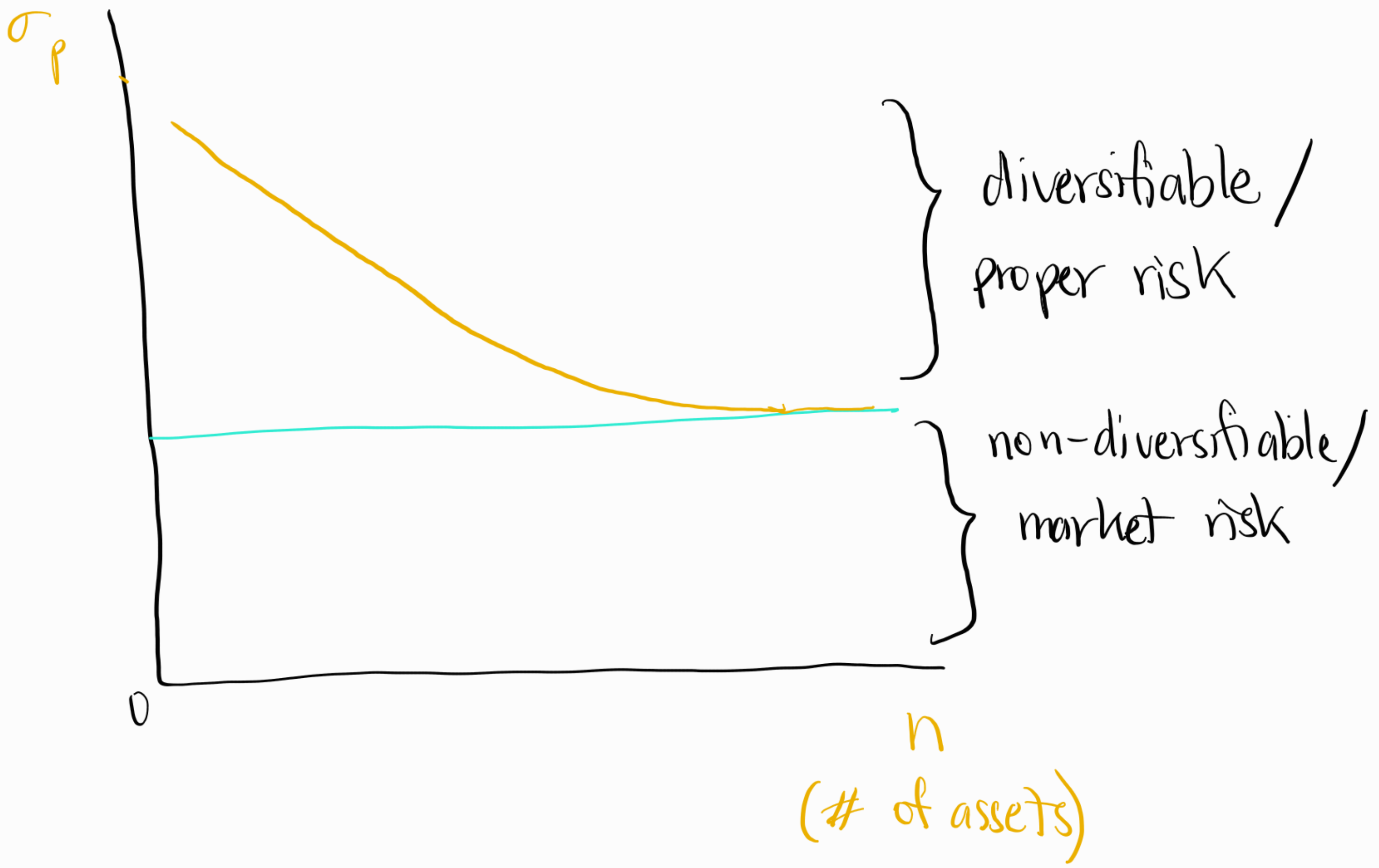
Asset Allocation (Risk Diversification)



Suppose we have 2 assets: debt (D) & equity (E). The weight we put on debt is W_D and the weight on equity is W_E.

A general rule is that $\sum_{i=1}^{\#\text{assets}} W_i = 1$.

This rule allows for short sells. W does not need to be a positive value. The sum just needs to equal 1.

$$E(r_p) = w_D \, E(r_D) + w_E \, E(r_E)$$

r_i = return for asset i (and $E(r_i)$ is simply the expected return for asset i

p = risky portfolio

We need to find the variance of our risky portfolio σ_p^2 using the bordered variance-covariance matrix:

	w_D	w_E
w_D	$\text{cov}(r_D, r_D)$	$\text{cov}(r_D, r_E)$
w_E	$\text{cov}(r_E, r_D)$	$\text{cov}(r_E, r_E)$

Properties of variances:

1. $\text{cov}(x, x) = \text{var}(x) = \sigma_x^2$

2. $\text{cov}(x, y) = \text{cov}(y, x)$

3. $\rho_{x,y} = \dfrac{\text{cov}(y, x)}{\sigma_x \sigma_y}$

(correlation between x & y)

$-1 \leq \rho \leq +1$ where difference from 0 shows magnitude and sign shows direction. $\rho = 0$ indicates linear independence.

