Practice Exam for Math Camp Final

Part I. Calculus (55 points)

1. (6 points) Given the two functions

$$y_1 = f_1(x_1, x_2) = (x_1^2 - 3x_2)(x_1 - 2)$$
$$y_2 = f_2(x_1, x_2) = 3x_1 ln x_2 + e^{x_1 x_2}$$

(a) (4 points) Compute the gradient of f_1, f_2, respectively.

$$f_1(x_1, x_2) = x_1^3 - 3x_1 x_2 + 6x_2 - 2x_1^2$$

$$\nabla f_1 = (f_{1x_1}, f_{1x_2}) = (3x_1^2 - 3x_2 - 4x_1, -3x_1 + 6)$$

$$\nabla f_2 = (f_{2x_1}, f_{2x_2}) = \left(3\ln x_2 + x_2 e^{x_1 x_2}, \frac{3x_1}{x_2} + x_1 e^{x_1 x_2}\right)$$

(b) (2 points) Form the Jacobian matrix and find the determinant of it. Are the two functions dependent?

$$|J| = \left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{x_2} \end{vmatrix} = \begin{vmatrix} \vec{\nabla} f_1 \\ \vec{\nabla} f_2 \end{vmatrix}$$

$$= \begin{vmatrix} 3x_1^2 - 3x_2 - 4x_1 & -3x_1 + 6 \\ 3\ln x_2 + x_2 e^{x_1 x_2} & \frac{3x_1}{x_2} + x_1 e^{x_1 x_2} \end{vmatrix}$$

$$= (3x_1^2 - 3x_2 - 4x_1)\left(\frac{3x_1}{x_2} + x_1 e^{x_1 x_2}\right) - (3\ln x_2 + x_2 e^{x_1 x_2})(6 - 3x_1)$$

$\neq 0 \Rightarrow f_1, f_2$ are not fn dependent.

Potential Substitute Questions.

$\boxed{\text{limit}}$

$\boxed{\text{Convergence}}$ $\varepsilon - N$

$\boxed{\text{Continuous}}$ $\varepsilon - \delta$

- $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

  $= e$

- $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

$\Downarrow$

L'Hospital Rule

$\boxed{\frac{0}{0}}$ $\boxed{\frac{\pm\infty}{\pm\infty}}$

$$\lim_{\substack{x\to x_0 \\ (\pm\infty)}}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

2. (4 points) Determine the total derivative $\frac{dz}{dt}$ for the following function

$$z = x^2 - 8xy - y^3$$

where $x = 2t, y = 1 - 2t$.

Total derivative $$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$ $z(x, y)$

$$\downarrow \qquad \downarrow$$

$$z_x \qquad z_y$$

Chain Rule $\Rightarrow$ $$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$= (2x-8y)\cdot 2 + (-8x-3y^2)(-2)$$

$$= 4x - 16y + 16x + 6y^2$$

$$= 20x - 16y + 6y^2$$

3. (6 points) Solve the following constrained optimization

Find extrema for.

$$\min.\ \cancel{\max} U(c_1, c_2) = (5c_1 - 2)^2 c_2^4$$
$$s.t. c_1 + c_2 = 30\ (unit - 1{,}000\ USD)$$

Where $U(c_1, c_2)$ is the utility function of consumptions at time $t = 1$ and $t = 2$.

Step 1. $\mathcal{L}(c_1, c_2; \lambda) = 5(c_1-2)^2 c_2^4 + \lambda (c_1 + c_2 - 30)$

Step 2. F.O.C.

$$\frac{\partial \mathcal{L}}{\partial c_1} = 5c_2^4\, 2(c_1-2) + \lambda \overset{set}{=} 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = 5(c_1-2)^2 \cdot 3c_2^3 + \lambda \overset{set}{=} 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = c_1 + c_2 - 30 \overset{set}{=} 0 \qquad (3)$$

Step 3.

