S21 Midterm Ans Key

Macroeconomic Theory I, Midterm Exam, Spring 2021.

You may consult the book, your notebook, or any other materials, but you may not work with another student or consult any other individual; all of the work must be yours and yours alone.

Answer any three of the following four questions. All questions receive equal weight.

Question 1. Consider an economy in which risk-neutral agents can buy either a bond or an equity share. The bond has a price of one and pays a gross return of $1+r$ next period with certainty, where $r > 0$. The equity share has a price of P_t and, beginning next period, pays the owner a stream of random dividend payments, $\{D_{t+1}, D_{t+2}, D_{t+3}, \ldots\}$.

A.) Use arbitrage to derive an expression that gives the current equilibrium value of P_t as a function of the expected values of P_{t+1} and D_{t+1}. Explain how and why arbitrage will cause that expression to be true.

B.) Suppose that the random dividends follow an MA(1) process so that $D_{t+1} = \xi_{t+1} + \theta\xi_t$, where ξ_{t+i} is $i.i.d.(0,\sigma^2)$. Derive the rational expectations equilibrium value of P_t as a function of the current information set.

C.) Suppose instead that the dividend is a random variable that equals d_1 with probability π and equals d_2 with probability $1-\pi$. Assume that $d_1 < d_2$. Solve for the rational expectations equilibrium value of P_t.

D.) Continuing from your solution in part C, how will the equilibrium P_t be affected by an increase in π. **Economically**, why does an increase in π have this effect?

Question 2. Suppose that the money demand function has the Cagan form. That is

$$\frac{M_t^d}{P_t} = A_0 Y_t^{\theta} \exp(-\alpha i_t), \quad \text{where } A_0 > 0, \theta > 0, \text{ and } \alpha > 0.$$

Assume that the Fisher equation holds and that the real interest rate is constant. (In deriving your answers you should feel free to simplify by normalizing constants appropriately.)

A.) Suppose that Y is constant. Let $m_t \equiv \ln M_t^s$ and let $p_t \equiv \ln P_t$. Suppose that m_t follows and AR(1) process. That is, $m_{t+1} = \rho_m m_t + \varepsilon_{t+1}$ where $|\rho_m| < 1$ and where ε_{t+1} is i.i.d. $(0,\sigma^2)$. Solve for the rational expectations equilibrium value of $p_t \equiv \ln P_t$.

B.) Now, suppose instead that the **money supply is constant** but that y_t follows an AR(1) process. That is, $y_{t+1} = \rho_y y_t + \varepsilon_{t+1}$ where $|\rho_y| < 1$ and where ε_{t+1} is i.i.d. $(0,\sigma^2)$. Once again, solve for the rational expectations equilibrium value of $p_t \equiv \ln P_t$.

Question 3. Consider a representative household that seeks to maximize

$$\sum_{s=0}^{\infty} \beta^s U(c_{t+s}) \qquad (1)$$

Where $\beta = \frac{1}{1+\theta}$, $\theta > 0$,

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \ \gamma \neq 1. \qquad (2)$$

Let W_t denote the household's wealth in the initial period. Assume that the household's only source of income is interest on its wealth. Let $r \geq 0$ denote the real interest rate. Household wealth therefore evolves according to

$$W_{t+s+1} = (1+r)W_{t+s} - c_{t+s} \qquad (3)$$

A.) Set up the Lagrangian for this problem and derive the intertemporal optimality condition (i.e., the intertemporal Euler equation).

B.) Derive the growth rate of consumption, $\frac{\Delta c_{t+1}}{c_t}$. Is consumption increasing, decreasing, or constant through time?

C.) What is the coefficient of relative risk aversion for the stated utility function? (Use the definition of the coefficient of relative risk aversion to show how you derive your answer).

Question 4. This question requires that you construct the phase diagram for the model of optimal capital accumulation discussed in lecture and in Chapter 2 of Wickens. Assume that capital is the only input to production and that the index of technology (total factor productivity) is constant. Assume the general form of the aggregate production function, that is, $Y_t = F(A, k_t)$ where $\frac{\partial F}{\partial k} > 0$ and $\frac{\partial^2 F}{\partial k^2} < 0$. Also, the production function satisfies the Inada conditions and $F(A, 0) = 0$.

A.) Write down the capital accumulation equation and the intertemporal optimality condition and use them to **DERIVE** the shape and location of the $\Delta c_{t+1} = 0$ and $\Delta k_{t+1} = 0$ loci.