Example: $\rho = -0.5$

$x \uparrow 10\% \rightarrow y \downarrow 5\% = (10\% \times -0.5)$

Given these properties:

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \, \text{cov}(r_D, r_E)$$

Note: We just multiplied across variance covariance matrix

Via property 3, we get that

$$cov(r_D, r_E) = \sigma_D \sigma_E \rho$$

Plugging this into the formula for σ_p^2, we get:

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \sigma_D \sigma_E \rho$$

* Risk diversification happens if the portfolio volatility is strictly smaller than the weighted volatility of its components. This will solely depend on ρ in the above specification.

Suppose $\rho = 1$ (perfect positive correlation)

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \sigma_D \sigma_E$$

Note: This follows the algebra rule formula $(x+y)^2 = x^2 + 2xy + y^2$ where $x = w_D \sigma_D$ and $y = w_E \sigma_E$

So we can rewrite σ_p^2 as:

$$\sigma_p^2 = [w_D \sigma_D + w_E \sigma_E]^2 \quad \text{when } \rho = 1$$

And thus : $\sigma_p = w_D \sigma_D + w_E \sigma_E$

Essentially, we have no diversification when $\rho = 1$. Having 1x and 1y is the same as having 2x or 2y since x and y are perfectly positively correlated. The portfolio risk exactly equals the weighted sum of its components.

Suppose $\rho = -1$ (perfectly negative correlation)

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 - 2 w_D w_E \sigma_D \sigma_E$$

Note that this follows the form $(x-y)^2 = x^2 - 2xy + y^2$

So $\sigma_p^2 = \left[w_D \sigma_D - w_E \sigma_E \right]^2$ when $\rho = -1$

and $\sigma_p = w_D \sigma_D - w_E \sigma_E$

There is a value of w_D & w_E that should make $\sigma_p = 0$. This is ideal (we want to reduce volatility as much as possible).

$$w_D \sigma_D - w_E \sigma_E = 0$$

Since $\sum w_i = 1$, we can write $w_E = 1 - w_D$

$$w_D \sigma_D - (1 - w_D)\sigma_E = 0$$

$$w_D(\sigma_D + \sigma_E) - \sigma_E = 0$$

$$\boxed{w_D^* = \frac{\sigma_E}{\sigma_D + \sigma_E} \qquad w_E^* = 1 - w_D^*}$$

This is a perfect hedge portfolio and only occurs when $\rho = -1$.

Example:

$E(r_D) = 8 \quad \sigma_D = 12$

$E(r_E) = 13 \quad \sigma_E = 20$

$\rho = -1$

w_D^*, w_E^*? $\quad E(r_p)$? $\quad \sigma_p$?

$$w_D^* = \frac{\sigma_E}{\sigma_D + \sigma_E} = \frac{20}{12+20} = \boxed{0.625}$$

