ECON 7020
Philip Shaw
Problem Set 6
Due date: May 6, 2022

Problems from McCandless and Wallace:
Chapter 3 Exercises:
3.3, 3.6, 3.7

Problems from McCandless and Wallace:
Chapter 9 Exercises:
9.1-9.6

Problem 1. Take a two-period OLG model with production. Agents maximize their discounted stream of utility over the two periods of their life. The budget constraint of the young is given by: $c_t^h(t) = w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1)$ and the budget constraint of the old is given by: $c_t^h(t+1) = w(t+1)\Delta_t^h(t+1) + r^l(t)l^h(t) + r^k(t+1)k^h(t+1)$. Suppose utility is given by: $u_t^h = c_t^h(t)[c_t^h(t+1)]^\beta$. Also assume that factor markets are prefectly competitive and the production function is given by $Y(t) = \gamma(t)L(t)^{1-\alpha}K(t)^\alpha$. In addition, let the population be constant so that $N(t) + N(t-1) = 1$ for all t and $\gamma(t+1) = (1+g)\gamma(t)$.

a. Derive the lifetime budget constraint (LBC) and state the no arbitrage condition. Why must the arbitrage condition hold in equilibrium?

b. Derive the individual savings function for an arbitrary agent h.

c. Define a perfect foresight competitive equilibrium.

d. Using the fact that $L(t) = N(t)\Delta_t^h(t) + N(t-1)\Delta_{t-1}^h(t)$ solve for the steady state capital stock.

e. Assuming $g > 0$ find the steady state growth rate of the capital stock. What is the growth rate of output?

Problem 1. Take a two-period OLG model with production. Agents maximize their discounted stream of utility over the two periods of their life. The budget constraint of the young is given by: $c_t^h(t) = w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1)$ and the budget constraint of the old is given by: $c_t^h(t+1) = w(t+1)\Delta_t^h(t+1) + r^l(t)l^h(t) + r^k(t+1)k^h(t+1)$. Suppose utility is given by: $u_t^h = c_t^h(t)[c_t^h(t+1)]^\beta$. Also assume that factor markets are prefectly competitive and the production function is given by $Y(t) = \gamma(t)L(t)^{1-\alpha}K(t)^\alpha$. In addition, let the population be constant so that $N(t)+N(t-1) = 1$ for all t and $\gamma(t+1) = (1+g)\gamma(t)$.

a. Derive the lifetime budget constraint (LBC) and state the no arbitrage condition. Why must the arbitrage condition hold in equilibrium?

$\because\ c_t^h(t) = w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1)$

$c_t^h(t+1) = w(t+1)\Delta_t^h(t+1) + r^l(t)l^h(t) + r^k(t+1)k^h(t+1)$

$\therefore\ \dfrac{c_t^h(t+1)}{r(t)} = \dfrac{w(t+1)\Delta_t^h(t+1)}{r(t)} + l^h(t) + \dfrac{r^k(t+1)}{r(t)}k^h(t+1)$

$\therefore$ LBC $:=\ c_t^h(t) + \dfrac{c_t^h(t+1)}{r(t)}$

$= w(t)\Delta_t^h(t) - \cancel{l^h(t)} - k^h(t+1) + \dfrac{w(t+1)\Delta_t^h(t+1)}{r(t)} + \cancel{l^h(t)} + \dfrac{r^k(t+1)}{r(t)}k^h(t+1)$

$= w(t)\Delta_t^h(t) + \dfrac{w(t+1)\Delta_t^h(t+1)}{r(t)} + \dfrac{r^k(t+1) - r(t)}{r(t)}K_t^h(t+1).$

The no arbitrage is for \$ Mkt, i.e., no arbitrage between loan and capital stock.

$\therefore\ r^k(t+1) = r^l(t)$

Reason: if $r^k(t+1) < r^l(t) \Rightarrow$ borrow stock in the rate of $r^k(t+1)$ and put the \$ into loan Mkt, to earn $r^l(t) \Rightarrow$ no risk gain,

if $r^k(t+1) > r^l(t) \Rightarrow$ borrow from the loan mkt, and use the \$ to stock $\Rightarrow$ to earn $r^k(t+1) \Rightarrow$ positive return w/o taking risk.

$\therefore$ in equilibrium $\quad r^k(t+1) = r^l(t)$

b. Derive the individual savings function for an arbitrary agent h.