B.) Use the equations for the Δc_{t+1} and Δk_{t+1} to **DERIVE** the dynamic behavior of c_t and k_t in all regions of the phase diagram. Use these dynamics to locate and draw the saddle path.

Question 1:

A.) Since agents are risk-neutral, arbitrage requires equal expected returns:

$$(1+r) = \frac{\underset{t}{E} P_{t+1} + \underset{t}{E} D_{t+1}}{P_t} \qquad (1)$$

Suppose not:

if $(1+r) < \dfrac{\underset{t}{E} P_{t+1} + \underset{t}{E} D_{t+1}}{P_t}$ All sell bond, buy equity so $\uparrow P_t$ moving returns to equality

or if $(1+r) > \dfrac{\underset{t}{E} P_{t+1} + \underset{t}{E} D_{t+1}}{P_t}$ All buy bond, sell equity so $\downarrow P_t$ moving returns to equality

B.) Rewrite (1) as

$$R P_t = \underset{t}{E} P_{t+1} + \underset{t}{E} D_{t+1} \quad \text{where } R \equiv (1+r) > 1$$

or $\underset{t}{E} P_{t+1} - R P_t = -\underset{t}{E} D_{t+1}$

or $(1 - RL)\underset{t}{E} P_{t+1} = -\underset{t}{E} D_{t+1} \qquad (2)$

Since $R > 1$, solve (2) forward. Eqn (2) gives

$$\underset{t}{E} P_{t+1} = \frac{1}{1-RL}\left(-\underset{t}{E} D_{t+1}\right)$$

or, solving forward

$$\underset{t}{E} P_{t+1} = \left[\frac{-R^{-1}L^{-1}}{1-R^{-1}L^{-1}}\right]\left(-\underset{t}{E} D_{t+1}\right).$$

Cancelling Negative Signs + Multiplying Through by L gives

$$P_t = R^{-1}\left[\frac{1}{1-R^{-1}L^{-1}}\right]\underset{t}{E} D_{t+1} \quad \text{or}$$

$$\boxed{P_t = R^{-1}\left[\sum_{j=0}^{\infty} R^{-j}\underset{t}{E} D_{t+1+j}\right] \qquad (3)}$$

Since $D_{t+1} = \xi_{t+1} + \theta\xi_t$ where $\xi_{t+i} \sim iid(0,\sigma^2)$ it follows That

$$\left.\begin{aligned}
&\underset{t}{E} D_{t+1} = \underset{t}{E}\left[\xi_{t+1} + \theta\xi_t\right] = \theta\xi_t \\
&\underset{t}{E} D_{t+2} = \underset{t}{E}\left[\xi_{t+2} + \theta\xi_{t+1}\right] = 0 \\
&\quad \text{AND} \\
&\cancel{\underset{t}{E} D_{t+j} = 0\,F} \\
&\underset{t}{E} D_{t+1+j} = 0 \text{ for } j = 2, 3, \ldots
\end{aligned}\right] \qquad (4)$$

Using (4) in (3) gives

$$P_t = R^{-1}\left[\theta\xi_t + R^{-1}\cdot 0 + R^{-2}\cdot 0 + \cdots\right]$$

$$\text{or} \quad P_t = \frac{1}{1+r}\theta\xi_t$$

$$\text{or} \quad \boxed{P_t = \left(\frac{\theta}{1+r}\right)\xi_t \qquad (5)}$$

Eqn (5) is REE Value of P_t

C.) Now, we have $D_{t+1+j} = d_1$ w/ PROB π and $D_{t+1+j} = d_2$ w/ PROB $1-\pi$. Thus

$$\boxed{E_t D_{t+1+j} = \pi d_1 + (1-\pi) d_2 \quad \text{for } j = 0, 1, 2, \dots \quad (6)}$$

using (6) in (3) gives

$$P_t = R^{-1} \sum_{j=0}^{\infty} R^{-j} \left[\pi d_1 + (1-\pi) d_2\right]$$

$$= R^{-1} \left[\frac{1}{1-R^{-1}}\right] \left[\pi d_1 + (1-\pi) d_2\right]$$

$$= \frac{1}{1+r} \left[\frac{1}{\left(\frac{r}{1+r}\right)}\right] \left[\pi d_1 + (1-\pi) d_2\right]$$

or
$$\boxed{P_t = \frac{1}{r}\left[\pi d_1 + (1-\pi) d_2\right] \quad (7)}$$

Eqn (7) gives the REE value of P_t

D.) From (7) it follows that

$$\frac{\partial P_t}{\partial \pi} = \frac{1}{r}\left[d_1 - d_2\right]. \quad \text{Since } d_1 < d_2$$

$$\boxed{\frac{\partial P_t}{\partial \pi} < 0. \quad (8)}$$

Inequality (8) Shows That an increase in π will cause a Decline in P_E, The equilibrium Equity Price.

