Exercises #10

12.B.1 Suppose that the inverse demand function in a monopolist market is $p(q) = a - bq$ and the monopolist's cost function is $c(q) = cq$, where $a > c \geq 0$ (so that $p(0) > c'(0)$) and $b > 0$. Calculate the monopolist's optimal quantity and price and contrast this to the socially optimal (competitive) output level and price.

12.B.1 (a) The monopolists maximizes $x(p)p - c(x(p))$, which leads to the following first order condition: $x'(p^m)\cdot p^m + x(p^m) = c'(x(p^m))\cdot x'(p^m)$.

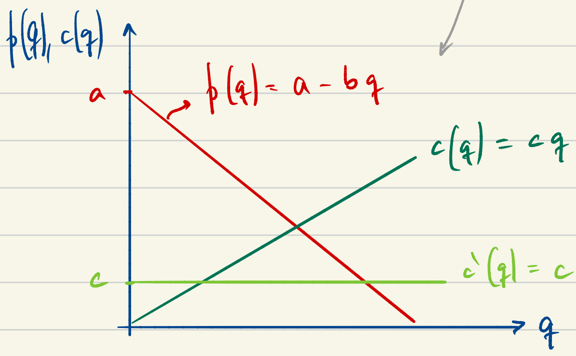
Rearranging this we get $\dfrac{[p^m - c'(x(p^m))]}{p^m} = \dfrac{1}{\varepsilon(p^m)}$, where the elasticity of demand is $\varepsilon(p^m) = -x'(p^m)\cdot\dfrac{p^m}{x(p^m)}$.

(b) If $c'(x(p^m)) > 0$ always, then $\dfrac{[p^m - c'(x(p^m))]}{p^m} < \dfrac{[p^m - 0]}{p^m} = 1$. Therefore $1/\varepsilon(p^m) < 1$, or $\varepsilon(p^m) > 1$.

Inverse demand function: $p(q) = a - bq$

Monopolist's cost function: $c(q) = cq$

with $a > c \geq 0$ $\left(\text{so } p(0) > c'(0)\right)$ and $b > 0$



Find: Monopolist's optimal quantity and price versus Socially optimal (competitive) output level and prices

Monopolist's optimal quantity and price:

$$\max_{\{q\}} \; p(q) \cdot q - c(q) = (a - bq)\, q - cq$$

FOC: (q): $a - bq - bq - c = 0 \Leftrightarrow 2bq = a - c$

$$\Leftrightarrow q^M = \frac{a-c}{2b}$$

$$p^M = p(q^M) = a - b\, q^M = a - \cancel{b}\left(\frac{a-c}{2\cancel{b}}\right) = a - \frac{a-c}{2} =$$

$$= \frac{2a - a + c}{2} = \frac{a+c}{2}$$

Monopolist's optimal q^M and p^M:

$$q^M = \frac{a-c}{2b} \qquad p^M = \frac{a+c}{2}$$

Socially optimal quantity and price:

$$\max_{\{q\}} \quad p \cdot q - c(q) = pq - cq$$

FOC: (q): $\quad p - c = 0 \Leftrightarrow p^* = c$

$$p(q^*) = a - bq^* \Leftrightarrow c = a - bq^* \Leftrightarrow bq^* = a - c$$

$$\Leftrightarrow q^* = \frac{a-c}{b}$$

Socially optimal q^* and p^*:

$$q^* = \frac{a-c}{b} \qquad p^* = c$$

Comparing the two outcomes:

$$q^M = \frac{a-c}{2b} \qquad p^M = \frac{a+c}{2}$$

$$q^* = \frac{a-c}{b} \qquad p^* = c$$

Therefore: $q^M < q$ $\qquad$ $p^M > p^*$

$\frac{a-c}{2b} < \frac{a-c}{b}$ as long as $a-c > 0$ or $a > c$

which holds by assumption (recall: $a > c \geq 0$)

$\frac{a+c}{2} > c \iff a+c > 2c \iff a > c$ which also

holds by assumption

12.B.7 Consider the widget market. The total demand by men for widgets is given by $x_m(p) = a - \theta_m p$ and the total demand by women is $x_m(p) = a - \theta_w p$, where $\theta_w < \theta_m$. The cost of production is c per widget.

(a) Suppose the widget market is competitive. Find the equilibrium price and quantity sold.

(b) Suppose, instead, that firm A is a monopolist of widgets. (Also make this assumption in c and d.) If firm is prohibited from charging different prices to men and women, what is the profit-maximizing price? Under what conditions do both men and women consume a positive level of widgets in this solution?