$\because U_t^h = C_t^h(t)\left[C_t^h(t+1)\right]^{\beta}$

$\therefore r(t) = MRS = \dfrac{(C_t^h(t+1))^{\beta}}{C_t^h(t)\,\beta\, C_t^h(t+1)^{\beta-1}} = \dfrac{C_t^h(t+1)}{\beta C_t^h(t)}$

$$C_{t+1}^h(t+1) = w(t+1)\Delta_t^h(t+1) + r^l(t)\underbrace{l^h(t)}_{w(t)\Delta_t^h(t) - C_t^h(t) - k^h(t+1)} + r^k(t+1)k^h(t+1)$$

$$\therefore r(t) = \frac{w(t+1)\Delta_t^h(t+1) + r^l(t)\left[w(t)\Delta_t^h(t) - C_t^h(t) - k^h(t+1)\right] + r^k(t+1)k^h(t+1)}{\beta C_t^h(t)}$$

$$\underline{\beta r(t)C_t^h(t)} = w(t+1)\Delta_t^h(t+1) + r^l(t)w(t)\Delta_t^h(t) \underline{- r^l(t)C_t^h(t)} - r^l(t)k^h(t+1) + r^k(t+1)k^h(t+1)$$

In general $r^l(t) = r(t)$

$$\Rightarrow (\beta+1)r(t)C_t^h(t) = w(t+1)\Delta_t^h(t+1) + r(t)w(t)\Delta_t^h(t) - r(t)k^h(t+1) + r^k(t+1)k^h(t+1)$$

$$= w(t+1)\Delta^h(t+1) + r(t)w(t)\Delta_t^h(t) + \left(r^k(t+1) - r(t)\right)k^h(t+1)$$

$$\overset{\text{by (a)}}{=} w(t+1)\Delta_t^h(t+1) + r(t)w(t)\Delta_t^h(t)$$

$$\Rightarrow C_t^h(t) = \frac{\cancel{r(t)}\,w(t)\Delta_t^h(t)}{(1+\beta)\cancel{r(t)}} + \frac{w(t+1)\Delta_t^h(t+1)}{(1+\beta)r(t)}$$

The general saving for individual is $w(t)\Delta(t) - C_t^h(t)$

$$\therefore S_t^h(t) = w(t)\Delta(t) - \frac{w(t)\Delta_t^h(t)}{(1+\beta)} + \frac{w(t+1)\Delta_t^h(t+1)}{(1+\beta)r(t)}$$

$$= \frac{\beta w(t) \Delta_t^h(t)}{1+\beta} + \frac{w(t+1)\Delta_t^h(t+1)}{(1+\beta) r(t)}$$

c. Define a perfect foresight competitive equilibrium.

A CE is an equilibrium that solves two mkts, goods and capital mkt in this question

in goods: consumption earn in capital:

① goods Mkt:

$$\sum_{h=1}^{N(t)} C_t^h(t) + \sum_{h=1}^{N(t)} C_t^h(t+1) = \sum_{h=1}^{N(t)} W(t)\Delta(t) + \sum_{h=1}^{N(t)} W(t+1)\Delta(t+1)$$

② Capital Mkt.

$$S_t(r(t)) = K(t+1)$$

$$\sum_{h=1}^{N(t)} l^h(t) + K^h(t+1) = 0$$

$r(t) = r^K(t+1)$ as in (b)

$$\text{wage} = W(t) = \frac{\partial Y(t)}{\partial L(t)}$$

$$r(t) = r^K(t+1) = \frac{\partial Y(t)}{K(t)}$$

d. Using the fact that $L(t) = N(t)\Delta_t^h(t) + N(t-1)\Delta_{t-1}^h(t)$ solve for the steady state capital stock.