$$w_E^* = 1 - w_D^* = 1 - 0.625 = \boxed{0.375}$$

$$E(r_p) = w_D^* E(r_D) + w_E^* E(r_E)$$

$$E(r_p) = 0.625 \cdot 8 + 0.375(13)$$

$$\boxed{E(r_p) = 9.875}$$

$$\sigma_p = w_D \sigma_D - w_E \sigma_E$$

$$\sigma_p = 0.625(12) - 0.375(20)$$

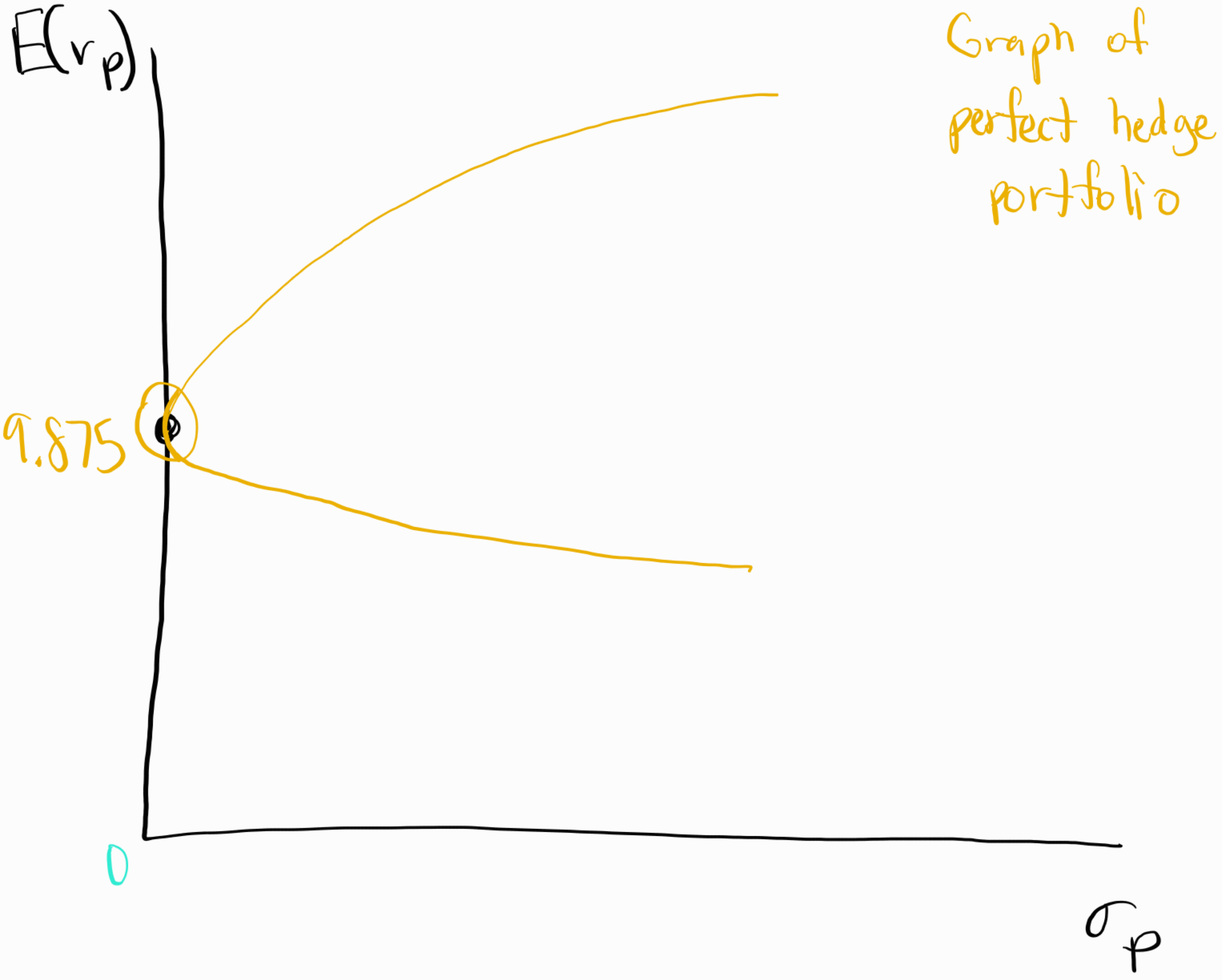
$$\boxed{\sigma_p = 0}$$

or $\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 - 2 w_D \sigma_D w_E \sigma_E$

$$\sigma_p^2 = (0.625^2 \times 12^2) + (0.375^2 \times 20^2) - 2(0.625 \times 12)(0.375 \times 20)$$

