Session 4

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Metric spaces, Normal Spaces, Topological Spaces

§1. Basic Concepts

1.1. Def. (metric space).

Let X be a nonempty subset of $\mathbb{R}$.

A function $\boxed{d}: X \times X \to \mathbb{R}$ is called a metric on X, if $x, y, z \in X$

(i) $d(x, y) \geq 0$ $d(x,y) = 0$ if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

(X, d) is called a metric space.

1.2. example

Let $X = \mathbb{R}$. Consider $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$d(x, y) = |x - y|$$

Then d is a metric on $\mathbb{R}$. $(\mathbb{R}, d)$ is a metric space.

1.3. example

$X = \mathbb{R}^2$ $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

d is a metric on $\mathbb{R}^2$

$\downarrow$

Euclidean distance on $\mathbb{R}^2$

$$d_1((x_1,y_1),(x_2,y_2)) = |x_1-x_2| + |y_1-y_2|$$

$$d_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$$

1.4. Ex. discrete metric, $x, y \in X$

$$d(x,y) = \begin{cases} 0 & x=y \\ 1 & \text{o.w} \end{cases}$$

1.3. (Open & closed balls in metric space)

Let (X, d) be metric space.

(i) The $\boxed{\text{open}}$ ball in X of center x_0 and radius r

$$B(x_0; r) = \{x \in X \mid d(x, x_0) < r\}$$

X can $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ ...

$\downarrow$ segment, $\downarrow$ circle, $\downarrow$ ball

(ii) The $\boxed{\text{closed}}$ ball ...

$$B'(x_0; r) = \{x \in X \mid d(x_0, x) \leq r\}$$

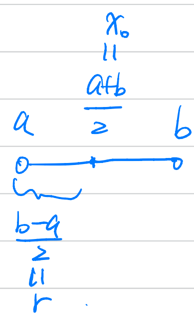
1.6 Ex

$$B(x_0, \delta) = (x_0-\delta, x_0+\delta) \qquad X = \mathbb{R}$$

$$B'(x_0, \delta) = [x_0-\delta, x_0+\delta)$$

$$(a,b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$$

$$[a,b] = B'\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$$



1.7. Ex. X with discrete metric

$$\underbrace{B(x_0;r)} = \{x \in X \mid \underbrace{d(x_0, x) < r}\}$$

$$= \begin{cases} \{x_0\} & , \text{ if } \underbrace{r \le 1} \\ X & \quad\quad r > 1 \end{cases}$$

1.8. proposition.

Let d be a metric on X.

$\Rightarrow$

(i) $P_1, P_2 \cdots P_n \in X$, then

$$d(P_1, P_n) \le d(P_1, P_2) + d(P_2, P_3) + \cdots d(P_{n-1}, P_n)$$

(ii) If $x, y, z \in X$

$$\Rightarrow |d(x,z) - d(y,z)| \le d(x,y)$$

1.9. Def (subspace) Let (X, d) metric space. Then for $Y \subset X$, we can defined a metric on Y

$$d: Y \times Y \to Y$$
$$(x, y) \mapsto d(x, y)$$

Y is called a subspace.

1.10. ex. $X = \mathbb{R}$, $Y = [0, 1]$. $Y \subset X$

$$d(y, y') = |y - y'|$$

(Y, d) is subspace of (X, d)

1.10. Cartesian product (X_1, d_1). (X_2, d_2)

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2).$$

$(X_1 \times X_2, d)$ is a metric space.

1.13. Def (equivalence) Two metrics $d_1 . d_2$ on X are called equivalence if $\exists c_1 . c_2 \in \mathbb{R}_+ \ni .$

$$c_1 d_1(x_1, x_2) \le d_2(x_1, x_2) \le c_2 d_1(x_1, x_2)$$

1.14. (Linear space)

A linear space is a combination of X, and a field F ($\mathbb{R}$ or $\mathbb{C}$) along w/. a binary operation $+ : X \times X \to X$ and scalar multiplication $\cdot : F \times X \to X \ni$

(i) $(x+y)+z = x+(y+z)$

(ii) $\exists\, 0 \in X \ni x+0 = 0+x = 0$

(iii) $\forall x \in X, \exists -x \in X \ni x+(-x) = (-x)+x = 0$

(iv) $x+y = y+x$

(v) $(\alpha\beta)x = \alpha(\beta x)$

(vi) $(\alpha+\beta)x = \alpha x + \beta x$

(vii) $1x = x$

(viii) $\alpha(x+y) = \alpha x + \alpha y$

1.15. (Norm & normed space) Let X a linear space over a Field F ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm if $\forall x, y \in X$, $\alpha \in F$

(i) $\|x\| \ge 0$, $\|x\| = 0$ iff $x = 0$ nonnegativity

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$

(iii) $\|x+y\| \le \|x\| + \|y\|$ triangular ineq.

