Session 8

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6.4. Theorem. $\begin{bmatrix} a_{11} & & U \\ & a_{22} & \\ L & & \ddots \\ & & & a_{nn} \end{bmatrix}$ $\leftarrow$ triangle matrix

$$\lambda_1 = a_{11}, \lambda_2 = a_{22}, \cdots \lambda_n = a_{nn}$$

$$\left| [A - \lambda I \mid 0] \right|$$

$$\left| \begin{bmatrix} a_{11}-1 & & 0 \\ & a_{22}-1 & \\ 0 & & \ddots & a_{nn}-1 \end{bmatrix} \right| \overset{set}{=} 0 .$$

$$(a_{11}-1)(a_{22}-1)\cdots(a_{nn}-1) \overset{set}{=} 0$$

ex. 6.5 Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 3 & 3 & 0 \\ 1 & 0 & 5 & -2 \end{bmatrix}$$

$$\lambda_1 = 2 \quad \lambda_2 = 1 \quad \lambda_3 = 3 \quad \lambda_4 = -2$$

6.6. Theorem $\exists A^{-1}_{n\times n}$ iff. $\lambda_i \neq 0$ $(\forall i \in \{1, \cdots n\})$

6.7. Theorem. Let $A_{n\times n}$ have eigenpair $(\lambda_i, \vec{v}_i)$ $i = 1, 2 \cdots n$

$\Rightarrow$ (a) λ^n is eigenvalue of A^n w/. the same $\vec{x}$. $(\lambda^n, \vec{x})$ is a eigenpair of A^n

(b) $\frac{1}{\lambda}$ is eigenvalue of A^{-1} with the $\vec{x}$. $(\frac{1}{\lambda}, \vec{x})$ is a eigenpair of A^{-1}

(c) $\alpha\lambda$ is eigenvalues of αA

(d) $q(\lambda)$ is eigenvalue of $q(A)$, q is polynomial

Theorem 6.8. $A_{n\times n}$ has pairs $(\lambda_1, \vec{v}_1)(\lambda_2, \vec{v}_2) \cdots (\lambda_n, \vec{v}_n)$. If $\vec{x} \in \mathbb{R}^n$, $\exists$

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \quad *$$

$\Rightarrow \forall k \in \mathbb{N}$

$$A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots + c_n\lambda_n^k\vec{v}_n$$

6.7 ex Compute $\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}^{10}\begin{bmatrix}5\\1\end{bmatrix}$

Step 1. solve for λ's.

$$|A-\lambda I| = 0.$$

$$\begin{vmatrix}-\lambda & 1\\ 2 & 1-\lambda\end{vmatrix} = 0.$$

$$(-\lambda)(1-\lambda) - 2 = 0.$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda-2)(\lambda+1) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = 2$$

Step 2. solve for eigenvectors w.r.t $\lambda_1 = -1$, $\lambda_2 = 2$.

(i) when $\lambda_1 = -1$

solve for $[A-\lambda I \mid 0]$

$$\left[\begin{array}{cc|c}1 & 1 & 0\\ 2 & 2 & 0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$ set $x_2 = -t \in \mathbb{R}$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}+t\\-t\end{bmatrix} = t\begin{bmatrix}+1\\-1\end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix}+1\\-1\end{bmatrix}$$

(ii) $\lambda_2 = 2$

$$\left[\begin{array}{cc|c}-2 & 1 & 0\\ 2 & -1 & 0\end{array}\right]$$

$$\left[\begin{array}{cc|c}-2 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$-2x_1 + x_2 = 0$$

$x_1 = t \quad x_2 = 2x_1 = 2t$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}t\\2t\end{bmatrix} = t\begin{bmatrix}1\\2\end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$$

Step 3. Solve for c_1, c_2.

$$c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}$$

Solve for $\left[\begin{array}{cc|c}1 & 1 & 5\\ -1 & 2 & 1\end{array}\right]$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{cc|c}1 & 1 & 5\\ 0 & 3 & 6\end{array}\right] \xrightarrow{R_2/3} \left[\begin{array}{cc|c}1 & 1 & 5\\ 0 & 1 & 2\end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{cc|c}1 & 0 & 3\\ 0 & 1 & 2\end{array}\right]$$

$c_1 = 3 \quad c_2 = 2$

$$\vec{x} = 3\vec{v}_1 + 2\vec{v}_2$$

Step 4

$$\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}^{10}\begin{bmatrix}5\\1\end{bmatrix}$$

$$= 3\cdot(-1)^{10}\begin{bmatrix}1\\-1\end{bmatrix} + 2\cdot 2^{10}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3+2^{11}\\-3+2^{12}\end{bmatrix}$$

Algorithm for computing $A^k\vec{x}$

step1. solve for λ's.