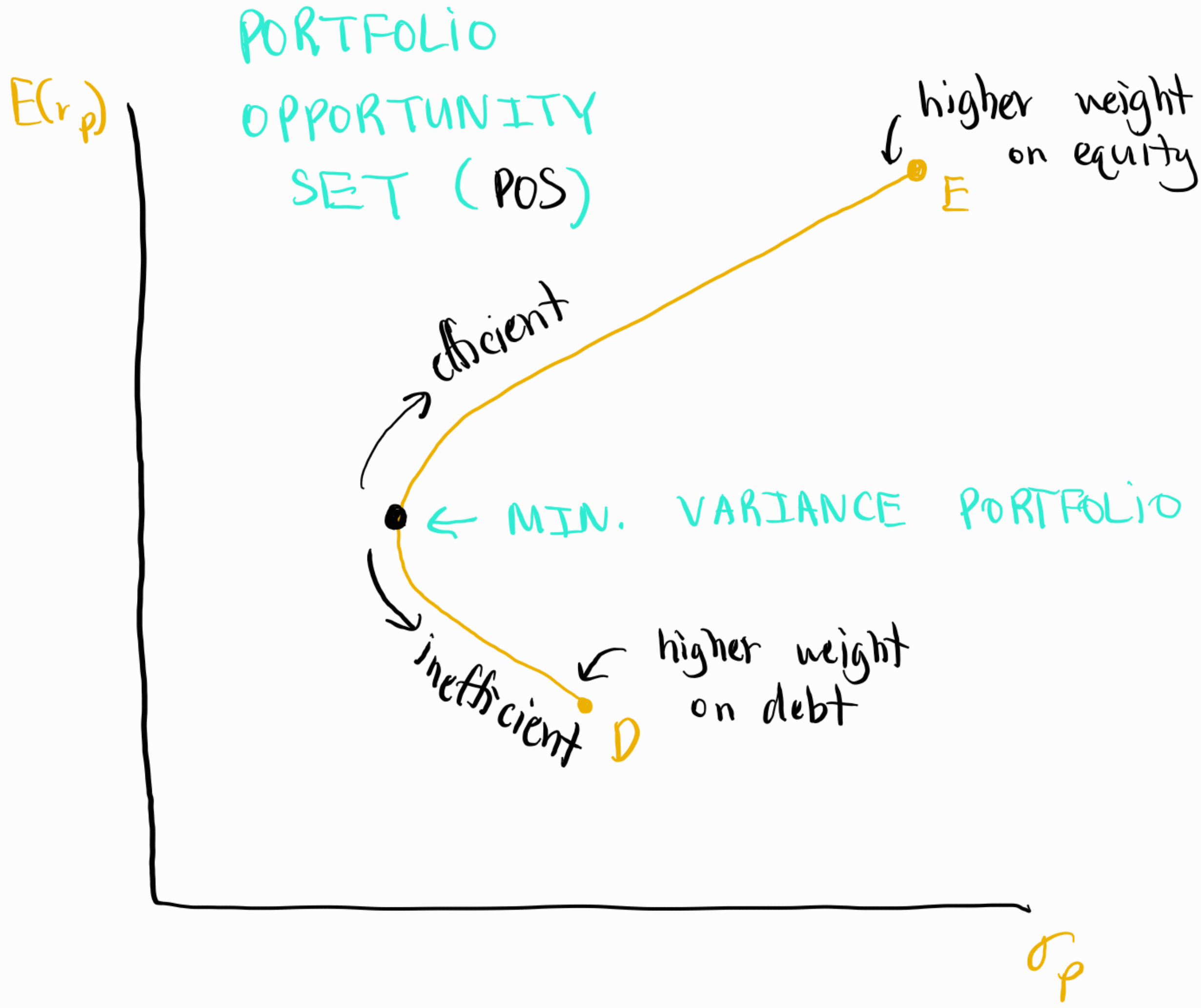
$$\boxed{\sigma_p^2 = 0}$$

When $\rho = -1$ and we have w_D^* & w_E^*, then there is no risk in our portfolio.



This will be the general shape of the $E(r_p)$ against σ_p line for a $\rho < 1$. However, we will only ever touch $\sigma_p = 0$ when $\rho = -1$.

As we change our weights, we move along the line below:



To get to the minimum variance portfolio, we optimize for the minimum σ_p^2.

$$\min_{w_D} \; \sigma_p^2 = w_D^2\sigma_D^2 + (1-w_D)^2\sigma_E^2 + 2w_D(1-w_D)\,cov(r_D, r_E)$$

$$\frac{\partial \sigma_p^2}{\partial w_D} = 2w_D\sigma_D^2 + 2(1-w_D)(-1)\sigma_E^2 + 2(1-w_D)\,cov + 2w_D(-1)\,cov \overset{\text{set}}{=} 0$$

$$2w_D\sigma_D^2 - 2\sigma_E^2 + 2w_D\sigma_E^2 + 2\text{cov} - 2w_D\text{cov} - 2w_D\text{cov} = 0$$

$$\cancel{2}\, w_D\left[\sigma_D^2 + \sigma_E^2 - 2\text{cov}\right] = \cancel{2}\left[\sigma_E^2 - \text{cov}\right]$$

$$\boxed{w_D^{MV} = \frac{\sigma_E^2 - \text{cov}(r_E, r_D)}{\sigma_D^2 + \sigma_E^2 - 2\text{cov}(r_E, r_D)}}$$