$$(1)(2) \Rightarrow \cancel{10}^{\,2}\, \cancel{c_2^4}^{\,c_2} \cancel{(c_1-2)} = -\lambda = \cancel{15}^{\,3}\, (c_1-2)^{\cancel{2}}\, \cancel{c_2^3}.$$

$$\Rightarrow \quad 2c_2 = 3(c_1-2)$$

$$\Rightarrow \quad c_2 = \frac{3(c_1-2)}{2} \qquad (4)$$

Bring (4) into (3) $\quad c_1 + \frac{3(c_1-2)}{2} - 30 = 0$

$$c_1 + \frac{3c_1}{2} - 3 = 30 \Rightarrow \frac{5c_1}{2} = 33$$

$$\Rightarrow c_1^* = \frac{66}{5} \qquad (5)$$

Bring (5) into (4) $\quad c_2^* = \frac{3}{2} \cdot \left(\frac{66}{5} - 2\right) = \frac{3}{\cancel{2}} \cdot \frac{\cancel{56}^{\,28}}{5} = \frac{84}{5} \qquad (6)$

$$\Rightarrow (c_1^*, c_2^*) = \left(\frac{66}{5}, \frac{84}{5}\right)$$

Second Derivative Test.

$f(x,y)$. find extrema

$\boxed{\text{step1.}}$ Find critical values.

$$f_x \overset{\text{set}}{=} 0 \quad . \quad f_y \overset{\text{set}}{=} 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x^* \qquad\qquad y^*$$

critical points (x^*, y^*)'s.

$\boxed{\text{step2.}}$ f_{xx}, f_{yy}, f_{xy}.

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$$

$\boxed{\text{Step 3}}$

(x_1^*, y_1^*)	$D(x_1^*, y_1^*)$ $D>0$ $f_{xx}>0$	local min
(x_2^*, y_2^*)	$D>0$ $f_{xx}<0$	local max
(x_3^*, y_3^*)	$D<0$	saddle point.
(x_4^*, y_4^*)	$D=0$	no information.

Will be given!

4. (4 points) Use Taylor's expansion to express a second order approximation around $x_0 = 1$ for the following function:

$$f(x) = xe^x$$

$$\boxed{f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2} + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \begin{cases} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} & (\xi \text{ between } x_0, x) \to \text{Lagrange Remainder} \\ o\left((x-x_0)^n\right) & \to \text{Peano Remainder} \end{cases}$$

$$f'(x) = \underline{e^x + xe^x}$$

$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$$

$x_0 = 1$

$$f(x) = e + (e+e)(x-1) + \frac{(2e+e)}{2}(x-1)^2 = e + 2e(x-1) + \frac{3}{2}e(x-1)^2$$

5. (15 points) Compute the integral in each case

(a) (4 points) $\int_1^6 \frac{dx}{x-2}$; (b) (5 points) $\int \ 4xe^{x^2+3}dx$; (c) (6 points) $\int_3^{x^2} \frac{dt}{t}$

(a).
$$\ln|x-2| \Big|_1^6$$
$$= \ln 4 - \ln 1$$
$$= \ln 4$$

$\ln 1 = 0$

(b). Step1. $2\int e^{x^2+3}(2x\,dx)$

$$2x\,dx = dx^2$$

let $u = x^2$

$$2\int e^{u+3}\,du$$
$$= 2e^{u+3} + C$$

Step 2 $\xrightarrow{u=x^2}$ $2e^{x^2+3} + C$

(c) Fundamental Theorem of Calculus.

$$g(x) = \int_a^x f(t)\,dt$$
$$\Rightarrow g'(x) = f(x)$$

$\Rightarrow$ Theorem (Chain Rule of FTC)

$$g(x) \overset{\text{def}}{=} \int_a^{u(x)} f(t)\,dt$$
$$g'(x) = \frac{d}{dx}\int_a^{u(x)} f(t)\,dt$$
$$\boxed{= f(u)\cdot u'(x)}$$

will be given.

$$g(x) = \int_3^{x^2} \frac{1}{t}\,dt$$

$u(x) = x^2$ $\quad f(t) = \frac{1}{t}$

$$g'(x) = f(u)\cdot u'(x)$$
$$= \frac{1}{x^2}\cdot 2x = \frac{2}{x}$$

6. (5 points) Derive the relative extrema of the following by the ~~second derivative test~~ and note whether you have found a relative minimum or maximum (and briefly note why it is a min or max):

$$y = 2x^3 - x^2 + 3$$

$$\boxed{\begin{cases} f'(x) = 0 & \\ f''(x) > 0 & \text{min.} \\ f''(x) < 0 & \text{max} \end{cases}}$$

$f'(x) = 0$

Step 1. $y' = 6x^2 - 2x = 2x(3x-1)$

$x_1 = 0$, $x_2 = \frac{1}{3}$

critical points.