$\Updownarrow$

solve $|A-\lambda I|=0$

Step2. Solve for eigenvector $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n$

$\downarrow \quad \downarrow \quad \downarrow$

$\lambda_1 \quad \lambda_2 \quad \lambda_n$

step3. solve for $c_1 \cdots c_n$

$\Updownarrow$

solve for $[\vec{v}_1 \vdots \vec{v}_2 \vdots \cdots \vec{v}_n \mid \vec{x}] \Leftrightarrow c_1\vec{v}_1 + \cdots c_n\vec{v}_n = \vec{x}$.

hint: RE form.

Step 4. Compute $A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots c_n\lambda_n^k\vec{v}_n$

7). Deteminants.

7.1. Def. For 3×3 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A = |A| = \boxed{a_{11}}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

where $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \overset{det}{=} a_{11}\cdot a_{22} - a_{12}a_{21}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a$$

7.2. ex. $A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

$$\det A = 5 \cdot \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= 5 \cdot (0+2) + 3 \cdot (3-4) + 2(-1)$$

$$= 10 - 3 - 2 = 5$$

7.3. Def. $A_{ij} \overset{\text{def}}{=}$ matrix A w.o. ith row and jth colum

$$|A| \overset{\text{def}}{=} a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots + (-1)^{1+n} a_{1n} |A_{1n}|$$

$$= -a_{21}|A_{21}| + a_{22}|A_{22}| - \cdots - (-1)^{1+n} a_{2n} |A_{2n}|$$

$+ \quad - \quad + \cdots +$

$- \quad + \quad - \cdots -$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

7.4. The Laplace Expansion Theorem.

$$\det A = a_{i1} C_{i1} + \cdots a_{in} C_{in}$$

$$C_{ij} = (-1)^{i+j} |A_{ij}|$$

$$C_{21} = (-1)^{2+1} A_{21} = -A_{21}$$

7.5 ex. Compute $|A|$. $A=\begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|A| = -(-3)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 3(3-4) + (10-2) = -3+8=5.$$

7.6. Thm. $A=\begin{bmatrix} a_{11} & & O \\ * & * & \\ * & ** & \\ \vdots & \vdots & a_{nn} \end{bmatrix}$ $A=\begin{bmatrix} a_{11} & * & \cdots & \\ & \ddots & & \vdots \\ 0 & & \ddots & * \\ & & & a_{nn} \end{bmatrix}$

$$|A| = \prod_{i=1}^{n} a_{ii} = \prod_{i=1}^{n} \lambda_i$$

7.7. Theorem. $A_{n\times n}$

a. If A has a zero $\left\{\begin{matrix}\text{column}\\ \text{row}\end{matrix}\right.$ $\Rightarrow |A|=0$

b. $A=\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$ $A^*=\begin{bmatrix} \vec{a}_1^T \\ \vec{a}_n^T \\ \vdots \\ \vec{a}_2^T \end{bmatrix}$

$|A| = -|A^*|$

c. If A has two identical $\left\{\begin{matrix}\text{column}\\ \text{rows}\end{matrix}\right.$ $\Rightarrow |A|=0$

d. $B=\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ k\vec{a}_s^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$ $|B| = k|A| \quad (\forall k)$ $A=\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_s^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$

e. $A=\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$ $B=\begin{bmatrix} \vec{b}_1^T \\ \vdots \\ \vec{b}_n^T \end{bmatrix}$ $C=\begin{bmatrix} \vec{c}_1^T \\ \vdots \\ \vec{c}_{k-1}^T \\ \vec{a}_k^T+\vec{b}_k^T \\ \vdots \\ \vec{c}_n \end{bmatrix}$

