Exercises #11

12.D.2 Show that with J firms, repeated choice of any price $p \in (c, p^m]$ can be sustained as a stationary SPNE outcome path of the infinitely repeated Bertrand game using Nash reversion strategies if and only if $\delta \geq (J-1)/J$. What does this say about the effect of having more firms in the market on the difficulty of sustaining collusion?

12.D.2. Let $\Pi^* > 0$ be the firms' equilibrium joint profits, which in equilibrium are split equally among firms. The best deviation for a firm is to undercut the rivals by a small ε, in which case it can steal all the demand and obtain almost as much as Π^* in that period. In the following punishment phase, the deviator will get zero forever after. If the firm does not deviate, its payoff is $\sum_{t=0}^{\infty} \delta^t \Pi^*/J$.

Therefore, deviation is not profitable if and only if $1/(1-\delta)\Pi^*/J \geq \Pi^m$, or $\delta \geq (J-1)/J$.

Since $(J-1)/J$ increases in J, as J increases, δ has to increase in order for collusion to be still sustainable. Therefore, as the number of firms increases, it is harder to sustain a collusive outcome.

We have: Infinitely Repeated Bertrand game
J firms

Show: any price $p \in (c, p^m]$ can be sustained as a stationary Subgame Perfect Nash Equilibrium (SPNE) iff $\delta \geq \frac{J-1}{J}$

For a certain $p^* \in (c, p^m]$, consider the following strategy for the infinitely repeated game:

- play p^* in the first period of the game OR, if we are not in the first period of the game, play p^* if every firm has played p^* in all previous histories of the game.

- play c otherwise (forever).

If all firms cooperate and play p^*, the joint profit of the firms is $\Pi^* > 0$ and each firm will obtain $\frac{\Pi^*}{J}$.

If one firm deviates, the best deviation strategy is to slightly undercut the rivals by a small amount, say $p^* - \varepsilon$, with ε infinitesimally small, and obtain almost as much as Π^* in that period. Assume the profit from deviating is Π^* for the deviating firm.

This begins the punishment phase of the trigger strategy. All the other firms are now charging price equal to marginal cost c. The deviating firm will obtain zero profit forever (regardless of its pricing decision, for any price $p \geq c$).

The profit of an individual firm from cooperating is:

$$\Pi^{cooperate} = \frac{\Pi^*}{J} + \delta \frac{\Pi^*}{J} + \delta^2 \frac{\Pi^*}{J} + \dots$$

$$= \underbrace{\left(1 + \delta + \delta^2 + \dots\right)}_{\frac{1}{1-\delta}} \frac{\Pi^*}{J}$$

$$= \left(\frac{1}{1-\delta}\right) \frac{\Pi^*}{J}$$

where $\delta < 1$ is the discount factor

The profit from deviating is given by:

$$\Pi^{deviate} = \Pi^{*} + \delta \times 0 + \delta^{2} \times 0 + \dots$$

$$= \Pi^{*}$$

(if the firm is going to deviate, it might as well deviate in the first period, because by deviating in subsequent period entails receiving a "discounted" profit)

So this collusion strategy is sustainable as long as:

$$\Pi^{cooperate} \geq \Pi^{deviate}$$

assume that, if indifferent, the firm will choose to cooperate

$$\Leftrightarrow \left(\frac{1}{1-\delta}\right) \frac{\cancel{\Pi^{*}}}{J} \geq \cancel{\Pi^{*}}$$

$$\Leftrightarrow J \leq \frac{1}{1-\delta} \Leftrightarrow (1-\delta) J \leq 1$$

$$\Leftrightarrow J - \delta J \leq 1 \Leftrightarrow \delta J \geq J - 1$$

$$\Leftrightarrow \delta \geq \frac{J-1}{J}$$

The more firms discount future profits (ie, the higher δ), the more likely it is that firms will choose to cooperate.