~~Note That Since~~

Note from (6) That an increase in π, which is an increase in The probability of The low dividend, d_1, and a corresponding decrease in $(1-\pi)$, The probability of The High dividend, d_2, lowers Expected Future Dividends.

Thus, The Decline in Expected Future Dividends causes a decline in The Equity Price.

Question 2 Part A.

$$\frac{M_t^d}{P_t} = A_0 Y_t^{\theta} \exp(-\alpha i_t) \qquad (1)$$

Fisher: $i_t = r + \pi_t^e \qquad (2)$

Where r is a constant

Use Money Market Equil to get

$$M_t^s = M_t^d \qquad (3)$$

Also, use Rational Expectations and the log Approx to percentages to get

$$\pi_t^e = E_t\left[\frac{P_{t+1} - P_t}{P_t}\right] \cong E_t p_{t+1} - p_t \qquad (4)$$

Use (2), (3), and (4) in (1) to get

$$\frac{M_t^s}{P_t} = A_0 \exp(-\alpha r) Y_t^{\theta} \exp[-\alpha(E_t p_{t+1} - p_t)] \qquad (5)$$

PART A. ~~Asser~~ Assume That $Y_t = Y$ a CONSTANT.

Eqn (5) Gives

$$\frac{M_t^S}{P_t} = \left[A_0 Y^\theta \exp(-\alpha r)\right] \cdot \exp\left[-\alpha\left(\underset{t}{E}\, p_{t+1} - p_t\right)\right] \quad (6)$$

Take logs of both sides and Normalize $\ln\left[A_0 Y^\theta \exp(-\alpha r)\right] = 0$ to get

$$m_t - p_t = -\alpha\left[\underset{t}{E}\, p_{t+1} - p_t\right] \quad (7)$$

Write as a First-order Difference Eqn + Apply Sargent's Method. Eqn (7) gives

$$m_t - (1+\alpha)\, p_t = -\alpha\, \underset{t}{E}\, p_{t+1} \quad \text{or}$$

$$\underset{t}{E}\, p_{t+1} - \left(\frac{1+\alpha}{\alpha}\right) p_t = \frac{-1}{\alpha}\, m_t \quad \text{or}$$

$$\left(1 - \left(\frac{1+\alpha}{\alpha}\right) L\right) \underset{t}{E}\, p_{t+1} = \frac{-1}{\alpha}\, m_t \quad (8)$$

Since $\frac{1+\alpha}{\alpha} > 1$ Solve (8) Forward

$$\underset{t}{E}\, p_{t+1} = \left[\frac{1}{1-\left(\frac{1+\alpha}{\alpha}\right)L}\right]\frac{-1}{\alpha}\, m_t \quad \text{or}$$

$$\underset{t}{E}\, p_{t+1} = \left[\frac{-\left(\frac{1+\alpha}{\alpha}\right)^{-1} L^{-1}}{1-\left(\frac{1+\alpha}{\alpha}\right)^{-1} L^{-1}}\right]\left(\frac{-1}{\alpha}\right) m_t \quad \text{or}$$

$$\underset{t}{E}\, p_{t+1} = \frac{1}{1+\alpha}\left[\frac{L^{-1}}{1-\left(\frac{1+\alpha}{\alpha}\right)^{-1} L^{-1}}\right] m_t$$

or, multiplying through by L

$$p_t = \left(\frac{1}{1+\alpha}\right)\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j \underset{t}{E}\, m_{t+j} \qquad (9)$$

Since

$$\underset{t}{E}\, m_{t+j} = \rho_m^j\, m_t \quad \text{for} \quad j = 0, 1, 2, \ldots$$

we have

$$p_t = \left(\frac{1}{1+\alpha}\right)\sum_{j=0}^{\infty}\left(\frac{\alpha}{1+\alpha}\right)^j \rho_m^j\, m_t$$

or, since $\sum \left(\frac{\alpha}{1+\alpha}\right)^j p_m^j = \sum \left(\frac{\alpha p_m}{1+\alpha}\right)^j = \frac{1}{1-\left(\frac{\alpha p_m}{1+\alpha}\right)}$

$$= \frac{1}{\left[\frac{1+\alpha-\alpha p_m}{1+\alpha}\right]} = \left[\frac{1+\alpha}{1+\alpha-\alpha p_m}\right]$$

$$p_t = \left(\frac{1}{1+\alpha}\right)\left[\frac{1+\alpha}{1+\alpha-\alpha p_m}\right] m_t \quad \text{or}$$

$$\boxed{p_t = \left[\frac{1}{1+\alpha-\alpha p_m}\right] m_t \qquad (10)}$$

Eqn (10) gives the REE ~~value~~ solution for p_t.