$\because L(t) = N(t)\Delta_t^h(t) + N(t-1)\Delta_{t-1}^h(t)$

From (b)

$\because S_t^h(t) = \dfrac{\beta w(t)\Delta_t^h(t)}{1+\beta} + \dfrac{w(t+1)\Delta_t^h(t+1)}{(1+\beta)r(t)}$

$\therefore S(t) = \dfrac{\beta w(t) L_t(t)}{1+\beta} + \dfrac{w(t+1) L_t(t+1)}{(1+\beta)r(t)}$

$\because w(t) = \dfrac{\partial Y(t)}{\partial L(t)} = \gamma(t)(1-\alpha)L(t)^{-\alpha}K(t)^{\alpha}$

$\therefore w(t+1) = \gamma(t+1)(1-\alpha)L(t+1)^{-\alpha}K(t+1)^{\alpha}$

$\therefore S(t) = \dfrac{\gamma(t)(1-\alpha)L(t)^{-\alpha}K(t)^{\alpha}}{(1+\beta)} + \dfrac{\gamma(t+1)(1-\alpha)L(t+1)^{-\alpha}K(t+1)^{\alpha}}{(1+\beta)r(t)}$

$= K(t+1)$

$(1+\beta)r(t)K(t+1) = r(t)\gamma(t)(1-\alpha)L^{-\alpha}(t)K^{\alpha}(t) + \gamma(t+1)(1-\alpha)L^{-\alpha}(t+1)K^{\alpha}(t+1)$

at steady state

$K(t+1) = K(t) = \bar{K}$

$\therefore (1+\beta)r(t)\cancel{\bar{K}} = \left[r(t)\gamma(t)(1-\alpha)L^{-\alpha}(t) + \gamma(t+1)(1-\alpha)L^{-\alpha}(t+1)\right]\bar{K}^{\alpha-1}$

$$\therefore \quad \bar{K}^{\alpha-1} = \underbrace{\frac{(1+\beta)\, r(t)}{(1-\alpha)\left[r(t)\gamma(t)L^{-\alpha}(t) + \gamma(t+1)L^{-\alpha}(t+1)\right]}}_{= M}$$

$$\therefore \quad \bar{K}^{*} = M^{\frac{1}{\alpha-1}}$$

Problem 1. Take a two-period OLG model with production. Agents maximize their discounted stream of utility over the two periods of their life. The budget constraint of the young is given by: $c_t^h(t) = w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1)$ and the budget constraint of the old is given by: $c_t^h(t+1) = w(t+1)\Delta_t^h(t+1) + r^l(t)l^h(t) + r^k(t+1)k^h(t+1)$. Suppose utility is given by: $u_t^h = c_t^h(t)[c_t^h(t+1)]^{\beta}$. Also assume that factor markets are prefectly competitive and the production function is given by $Y(t) = \gamma(t)L(t)^{1-\alpha}K(t)^{\alpha}$. In addition, let the population be constant so that $N(t)+N(t-1) = 1$ for all t and $\gamma(t+1) = (1+g)\gamma(t)$.

e. Assuming $g > 0$ find the steady state growth rate of the capital stock. What is the growth rate of output?

$$\bar{K}^{\alpha-1} = \frac{(1+\beta)\, r(t)}{(1-\alpha)\left[r(t)\gamma(t)L^{-\alpha}(t) + (1+g)\gamma(t)L^{-\alpha}(t+1)\right]}$$

$$= \frac{(1+\beta)\, r(t)}{(1-\alpha)\gamma(t)\left[r(t)L^{-\alpha}(t) + (1+g)L^{-\alpha}(t+1)\right]}$$

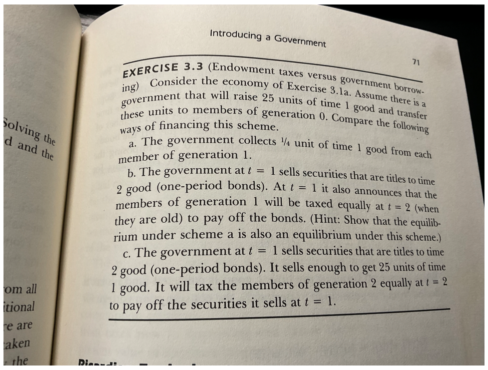
$\therefore$

$$r(t)L^{-\alpha}(t) + (1+g)L^{-\alpha}(t+1) = \frac{(1+\beta)\, r(t)}{(1-\alpha)\gamma(t)\bar{K}^{\alpha-1}}$$

$$\therefore\quad (1+g)L^{-\alpha}(t+1) = \frac{(1+\beta)r(t)}{(1-\alpha)\sigma(t)\bar{k}^{\alpha-1}} - r(t)L^{-\alpha}(t)$$

$$\therefore\quad 1+g = \underbrace{\frac{(1+\beta)r(t)L(t+1)}{(1-\alpha)\sigma(t)\bar{k}^{\alpha-1}} - r(t)\frac{L(t+1)}{L(t)}}_{=P}$$

$$\therefore\quad g = P-1.$$



Introducing a Government

71

EXERCISE 3.3 (Endowment taxes versus government borrowing) Consider the economy of Exercise 3.1a. Assume there is a government that will raise 25 units of time 1 good and transfer these units to members of generation 0. Compare the following ways of financing this scheme.