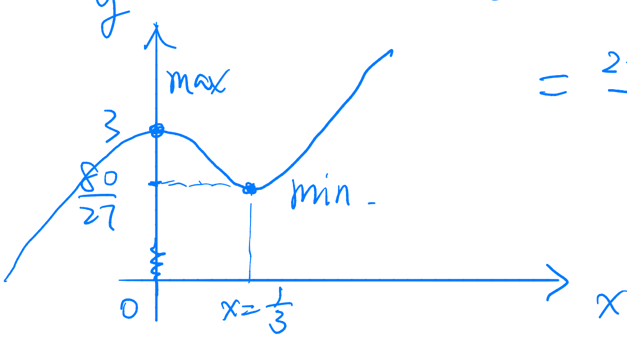
Step 2. $\overset{+}{2x}\,(\overset{-1}{3x-1})$

	$(-\infty, 0)$	0	$(0, \frac{1}{3})$	$\frac{1}{3}$	$(\frac{1}{3}, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
	$\uparrow$	max	$\downarrow$	min	$\uparrow$
$f(\cdot)$		3		$\frac{80}{27}$	

$3 = f(3)$

$f(\frac{1}{3}) = 2 \cdot \frac{1}{27} - \frac{1}{9} + 3$

$= \frac{2-3+81}{27} = \frac{80}{27}$



7. (5 points) Find the implicit differentiation $\frac{dy}{dx}$

$$F(x,y) = 5x^3 + x^2y + 5y^2 \boxed{= 20}.$$

Take derivative on both sides $y = f(x)$

$$15x^2 + 2x \cdot y + \underbrace{x^2 \cdot y' + 10y \cdot y'}_{} = 0$$

$$(x^2+10y)\,y' = -15x^2 - 2xy.$$

$$\frac{dy}{dx} = y' = -\frac{15x^2+2xy}{x^2+10y}$$

8. (5 points) Determine if $\sum_{n=1}^{\infty} \frac{5^{n+1}}{n^2}$ is a convergent series.

Ratio. $\sum a_n$

(i) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges.

(ii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow$ ... diverges

(iii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow$ no information

will be given.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{5^{n+2}}{(n+1)^2}}{\frac{5^{n+1}}{n^2}} = \lim_{n\to\infty} \frac{5^{n+2}}{5^{n+1}} \cdot \left(\frac{n}{n+1}\right)^2$$

$$= 5 \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^2$$

$$= 5 \cdot \lim_{n\to\infty} \left(1 - \frac{2}{n+1} + \frac{1}{(n+1)^2}\right)$$

$$= 5 \cdot 1 - 5 \cdot \lim_{n\to\infty} \underset{\substack{\downarrow \\ 0}}{\frac{2}{n+1}} + 5 \cdot \lim_{n\to\infty} \underset{\substack{\downarrow \\ 0}}{\frac{1}{(n+1)^2}}.$$

$$= 5 > 1$$

$\sum \frac{5^{n+1}}{n^2}$ diverges.

9. (5 points) What is the degree of homogeneity of the following function:

$$f(x_1, x_2) = \frac{1}{2} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

$HMG(k): \quad f(t\vec{x}) = t^k f(\vec{x})$

$$f(tx_1, tx_2) = \frac{1}{2}(tx_1)^{\frac{1}{3}}(tx_2)^{\frac{2}{3}}$$

$$= \frac{1}{2} t^{\frac{1}{3}+\frac{2}{3}} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}.$$

$$= \frac{1}{2} t \, x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}.$$

$$= t^{1} f(x_1, x_2)$$

$f(x_1, x_2)$ is $HMG(1)$

Part II. Real Analysis (10 points)

10. (15 points) Examine the following claims.

(a) (5 points) Show that $(\mathbb{R}^n, d_\infty)$, where $d_1(x, y) = \cancel{\max(x, y)}$ $|x-y|$ is a metric space.

(X, d) is called a metric space

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

(i). $|x-y| \geq 0 \quad \forall x, y \in \mathbb{R}$.

(ii) $|x-y| = |y-x| \quad \forall x, y \in \mathbb{R}$

(iii) $\underset{\parallel}{|x-z|} \leq |x-y| + |y-z| \quad \forall x, y, z \in \mathbb{R}$

$LHS = \underline{|x-y+y-z|} \leq \underline{|x-y| + |y-z|} = RHS.$

(b) (5 points) True or False.