(minimum variance)

$$w_E^{MV} = 1 - w_D^{MV}$$

Example: If $\text{cov}(r_D, r_E) = 72$

Find: w_D^{MV}, w_E^{MV}, $E(r_p)$, σ_p

From before: $E(r_D) = 8$, $E(r_E) = 13$, $\sigma_D = 12$, $\sigma_E = 20$

$$w_D^{MV} = \frac{20^2 - 72}{12^2 + 20^2 - 2(72)} = \frac{328}{400} = \boxed{0.82}$$

$$w_E^{MV} = 1 - w_D^{MV} = \boxed{0.18}$$

$$E(r_p) = w_D^{MV} E(r_D) + w_E^{MV} E(r_E)$$

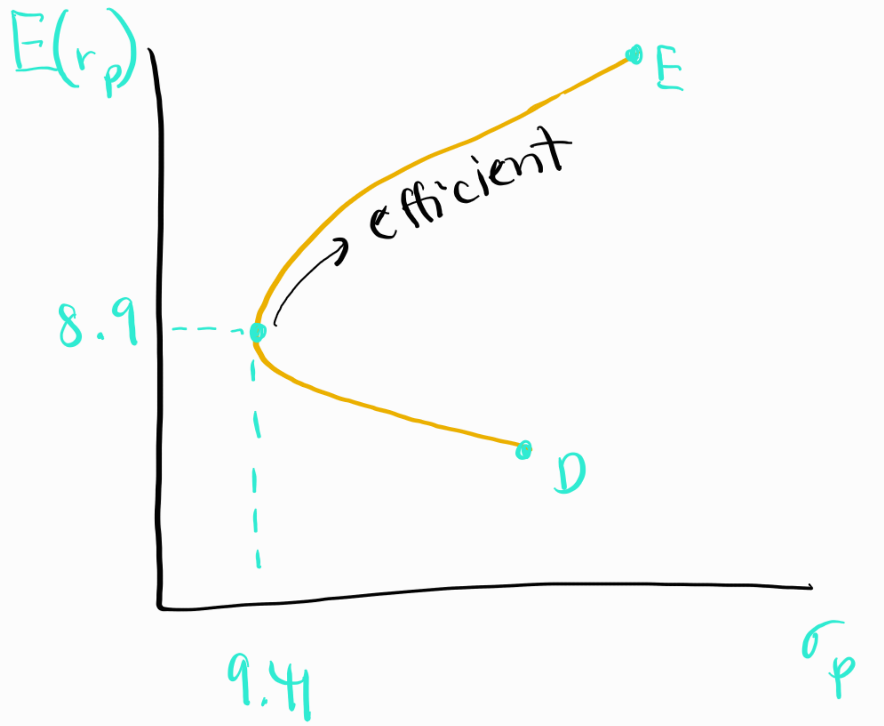
$$= 0.82 \times 8 + 0.18 \times 13 = \boxed{8.9}$$

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \operatorname{cov}(r_D, r_E)$$

$$\sigma_p^2 = (0.82)^2 12^2 + (0.18)^2 20^2 + 2(0.82)(0.18)\,72$$

$$= 96.8256 + 12.96 + 21.2544$$

$$\sigma_p^2 = 131.04 \rightarrow \boxed{\sigma_p = 11.45}$$



From this point on, we only focus on the efficient portion of the graph.

Back to the utility specification (another way to get to w_D)

$$U = E(r_p) - 0.005 A \sigma_p^2$$

We want to maximize U with respect to w_D,

where $E(r_p) = w_D E(r_D) + (1 - w_D) E(r_E)$

and $\sigma_p^2 = w_D^2 \sigma_D^2 + (1-w_D)^2 \sigma_E^2 - 2 w_D (1-w_D) \text{cov}$

After some algebra, we find that:

$$\frac{\partial U}{\partial w_D} = 0 = E(r_D) - E(r_E) - 0.01 A \cdot (w_D(\sigma_D^2 + \sigma_E^2 - 2\text{cov}) - \sigma_E^2 + \text{cov})$$

and therefore:

$$\boxed{w_D^* = \frac{E(r_D) - E(r_E) + 0.01 A[\sigma_E^2 - \text{cov}]}{0.01 A[\sigma_D^2 + \sigma_E^2 - 2\text{cov}]}}$$

(no risk free asset)

We use this formula for w_D^* when we are not given an option for a risk free asset. $r_c = r_p$ only. No r_f.

Example:

$A = 4 \quad cov = 72 \quad E(r_D) = 8 \quad E(r_E)\, 13$

$\sigma_D = 12 \quad \sigma_E = 20$

$$w_D = \frac{8 - 13 + 0.01(4)\left[20^2 - 72\right]}{0.01(4)\left[12^2 + 20^2 - 2(72)\right]} = \frac{8.12}{16}$$