(c) If firm A has produced some level of output X, what is the welfare-maximizing way to distribute it between the men and women?

(d) Suppose that firm A is allowed to charge men and women different prices. What prices does it charge? In the case where the nondiscriminatory solution in b has positive consumption of widgets by both men and women, does aggregate welfare as measured by Marshallian aggregate surplus rise or fall relative to when discrimination is allowed? What if the nondiscriminatory solution in b has only one type of consumers being served?

12.B.7 **(a)** In a competitive equilibrium we must have $p^m = p^w = c$, and therefore $x_m = a - \theta_m c$, and $x_w = a - \theta_w c$.

(b) If the monopolist will serve both markets it solves,

$$\text{Max } (a - \theta_m p)(p - c) + (a - \theta_w p)(p - c) .$$

The FOC yields $p^* = \frac{a}{\theta_m + \theta_w} + \frac{c}{2}$. However, since the women are willing to pay higher prices than men for the same quantities ($\theta_m > \theta_w$) then it may be better to charge a price above $\frac{a}{\theta_m}$. This cannot be captured in the program above since it causes negative quantities in the men's demand, so we have to compare the solution p^* above to the solution of Max $(a - \theta_w p)(p - c)$ which yields $\hat{p} = \frac{a}{2\theta_w} + \frac{c}{2}$. Figure 12.B.7 shows the two cases.

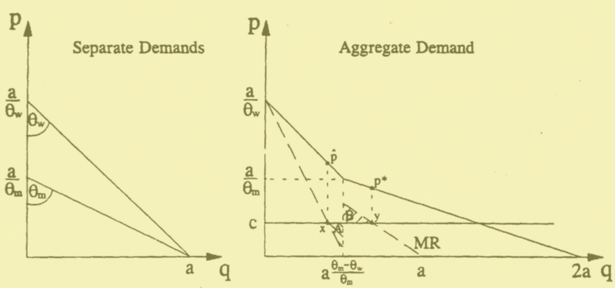


Figure 12.B.7

Due to the kink in the demand curve, the marginal revenue curve MR has a jump at that point (the dashed line). When equating the marginal cost, c, to MR, we could have one or two solutions. When we have one solution, it means that c cuts MR only to the left of the discontinuity, or only to the right. If c cuts MR only to the left, then the optimal price is above a/θ_m and we only serve the women. If c cuts MR only to the right, then the optimal price is below a/θ_m and we serve both markets. When we have two solutions, then c cuts MR both to the left and to the right of the discontinuity, and this is shown in the figure by points x and y. When the monopolist moves from point x to y (i.e., from $\hat{p}$ to p^*) it loses profits equal to the triangle A, and gains profits equal to the triangle B. The costs and benefits of such a move will determine if $\hat{p}$ is optimal (only the women are served) or if p^* is (both markets are served).

(c) Given quantity X, we maximize consumer surplus by solving,

$$\text{Max} \int_0^{q_m} p_m(x)dx + \int_0^{q_w} p_w(x)dx$$

$$\text{s.t. } q_m + q_w = X$$

and letting λ denote the Lagrange multiplier we get the FOCs $p_m(q_m) = \lambda$ and $p_w(q_w) = \lambda$, which yield $p_m(q_m) = p_w(q_m)$, or, $\dfrac{(a - q_m)}{\theta_m} = \dfrac{(a - q_w)}{\theta_w}$. Together with the constraint we solve for q_m and q_w.

(d) The discriminatory monopolist solves

$$\max_{p_m, p_w} (a - \theta_m p_m)(p_m - c) + (a - \theta_w p_w)(p_w - c) ,$$

which FOC's yield $p_m = \dfrac{a+c\theta_m}{2\theta_m}$, and $p_w = \dfrac{a+c\theta_w}{2\theta_w}$. These prices yield quantities of $q_m = (a - c\theta_m)/2$, and $q_w = (a - c\theta_w)/2$. Note that aggregate output under price discrimination is equal to the aggregate output <u>without</u> price discrimination if <u>both</u> markets were served ($Q = a - \dfrac{(\theta_m + \theta_w)c}{2}$). From part (c), we know that the welfare maximizing distribution for a given output is to set $p_m(q^m) = p_w(q^w)$, which is the case without price discrimination when both markets are served. Notice that under price discrimination $p_m(q^m) \neq p_w(q^w)$ if $\theta_m \neq \theta_w$, which implies that welfare is lower under price discrimination. If, however, without price discrimination only the women were served, then by allowing the monopolist to discriminate it will open the men's market (assuming that $c < \dfrac{a}{\theta_m}$) without changing the price in the women's market. This means that there is added surplus from the men's market and welfare under price discrimination is higher than welfare when price discrimination is forbidden.

