Homework 3

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QF8915 - Stochastic Calculus

Due on Nov 27, 2022

Problem1

Show $W(t)^3 = 3\int_0^t W(s)^2 dW(s) + 3\int_0^t W(s)ds$

Solution:

Since $f(x) = x^3$, so we can get $f'(x) = 3x^2$, and $f''(x) = 6x$.

By Ito's formula, we know:

$$W(t)^3 - W(0)^3 = \int_0^t 3W(s)^2 dW(s) + \frac{1}{2}\int_0^t 6W(s)ds$$

Therefore,

$$W(t)^3 = 3\int_0^t W(s)^2 dW(s) + 3\int_0^t W(s)ds$$

Problem2

Compute $d(S(t)^p)$

Solution:

$$\begin{aligned}
d(S(t)^p) &= (S^p(t))' dS(t) + \frac{1}{2}(S^p(t))'' dS(t)dS(t) \\
&= pS^{p-1}(t)[\alpha S(t)dt + \sigma S(t)dW(t)] + \frac{1}{2}(p(p-1)S^{p-2}(t))[\alpha S(t)dt + \sigma S(t)dW(t)][\alpha S(t)dt + \sigma S(t) \\
&= p\alpha S^p(t)dt + p\sigma S^p(t)dW(t) + \frac{1}{2}p(p-1)S^p(t)dt \\
&= pS^p(t)[\alpha dt + \sigma dW(t) + \frac{1}{2}(p-1)dt] \\
&= pS^p(t)[(\alpha + \frac{1}{2}(p-1))dt + \sigma dW(t)]
\end{aligned}$$

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Problem3

Solution

(i)

$$\begin{aligned} d(W^(t)^4) &= 4W(t)^3 dW(t) + \frac{1}{2} 4 \cdot 3W(t)^2 dt \\ &= 4W(t)^3 dW(t) + 6W(t)^2 dt \end{aligned}$$

So:

$$W(t)^4 - \underbrace{W(0)^4}_{=0} = \int_0^t 4W(s)^3 dW(s) + \int_0^t 6W(s)^2 ds$$

(ii)

$$\begin{aligned} EW(t)^4 &= 4E \int_0^t W(s)^3 dW(s) + 6E \int_0^t W(s)^2 ds \\ &= 4 \sum_0^t E(s)^3 E(W(s) - W(s-1)) + 6 \int_0^t EW(s)^2 ds \\ &= 0 + 6 \int_0^t s^2 ds \\ &= 6 \cdot \frac{1}{2} s^2 |_0^t \\ &= 3t^2 \end{aligned}$$

(iii)

$$dW(t)^6 = 6W(t)^5 dW(t) + 15W(t)^4 dt$$

Then use Ito's lemma:

$$\begin{aligned} W(t)^6 - \underbrace{W(0)^6}_{=0} &= \int_0^t 6W(s)^5 dW(s) + \int_0^t 15W(s)^4 ds \\ &= 6 \int_0^t W(s)^5 dW(s) + 15 \int_0^t W(s)^4 ds \end{aligned}$$

$$\begin{aligned} EW(t)^6 &= 6E \int_0^t W(s)^5 dW(s) + 15E \int_0^t W(s)^4 ds \\ &= 6 \int_0^t E(W(s)^5) dW(s) + 15 \int_0^t E(W(s)^4) dS \\ &= 15 \int_0^t 3s^2 ds \\ &= 45 \cdot \frac{1}{3} s^3 |_0^t \\ &= 15t^3 \end{aligned}$$

Problem4

Solution:

Let $f(t,x) = \exp(ct + \alpha x)$, and $X_t = f(t, W_t)$. Thus: $f_t(t, W_t) = c\exp(ct + \alpha W(t))$, $f_x(t, W(t)) = \alpha \exp(ct + \alpha W(t))$, $f_{xx}(t, W(t)) = \alpha^2 \exp(ct + \alpha W(t))$

Then, by Ito's formula:

$$
\begin{aligned}
dX_t &= f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \underbrace{f_{tx}(t, W(t))dtdW(t)}_{=0} + \frac{1}{2}f_{xx}(t, W(t))dW(t)dW(t) \\
&= c\exp(ct + \alpha W(t))dt + \alpha \exp(ct + \alpha W(t))dW(t) + \frac{1}{2}\alpha^2 \exp(ct + \alpha W(t))dt \\
&= (c + \frac{1}{2}\alpha^2)X_t dt + \alpha X_t dW(t)
\end{aligned}
$$

Problem5

Solution:

Since $Y(t) = \log(S(t))$, $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$. There is an easier method to solve this question of $dY(t)$ compared with the hint.

$$
\begin{aligned}
dY(t) &= \frac{1}{S(t)}dS(t) \\
&= \frac{1}{S(t)}(\alpha S(t)dt + \sigma S(t)dW(t)) \\
&= \alpha dt + \sigma dW(t)
\end{aligned}
$$