Session 6

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2.5. Ex. Find the normal and general form of the equations of the plane that : (i) contains $p=(6,0,1)$ (ii) has a normal vector $\vec{n}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

$$ax+by+cz=d$$

$$\vec{n}\cdot(\vec{x}-\vec{p})=0.$$

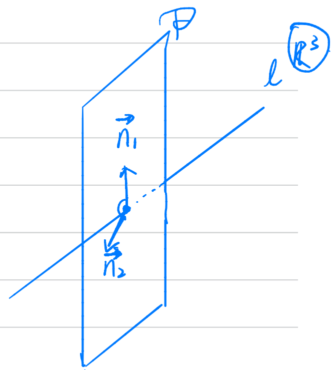
(1) Normal $\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\left[\vec{x}-\begin{bmatrix}6\\0\\1\end{bmatrix}\right]=0$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\underbrace{\vec{x}}_{\begin{bmatrix}x\\y\\z\end{bmatrix}}=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}6\\0\\1\end{bmatrix}$$

$$x+2y+3z = 6+2+3=9$$

$$x+2y+3z=9$$

(2) general form.



2.6. Summary

		Normal form	General form	Vector form
$\mathbb{R}^2$	l	$\vec{n}\cdot(\vec{x}-\vec{p})=0$ $\vec{n}\cdot\vec{x}=\vec{n}\cdot\vec{p}$	$ax+by=c$	$\vec{x}=\vec{p}+t\vec{d}$, $t\in\mathbb{R}^1$
$\mathbb{R}^3$	l	$\vec{n}_1\cdot\vec{x}=\vec{n}_1\cdot\vec{p}$ $\vec{n}_2\cdot\vec{x}=\vec{n}_2\cdot\vec{p}$	$a_1x+b_1y+c_1z=d_1$ $a_2x+b_2y+c_2z=d_2$	$\vec{x}=\vec{p}\cdot t\vec{d}$
	$\mathcal{P}$	$\vec{n}\cdot\vec{x}=\vec{n}\cdot\vec{p}$	$ax+by+cz=d$	$\vec{x}=\vec{p}+s\vec{u}+t\vec{v}$

2.7. Def (vector space). Let V be a set on which two operations "+" "c·" have been defined. If $\vec{u}, \vec{v} \in V$, the sum is "$u+v$", scalar multiplication is "cu". If the following hold for $\vec{u}, \vec{v}, \vec{w} \in V$, the V is called a vector space

(i) $\vec{u}+\vec{v} \in V$

(ii) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(iii) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(iv) $\exists\ 0 \in V \ni \vec{u}+0 = u$

(v) $\forall \vec{u} \in V,\ \exists -\vec{u} \ni \vec{u}+(-\vec{u}) = 0$

(vi) $c\vec{u} \in V$

(vii) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

(viii) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

(ix) $c(d\vec{u}) = (cd)\vec{u}$

(x) $1\vec{u} = \vec{u}$

2.8. Def (span). $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ set of vectors. If $S \subset V$, then the set of all L.C. of $\vec{v}_1 \cdots \vec{v}_k$ is called the span of $\vec{v}_1 \cdots \vec{v}_k$ denoted as $\boxed{\text{span}(\vec{v}_1 \cdots \vec{v}_k)}$. If $V = \text{span}\{S\}$ then S is called a spanning set for V, V is said to be spanned by S.

$\vec{v}_1 = \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k\times 1}$

$\vec{v}_2 = \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vdots$

$\vec{v}_k = \vec{e}_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

$V = \mathbb{R}^k$

2.9. (Linearly independent) $\{\vec{v}_1 \cdots \vec{v}_k\} \subset V$ is Linearly dependent if $\exists\, c_1 \cdots c_k \in \mathbb{R}$ at least one of which is not 0 $\ni$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_k\vec{v}_k = \vec{0}$$

$c_1 = c_2 = \cdots = c_k = 0 \Rightarrow$ Linear IND

If $\exists$ at one, say c_i's $(i \in \mathbb{N}) \Rightarrow$ Dependent

example $c_{10} \neq 0$

$$-(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_9\vec{v}_9 + c_{11}\vec{v}_{11} + \cdots c_k\vec{v}_k) = c_{10}\vec{v}_{10}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2.10. Theorem $\{v_1 \cdots v_k\} \subset V$ is linearly dependent iff. at least one of $\{\vec{v}_1 \cdots \vec{v}_k\}$ can be expressed as the LC of the others.