What happens as J increases?

$$\frac{\partial \overbrace{\left(\frac{J-1}{J}\right)}^{= J^{-1}(J-1)}}{\partial J} = -J^{-2}(J-1) + J^{-1} = \frac{1-J}{J^2} + \frac{1}{J} = \frac{1-J+J}{J^2} = \frac{1}{J^2} > 0$$

Since $\frac{J-1}{J}$ is increasing in J, as the number of firms increases, δ has to increase in order for collusion to still be sustainable.

Therefore, as the number of firms increases, it is harder to sustain a collusive outcome.

12.D.3 Consider an infinitely repeated Cournot duopoly with discount factor $\delta < 1$, unit costs of $c > 0$, and inverse demand function $p(q) = a - bq$, with $a > c$ and $b > 0$.

(a) Under what conditions can the symmetric joint monopoly outputs $(q_1, q_2) = (q_1^m/2, q_2^m/2)$ be sustained with strategies that call for $(q_1^m/2, q_2^m/2)$ to be played if no one has yet deviated and for the single-period Cournot (Nash) equilibrium to be played otherwise?

(b) Derive the minimal level of δ such that the output levels $(q_1, q_2) = (q, q)$ with $q \in [(a-c)/2b, (a-c)/b]$ are sustainable through Nash reversion strategies. Show that this level of δ, $\delta(q)$, is an increasing, differentiable function of q.

12.D.3. (a) Example 12.C.1 in the textbook solves for the static Nash equilibrium in the game. The equilibrium yields the Cournot outcome in which each firm makes a profit of $(a-c)^2/9b$. The maximum gain from deviation can be obtained by playing the best response to $q^m/2 = (a-c)/4b$, which is $3(a-c)/8b$. This maximum gain is $9/64\ (a-c)^2/b$. Monopoly profit can be calculated to be $(a-c)^2/4b$. The payoff from deviating is

$$9(a-c)^2/64b + \sum_{t=1}^{\infty} \delta^t \ (a-c)^2/9b = 9(a-c)^2/64b + \delta/(1-\delta)\ (a-c)^2/9b.$$

The payoff from not deviating is

$$\sum_{t=0}^{\infty} \delta^t\ (a-c)^2/8b = 1/(1-\delta)\ (a-c)^2/8b.$$

Therefore, deviation is not profitable if and only if

$1/(1-\delta)\ (a-c)^2/8b \geq 9(a-c)^2/64b + \delta/(1-\delta)\ (a-c)^2/9b$, or $\delta \geq 9/17$.

(b) The maximum gain from deviation can be obtained by playing the best response to q, which is $(a-c)/2b - q/2$. This maximum gain is $b[(a-c)/2b - q/2]^2$.

The payoff from deviating is

$$b[(a-c)/2b - q/2]^2 + \sum_{t=0}^{\infty} \delta^t\ (a-c)^2/9b =$$

$$= b[(a-c)/2b - q/2]^2 + \delta/(1-\delta)(a-c)^2/9b.$$

The payoff from not deviating is

$$\sum_{t=0}^{\infty} \delta^t \quad (a-c)^2/4b = 1/(1-\delta)\ (a-c)^2/4b.$$

Therefore, deviation is not profitable if and only if

$1/(1-\delta)\ \ (a-c)^2/4b = b[(a-c)/2b - q/2]^2 + \delta/(1-\delta)\ \ (a-c)^2/9b.$

Thus,

$\delta(q) = [(a-c)^2/4b - b[(a-c)/2b - q/2]^2]/[(a-c)^2/9b - b[(a-c)/2b - q/2]^2]$,

which is a decreasing, differentiable function of q.

We have: Infinitely Repeated Cournot duopoly

$\delta < 1$: discount factor

$c > 0$: unit cost

$p(q) = a - bq$: inverse demand function

$a > c, \quad b > 0$

a) Find: Conditions for δ under which a collusive outcome is sustainable.