Total demand by men: $x_m(p) = a - \theta_m p$

Total demand by women: $x_w(p) = a - \theta_w p$

where $\theta_w < \theta_m$

cost of production $= c$

a) Find: Competitive equilibrium

$$\max_{\{x_m, x_w\}} (p-c)(x_m + x_w)$$

FOC: (x_m): $p - c = 0 \iff p = c$

(x_w): $p - c = 0 \iff p = c$

so $p^*_m = p^*_w = p^* = c$

$x^*_m = x_m(c) = a - \theta_m c$

$x^*_w = x_w(c) = a - \theta_w c$

b) Find: Monopolist equilibrium (no price discrimination).
Conditions for positive demand for men and women.

$$\max_{\{p\}} (p-c)\left(x_m(p) + x_w(p)\right) = (p-c)\left(2a - \theta_m p - \theta_w p\right)$$

FOC: (p): $\quad 2a - \theta_m p - \theta_w p + (p-c)\left(-\theta_m - \theta_w\right) = 0$

$$\Leftrightarrow \quad 2a - \theta_m p - \theta_w p = (p-c)(\theta_m + \theta_w)$$

$$\Leftrightarrow \quad 2a - (\theta_m + \theta_w) p = (\theta_m + \theta_w) p - (\theta_m + \theta_w) \cdot c$$

$$\Leftrightarrow \quad 2a + (\theta_m + \theta_w) c = 2 (\theta_m + \theta_w) p$$

$$\Leftrightarrow \quad p = \frac{a}{\theta_m + \theta_w} + \frac{c}{2}$$

So $\quad p_m^M = p_w^M = p^M = \dfrac{a}{\theta_m + \theta_w} + \dfrac{c}{2} = \dfrac{2a + (\theta_m + \theta_w) c}{2(\theta_m + \theta_w)}$

$$x_m^M(p^M) = a - \theta_m\left(\frac{a}{\theta_m+\theta_w} + \frac{c}{2}\right) = \frac{\cancel{\theta_m a} + \theta_w a - \cancel{\theta_m a}}{\theta_m + \theta_w} - \theta_m \frac{c}{2}$$

$$= \frac{\theta_w a}{\theta_m+\theta_w} - \theta_m \frac{c}{2} = \frac{2\theta_w a - \theta_m c(\theta_m+\theta_w)}{2(\theta_m+\theta_w)}$$

$$x_w^M(p^M) = a - \theta_w\left(\frac{a}{\theta_m+\theta_w} + \frac{c}{2}\right) = \frac{\theta_m a + \cancel{\theta_w a} - \cancel{\theta_w a}}{\theta_w + \theta_w} - \theta_w \frac{c}{2}$$

$$= \frac{\theta_m a}{\theta_m+\theta_w} - \theta_w \frac{c}{2} = \frac{2\theta_m a - \theta_w c(\theta_m+\theta_w)}{2(\theta_m+\theta_w)}$$

Under what conditions will we have positive demand by men and women?

$$x_m(p) > 0 \iff a - \theta_m p > 0 \iff p < \frac{a}{\theta_m}$$

$$x_w(p) > 0 \iff a - \theta_w p > 0 \iff p < \frac{a}{\theta_w}$$

Given that $\theta_w < \theta_m$, for demand in both markets to be positive we need: $p < \frac{a}{\theta_m}$.

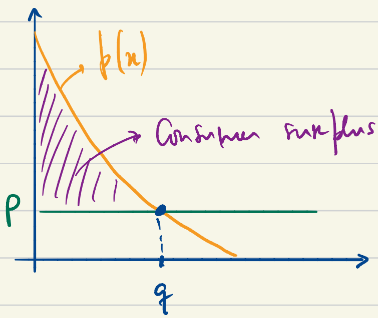
c) Find: Welfare-maximizing way to distribute output X between the men and women.