(i) $\emptyset$ is both open and closed. ✓

(ii) The union of ~~finite~~ any collection of open subsets of $\mathbb{R}^n$ is open ✓

(iii) The intersection of ~~any~~ finite collection of open subsets of $\mathbb{R}^n$ is open. ✗

(iv) The closure and interior of $\mathbb{R}^n$ is $\mathbb{R}^n$. ✓

$\{(-\frac{1}{n}, \frac{1}{n})\}_{n=2}^{\infty} = I_n$. $\bigcap_{n=2}^{\infty} I_n = \{0\}$.

(v) A Cauchy sequence is convergent. ✗

Convergent $\Rightarrow$ Cauchy sequence

Part III. Linear Algebra (55 points)

11. (23 points) Use matrices A through D to answer the following questions:

$$A = \begin{pmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 3 \\ 3 & 2 \\ 1 & 5 \end{pmatrix}$$

(a) (2 points) Compute BC^T.

$$\underbrace{\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}}_{2\times2} \underbrace{\begin{pmatrix} -2 & 3 & 1 \\ 3 & 2 & 5 \end{pmatrix}}_{2\times3} = \begin{pmatrix} -2+9 & 3+6 & 1+15 \\ -8+6 & 12+4 & 4+10 \end{pmatrix}_{2\times3}$$

$$= \begin{pmatrix} 7 & 9 & 16 \\ -2 & 16 & 14 \end{pmatrix}$$

(b) (3 points) Compute $det\ A$ using Laplace Expansion Theorem.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad A = 3 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 3 \cdot (4-2) = 6$$

$[A|I] \rightarrow [I \mid A^{-1}]$

(c) (5 points) Compute A^{-1}. What is $trace(A), rank(A)$?

$$\left[\begin{array}{ccc|ccc} 5 & 2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xRightarrow{R_1/5} \left[\begin{array}{ccc|ccc} 1 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{7}{5} & \frac{12}{5} & \frac{1}{5} & 1 & 0 \\ 0 & \frac{6}{5} & \frac{6}{5} & \frac{3}{5} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-R_2\cdot\frac{6}{7}} \left[\begin{array}{ccc|ccc} 1 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{7}{5} & \frac{12}{5} & \frac{1}{5} & 1 & 0 \\ 0 & 0 & -\frac{6}{7} & \frac{3}{7} & -\frac{6}{7} & 1 \end{array}\right] \xrightarrow[\frac{5}{7}R_2]{-\frac{7}{6}R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{2}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{12}{7} & \frac{1}{7} & \frac{5}{7} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{7}{6} \end{array}\right] \xrightarrow[R_1-\frac{2}{5}R_3]{R_2-R_3\cdot\frac{12}{7}} \left[\begin{array}{ccc|ccc} 1 & \frac{2}{5} & 0 & 0 & -\frac{2}{5} & \frac{7}{15} \\ 0 & 1 & 0 & -\frac{5}{7} & -1 & 2 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{7}{6} \end{array}\right]$$

(d) (4 points) Give $adj\ B$.

$adj \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$$\boxed{adj\ B = [C_{ji}] = [C_{ij}]^T} = \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

(e) (4 points) Give two equivalent statements to the claim that an $n \times n$ square matrix is invertible.

(FTMI)

$\exists A^{-1}$ $\iff$ (i) $A\vec{x}=\vec{b}$ has unique solution

$\iff$ (ii) $A\vec{x}=\vec{0}$ has only trivial solution

$\iff$ (iii) A's row/column vectors are IND

$\iff$ (iv) $rank(A)=n$ $\iff$ (v) $nullity(A)=0$

(f) (5 points) Use Cramer's rule to solve the system again. Compare with part (e) to verify your answer.

$$5x_1 - 3x_2 + 4x_3 = 4$$
$$x_1 + 2x_2 = 7$$
$$-x_2 + 3x_3 = 3$$

Step 1. $[A|b] = \left[\begin{array}{ccc|c} 5 & -3 & 4 & 4 \\ 1 & 2 & 0 & 7 \\ 0 & -1 & 3 & 3 \end{array}\right]$

