a competitive equilibrium.

CE: No matter it's high ability or low ability, they should be able to max their utility given their budget constraint, since there is only goods market, ⇒ goods market should be clear in competitive equilibrium environment.

$$C = W \quad [\text{in aggregate way}]$$

$$S(t) = 0$$

b. Assuming $N^h = 100$ and $N^l = 50$, solve the the competitive equilibrium. What is the equilibrium interest rate? What the the savings of the high ability and low ability agents? What are the consumption levels for each type of agent?

⇒ my Q: is there explicit setting in the PS to say there is lending market?

∴ from the lecture notes

$$s_t^h(t) = \frac{1}{2} a_i \tilde{w}_t^h(t) - \frac{\tilde{w}_t^h(t+1)}{2r(t)}$$

$$S(t) = N^h \cdot \left[\frac{1}{2} a_h \tilde{w}_t^h(t) - \frac{w_t^h(t+1)}{2r(t)}\right] + N^l \cdot \left[\frac{1}{2} \cdot a_l \hat{w}_t^h(t) - \frac{w_t^h(t+1)}{2r(t)}\right]$$

$$= 100\left[\frac{1}{2} \cdot a_h - \frac{1}{2r(t)}\right] + 50\left[\frac{1}{2} \cdot a_l - \frac{1}{2r(t)}\right]$$

$$= 50a_h - \frac{50}{r(t)} + 25a_l - \frac{25}{r(t)}$$

$$= 50a_h + 25a_l - \frac{75}{r(t)} = 0$$

$\therefore \quad 50a_h + 25a_\ell = \frac{75}{r(t)}$

$\therefore \quad \boxed{r^*(t) = \frac{75}{50a_h + 25a_\ell}}$

For individual high ability agent:

$$S_t^{hh}(t) = \frac{1}{2} a_h \cdot 1 - \frac{1}{\frac{150}{(50a_h + 25a_\ell)}}$$

$$= \frac{1}{2} a_h - \frac{\cancel{50}^{2}a_h + \cancel{25}\,a_\ell}{\cancel{150}\; 6}$$

$$= \frac{a_h}{2} - \frac{2a_h + a_\ell}{6} = \boxed{\frac{a_h - a_\ell}{6}}$$

$$S_t^{hl}(t) = \frac{1}{2} a_\ell \cdot 1 - \frac{2a_h + a_\ell}{6}$$

$$= \boxed{\frac{a_\ell - a_h}{6}}$$

$$C_t^{hh}(t) = a_h w_t^h(t) - S_t^{hh}(t)$$

$$= a_h - \frac{a_h - a_\ell}{6} = \boxed{\frac{5}{6} a_h + \frac{a_\ell}{6}}$$

$$C_t^{hl}(t) = a_\ell w_t^h(t) - S_t^{hl}(t)$$

$$= a_l - \frac{a_l - a_h}{6}$$

$$= \boxed{\frac{5}{6} a_l + \frac{a_h}{6}}$$

c. Describe in words the trading arrangements between the high and low ability agents. Do they make sense?

For high abity agent, he/she lends $S_t^{hh}(t)$ to lower ability w/ the interest rate $r(t)$, in $t+1$, the low ability agent pays high one w/ $r(t) S_t^{hh}(t)$.

Yes, it smooths consumption.

$\because$ [$a_h > a_l$ by the setting]

d. Now suppose we fix $N^l = 50$ but allow the population of high ability agents to remain unspecified at N^h. What is the limiting behavior of the interest rate as population of high ability agents becomes arbitrarily large or small? What is the limiting behavior of the individual savings functions? Explain your results intuitively. (Note: You should be able to say exactly what the limiting behavior is.)

$$S(t) = 50\left[\frac{1}{2}a_l - \frac{1}{2r(t)}\right] + N^h\left[\frac{1}{2}a_h - \frac{1}{2r(t)}\right]$$

$$= 25a_l - 25\frac{1}{r(t)} + \frac{1}{2}a_h N^h - \frac{N^h}{2r(t)} = 0$$

$$= 25a_l + \frac{1}{2}a_h N^h - \frac{50+N^h}{2r(t)} = 0$$

$$\therefore \quad 2r(t) = \frac{50+N^h}{25a_l + \frac{1}{2}a_h N^h}$$

$$r(t) = \frac{25+\frac{1}{2}N^h}{25a_l + \frac{1}{2}a_h N^h}$$

$$\lim_{N^h \to \infty} r(t) = \frac{1}{a_h}$$

$$\lim_{N^h \to \infty} S_t^{hh}(t) = \frac{1}{2}a_h - \frac{1}{2\frac{1}{a_h}} = \frac{a_h}{2} - \frac{a_h}{2} = 0$$

$$\lim_{N^h \to \infty} S_t^{hl}(t) = \frac{1}{2}a_l - \frac{a_h}{2} = \frac{a_l - a_h}{2}$$

Theoretically, there are infinity lenders, but not enough borrowers, $\therefore$ lim. for each lender on lending is 0

For finite borrower, they can borrow the amount of
each of commodity due to ability difference.

For interest rate, it's only the fact of ~~Cn~~,
∴ it's probably b/c marginal benefits of having additional
wealth.

Problem 2. Building on the model presented in the first problem, assume that generations transition over time in ability according to the following transition matrix:

$$P = \begin{array}{c|cc} & h & l \\ h & p_{hh} & p_{hl} \\ l & p_{lh} & p_{ll} \end{array} \tag{1}$$

where p_{hh} gives the probability of high ability agents giving "birth" to high ability agents and p_{hl} is the probability of high ability agents giving "birth" to low ability agents. We can think of ability as following a Markov chain (a, P, π_0) where a is the ability type, P is a transition matrix, and π_0 is the initial distribution of each type of agent. Assume that $N^h + N^l = 1$ where the population of agents at time t is given by the proportion of each type of agent contained in $\pi_t = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$. Furthermore assume that $p_{ij} = .5$ for all i and j.