$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$c_1\vec{v}_1 + c_2\vec{v}_2 = 0$

$c_1 = c_2 =$

§.3. Systems of linear equations.

3.1. Def. $a x_1 + a x_2 + \cdots a_n x_n = b$.

3.2. ex Solve the system

$$\begin{cases} x - y - z = 2 \\ y + 3z = 5 \\ 5z = 10 \end{cases}$$

$x = 2 + (-1) + 2 = 3$

$y = 5 - 6 = -1$

$z = 2$

Back substitution.

3.3. ex. Solve

$$
\begin{aligned}
x - y - z &= 2 \quad ①\\
3x - 3y + 2z &= 16 \quad ②\\
2x - y + z &= 9 \quad ③
\end{aligned}
$$

Step 1.
$$
\underbrace{\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2\\ 3 & -3 & 2 & 16\\ 2 & -1 & 1 & 9 \end{array}\right]}_{\text{augmented matrix}} \xrightarrow{\neq 0} \cancel{HMC}
$$

$$
\left[\begin{array}{ccc} 1 & -1 & -1\\ 3 & -3 & 2\\ 2 & -1 & 1 \end{array}\right]
$$

matrix

Step 2. subtraction

$$
\begin{cases}
x - y - z = 2 & ①\\
\cancel{3x} - \cancel{3y} + 2z - 3(\cancel{x} - \cancel{y} - z) = 16 - 3\times 2 & \\
5z = 10 & ②'\\
y + 3z = 5 & ③'
\end{cases}
$$

Step 3. Interchange ②' ③'

$$
\begin{cases}
x - y - z = 2\\
y + 3z = 5\\
5z = 10
\end{cases}
$$

3.4. Matrices & Echelon form

If a matrix satisfies

(i) Any rows of all 0's are at the bottom

(ii)
$$
\left[\begin{array}{cccccc} 0 & 1 & * & * & * & *\\ \cancel{0} & & 1 & & & \\ \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Step 1

②−①×3 } remove x

③−①×2

$$
\begin{cases}
x - y - z = 2 & ①\\
5z = 10 & ②'\\
y + 3z = 5 & ③'
\end{cases}
$$

Step 2.

$$
\begin{cases}
x - y - z = 2\\
y + 3z = 5\\
5z = 10
\end{cases}
$$

$$
\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2\\ 3 & -3 & 2 & 16\\ 2 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \times(-3)\\ \times(-2) \end{array}
$$

$$
\xrightarrow[③-①\times 2]{②-①\times 3}
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2\\ 0 & 0 & 5 & 10\\ 0 & 1 & 3 & 5 \end{array}\right]
$$

$$
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2\\ 0 & 1 & 3 & 5\\ 0 & 0 & 5 & 10 \end{array}\right] /5
$$

$$
\Downarrow
$$

$$
\left[\begin{array}{ccc|c} \triangle & -1 & 1 & 2\\ 0 & \triangle & 3 & 5\\ 0 & 0 & \triangle & 2 \end{array}\right]
$$

Echelon Form $\Leftrightarrow$ Gaussian Elimin.

