ECON 6700 Math II, Final Exam, Fall 2020

You may consult the book, your notebook, or any other materials, but you may not work with another student or consult any other individual; all of the work must be yours and yours alone.

Answer **any three** of the following four questions. All questions receive equal weight.

Question 1. Consider the following stochastic optimal-employment problem. A firm seeks to maximize the present discounted value of its profits over an infinite horizon. It produces output, y_t, using labor inputs, n_t, according to a linear production function, $y_t = An_t$, where A is a positive constant. It sells each unit of output for the exogenous price p_t where p_t is $i.i.d.(\bar{p},\sigma^2)$. There are two costs: a quadratic cost of training new workers and a constant wage, $w>0$, per unit of labor employed. Thus, the firm's profit function is

$$R(H_{t+i},n_{t+i}) = p_{t+i}An_{t+i} - \tfrac{\theta}{2}H_{t+i}^2 - wn_{t+i}, \text{ where } \theta>0, \tag{1}$$

and where H_t denotes new hires. Labor leaves the firm at rate ω where $0<\omega<1$, so that

$$n_{t+1+i} = (1-\omega)n_{t+i} + H_{t+i}. \tag{2}$$

Let β denote the discount factor, where $0<\beta<1$. The firm's problem is to maximize $E_t\sum_{i=0}^{\infty}\beta^i R(H_{t+i},n_{t+i})$, subject to the constraint in equation (2).

Set up the dynamic programming problem. Derive the Euler equation using the method discussed in class. Be careful to write out Bellman's equation and explain the value function. Also, be explicit about how the Envelope Theorem is used in your derivation of the Euler equation. Appropriate side conditions apply but need not be discussed in your answer to this question.

Question 2. Consider the following maximization problem:

Maximize $f(x,y) = xy$

subject to $x+y^2 \le 0$, $x\ge 0$, $y\ge 0$.

A.) Formulate the Lagrangian.
B.) Write out the Kuhn-Tucker conditions (marginal, complementary slackness, and non-negativity conditions).
C.) Find the maximizer for this problem.

Question 3. Consider the following version of the optimal capital accumulation problem. The problem is to maximize $\sum_{t=0}^{\infty} \beta^t \ln C_t$ subject to $C_t + K_{t+1} \le K_t^{\alpha}$. Here C_t denotes consumption, K_t denotes the capital stock, $0 < \beta < 1$, and $0 < \alpha < 1$. Bellman's equation gives

$$V_{j+1}(K) = \max_{C} \left\{ \ln C + \beta V_j(\tilde{K}) \right\} \qquad (1)$$

where a tilde denotes the next-period value. What restriction should be placed on $V_0(K)$? Explain why this restriction is reasonable.

Use recursions on Bellman's equation, equation (1), to derive the value function $V_2(K)$ as an explicit function of α, β, and K.

Question 4. For this question let R^1 denote the set of (scalar) real numbers. You may take the following two inequalities as given:

$$|x+y| \le |x| + |y| \qquad (1)$$

$$|x| - |y| \le |x-y| \qquad (2)$$

where $x,\ y \in \mathrm{R}^1$.

Prove the following propositions.

A.) *Proposition 1:* A sequence of vectors in R^{M} converges if all M sequences of its components converge in R^1.

B.) *Proposition 2:* Every Cauchy sequence in R^1 is bounded.

C.) *Proposition 3:* If a Cauchy sequence, $\{x_n\}_{n=1}^{\infty}$, has a subsequence converging to the limit y then the whole sequence converges to y.

Question 1

1.) State the problem

The Firm Seeks to

$$\text{MAX} \sum_{i=0}^{\infty} \beta^i R(H_{t+i}, M_{t+i}) \quad (3)$$

~~or MAX $\sum_{i=0}^{\infty} \beta^i [P_{t+i} A M_{t+i} - \frac{\theta}{2} H_{t+i}^2]$~~

$$\text{s.t. } M_{t+1+i} = (1-\omega) M_{t+i} + H_{t+i} \quad (2)$$

(Appropriate Side Conditions Apply.)

2.) Decide State + Control Variables, write out Transition eqn:

a.) State Variable(s): A natural choice of state variable is M_t as eqn (2) implies

$$M_t = (1-\omega) M_{t-1} + H_{t-1} \quad (2')$$

So that M_t is predetermined in period t.

(P_t Could also be included as a state Variable but, as it is exogenous and iid it need Not be included for full credit.)

b.) Control Variable: A natural choice for control variable is H_t.

c.) The transition eqn is, then, eqn (2), which gives next period's state variable, M_{t+1}, as a function of this period's state and control, M_t and H_t, respectively.

3.) Bellman's Eqn: Define the value function

$V(M_t)$ = The maximum attainable value of the PdV of profits given the current state, M_t.

