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Part VII: Modern Portfolio theory

1. What is the difference between Capital, Asset and Security Allocation?
2. Is the standard deviation of a risky portfolio equal to the linear combination of the standard deviations of the securities included in the portfolio? When this is true?
3. When you can get a perfect hedging portfolio? Derive its formula.
4. Explain, using the CAL and any other material presented in class, what is the expected effect of raising the FED rates for emerging markets? Does this depend on the size of the increase? Does it make sense for local governments (emerging markets' central banks) to increase their local interest rates? Why?
5. What is the minimum variance portfolio? Derive its formula. Explain the efficient vs the non-efficient portions of the Portfolio Opportunity Set (POS).
6. If you have a risk-free financial instrument, is the minimum variance portfolio part of the best CAL? If not, explain the way to obtain the efficient one.
7. Replicate, using excel, the capital and security allocation model. Be sure you understand how to arrive to each of the values presented in class (including the use of solver). Moreover, be sure that you understand how the model works (formulas and so on) and how you can implement them. Be sure that you understand the different between capital allocation and asset allocation.
8. Select 6 securities, 5 years of monthly data. Compute the monthly average returns, standard deviations and correlations. Assume a risk-free rate of 0.5% annual (transform to monthly). Using Excel create the optimal portfolio allocation for an individual with risk aversion coefficient of 3. Provide the risky portfolio weights and the weights between risk free and risky portfolio.
9. Explain the "Two Fund Separation Theorem". Who is more likely to perform each of them (the technical and final decision about the portfolio allocation)?
10. In the Markovitz model, what is the impact of restricting the portfolio weights to be all non-negative (no short sales).
11. Why is the CAPM a general equilibrium model? How does the pricing mechanism work in order to achieve the equilibrium market portfolio?
12. Why the market portfolio should have a weight of 1 in:

$$y^m = \frac{E(r_m) - r_f}{0.01\bar{A}\sigma_m^2}$$

13. Based on the equation presented in question 12, what should be the impact on the market risk premium of:
    a. An increase in market volatility?
    b. An increase in negative market sentiment that increases the average marker risk aversion.
14. What is the relationship/difference between the Capital Allocation Line (CAL) and the Capital Market Line (CML).
15. Why what matters in portfolio allocation is the marginal contribution of a given asset to the overall risk of the portfolio and not the individual risk? How is this idea incorporated in the CAPM?

16. Show:

$$\sum_{j=1}^{n} w_j cov(r_i, r_j) = cov(r_i, r_m)$$

Where r_m represents the market returns.
17. Derive the expected return-beta relationship.
18. How can you use the security market line (SML) and how you can use it to determine whether (under the assumptions of the model) an asset is over- or under-valued.
19. Show that indeed the beta of the market equals 1. Use $\sum_i^n w_i \beta_i$ with n representing all available financial securities in the market.
20. Assume that you have 4 securities and that each enter your portfolio with the following weights: $w_1 = 0.2$, $w_2 = 0.4$, $w_3 = 0.1$, $w_4 = 0.3$. The corresponding betas are: $\beta_1 = 0.75$, $\beta_1 = 0.90$, $\beta_1 = 1.25$, $\beta_1 = 1.50$. What is the beta of your portfolio?
21. Using the data provided in question 20, determine the portfolio weights that allow you to get a portfolio beta of 1.
22. Using Excel, test of the Capital Asset Pricing Model (CAPM) using a simple ordinary least squares. Explain your results in terms of the alpha (intercept) and beta. For this, select any stock (yahoo finance is a good source for data), at least 2 years of daily data.

Part VIII: Options

1. What is the lower value for the price of an European put option with the following characteristics:
   Stock price = US$ 12
   Strike price = US$ 15
   Duration = 1 month
   Dividends = No
   Rf = 6% annual
2. A 1-month European put on a stock that pays no dividends is sold in the market at US$ 2.5. The actual price of the stock is 47, the strike price is US$ 50 and rf is 6% annual. Is there any arbitrage opportunity? If there is one, how can you profit from it?
3. A 4-month European call on a stock that will pay dividends equal to US$ 0.5 at the end of the second month is sold in the market at US$ 2.8. The actual price of the stock is 28, the strike price is US$ 25 and rf is 8% annual. Is there any arbitrage opportunity? If there is one, how can you profit from it?
4. Compute the value of a 9-month European put on a stock with current price of US$ 25 and strike price of US$ 27, where the value of an European call on the same stock and with the same strike price is US$ 2.5 and the rf equals 10% annual.
   If the market price of the put is US$ 2.35, identify the existence of an arbitrage opportunity and explain how you can profit from it.
5. Derive the price formula of the European put on a stock that pays no dividends. Use the portfolio replica as in the European call presented in class.
6. Using the binomial model compute the price of an American call on a stock that pays no dividends. The information that you have is the following:
   Actual value of the stock = US$ 100
   Strike price = US$ 95
   Duration = 1 period
   Rf = 10% per period
   u = 1.25
   d = 0.8
   Do you expect any difference between this price and the one of a European call option on a stock that pays no dividends? Explain.
7. Re-do exercise six, but now assume that the duration is two periods.
8. Using the same data as in question six, what is the price of an American put? Can an American put worth more than a European one? Explain.
9. Re-do exercise eight, but now assume that the duration is two periods.
10. Assuming the following data, what should be the price of an European call?
    Actual value of the stock = US$ 60
    Strike price = US$ 55
    Duration = 2 periods
    Rf = 6% per period
    u1 = 1.1 u2 = 1.15
    d1 = 0.9 d2 = 0.8

11. Assuming the following data, what should be the price of an European call if the stock will pay a dividend proportional to the actual price of the asset, established at 0.1, payable at t1?

    Actual value of the stock = US$ 60
    Strike price = US$ 55
    Duration = 2 periods
    Rf = 6% per period
    u = 1.15
    d = 0.8

12. Assuming the following data, what should be the price of an European put if the stock will pay a constant dividend of US$ 0.5 payable at t1?

    Actual value of the stock = US$ 70
    Strike price = US$ 72
    Duration = 2 periods
    Rf = 5% per period
    u = 1.2
    d = 0.9

Part VII

1) Capital allocation dictates what percentage of the agent's capital should go to risky assets versus fixed (risk free) assets.

Asset allocation dictates the weights given to different assets (stocks vs debt vs gold, etc) within the risky portion of a complete portfolio.

Security allocation decides the weight that should be given to each individual security within an asset class (stocks: AMZN, GOOG, etc)

All of these allocations depend on the agents level of risk aversion (A).

2) The standard deviation of a risky portfolio in only equal to the linear combination of its components standard deviations when the correlation coefficient between its components is 1 (so $\rho = 1$).

The general formula for the variance of r_p is:

$$\sigma^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \operatorname{cov}(D, E)$$

where $\operatorname{cov}(D, E) = \rho \sigma_D \sigma_E$

when $\rho = 1$, we have

$$\sigma^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2\,w_D\sigma_D\,w_E\sigma_E$$

which follows the format in the algebra rule:

$$(x+y)^2 = x^2 + 2xy + y^2$$

Thus $\sigma^2 = (w_D\sigma_D + w_E\sigma_E)^2$

and taking the square root to find the standard deviation gives:

$$\sigma = w_D\sigma_D + w_E\sigma_E$$

which is our weighted average of st. deviations.