$$\boxed{w_D = 0.5075 \qquad w_E = 0.4925}$$

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$

$$= 0.5075(8) + 0.4925(13)$$

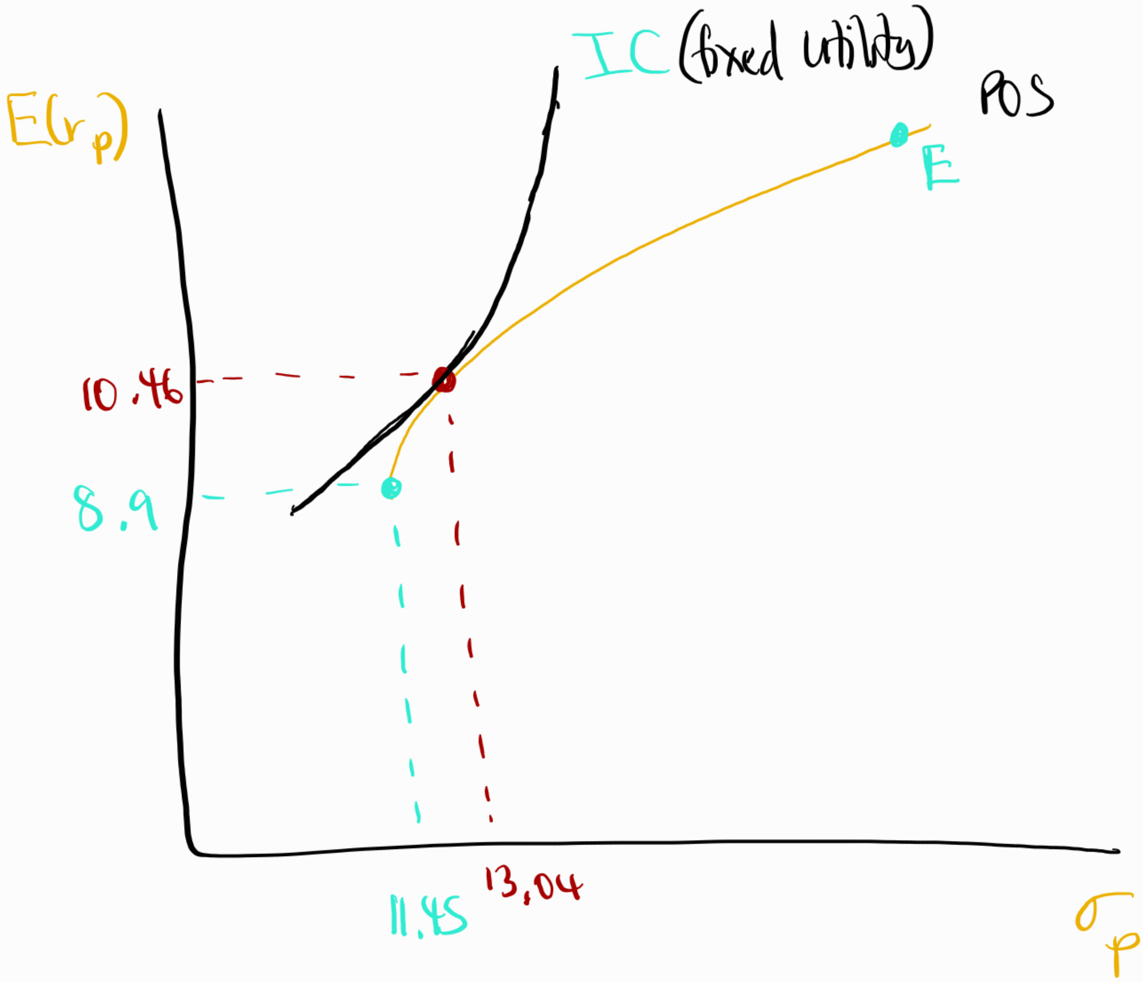
$$\boxed{E(r_p) = 10.4625}$$

$$\sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \, cov(r_D, r_E)$$

$$= (0.5075)^2 (12)^2 + (0.4925)^2 20^2 + 2(0.5075)(0.4929)72$$

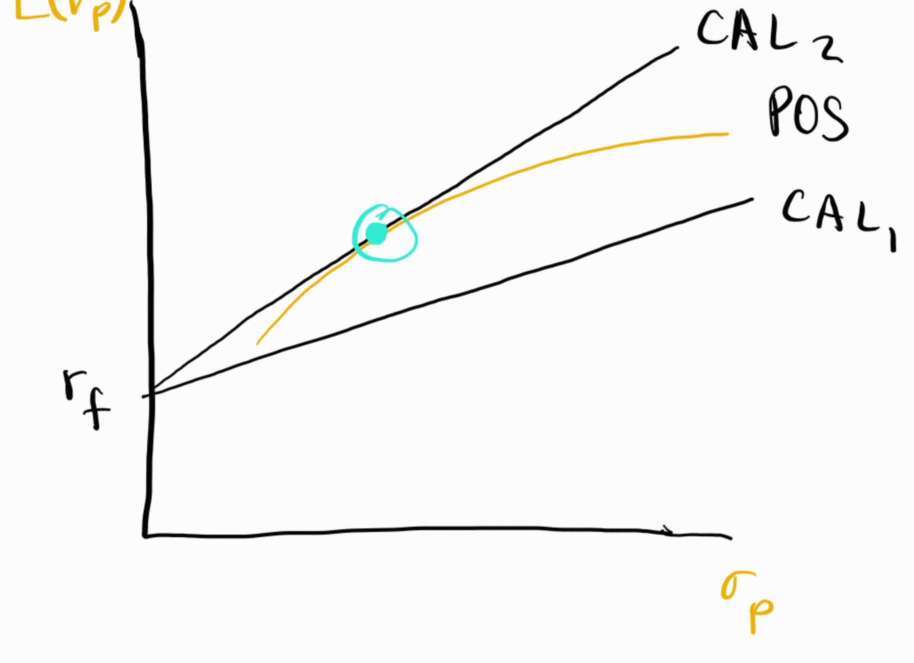
$$\sigma_p^2 = 37.0881 + 97.0225 + 35.9913$$

