Homework #2 Stochastic Calculus Solutions

Ito's Integral

Important: This is intended to show the method for the solutions. There could be small calculation erros that I may have made.

Note: *These solutions contain more details than necessary to help some students. The suggested solutions have not been thoroughly checked. Use them as hints for solving the problems.*

Problem 1. Let $W(t)$ be a standard Brownian Motion $[0,T]$, and $0 \leq s < t \leq T$. What is the conditional density of $W(s)$ given $W(t) = y$?

Answer: The following is sort of the standard way of solving the problem, although it is not the shortest way. From the properties of BM, the joint density of $W(s)$, and $W(t)$ is given by

$$f_{W(s),W(t)}(u,v) = \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2(1-s/t)}\left[\frac{u^2}{s} + \frac{v^2}{t} - \frac{2uv}{s/t}\right]\right).$$

We also know that the marginal density of $W(t)$ is

$$f_{W(t)}(v) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{v^2}{t}\right).$$

Hence the conditional density of $W(s)$ given $W(t) = y$ is

$$\begin{aligned}
f_{W(s)|W(t)=y}(u) &= \frac{f_{W(s),W(t)}(u,v)}{f_{W(t)}(v)}|_{v=y} \\
&= \frac{\sqrt{2\pi t}}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2(1-s/t)}\left[\frac{u^2}{s} + \frac{v^2}{t} - \frac{2uv}{s/t}\right] + \frac{1}{2}\frac{v^2}{t}\right)|_{v=y} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{s(t-s)/t}} \exp\left(-\frac{1}{2s(t-s)/t}\left[u - \frac{s}{t}v\right]^2\right)|_{v=y} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{s(t-s)/t}} \exp\left(-\frac{1}{2s(t-s)/t}\left[u - \frac{s}{t}y\right]^2\right).
\end{aligned}$$

This means the conditional distribution of $W(s)$ given $W(t) = y$ is $N(sy/t, s(t-s)/t)$. The purpose of this problem is to remind you how conditional densities are calculated for continuous random variables.

Problem 2. Show that $W(t)^3 - 3tW(t)$ is a martingale. Here $W(t)$ is standard Brownian Motion.

Answer: We need only to verify that the martingale property holds: (here $s < t$)

$$\mathbb{E}\left[W(t)^3 - 3tW(t)|F(s)\right] = W(s)^3 - 3sW(s).$$

Obviously, $\mathbb{E}\left[3tW(t)|F(s)\right] = 3tW(s),$ due to the fact that $W(t)$ is a martingale. Hence the task reduces to verifying that

$$\mathbb{E}\left[W(t)^3|F(s)\right] = W(s)^3.$$

This is easily shown using the independent increment property of BM:

$$\mathbb{E}\left[W(t)^3|F(s)\right] = \mathbb{E}\left[\{W(t) - W(s) + W(s)\}^3|F(s)\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\{W(t)-W(s)\}^3 + 3\{W(t)-W(s)\}^2 W(s) + 3\{W(t)-W(s)\} W(s)^2 + W(s)^3 | F(s)\right] \\
&= \mathbb{E}[\{W(t)-W(s)\}^3 |F(s)] + 3\mathbb{E}\left[\{W(t)-W(s)\}^2 W(s)|F(s)\right] \\
&\quad +3\mathbb{E}\left[\{W(t)-W(s)\} W(s)^2|F(s)\right] + \mathbb{E}[W(s)^3|F(s)] \\
&= \mathbb{E}\{W(t)-W(s)\}^3 + 3W(s)\mathbb{E}\{W(t)-W(s)\}^2 \quad \text{[what properties are used here?]} \\
&\quad +3W(s)^2\mathbb{E}\left[\{W(t)-W(s)\}\right] + W(s)^3 \\
&= 0 + 3W(s)(t-s) + 0 + W(s)^3 \\
&= 3W(s)(t-s) + W(s)^3.
\end{aligned}
$$

Putting together, we obtain

$$
\begin{aligned}
\mathbb{E}\left[W(t)^3 - 3tW(t)|F(s)\right] &= 3W(s)(t-s) + W(s)^3 - 3tW(s) \\
&= W(s)^3 - 3sW(s).
\end{aligned}
$$

This is what we need to show.