1.16. Ex. The space $\mathbb{R}^n$ is a linear space w.r.t. operation mentioned before.

For $\underset{\sim}{x} = (x_1, x_2 \cdots x_n) \in \mathbb{R}^n$

define

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

$$\|x\|_\infty = \max\{|x_1|, \cdots |x_n|\}$$

these norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ can be verified.

1.17. Def. (equivalent norms). Two norms $\|\cdot\|_1, \|\cdot\|_2$. on a linear space X. They are equivalent if $\exists\ C_1, C_2 \in \mathbb{R}_+\ F \ni.$

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$$

for all $x \in X$.

$$\left.\begin{aligned} d_1(x,y) &= \|x-y\|_1 \\ d_2(x,y) &= \|x-y\|_2 \end{aligned}\right\} \text{strongly equivalent on } \underline{X}$$

Micro II

§ Open Sets. Closed Sets. Convergence, Completeness.

2.1. Def. A subset $A \subset X$ is open if $\forall a \in A$, $\exists B(a,\delta) \ni$ $B(a,\delta) \subset A$

2.2. ex. i) In $\mathbb{R}$ ($d = |\cdot|$, $(\mathbb{R}, |\cdot|)$ — metric space) $A = (c, d)$ is open

(ii) In $\mathbb{R}$ $(-\infty, c)$, $(c, +\infty)$ are both open. $[c, \infty)$ is not open.
$\exists\, c \in A$, $\nexists\, B(a,\delta) \ni B(c,\delta) \subset A$

2.3. Theorems. Let X be a metric space. The following hold:

(i) ϕ is open

(ii) X is open

(iii) The union of any collection of open subsets of X is open

$$G = \bigcup_{\alpha \in I} G_\alpha$$

$\boxed{\bigcup_{n=1}^{\infty} G_n}$ $|\mathbb{N}|$ is countable $\begin{cases} \text{finite} \\ \text{countably infinite } |\mathbb{N}| \end{cases}$

$|\mathbb{R}| = c / \aleph_1$

(iv) The intersection of finite number of open subsets of X is open

$$G = \bigcap_{i=1}^{n} G_i$$

2.4. proposition. open ball is an open set.

2.5. Def. (closed set) A subset S of metric space X is closed if its complement $S^c = X - S$ is open.

2.6. proposition closed ball is a closed set

2.7. Theorems. Let X be metric space.

(i). ϕ is closed

(ii) X is closed

(iii) $S = \bigcap_{\alpha \in I} S_\alpha$ S_α is closed $(\forall \alpha \in I)$

any collection $\Rightarrow S$ is closed.

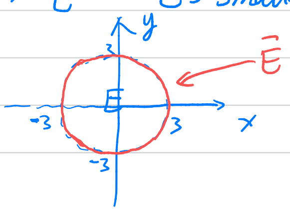
finite number

(iv) $S = \bigcup_{i=1}^{n} S_\alpha$ S_α is closed $\Rightarrow$ S is closed.

§ 2.2. Closure interior.

2.8. Def (closure). Let $E \subset X$. The closure of E, denoted $\bar{E}$ is the intersection of all closed sets that contains E.

$\Rightarrow \bar{E}$ — E's smallest closed container.



2.10. proposition. Let $A, B \subset X$.

(i) For a set $F \ni A \subset F$.
F is closed $\Rightarrow \bar{A} \subset F$

(ii) $A \subset B$, $\bar{A} \subset \bar{B}$

(iii) $\bar{A} = A$ iff. A is closed

(iv) $\bar{\bar{A}} = \bar{A}$

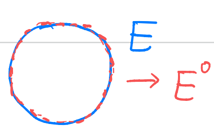
(v) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

2.11. (Interior point) Let E be a subset of a metrix space X. A point $x \in X$ is called interior point if.
$\exists\, r > 0 \ni B(x, r) \subset E$

The set of all interior points of, denoted E°, is called interior of E.