$$\sigma_p^2 = 169.98 \longrightarrow \boxed{\sigma_p = 13.04}$$



The maximizing point occurs where U is tangent to the POS.

Now suppose we bring back the risk free asset (D & E are both part of the risky portfolio).



In the graph above, we include both the POS and the CAL. The optimal point for the entire portfolio C (risk and no risk) occurs when the CAL is tangent to the POS. We are maximizing the slope of the CAL or the Sharpe Ratio.

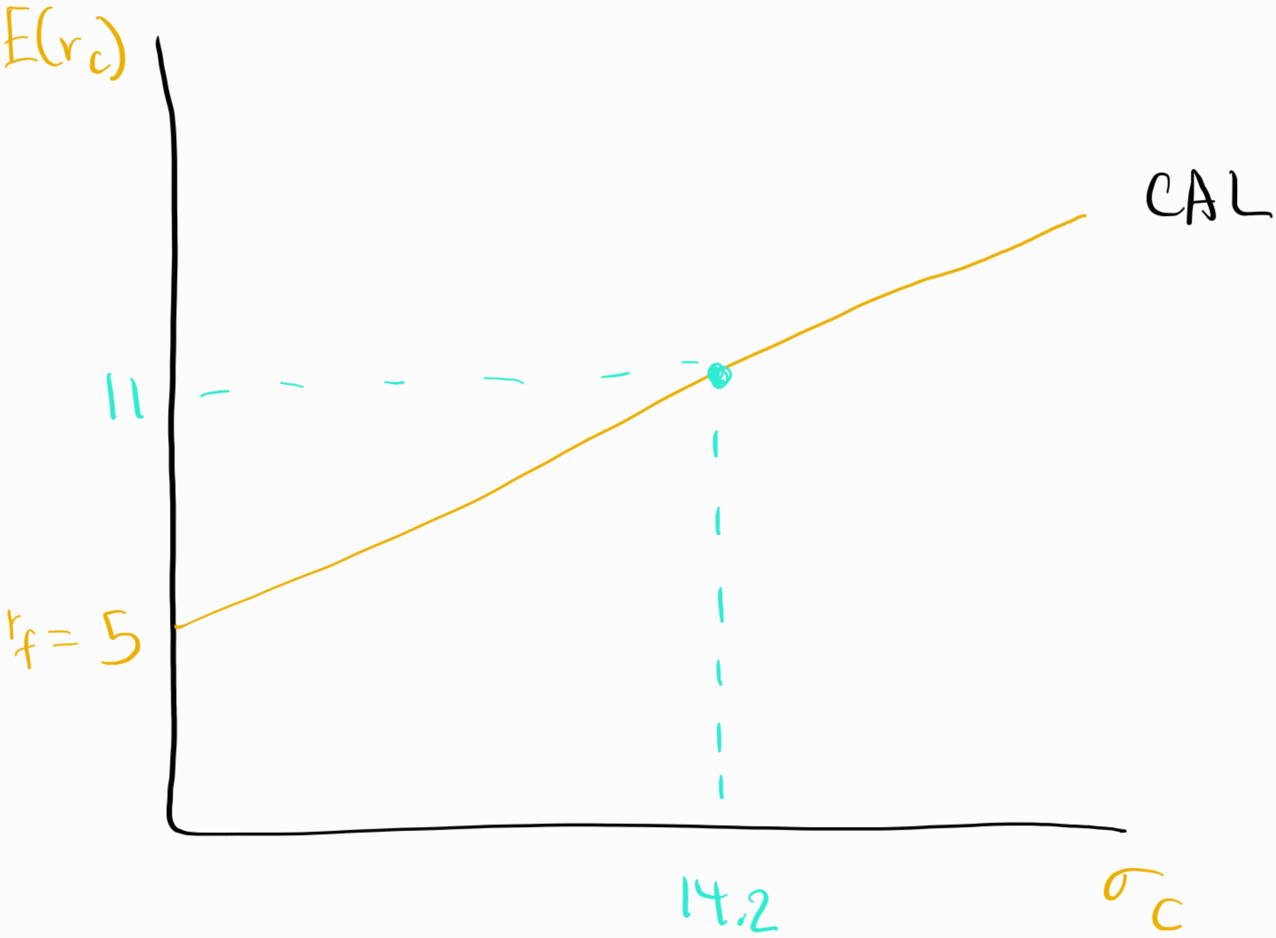
$$\max_{w_D} \; S = \frac{E(r_p) - r_f}{\sigma_p}$$