① 0-row at bottom
② leading ones (1st non-zero one)
③ lower leading 1 is to the right of previous leading 1's

Reduced Form

④ all the elements above or below leading 1 are zero

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{(Row) E.F.}$$

$\Downarrow$

$$\begin{array}{l} R_1 - R_3 \\ R_2 - 3 \times R_3 \\ \\ \end{array} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{Reduced E.F.}$$

$\Leftrightarrow$ Gauss-Jordan Eliminatio

$\Downarrow$

$$\begin{array}{l} R_1 + 3 \times R_2 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row Echelon Elimination

3.5. ex. Reduce $A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$

to the $\boxed{\text{echelon form}}$

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix} \overset{\substack{R_2-2R_1 \\ R_3-2R_1}}{=\!=} \underset{R_1+R_4}{} \begin{bmatrix} 1 & 2 & -4 & 4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\overset{R_2-R_3}{=\!=} \begin{bmatrix} 1 & 2 & -4 & 4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix} \overset{R_4+3R_2}{=\!=} \begin{bmatrix} 1 & 2 & -4 & 4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

$$\overset{R_3/8}{=\!=} \begin{bmatrix} 1 & 2 & -4 & 4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix} \overset{R_4-29\times R_3}{=\!=} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

$$\overset{R_4/24}{=\!=} \underset{R_2\times(-1)}{} \begin{bmatrix} 1 & 2 & -4 & -4 & \overset{0}{\cancel{5}} \\ 0 & 1 & -10 & -9 & \overset{0}{\cancel{5}} \\ 0 & 0 & 1 & 1 & \overset{0}{\cancel{-1}} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{R_3+R_4}{\overset{R_1-5R_4,\ R_2-5R_4}{=}} \begin{bmatrix} 1 & 2 & -4 & \boxed{4} & \boxed{0} \\ 0 & 1 & -10 & \boxed{-9} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{R_1-4R_3}{\overset{R_2+9R_3}{=}} \begin{bmatrix} 1 & 2 & -8 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We cannot generate leading 1 for 3rd-column w.o. affecting 1's in the 4th/5th column.

$$\overset{R_1-2\cdot R_2}{=} \begin{bmatrix} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(A) = 4$

Reduced echelon form

3.5. Recipe for G-J elimination

Step 1. write the augmented matrix (of the system of linear equations)

$$\begin{cases} ax + by + cz = A \\ dx + ey + fz = B \\ ix + hy + jz = C \end{cases} \Rightarrow \left[\begin{array}{ccc|c} a & b & c & A \\ d & e & f & B \\ i & h & j & C \end{array}\right]$$

$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ i & h & j \end{bmatrix}}_{= A}$

Step 2. Use the element row operations to reduce the Augmented matrix to the $\boxed{\text{reduced echelon form.}}$

Step 3. make sure each column containing a $\triangle$ has zeros everywhere else.

→ solve the equation system

$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_3$

3.6. Def (rank) $\boxed{1}$ The rank of a matrix is the # of nonzero rows in its echelon form

Rank(A) = 3

$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & \triangle & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Rank(B) = 2

$B \to \begin{bmatrix} \triangle & 1 & 0 \\ 0 & \triangle & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3.7. Theorem.

A is the coefficient matrix of system of linear equations w/ n variables

of $\underbrace{\text{free variables}}_{}$ = n − rank(A).

variables not restricted by any conditions (can be any # in $\mathbb{R}$)

3.8. ex. show that $\mathbb{R}^2 = \text{span}\Big(\underbrace{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}}_{S}\Big)$

$\mathbb{R}^n$ can be spanned by n IND.

$\Updownarrow$

We need to show an arbitrary vector $\begin{bmatrix}a\\b\end{bmatrix}$ is a L.C. of $\begin{bmatrix}2\\-1\end{bmatrix}$ & $\begin{bmatrix}1\\3\end{bmatrix}$

$\Updownarrow$

show $\underbrace{x}_{\text{scalars}} \begin{bmatrix}2\\-1\end{bmatrix} + \underbrace{y}_{\text{scalars}} \begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

can always be solved for $x, y \in \mathbb{R}$

step 1 $\left[\begin{array}{cc|c} 2 & 1 & a \\ -1 & 3 & b \end{array}\right]$

step 2. $\left[\begin{array}{cc|c} -1 & 3 & b \\ 2 & 1 & a \end{array}\right] \overset{R_2+2R_1}{=} \left[\begin{array}{cc|c} -1 & 3 & b \\ \hline 0 & 7 & a+2b \end{array}\right]$