3) We can get a perfect hedging portfolio only when $\rho_{E,D} = -1$. This allows us to choose weights for our risky portfolio such that the portfolio volatility $\sigma_p = 0$.

$$\sigma^2 = w_D^2\sigma_p^2 + w_E^2\sigma_E^2 + 2\,w_D w_E\overbrace{(\rho\sigma_D\sigma_E)}^{\text{cov}(D,E)}$$

with $\rho = -1$, the equation becomes

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 - 2\,w_D\sigma_D w_E\sigma_E$$

which follows the algebra rule:

$$(x-y)^2 = x^2 - 2xy + y^2$$

We rewrite the equation as

$$\sigma_p^2 = (w_D\sigma_D - w_E\sigma_E)^2 \quad \text{or} \quad \sigma_p = w_D\sigma_D - w_E\sigma_E$$

Setting σ_p^2 to zero and rewriting w_E as $(1-w_D)$ gives

$$w_D\sigma_D - (1-w_D)\sigma_E = 0$$

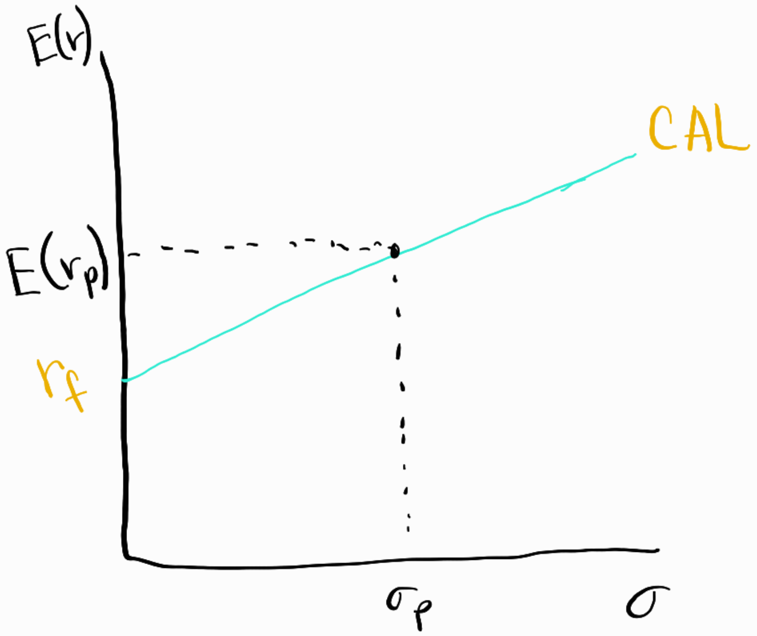
$$w_D\sigma_D - \sigma_E + w_D\sigma_E = 0$$

$$w_D(\sigma_D+\sigma_E) - \sigma_E = 0$$

So $w_D = \dfrac{\sigma_E}{\sigma_D+\sigma_E}$ and $w_E = 1-w_D$

in our perfect hedging portfolio.

4)

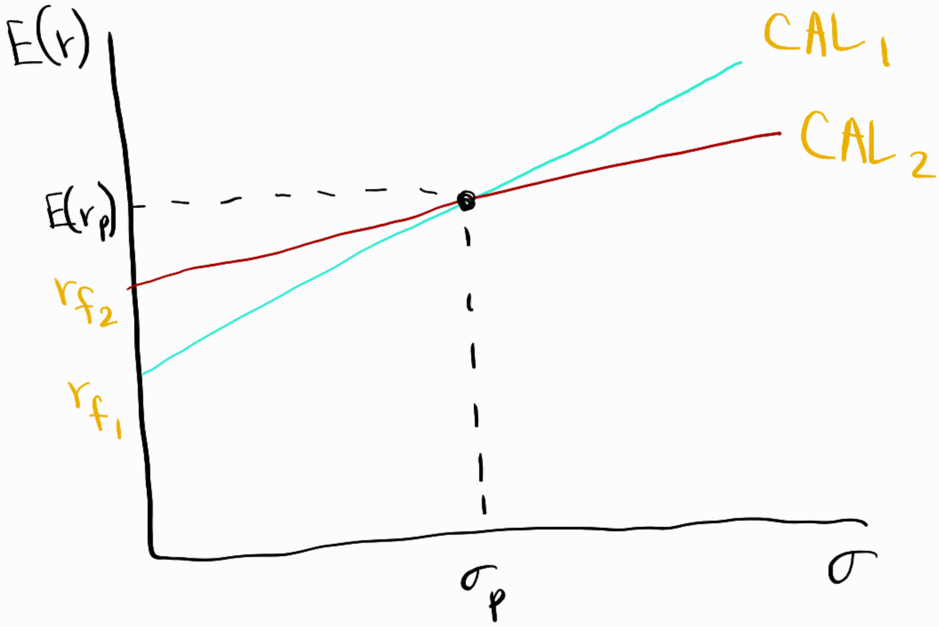


The equation of the capital allocation line is:

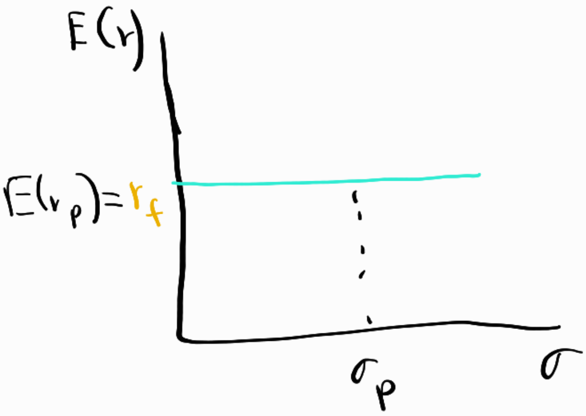
$E(r_c) = r_f + S\sigma_c$ where the slope or Sharpe Ratio is

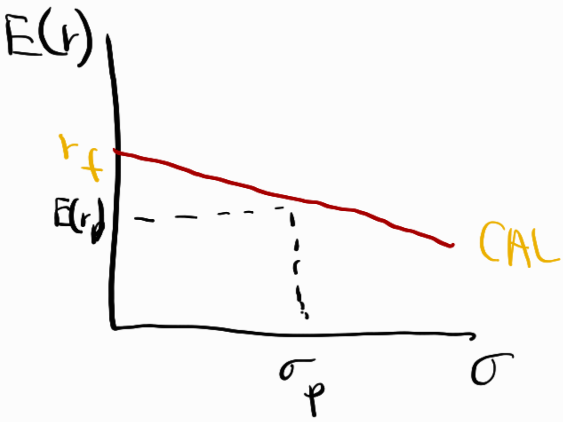
$S = \dfrac{E(r_p) - r_f}{\sigma_p}$. Larger S is preferred.

The rate set by the FED can be thought of as r_f. If r_f increases, the intercept of the CAL increases, but the slope S decreases.



The impact does depend on the magnitude of the change in rates. If r_f is raised until it is equal to $E(r_p)$, then $E(r_c)$ is just r_f such that persons exclusively invest in the fixed asset. If r_f is greater than $E(r_p)$, the CAL has a negative slope.





$$S = \frac{E(r_p) - r_f}{\sigma_p}$$

$$S = 0$$

$$S = \frac{E(r_p) - r_f}{\sigma_p}$$

S is negative

It would only make sense for local governments to raise their rates if their finances are stable enough for their bills to be considered truly risk free.

