Homework #5 Stochastic Calculus Solutions
Black-Scholes and Feyman-Kac

Note: The calculations in the solutions have not been thoroughly checked and may contain errors. Use the solutions as an extended hint on solving the problems.

Requirement: For the following problems, you need to provide the dynamic replication/hedging argument to derive the pricing partial equation, invoke the Feyman-Kac to obtain the solution, and then evaluate the expection to obtain the final answer.

Problem 1 [Pricing the log **Contract].** Suppose we have the assumptions of the Black-Scholes Model. Find the pricing formula for the European Style derivative whose payoff is given by function $\log(S_T)$, where S_T is the stock price in the BS model on date T.

Answer: In the BS model assumes the the risk-free bond which follows

$$dB_t = rB_t dt,$$

where r is the *continuously compounded* interest rate. And a stock whose price follows Geometric BM [Note that the drift is μ]:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Note that our log contract's payoff is $\log(S_T)$. Our goal is to derive the pricing formula for this contract. We achieve this by assuming the formula is given by a function $C(x,t)$ (which gives the price of the log contract to be $C(S_t,t)$ when the stock price is S_t at time t), and then find what conditions $C(x,t)$ has to satisfy.

With this assumed pricing formula $C(S_t,t)$, we know the dynamics of the price of this log contract is given by Ito's formula as follows

$$\begin{aligned}
dC(S_t,t) &= C_t(S_t,t)\,dt + C_x(S_t,t)\,dS_t + \frac{1}{2}C_{xx}(S_t,t)\,\sigma^2 S_t^2 dt \\
&= C_t(S_t,t)\,dt + C_x(S_t,t)\,[\mu S_t dt + \sigma S_t dW_t] + \frac{1}{2}C_{xx}(S_t,t)\,\sigma^2 S_t^2 dt \\
&= \left[C_t(S_t,t) + C_x(S_t,t)\,\mu S_t + \frac{1}{2}C_{xx}(S_t,t)\,\sigma^2 S_t^2\right] dt + C_x(S_t,t)\,\sigma S_t dW_t.
\end{aligned}$$

Now we creat a riskless portfolio by going long one of this log contract and shorting $C_x(S_t,t)$ share of stock. The dynamics of this portfolio is given by

$$\begin{aligned}
dC(S_t,t) - C_x(S_t,t)\,dS_t &= \left[C_t(S_t,t) + C_x(S_t,t)\,\mu S_t + \frac{1}{2}C_{xx}(S_t,t)\,\sigma^2 S_t^2\right] dt + C_x(S_t,t)\,\sigma S_t dW_t \\
&\quad - C_x(S_t,t)\,\mu S_t dt - C_x(S_t,t)\,\sigma S_t dW_t \\
&= \left[C_t(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\,\sigma^2 S_t^2\right] dt
\end{aligned}$$

Note that the dW_t term vanished, and hence this portfolio is instantanesously riskless. Hence it should earn risk-free rate if we assume no-arbitrage or the Law of One Price:

$$\frac{\left[C_t(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\,\sigma^2 S_t^2\right]}{C(S_t,t) - C_x(S_t,t)\,S_t} = r$$

which yields the Black-Scholes PDE for derivatives pricing: [Note that this equation is the same for any derivatives.]

$$C_t\left(S_t,t\right)+rS_tC_x\left(S_t,t\right)+\frac{1}{2}\sigma^2S_t^2C_{xx}\left(S_t,t\right)=rC\left(S_t,t\right). \tag{1}$$

Of course, on the maturity date T, $C(S_t,t)$ is equal to the payoff function which is a boundary condition

$$C(S_T,t)=\log(S_T). \tag{2}$$

The above equations (1) and (2) are written in terms of S_t. When S_t runs through all sample paths, the (1) and (2) is equivalent to

$$C_t\left(x,t\right)+rxC_x\left(x,t\right)+\frac{1}{2}\sigma^2x^2C_{xx}\left(x,t\right)=rC\left(x,t\right). \tag{3}$$

with the boundary condition

$$C(x,t)=\log(x). \tag{4}$$

The PDE in (3) and (4) can be solved in Feyman-Kac Theorem. The Feyman-Kac Theorem says that the solutions to (3) and (4) is given by the following expectation

$$C(x,t)=E^{x,t}\left[e^{-r(T-t)}\log(X_t)\right] \tag{5}$$

where the process X_t solves the following SDE [How are the drift and diffusion coefficients obtained? Note that the drift term is r now from (3) and not μ!]

$$dX_t=rX_tdt+\sigma X_tdW_t.$$

But such X_t is a geometric BM what can be easily solved:

$$\log(X_T)-\log X_t=(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\left[W_T-W_t\right]$$

This means that $\log(X_T)$ has normal distribution with mean $\log x+(r-\frac{1}{2}\sigma^2)(T-t)$, and variance of $\sigma^2\left(T-t\right)$ conditional on $X_t=x$:

$$\log(X_T)\sim\log x+(r-\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}Z\,.$$

[here Z is the standard normal random variable] It follows that

$$\begin{aligned}
C(x,t) &= E^{x,t}\left[e^{-r(T-t)}\log(X_t)\right] \\
&= e^{-r(T-t)}E^{x,t}\left[\log(X_T)\right] \\
&= e^{-r(T-t)}\int\left[\log x+(r-\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}z\right]\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= e^{-r(T-t)}\left[\log x+(r-\frac{1}{2}\sigma^2)(T-t)\right].
\end{aligned}$$

So the log contract's price at time t is given by

$$C(S_t,t)=e^{-r(T-t)}\left[\log S_t+(r-\frac{1}{2}\sigma^2)(T-t)\right],$$

where S_t is the underlying stock price at time t.