$(-1)x + 3y = b$

$7y = a+2b$

$y = \frac{a+2b}{7}$.

step 3. solve for x, y gives

$$\begin{cases} x = \frac{3a-b}{7} \in \mathbb{R} \\ y = \frac{2a+b}{7} \in \mathbb{R} \end{cases}$$

$x = 3y - b$

$= \frac{3a+6b}{7} - \frac{7b}{7}$

$= \frac{3a-b}{7}$

$\therefore$ $\mathbb{R}^2$ can be spanned by $\begin{bmatrix}2\\-1\end{bmatrix}$ & $\begin{bmatrix}1\\3\end{bmatrix}$

3.9.ex. Determine if the vectors are IND.

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

step 1. If $\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\1 & 1 & 0 & 0\\0 & 1 & 1 & 0\end{array}\right]$ can be $\rightarrow$ HMG

reduced to the echelon form

that has 3 nonzero rows

then $c_1 = c_2 = c_3 = 0 \Rightarrow$ IND.

step 2.

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 1 & 1 & 0\end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 2 & 0\end{array}\right] \overset{\dots}{=} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

Step 3.

$$\Leftrightarrow \begin{cases} c_1 x = 0 \\ c_2 y = 0 \\ c_3 z = 0 \end{cases} \Rightarrow \quad c_1 = c_2 = c_3 = 0$$

$$\Rightarrow \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ are IND.}$$

3.10. (Def) homogeneous. A system of linear equations is HMG if the constant term on RHS $= 0$.

3.11. theorem. If $[A|\vec{0}]$ is HMG system of m linear equations w/. n variables.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

$m \times n$

If $m < n$

$\Rightarrow$ system has infinitely many solutions.

3.12. Theorem. Let $\vec{v}_1 \cdots \vec{v}_m \in \mathbb{R}^n$.

Let A be $n \times m$ $[\vec{v}_1 \vdots \vec{v}_2 \vdots \cdots \vdots v_m]$

$\Rightarrow$. $\vec{v}_1 \cdots \vec{v}_m$ are linearly dependent

iff.

$[A \mid \vec{0}]$ has a nontrival solution (nonzero)

§ 4 Matrix

1). Matrix Operations.

$A = [a_{ij}]^{m \times n}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

1.1. (1) $A + B = [a_{ij} + b_{ij}]$

(2) $cA = c[a_{ij}] = [ca_{ij}]$

(3) $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$ $\quad B = \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} \end{bmatrix}_{n \times k}$

$AB = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{m \times k}$

$\parallel$

C

$\boxed{m \times n}$ $\boxed{n \times k}$ n be consistent

$\boxed{m \times s}$ $\boxed{n \times k}$ $s \neq n$ $AB \not\exists$

Rule 1: $\boxed{m \times n}$ multiply be $\boxed{n \times k}$ gives $\boxed{m \times k}$

Rule 2: $c_{11} = \sum_{i=1}^{n} a_{1i} b_{i1} = a_{11} b_{11} + a_{12} b_{21} + \cdots a_{1n} \cdot b_{n1}$

c_{12} = 1st row of A $\times$ 2nd column of B

$\vdots$

c_{ij} = ith row of A $\times$ jth column of B.