2.12. Remark: E° is the union of all open sets contained in E

$\Rightarrow$ interior — largest open subset of E
E°



2.13. proposition. Let $A, B \subset X$.

(i) $U \subset A$. U is open $\Rightarrow U \subset A^\circ$.

(ii) $A \subset B \Rightarrow A^\circ \subset B^\circ$

(iii) $A^\circ = A$ iff. A is open

(iv) $(A^\circ)^\circ = A^\circ$

(v) $(A \cap B)^\circ = A^\circ \cap B^\circ$

Closed $\leftarrow$ open   closure   interior

$E^c \leftarrow E$ $\bar{E}$ E°

E's smallest closed container   E's largest open subset.

§ 2.3. Convergence & Completeness.

2.14. (Convergence). Let (X, d) be a metric space.
A sequence $\{x_n\}$ of points in X is convergent to a point a $|x_n - a| < \varepsilon$,
if $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} \ni d(x_n, a) < \varepsilon$

$(\forall n \geq N(\varepsilon))$

Remark 2.15: $\lim_{n\to\infty} x_n = x \iff d(x_n, a) \to 0 \; (n \to \infty)$

2.16. (Uniqueness) $\{x_n\}$ in X has at most one limit.

2.17. (Subsequence)

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5\ \cdots\ x_{100},\ x_{101}\cdots x_{200},\cdots$$

$$x_{n_1},\quad x_{n_2},\quad x_{n_3},\ x_{n_4}\cdots x_{n_{10}},\cdots$$

Let $\{x_n\}$ be a sequence of metric space X. $n_1 < n_2 < n_3 < \cdots$ $(\in \mathbb{N})$
Then $\{(x_{n_k})\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}$ in X.

ex. $x_n = (-1)^n$.

$$-1,\ +1,\ -1,\ +1\ \cdots$$

1st 2nd 3rd 4th

$$\boxed{x_{2k-1}} = -1,\quad \boxed{x_{2k}} = 1$$

Subsequence of $\{x_n\}$.

Remark. Let $(n_k)_k$, $n_k \geq k$

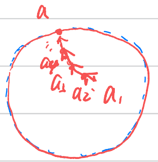
2.18. proposition. If a sequence $\{x_k\}$ converges to a. then any subsequence of $\{x_k\}$ also converges to a.

$$\lim_{k\to\infty} x_k = a \Rightarrow \lim_{k\to\infty} x_{n_k} = a.$$

2.19. (Bounded). A subset A of a metric space X is bounded if $\exists\, B(a;r)$ $\ni$ $A \subset B(a;r)$

2.20. proposition. Let X be a metric space. Let $A \subset X$. Then.

$a \in \bar{A}$ iff. $\exists \{a_n\}_{n=1}^{\infty} \subset A \ni \lim_{n\to\infty} a_n = a$



A $\quad a \in \bar{A}$

$a_{100}, a_{1000000} \in A$

$a_1, a_2, a_3 \cdots a_{10000} \to a$ $\quad \bar{A}$

※ 2.21 Theorem. Let A be a subset of X.

$\boxed{A \text{ is closed}} \Leftrightarrow \boxed{\text{whenever } \{a_n\}_{n=1}^{\infty} \text{ in } A \text{ converges to a point } a,\ a \in A}$

a closed set contains all of its subsequences' limit points.

2.22. (Cauchy sequence). Let (X, d) be a metric space. $\{x_n\} \subset X$ is called Cauchy if. $\forall \varepsilon > 0,\ \exists N(\varepsilon) \in \mathbb{N},\ \ni \forall\ d(x_m, x_n) < \varepsilon$ for any $m, n \geq N(\varepsilon)$ $\quad d(x_n, a) < \varepsilon$

Remark: convergent $\overset{\nLeftarrow}{\Rightarrow}$ Cauchy

example: $\boxed{\frac{1}{x}}$ defined in $((0,1), |\cdot|)$

$\frac{1}{x} \to 0 \notin (0,1)$

2.23. (Completeness) Def. A metric space (X, d). It is complete if every Cauchy sequence in X is convergent.

