Homework #2 Stochastic Calculus

Ito's Integral

Due Date: November 22, 2022 (in class)

Please email HW to the dedicated email:

`msqfeconometrics2015@gmail.com`

Note: *Your homework should be typped. No late HW will be accepted.*

Problem 1. Let $W(t)$ be a standard Brownian Motionon $[0,T]$, and $0 \leq s < t \leq T$. What is the conditional density of $W(s)$ given $W(t) = y$?

Hint: First determine the joint density of $W(s)$ and $W(t)$, then use conditional density formula.

Problem 2. Show that $W(t)^3 - 3tW(t)$ is a martingale. Here $W(t)$ is standard Brownian Motion.

Hint: Use the method in the lecture notes where we showed $W(t)^2 - t$ is a margingale.

Problem 3 (Stochastic Integrals with Determinic Integrand). Evaluate $\int_0^t s dW(s)$ using the defining approximations to show that

$$\int_0^t s dW(s) = tW(t) - \int_0^t W(s) ds.$$

Here $W(t)$ is standard Brownian Motion. **Hint:** The definition of the left-hand side gives you (here $\{s_j\}$ is the grid on $[0,t]$)

$$\int_0^t s dW(s) = \lim \sum_j s_j \Delta W_j,$$

and the sum here can can be re-written as

$$\sum_j s_j \Delta W_j = \sum_j \Delta (s_j W_j) - \sum_j W_j \Delta s_j. \text{ [to see this is true, expand } \sum_j \Delta (s_j W_j) \text{]}$$

Problem 4. (Riemann Integrals with Stochastic Integrand). Let $W(t) \equiv W(\omega, t)$ be the standard

Brownian Motion on the interval $[0, 1]$. Define random variable Z as

$$Z(\omega) = \int_0^1 W(\omega, t)dt.$$

Because the BM has continuous sample path, the above Z is defined for every ω in the classical sense of Riemann. For such defined Z, find the distribution of Z? **Hint:** *This integral is defined path-wise as*

$$\int_0^1 W(\omega, t)dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} W\left(\frac{i}{n}\right) \cdot \frac{1}{n} \tag{1}$$

Then observe that the summation is a sum of normal random variables, and hence it is also normally distributed. As a consequence, the limit is also normally distributed. Your job is to figure out the relavant parameters of the limit (a random variable) that determine its distribution. This can be done from first principle using the covariances for $W(t)$ that you obtained in HW#1.

Problem 5. (Finding Ditribution of Integrals using Ito Isometry). Find the distribution for

$$\int_0^T e^t dW(t).$$

Hint: The integral here is similar to the one in Problem 3. Recognize that this integral is a normal random variable using the definition similar to the case in (1).

Problem 6. (Finding Ditribution of Integrals using Ito Isometry). Find mean and variance for

$$\int_0^T W(t)^3 dW(t).$$

Hint: To compute the variance you will need the following change of order of integration (the so called Fubini's Theorem. you can assume this is true without proof):

$$\mathbb{E}\left[\int_0^T X(t)dt\right] = \int_0^T \left[\mathbb{E}X(t)\right] dt. \text{ for process } X(t)$$

Then you will find that you need to evaluate the 6-th moment of normal random variable similar to what you did in HW#1.

Bonus Problems

Problem 7. (Stochastic Integrand). Show the following result by evaluating the stochastic integral on the left using the definition:

$$\int_0^t W(s)^2 dW(s) = \frac{1}{3}W(t)^3 - \int_0^t W(t)ds.$$

Hint: Note that the integral on the right is Riemann integral defined path-wise as in Problem 3. The left-hand side is

$$\int_0^t W(t)^2 dW(t) = \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i)^2 \left[W(t_{i+1}) - W(t_i)\right].$$

Try to connect it to the right-hand side.

Problem 8 (Integrals with Deterministic Integrand). One can generalize Problem 4 to the following: what is the distribution of

$$\int_0^T g(s)dW(t),$$

where g is a "nice" determinstic function?