a. The government collects ¼ unit of time 1 good from each member of generation 1.

b. The government at $t = 1$ sells securities that are titles to time 2 good (one-period bonds). At $t = 1$ it also announces that the members of generation 1 will be taxed equally at $t = 2$ (when they are old) to pay off the bonds. (Hint: Show that the equilibrium under scheme a is also an equilibrium under this scheme.)

c. The government at $t = 1$ sells securities that are titles to time 2 good (one-period bonds). It sells enough to get 25 units of time 1 good. It will tax the members of generation 2 equally at $t = 2$ to pay off the securities it sells at $t = 1$.

Exercise 3.3

$N(t) = 100$, $\quad u_t^h = c_t^h(t)\, c_t^h(t+1)^{\beta}$ w/ $\beta = 1$ $\quad [w_t^h(t), w_t^h(t+1)] = [2, 1]$

Young: tax 1 unit. old: transfer 1 unit.

4 knows the scheme.

a. The govt collects $\frac{1}{4}$ unit of time 1 good from each member of generation

$$c_t^h(t) = \frac{w_t^h(t) - \tau(t)}{1+\beta} + \frac{w_t^h(t+1) + \tau(t+1)}{r(t)(1+\beta)}$$

$$= \frac{w_t^h(t) - \frac{1}{4}}{2} + \frac{w_t^h(t+1) + \frac{1}{4}}{2r(t)}$$

$$= \frac{2 - \frac{1}{4}}{2} + \frac{1 + \frac{1}{4}}{2r(t)}$$

$$= \frac{7}{8} + \frac{5}{8r(t)}$$

$$s_t^h(t) = 2 - \frac{1}{4} - \frac{7}{8} - \frac{5}{8r(t)} = 0$$

$$\Rightarrow \frac{7}{8} = \frac{5}{8r(t)}$$

$$\therefore r(t) = \frac{5}{7}$$

$$\Rightarrow c_t^h(t) = \frac{7}{8} + \frac{5}{8} \times \frac{7}{5} = \frac{7}{8} + \frac{7}{8} = \frac{7}{4}$$

$s_t^h(t) = 0$

b. The government at $t = 1$ sells securities that are titles to time 2 good (one-period bonds). At $t = 1$ it also announces that the members of generation 1 will be taxed equally at $t = 2$ (when they are old) to pay off the bonds. (Hint: Show that the equilibrium under scheme a is also an equilibrium under this scheme.)

Since

$$s_t^h(t) = \frac{w_t^h(t) - t - \beta B}{(1+\beta)} - \frac{w_t^h(t+1) + B - t}{(1+\beta) r(t)}$$

$\because$ $\beta = 1$ $w_t^h(t) = 2$ $w_t^h(t+1) = 1$

$$\therefore s_t^h(t) = \frac{2 - t - B}{2} - \frac{1 + B - t}{2 r(t)}$$

$N(t)\, s_t^h(t) = 0$

$$\therefore \frac{2 - t - B}{2} = \frac{1 + B - t}{2 r(t)}$$

$\therefore$ $2 + 2B - 2t = 4r(t) - 2r(t) \cdot t - 2r(t) B$

$1 + B - t = 2r(t) - r(t) \cdot t - r(t) \cdot B$

$$\therefore r(t) = \frac{1 + B - t}{2 - t - 1}$$

c. The government at $t = 1$ sells securities that are titles to time 2 good (one-period bonds). It sells enough to get 25 units of time 1 good. It will tax the members of generation 2 equally at $t = 2$ to pay off the securities it sells at $t = 1$.