Given quantity X, we maximize consumer surplus by solving:

$$\max \int_0^{q_m} p_m(x)\,dx + \int_0^{q_w} p_w(x)\,dx$$

$$\text{s.t.: } q_m + q_w = X$$

typically:



Consumer surplus = $\int_0^{q} [p(x) - P]\,dx$

but we are assuming the quantity X is distributed for "free" $(P=0)$

$$\mathcal{L} = \int_0^{q_m} p_m(x)\,dx + \int_0^{q_m} p_w(x)\,dx + \lambda\left[X - q_m - q_w\right]$$

FOC: (q_m): $p_m(q_m) = d$

(q_w): $p_w(q_w) = d$

From (q_m) + (q_w): $p_m(q_m) = p_w(q_w)$

$$x_m = a - \theta_m p_m \Leftrightarrow p_m(x_m) = \frac{a - x_m}{\theta_m}$$

$$x_w = a - \theta_w p_w \Leftrightarrow p_w(x_w) = \frac{a - x_w}{\theta}$$

So: $p_m(q_m) = p_w(q_w) \Leftrightarrow \dfrac{a - q_m}{\theta_m} = \dfrac{a - q_w}{\theta_w}$

since $q_m + q_w = X$ we have: $q_m = X - q_w$

$$\Leftrightarrow \frac{a - X + q_w}{\theta_m} = \frac{a - \theta q_w}{\theta_w} \Leftrightarrow a - X + q_w = \frac{\theta_m}{\theta_w} a - \frac{\theta_m}{\theta_w} q_w$$

$$\Leftrightarrow \left(1+\frac{\theta_m}{\theta_w}\right) q_w = X - \left(\frac{\theta_m}{\theta_w} - 1\right) a$$

$$\Leftrightarrow \left(\frac{\theta_m + \theta_w}{\cancel{\theta_w}}\right) q_w = \frac{\theta_w X - \theta_m a + \theta_w a}{\cancel{\theta_w}}$$

$$\Leftrightarrow (\theta_m + \theta_w)\, q_w = \theta_w X - \theta_m a + \theta_w a$$

$$\Leftrightarrow \quad q_w = \frac{\theta_w X - \theta_m a + \theta_w a}{\theta_m + \theta_w}$$

and $$q_m = X - q_w = \frac{(\theta_m + \theta_w) X - \theta_w X + \theta_m a - \theta_w a}{\theta_m + \theta_w}$$

$$= \frac{\theta_m X + \theta_m a - \theta_w a}{\theta_m + \theta_w}$$

d) Find: Equilibrium when firm is allowed to price discriminate.

$$\max_{\{p_m, p_w\}} (p_m - c)\, x_m(p_m) + (p_w - c)\, x_w(p_w)$$

$$= (p_m - c)(a - \theta_m p_m) + (p_w - c)(a - \theta_w p_w)$$

(p_m): $a - \theta_m p_m - \theta_m (p_m - c) = 0 \Leftrightarrow 2\theta_m p_m = a + \theta_m c$

$$\Leftrightarrow \quad p_m = \frac{a + \theta_m c}{2\theta_m}$$

(p_w): $a - \theta_w p_w - \theta_w (p_w - c) = 0 \Leftrightarrow 2\theta_w p_w = a + \theta_w c$

$$\Leftrightarrow \quad p_w = \frac{a + \theta_w c}{2\theta_w}$$

$$q_m = a - \cancel{\theta_m}\left(\frac{a - \theta_m c}{2\cancel{\theta_m}}\right) = a - \frac{a}{2} - \frac{\theta_m c}{2} = \frac{a - \theta_m c}{2}$$

$$q_w = a - \theta_w \left(\frac{a - \theta_w c}{2\theta_w}\right) = a - \frac{a}{2} - \frac{\theta_w c}{2} = \frac{a - \theta_w c}{2}$$

$$Q = q_m + q_w = \frac{a - \theta_m c}{2} + \frac{a - \theta_w c}{2} = a - \frac{(\theta_m + \theta_w)c}{2}$$

Aggregate output under price discrimination is equal to the aggregate output without price discrimination if both markets are served

$$x_m^M(p^M) = \frac{2\theta_w a - \theta_m c(\theta_m + \theta_w)}{2(\theta_m + \theta_w)}$$

$$x_w^M(p^M) = \frac{2\theta_m a - \theta_w c(\theta_m + \theta_w)}{2(\theta_m + \theta_w)}$$

$$Q^M = x_m^M + x_w^M = \frac{2\theta_w a - \theta_m c(\theta_m + \theta_w)}{2(\theta_m + \theta_w)} + \frac{2\theta_m a - \theta_w c(\theta_m + \theta_w)}{2(\theta_m + \theta_w)}$$

$$= \frac{\cancel{2(\theta_m + \theta_w)}a}{\cancel{2(\theta_m + \theta_w)}} - \frac{(\theta_m + \theta_w)c\cancel{(\theta_m + \theta_w)}}{2\cancel{(\theta_m + \theta_w)}} = a - \frac{(\theta_m + \theta_w)c}{2}$$

From part (c) we know that the welfare maximizing distribution for a given output is to set $p_m(q_m) = p_w(q_w)$ which is the case without price discrimination when both markets are served.