Part B Return to Eqn (5). Assume that $M_t^S = M^S$, a constant. We then have

$$\frac{M^S}{P_t} = A_0 \exp(-\alpha r)\, Y_t^{\theta} \exp\left[-\alpha\left(E_t\, p_{t+1} - p_t\right)\right] \quad (11)$$

or

$$\frac{1}{P_t} = \left[\frac{A_0 \exp(-\alpha r)}{M^S}\right] Y_t^{\theta} \exp\left[-\alpha\left(E_t\, p_{t+1} - p_t\right)\right] \quad (12)$$

↘

Take logs of Both sides and Normalizing So That

$$\ln\left[\frac{A_0 \exp(-\alpha r)}{Ms}\right] = 0.$$ The Result is

$$-p_t = \Theta y_t - \alpha\left(\underset{t}{E}\, p_{t+1} - p_t\right)$$ or

$$\alpha \underset{t}{E}\, p_{t+1} - (1+\alpha) p_t = \Theta y_t$$ or

$$\underset{t}{E}\, p_{t+1} - \left(\frac{1+\alpha}{\alpha}\right) p_t = \left(\frac{1}{\alpha}\right) \Theta y_t$$ or

$$\left[1 - \left(\frac{1+\alpha}{\alpha}\right) L\right] \underset{t}{E}\, p_{t+1} = \frac{1}{\alpha} \Theta y_t \qquad (13)$$

Solve (13) forward

$$\underset{t}{E}\, p_{t+1} = \left[\frac{1}{1 - \left(\frac{1+\alpha}{\alpha}\right) L}\right] \frac{1}{\alpha} \Theta y_t = \left[\frac{-\left(\frac{1+\alpha}{\alpha}\right)^{-1} L^{-1}}{1 - \left(\frac{1+\alpha}{\alpha}\right)^{-1} L^{-1}}\right] \frac{1}{\alpha} \Theta y_t$$ or

$$\underset{t}{E}\, p_{t+1} = \left(\frac{1}{1+\alpha}\right) \left[\frac{L^{-1}}{1 - \left(\frac{1+\alpha}{\alpha}\right)^{-1} L^{-1}}\right] \left(-\Theta y_t\right)$$

or, multiplying Through by L

$$p_t = \left(\frac{-\theta}{1+\alpha}\right) \sum_{j=0}^{\infty} \left(\frac{\alpha}{1+\alpha}\right)^j E_t\, y_{t+j} \qquad (14)$$

Since $E_t\, y_{t+j} = \rho_y^j\, y_t$ for $j = 0, 1, 2, \ldots$ we have

$$p_t = \left(\frac{-\theta}{1+\alpha}\right) \sum_{j=0}^{\infty} \left(\frac{\alpha \rho_y}{1+\alpha}\right)^j y_t \qquad \text{or}$$

$$p_t = \frac{-\theta}{1+\alpha} \left[\frac{1+\alpha}{1+\alpha-\alpha\rho_y}\right] y_t \qquad \text{or}$$

$$\boxed{p_t = \left[\frac{-\theta}{1+\alpha-\alpha\rho_y}\right] y_t \qquad (15)}$$

Eqn (15) gives the REE Solution for p_t.

QUESTION 3

PROBLEM: $$\text{MAX} \sum_{s=0}^{\infty} \beta^s U(C_{t+s}) \qquad (1)$$

where $\beta = \frac{1}{1+\theta}$, $\theta > 0$ and where

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1 \qquad (2)$$

SUBJECT TO

$$W_{t+s+1} = (1+r) W_{t+s} - C_{t+s} \qquad (3)$$

A. To Derive The Intertemporal Euler Eqn, Set up Lagrangian:

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \left\{ \beta^s U(C_{t+s}) + \lambda_{t+s}\left[(1+r)W_{t+s} - C_{t+s} - W_{t+s+1}\right] \right\}$$