$$c_t^h(t) = \frac{w_t^h(t) + \frac{1}{4}}{2} + \frac{w_t^h(t+1) - \frac{1}{4}}{2r(t)}$$

$$s_t^h(t) = w_t^h(t) - c_t^h(t)$$

$$= \frac{w_t^h(t) - \frac{1}{4}}{2} - \frac{w_t^h(t+1) - \frac{1}{4}}{2r(t)} = 0$$

$$\frac{w_t^h(t) - \frac{1}{4}}{2} = \frac{w_t^h(t+1) - \frac{1}{4}}{2r(t)}$$

$$\frac{\frac{7}{4}}{2} = \frac{\frac{3}{4}}{2r(t)}$$

$$\frac{7}{8} = \frac{3}{8r(t)}$$

$$24 = 56r(t)$$

$$r(t) = \frac{\cancel{24}\ 3}{\cancel{56}\ 7}$$

$$\therefore c_t^h(t) = \frac{\frac{9}{4}}{2} + \frac{\frac{3}{4}}{\frac{6}{7}}$$

$$= \frac{9}{8} + \frac{\cancel{3}}{4} \times \frac{7}{\cancel{6}\,2}$$

$$= \frac{9}{8} + \frac{7}{8} = \frac{16}{8} = 2.$$

50 is bo... path as was followed for 51 ... above occurs. Only 50 and 0 are stationary state borrowings in this economy.

In an economy in which all individuals are identical in preferences and in endowments, stationary state equilibria occur whenever the gross interest rate and the consumption pair are the same for every member of every generation (except the current old, for whom we may not know the consumption when young). This statement is true even if the economy is growing. In the next exercise, you are asked to find the stationary state equilibrium for a growing economy.

EXERCISE 3.6 Consider an economy where all individuals h of all generations t have the endowment [2, 1] and the utility function

$$u_t^h = c_t^h(t)[c_t^h(t+1)]^\beta$$

for $\beta = 1$. The population is growing with $N(1) = 100$ and $N(t) = 1.5N(t-1)$. Find the amount of government borrowing and the number of bonds that the government must issue and continually roll over to generate a stationary state equilibrium for this economy. The absolute number of bonds the government needs to issue each period will grow. In solving this exercise, find the stationary state gross interest rate and the consumption pair that everyone has.

Equivalence between Equilibria with Bonds and Tax-Transfer Schemes

In our economies with bonds, the government operates under a budget constraint. With no direct government expenditures,[1] this budget constraint is

$$\sum^{N(t)} t_t^h(t) + \sum^{N(t-1)} \ldots$$

Propositio... consumptio... ment's budg...

The go... at time t... bonds it... t old. It... roll ove... Assume... exactly... had be... either... is the s... two m... are fo... mem...

$$\therefore \quad s_t^h(t) = \frac{\beta w_t^h(t)}{1+\beta} - \frac{w_t^h(t+1)}{(1+\beta)r(t)}$$

$$= \frac{w_t^h(t)}{2} - \frac{w_t^h(t+1)}{2r(t)}$$

$$= 1 - \frac{1}{2r(t)}$$

$$\therefore \quad S(1) = N(t) \cdot s_t^h(t) = N(1)\left[1 - \frac{1}{2r(t)}\right]$$

$$= 100\left[1 - \frac{1}{2r(t)}\right]$$

$$= 100 - \frac{50}{r(t)}$$

$$S(2) = 150\left[1 - \frac{1}{2r(t)}\right] = 150 - \frac{75}{r(t)}$$

$$\therefore \quad S(1) = B(1) = \frac{B(2)}{r(t)}$$

$$\therefore \quad \left(100 - \frac{50}{r(t)}\right) = \frac{\left(150 - \frac{75}{r(t)}\right)}{r(t)}$$

$$\therefore \quad r(t) = \frac{3}{2}$$

$$\therefore \quad s_t^h(t) = \frac{2}{3}$$

$$\therefore \quad c_t^h(t) = \left[\frac{4}{3}, 2\right]$$

This … sions it illustrates … as being at point A (the after-tax point) and is trading to point C from there. … individual h whose deci- …

EXERCISE 3.7 Consider an economy with $N(t) = 100$, where all individuals h of all generations t have the utility function

$$u_t^h = c_t^h(t)c_t^h(t+1),$$

and the endowments of

$$\omega_t^h = \begin{cases} [2, 1], & \text{for } h = 1, 3, 5, \ldots, 99, \quad \text{and} \\ [1, 1], & \text{for } h = 2, 4, 6, \ldots, 100. \end{cases}$$

The government issues 30 one-period bonds at time 1 and taxes the young of time 2 to pay off these bonds. Find a tax-transfer scheme (without bonds) that gives the same equilibrium consumption allocation and the same gross interest rate as the equilibrium with bonds.