5) The minimum variance portfolio is the portfolio with the weights that give the smallest possible volatility for that portfolio. These weights are found by optimizing the variance formula by choice of w_D.

$$\min_{w_D} \sigma^2 = w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\,cov(D,E)$$

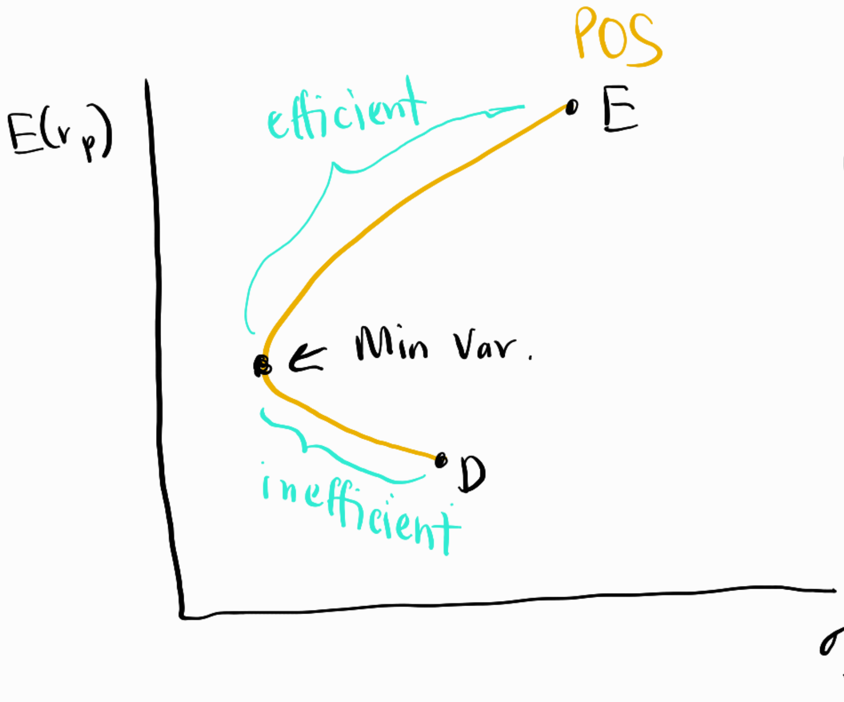
$$\sigma^2 = w_D^2\sigma_D^2 + (1-2w_D+w_D^2)\sigma_E^2 + 2w_D cov - 2w_D^2 cov$$

$$\frac{d\sigma^2}{dw_D} = 0 = 2w_D\sigma_D^2 - 2\sigma_E^2 + 2\sigma_E^2 w_D + 2cov - \underbrace{4w_D cov}_{-2w_D cov - 2w_D cov}$$

$$2w_D\left[\sigma_D^2 + \sigma_E^2 - 2cov\right] = 2\left[\sigma_E^2 - cov\right]$$

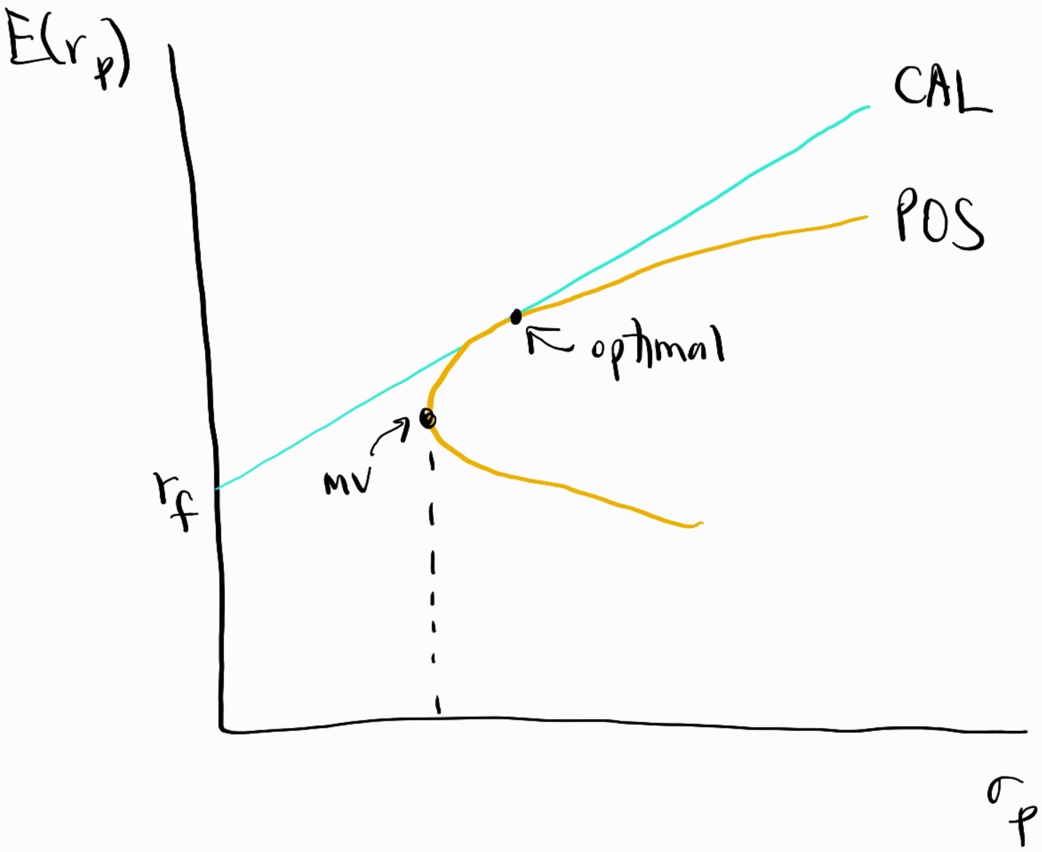
$$\boxed{w_D^{MV} = \frac{\sigma_E^2 - cov(r_E, r_D)}{\sigma_D^2 + \sigma_E^2 - 2cov(r_E, r_D)}}$$

The POS is the line formed by changing the weights between D and E.



The part of the graph under the minimum variance is called inefficient because there exists a w_D and w_E that gives a higher $E(r_p)$ for the same level of volatility.

6) If there is a risk-free instrument, the minimum variance portfolio is not a part of the best CAL. The optimal point occurs where the CAL and POS are tangent.