Bellman's Eqn is

$$V(M_t) = \max_{H_t} E_t \left\{ R(H_t, M_t) + \beta V(M_{t+1}) \right\} \qquad (4)$$

$$\text{where } R(H_t, M_t) = P_t A M_t - \frac{\theta}{2} H_t^2 - W M_t \qquad (1')$$

$$\text{and where } M_{t+1} = (1-\omega) M_t + H_t \qquad (2'')$$

That is, the second term on the RHS of (4) is

$$\beta V\left[(1-\omega) M_t + H_t\right].$$

4.) <u>FOC</u>. The maximization on the RHS of (4) requires

$$E_t\left\{\frac{\partial R(H_t,M_t)}{\partial H_t} + \beta V_m(M_{t+1})\frac{\partial M_{t+1}}{\partial H_t}\right\}=0 \qquad (5)$$

or, since $\frac{\partial R(H_t,M_t)}{\partial H_t} = -\theta H_t$ and $\frac{\partial M_{t+1}}{\partial H_t} = 1$

$$E_t\left[-\theta H_t + \beta V_m(M_{t+1})\right] = 0 \qquad (6)$$

or, since H_t is known in period t,

$$H_t = \frac{1}{\theta}\beta E_t V_m(M_{t+1}) \qquad (7)$$

We must now evaluate $E_t V_m(M_{t+1})$

5.) Use the Envelope Theorem to evaluate $V_m(M_{t+1})$

a.) First, evaluate $V_m(M_t)$. Note that the value of H_t on the RHS of (4) is the maximizing value of H_t taking the state, M_t, as given. Write

$$H_t^* = H^*(M_t) \qquad (8)$$

b) Then

$$\frac{\partial V(M_t)}{\partial M_t} = \underset{t}{E}\left\{\frac{\partial R(H_t, M_t)}{\partial H_t} \cdot \frac{\partial H_t^*}{\partial M_t} + \frac{\partial R(H_t, M_t)}{\partial M_t}\right.$$

$$\left. + \beta V_m(M_{t+1}) \frac{\partial M_{t+1}}{\partial H_t} \frac{\partial H_t^*}{\partial M_t} + \beta V_m(M_{t+1}) \left.\frac{\partial M_{t+1}}{\partial M_t}\right|_{H_t \text{ fixed}}\right\} \quad (9)$$

Note that the 1st and 3rd terms on the RHS (9) are

$$\frac{\partial H_t^*}{\partial M_t} \underset{t}{E}\left[\frac{\partial R(H_t, M_t)}{\partial H_t} + \beta V_m(M_{t+1}) \frac{\partial M_{t+1}}{\partial H_t}\right] = 0$$

via eqn (5)

Thus, (9) becomes

$$V_m(M_t) = \underset{t}{E} \frac{\partial R(H_t, M_t)}{\partial H_t} + \beta \underset{t}{E} V_m(M_{t+1}) \left.\frac{\partial M_{t+1}}{\partial M_t}\right|_{H_t \text{ fixed}}$$

or, since $\frac{\partial R(H_t, M_t)}{\partial H_t} = (A p_t - w)$ and $\left.\frac{\partial M_{t+1}}{\partial M_t}\right|_{H_t \text{ fixed}} = (1-\omega)$

$$\boxed{V_m(M_t) = A p_t - w + (1-\omega)\beta \underset{t}{E} V_m(M_{t+1}) \quad (10)}$$

The simplification of (9) using (5) is an application of the Envelope Theorem

Now, using (7) in (10) gives

$$V_m(m_t) = Ap_t - w + (1-\omega)\theta H_t \quad (11)$$

and, thus,

$$V_m(m_{t+1}) = Ap_{t+1} - w + (1-\omega)\theta H_{t+1} \quad (12)$$

6.) Use (12) in (7) to get

$$H_t = \frac{1}{\theta}\beta E_t\left[Ap_{t+1} - w + (1-\omega)\theta H_{t+1}\right] \quad (13)$$

or

$$H_t = \frac{\beta}{\theta}AE_t p_{t+1} - \frac{\beta}{\theta}w + \beta(1-\omega)E_t H_{t+1} \quad (13')$$

Eqn (13) or (13') is the (intertemporal) Euler Eqn.

a) FORMULATE LAGRANGIAN

$$\mathcal{L}(x,y,\lambda) = xy - \lambda(x+y^2-2) \qquad (1)$$

b) MARGINAL CONDITIONS

$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda \le 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda y \le 0 \qquad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x+y^2-2) = 2-x-y^2 \ge 0 \qquad (4)$$