Reprise

$\frac{15^3}{20\beta}$ [illegible]

$$\frac{30}{100} = \frac{3}{10} = t_t(t)$$

$\therefore$ odd.

$(1+\beta)(w_t^h + \frac{3}{10})$

$$c_t^h(t) = \frac{w_t^h(t) + \frac{3}{10}}{(1+\beta)} + \frac{w_t^h(t+1) - \frac{3}{10} r(t)}{(1+\beta) r(t)}$$

$$\therefore s_t^h(t) = \frac{\beta(w_t^h(t) + \frac{3}{10})}{1+\beta} - \frac{w_t^h(t+1) - \frac{3}{10} r(t)}{(1+\beta) r(t)}$$

$$= \frac{\beta(\frac{23}{10})}{1+\beta} - \frac{\frac{7}{10} r(t)}{(1+\beta) r(t)}$$

$$S_{odd}(t) = \frac{\frac{23}{10}\beta r(t) - \frac{7}{10} r(t)}{(1+\beta) r(t)}$$

$$S_{t\,even}^h = \frac{\beta(\frac{13}{10})}{1+\beta} - \frac{\frac{7}{10} r(t)}{(1+\beta) r(t)}$$

$$= \frac{\frac{13}{10}\beta r(t) - \frac{7}{10} r(t)}{(1+\beta) r(t)}$$

$$S(t) = 50\left[\frac{\frac{23}{10}\beta r(t) - \frac{7}{10}}{(1+\beta) r(t)}\right] + 50\left[\frac{\frac{13}{10}\beta - \frac{7}{10} r(t)}{(1+\beta) r(t)}\right]$$

?

$$= \frac{115\beta r(t) - 35 + 85\beta r(t) - 35}{(1+\beta) r(t)}$$

$$\therefore 2\cancel{0}0\beta r(t) = 7\cancel{0} \qquad r(t) = \frac{7}{20\beta}$$

graph. The shape of both the production function and the marginal product of capital curve in this figure are characteristic of production functions with diminishing marginal product of capital.

EXERCISE 9.1 Consider an economy with the Cobb-Douglas production function given in Equation (9.1), where $\gamma(t) = 6$, $L(t) = 9$, $K(t) = 15$, and $a = .4$. What is the output in this economy? Suppose that both capital and labor increase by one third, to $L(t) = 12$ and $K(t) = 20$. What is the new level of output? What is the percentage increase of this level of output over the initial level of output?

Goods produced in one period can only be carried over to the next period as capital. This assumption means that the feasibility condition

Eqn1: $Y(t) = \underbrace{\gamma(t)}_{6} \underbrace{L(t)}_{9}^{1-\alpha} \underbrace{K(t)}_{15}^{\alpha}$

$Y(t) = 6 \cdot 9^{0.6} \cdot 15^{0.4} = 66.24$

$Y'(t) = 6 \cdot 12^{0.6} \cdot 20^{0.4} = 88.32$

$\dfrac{Y'(t) - Y(t)}{Y(t)} = \dfrac{88.32 - 66.24}{66.24}$

$= 33.3333\%$

...al of capital equals the rental times $K(t)$, or

$$\text{rental}(t)K(t) = a\gamma(t)\left[\frac{K(t)}{L(t)}\right]^{a-1} K(t)$$
$$= aY(t).$$

The total payment to capital is a of total output.

Notice that the firms gain no profits from producing goods. All of the output of the firms goes to pay the wage and the rental bills. The wage bill is $(1-a)Y(t)$ and the rental bill is $aY(t)$; adding these together totals $Y(t)$.

EXERCISE 9.2 For the economy in Exercise 9.1, what are the wage and rental rates before and after the increase in labor and capital? What are the total wage bills and total return on capital both before and after the increase in capital and labor?

The Individual Choice Decision under Perfect Foresight

Individual h of generation t wishes to maximize utility subject to the budget constraints...

$$Y(t) = \gamma(t) L(t)^{1-\alpha} K(t)^{\alpha}$$

before:
$$\frac{\partial Y(t)}{\partial L(t)} = (1-\alpha)\,\gamma(t)\, L(t)^{-\alpha} K(t)^{\alpha}$$
$$= 0.6 \cdot 6 \cdot 9^{-0.4}\ 15^{0.4}$$
$$= 4.42 = \text{wage}(t)$$

$$\frac{\partial Y(t)}{\partial K(t)} = \alpha\,\gamma(t)\, L^{1-\alpha}(t)\, K(t)^{\alpha-1}$$
$$= 0.4 \cdot 6 \cdot 9^{0.6}\ 15^{0.4}$$
$$= 1.77 = \text{rental}(t)$$

∵ The production function is Homogeneous degree 0.