To find the weights w_D & w_E for this optimal point, we maximize the slope of the CAL by choice of w_D:

$$\max_{w_D} \quad S = \frac{E(r_p) - r_f}{\sigma_p}$$

where $E(r_p) = w_D E(r_D) + (1-w_D) E(r_E)$

and $\sigma_p^2 = w_D^2 \sigma_D^2 + (1-w_D)^2 \sigma_E^2 - 2 w_D (1-w_D) \mathrm{cov}(r_D, r_E)$

This gives us the following formula for w_D:

$$w_D^* = \frac{[E(r_D) - r_f]\sigma_E^2 - [E(r_E) - r_f]\,\mathrm{cov}(r_D, r_E)}{[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 - [E(r_D) + E(r_E) - 2r_f]\,\mathrm{cov}(r_D, r_E)}$$

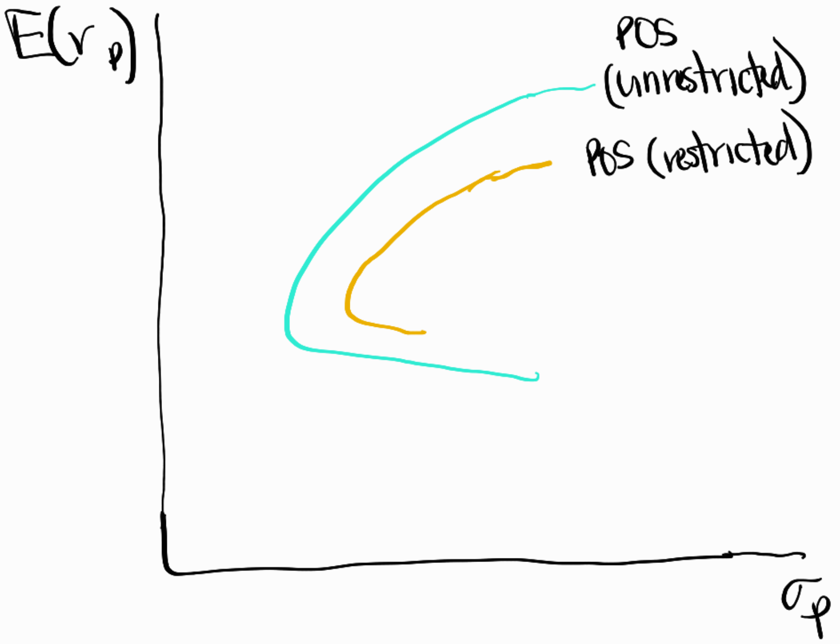
7) Review Excel worksheets (capital & asset)

8) Review Excel worksheets (security allocation)

9) The two fund seperation theorem states that under conditions where all investors borrow and lend at the riskless rate, all investors will choose to possess either the risk free portfolio or the market portfolio. The chosen portfolio depends on the investors

risk aversion levels

10) When we only allow positive weights (no short sales), the area covered by the POS shrinks. The end include putting everything into 1 asset / security.



11) The capital asset pricing model is a general equilibrium model because, under the model's assumptions, we end up with a single risky portfolio across all investors. This is called the market portfolio.

For each asset i, the weight assigned to it is:

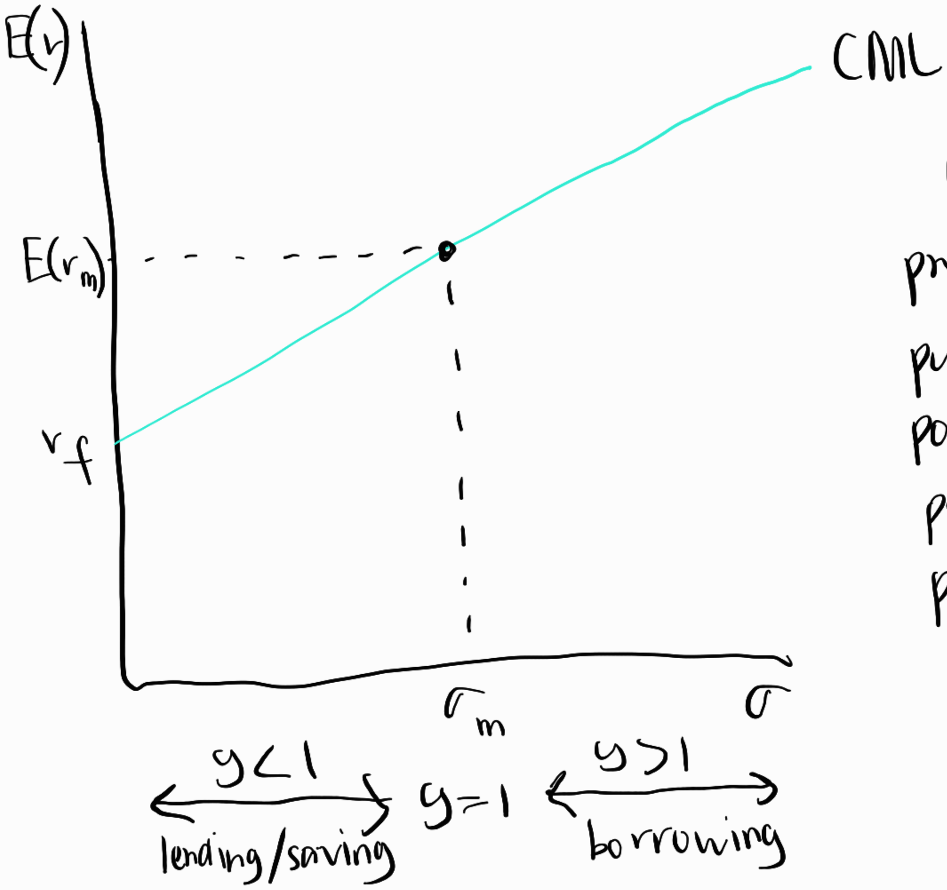
$$w_i = \frac{\text{market cap. of asset } i}{\text{total market cap}}$$ where market cap = # of shares $\cdot$ share price

If the price of any particular asset was so high that no one wants to purchase, then the price would fall such that the market cap for asset i falls. This in turn reduces the weight given to that asset. The price falls until it becomes attractive enought for investors to purchase the asset again.

12) In the CAPM, the risky market portfolio should have a weight of 1, so:

$$y^m = \frac{E(r_m) - r_f}{0.01 A \sigma_m^2} = 1$$

This is because in the aggregate, borrowing and lending across all investors should be equal. If borrowing equals lending, then $y = 1$. $y > 1$ suggest borrowing and $y < 1$ suggests lending/saving.



Recall that y is the proportion of capital to put into the risky portfolio. $1-y$ is the portion in the risk-free portfolio.

13) Since $y^m = 1$ always, any change in the denominator must be compensated for in the numerator.

$$y^m = \frac{E(r_m) - r_f}{0.01\bar{A}\,\sigma_m^2}$$

where $E(r_m) - r_f$ is the market risk premium.

a) If market volatility increases, the overall value of the fraction would be smaller. Thus, the market risk premium must increase to compensate. So $E(r_m)$ must increase in response to an increase in σ_m, which is what we see in the CML graph.

b) If the average market risk aversion $\bar{A}$ increases, the market risk premium must increase to compensate. So $E(r_m)$ must increase in response to higher risk aversion in the market for investors to maintain the same proportion of capital in their risky portfolio.

14) Both the CML and CAL map the expected returns of a portfolio to the volatility of that return, and the general relationship is that greater volatility requires a higher expected return. However, the CAL looks at the $E(r_c)$ for one investor based on only their level of risk aversion. Also, since we only focus on one person instead of the aggregate, they can choose to borrow or lend/save. So $y \geq 1$ or $y \leq 1$.

For CML, we look at the aggregate such that the required $E(r_m)$ and y_m are based on the average market risk aversion $\bar{A}$. So CML is based on the CAPM general equilibrium model. $y_m = 1$ since borrowing and lending cancel each other out in the aggregate.

