stochastic Calculus

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Problem 1

(1) since W_t is a standard BM $W_t \sim N(0,t)$.

$$Z_t = W_t - \frac{t}{T} W_T$$

$$E(Z_t) = E(W_t) - E\left(\frac{t}{T} W_T\right)$$

$$= 0 - \frac{t}{T} E(W_T) = 0 - 0 = 0$$

$$Var(Z_t) = E\left(W_t - \frac{t}{T} W_T\right)^2 - \underbrace{\left[E\left(W_t - \frac{t}{T} W_T\right)\right]^2}_{0}$$

$$= E\left[W_t^2 - 2\frac{t}{T} W_T W_t + \frac{t^2}{T^2} W_T^2\right]$$

$$= E[W_t^2] - 2\frac{t}{T} E(W_t W_T) + \frac{t^2}{T^2} E(W_T)^2 \quad \because t < T$$

$$= t - 2\frac{t}{T} \cdot t + \frac{t^2}{T^2} \cdot T$$

$$= t - \frac{2t}{T} \cdot t + \frac{t^2}{T}$$

$$= t - \frac{2t^2}{T} + \frac{t^2}{T} = t - \frac{t^2}{T}$$

$$\therefore Z_t \sim N\left(0, t - \frac{t^2}{T}\right)$$

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Problem 1

(ii)

$$\operatorname{Cov}(Z_s, Z_t)$$

$$= E(Z_s Z_t) - E(Z_s)E(Z_t)$$

$$= E(Z_s Z_t)$$

$$= E\left[(W_s - \tfrac{s}{T}W_T)(W_t - \tfrac{t}{T}W_T)\right]$$

$$= E\left[W_s W_t - \tfrac{t}{T}W_s W_T - \tfrac{s}{T}W_t W_T + \tfrac{st}{T^2}W_T^2\right]$$

$$= s - \tfrac{t}{T}s - \tfrac{s}{T}t + \tfrac{st}{T}$$

$$= s + \frac{-2st + st}{T} = s - \frac{st}{T}$$

Problem 1

(iii)

var-cov matrix of Z_s, Z_t

$$= \begin{pmatrix} s + \frac{s^2}{T} & s + \frac{st}{T} \\ s - \frac{st}{T} & t - \frac{t^2}{T} \end{pmatrix}$$

$$\rho = \frac{\operatorname{cov}(Z_s, Z_t)}{\sigma_{Z_s} \sigma_{Z_t}} = \frac{s - \frac{st}{T}}{\sqrt{s - \frac{s^2}{T}}\sqrt{t - \frac{t^2}{T}}}$$

$$\therefore \rho^2 = \frac{\left(s - \frac{st}{T}\right)^2}{\left(s - \frac{s^2}{T}\right)\left(t - \frac{t^2}{T}\right)}$$

$f_{Z_s Z_t}(s, t) =$

$$\frac{1}{2\pi \sqrt{s - \frac{s^2}{T}}\sqrt{t - \frac{t^2}{T}}\sqrt{1 - \rho^2}} \exp\left[-\frac{1}{2(1 - \rho^2)} \left[\frac{s^2}{s + \frac{s^2}{T}} - 2\rho \cdot \left(\frac{s}{\sqrt{s - \frac{s^2}{T}}}\right)\left(\frac{t}{\sqrt{t - \frac{t^2}{T}}}\right) + \frac{t}{t - \frac{t^2}{T}} \right. \right.$$

Problem 2

Since the deterministic function $f(x) = -x^4$ $\qquad f'(x) = -4x^3$

$\therefore$ total variation would be $\int_{-3}^{3} |-4x^3| \, dx$

$= 2\int_0^3 4x^3 \, dx$

$= 8\int_0^3 x^3 dx = 8 \cdot \frac{1}{4} x^4 \Big|_0^3$

$= 2x^4 \Big|_0^3 = 2 \cdot [81 - 0] = 162$

quadratic variation.