Under price discrimination, $p_m(q_m) \neq p_w(q_w)$ if $\theta_m \neq \theta_w$ which implies that welfare is lower with price discrimination.

If, however, without price discrimination, only the women are served, then by allowing the monopolist to discriminate it will open the men's market $\left(\text{assuming } c < \frac{a}{\theta_m}\right)$ without changing the price in the women's market.

This means that there is added surplus from the men's market and welfare under price discrimination is higher than welfare when price discrimination is not allowed.

12.C.1 Show that in any Nash equilibrium of the Bertrand model with $J > 2$ firms, all sales take place at a price equal to cost.

12.C.1 Following the arguments in the textbook, we cannot have a Nash Equilibrium (NE) with $p_i > c$ for all i. Note also that $p_1 = p_2 = \dots = p_J = c$ is a NE - none of the firms can gain by raising its price and it will lose money if it lowers its price. Extending the argument given in the textbook one can show that if at least two firms charge $p = c$ and all the other firms charge at least c we have a NE. Therefore there will be multiple Nash equilibrium, all yielding the competitive price.

Show: Nash Equilibrium of Bertrand model with $J > 2$ firms occurs at price equals cost.

We cannot have a Nash Equilibrium with $p_i > c$ for all i.

$p_1 = p_2 = \dots = p_j = c$ is a NE. None of the firms can gain by raising its price and it will lose money if it lowers the price.

If at least two firms charge $p = c$ and all other firms charge at least c we have a NE.

Therefore there will be multiple NE, all yielding the competitive price.

12.C.6 Consider a Cournot duopoly in which firms have a cost per unit produced of c and the inverse demand function is $p(q) = a - bq$, with $a > c \geq 0$ and $b > 0$. Find firm j's best response function and find the output pair (q_1^*, q_2^*) that supports a Nash equilibrium. Verify that the Nash equilibrium price is between the monopolist's and socially optimal (competitive) price calculated in 12.B.1.

12.C.6 Each firm j maximizes $[a - b(q_j + q_k) - c]\cdot q_j$, and the FOC is $a - 2bq_j - bq_k - c = 0$ (when $q_j > 0$), which yields the best response function $b_j(q_k) = \text{Max}\{0, (a - c - bq_k)/(2b)\}$. To calculate the Nash equilibrium we have two equations (both BR functions) with two variables since we set $b_j(q_k) = q_j$ to get the equilibrium. Straightforward algebra yields the results.

We have: Cournot duopoly

Cost per unit $= c$

$p(q) = a - bq$ with $a > c \geq 0$, $b > 0$

Find: Firm j's best response function.

NE output.

NE price (between monopolist's and socially optimal price)

Firm j's optimization problem.

$$\max_{\{q_j\}} \Pi_j(q_j, q_k) = (p - c)\, q_j = \Big(a - b\underbrace{(q_j + q_k)}_{q} - c\Big)\, q_j$$

FOC: (q_j): $a - b q_j - b q_k - c - b q_j = 0$

$$\Leftrightarrow q_j = \frac{a - b q_k - c}{2b}$$

Best-response of firm j:

$$b_j(q_k) = \frac{a - b q_k - c}{2b}$$

By symmetry, best response of firm k:

$$b_k(q_j) = \frac{a - b q_j - c}{2b}$$

Nash equilibrium: $q_j = b_j\left(b_k(q_j)\right)$ → plug k's BR into j's BR

$$q_j = \frac{a - \cancel{b}\left(\frac{a - b q_j - c}{2\cancel{b}}\right) - c}{2b}$$

$$\Leftrightarrow \quad 2b\, q_j = a - \frac{a - b q_j - c}{2} - c$$

$$\Leftrightarrow \quad 2b\, q_j = \frac{a}{2} + \frac{b}{2} q_j - \frac{c}{2}$$

$$\Leftrightarrow \quad 4b\, q_j = a + b q_j - c \quad \Leftrightarrow \quad 3b\, q_j = a - c$$