FOC

$$\frac{\partial \mathcal{L}_t}{\partial C_{t+s}} = \beta^s U'(C_{t+s}) - \lambda_{t+s} = 0 \qquad (4)$$

$$\frac{\partial \mathcal{L}_t}{\partial W_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}(1+r) = 0 \qquad (5)$$

$$\frac{\partial \mathcal{L}_t}{\partial \lambda_{t+s}} = 0$$ RETURNS EQN (3)

FROM (4) $$\lambda_{t+s} = \beta^s U'(C_{t+s}) \qquad (6)$$

using (6) in (5) gives

$$\beta^{s} u'(C_{t+s}) = \beta^{s+1} u'(C_{t+s+1})(1+r) \quad \text{or}$$

$$\boxed{u'(C_{t+s}) = \beta(1+r)\, u'(C_{t+s+1}) \qquad (7)}$$

Using $\beta = \frac{1}{1+\theta}$, (2), and evaluating at $s=0$ gives

$$\boxed{C_t^{-\gamma} = \left[\frac{1+r}{1+\theta}\right] C_{t+1}^{-\gamma} \qquad (8)}$$

Equations (7) and (8) are two versions of the <u>intertemporal Euler equation.</u>

B.) <u>Derive the growth rate of consumption</u>, $\frac{\Delta C_{t+1}}{C_t}$.

Begin from a 1st order Taylor Series Expansion of $u'(C_{t+1})$ around C_t:

$$u'(C_{t+1}) \cong u'(C_t) + u''(C_t)\cdot(C_{t+1} - C_t) \qquad (9)$$

Divide (9) by $u'(C_t)$ to get

$$\frac{u'(C_{t+1})}{u'(C_t)} = 1 + \frac{u''(C_t)}{u'(C_t)}\Delta C_{t+1} \qquad (10)$$

Next from eqn (7) with $s=0$

$$u'(C_t) = \beta(1+r)\, u'(C_{t+1}) \quad \text{or}$$

$$\frac{u'(C_{t+1})}{u'(C_t)} = \frac{1}{\beta(1+r)}.$$ This in LHS (10) gives

$$\frac{1}{\beta(1+r)} - 1 = \frac{u''(C_t)}{u'(C_t)} \Delta C_{t+1} \quad \text{or}$$

$$\Delta C_{t+1} = \frac{u'(C_t)}{u''(C_t)}\left[\frac{1}{\beta(1+r)} - 1\right] \quad \text{or}$$

$$\frac{\Delta C_{t+1}}{C_t} = -\left[\frac{u'(C_t)}{u''(C_t)\cdot C_t}\right]\left[1 - \frac{1}{\beta(1+r)}\right] \qquad (12)$$

Since $u'(C_t) = C_t^{-\gamma}$, $u''(C_t) = -\gamma C_t^{-(\gamma+1)}$, we have

$$-\frac{u'(C_t)}{u''(C_t)\cdot C_t} = \frac{-C_t^{-\gamma}}{-\gamma C_t^{-(\gamma+1)} C_t} = \frac{1}{\gamma} \qquad (13)$$

Use (13) in (12) to get

$$\frac{\Delta C_{t+1}}{C_t} = \frac{1}{\gamma}\left[1 - \frac{1}{\beta(1+r)}\right] \qquad (14)$$

Since $1 - \frac{1}{\beta(1+r)} = 1 - \frac{1}{\frac{1+r}{1+\theta}} = 1 - \frac{1+\theta}{1+r}$

$= \frac{(1+r) - (1+\theta)}{1+r} = \left[\frac{r-\theta}{1+r}\right]$ This in (14) gives

$$\boxed{\frac{\Delta C_{t+1}}{C_t} = \frac{1}{\gamma}\left[\frac{r-\theta}{1+r}\right] \qquad (15)}$$

Since $\frac{\Delta C_{t+1}}{C_t}$ is the growth rate of consumption, it follows from (15) that

if $r > \theta$ then $\frac{\Delta C_{t+1}}{C_t} > 0$ and consumption is increasing through time

if $r < \theta$ then $\frac{\Delta C_{t+1}}{C_t} < 0$ and consumption is decreasing through time

if $r = \theta$ $\frac{\Delta C_{t+1}}{C_t} = 0$ and consumption is constant through time

C. The Coefficient of Relative Risk Aversion

is Defined as $\quad -\dfrac{u''(C_t)}{u'(C_t)} \cdot C_t$

Since, from eqn (2),

$u'(C_t) = C_t^{-\gamma}$, $u''(C_t) = -\gamma\, C_t^{-(\gamma+1)}$, we have

$$-\frac{u''(C_t)}{u'(C_t)} \cdot C_t = -\left[\frac{-\gamma C_t^{-(\gamma+1)}}{C_t^{-\gamma}}\right] C_t = \gamma$$

The Coefficient of Relative Risk Aversion is γ.