COMPLEMENTARY SLACKNESS CONDITIONS

$$\frac{\partial \mathcal{L}}{\partial x}\cdot x = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}}{\partial x}\cdot x = (y-\lambda)x = 0 \qquad (5)$$

$$\frac{\partial \mathcal{L}}{\partial y}\cdot y = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}}{\partial y}\cdot y = (x-2\lambda y)y = 0 \qquad (6)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}\cdot \lambda = 0 \quad \text{or} \quad \frac{\partial \mathcal{L}}{\partial \lambda}\cdot \lambda = (2-x-y^2)\lambda = 0 \qquad (7)$$

Non-Negativity Conditions

$$x \geq 0 \quad , y \geq 0 \quad , \lambda \geq 0 \qquad (8)$$

c) As $\lambda \geq 0$

(i) When $\lambda = 0$

From (5) and (6), if $\lambda = 0$ Then $x\, y = 0$
and

$$f(x, y) = 0 \qquad (9)$$

(ii) When $\lambda > 0$

From (7), $\cancel{2 - y}$ $2 - x - y^2 = 0 \quad (10)$

From (10) it is not possible for both $x = 0$ and $y = 0$.

① Suppose $y > 0$ and $x = 0$. Then (10) gives $y = \sqrt{2}$
and $f(x, y) = 0 \qquad (11)$

② Suppose instead $x > 0$ and $y = 0$. Then (10) gives
$x = 2$ and

$$f(x, y) = 0 \qquad (12)$$

3 finally suppose $x > 0$ and $y > 0$.

From (2) $y = \lambda$

From (3) $x = 2\lambda y$ or $\lambda = \dfrac{x}{2y}$

so $\dfrac{x}{2y} = y$ which gives $x = 2y^2$ (13)

use (13) in (10)

$$2 - 2y^2 - y^2 = 0$$

$$2 = 3y^2$$

$$y^2 = \frac{2}{3}$$

$$y = \sqrt{\frac{2}{3}} \qquad (14)$$

use (14) in (13)

$$x = 2\left(\sqrt{\frac{2}{3}}\right)^2 = 2 \cdot \frac{2}{3}$$

$$x = \frac{4}{3} \qquad (15)$$

$$f(x,y) = \frac{4}{3} \cdot \sqrt{\frac{2}{3}} \qquad (16)$$

Comparing the possibilities $f(x,y)$ is maximized

at $(x,y) = \left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right)$ where $f(x,y) = \frac{4}{3}\sqrt{\frac{2}{3}}$.

at $(x,y) = \left(\frac{4}{3}, \frac{\sqrt{6}}{3}\right)$ where $f(x,y) = \dfrac{4\sqrt{6}}{9}$

In general, let K and C denote current capital and consumption, respectively, and let $\tilde{K}$ denote next period's capital. Thus, the constraint is

$$C + \tilde{K} \leq AK^{\alpha} \qquad (2)$$

A reasonable restriction on $V_0(K)$ is that $V_0(K) = 0$ b/c this is the value of capital in the period after the terminal period.

To find $V_1(K)$ we have from eqn (1)

$$V_1(K) = \max_{C} \{ \ln C + V_0(\tilde{K}) \}$$ which, using $V_0(\tilde{K}) = 0$

gives $$V_1(K) = \max_{C} \{ \ln C \} \qquad (3)$$

Looking at (2) it is clear that the soln to the RHS of (3) is

$$C = AK^{\alpha} \qquad (4) \qquad (\text{hence } \tilde{K} = 0)$$

Using (4) in RHS(3) gives

$$V_1(K) = \ln(AK^{\alpha}) \text{ or}$$
$$V_1(K) = \ln A + \alpha \ln K \qquad (5)$$

Now, using (1), consider $V_2(K)$

$$V_2(K) = \max_{C} \{\ln C + \beta V_1(\tilde{K})\} \qquad (6)$$

where $C + \tilde{K} \leq AK^{\alpha}$ (2)

Since utility, $\ln C$, is monotonically increasing in C [and $V_1(\tilde{K})$ is monotonically increasing in $\tilde{K}$] it is clear that (2) will hold with equality and, thus,

$$\tilde{K} = AK^{\alpha} - C \qquad (7)$$

Use (7) and (5) in (6) to get

$$V_2(K) = \underset{C}{MAX}\left\{\ln C + \beta\left[\ln A + \alpha \ln(AK^\alpha - c)\right]\right\} \quad (8)$$

The FOC for the maximization on RHS of (8) is

$$\frac{1}{c} + \beta\alpha\left[\frac{1}{AK^\alpha - c}\right](-1) = 0 \quad \text{or}$$

$$\frac{1}{c} = \left[\frac{\beta\alpha}{AK^\alpha - c}\right] \quad \text{or} \quad c = \frac{1}{\beta\alpha}\left[AK^\alpha - c\right]$$

$$\text{or} \quad \left(1 + \frac{1}{\beta\alpha}\right)c = \frac{1}{\beta\alpha}AK^\alpha$$

$$\text{or} \quad C = (1+\beta\alpha)^{-1}AK^\alpha \quad (9)$$

(Note that (9) in (7) implies

$$\tilde{K} = \left(\frac{\beta\alpha}{1+\beta\alpha}\right)AK^\alpha \quad (10)$$
)