$\Rightarrow$ if $\{x_n\}_{n=1}^{\infty} \subset X$, $\exists\, x \in X$, $\ni \lim_{n\to\infty} x_n = x$

Cauchy

A subset $E \subset X$ is called complete if (E, d) is a complete metric space.

Remark: $\mathbb{R}^n$ is complete.

Recap: (X, d) $\quad X$ is open & closed

ϕ is open & closed.

§3. Continuity.

3.1. Def. Let (X, d), (Y, d') be metric spaces.

Let $f: X \to Y$.

f is continuous at $x_0 \in X$ if

$\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, $|x - x_0| < \delta(\varepsilon)$

$\ni \forall x \in X$, and $d(x, x_0) < \delta(\varepsilon)$

then

$$d'(f(x), f(x_0)) < \varepsilon.$$

$$|f(x) - f(x_0)| < \varepsilon.$$

3.2. (Lipschitz Continuous) Def. $f: X \to Y$ is called L.C. w/. Lipschitz constant $k > 0$ if

$$d'(f(u), f(v)) \le k\, d(u, v) \quad (\forall u, v \in X)$$

L.C. $\overset{\not\Leftarrow}{\Rightarrow}$ Continuous

ex. (i) $X = [a, \infty)$ $a > 0$.

$f(x) = \sqrt{x}$ is Lipschitz continuous on X.

$$|f(u) - f(v)| = |\sqrt{u} - \sqrt{v}| = \left|\frac{(\sqrt{u}-\sqrt{v})(\sqrt{u}+\sqrt{v})}{\sqrt{u}+\sqrt{v}}\right|$$

$$= \frac{|u-v|}{\sqrt{u}+\sqrt{v}} \le \frac{1}{2\sqrt{a}} |u - v|$$

$u \ge a$ $v \ge a$

Lipschitz constant (>0)

(ii) $X = (0, \infty)$. f is not L.C. but its continuous on X

Suppose $\exists$ Lipschitz constant $l > 0$.

$$|f(u) - f(v)| = |\sqrt{u} - \sqrt{v}| \le l\, |u - v|$$

$$\Leftrightarrow \frac{\sqrt{u}-\sqrt{v}}{u - v} \le l$$

$$\frac{\cancel{\sqrt{u}-\sqrt{v}}}{\cancel{(\sqrt{u}-\sqrt{v})}(\sqrt{u}+\sqrt{v})} \le l$$

$$\frac{1}{\sqrt{u}+\sqrt{v}} \le l$$

$u, v \to 0^+$

l cannot be a finite number.

§4. Connected Metric Spaces

4.1. Def (Connectedness). Let X be a metric space. X is disconnected if

$\exists$ $U, V\ (\neq \emptyset)$ $\ni$. $U \cap V = \emptyset$

$U \cup V = X$

X is connected if it is not disconnected.

4.2. proposition. Let X be a metric space. Then X is disconnected iff. X contains a nonempty, open, closed subset E.

4.3. ex. Any connected subset of $\mathbb{R}$ is an open interval

4.4. Theorem. Let $f: X \to Y$, X, Y are two metric spaces. f is a continuous function.

X is connected $\Rightarrow$ $f(X)$ is connected

($f(X)$ is subset of Y)

§.5. Compact Spaces.

5.1. Def. A subset A of a metric space X is compact if whenever A contained in union of collection of open subsets of X, then A is contained in the union of a finite number of these open subsets.

Compact iff. $A \subset \bigcup_{\alpha \in I} G_\alpha$ is open in X $(\forall \alpha \in I)$

$\exists \alpha_i \quad i = 1, 2, \cdots n. \in \mathbb{N}.$

$\ni$

$A \subset \bigcup_{\alpha_i}^{n} G_\alpha$

5.2. ex. Any finite subset of a metric space is compact.

5.3. (Total boundness) X is totally bounded if $\forall \varepsilon > 0$, X is the union of finite number of closed balls of radius ε.