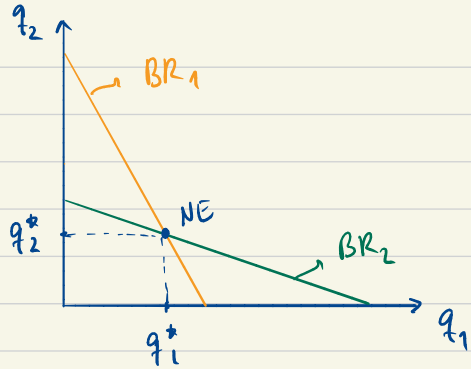
$$\Leftrightarrow \quad q_j = \frac{a-c}{3b}$$

$$q_1^* = q_2^* = \frac{a-c}{3b}$$

NE output: $\quad (q_1^*, q_2^*) = \left(\frac{a-c}{3b}, \frac{a-c}{3b} \right)$

NE price: $\quad p^* = a - b\left(q_1^* + q_2^*\right) = a - \cancel{b}\left(\frac{2(a-c)}{3\cancel{b}} \right)$

$$= a - \frac{2}{3} a + \frac{2}{3} c = \frac{a+2c}{3}$$



The Cournot duopoly NE price is between the monopolist and socially optimal price:

$$p^* = c \quad \rightarrow \text{Competitive}$$

$$p^C = \frac{a+2c}{3} \quad \rightarrow \text{Cournot}$$

$$p^M = \frac{a+c}{2} \quad \rightarrow \text{Monopoly}$$

Therefore: $p^* < p^C < p^M$

12.C.7 Derive the Nash equilibrium price and quantity levels in the Cournot model with J firms where each firm has a constant unit production cost of c and the inverse demand function in the market is $p(q) = a - bq$, with $a > c \geq 0$ and $b > 0$. Verify that when $J = 1$, we get the monopoly outcome; that output rises and price falls as j increases; and that as $J \to \infty$ and price and aggregate output in the market approach their competitive levels.

12.C.7 Using equation 12.C.6 and plugging in the linear demand curve we obtain

$$-bQ^*/J + (a - bQ^*) = c,$$

which yields $Q^* = [J/(J+1)]\cdot[(a-c)/b]$, and $p = (a+Jc)/(J+1)$. For $J = 1$, we get the monopoly quantity $(a-c)/2b$ and price $(a+c)/2$. Clearly the price falls and the quantity increases as J increases. In addition, as $J \to \infty$, the price goes to the competitive price c and the quantity goes to the competitively supplied quantity $(a-c)/b$.

We have: Cournot model with J firms

constant unit production cost $= c$

$p(q) = a - bq$ $\quad a > c \geq 0 \quad b > 0$

Find: NE price and quantity in Cournot model with J firms.

Verify: When $J = 1$ we have the monopoly outcome.
As $J \to \infty$ we have the competitive outcome.

$$q = q_i + Q_{-i}$$ → sum of all other firms' output except i

$$\max_{\{q_i\}} \pi_i(q_i, Q_{-i}) = \left(a - b\left(\overbrace{q_i + Q_{-i}}\right) - c\right) q_i$$

$$= \left(a - bq_i - bQ_{-i} - c\right) q_i$$

FOC: (q_i): $a - b q_i - b Q_{-i} - c - b q_i = 0$

$\Leftrightarrow \quad 2b q_i = a - b Q_{-i} - c$

$\Leftrightarrow \quad q_i = \dfrac{a - b Q_{-i} - c}{2b}$

$b_i(Q_{-i}) = \dfrac{a - b Q_{-i} - c}{2b}$

Symmetric NE: $q_1^* = q_2^* = \dots = q_J^* = q^*$

$q^* = \dfrac{a - b(J-1) q^* - c}{2b} \quad \Leftrightarrow \quad 2b q^* = a - b(J-1) q^* - c$

$\Leftrightarrow \quad \left(2b + b(J-1)\right) q^* = a - c \quad \Leftrightarrow \quad (2b + Jb - b) q^* = a - c$

$\Leftrightarrow \quad b(J+1) q^* = a - c \quad \Leftrightarrow \quad q^* = \dfrac{a-c}{b(J+1)}$

$p^* = a - b(J q^*) = a - \cancel{b} J \left(\dfrac{a-c}{\cancel{b}(J+1)} \right) =$

$= \dfrac{(J+1)a - Ja + Jc}{J+1} = \dfrac{\cancel{Ja} + a - \cancel{Ja} + Jc}{J+1} =$

$= \dfrac{a + Jc}{J+1}$

NE Cournot model with J firms:

$$q^*(J) = \frac{a-c}{b(J+1)}$$

$$p^*(J) = \frac{a+Jc}{J+1}$$

When $J=1$:

$$q^*(J=1) = \frac{a-c}{2b}$$

$$p^*(J=1) = \frac{a+c}{2}$$

As $J \to \infty$:

$$\lim_{J\to\infty} q^*(J) = 0$$

$$\lim_{J\to\infty} Q^*(J) = \lim_{J\to\infty} \left(\frac{J}{J+1}\right)\left(\frac{a-c}{b}\right) = \frac{a-c}{b}$$

$$\lim_{J\to\infty} p^*(J) = c$$

12.C.15 Derive the Nash equilibrium prices of the linear city model where a consumer's travel cost is quadratic in distance, that is, where the total cost of purchasing from firm j is $p_j + td^2$, where d is the consumers' distance from firm j. Restrict attention to the case where v is large enough that the possibility of nonpurchase can be ignored.