2x1

a. How does the stochastic nature of the model impact the decision of each individual?

Since π_t is not stochastic => known info at time t.

The only stochastic factor is transition matrix.

if the agent knows he has high prob to transit to high ability type, he will tend to lend his wealth at initial time, w/ high prob to low type is contrary.

b. What is the time t equilibrium interest rate?

from the lecture notes

$$s_t^h(t) = \frac{1}{2} a_i \tilde{W}_t^h(t) - \frac{\tilde{W}_t^h(t+1)}{2r(t)}$$

$$\pi_t = \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}_{2\times 1} \quad P = \begin{array}{c|cc} & h & l \\ h & 0.5 & 0.5 \\ l & 0.5 & 0.5 \end{array}_{2\times 2}$$

$$P \cdot \pi = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}_{2\times 2} \begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = P'$$

$$S_t^{hh}(t) = \frac{1}{2} N_L \left[\frac{1}{2} a_L \tilde{W}_t^h(t) - \frac{W_t^h(t+1)}{2r(t)}\right] + \frac{1}{2} N_h \left[\frac{1}{2} a_h \tilde{W}_t^h(t) - \frac{W_t^h(t+1)}{2r(t)}\right]$$

$$= \frac{1}{2} N_L \left[\frac{1}{2} a_L - \frac{1}{2r(t)}\right] + \frac{1}{2}[1-N_L]\left[\frac{1}{2} a_h - \frac{1}{2r(t)}\right]$$

$$= \frac{1}{4} a_L N_L - \cancel{\frac{N_L}{4r(t)}} + \frac{1}{4} a_h - \frac{1}{4r(t)} - \frac{1}{4} a_h N_L + \cancel{\frac{N_L}{4r(t)}} = 0$$

$$\Rightarrow \cancel{\frac{1}{4}} N_L [a_L - a_h] + \cancel{\frac{1}{4}} a_h = \frac{1}{\cancel{4} r(t)}$$

$$\Rightarrow r^*(t) = \frac{1}{N_L(a_L - a_h) + a_h}$$

hl

c. What is the stationary distribution for each type of agent?

$$\underset{2\times2}{P} \cdot \underset{2\times1}{\pi} = \pi$$

$$\therefore \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

$$\begin{cases} 0.5\pi_1 + 0.5\pi_2 = \pi_1 \\ 0.5\pi_1 + 0.5\pi_2 = \pi_2 \end{cases}$$

$\Rightarrow \pi_1 = \pi_2$

$\because \pi_1 + \pi_2 = 1 \Rightarrow \pi_1 = \pi_2 = 0.5$

d. Define a stationary competitive equilibrium.

CE : 2 CE: price + quantity

$S(t)=0$ [one agent lending is the other's borrowing] ⇒ money market

r = MU [Tradeoff between consumptn and lending]

e. Solve for the stationary equilibrium interest rate. How does this interest rate compare to the one computed in part b? What are the individual savings for each type of consumer? What are the individual consumption levels for each type of consumer?

$$\underset{2\times2}{P} \cdot \underset{2\times1}{\pi} = \pi$$

$$\therefore \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

$$\begin{cases} 0.5\pi_1 + 0.5\pi_2 = \pi_1 \\ 0.5\pi_1 + 0.5\pi_2 = \pi_2 \end{cases}$$

$\Rightarrow \pi_1 = \pi_2$

$\because \pi_1 + \pi_2 = 1 \Rightarrow \pi_1 = \pi_2 = 0.5$

$$S_t^{hh}(t) = \frac{1}{2} N_L \left[\frac{1}{2} a_L \tilde{W}_t^h(t) - \frac{W_t^h(t+1)}{2r(t)} \right] +$$

$$\frac{1}{2} N_h \left[\frac{1}{2} a_h \tilde{W}_t^h(t) - \frac{W_t^h(t+1)}{2r(t)} \right]$$

$$= \frac{1}{2} N_L \left[\frac{1}{2} a_L - \frac{1}{2r(t)} \right] + \frac{1}{2} [1 - N_L] \left[\frac{1}{2} a_h - \frac{1}{2r(t)} \right]$$

$$= \frac{1}{4} a_L N_L - \cancel{\frac{N_L}{4r(t)}} + \frac{1}{4} a_h - \frac{1}{4r(t)} - \frac{1}{4} a_h N_L$$

$$+ \cancel{\frac{N_L}{4r(t)}} = 0$$

$$\Rightarrow \cancel{\frac{1}{4}} N_L [a_L - a_h] + \cancel{\frac{1}{4}} a_h = \frac{1}{\cancel{4} r(t)}$$

$$\Rightarrow r^*(t) = \frac{1}{N_L(a_L - a_h) + a_h}$$

$$S_t^{hL}(t) = \frac{1}{2} a_L - \frac{1}{2N_L(a_L - a_h) + a_h}$$

$$S_t^{hh}(t) = \frac{1}{2} a_h - \frac{1}{2N_L(a_L - a_h) + a_h}$$

$$C_t^{hL}(t) = 1 - \frac{1}{2}a_L + \frac{1}{2N_L(a_L - a_n) + a_n}$$

$$C_t^{hh}(t) = 1 - \frac{1}{2}a_n + \frac{1}{2N_L(a_L - a_n) + a_n}$$