Problem 3 (Stochastic Integrals with Deterministic Integrand). Evaluate $\int_0^t sdW(s)$ using the defining approximations to show that

$$
\int_0^t sdW(s) = tW(t) - \int_0^t W(s)ds.
$$

Here $W(t)$ is standard Brownian Motion.

Answer: The definition of the left hand side is

$$
\int_0^t sdW(s) = \lim \sum_j s_j \Delta W_j
$$

for a given grid $\{s_j\}_{j=0}^{j=n}$ on $[0,t]$. To handle this limit, we need to write it in a more convenient form. To this end, use the following identity:

$$
\sum_j s_j \Delta W_j = \sum_j \Delta(s_j W_j) - \sum_j W_j \Delta s_j. \text{ [you should verify this is true]}
$$

we obtain that

$$
\begin{aligned}
\int_0^t sdW(s) &= \lim_n \left[\sum_j \Delta(s_j W_j) - \sum_j W_j \Delta s_j\right] \\
&= \lim_n \sum_j \Delta(s_j W_j) - \lim_n \sum_j W_j \Delta s_j.
\end{aligned}
$$

Then the first term terms on the right hand side are just

$$
\lim \sum_j \Delta(s_j W_j) = \lim s_n W_n = tW(t), \text{ [write out } \sum_j \Delta(s_j W_j) \text{ to see the terms cancel.]}
$$

The second term is just the Riemann integral by definition

$$
\lim \sum_j W_j \Delta s = \int W(s)ds.
$$

The proof is completed.

Problem 4. (Riemann Integrals with Stochastic Integrand). Let $W(t) \equiv W(\omega, t)$ be the standard Brownian Motion on the interval $[0, 1]$. Define random variable Z as

$$Z(\omega) = \int_0^1 W(\omega, t)dt.$$

Because the BM has continuous sample path, the above Z is defined for every ω in the classical sense of Riemann. For such defined Z, find the distribution of Z?

Answer: This integral is defined path-wise as using grid $\{t_i\} = \{\frac{i}{n}\}_{i=0}^n$ on $[0, 1]$:

$$\int_0^1 W(\omega, t)dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} W\left(\frac{i}{n}\right) \cdot \frac{1}{n} \tag{1}$$

Then observe that the summation is a sum of normal random variables, and hence it is normally distributed. As a consequence, the limit is also normally distributed. So our job is to figure out the mean and variance of this normal distribution. Due to the fact that $W(t)$ is a martingale, $W(t)$ has mean 0 for all t. Hence

$$\mathbb{E}\sum_{i=0}^{n-1} W\left(\frac{i}{n}\right) \cdot \frac{1}{n} = \sum_{i=0}^{n-1} 0 \cdot \frac{1}{n} = 0.$$

and variance is

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=0}^{n-1} W\left(\frac{i}{n}\right) \cdot \frac{1}{n}\right]^2 &= \frac{1}{n^2}\mathbb{E}\sum_{i=0}^{n-1}\left[W\left(\frac{i}{n}\right)\right]^2 + 2\frac{1}{n^2}\mathbb{E}\sum_{0\le i<j\le n-1}\left[W\left(\frac{i}{n}\right)\right]\left[W\left(\frac{j}{n}\right)\right] \\
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\mathbb{E}\left[W\left(\frac{i}{n}\right)\right]^2 + 2\frac{1}{n^2}\sum_{0\le i<j\le n-1}\mathbb{E}\left[W\left(\frac{i}{n}\right)\right]\left[W\left(\frac{j}{n}\right)\right] \\
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\frac{i}{n} + 2\frac{1}{n^2}\sum_{0\le i<n-1}\sum_{i\le j\le n-1}\mathbb{E}\left[W\left(\frac{i}{n}\right)\right]\left[W\left(\frac{j}{n}\right)\right] \\
&= \frac{1}{n^2}\frac{(n-1)n}{2n} + 2\frac{1}{n^2}\sum_{0\le i<n-1}\sum_{i<j\le n-1}\frac{i}{n} \\
&= \frac{n-1}{2n^2} + 2\frac{1}{n^3}\sum_{0\le i<n-1}\sum_{i<j\le n-1} i
\end{aligned}$$