$|A| = 1\begin{vmatrix} 5 & 4 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 5 & -3 \\ 1 & 2 \end{vmatrix} = -4 + 3\cdot(10+3) = -4+39 = 35$

Step 2. $A_i(\vec{b})$

$A_1(\vec{b}) = \begin{bmatrix} 4 & -3 & 4 \\ 7 & 2 & 0 \\ 3 & -1 & 3 \end{bmatrix}$ $\quad |A_1(\vec{b})| = -7\cdot\begin{vmatrix} -3 & 4 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 4 & 4 \\ 3 & 3 \end{vmatrix} = -7\cdot(-9+4) = 35$

$A_2(\vec{b}) = \begin{bmatrix} 5 & 4 & 4 \\ 1 & 7 & 0 \\ 0 & 3 & 3 \end{bmatrix}$ $\quad |A_2(\vec{b})| = -1\begin{vmatrix} 4 & 4 \\ 3 & 3 \end{vmatrix} + 7\begin{vmatrix} 5 & 4 \\ 0 & 3 \end{vmatrix} = 7\cdot 15 = 105$

$A_3(\vec{b}) = \begin{bmatrix} 5 & -3 & 4 \\ 1 & 2 & 7 \\ 0 & -1 & 3 \end{bmatrix}$ $\quad |A_3(\vec{b})| = 1\cdot\begin{vmatrix} 5 & 4 \\ 1 & 7 \end{vmatrix} + 3\begin{vmatrix} 5 & -3 \\ 1 & 2 \end{vmatrix} = 31 + 3(10+3) = 70$

Step 3. $x_i = \frac{|A_i(\vec{b})|}{|A|}$ $\quad x_1 = \frac{35}{35} = 1, \; x_2 = \frac{105}{35} = 3, \; x_3 = \frac{70}{35} = 2.$

12. (32 points) Suppose matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$, compute

(a) (3 points) Find $null(A)$.

Solve $\left[\begin{array}{cc|c} 3 & 4 & 0 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad null(A) = \{\vec{0}\}$.

(b) (3 points) Determine if A is positive definite.

2x2. (i) A symmetric

(ii) $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\begin{cases} ac - b^2 > 0 \\ a > 0 \end{cases}$

$\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ is not sym.

$\Rightarrow$ not PD.

(c) (3 points) If a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ can be defined as $T(\vec{x}) = A\vec{x}, \forall \vec{x} \in \mathbb{R}^2$. Find the standard matrix of its inverse T^{-1}.

* If $T_A \vec{x} \overset{def}{=} A\vec{x} \Rightarrow A^{-1}$ is the standard matrix for T^{-1}

$T_A^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{9-4}\begin{bmatrix} 3 & -4 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ -1 & 3 \end{bmatrix}$

(d) (3 points) Find $ker(A)$

Ker(T) A is standard matrix for T, ker(A) $\Leftrightarrow$ ker(T)

$(T\vec{v} = A\vec{v})$ $\quad ker(A) = \boxed{\{\vec{v} \in \mathbb{R}^2 : A \cdot \vec{v} = \vec{0}\}}$ $\quad \boxed{\text{Def}}$ $ker(T) = \{\vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0}\}$

Def of null space.

$= \{\vec{v} : \text{solution of } [A|0]\} = \boxed{null(A)} = \{\vec{0}\}$.

(e) (4 points) Find the eigenvalues and eigenvectors of A.

$\vec{v}$ is the solution to $A\vec{v} = 0$

Step 1. $|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 4 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)^2 - 4 = 0 \Rightarrow 9 - 6\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow (\lambda+1)(\lambda+5) = 0$

Step 2. $\lambda_1 = -1$

$[A - \lambda I | 0]$

$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -12 & 0 \end{array}\right]$

$\vec{e}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_2 = -5$

$\left[\begin{array}{cc|c} 8 & 4 & 0 \\ 1 & 8 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 16 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -15 & 0 \end{array}\right]$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(f) (3 points) Diagonalize A.

Recap $A = PDP^{-1}$

$$= \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \boxed{\begin{bmatrix} e_1 & e_2 \end{bmatrix}^{-1}}$$

(g) (3 points) Find A^8

$$A^8 = PD^8P^{-1}$$

$$D^8 = \begin{bmatrix} -1 & \\ & -5 \end{bmatrix}^8 = \begin{bmatrix} 1 & \\ & 5^8 \end{bmatrix}$$

(h) (4 points) By Grant-Schmidt process, find the orthonormal set of column vectors of A.