$\vec{a}_i^T = \vec{b}_i^T = \vec{c}_i^T$ except $c_k^T = a_k^T + b_k^T$

$|C| = |A| + |B|$

Ex1 $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 1 & 5 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 6 \\ 1 & 5 & 1 \end{vmatrix}$

g. $B=\begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_k + k\vec{a}_j \\ \vdots \\ \vec{a}_n \end{bmatrix}$ $|A|=|B|$

Ex2 $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

7.8. Ex.

(i) $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$|A| = 0$

(ii) $B = \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}$

Compute determinant of $A_{n\times n}$ $(n \geq 4)$

Use the following operations as done w/ R.E. form

(i) Interchange R_i w/ R_j: $\boxed{\times(-1)}$

(ii) Add kR_i to R_j

(iii) multiply c on R_i: $\boxed{\text{divide } c \text{ on outside}}$

$A \to L$ or U.

Using $|L| = L_{11} \cdot L_{22} \cdots L_{nn}$

$B \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 0 & -3 & 6 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}$ b. (-1) $\xrightarrow{R_1/3}$ -3 d. $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}$

$\xrightarrow[R_4-5R_1]{R_3-2R_1}$ -3 g. $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 4 & 7 & 3 \\ 0 & -1 & 2 & -9 \end{bmatrix}$ $\xrightarrow{R_2 \leftrightarrow R_4}$ $-(-3)$ $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{bmatrix}$

$\xrightarrow[R_4+2R_2]{R_3+4R_2}$ 3 g. $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 0 & 15 & -33 \\ 0 & 0 & 0 & -13 \end{bmatrix} \to 3(15)$ d. $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 0 & 1 & \frac{-33}{15} \\ 0 & 0 & 0 & -13 \end{bmatrix}$

$|B| = 3 \cdot 1 \cdot (-1) \cdot 15 \cdot (-13) = 3 \cdot 15 \cdot 13$

$\parallel$

$3 \cdot 15 \cdot 1(-1) \cdot 1 \cdot (-13)$

7.9. Theorem

a. $A_{n\times n}$ is invertible iff $|A| \neq 0$.

b. $|kA_{n\times n}| = k^n |A|$

c. $|A_{n\times n} B_{n\times n}| = |A_{n\times n}|\,|B_{n\times n}|$

d. A is invertible $\Rightarrow |A^{-1}| = \dfrac{1}{|A|}$

e. $|A| = |A^T|$

7.10. Cramer's Rule

$A_{n\times n}$, $\vec{b} \in \mathbb{R}^n$. Then the unique solution of $A\vec{x} = \vec{b}$ is given as

$$\vec{x} = [x_1 \cdots x_n]^T$$

$$x_i = \frac{\det(A_i(\vec{b}))}{\det A}$$

$$A_i(\vec{b})_{n\times n} = [a_1 \ \cdots \ a_i \ \cdots \ a_n]$$

$i = 1, \cdots n$.

$a_i \to \vec{b}$

$$[a_1 | \cdots | a_{i-1} | \vec{b} | \vec{a}_{i+1} \cdots a_n]$$

7.11 ex Use Cramer's Rule to solve

$$x_1 + 2x_2 = 2 \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$-x_1 + 4x_2 = 1$$

Find $|A|$ $A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 4 - (-2) = 6$

$$|A_1(\vec{b})| = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8 - 2 = 6$$

$$|A_2(\vec{b})| = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3.$$

$$x_1 = \frac{|A_1(\vec{b})|}{|A|} = \frac{6}{6} = 1 \qquad x_2 = \frac{|A_2\vec{b}|}{|A|} = \frac{3}{6} = \frac{1}{2} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Compute determinant of

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -3 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{3R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2]{R_2 + R_4} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 4 & 7 \end{bmatrix}$$

$$\xrightarrow{+\frac{4}{3} \cdot R_3 + R_4} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$1 \cdot 1 \cdot (-3) \cdot \frac{1}{3} \ -$$