The evaluation of the double summation above is done as follows

$$\begin{aligned}
\sum_{0\le i<n-1}\sum_{i<j\le n-1} i &= \sum_{0\le i<n-1} i(n-i) \\
&= n\sum_{0\le i<n-1} i - \sum_{0\le i<n-1} i^2 \\
&= n\cdot\frac{n^2}{2} - \frac{(n-1)n(2n-1)}{6} \\
&= \frac{n^3}{2} - \frac{(n-1)n(2n-1)}{6}
\end{aligned}$$

here we used the well-known result

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Combine the above results we see that variance

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=0}^{n-1} W\left(\frac{i}{n}\right)\cdot\frac{1}{n}\right]^2 &= \frac{n-1}{2n^2} + 2\frac{1}{n^3}\left[\frac{n^3}{2} - \frac{(n-1)n(2n-1)}{6}\right] \\
&= \frac{n-1}{2n^2} + \left[1 - \frac{(n-1)(2n-1)}{3n^2}\right] \\
&\to 0 + \left[1 - \frac{2}{3}\right] = \frac{1}{3} \text{ as } n \to \infty.
\end{aligned}$$

Therefore we know that the $\int_0^1 W(\omega,t)dt$ is normally distributed as $N(0,1/3)$.

Remark: Some students solved the problem differently. The method is concise but requires more sophistication in Probability/Measure Theory: the mean and variance of $\int_0^1 W(\omega,t)dt$ is calculated as follows

$$\begin{aligned}
\mathbb{E}\int_0^1 W(\omega,t)dt &= \int_0^1 \left[\mathbb{E}W(\omega,t)\right]dt = \int_0^1 0dt = 0 \\
\mathbb{E}\left[\int_0^1 W(\omega,t)dt\right]^2 &= \mathbb{E}\left[\int_0^1 W(\omega,s)ds \int_0^1 W(\omega,t)dt\right] \text{ Note } s \text{ and } t \text{ are just dummies} \\
&= \mathbb{E}\int_0^1\int_0^1 W(\omega,s)W(\omega,t)dsdt \\
&= \int_0^1\int_0^1 \mathbb{E}\left[W(\omega,s)W(\omega,t)\right]dsdt \\
&= \int_0^1\int_0^1 \min(s,t)dsdt \quad \text{[Note: } \mathbb{E}\left[W(s)W(t)\right] = \min(s,t)\text{]} \\
&= \int_0^1\left[\int_0^1 \min(s,t)ds\right]dt \\
&= \int_0^1\left[\int_0^t sds + \int_t^1 tds\right]dt = \int_0^1\left[\frac{t^2}{2} + t(1-t)\right]dt \\
&= \frac{1}{3}.
\end{aligned}$$

Those who are ambitious may learn the skills shown above in handling the expectations.

Problem 5. (Finding Distribution of Integrals using Ito Isometry). Find the distribution for

$$\int_0^T e^t dW(t).$$

Answer: Writing out the integral using the definition, we can see that integral is the limit of sums of normally distributed random variables so it is also normally distributed. Thus we only need to know the mean and variance parameters.

The mean is obviously zero due to the martingale property. The variance can be computed using the Ito Isometry

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^t dW(t)\right]^2 &= \int_0^T \left[e^t\right]^2 dt \\
&= \int_0^T e^{2t}dt
\end{aligned}$$

$$= \frac{1}{2}\left(e^{2T} - 1\right).$$

Problem 6. (Finding Distribution of Integrals using Ito Isometry). Find mean and variance for

$$\int_0^T W(t)^3 dW(t).$$

Answer: The mean is zero due to martingale property of stochastic integrals. The variance is computed using Ito Isometry as follows

$$\begin{aligned}
\mathbb{E}\left[\int_0^T W(t)^3 dW(t)\right]^2 &= \mathbb{E}\int_0^T \left[W(t)^3\right]^2 dt \\
&= \mathbb{E}\int_0^T \left[W(t)\right]^6 dt \\
&= \int_0^T \left[\mathbb{E}W(t)^6\right] dt \quad \text{Change order of integration} \\
&= \int_0^T \left[15t^3\right] dt = \frac{15}{4}T^4.
\end{aligned}$$