Then, if both of the input increase the same range

The marginal rate doesn't change.

∴ wage(t) = 4.42 rental(t) = 1.77

Total wage bill before:

$w(t) \cdot L(t) = 4.42 \times 9 = 37.8$

Total rental bill before:

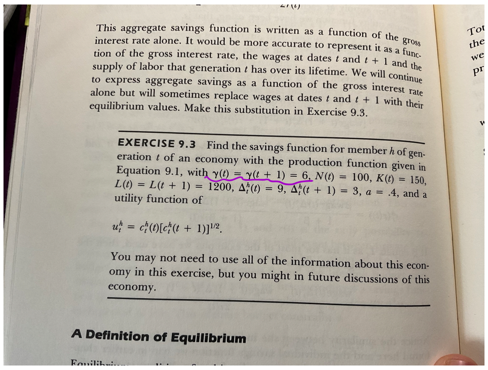
$\text{rental}(t) \cdot K(t) = 1.77 \times 15 = 26.55$

Total wage bill after:

$$w(t)\cdot L(t) = 4.42 \cdot 12 = 53.04$$

Total rental bill after:

$$K(t)\ \text{rental}(t) = 1.77 \times 20 = 35.4$$



This aggregate savings function is written as a function of the gross interest rate alone. It would be more accurate to represent it as a function of the gross interest rate, the wages at dates t and $t+1$ and the supply of labor that generation t has over its lifetime. We will continue to express aggregate savings as a function of the gross interest rate alone but will sometimes replace wages at dates t and $t+1$ with their equilibrium values. Make this substitution in Exercise 9.3.

EXERCISE 9.3 Find the savings function for member h of generation t of an economy with the production function given in Equation 9.1, with $\gamma(t) = \gamma(t+1) = 6$, $N(t) = 100$, $K(t) = 150$, $L(t) = L(t+1) = 1200$, $\Delta_t^h(t) = 9$, $\Delta_t^h(t+1) = 3$, $a = .4$, and a utility function of

$$u_t^h = c_t^h(t)[c_t^h(t+1)]^{1/2}.$$

You may not need to use all of the information about this economy in this exercise, but you might in future discussions of this economy.

A Definition of Equilibrium

Eq. 9.1

$$Y(t) = \gamma(t) L(t)^{1-a} K(t)^{a}$$

$$\therefore\ c_t^h(t) = \frac{w_t^h(t)}{1+\beta} + \frac{w_t^h(t+1)}{(1+\beta)r(t)}$$

$$s_t^h(t) = \frac{\beta w_t^h(t)}{1+\beta} + \frac{w_t^h(t+1)}{(1+\beta)r(t)}$$

if only labor move as the resources

$$\Rightarrow s_t^h(t) = \frac{\beta\cdot \text{wage}(t)\Delta(t)}{1+\beta} + \frac{\text{wage}(t+1)\Delta(t+1)}{(1+\beta)r(t)}$$

$$= \frac{\frac{1}{2}\,\text{wage}(t)\cdot 9}{1+\frac{1}{2}} + \frac{\text{wage}(t+1)\cdot 3}{\frac{3}{2}\cdot r(t)}$$

$$\because\ \text{wage}(t) = MPL_t = 1.57$$

$$\text{wage}(t+1) = 0.211 \cdot K(t+1)^{0.4}$$

$$\therefore\ s_t^h(t) = 4.71 + \frac{0.422\cdot K(t+1)^{0.4}}{r(t)}$$

B/C in equi:

$r(t) = \text{rental}(t+1)$

$\therefore r(t) = MP_K = a \cdot \sigma(t+1) \left[\frac{K(t+1)}{L(t+1)} \right]^{a-1}$

$= 168.94\, K(t+1)^{-0.6}$

$\therefore S_t^h(t) = 4.71 + \frac{K(t+1)^{0.4}}{168.94\, K(t+1)^{0.6}}$

$= 4.71 + \frac{0.0025}{K(t+1)^{0.2}}$

EXERCISE 9.4 For the economy in Exercise 9.3, find the stationary state capital stock, output, and consumption allocations.