$[f, f](x) = \int_{-3}^{3} |f'(x)|^2 dx$

$= \int_{-3}^{3} 16x^6 \, dx$

$= 16 \cdot \frac{1}{7} x^7 \Big|_{-3}^{3}$

$= \frac{16}{7} [3^7 - (-3)^7] = 0$

Problem 3

(1) $f'(x) = x^2$ $\quad f''(x) = 2x$

$f(x) = \frac{1}{3}x^3$

$\therefore f(W_t) - f(W_0) = \int_0^t [W_s^2]\, dW_s + \frac{1}{2}\int_0^t 2W_s\, (dW_s)^2$

$\Rightarrow \frac{1}{3}W_t^3 - \underbrace{\frac{1}{3}W_0^3}_{0} = \int_0^t (W_s^2)\, dW_s + \frac{1}{2}\int_0^t 2W_s\, ds.$

$\Rightarrow \int_0^t (W_s)^2\, dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s\, ds.$

(2) $f_x(t,w) = t^2 + e^{W_t}$ $\quad f(t,w) = t^2 W_t + e^{W_t}$

$f_{xx}(t,W) = e^{W_t}$ $\quad f_t(t,w) = 2tW_t$

$\therefore$ By Ito's formula:

$f(t,W_t) - f(0,W_0) = \int_0^t f_t(t,w)\, ds + \int_0^t f_x(t,w)\, dW_s + \frac{1}{2}\int_0^t f_{xx}(t,w)\, ds.$

$t^2 + e^{W_t} - [0^2 \cdot 0 + e^0] = \int_0^t 2sW_s\, ds + \int_0^t (s^2 + e^{W_s})\, dW_s + \frac{1}{2}\int_0^t e^{W_s}\, ds$

$t^2 + e^{W_t} - 1 = 2\int_0^t sW_s\, ds + \int_0^t (s^2 + e^{W_s})\, dW_s + \frac{1}{2}\int_0^t e^{W_s}\, ds$

$\Rightarrow \int_0^t (s^2 + e^{W_s})\, dW_s = t^2 + e^{W_t} - 1 - 2\int_0^t sW_s\, ds - \frac{1}{2}\int_0^t e^{W_s}\, ds$

$= t^2 + e^{W_t} - 1 - \int_0^t \left(2sW_s + \frac{1}{2}e^{W_s}\right) ds.$

Problem 4

$$E(Y) = E\int_0^t (s+(W_s)^2)\,dW_s$$

$$= \int_0^t E(s+W_s^2)\,dW_s$$

$$= 0$$

$$var(Y) = E(Y^2) - [E(Y)]^2$$

$$= E(Y^2).$$

Ito's Isometry

$$= E\int_0^t (s+W_s^2)^2\,ds$$

$$= \int_0^t s^2+2sW_s^2+W_s^4\,ds$$

$$= \int_0^t s^2\,ds + 2\int_0^t sW_s^2\,ds + \int_0^t W_s^4\,ds.$$

$$= \frac{1}{3}s^3\Big|_0^t + 2\int_0^t sW_s^2\,ds + \int_0^t W_s^4\,ds$$

$$= \frac{1}{3}t^3 + 2\int_0^t sW_s^2\,ds + \int_0^t W_s^4\,ds.$$

$$= \frac{1}{3}t^3 + \int_0^t (2sW_s^2 + W_s^4)\,ds$$

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Problem 5

$$Y_t = X_t^3 = f(X_t)$$

$$f'(X_t) = 3X_t^2 \qquad f''(X_t) = 6X_t$$

$$d(Y_t) = 3X_t^2\, dX_t + \frac{1}{2} \cdot 6X_t (dX_t)^2$$

$$= 3X_t^2 (W_t - [W_t]^2)dt + 3\sqrt{W_t}\, dW_t) + 3X_t \cdot 9W_t\, dt$$

$$= 3X_t^2 (W_t - [W_t]^2)dt + 9X_t^2\sqrt{W_t}\, dW_t + 27 X_t W_t\, dt$$

$$= [3X_t^2 W_t - 3X_t^2 [W_t]^2 + 27 X_t W_t]\, dt + 9X_t^2 \sqrt{W_t}\, dW_t.$$

$$\therefore Y_t = \int_0^t (3X_s^2 W_s - 3X_s^2 \overline{W_s}^2 + 27X_s W_s) ds + 9\int_0^t X_s^2 \sqrt{W_s}\, dW_s$$

Problem 6

$$\mathrm{cov}\left(W_t, \int_0^t W_s\,ds\right)$$

$$= E\left[W_t \cdot \int_0^t W_s\,ds\right] - E(W_t)\,E\left(\int_0^t W_s\,ds\right)$$

$$= E\left[W_t \cdot \int_0^t W_s\,ds\right] - 0 \cdot \int_0^t E(W_s)\,ds$$

$$= E\left[W_t \cdot \int_0^t W_s\,ds\right]$$

$$= E\left[W_t \cdot \frac{1}{2} W_s^2 \Big|_0^t\right]$$

$$= E\left[\frac{1}{2} W_t^3\right] = \frac{1}{2} E(W_t^3) = 0$$

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