7.11. Adjoint $\quad C_{ji} \equiv (-1)^{j+i}|A_{ji}| = $

$$\begin{array}{ccccccc} & & & i\text{th column} & & & \\ & & & \uparrow & & & \\ \left|\begin{matrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \\ a_{j1} & a_{j2} & \cdots \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots \end{matrix}\right. & & & \begin{matrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{matrix} & & & \left.\begin{matrix} \cdots & a_{1n} \\ \cdots & a_{2n} \\ & \vdots \\ \cdots & a_{jn} \\ & \vdots \\ \cdots & a_{nn} \end{matrix}\right| \begin{matrix} \\ \\ \\ \to j\text{th row} \\ \\ \\ \end{matrix} \end{array}$$

is called the (j,i)th − cofactor of A

$$[C_{ji}] = [C_{ij}]^T \quad \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

is called adjoint (adjugate), $C = \text{adj } A$

7.12. Theorem $\quad A^{-1} = \dfrac{1}{|A|} \cdot \text{adj } A$

7.13. ex. $\quad A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ & & \\ & & \end{bmatrix}$

$$|A| = 1\begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= -6 - 12 + 2((+6) + 4) + (-1)(6 - 2)$$

$$= -18 + 20 - 4 = -2$$

$$C_{ij} = \begin{bmatrix} C_{11} = +\begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = -18 & C_{12} = -\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = 10 & C_{13} = +\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 \\ C_{21} = -\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3 & C_{22} = +\begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2 & C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1 \\ C_{31} = +\begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10 & C_{32} = -\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6 & C_{33} = +\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & 2 \end{bmatrix}^T \Rightarrow A^{-1} = \frac{1}{-2}\begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

Summary

1. 2x2. $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

2. $[A \mid I] \xrightarrow{R.E.} [I \mid A^{-1}]$

3. $A^{-1} = \frac{1}{|A|}(\operatorname{adj} A)$

8). Diagonalization

8.1. Def. $A_{n\times n}$, $B_{n\times n}$ A is similar to B
if $\exists P_{n\times n} \ni P^{-1}AP = B$
denoted, $A \sim B$

$\Rightarrow A \overset{R.E}{\sim} B$.

8.2. Theorem. a. $A \sim A$
b. $A \sim B$, $B \sim C \Rightarrow A \sim C$
c. $A \sim B \Rightarrow B \sim A$

8.3. Properties $A_{n\times n} \sim B_{n\times n} \Rightarrow$

a. $|A| = |B|$

b. $\exists A^{-1}$ iff $\exists B^{-1}$

c. $\text{rank}(A) = \text{rank}(B)$

d. A, B have the same eigenvalues

e. $A^m \sim B^m \quad m \in \mathbb{N}$

$\Updownarrow$

f. $\exists A^{-1} \quad A^m \sim B^m$ for any $m \in \mathbb{Z}$

$P^{-1}AP = B$

$PP^{-1}APP^{-1} = PBP^{-1}$

$\underbrace{(PP^{-1})}_{I} A \underbrace{(PP^{-1})}_{I} = PBP^{-1}$

$A = PBP^{-1}$

8.4. Def. $A_{n\times n}$ is diagonalizable if

$\boxed{\exists D \ni A \sim D}$ D is diagonal matrix

$\Updownarrow$

$\boxed{\text{i.e. } \exists P_{n\times n} \ni P^{-1}AP = D}$

8.5. Theorem. A is diag'lizable iff A has n IND eigenvectors

Moreover A is diag'lizable $\Rightarrow \exists P \ni P^{-1}AP = D$

where $P = [\vec{v}_1 \cdots \vec{v}_n]$, $\underbrace{\vec{v}_1 \cdots \vec{v}_n}_{\text{eigenvectors of } A}$ are IND.

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$\lambda_1 \cdots \lambda_n$ are eigenvalues

8.6. ex. $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

step 1. solve for eigenvalues.

solve $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0.$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$+(-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\lambda \left[(1+\lambda)^2 - 1 \right] = 0.$$

$$\lambda \left[\lambda^2 + 2\lambda \right] = 0$$

$$\lambda^2 (\lambda + 2) = 0$$

$$\lambda_1 = \lambda_2 = 0 \qquad \lambda_2 = -2$$

step 2. solve for eigenvectors.

solve. $[A - \lambda I \mid 0]$

$\lambda_1 = 0$
$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 + 3R_1} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow -x_1 + x_3 = 0$$

$x_1 = x_3$.

x_3 can be any real number

set $x_3 = t$.

x_2 can be any real number

set $x_2 = s$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{x_3 = t}{\underset{x_2 = s}{=}} \begin{bmatrix} t \\ s \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$$

$$= t \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v_1}} + s \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v_2}}$$

$\lambda_3 = -2$.
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0 \qquad x_1 = -x_3$

$x_2 - 3x_3 = 0 \qquad x_2 = 3x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{x_3 = t}{=\!=} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} t \quad \vec{v_3}$$

Step 3. $P \equiv [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$

$$D \equiv \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$$D = P^{-1} A P$$

$$\boxed{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = P^{-1} A [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]} \longrightarrow \boxed{\text{Algorithm to diagonalize matrix } A}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = P^{-1} \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^{-1} \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \quad P$$

Remark: to find eigenvectors, which equivalent to $[A - \lambda I \mid 0]$.