? 15) We focus on the contributions to the overall market risk rather than the individual asset risk because it may be that this individual asset risk is cancelled out in the aggregate. Thus we need to take into account the covariance of asset i and each of the other $j-1$ assets.

We can determine this by summing across the row for asset i in the border multiplied variance covariance matrix.

$$\begin{array}{c|ccccc} & w_1 & \cdots & w_i & \cdots & w_j \\ \hline w_1 & var(w_1) & \cdots & cov(w_1,w_i) & \cdots & cov(w_1,w_j) \\ \vdots & & \ddots & & & \\ w_i & cov(w_i,w_1) & \cdots & var(w_i) & \cdots & cov(w_i,w_j) \\ \vdots & & & & \ddots & \\ w_j & & & & & var(w_j) \end{array}$$

$$w_i \cdot \sum_{j=1}^{j} w_j \, cov(w_i,w_j)$$

16) Show that

$$\sum_{j=1}^{n} w_j \operatorname{cov}(r_i, r_j) = \operatorname{cov}(r_i, r_m)$$

We know that in all our allocation problems, the sum of our weights always add up to 1. Thus, $\sum_{j=1}^{n} w_j = 1$. Also $\sum_{j=1}^{n} \operatorname{cov}(r_i, r_j) = \operatorname{cov}(r_i, \sum_{j=1}^{n} r_j)$.

We know that the sum of all returns for the individual assets is just the market return. Thus $\sum_{j=1}^{n} r_j = r_m$. So:

$$\sum_{j=1}^{n} \cancel{w_j} \underbrace{\operatorname{cov}(r_i, r_j)}_{\Sigma = r_m} \longrightarrow \boxed{\operatorname{cov}(r_i, r_m)}$$

17) Expected return Beta relationship:

Summing across the asset i row in the B-M cov matrix gives $w_i \cdot \sum_{j=1}^{n} w_j \operatorname{cov}(r_i, r_j)$. The answer in 16 shows us that this can be rewritten as

$w_i \cdot \operatorname{cov}(r_i, r_m)$ which tells us how volatility in asset i impacts the market volatility.

To derive the Beta relationship, we first want to compare the benefit of asset i to overall market return against any changes in market volatility that asset i causes.

$$\frac{i \text{ contribution to } r_m}{i \text{ contribution to } \sigma_m} = \frac{\cancel{w_i}\,[E(r_i) - r_f]}{\cancel{w_i}\, cov(r_i, r_m)}$$

If we wrote the same formula for the market, we get:

$$\frac{\Sigma w_j\,[E(r_m) - r_f]}{\Sigma w_j\, cov(r_m, r_m)} \quad \text{or} \quad \frac{E(r_m) - r_f}{\sigma_m^2}$$

Setting the two formulas equal to each other gives:

$$\frac{E(r_i) - r_f}{cov(r_i, r_m)} = \frac{E(r_m) - r_f}{\sigma_m^2}$$

$$[E(r_i) - r_f] = \underbrace{\boxed{\frac{cov(r_i, r_m)}{\sigma_m^2}}}_{\beta} [E(r_m) - r_f]$$

so market premium for asset i = $\beta \times$ market premium for the market

$B > 1$ aggressive & $B < 1$ defensive.

If we write out this relationship for every asset multiplied by its weight, we get the following:

$$w_1 E(r_1) = w_1 r_f + w_1 \beta_1 [E(r_m) - r_f]$$

$$w_2 E(r_2) = w_2 r_f + w_2 \beta_2 [E(r_m) - r_f]$$

$$\vdots$$

$$w_j E(r_j) = w_j r_f + w_j \beta_j [E(r_m) - r_f]$$

Summing across these equations gives:

$$\underbrace{\sum_{j=1}^{n} w_j E(r_j)}_{E(r_p)} = \underbrace{\sum_{j=1}^{n} w_j}_{1} r_f + \underbrace{\sum_{j=1}^{n} w_j \beta_j}_{\beta_p} [E(r_m) - r_f]$$

So $E(r_p) = r_f + \beta_p [E(r_m) - r_f]$

Note: for $E(r_m)$, $\beta = 1$ always.

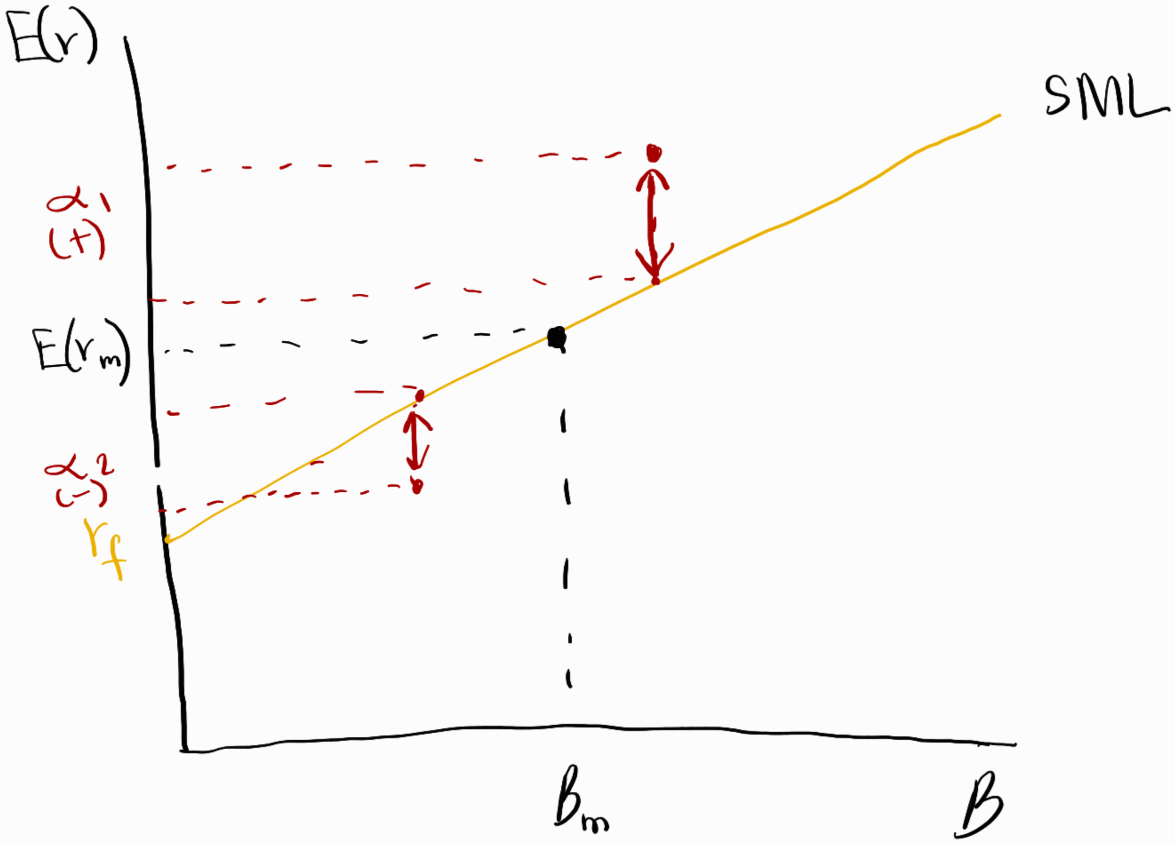
18) $E(r_p) = r_f + \beta_p [E(r_m) - r_f]$ gives us the security markets line over different values for β. The SML essentially shows the expected return of asset i given its parameter β_i.