$$\boxed{w_D^* = \frac{[E(r_D) - r_f]\,\sigma_E^2 - [E(r_E) - r_f]\,\text{cov}(r_D, r_E)}{[E(r_D) - r_f]\,\sigma_E^2 + [E(r_E) - r_f]\,\sigma_D^2 - [E(r_D) + E(r_E) - 2r_f]\,\text{cov}(r_D, r_E)}}$$

we use this formula when you have the option of a risk free asset.

Using the values of $E(r_D)$, $E(r_E)$, σ_D, σ_E, & cov from other examples with $rf = 5$, we get

$$\boxed{\begin{array}{lll} w_D^* = 0.4 & w_E^* = 0.6 & \\ E(r_p) = 11 & \sigma_p = 14.20 & S = 0.42 \end{array}}$$



Recap:

$$U = E(r_c) - 0.005\, A\, \sigma_c^2$$

where $E(r_c) = r_f + y[E(r_p) - r_f] = \text{CAL}$

$$\sigma_c = y\,\sigma_p$$

$$y^* = \frac{E(r_p) - r_f}{0.01\, A\, \sigma_p^2}$$

So for our class example (with $A = 5$):

$$y^* = \frac{11 - 5}{0.01(5)(14.2)^2} = 0.595$$

$E(r_c) = 5 + 0.595(11-5) = 8.57$

$\sigma_c = 0.595(14.2) = 8.449$

Final Total Allocation:

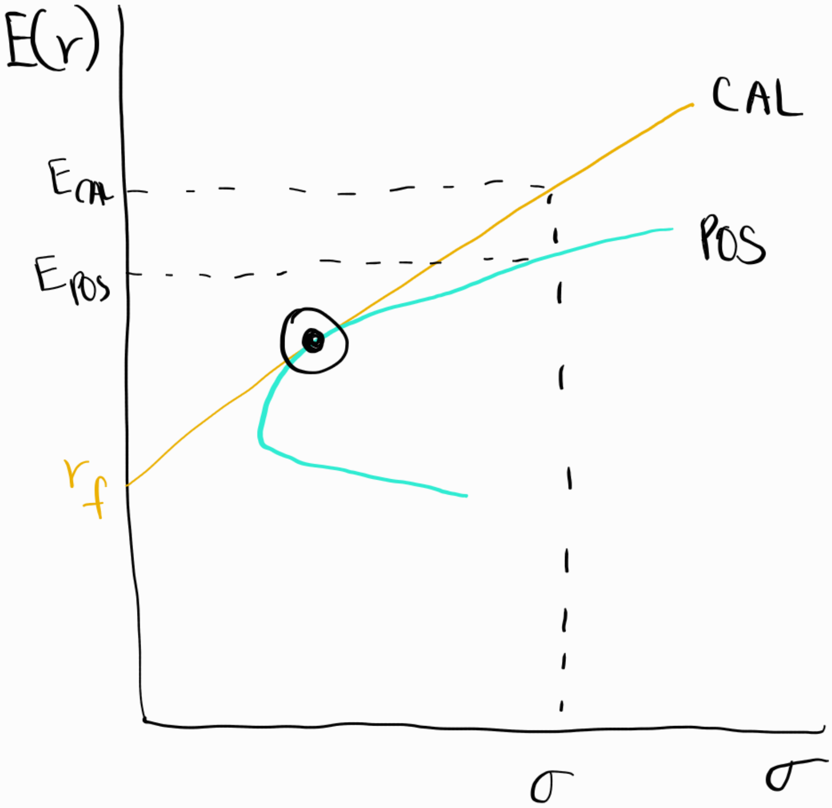
(risk free) $R_F \rightarrow 0.405$

(risky) $R_P \rightarrow 0.595$

D $\rightarrow (0.4)(0.595) = 0.238$

E $\rightarrow (0.6)(0.595) = 0.357$

Note:



Having the risk-free assets opens up better investment options for the agent.

The CAL allows higher E(r) for each value of σ than the POS except at the tangent point.