Bonus Problems

Problem 7. (Stochastic Integrand). Show the following result by evaluating the stochastic integral on the left using the definition:

$$\int_0^t W(s)^2 dW(s) = \frac{1}{3}W(t)^3 - \int_0^t W(t)ds.$$

Answer: Use the definition of stochastic integrals, we know that the left-hand side is defined to be

$$\int_0^t W(t)^2 dW(t) = \lim_{n\to\infty}\sum_{i=0}^{n-1} W(t_i)^2\left[W(t_{i+1}) - W(t_i)\right]. \tag{2}$$

This sum is not very convenient for the evaluation. How to change it to a better form? You can use the following formula:

$$3a^2(a-b) = (b^3 - a^3) \ - 3a(b-a)^2 - (b-a)^3$$

Or

$$a^2(a-b) = \frac{1}{3}(b^3 - a^3) \ - a(b-a)^2 - \frac{1}{3}(b-a)^3.$$

Using this equation for (2) with $a = W(t_i)$ and $b = W(t_{i+1})$ we get

$$\begin{aligned}
\int_0^t W(t)^2 dW(t) &= \lim_{n\to\infty}\sum_{i=0}^{n-1} W(t_i)^2\left[W(t_{i+1}) - W(t_i)\right] \\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{3}\left[W(t_{i+1})^3 - W(t_i)^3\right]
\end{aligned}$$

$$
\begin{aligned}
& - \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i) \left[W(t_{i+1}) - W(t_i)\right]^2 \\
& - \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{3} \left[W(t_{i+1}) - W(t_i)\right]^3 \\
\equiv \;& I + II + III.
\end{aligned}
$$

It is easy to see that I is just $\frac{1}{3}W(t)^3$, and II is $-\int_0^t W(t)ds$. The third term III is zero since BM has finite quadratic variation. This proves the result.

Problem 8 (Integrals with Deterministic Integrand). We can generalize Problem 4 to the following: what is the distribution of

$$\int_0^T g(s)dW(t),$$

where g is a "nice" deterministic function?

Answer: This is similar to Problem 5. The defining approximation

$$\int_0^T g(s)dW(t) = \lim \sum_{i=0}^{n} g(t_i) \left[W(t_{i+1}) - W(t_i)\right]$$

indicates that it is a limit of sum of normal random variables, so it has normal distribution. The mean parameter is zero

$$
\begin{aligned}
\mathbb{E} \int_0^T g(s)dW(t) &= \mathbb{E} \lim \sum_{i=0}^{n} g(t_i) \left[W(t_{i+1}) - W(t_i)\right] \\
&= \lim \mathbb{E} \sum_{i=0}^{n} g(t_i) \left[W(t_{i+1}) - W(t_i)\right] \\
&= \lim \sum_{i=0}^{n} g(t_i) \mathbb{E} \left[W(t_{i+1}) - W(t_i)\right] \\
&= 0.
\end{aligned}
$$

The variance parameter is obtained by Ito Isometry

$$\mathbb{E}\left[\int_0^T g(s)dW(t)\right]^2 = \mathbb{E}\int_0^T g(s)^2 ds = \int_0^T g(s)^2 ds.$$

Or, equivalently, the variance parameter is the limit of the variances of the defining approximation:

$$
\begin{aligned}
\lim \mathbb{E} \left(\sum_{i=0}^{n} g(t_i) \left[W(t_{i+1}) - W(t_i)\right]\right)^2 &= \lim \sum_{i=0}^{n} g(t_i)^2 \mathbb{E} \left[W(t_{i+1}) - W(t_i)\right]^2 \text{ Due to independent increment of BM} \\
&= \lim \sum_{i=0}^{n} g(t_i)^2 \left[t_{i+1} - t_i\right] \\
&= \int_0^T g(t)^2 dt.
\end{aligned}
$$