Question 4.

A.) The capital accumulation Eqn is

$$K_{t+1} = (1-\delta)K_t + F(A, K_t) - C_t \qquad (1)$$

The intertemporal optimality condition is

$$u'(C_t) = \beta u'(C_{t+1})\left[F_K(A, K_{t+1}) + 1 - \delta\right] \qquad (2)$$

① First consider $\Delta K_{t+1} = 0$. From (1)

$$\boxed{\Delta K_{t+1} = F(A, K_t) - C_t - \delta K_t \qquad (3)}$$

so $\Delta K_{t+1} = 0$ where

$$\boxed{C_t = F(A, K_t) - \delta K_t \qquad (4)}$$

To draw the function described by (4) in the phase diagram (C_t vs. K_t space) ~~Note~~ Note that it has the following properties:

(a) Since $F(A, 0) = 0$ it follows from (4) that $C_t = 0$ when $K_t = 0$. Therefore, the $\Delta K_{t+1} = 0$ locus passes through the origin.

(b) The slope of the $\Delta K_{t+1} = 0$ locus is given by

$$\left.\frac{dC_t}{dK_t}\right|_{\Delta K = 0} = F_K(A, K_t) - \delta \qquad (5)$$

The Inada conditions imply that

$$\lim_{K \to 0} \left.\frac{dc_t}{dK_t}\right|_{\Delta K=0} = \lim_{K \to 0}\left[F_K(A,K)-\delta\right] = +\infty$$

So the slope of the $\Delta K_{t+1}=0$ locus approaches $+\infty$ at the origin.

(c) Also, the INADA ~~D~~ conditions imply that

$$\lim_{K \to \infty} \left.\frac{dc_t}{dK_t}\right|_{\Delta K=0} = \lim_{K \to \infty}\left[F_K(A,K)-\delta\right] = -\delta$$

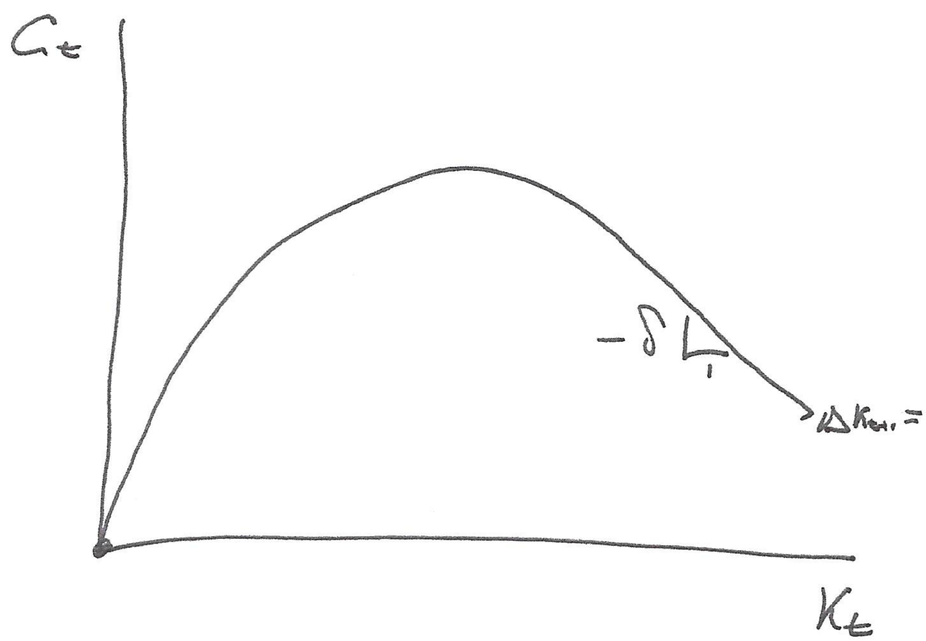
So as $K \to \infty$ the slope of the $\Delta K_{t+1}=0$ locus approaches $-\delta$.

(d) The $\Delta K_{t+1}=0$ locus is concave since, using (5),

$$\frac{d^2 c_t}{dK_t^2} = F_{KK}(A, K_t) < 0 \qquad (6)$$

Collecting, the $\Delta K_{t+1}=0$ locus is shown in

Figure 1.



