Homework 0

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QF8915 - Stochastic Calculus

Due on Nov 8, 2022

Problem 1

(1) Let $g(X) = Y = e^X$. From the question, X is a r.v distributed as $\mathcal{N}(\mu, \sigma^2)$ with $\mu = 0.06, \sigma = 0.25$. And $g^{-1}(Y) = \ln Y$

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))|\frac{dg^{-1}(y)}{dy}| \\ &= f_X(\ln y)|\frac{d\ln y}{dy}| \\ &= [\frac{4}{\sqrt{2\pi}}e^{-8(\ln y - 0.06)^2}]\frac{1}{y} \\ &= \frac{4}{y\sqrt{2\pi}}e^{-8(\ln y - 0.06)^2} \end{aligned}$$

(2)

$$\begin{aligned} EY &= \int_0^\infty y f_Y(y) dy \\ &= \int_0^\infty y \cdot \frac{4}{y\sqrt{2\pi}}e^{-8(\ln y - 0.06)^2} dy \\ &= \frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-8(\ln y - 0.06)^2} dy \\ &= e^{\frac{73}{800}} \end{aligned}$$

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(3)

$$
\begin{aligned}
EY &= E\exp(X) \\
&= \int_{-\infty}^{\infty} e^x f_X(x)dx \\
&= \int_{-\infty}^{\infty} e^x \frac{4}{\sqrt{2\pi}} e^{-8(x-0.06)^2} dx \\
&= \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x-8(x-0.06)^2} dx \\
&= e^{\frac{73}{800}}
\end{aligned}
$$

Note: I used online calculator[1] to compute the last step with respect to integral. like in (3), the integral result is $\frac{e^{\frac{73}{800}}\sqrt{\pi}}{2^{\frac{2}{3}}}$, then we multiply $\frac{4}{\sqrt{2\pi}}$ to get our result.

It's always easier to compute integral of x or x^2 instead of ln, thus, as in the comment, the second way is better in term of computation[2]

Problem 2

(1) Since $\mathcal{F}$ is σ-algebra of Ω, so the sets in $\mathcal{F}$ is as below(the num is 2^5):

$$
\begin{aligned}
\{\varnothing, \Omega, \{a\}\,\{b\}, \{c\}, \{d\}, \{e\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{c,e\}, \{d,e\} \\
\{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}, \{b,c,d\}, \{b,c,e\}, \{b,d,e\}, \{c,d,e\}, \\
\{a,b,c,d\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,c,d,e\}, \{b,c,d,e\}\}
\end{aligned}
$$

(2) Since x is a r.v, so the sets in $\sigma(x)$ is:

$$\{\varnothing, \Omega, \{a,b,c\}, \{d,e\}\}$$

(3) To derive $E(Y|X)$, let V be conditional expectation $E(Y|X)$. Since it's on σ_x-algebra, so we can assume $\alpha = V(a) = V(b) = V(c)$, and $\beta = V(d) = V(e)$.

By partial averaging, we can easily get:

$$E(V \cdot \mathcal{I}_A) = E(Y \cdot \mathcal{I}_A) \qquad A \in \sigma(X)$$

Therefore:

$$V(a)P(a) + V(b)P(b) + V(c)P(c) = Y(a)P(a) + Y(b)P(b) + Y(c)P(c)$$

$$\alpha(P(a) + P(b) + P(c)) = Y(a)P(a) + Y(b)P(b) + Y(c)P(c)$$

[1] https://www.integral-calculator.com/

[2] for (2) and (3), it can be computed in any online integral calculator. It's not wiseful to compute by hand.

The α will be:

$$\begin{aligned}\alpha &= \frac{Y(a)P(a) + Y(b)P(b) + Y(c)P(c)}{P(a) + P(b) + P(c)} \\ &= \frac{P(a) - 2P(b) + P(c)}{\frac{1}{6} + \frac{1}{6} + \frac{1}{4}} \\ &= \frac{\frac{1}{12}}{\frac{7}{12}} \\ &= \frac{1}{7}\end{aligned}$$

Now, we derive β:

$$V(d)P(d) + V(e)P(e) = Y(d)P(d) + Y(e)P(e)$$

$$\beta(P(d) + P(e)) = -2P(d) - 2P(e)$$

$$\begin{aligned}\beta &= \frac{-2P(d) - 2P(e)}{P(d) + P(e)} \\ &= \frac{-\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} \\ &= \frac{-\frac{5}{6}}{\frac{5}{12}} \\ &= -2\end{aligned}$$

(4) $Y^2(a) = 1, Y^2(b) = 4, Y^2(c) = 1, Y^2(d) = 4, Y^2(e) = 1$. Let $V = E(Y^2|X)$, Same as (3) by partial averaging property, we can get $V(a) = V(b) = V(c) = \alpha$, $V(d) = V(e) = \beta$. Now, begin deriving α:

$$\alpha(P(a) + P(b) + P(c)) = Y^2(a)P(a) + Y^2(b)P(b) + Y^2(c)P(c)$$

$$\begin{aligned}\alpha &= \frac{P(a) + 4P(b) + P(c)}{P(a) + P(b) + P(c)} \\ &= \frac{\frac{1}{6} + \frac{1}{4} + \frac{4}{6}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{4}} \\ &= \frac{13}{7}\end{aligned}$$

Now, begin deriving β:

$$V(d)P(d) + V(e)P(e) = Y^2(d)P(d) + Y^2(e)P(e)$$

$$\begin{aligned}\beta &= \frac{4(P(d) + P(e))}{P(d) + P(e)} \\ &= 4\end{aligned}$$