Homework #3 Stochastic Calculus Solutions

Ito's Formula

Problem 1. Specialize the derivation of Ito's formula in the lecture notes to function $f(x) = x^3$ to show that

$$W(t)^3 = 3\int_0^t W(s)^2 dW(s) + 3\int_0^t W(s)ds.$$

Answer: This should be an easier task. Note that the purpose here is not to use Ito's formula but to understand the "proof" of it. Note that

$$f'(x) = 3x^2, f''(x) = 6x, f'''(x) = 6.$$

The Taylor expansion of $f(x)$ around y is

$$\begin{aligned} f(x) - f(y) &= f'(y)(x-y) + \frac{1}{2}f''(y)(x-y)^2 + \frac{1}{6}f'''(y)(x-y)^3. \\ &= 3y^2(x-y) + 3y(x-y)^2 + (x-y)^3. \end{aligned}$$

The key for the derivation is to write $W(t)^3$ as a sum of small incremements for a given grid $\{t_i\}$:

$$\begin{aligned} W(t)^3 &= W(t)^3 - W(0) \\ &= \sum_{i=0}^{n-1} \left[W(t_{i+1})^3 - W(t_i)^3\right] \\ &= \sum_{i=0}^{n-1} 3W(t_i)^2 \left[W(t_{i+1}) - W(t_i)\right] + \sum_{i=0}^{n-1} 3W(t_i)\left[W(t_{i+1}) - W(t_i)\right]^2 \\ &\quad + \sum_{i=0}^{n-1} \left[W(t_{i+1}) - W(t_i)\right]^3. \end{aligned}$$

[**Note:** Apply Taylor with $x = W(t_{i+1}), y = W(t_i)$]. Obviously

$$\begin{aligned} \sum_{i=0}^{n-1} 3W(t_i)^2 \left[W(t_{i+1}) - W(t_i)\right] &\to 3\int_0^t W(s)^2 dW(s) \text{ by definition of stochastic integral} \\ \sum_{i=0}^{n-1} 3W(t_i)\left[W(t_{i+1}) - W(t_i)\right]^2 &\to 3\int_0^t W(s)ds \text{ [This part is related to quadratic variation of BM]} \\ \sum_{i=0}^{n-1} \left[W(t_{i+1}) - W(t_i)\right]^3 &\to 0 \text{ [The 3rd order variation of BM is zero, Why?].} \end{aligned}$$

Puting these together, we obtain the result for $W(t)$.

Problem 2. Let $S(t) = S(0)\exp\left\{\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t\right\}$ be a geometric Brownian motion. Let p be a constant. Compute $d(S(t)^p)$.

Answer: Note that $S(t) = S(0)\exp\left\{\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t\right\}$. So

$$S(t)^p = S(0)^p \exp\left(p\sigma W(t) + p(\alpha - \frac{1}{2}\sigma^2)t\right).$$

For ease of notation, denote the left hand-side by Y_t. Then Y_t is just

$$Y_t = f(t, W(t))$$

for

$$f(t,x) = S(0)^p \exp\left(p\sigma x + p(\alpha - \frac{1}{2}\sigma^2)t\right).$$

To apply Ito's formula, note that

$$\begin{aligned}
f_t(t,x) &= p(\alpha - \frac{1}{2}\sigma^2)S(0)^p \exp\left(p\sigma x + p(\alpha - \frac{1}{2}\sigma^2)t\right) = p(\alpha - \frac{1}{2}\sigma^2)f(t,x) \\
f_x(t,x) &= p\sigma S(0)^p \exp\left(p\sigma x + p(\alpha - \frac{1}{2}\sigma^2)t\right) = p\sigma f(t,x) \\
f_{xx}(t,x) &= p^2\sigma^2 S(0)^p \exp\left(p\sigma x + p(\alpha - \frac{1}{2}\sigma^2)t\right) = p^2\sigma^2 f(t,x).
\end{aligned}$$

Hence

$$\begin{aligned}
dY_t &= f_t(t,W(t))dt + f_x(t,W(t))dW(t) + \frac{1}{2}f_{xx}(t,W(t))dt \qquad (1) \\
&= p(\alpha - \frac{1}{2}\sigma^2)\ f(t,W_t)dt + p\sigma f(t,W_t)dW(t) + \frac{1}{2}p^2\sigma^2 f(t,W_t)dt.
\end{aligned}$$

We can rewrite the above result in terms of Y_t

$$dY_t = p(\alpha - \frac{1}{2}\sigma^2)[Y_t]dt + p\sigma[Y_t]dW(t) + \frac{1}{2}p^2\sigma^2[Y_t]dt.$$

That is

$$dY_t = \left[p(\alpha - \frac{1}{2}\sigma^2) + \frac{1}{2}p^2\sigma^2\right][Y_t]dt + p\sigma[Y_t]dW(t).$$

Or

$$dY_t = \left[p\alpha + \frac{1}{2}p(p-1)\sigma^2\right][Y_t]dt + p\sigma[Y_t]dW(t).$$

Remark: I have provided details more than necessary. One conclusion is that the power of geometric BM is still a geometric BM, which should not come as a surprise due to the property of exponential function.

