Homework #3 Stochastic Calculus Solutions

Ito's Formula

Due Date: November 27, 2022

Important: Note the above date is a Sunday!

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Note: Problem 1, 3, and 4 can be done with your current knowledge of Nov 17 Lecture. Problem 2 & 5 will need the Ito's formula in the upcoming Nov 22 Lecture.

Problem 1. Specialize the derivation of Ito's formula in the lecture notes to function $f(x) = x^3$ to show that

$$W(t)^3 = 3\int_0^t W(s)^2 dW(s) + 3\int_0^t W(s)ds.$$

Problem 2. Let $S(t) = S(0)\exp\left\{\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t\right\}$ be a geometric Brownian motion. Let p be a constant. Compute $d(S(t)^p)$.

Hint: The expresion for $S(t)$ can be expressed in differential form. Then you can compute $d\left(S(t)^p\right)$ using Ito's formula for function $f(x) = x^p$.

Problem 3.

(i) Compute $d(W(t)^4)$, and the write $W(t)^4$ as the sum of an ordinary integral and an Ito integral

(ii) Take expectation of both sides of the formula you obtained in (i), use the fact that $\mathbb{E}[W(t)^2] = t$, and derivative the formula $\mathbb{E}W(t)^4 = 3t^2$.

(iii) Use the method of (i) and (ii) to derivative a formula for $\mathbb{E}W(t)^6$.

Problem 4. Let $X_t = \exp\left(ct + \alpha W(t)\right)$. Show that X_t satisfies $dX_t = (c + \frac{1}{2}\alpha^2)X_t dt + \alpha X_t dW(t)$.

Hint: Let $f(t, x) = \exp(ct + \alpha x)$, then $X_t = f(t, W_t)$. Use Ito's formula.

Problem 5. Suppose that $S(t)$ satisfy

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t).$$

Set $Y(t) = \log(S(t))$. What equation does $dY(t)$ satisfy?

Hint: This is just asking you to express $dY(t)$ using dt and dW_t by applying Ito's formula.