12.C.15. The consumer who is indifferent between buying from firm i and firm j is located at distance x from firm i, where

$$p_i + tx^2 = p_j + t(1-x)^2.$$

From here we obtain the demand for firm i's product:

$$x_i(p_i,p_j) = [1/2 + (p_j - p_i)/2t]\ M.$$

(Observe that the demand is the same as for linear transportation costs. The equilibrium is therefore also going to be the same.)

We are looking for an equilibrium (p_i^*,p_j^*). In equilibrium, firm i solves

$$\underset{p_i}{\text{Max}}\ (p_i - c)\ x_i(p_i,p_j^*) = (p_i - c)\ [1/2 + (p_j^* - p_i)/2t]\ M$$

The first-order condition for this maximization program is

$$[t + p_j^* - 2p_i^* + c]\ M/2t = 0.$$

A symmetric derivation for firm j yields

$$[t + p_i^* - 2p_j^* + c]\ M/2t = 0.$$

Solving these two equations, one obtains $p_i^* = p_j^* = c + t$.

We have: Linear City model

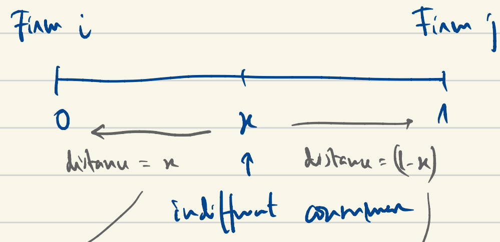
total cost of purchasing from firm j = $p_j + t d^2$

$d \to$ consumer's distance from firm j

$t \to$ travel cost

Find: NE price

Indifferent Consumer (consumer indifferent between buying from firm i and j)



$$p_i + t x^2 = p_j + t(1-x)^2$$

$$\Leftrightarrow \quad p_i + t x^2 = p_j + t(1 - 2x + x^2)$$

$$\Leftrightarrow \quad p_i + t\cancel{x^2} = p_j + t - 2tx + t\cancel{x^2}$$

$$\Leftrightarrow \quad 2tx = p_j - p_i + t$$

$$\Leftrightarrow \quad x = \frac{p_j - p_i + t}{2t}$$

Demand from firm i's product:

$$x_i(p_i, p_j) = \left(\frac{p_j - p_i + t}{2t}\right) M$$

M → mass of total consumers

Firm i's maximization problem:

$$\max_{\{p_i\}} (p_i - c)\, x_i(p_i, p_j) = (p_i - c)\left(\frac{p_j - p_i + t}{2t}\right) N$$

FOC: (p_i): $\left(\frac{p_j - p_i + t}{2t}\right) N - \frac{1}{2t}(p_i - c) N = 0$

$$\Leftrightarrow p_j - p_i + t - p_i + c = 0 \Leftrightarrow p_i = \frac{p_j + t + c}{2}$$

Firm i's best response function:

$$b_i(p_j) = \frac{p_j + t + c}{2}$$

Firm j's best response function:

$$b_j(p_i) = \frac{p_i + t + c}{2}$$

NE: $p_i = b_i(p_j)$

$$p_i = \frac{\left(\frac{p_i + t + c}{2}\right) + t + c}{2}$$

$$\Leftrightarrow \quad 2p_i = \frac{p_i + t + c}{2} + t + c$$

$$\Leftrightarrow \quad 4p_i = p_i + t + c + 2t + 2c$$

$$\Leftrightarrow \quad 3p_i = 3t + 3c$$

$$\Leftrightarrow \quad p_i^* = t + c$$

by symmetry: $p_i^* = p_j^* = t + c$

NE $(p_i^*, p_j^*) = (t + c, \ t + c)$

12.C.18 There are two firms in a market. Firm 1 is the "leader" and picks its quantity first. Firm 2, the "follower", observes firm 1's choice and then chooses its quantity. Profits for firm j given quantity choices q_1 and q_2 are $p(q_1 + q_2)q_j - cq_j$, where $p'(q) < 0$ and $p'(q) + p''(q)q < 0$ at all $q \geq 0$.