Problem 3.

(i) Compute $dW(t)^4$, and the write $W(t)^4$ as the sum of an ordinary integral and an Ito integral

(ii) Take expectation of both sides of the formula you obtained in (i), use the fact that $\mathbb{E}W(t)^2 = t$, and derivative the formula $\mathbb{E}W(t)^4 = 3t^2$.

(iii) Use the method of (i) and (ii) to derivative a formula for $\mathbb{E}W(t)^6$.

Answer:

(i) Let $f(x) = x^4$. Our task is to find dY_t for $Y_t = f(W(t))$. The derivatives of f is easy to calculate

$$f'(x) = 4x^3, f'' = 12x^2.$$

Hence Ito's formula gives us

$$\begin{aligned} dY_t &= f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt \\ &= 4W(t)^3 dW(t) + \frac{1}{2} \cdot 12W(t)^2 dt \\ &= 4W(t)^3 dW(t) + 6W(t)^2 dt. \end{aligned}$$

The equivalent integral form is

$$Y_t - Y_0 = \int_0^t 4W(s)^3 dW(s) + \int_0^t 6W(s)^2 ds.$$

That is

$$W(t)^4 = 4\int_0^t W(s)^3 dW(s) + 6\int_0^t W(s)^2 ds.$$

(ii) Taking expectations of the above result gives

$$\begin{aligned} \mathbb{E}W(t)^4 &= 4\mathbb{E}\int_0^t W(s)^3 dW(s) + 6\mathbb{E}\int_0^t W(s)^2 ds \\ &= 6\mathbb{E}\int_0^t W(s)^2 ds \quad \text{[Why the first term above vanishes?]} \\ &= 6\int_0^t \mathbb{E}W(s)^2 ds = 6\int_0^t s ds = 3t^2. \end{aligned}$$

(iii) The calculation is similar to (ii) so we are brief here

$$d\left[W(t)^6\right] = 6[W(t)^5]dW(t) + 15W(t)^4 dt.$$

The integral form is

$$\begin{aligned} W(t)^6 &= \int_0^t 6[W(s)^5]dW(s) + \int_0^t 15W(s)^4 ds \\ \mathbb{E}W(t)^6 &= \mathbb{E}\int_0^t 15W(s)^4 ds = 15\int_0^t \left[\mathbb{E}W(s)^4\right] ds = 15 \cdot \int_0^t \left[3s^2\right] ds = 15t^3. \end{aligned}$$

Problem 4. Let $X_t = \exp\left(ct + \alpha W(t)\right)$. Show that X_t satisfies $dX_t = \left(c + \frac{1}{2}\alpha^2\right) X_t dt + \alpha X_t dW(t)$.

Answer: Here $X_t = f(t, W(t))$ for $f(t,x) = \exp(ct + \alpha x)$. The Ito's formula gives

$$\begin{aligned} dX_t &= f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2} f_{xx}(t, W(t))dt \\ &= c\exp\left(ct + \alpha W(t)\right) dt + \alpha \exp\left(ct + \alpha W(t)\right) dW(t) + \frac{1}{2}\alpha^2 \exp\left(ct + \alpha W(t)\right) dt \\ &= cX_t dt + \alpha X_t dW(t) + \frac{1}{2}\alpha^2 X_t dt \\ &= \left(c + \frac{1}{2}\alpha^2\right) X_t dt + \alpha X_t dW(t). \end{aligned}$$

Problem 5. Suppose that $S(t)$ satisfy

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t).$$

Set $Y(t) = \log(S(t))$. What stochastic differential equation does $Y(t)$ satisfy?

Answer: Direct application of Ito's formula to $\log(S(t))$ gives (here $f(x) = \log(x)$)

$$\begin{aligned} dY_t &= f_x(S(t))dS(t) + \frac{1}{2} f_{xx}(S(t))\left(dS(t)\right)^2 \\ &= \frac{1}{S(t)} dS(t) + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right) d(S(t))^2 \\ &= \frac{1}{S(t)} dS(t) - \frac{1}{2}\left(\frac{1}{S(t)^2}\right)\sigma^2 S(t) dt \qquad \text{[Note: } d(S(t))^2 = \sigma^2 S(t)^2 dt\text{]} \\ &= \frac{1}{S(t)} dS(t) - \frac{1}{2}\sigma^2 dt \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t). \end{aligned}$$