Homework #5 Stochastic Calculus Solutions

Black-Scholes PDE, Feymann-Kac, Risk-Neutral Pricing

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Due Date: (Saturday !) December 17, 2020 (11:59PM)

Note: HW handed in after the deadline will not be credited (I will post the solutions to prepare you for the Final Exam). Problem 1 involves using Feymann-Kac Theorem – watch my lecture recording for how to do it.

Note: For Problems 2 and 3 you will need to provide the dynamic replication argument to derive the pricing partial equation (PDE) along with the associated boundary condition (i.e., terminal condition). You then solve the pricing PDE by invoking the Feyman-Kac to obtain the solution, and then evaluate the expection to obtain the final answer. Of course, you can also use risk-neutral pricing method.

Problem 1. In the class, we derived the pricing PDE for European call option:

$$C_t + rxC_x + \frac{1}{2}\sigma^2 x^2 C_{xx} \ = rC, \tag{1}$$

with boundary condition

$$C(x,T) = (x-K)^+ . \tag{2}$$

The solution to (1) is the Black-Scholes option formula

$$\begin{aligned} C(x,t) &= xN(d_1) - Ke^{-r(T-t)}N(d_2) \\ d_{1,2} &= \frac{\ln(x/K) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Verify by calculating partial derivatives of the Black-Scholes formula that it indeed solves the pricing PDE (1). Also, either demonstrate mathematically or demonstrate using Matlab computation, the terminal condition (2) is also satisfied.

Bonus Problem 2 [Pricing the log **Contract].** Suppose we have the assumptions of the Black-Scholes Model. Find the pricing formula for the European Style derivative whose payoff is given by function $\log(S_T)$, where S_T is the stock price in the BS model on date T.

Bonus Problem 3 [Pricing the Variance Contract]. Suppose we have the assumptions of the Black-Scholes Model. Find the pricing formula for the European Style derivative whose payoff is given by function $[\max(S_T - K), 0]^2$, where S_T is the stock price in the BS model on date T.