(a) Prove formally that firm 1's quantity choice is larger than its quantity choice would be if the firms chose quantities simultaneously and that its profits are larger as well. Also show that aggregate output is larger and that firm 2's profits are smaller.

(b) Draw a picture of this outcome using best-response functions and isoprofit contours.

12.C.18. (a) Firm 1 chooses q_1 knowing that given its choice, firm 2 will produce $b_2(q_1)$, where $b_2(q_1)$ is firm 2's best-response function. Firm 1's program, therefore, is

$$\operatorname*{Max}_{q_1} \ \Pi^1(q_1, b_2(q_1)).$$

The first-order condition for this program can be written as

$$\Pi^1_1(q_1, b_2(q_1)) = -\Pi^1_2(q_1, b_2(q_1))\, b_2'(q_1) < 0$$

(since $\Pi^1_2(q_1, b_2(q_1)) < 0$ and $b_2'(q_1) < 0$). If the firms instead choose quantities simultaneously, the first-order condition becomes

$$\Pi^1_1(q_1, b_2(q_1)) = 0.$$

Since $\Pi^1_{11}(q_1, b_2(q_1)) < 0$, this implies that the Stackelberg leader picks a larger quantity in equilibrium than in the Cournot game. Since the best response function of firm 2 is downward-sloping with a slope larger than - (see Exercise 12.C.8(c)), this implies that the follower picks a smaller quantity and aggregate output increases (and therefore price decreases). Since the leader could have chosen the Cournot quantity, we know that his profits as a Stackelberg leader are higher. The follower produces less and obtains a lower price than in the Cournot outcome, which implies that his profits are lower.

(b) See Figure 12.C.18. *N* denotes the Nash equilibrium outcome, while *S* denotes the equilibrium of the Stackelberg game.

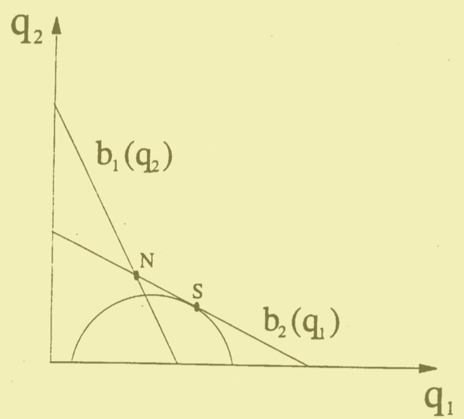


Figure 12.C.18

We have: 2 firms, sequential game (Stackelberg model)

Firm 1 is leader

Firm 2 is follower

$$\text{profit for firm } j = p(q_1 + q_2)\, q_j - c q_j$$

where $p'(q) < 0$ and $p'(q) + p''(q)\, q < 0 \quad \forall q \geq 0$

a) Prove: q_1 in sequential is larger than in simultaneous

$q_1 + q_2$ is larger

π_2 are smaller

Firm 1 chooses q_1 knowing that, given its choice q_1, firm 2 will produce $b_2(q_1)$, where $b_2(q_1)$ is firm 2's best response function.

Firm 1 therefore solves:

$$\max_{\{q_1\}} \Pi_1\left(q_1, b_2(q_1)\right)$$

FOC: (q_1):

$$\frac{\partial \Pi_1\left(q_1, b_2(q_1)\right)}{\partial q_1} + \frac{\partial \Pi_1\left(q_1, b_2(q_1)\right)}{\partial b_2(q_1)} \cdot \frac{\partial b_2(q_1)}{\partial q_1} = 0$$

If firms choose simultaneously then:

$$\frac{\partial \pi_1 \left(q_1, b_2(q_1)\right)}{\partial b_2(q_1)} \cdot \frac{\partial b_2(q_1)}{\partial q_1} = 0$$

and the FOC becomes:

$$\frac{\partial \pi_1 \left(q_1, b_2(q_1)\right)}{\partial q_1} = 0$$

This implies that the Stackelberg leader picks a larger quantity in equilibrium than in the Cournot game.

Since $b_2'(q_1) < 0$, this implies that the follower picks a smaller quantity and aggregate output increases (and therefore price decreases).

Since the leader could have chosen the Cournot quantity, we know that the profit as a Stackelberg leader is higher.

The follower produces less and obtains a lower price than in the Cournot outcome which implies that the profit is lower.

b) Graph: outcome using BR functions and isoprofit contours