5.4. Theorem. Let X be a metric space.

$\boxed{X \text{ is compact}} \Leftrightarrow \boxed{\begin{array}{l} X \text{ is complete} \\ \text{and totally bounded.} \end{array}}$

5.5. Remark Any closed bounded subset of $\mathbb{R}^n$ is compact.

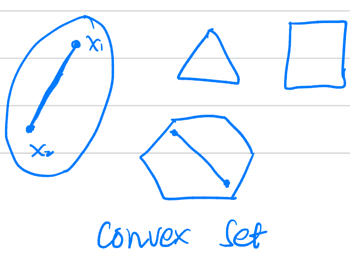
※ 5.6. Brower Fixed-point Theorem.

Let $S \subset \mathbb{R}^n$. $S \neq \emptyset$, S is compact. S is convex. Let $f: S \to S$. f is continuous.

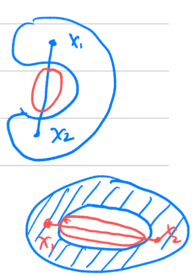
$\Rightarrow$ $\exists$ at least one fixed point f in S. $\ni$. $\exists$ at least one $x^* \in S$. where $x^* = f(x^*)$

§6. Convex Set.

6.1 $E \subset \mathbb{R}^n$ is a convex set, if $\forall x_1, x_2 \in E$ we have $\underbrace{t x_1 + (1-t) x_t}_{} \in E$ $\quad t \in (0,1)$



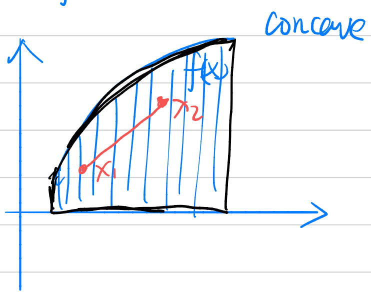
Convex Set



6.2. Thereom. Points on and below the graph of a concave function form a convex set.

Let $A = \{(x,y) \mid x \in D, f(x) \geq y\}$ be the set of points "on and below"

$f: D \to \mathbb{R}$. $\Rightarrow$ A is a convex set iff. f is concave



*. Homogeneous functions.

Def.. $f(x)$ is called $HMG(k)$

$$f(tx) = t^k f(x), \quad \forall t > 0$$

linear homogeneous : $f(tx) = t f(x)$

$HMG(0)$: $f(tx) = f(x)$

ex. Determine homogeneity of $f(\cdot)$

Then $f(x_1, x_2) = A x_1^{\alpha} x_2^{\beta}$, $A > 0$

$\alpha, \beta > 0$.

$* \ \alpha + \beta = 1$

— Cobb-Douglas function.

$$f(tx_1, tx_2) = A(tx_1)^{\alpha}(tx_2)^{\beta}$$

$$= \underline{A}\, t^{\alpha+\beta}\, \underline{x_1^{\alpha}}\, \underline{x_2^{\beta}}$$

$f(x_1, x_2)$

$$= t^{\alpha+\beta} f(x_1, x_2)$$

$HMG(\alpha+\beta) = HMG(1)$.

$\Downarrow$

linear homogeneous.

Theorem. If $f(x)$ is $HMG(k)$

$$\Rightarrow \frac{\partial f(x)}{\partial x} \text{ is } HMG(k-1)$$

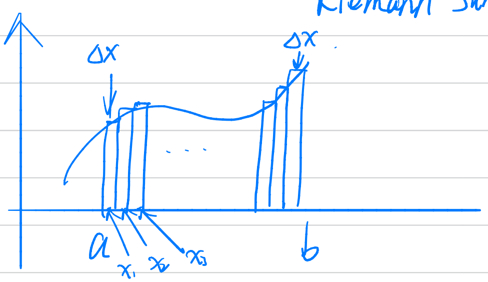
Integration.

§1. Antiderivative & Indefinite Integral

1.1. Def (Definit integral) Suppose $f \in C[a,b]$

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{\infty} f(x_i)\Delta x$$

$\| $ Riemann Sum.