Case 1: If $\lambda_1, \lambda_2, \cdots \lambda_n$ are distinct

plug in λ_i $[A - \lambda_i I \mid 0]$

and solve V_i one-by-one

Case 2: If $\lambda_i = \lambda_j$ $\exists\, i, j \in \{1, 2 \cdots, n\}$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = t\vec{V}_i + s\vec{V}_j$$

we extract $\vec{V}_i$ as eigenvector for λ_i

$\vec{V}_j$ . . . for λ_j

Case 3: $\lambda_i = \lambda_j = \lambda_k$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = t\vec{V}_i + s\vec{V}_j + h\vec{V}_k$$

8.7. Theorem. $A_{n\times n}$ w/ n distinct eigenvalues

$\Rightarrow$ (1) $A_{n\times n}$ is diagonalizable

(2) $V_1 \cdots V_k$ are IND.

8.8. Algorithm to compute A^k

Corrallory if $A = P D P^{-1}$

$$\Rightarrow \quad A^k = P D^k P^{-1}$$

8.9 ex. $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ Find A^{10}.

step 1. Solve for λ_1, λ_2

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(1-\lambda) - 2 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = 2$$

Step 2. solve $\vec{V}_1, \vec{V}_2$

$\lambda_1 = -1$. $\quad [A - \lambda I \mid 0]$

$$\rightarrow \left[\begin{array}{cc|c} +1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0 \qquad x_1 = -x_2 \overset{\text{set}}{=} t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{v}_1}$$

$\lambda_2 = 2 \quad \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad -2x_1 + x_1 = 0$$

$2x_1 = x_2 \overset{\text{set}}{=} 2t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}_2} t$

Step 3. $P = [\vec{V}_1 \ \vec{V}_2] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

step 4. $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^{10}} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^{10} & \\ & 2^{10} \end{bmatrix} \frac{1}{2+1} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Additional Theorems

1. Let $A_{m\times n}$, $B_{n\times m}$. Then AB is $m\times m$ BA is $n\times n$

   Suppose $m \le n$

   $\Rightarrow$ eigenvalues of BA are the m eigenvalues of AB w/. extra eigenvalues being 0.

2. Def. A symmetric matrix A is idempotent if $A = AA$

   Thm. If A is idempotent iff. all its eigenvalues are either 0 or 1.

   The # of eigenvalues equal to 1 = Trace (A)

3. Thm. All eigenvalues of a symmetric matrix are real

4. Thm. Any symmetric matrix is diag'able

5. Def Matrix Exponential

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

properties (i) $e^0 = I$

(ii) $e^{-A} = (e^A)^{-1}$

(iii) $e^{(a+b)A} = e^{aA} e^{bA}$

(iv) if $AB = BA$, $e^{A+B} = e^A e^B$

(v) If $\exists A^{-1}$, $e^{A^{-1}BA} = A^{-1} e^B A$

(vi) $|e^A| = e^{tr(A)}$

$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$$\underbrace{\begin{bmatrix} 1 & LM_1 & LT_1 & K_1 \\ 1 & LM_2 & LT_2 & K_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & LM_n & LT_n & K_n \end{bmatrix}}_{n \times 4} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{4 \times 1}$$

6. Condition number VIF

$\lambda_1, \dots, \lambda_n$ are eigenvalues of $X^T X$

$$k = \frac{\lambda_{max}}{\lambda_{min}} \begin{cases} (0, 100] & \text{Good} \\ [100, 1000] & \text{moderate multicollinearity} \\ (1000, \infty) & \text{serious mult.} \end{cases}$$

Multicollinearity problems

(1) large variance and covariances

(2) explosive and sensitive β's

Treatment of multicollinearity

(1) More data

(2) Model Respecification

(3) Ridge Regression

(4) PCA

(5) X_i / X_j | Delete x