Problem 2 [Bonus][Pricing the Variance Contract]. Suppose we have the assumptions of the Black-Scholes Model. Find the pricing formula for the European Style derivative whose payoff is given by function $[\max(S_T - K, 0]^2$, where S_T is the stock price in the BS model on date T.

Answer: Repeating the argument in Problem 1, we are lead to that the price function $C(x,t)$ for this variance contract satisfies

$$C_t(x,t) + rxC_x(x,t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x,t) = rC(x,t).$$

with the boundary condition

$$C(x,t) = [\max(x-K,0)]^2.$$

The above PDE can be solved by Feyman-Kac Theorem. The Feyman-Kac Theorem says that the solution is given by the following expectation

$$C(x,t) = E^{x,t}\left[e^{-r(T-t)}[\max(X_T-K,0)]^2\right]$$

where the process X_t solves the following SDE [How are the drift and diffusion coefficients obtained? Note that the drift term is r and not μ!]

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

with the initial condition $X_t = x$. But such an X_t is a geometric BM what can be easily solved (c.f. lecture notes and prior HW):

$$\log(X_T) - \log X_t = (r - \frac{1}{2}\sigma^2)(T-t) + \sigma[W_T - W_t]$$

This means that $\log(X_T)$ has normal distribution with mean $\log x + (r - \frac{1}{2}\sigma^2)(T-t)$, and variance of $\sigma^2(T-t)$ conditional on $X_t = x$ [here Z is the standard normal random variable]:

$$\log(X_T) \sim \log x + (r - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}Z$$

It follows that

$$\begin{aligned}
C(x,t) &= E^{x,t}\left[e^{-r(T-t)}[\max(X_T-K,0)]^2\right] \\
&= e^{-r(T-t)}\int [\max(X_T-K,0)]^2 \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= e^{-r(T-t)}E^{x,t}\left[\max\left(e^{\log x+(r-\frac{1}{2}\sigma^2)(T-t)-\sigma\sqrt{T-t}Z} - K, 0\right)\right]^2 \\
&= e^{-r(T-t)}\int_{z<d^-}\left(e^{\left[\log x+(r-\frac{1}{2}\sigma^2)(T-t)\right]-\sigma\sqrt{T-t}z} - K\right)^2 \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= e^{-r(T-t)}\int_{z<d^-} e^{2\left[\log x+(r-\frac{1}{2}\sigma^2)(T-t)\right]-2\sigma\sqrt{T-t}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&\quad -2Ke^{-r(T-t)}\int_{z<d^-} e^{\left[\log x+(r-\frac{1}{2}\sigma^2)(T-t)\right]-\sigma\sqrt{T-t}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&\quad +K^2e^{-r(T-t)}\int_{z<d^-}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&\equiv I + II + III.
\end{aligned}$$

The three terms can be evaluated easily using properties of normal distribution.

$$
\begin{aligned}
I &= e^{-r(T-t)} \int_{z<d^-} e^{2\left[\log x + (r - \frac{1}{2}\sigma^2)(T-t)\right] - 2\sigma\sqrt{T-t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= e^{-r(T-t)} \int_{z<d^-} e^{2\left[\log x + (r - \frac{1}{2}\sigma^2)(T-t)\right] + 2\sigma^2(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+2\sigma\sqrt{T-t})^2}{2}} dz \\
&= e^{-r(T-t)} \cdot e^{2\left[\log x + (r - \frac{1}{2}\sigma^2)(T-t)\right] + 2\sigma^2(T-t)} \int_{z<d^-} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+2\sigma\sqrt{T-t})^2}{2}} dz \\
&= x^2 e^{\sigma^2(T-t)} \int_{y<d^- + 2\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= x^2 e^{\sigma^2(T-t)} N(d^- + 2\sigma\sqrt{T-t}),
\end{aligned}
$$

where d^- is given by

$$
d^- = \frac{\log(x/K) + (r - \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{(T-t)}}.
$$

$$
\begin{aligned}
II &= -2Ke^{-r(T-t)} \int_{z<d^-} e^{\left[\log x + (r - \frac{1}{2}\sigma^2)(T-t)\right] - \sigma\sqrt{T-t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= -2Ke^{-r(T-t)} \int_{z<d^-} e^{\left[\log x + (r - \frac{1}{2}\sigma^2)(T-t)\right] + \frac{1}{2}\sigma^2(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+\sigma\sqrt{T-t})^2}{2}} dz \\
&= -2Ke^{-r(T-t)} \cdot e^{\left[\log x + (r - \frac{1}{2}\sigma^2)(T-t)\right] + \frac{1}{2}\sigma^2(T-t)} \int_{z<d^-} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+\sigma\sqrt{T-t})^2}{2}} dz \\
&= -2Kx \int_{y<d^- + \sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= -2KxN(d^- + \sigma\sqrt{T-t}).
\end{aligned}
$$

$$
\begin{aligned}
III &= K^2 e^{-r(T-t)} \int_{z<d^-} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= K^2 e^{-r(T-t)} N(d^-).
\end{aligned}
$$

Hence the pricing function is given by

$$
C(x,t) = x^2 e^{\sigma^2(T-t)} N(d^- + 2\sigma\sqrt{T-t}) - 2KxN(d^- + \sigma\sqrt{T-t}) + K^2 e^{-r(T-t)} N(d^-),
$$

where d^- is defined in the Black-Scholes option pricing formula

$$
d^- = \frac{\log(x/K) + (r - \frac{1}{2}\sigma^2(T-t))}{\sigma\sqrt{(T-t)}}.
$$

The derivative's price at time t when the underlying stock price is S_t is just $C(S_t, t)$.