To see whether the asset is over or under-valued, we test for α in the below equation:

$$E(r_i) - r_f = \alpha + \beta_i [E(r_m) - r_f]$$

$\alpha > 0 \rightarrow$ undervalued {higher real $E(r_i)$}

$\alpha < 0 \rightarrow$ overvalued {lower real $E(r_i)$}



19) Show that $\beta_m = 1$

From 17, we saw that

$$E(r_p) = r_f + \beta_p \left[E(r_m) - r_f\right]$$

If we used $E(r_m)$ instead of $E(r_p)$ →

$$E(r_m) = r_f + \beta \left[E(r_m) - r_f\right]$$

$$E(r_m) - r_f = \beta \left[E(r_m) - r_f\right]$$

$$\beta = \frac{E(r_m) - r_f}{E(r_m) - r_f} = \underline{1} \quad \text{so} \quad \beta_m = \underline{1}$$

20) 4 securities

$w_1 = 0.2 \quad w_2 = 0.4 \quad w_3 = 0.1 \quad w_4 = 0.3$

$B_1 = 0.75 \quad B_2 = 0.9 \quad B_3 = 1.25 \quad B_4 = 1.5$

The overall B is the weighted average of the individual Betas:

$$w_1 B_1 + w_2 B_2 + w_3 B_3 + w_4 B_4 = \underline{\underline{B}}$$

$$0.2(0.75) + 0.4(0.9) + 0.1(1.25) + 0.3(1.5) = 1.085$$

$$\boxed{\underline{\underline{B}} = 1.085}$$

21) Using data from 20, find the weights that give $\underline{\underline{B}} = 1$.

$w_1 = 0.15 \quad w_2 = 0.60 \quad w_3 = 0.1 \quad w_4 = 0.15$

22) Do in Excel

Part VIII

1) For there to be no arbitrage opportunity, the lower bound of a European put with no dividends is:

$$\boxed{P > Ke^{-r(T-t)} - S}$$

Derivation (portfolio comparisons):

	t_0	t_1: $S_T > K$	t_1: $S_T < K$
Portfolio 1 (put, stock)	$-P, -S$	0	$K - S_T$
		S_T	S_T
		S_T	K
Portfolio 2 (cash)	$-Ke^{-r(T-t)}$	K	K

When $S_T > K$, we prefer port 1. When $S_T < K$, we are indifferent. Thus, for no arbitrage, portfolio 1 must cost more than portfolio 2. So

$-P - S < -Ke^{-r(T-t)}$ or $P > Ke^{-r(T-t)} - S$

For $S=12$ $K=15$ $T-t=1$ month $r=6\%$ (annual)

$$p > (15)e^{-\frac{0.06}{12}(1)} - 12 \rightarrow \boxed{p > 2.93}$$

2) European put, no dividends

$p = 2.5$ $S=47$ $K=50$ $r_f = 6\%$ (annual)
$T-t = 1$ mont

Arbitrage opportunity and strategy?

For there to be no arbitrage opportunity, $p > Ke^{-r(T-t)} - S$

$$p > 50e^{-0.06(1/12)} - S \rightarrow p > 2.75$$

$p = 2.5 < 2.75$ $\boxed{\text{There is an arbitrage opportunity}}$

Strategy for profiting from arbitrage opportunity	buy put	(-2.5)
	buy stock	(-47)
	borrow money to buy put & stock	$(+49.5)$

Future value of borrowed money =

$$-\left[49.5e^{0.06(\frac{1}{12})}\right] = -49.75$$ to be returned in $t=1$

If $S_T > K$ in $t=1$	If $S_T < K$ in $t=1$
→ do not exercise put (0) ↳ sell stock in market (S_T) → pay loan (-49.75)	→ excercise put (50) → pay loan (-49.75)
payoff: $S_T - 49.75 > 0$ also > 0.25	$50 - 49.75 = 0.25 > 0$

3) European call with dividends

$D = 0.5$ (paid at end of second month)

$c = 2.8 \quad S = 28 \quad K = 25 \quad r_f = 8\%$ (annual)

$(T-t)_c = 4$ months $\quad (T-t)_D = 1$ month

Arbitrage opportunity & strategy?

For there to be no arbitrage opportunity:

$$c > S - Ke^{-r(T-t)} - De^{-r(T-t)}$$

$$28 - (25)e^{-0.08(4/12)} - (0.5)e^{-0.08(1/12)} = 3.16$$

$c = 2.8 < 3.16$

$\boxed{\text{So we have an arbitrage opportunity}}$

Strategy to profit from opportunity $\begin{cases} \text{buy call} & (-2.8) \\ \text{short sell stock} & (28) \\ \text{lend the difference} & (-25.2) \end{cases}$

future value of loans : $25.2e^{0.08(4/12)} = 25.88$

$t = 1$

$S_T > k$	$S_T < k$
→ exercise call (−25)	→ do not exercise call
→ collect dividend (+0.5)	→ buy stock at market price $(-S_T)$
→ take stock and return to lender	→ collect dividend (+0.5)
→ recover loans (+25.88)	→ return to lender
	→ recover loans (+25.88)
payoff: $1.38 > 0$	$26.38 - S_T > 0$ also > 1.38

4) Price of European put with no dividends?
$S = 25 \quad k = 27 \quad r = 10\%$ (annual) $T - t = 9$ months

$c_E = 2.5$

Also if market $p = 2.35$, is there an arbitrage opportunity and what is the strategy to profit from it?

Put-Call Parity Derivation:

	t_0	t_1: $S_T > k$	t_1: $S_T < k$
Port. 1 (call, cash)	$-C$	$S_T - k$	0
	$-ke^{-r(T-t)}$	k	k
		payoff: S_T	k
Port. 2 (put, stock)	$-P$	0	$k - S_T$
	$-S$	S_T	S_T
		payoff: S_T	k

We are indifferent between both portfolios in both instances. Thus the cost of the portfolios must be equal:

$$-C - ke^{-r(T-t)} = -P - S$$

or $$C + ke^{-r(T-t)} = P + S$$

→ calculating the put price using formula:

$$P = C + ke^{-r(T-t)} - S = 2.5 + 27e^{-0.1(9/12)} - 25$$

$P = 2.55$ is our no arbitrage condition.

$P_{market} = 2.35 \neq 2.55$ so there is an arbitrage opportunity

$P_{market} < P_{parity}$ so the put is undervalued, which means we should buy it.

strategy to profit from arbitrage

- → buy the put (−2.35)
- → buy the stock (−25)
- → sell the call (2.5)
- → borrow the difference (to buy put & stock) (24.85)

future value of borrowed money:

$$-\left[24.85e^{0.1(9/12)}\right] = -26.79$$

$S_T > k$	$S_T < k$
→ do not exercise put (0)	→ exercise put (27)
→ the person who you sold the call to exercises call (27)	→ person we sold call to does not exercise call (0)
→ pay loans (−26.79)	→ pay loans (−26.79)
payoff: $0.21 > 0$	payoff: $0.21 > 0$

Note: Arbitrage opportunity is equal between both S_T scenarios.