Subst (9) into (8) to get

$$V_2(K) = \ln\left[\left(\frac{1}{1+\beta\alpha}\right)AK^\alpha\right] + \beta\ln A + \beta\alpha\ln\left[\left(\frac{\beta\alpha}{1+\beta\alpha}\right)AK^\alpha\right]$$

or ↘

$$V_2(K) = \ln\left(\frac{A}{1+\beta\alpha}\right) + \alpha \ln K + \beta \ln A + \beta\alpha \ln\left(\frac{\beta\alpha A}{1+\beta\alpha}\right)$$
$$+ \beta\alpha^2 \ln K$$

or

$$\boxed{V_2(K) = \left[\ln\left(\frac{A}{1+\beta\alpha}\right) + \beta \ln A + \beta\alpha \ln\left(\frac{\beta\alpha A}{1+\beta\alpha}\right)\right] + \alpha(1+\beta\alpha)\ln K} \qquad (11)$$

Eqn (11) is the Answer: $V_2(K)$ as an Explicit function of α, β, and K.

A. **Prop1** A sequence of vectors in $\mathbb{R}^M$ converges if all M sequences of its components converges in $\mathbb{R}^1$

Proof: Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of vectors in $\mathbb{R}^M$. Write $X_m \equiv \{x_{1m}, x_{2m}, \ldots, x_{Mm}\}$. Suppose that each of the M sequences, $\{x_{im}\}_{m=1}^{\infty}$, converges to a limit, x_i^*. Let $X^* = \{x_1^*, x_2^*, \ldots, x_M^*\}$. Choose and fix $\varepsilon > 0$. For each i from 1 to M $\exists$ an integer, $N_i \ni$ for $m \geq N_i$ $|x_{im} - x_i^*| < \frac{\varepsilon}{\sqrt{M}}$.

Let $N = \max\{N_1, N_2, \ldots, N_M\}$. Suppose $m \geq N$.

Then

$$\|X_m - X^*\| = \sqrt{(x_{1m} - x_1^*)^2 + (x_{2m} - x_2^*)^2 + \cdots + (x_{Mm} - x_M^*)^2}$$

$$< \sqrt{\frac{\varepsilon^2}{M} + \frac{\varepsilon^2}{M} + \cdots + \frac{\varepsilon^2}{M}} = \sqrt{\varepsilon^2} = \varepsilon$$

Thus $X_m \rightarrow X^*$

QED

B. Prop 2 Every Cauchy Sequence in $\mathbb{R}^1$ is Bounded.

Proof: Let $\{x_m\}_{m=1}^{\infty}$ be Cauchy in $\mathbb{R}^1$. Choose + Fix $\varepsilon > 0$.

Since The Sequence is Cauchy $\exists$ an integer $N \ni$

$|x_i - x_j| < \varepsilon \quad \forall\, i, j \geq N$. In particular

$|x_N - x_i| < \varepsilon \quad \forall\, i \geq N$. Using (2), As given,

$$|x_i| - |x_N| \leq |x_i - x_N| = |x_N - x_i| \text{ .}$$

Thus, $x_i \leq |x_N| + |x_N - x_i| < |x_N| + \varepsilon \quad \forall\, i \geq N$.

So $|x_N| + \varepsilon$ is ~~AB~~ a bound for All but The first $N-1$ terms in The Sequence.

Let $b = \max\{|x_1|, |x_2|, \ldots, |x_N|\}$. It follows That

$$|x_i| \leq b + \varepsilon \quad \forall\, i \text{ (not just } i \geq N)$$

QED

Prop 3: If a Cauchy sequence, $\{x_n\}_{n=1}^{\infty}$, has a subsequence converging to y then the whole sequence converges to y.

Proof: Choose $\varepsilon > 0$. Since the sequence is Cauchy, $\exists$ an integer $N \ni |x_i - x_j| < \frac{\varepsilon}{2} \ \forall i,j \geq N$.

Choose an element of the convergent subsequence, x_K, where $K \geq N$ and where K is large enough that $|x_K - y| < \frac{\varepsilon}{2}$. It follows that $\forall i \geq N$

$$|x_i - y| = |(x_i - x_K) + (x_K - y)| \leq |x_i - x_K| + |x_K - y|$$

via eqn (1) as given

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

~~QED~~

Thus, $\forall i \geq N$, $|x_i - y| < \varepsilon$ and, therefore $x_n \longrightarrow y$

QED