Consider the following strategy:

- play the collusive quantity $q^{coop} = \frac{q^M}{2}$ (half the monopoly quantity) in the first period of the game on, if we are not at the first period of the game, play q^{coop} if everyone has played q^{coop} in all previous histories of the game.

- play the cournot quantity q^c otherwise.

This collusion strategy will be sustainable if:

$$\underbrace{\Pi^{coop}}_{\text{PDV of profits from colluding}} \geq \underbrace{\Pi^{deviating}}_{\text{PDV of profits from deviating in the first period, obtaining the deviation profit once and the Cournot profit henceforth}}$$

So we need to find: q^M, Π^M (monopoly)

q_i^D, q_j^D, Π_i^D when firm i deviates

q^c, Π^c (Cournot)

Monopoly case:

$$\max_{q} \; p(q)q - cq = (p(q)-c)q = (a-bq-c)q$$

FOC: (q): $a - bq - c - bq = 0 \Leftrightarrow q^M = \dfrac{a-c}{2b}$

$$p^M = p(q^M) = a - \cancel{b}\left(\frac{a-c}{2\cancel{b}}\right) = a - \frac{(a-c)}{2} = \frac{2a-a+c}{2}$$

$$= \frac{a+c}{2}$$

$$\pi^M = (p^M - c)q^M = \left(\frac{a+c}{2} - c\right)\left(\frac{a-c}{2b}\right) = \left(\frac{a+c-2c}{2}\right)\left(\frac{a-c}{2b}\right)$$

$$= \frac{(a-c)^2}{4b}$$

Cournot case:

Firm i's optimization problem: with $i, j = 1, 2$ and $i \neq j$

$$\max_{\{q_i\}} \pi_i(q_i, q_j) = p(q)\, q_i - c\, q_i = \left(p(q) - c\right) q_i$$

$$= \left(a - b(q_i + q_j) - c\right) q_i$$

FOC: (q_i): $a - b(q_i + q_j) - c - b\, q_i = 0$

$\Leftrightarrow$ $a - 2b\, q_i - b\, q_j - c = 0$ $\Leftrightarrow$ $2b\, q_i = a - b\, q_j - c$

$\Leftrightarrow$ $q_i = \dfrac{a - b\, q_j - c}{2b}$

Best response function for firm i:

$$b_i(q_j) = \frac{a - b\, q_j - c}{2b}$$

By symmetry, the BR function for firm j is:

$$b_j(q_i) = \frac{a - b q_i - c}{2b}$$

To find the (Competitive) Cournot NE, we impose the symmetric NE:

$$q_i^c = q_j^c = q^c$$

(alternatively plug $b_j(q_i)$ into $b_i(q_j)$ to find the NE)

$$q^c = \frac{a - b q^c - c}{2b} \Leftrightarrow 2b\, q^c = a - b\, q^c - c$$

$$\Leftrightarrow 3b\, q^c = a - c \Leftrightarrow q^c = \frac{a-c}{3b}$$

$$p^c = p(q^c + q^c) = a - \cancel{b}\left(\frac{2(a-c)}{3\cancel{b}}\right) = \frac{3a - 2a + 2c}{3}$$

$$= \frac{a+2c}{3}$$

$$\pi^c = (p^c - c)\, q^c = \left(\frac{a+2c}{3} - c\right)\left(\frac{a-c}{3b}\right) = \left(\frac{a+2c-3c}{3}\right)\left(\frac{a-c}{3b}\right)$$

$$= \frac{(a-c)^2}{9b}$$

Deviation Case:

To find the optimal deviating quantity for firm i, simply plug $q_j = \frac{q^m}{2}$ into the best response function for firm i.