(2) Next Consider $\Delta C_{t+1} = 0$. Rewrite eqn (2) as

$$\frac{u'(C_{t+1})}{u'(C_t)} = \frac{1}{\beta\left[F_K(A, K_{t+1}) + 1 - \delta\right]} \qquad (7)$$

Next, Treat $u'(C_{t+1})$ as a function of C_{t+1} and take a 1^{st} order Taylor Series approximation Around C_t to get

$$u'(C_{t+1}) \cong u'(C_t) + u''(C_t)\left[C_{t+1} - C_t\right] \qquad (8)$$

Divide Through by $u'(C_t)$ to get

$$\frac{u'(C_{t+1})}{u'(C_t)} = 1 + \frac{u''(C_t)}{u'(C_t)}\left[C_{t+1} - C_t\right] \quad \text{or}$$

$$\Delta C_{t+1} = \frac{u'(C_t)}{u''(C_t)}\left[\frac{u'(C_{t+1})}{u'(C_t)} - 1\right]$$

which, using (7) gives

$$\boxed{\Delta C_{t+1} = -\left(\frac{u'(C_t)}{u''(C_t)}\right)\left[1 - \frac{1}{\beta\left[F_K(A, K_{t+1}) + 1 - \delta\right]}\right] \qquad (9)}$$

From (9), $\Delta C_{t+1} = 0$ where $\beta\left[F_K(A, K_{t+1}) + 1 - \delta\right] = 1$

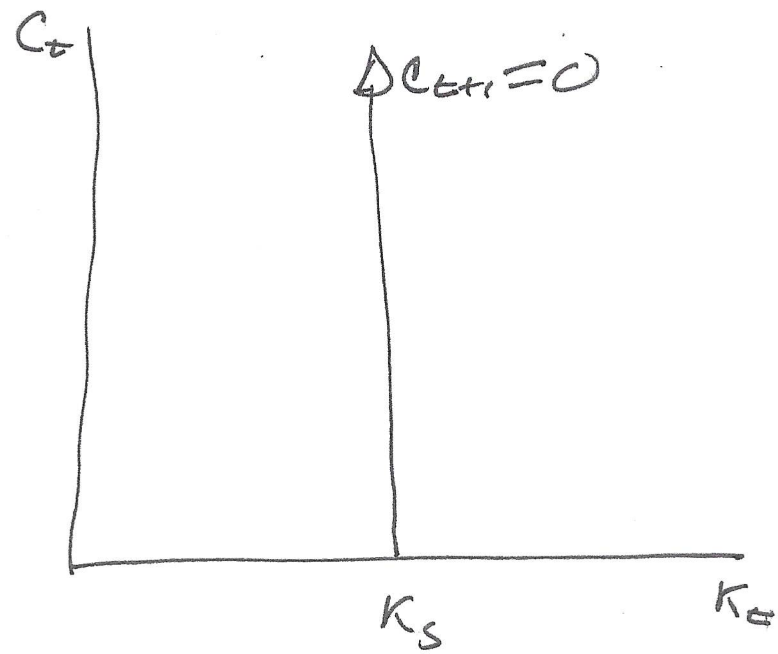
or, if $\beta = \frac{1}{1+\theta}$, where