$$K(t+1) = \frac{\dfrac{\beta(1-\tau)\,\sigma(t)\,L(t)^{-a} K(t)^{a}\, \Delta_t^{h}(t+1) N(t+1)}{1+\beta}}{1 + \dfrac{1-a}{(1+\beta)a} \dfrac{\Delta_t^h(t) N(t)}{L(t+1)}}$$

$= M$

$\therefore K(t+1) = M\, K(t)^{a}$

at the steady state $K = K(t+1) = K(t)$

$\therefore K = M^{\frac{1}{1-a}}$

$\therefore K^* = 694$

$Y^* = \sigma^* L^{*1-a} K^{*a}$

$= 5784$

$\because C(t) + K(t+1) = Y(t)$

∴ $C^t = Y^t - K^t = 5089.84$

∵ $\sum_{h=1}^{N(t)} C_t^h(t) = nage(t) L_t - \sum_{h=1}^{N(t)} K^h(t+1)$

∴ $C_t(t) = MPL(t) \cdot L_t(t) - K^*$

$= 1908.7$

∴ $C_{t-1}(t) = C_{t-1}(t) = 3181.14$

∴ $C_t^h(t) = \frac{1908.7}{100} = 19.087$

$C_t^h(t+1) = \frac{3184.14}{100} = 31.8$

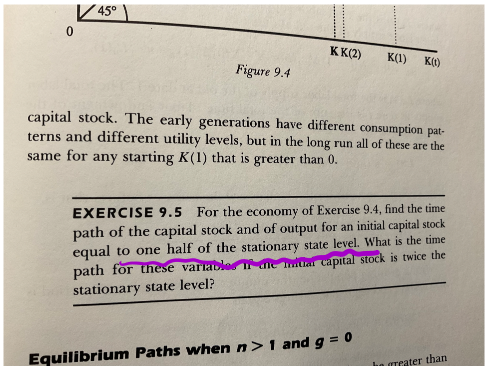


Figure 9.4

capital stock. The early generations have different consumption patterns and different utility levels, but in the long run all of these are the same for any starting $K(1)$ that is greater than 0.

EXERCISE 9.5 For the economy of Exercise 9.4, find the time path of the capital stock and of output for an initial capital stock equal to one half of the stationary state level. What is the time path for these variables if the initial capital stock is twice the stationary state level?

Equilibrium Paths when $n > 1$ and $g = 0$

$K(1) = \frac{1}{2} K^* = 347$

$$K(t+1) = \frac{\frac{1}{2}(1-a)\, r(t) \left(\frac{1}{L(t)}\right)^a \frac{L_t(t)}{1+B}}{1 + \frac{1-a}{(1+B)a} \left[\frac{L_t(t+1)}{L(t+1)}\right]} \cdot K(t)^a$$

$= 50.7$

$\therefore k(2) = 50.7 \times 347^{0.4} = 520.2$,

$k(3) = 50.7 \times 520.2^{0.4} = 621.5$

$k(4) = 50.7 \times 621.5^{0.4} = 664.3$

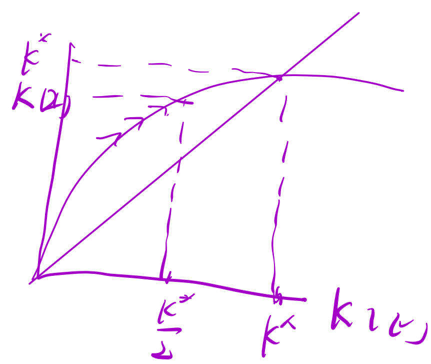
$k(5) = 50.7 \times 664.3^{0.4} = 682.3$

$k(6) = 50.7 \times 682.3^{0.4} = 689.6$

$k(7) = 50.7 \times 689.6^{0.4} = 692.5$

$k(8) = 50.7 \times 692.5^{0.4} = 693.7$

$k(9) = 50.7 \times 693.7^{0.4} = 694.2$.



9.6 if $k(1) = \frac{1}{2}k^*$, the seq will converge.

$\therefore k(1) = 2k^* = 1388$.

$k(t+1) = M \cdot k(t)^a$

$\therefore k(2) = 50.7 \times 1388^{0.4} = 916.1$

$k(3) = 775.9$

$\vdots$

$k(9) = 694.8$.

$\therefore$