That is, given that firm j is playing the collusion quantity, what is the profit maximizing quantity for firm i? That gives the optimal deviation quantity for firm i:

$$q_i^D = b_i\left(q_j = \frac{q^m}{2}\right) \quad \text{with} \quad q^m = \frac{a-c}{2b}$$

$$q_i^D = \frac{a - b\left(\frac{a-c}{4b}\right) - c}{2b} = \frac{\frac{3}{4}(a-c)}{2b} = \frac{3(a-c)}{8b}$$

notice that firm i deviates by increasing production!

$$q_i^D = \frac{3}{8}\frac{(a-c)}{b} > \frac{q^m}{2} = \frac{1}{4}\frac{(a-c)}{b} \qquad \frac{3}{8} = 0.375 > \frac{1}{4} = 0.25$$

$$p^D = p\left(q_i^D + \frac{q^m}{2}\right) = a - b\left(\frac{3(a-c)}{8b} + \frac{(a-c)}{4b}\right)$$

$$= a - \cancel{b}\left(\frac{3(a-c) + 2(a-c)}{8\cancel{b}}\right) = a - \frac{5(a-c)}{8}$$

$$= \frac{8a - 5a + 5c}{8} = \frac{3a + 5c}{8}$$

$$\pi_i^D = \left(p^D - c\right) q_i^D = \left(\frac{3a+5c}{8} - c\right)\frac{3(a-c)}{8b}$$

$$= \left(\frac{3a+5c-8c}{8}\right)\frac{3(a-c)}{8b} = \frac{(3a-3c)\,3(a-c)}{64b}$$

$$= \frac{9(a-c)^2}{64b}$$

<u>Condition for sustainable collusion:</u>

The collusion will be sustainable if:

$$\Pi^{coop} \geq \Pi^{deviating}$$

$$\Leftrightarrow \frac{\pi^M}{2} + \delta\frac{\pi^M}{2} + \delta^2\frac{\pi^M}{2} + \dots \geq \pi^D + \delta\pi^C + \delta^2\pi^C + \dots$$

$$\Leftrightarrow \underbrace{\left(1 + \delta + \delta^2 + \dots\right)}_{\frac{1}{1-\delta}}\frac{\pi^M}{2} \geq \pi^D + \underbrace{\left(\delta + \delta^2 + \dots\right)}_{\frac{\delta}{1-\delta}}\pi^C$$

$$\Leftrightarrow \left(\frac{1}{1-\delta}\right)\frac{\pi^M}{2} \geq \pi^D + \left(\frac{\delta}{1-\delta}\right)\pi^C$$

Recall: $\Pi^M = \frac{(a-c)^2}{4b}$ $\quad \Pi^C = \frac{(a-c)^2}{9b}$ $\quad \Pi^D = \frac{9(a-c)^2}{64b}$

$$\Leftrightarrow \left(\frac{1}{1-\delta}\right)\frac{(a-c)^2}{8b} \geq \frac{9(a-c)^2}{64b} + \left(\frac{\delta}{1-\delta}\right)\frac{(a-c)^2}{9b}$$

$$\Leftrightarrow \left(\frac{1}{1-\delta}\right)\frac{1}{8} \geq \frac{9}{64} + \left(\frac{\delta}{1-\delta}\right)\frac{1}{9} \Leftrightarrow \frac{1}{8(1-\delta)} - \frac{\delta}{9(1-\delta)} \geq \frac{9}{64}$$

$$\Leftrightarrow \frac{9-8\delta}{72(1-\delta)} \geq \frac{9}{64} \Leftrightarrow \frac{9-8\delta}{72} \geq \frac{9}{64}(1-\delta) \Leftrightarrow$$

$$\Leftrightarrow \frac{9-8\delta}{72} \geq \frac{9}{64} - \frac{9}{64}\delta \Leftrightarrow \left(\frac{9}{64} - \frac{8}{72}\right)\delta \geq \frac{9}{64} - \frac{9}{72}$$

$$\Leftrightarrow \frac{17}{576}\delta \geq \frac{1}{64} \Leftrightarrow \delta \geq \frac{9}{17}$$

So the collusive outcome is sustainable as long as $\delta \geq \frac{9}{17}$.

b) Find: Minimal δ such that $(q_1, q_2) = (q, q)$ with $q \in \left[\frac{a-c}{2b}, \frac{a-c}{b}\right]$ can be sustainable through Nash reversible strategies.