Recap $\vec{V_1} = \vec{x_1}$ $\quad \boxed{\vec{V_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$, $\vec{x_2} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\vec{V_2} = \vec{x_2} - \left(\frac{\vec{V_1}\cdot\vec{x}}{\vec{V_1}\cdot\vec{V_1}}\right)\cdot\vec{V_1}$$

$$= \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \frac{12+3}{9+1}\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\frac{9}{2} \\ 3-\frac{3}{2} \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{2} \\ +\frac{3}{2} \end{bmatrix}}$$

(i) (3points) QR factorize A.

Step 1 From (h) normalize $\vec{V_1}, \vec{V_2} \to \vec{q_1}, \vec{q_2}$

$$\vec{q_1} = \frac{1}{\sqrt{3^2+1^2}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \qquad \vec{q_2} = \frac{1}{\sqrt{\frac{1}{4}+\frac{9}{4}}}\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = \frac{2}{\sqrt{10}}\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

Step 2. $Q = \begin{bmatrix} \vec{q_1} & \vec{q_2} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

Step 3. $A = QR \Rightarrow Q^TA = \underbrace{Q^TQ}_{I}R \Rightarrow R = Q^TA = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}\cdot\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} = \square$

(j) (3 points) Compute A^+.

$$A^+ = (A^TA)^{-1}A^T = \left(\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix}}^{-1}\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} = \frac{1}{250-225}\begin{bmatrix} 25 & -15 \\ -15 & 10 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3-\frac{12}{5} & 1-\frac{9}{5} \\ -\frac{9}{5}+\frac{8}{5} & -\frac{3}{5}+\frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

Bonus (15 points)

For problem 12.

1. (3 points) LU factorize A.

Step 1. $\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \xrightarrow{\boxed{R_2 - \frac{1}{3}R_1}} \begin{bmatrix} 3 & 4 \\ 0 & 3-\frac{4}{3} \end{bmatrix} \to \begin{bmatrix} 3 & 4 \\ 0 & \frac{5}{3} \end{bmatrix} = U$

Step 2. $\begin{bmatrix} 1 & 0 \\ \boxed{+\frac{1}{3}} & 1 \end{bmatrix} \equiv L \qquad LU = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$

2. (5 points) Cholesky decompose A.

As A is not PD, A does not have LDL^T decomposition and so no Cholesky decomposition !!!

Step 1. $A = LU \Rightarrow U = \begin{bmatrix} 3 & 0 \\ 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & \boxed{?} \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 3 & 4 \\ 0 & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

$2\times2 \qquad 2\times2$

$4 = 3x \Rightarrow x = \frac{4}{3}$

$U = \begin{bmatrix} 3 & 0 \\ 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = D \cdot L^T$

3. (7 points) Find the singular value decomposition of A.

$A = U \Sigma V^T$

Step 1. $A^T A = \begin{bmatrix} 10 & 15 \\ 15 & 25 \end{bmatrix} \qquad \begin{vmatrix} 10-\lambda & 15 \\ 15 & 25-\lambda \end{vmatrix} = 0$

$(10-\lambda)(25-\lambda) - 225 = 0$

$250 + \lambda^2 - 35\lambda - 225 = 0$

$\lambda^2 - 35\lambda + 25 = 0 \qquad 1125 \quad 3$

$\lambda_{1,2} = \frac{35 \pm \sqrt{1225-100}}{2} = \frac{35 \pm 9\sqrt{5}}{2}$

$\lambda_1 = \frac{35+9\sqrt{5}}{2} \qquad \lambda_2 = \frac{35-9\sqrt{5}}{2}$

Step 2. $\vec{e}_1 \downarrow \vec{q}_1$ $\boxed{\text{G-S normalization}}$ $\vec{e}_2 \downarrow \vec{q}_2 \to V = [\vec{q}_1 \; \vec{q}_2]$

Step 3. $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & \\ & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$

Step 4. $\vec{u}_1 = \frac{1}{\sigma_1} A \vec{q}_1 \quad \vec{u}_2 = \frac{1}{\sigma_2} A \vec{q}_2 \quad U = [\vec{u}_1 \; \vec{u}_2]$