1.2. Anti-derivative. $f(x) \xrightarrow{A.P.} F(x)$

$$\ni f(x) = F'(x)$$

1.3. Indefinite integral $\int f(x)\,dx = F(x) + C$

1.4. ex Find antiderivative of $\frac{1}{x^4}$

$$\int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C.$$

$$(x^k)' = k\,x^{k-1}$$

$$(x^{-3})' = (-3)x^{-4}$$

$$(-\tfrac{1}{3}x^{-3})' = -\tfrac{1}{3}(-3)x^{-4}$$

1.5. ex. Find $\int e^{2x}dx = \frac{1}{2}e^{2x}+C$

$$(e^{2x})' = 2e^{2x}$$

1.6. ex. Find $\int 3^{-x}dx = -\frac{1}{\ln 3}3^{-x}+C$.

$$(3^{x})' = 3^{x}\ln 3.$$

$$(3^{-x})' = \underline{-3^{-x}\ln 3}.$$

§2. Techniques of integral

1). Integration by substitution.

$\boxed{2.1.}$ $\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

$$u = u(x).$$

2.2. ex. Compute $\int e^{\frac{x}{2}}dx$

$$u = \frac{x}{2} \Rightarrow du = \frac{dx}{2}$$

$$dx = 2du$$

$$\int e^{\frac{x}{2}}dx = \int e^{u}\cdot 2du = 2\int e^{u}\cdot du$$

$$= 2(e^{u}+C) = 2e^{u}+C$$

$$\overset{u=\frac{x}{2}}{=} 2e^{\frac{x}{2}}+C$$

2.3. ex. $\int (3x+2)^5 dx$

$u = 3x+2.$

$du = d(3x+2) = 3dx \Rightarrow dx = \frac{du}{3}.$

$LHS = \int u^5 \frac{du}{3} = \frac{1}{3}\int u^5 du$ $\quad (u^6)' = 6u^5$

$= \frac{1}{3}\cdot\left(\frac{1}{6}\cdot u^6 + C\right)$

$= \frac{u^6}{18} + C$

2). Integration by parts

$\int u\,dv = uv - \int v\,du$

2.4. ex. Find $\int x \boxed{\cos 2x\, dx}$

$LHS = \int \underbrace{x}_{u}\, d\underbrace{\left(\tfrac{1}{2}\sin 2x\right)}_{v}$ $\quad (\sin 2x)' = 2\cos 2x$

$= \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\, dx$

$= \frac{x\sin 2x}{2} + \frac{1}{2}\cdot\left(\frac{1}{2}\cos 2x + C\right)$ $\quad d\cos 2x = -(\sin 2x)\,2dx.$

$= \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + C$

2.5. ex $\int \ln x\,dx$ $\qquad \frac{1}{x}dx$

$= x\cdot\ln x - \int x\,\boxed{d\ln x}$

$= x\ln x - \int 1\cdot dx$

$= x\ln x - (x + C)$

$= x\ln x - x + C.$

2.6. ex $\int x\,\underbrace{e^{-x}dx}$

$= -\int \underbrace{x}_{u}\, d\underbrace{e^{-x}}_{v}$

$= -\left[x\cdot e^{-x} - \int e^{-x}dx\right]$

$= -\left[x\cdot e^{-x} - (-e^{-x})\right] + C$

$= -xe^{-x} - e^{-x} + C$

$= -e^{-x}(x+1) + C$

3). (1.) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

(2.) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}$

(3) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a}$

(4) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$

$\int k\,dx = kx + C$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x}dx = \ln|x| + C$

$\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

$\int \ln u\,du = u\ln u - u + C$

$\int e^u du = e^u + C.$

Direchlet. $f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \in \mathbb{Q} \end{cases}$ # $\not\Rightarrow$ Riemann Integrated

$\Rightarrow$ Lesbogues Integratal ✓

2.8. Properties of definite integral

(1) $\int_a^b 1 \cdot dx = b-a$

(2) $\int_a^b k f(x) dx = k \int_a^b f(x) dx$

(3) $\int_a^b (f(x)+g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

(4) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

(5) $0 \le f(x) \le g(x)$, $x \in [a,b]$

$0 \le \int_a^b f(x) dx \le \int_a^b g(x) dx$

(6) $\int_a^b f(x) dx = -\int_b^a f(x) dx$

(7) $\int_a^a f(x) dx = 0$

(8) $f(x) \geq 0$ $\int_a^b f(x) dx \geq 0$

(9) $\left| \int_a^b f(x) dx \right| \le \int_a^b |f(x)| dx$

(10) $m \le f(x) \le M$ for $x \in [a,b]$

$m(b-a) \le \int_a^b f(x) dx \le M(b-a)$