Show: $\delta(q)$ increasing differentiable function of q.
decreasing ?

Consider some $q \in \left[\frac{a-c}{2b}, \frac{a-c}{b}\right]$. Notice that $q^M = \frac{a-c}{2b}$ (monopoly case) and $q^{comp} = \frac{a-c}{b}$ (Perfect competition case).

Now, consider the following strategy:

- play q in the first period of the game and, if it is not the first period of the game, play q if all players have chosen q in all previous histories of the game.

- play q^C (Cournot quantity) otherwise (forever).

This collusion strategy is sustainable as long as:

$$\Pi^{cooperate} \geq \Pi^{Deviate}$$

and the minimal δ that can sustain this strategy is given at the equality:

$$\Pi^{cooperate} = \Pi^{Deviate}$$

If both firms cooperate:

$$q^{coop} = q$$

$$p^{coop} = p(2q) = a - 2bq$$

$$\pi^{coop} = \left(p^{coop} - c\right) q^{coop}$$

$$= \left(a - 2bq - c\right) q$$

If one firm deviates:

$$q^{D} = \frac{a - bq - c}{2b} \quad \left(\text{plug } q \text{ into } b_i(q_j)\right)$$

$$p^{D} = p\left(q^{D} + q\right) = a - b\left(\frac{a - bq - c}{2b} + q\right)$$

$$= a - \cancel{b}\left(\frac{a - bq - c + 2bq}{2\cancel{b}}\right) = \frac{2a - a + bq + c - 2bq}{2}$$

$$= \frac{a - bq + c}{2}$$

$$\pi^{D} = \left(p^{D} - c\right) q^{D} = \left(\frac{a - bq + c}{2} - c\right)\left(\frac{a - bq - c}{2b}\right)$$

$$= \left(\frac{a - bq + c - 2c}{2}\right)\left(\frac{a - bq - c}{2b}\right) = \frac{(a - bq - c)^2}{4b}$$

After one firm deviates, the game reverts to Cournot:

$$q^c = \frac{a-c}{3b}$$

$$p^c = p(2q^c) = \frac{a+2c}{3}$$

$$\pi^c = \frac{(a-c)^2}{9b}$$

The minimal δ for which the collusive outcome can be sustained is therefore given by:

$$\Pi^{cooperate} = \Pi^{Deviate}$$

$$\Leftrightarrow \quad \pi^{coop} + \delta\, \pi^{coop} + \delta^2\, \pi^{coop} + \dots = \Pi^D + \delta\, \pi^c + \delta^2\, \pi^c + \dots$$

$$\Leftrightarrow \quad \left(\frac{1}{1-\delta}\right) \pi^{coop} = \Pi^D + \left(\frac{\delta}{1-\delta}\right) \pi^c$$

$$\Leftrightarrow \quad \left(\frac{1}{1-\delta}\right)(a - 2bq - c)\, q = \frac{(a - bq - c)^2}{4b} + \left(\frac{\delta}{1-\delta}\right)\frac{(a-c)^2}{9b}$$

Solving for δ we obtain:

(you can use Wolframalpha.com or a similar math software for this part)

$$\delta(q) = \frac{q(a-3b-q)}{5a-3bq-c}$$

Differentiating $\delta(q)$ wrt to q we obtain:

$$\frac{\partial \delta(q)}{\partial q} = -\frac{108b\overbrace{(a-c)}^{>0}}{\underbrace{(5a-3bq-5c)^2}_{>0}} < 0$$

So $\delta(q)$ is <u>decreasing</u> in q. This makes sense because the higher the quantity q the firms agree to collude on, the more likely it is that firms will stick to the collusive agreement. Therefore the collusive agreement can be sustained for a lower value of δ.