5) Price formula for European put on a stock that pays no dividends.

$$
\begin{tikzcd}[row sep=small]
t_0 & t_1 \\
 & us \\
S \arrow[ur] \arrow[dr] & \\
 & ds
\end{tikzcd}
$$

$$
\begin{tikzcd}[row sep=small]
 & p_u = \max(X - us, 0) \\
p \arrow[ur] \arrow[dr] & \\
 & p_d = \max(X - ds, 0)
\end{tikzcd}
$$

Portfolio Replica;

$\phi s + B$ equivalent to p (it is constructed to mimic the movements of p)

where $\phi s > 0$ indicates long position
$\phi s < 0$ indicates short position
$B > 0$ indicates lending
$B < 0$ indicates borrowing

t_0 $\qquad\qquad\qquad$ t_1

$$\emptyset s + B \;\nearrow\; \emptyset us + B(1+r) \approx P_u$$

$$\emptyset s + B \;\searrow\; \emptyset ds + B(1+r) \approx P_d$$

By construction:

$$P_u = \emptyset us + B(1+r) \qquad (1)$$

$$P_d = \emptyset ds + B(1+r) \qquad (2)$$

$$P_u - P_d = \emptyset (us - ds) \qquad ((1) - (2))$$

$$\emptyset = \frac{P_u - P_d}{us - ds} \qquad (3)$$

$$\left(\frac{P_u - P_d}{u\cancel{s} - d\cancel{s}}\right) u\cancel{s} + B(1+r) = P_u \qquad ((3) \text{ in } (1))$$

$$\frac{uP_u - uP_d}{u - d} + B(1+r) = P_u$$

$$B(1+r) = P_u - \left(\frac{uP_u - uP_d}{u - d}\right)$$

$$B(1+r) = \frac{\cancel{u p_u} - d p_u - \cancel{u p_u} + u p_d}{u-d}$$

$$\boxed{B = \frac{u p_d - d p_u}{(u-d)(1+r)}} \quad (4)$$

Also by construction:

$$\phi s + B = p \quad (5)$$

$$\frac{p_u - p_d}{(u-d)\cancel{s}} \cdot \cancel{s} + \frac{u p_d - d p_u}{(u-d)(1+r)} = p \quad \left(\begin{array}{c} (3) \, \& \, (4) \\ \text{in } (5) \end{array} \right)$$

$$\frac{(1+r)(p_u - p_d) + u p_d - d p_u}{(u-d)(1+r)} = p$$

$$\frac{p_u\left[(1+r) - d\right] + p_d\left[u - (1+r)\right]}{(u-d)(1+r)} = p$$

If we define quasi-probabity q as:

$$\boxed{q = \frac{[(1+r) - d]}{(u-d)}} \quad \text{then} \quad \boxed{(1-q) = \frac{u-(1+r)}{(u-d)}}$$

So our price formula for the European put is:

$$\boxed{P = \frac{q\,P_u + (1-q)\,P_d}{(1+r)}}$$

6) Price of American call with no dividends

$S = 100 \qquad X = 95 \qquad t = 1$ period

$R_f = 10\%$ per period $\qquad u = 1.25 \quad d = 0.8$

$t_0 \qquad\qquad t_1$

$$\begin{tikzcd}[row sep=small] & uS = 1.25 \times 100 = \underline{125} \\ S \arrow[ur] \arrow[dr] & \\ & dS = 0.8 \times 100 = \underline{80} \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & C_u = \max(uS - X, 0) = \max(125 - 95, 0) = \underline{30} \\ C \arrow[ur] \arrow[dr] & \\ & C_d = \max(dS - X, 0) = \max(80 - 95, 0) = \underline{0} \end{tikzcd}$$

$$C = \frac{q\,C_u + (1-q)\,C_d}{(1+r)} = \frac{30q + 0(1-q)}{1.1}$$

$$q = \frac{[(1+r)-d]}{(u-d)} = \frac{[1.1-0.8]}{1.25-0.8} = \frac{0.3}{0.45} = \frac{2}{3}$$

$$q = \frac{2}{3} \qquad (1-q) = \frac{1}{3}$$

$$C = \frac{30(2/3)}{1.1} = \boxed{18.18}$$

In this case, we dont expect the price of an American and European call to be different since the is only one time period.

7) Same as 6 but with two periods:

$S=100$ $\quad X=95$ $\quad R_f = 10\%$ per period $\quad t = 2$ periods

$u = 1.25$ $\quad d = 0.8$

$$\begin{tikzcd}
t_0 & t_1 & t_2 \\
 & & u^2s = 156.25 \\
 & us = 125 \arrow[ur] \arrow[dr] & \\
s \arrow[ur] \arrow[dr] & & uds = 100 \\
 & ds = 80 \arrow[ur] \arrow[dr] & \\
 & & d^2s = 64
\end{tikzcd}$$

$$C \to C_u \ (t=1 \to 30) \to C_{u^2_3} = \max(156.25-95, 0) = \underline{61.25}$$

$$C_u \ (t=1 \to 30) \to C_{ud_3} = \max(100-95, 0) = \underline{5}$$

$$C \to C_d \ (t=1 \to 0) \to C_{ud_3}$$

$$C_d \ (t=1 \to 0) \to C_{d^2_3} = \max(64-95, 0) = \underline{0}$$

$q = 2/3$

$$C_u = \frac{q(61.25)+(1-q)(5)}{(1.1)} = 38.64$$

$$C_d = \frac{q(5)+(1-q)(0)}{(1.1)} = 3.03$$

$\underset{(t=2)}{C_u} = 38.64 > \underset{(t=1)}{C_u} = 30$ So we do not execute early when stock increases

$\underset{(t=2)}{C_d} = 3.03 > \underset{(t=1)}{C_d} = 0$ So we do not execute early when stock decreases.

$$C_{American} = \frac{q(38.64)+(1-q)(3.03)}{1.1} = \boxed{24.34}$$

Since we do not execute the call early, the call option price for an American call should be the same as a call option price for a European call (a general rule for call options pricing).

8) Price of an American put.

$S = 100 \quad X = 95 \quad t = 1$ period $\quad r = 10\% \quad u = 1.25 \quad d = 0.8$

$q = 2/3 \qquad (1-q) = 1/3$

$$\begin{array}{lll} t_0 & & t_1 \\ & \nearrow & uS = 125 \\ S & & \\ & \searrow & dS = 80 \end{array}$$

$$\begin{array}{ll} & \nearrow P_u = \max(X - uS, 0) = \max(95-125, 0) = \underline{0} \\ P & \\ & \searrow P_d = \max(X - dS, 0) = \max(95-80, 0) = \underline{15} \end{array}$$

$$P = \frac{q(0) + (1-q)15}{1.1} = \frac{\frac{1}{3}(15)}{1.1} = \boxed{4.54}$$

Here, the price of a European and American put is the same since we only have one time period. However, an American put can be worth more than a European put if there is more than one time period.