1.2. ex $A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & -1 & 1 \end{bmatrix}_{2\times3}$

$B = \begin{bmatrix} -4 & 0 & 3 & -1 \\ 5 & -2 & -1 & 1 \\ -1 & 2 & 0 & 6 \end{bmatrix}_{3\times4}$

$\exists\ AB$

$\boxed{2\times3} = \boxed{3\times4}$

$C = AB = \begin{bmatrix} -4+3\times5+1(-1) & 3(-2)+1\times2 & 3-3 & -1+3+6 \\ (-2)(-4)+(-1)5+1(-1) & (-1)(-2)+1\cdot2 & (-2)3+(-1)(-1) & (-2)(-1)+(-1)+1\cdot6 \end{bmatrix}_{2\times4}$

$2-1+6$

$= \begin{bmatrix} 10 & -4 & 0 & 8 \\ 2 & 4 & -5 & 7 \end{bmatrix}$

Matrix Multiplication

$A_{n\times m}$ $B_{m\times k}$

$C_{k\times n}$.

AB ✓ AC ✗

BC ✓

Step 1. To determintion the product's dimension

$AB \leftarrow n\times k$

$C = AB = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ \vdots & & C_{ij} & \\ C_{n1} & C_{n2} & \cdots & C_{nk} \end{bmatrix}_{n\times k}$

Step 2. Fill in elements.

C_{ij}: the sum of ith row of A inner product w/. jth column of B

1.3. Partional Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 7 & 0 \end{bmatrix}_{5\times5}$$

(I 3×3, 3×2, 0 2×3, 2×2.)

$= \left[\begin{array}{c|c} I_{3\times3} & B \\ \hline O_{2\times3} & C \end{array}\right] = \begin{bmatrix} I_{3\times3} & B \\ O_{2\times2} & C \end{bmatrix}$

1.4. ex $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 4 & -1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix} = \left[\begin{array}{c:c} 4 & -1 \\ 1 & 2 \\ 3 & 0 \end{array}\right] = \left[\begin{array}{c:c} B_1 & B_2 \end{array}\right]$

$C_1 \equiv AB_1 = \underset{2\times 3}{\begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}_{2\times 1}$

$C_2 \equiv AB_2 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}_{2\times 1}$

$AB = \begin{bmatrix} 13 & 5 \\ 2 & -2 \end{bmatrix} = \left[\begin{array}{c:c} C_1 & C_2 \end{array}\right]$

$\boxed{A \quad B}$

$B = \left[\begin{array}{c:c} B_1 & B_2 \end{array}\right]$

$AB_1 = C_1 \quad AB_2 = C_2$

$AB = A\left[\begin{array}{c:c} B_1 & B_2 \end{array}\right]$

$= \left[\begin{array}{c:c} C_1 & C_2 \end{array}\right]$

$= C$

1.5. Matrix Power.

$A^k = \underbrace{AA\cdots A}_{k \text{ times.}}$

$r, s \in \mathbb{N}$

(i) $A^r A^s = A^{r+s}$

(ii) $(A^r)^s = A^{rs}$

1.6 ex. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$A^3 = A^2 A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$

1.7. Def (transpose)

interchange rows and columns of .

1.8. symmetric. $A^T = A$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

2). Matrix algebra.

2.1. Algebraic properties. A, B are matrices.

(i) $A + B = B + A$

(ii) $(A+B)+C = A+(B+C)$

(iii) $A + 0 = A$

(iv) $A + (-A) = 0$

(v) $c(A+B) = cA + cB \qquad c \in \mathbb{R}$

(vi) $(c+d)A = cA + dA$

(vii) $c(dA) = (cd)A$

(viii) $IA = A$. $\quad I$ is identity.

$$\underset{\text{in}}{I}\ \underset{n\times n}{A} = A \qquad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

2.2. properties of matrix multiplication.

A, B, C matrices

(i) $A(BC) = (AB)C$

(ii) $A(B+C) = AB + AC$

(iii) $(A+B)C = AC + BC$

(iv) $k(AB) = (kA)B = A(kB)$

(v) $IA = AI$

2.3 Transpose properties

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(kA)^T = k(A^T)$

(iv) $(AB)^T = B^T A^T$

(v) $[A^r]^T = [A^T]^r \qquad r \in \mathbb{Z}$

Inverse

col(A)

row(A)

null(A)