$$F_K(A, K_{t+1}) = \delta + \theta \qquad (10)$$

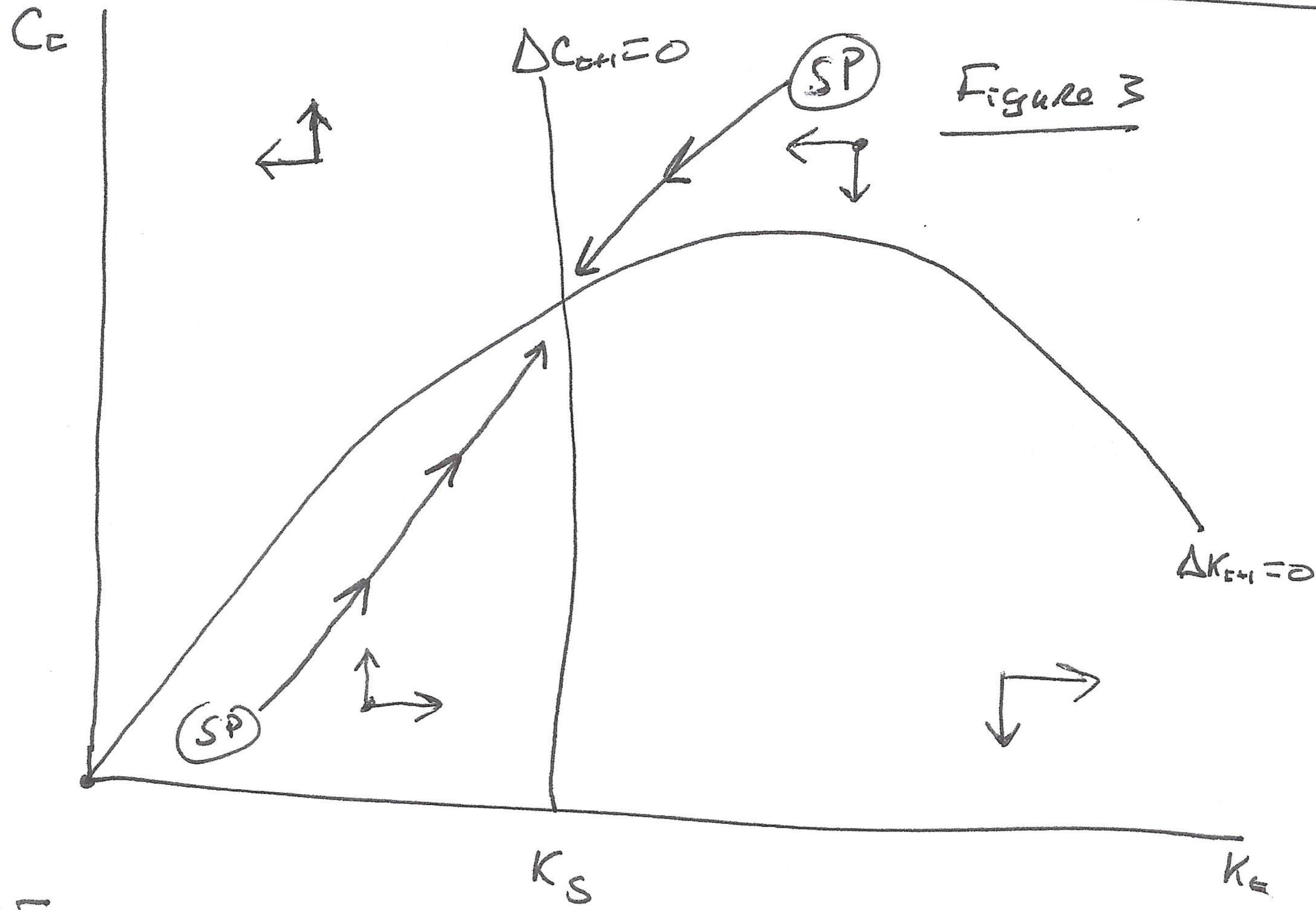
Thus, $\Delta C_{t+1} = 0$ at $K_{t+1} = K_t = K_s$ such that

$$F_K(A, K_s) = \delta + \theta.$$

Since $\Delta C_{t+1} = 0$ @ K_s regardless of C_t, the $\Delta C_{t+1} = 0$ locus is vertical at K_s, as shown in Figure 2.



B.) Note that the two loci divide the phase diagram into four regions as shown in Figure 3 (next page).



① From (3) it is clear that $\frac{\partial \Delta K_{t+1}}{\partial C_t} = -1 < 0$. Thus, if we begin on $\Delta K_{t+1}=0$ and increase C_t we have $\Delta K_{t+1} < 0$. Similarly, if we begin on $\Delta K_{t+1}=0$ locus and reduce C_t we have $\Delta K_{t+1} > 0$. These dynamics are illustrated by the horizontal arrows in Figure 3.

② From (9), since $F_{KK} < 0$, an increase in K_{t+1} causes a decline in F_K, which causes $\uparrow \frac{1}{\beta[F_K + 1 - \delta]}$, which, since $-\frac{u'}{u''} > 0$ causes a decline in ΔC_{t+1}.

So, if we begin on the $\Delta C_{t+1}=0$ locus and increase K_{t+1}

↘

~~we~~ we have $\Delta C_{t+1} < 0$. Similarly, if we begin on $\Delta C_{t+1} = 0$ and reduce K_{t+1} we have $\Delta C_{t+1} > 0$. These dynamics are illustrated by the vertical arrows in Figure 3.

(3) Given the dynamics summarized by the arrows in Figure 3, there is a unique dynamic path converging to the steady state, a unique saddle path, which is labelled "SP" in Figure 3.