9) Same as 8 but with 2 time periods

$S=100 \quad X=95 \quad t=2 \text{ periods} \quad r=10\% \quad u=1.25 \quad d=0.8$

$q = 2/3 \quad (1-q) = 1/3$

$t_0 \qquad t_1 \qquad t_2$

$$S \to uS = 125 \to u^2S = \underline{156.25}$$
$$uS = 125 \to udS = \underline{100}$$
$$S \to dS = 80 \to udS = \underline{100}$$
$$dS = 80 \to d^2S = \underline{64}$$

$$P \to \underset{(t=1 \to 0)}{P_u} \to P_{u^2S} = \max(95-156.25, 0) = \underline{0}$$
$$P_u \to P_{udS} = \max(95-100, 0) = \underline{0}$$
$$P \to \underset{(t=1 \to 15)}{P_d} \to P_{udS}$$
$$P_d \to P_{d^2S} = \max(95-64, 0) = \underline{31}$$

$$\underset{(t=2)}{P_u} = \frac{q(0) + (1-q)0}{1.1} = \underline{0}$$

$$\underset{(t=2)}{P_d} = \frac{q(0) + (1-q)31}{1.1} = \frac{1/3(31)}{1.1} = \underline{9.4}$$

$$\underset{(t=1)}{P_u} = \underset{(t=2)}{P_u} = 0$$

so we don't exercise early when the stock price increases in t_1

$P_d = 15 > P_d = 9.4$
$(t=1)$ $(t=2)$

so we do exercise the American put early when the stock price decreases in t_1

American Put Price

$P_A \to 0, \; 15$

$$P_A = \frac{q(0) + (1-q)15}{1.1}$$

$$\boxed{P_A = 4.54}$$

European Put Price

$P_E \to 0, \; 9.4$

$$P_E = \frac{q(0) + (1-q)9.4}{1.1}$$

$$P_E = 2.84$$

10) European Call with different changes in each period:

$S = 60$ $X = 55$ $t = 2$ periods $r_f = 6\%$ per period

$(u_1 = 1.1, d_1 = 0.9)$; $(u_2 = 1.15, d_2 = 0.8)$

$$q_1 = \frac{[(1+r) - d_1]}{u_1 - d_1} = \frac{(1.06 - 0.9)}{(1.1 - 0.9)} = \underline{0.8}$$

$$q_2 = \frac{[(1+r) - d_2]}{u_2 - d_2} = \frac{(1.06 - 0.8)}{(1.15 - 0.8)} = \underline{0.74}$$

t_0 t_1 t_2

$$
\begin{tikzcd}[column sep=small]
 & & u_1u_2S = \underline{75.9} \\
 & u_1S = 66 \arrow[ur] \arrow[r] & u_1d_2S = \underline{52.8} \\
S=60 \arrow[ur] \arrow[dr] & & d_1u_2S = \underline{62.1} \\
 & d_1S = 54 \arrow[ur] \arrow[r] & d_1d_2S = \underline{43.2}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & & C_{u_1u_2} = \max(75.9-55,0) = \underline{20.9} \\
 & C_{u_1} \arrow[ur] \arrow[r] & C_{u_1d_2} = \max(52.8-55,0) = \underline{0} \\
C \arrow[ur] \arrow[dr] & & C_{d_1u_2} = \max(62.1-55,0) = \underline{7.1} \\
 & C_{d_1} \arrow[ur] \arrow[r] & C_{d_1d_2} = \max(43.2-55,0) = \underline{0}
\end{tikzcd}
$$

$$C_{u_1} = \frac{q_2 C_{u_1u_2} + (1-q_2) C_{u_1d_2}}{(1+r)} = \frac{0.74(20.9)}{1.06} = \underline{\underline{14.59}}$$

$$C_{d_1} = \frac{q_2 C_{d_1u_2} + (1-q_2) C_{d_1d_2}}{(1+r)} = \frac{0.74(7.1)}{1.06} = \underline{\underline{4.96}}$$

$$C_E = \frac{q_1 C_{u_1} + (1-q_1) C_{d_1}}{(1+r)} = \frac{0.8(14.59) + 0.2(4.96)}{1.06} = \boxed{11.95}$$

11) Price of European call with 0.1 proportional dividend paid in t=1.

$S=60 \quad X=55 \quad t=2 \text{ periods} \quad r_f = 6\% \quad u=1.15 \quad d=0.8$

$$q = \frac{(1.06-0.8)}{(1.15-0.8)} = 0.74$$

t_0 | t_1 | t_2

$S=60 \rightarrow (1-\delta)us = 0.9us = \underline{62.1}$
$(us - \delta us)$

$S=60 \rightarrow (1-\delta)ds = 0.9ds = \underline{43.2}$
$(ds - \delta ds)$

$62.1 \rightarrow (1-\delta)u^2s = \underline{71.42}$
$(62.1 \times u)$

$62.1,\ 43.2 \rightarrow (1-\delta)uds = \underline{49.68}$

$43.2 \rightarrow (1-\delta)d^2s = \underline{34.56}$
$(43.2 \times d)$

$C \rightarrow C_u,\ C_d$

$C_u \rightarrow C_{u^2s} = \max(71.42-55, 0) = \underline{16.42}$

$C_u,\ C_d \rightarrow C_{uds} = \max(49.68-55, 0) = \underline{0}$

$C_d \rightarrow C_{d^2s} = \max(34.56-55, 0) = \underline{0}$

$$C_u = \frac{0.74(16.42)+0.26(0)}{1.06} = 11.46$$

$$C_d = \frac{q(0)+(1-q)(0)}{1.06} = 0$$

$C \rightarrow 12.15$
$C \rightarrow 0$

$$c = \frac{0.74(11.46)}{1.06} = \boxed{8}$$

12) European put with constant dividend of 0.5 paid in $t=1$.

$S=70 \quad X=72 \quad t=2$ periods $\quad r_f = 5\% \quad u=1.2 \quad d=0.9$

$$q = \frac{[(1+r)-d]}{u-d} = \frac{(1.05-0.9)}{(1.2)-0.9} = 0.5$$

$t_0 \qquad t_1 \qquad t_2$

$S=70$

$\rightarrow uS-D = \underline{83.5}$

$\qquad \rightarrow C_{u^2s} = (uS-D)u = \underline{100.2}$

$\qquad \rightarrow C_{uds} = (uS-D)d = \underline{75.15}$

$\rightarrow dS-D = \underline{62.5}$

$\qquad \rightarrow C_{dus} = (dS-D)u = \underline{75}$

$\qquad \rightarrow C_{d^2s} = (dS-D)d = \underline{56.25}$

$P \rightarrow P_u$

$\qquad \rightarrow P_{u^2s} = \max(72-100.2, 0) = \underline{0}$

$\qquad \rightarrow P_{uds} = \max(72-75.15, 0) = \underline{0}$

$P \rightarrow P_d$

$\qquad \rightarrow P_{dus} = \max(72-75, 0) = \underline{0}$

$\qquad \rightarrow P_{d^2s} = \max(72-56.25, 0) = \underline{15.75}$

$P_u \rightarrow 0, \; \rightarrow 0 \qquad P_u = \underline{0}$

$P_d \rightarrow 0, \; \rightarrow 15.75 \qquad P_d = \dfrac{q(0)+(1-q)15.75}{1.05} = \dfrac{0.5(15.75)}{1.05} = \underline{7.5}$

$P \rightarrow 0, \; \rightarrow 7.5 \qquad P = \dfrac{q(0)+(1-q)7.5}{1.05} = \dfrac{0.5(7.5)}{1.05} =$