Problem 1.

Let $f(x)=x^6$ $\quad f'(x)=6x^5$ $\quad f''(x)=30x^4$

$$\therefore\ d(W_t)^6 = 6(W_t)^5\, dW_t + \frac{1}{2}\cdot 30 (W_t)^4\, dt$$

$$= 6(W_t)^5\, dW_t + 15 (W_t)^4\, dt.$$

$$\therefore\ W_T^6 = 6\int_0^T (W_t)^5\, dW_t + 15\int_0^T (W_t)^4\, dt.$$

$$\therefore\ 6\int_0^T (W_t)^5\, dW_t = W_T^6 - 15\int_0^T (W_t)^4\, dt.$$

$$\Rightarrow\ \int_0^T (W_t)^5\, dW_t = \frac{1}{6} W_T^6 - \frac{5}{2}\int_0^T (W_t)^4\, dt.$$

Problem 2

(1) Since $X_t = \int_0^t \sin(1+3W_s)\,dW_s$, find expression of X_t.

$f'(x) = \sin(1+3x) \Rightarrow f(x) = -\frac{1}{3}\cos(1+3x)$

$f''(x) = 3\cos(1+3x)$

$\therefore -\frac{1}{3}\cos(1+3W_t) - (-\frac{1}{3}\cos(1+W_0)) = \int_0^t \sin(1+3W_s)\,dW_s + \frac{1}{2}\int_0^t 3\cos(1+3W_s)\,ds$

$\Rightarrow -\frac{1}{3}\cos(1+3W_t) + \frac{1}{3}\cos(1) = \int_0^t \sin(1+3W_s)\,dW_s + \frac{3}{2}\int_0^t \cos(1+3W_s)\,ds$

$\Rightarrow X_t = \int_0^t \sin(1+3W_s)\,dW_s = -\frac{1}{3}\cos(1+3W_t) + \frac{1}{3}\cos(1) - \frac{3}{2}\int_0^t \cos(1+3W_s)\,ds$

(2) $E(X_t) = E\int_0^t \sin(1+3W_s)\,dW_s$

$= 0$

$Var(X_t) = E(X_t)^2 - [E(X_t)]^2$

$= E(X_t)^2$

$= E\left[\int_0^t \sin(1+3W_s)\,dW_s\right]^2$

Ito's Isometry

$= E\int_0^t \sin^2(1+3W_s)\,ds$

$\because \sin^2(x) = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$\therefore \int_0^t \left(\frac{1}{2} - \frac{1}{2}\cos(2+6W_s)\right) ds$

$= \frac{1}{2}t - \underbrace{\frac{1}{2}\int_0^t \cos(2+6W_s)\,ds}$

It's Riemann Integral, it should be considered done.

Problem 3

Since $f(x) = \int_1^x \frac{1}{w^2}\,dw$

$$\therefore \quad = -\frac{1}{w}\Big|_1^x$$

$$= -\left[\frac{1}{x} - 1\right]$$

$$= 1 - \frac{1}{x}$$

$$\therefore Y_t = f(X_t) = 1 - \frac{1}{X_t}$$

$$f'(X_t) = \frac{1}{X_t^2} \qquad f''(X_t) = \frac{-2X_t}{X_t^4} = -\frac{2}{X_t^3}$$

$$\therefore dY_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,dX_t\,dX_t$$

$$= \frac{1}{X_t^2}\,dX_t - \frac{1}{X_t^3}\,dX_t\,dX_t$$

$$= \frac{1}{X_t^2}\left[\alpha X_t\,dt + \beta (X_t)^2 dW_t\right] - \frac{1}{X_t^3}\left[\beta^2 (X_t)^4\,dt\right]$$

$$= \alpha \frac{1}{X_t}\,dt + \beta\,dW_t - \beta^2 X_t\,dt$$

$$= \left(\alpha \frac{1}{X_t} - \beta^2 X_t\right) dt + \beta\,dW_t$$

Since $Y_0 = 1 - \frac{1}{X_0} = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ the initial for Y_t is $Y_0 = \frac{1}{2}$

pages

Problem 4

$$\begin{cases} dS_t = \mu S_t\,dt + \sigma S_t\,dW_t & S_0 = S \\ r_t = r \\ \text{Terminal condition: } (K - S_T)^2 \end{cases}$$

For solving this question, we the price formula for call option: is:

$C(S_t, t)$, then By Ito's formula:

$$dC(S_t,t) = C_t(S_t,t)\,dt + C_x(S_t,t)\,dS_t + \tfrac{1}{2}C_{xx}(S_t,t)\,dS_t\,dS_t.$$

$$= C_t(S_t,t)\,dt + C_x(S_t,t)\,[\mu S_t\,dt + \sigma S_t\,dW_t] + \tfrac{1}{2}C_{xx}(S_t,t)\,[\sigma^2 S_t^2\,dt]$$

$$= [C_t(S_t,t) + C_x(S_t,t)\mu S_t + \tfrac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2]\,dt + C_x(S_t,t)\sigma S_t\,dW_t$$

Now, we can construct a riskless portfolio by buy a call option and short $C_x(S_t,t)$ share of stock.

Thus,

$$dC(S_t,t) - C_x(S_t,t)\,dS_t = [C_t(S_t,t) + \cancel{C_x(S_t,t)\mu S_t} + \tfrac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2]\,dt + \cancel{C_x(S_t,t)\sigma S_t\,dW_t} - \cancel{C_x(S_t,t)[\mu S_t\,dt + \sigma S_t\,dW_t]}$$

$$= [C_t(S_t,t) + \tfrac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2]\,dt.$$

By no arbitrage condition.

$$\frac{C_t(S_t,t) + \frac{1}{2}C_{xx}(S_t,t)\sigma^2 S_t^2}{C(S_t,t) - C_x(S_t,t)\,S_t} = r_t = r$$

Therefore,

$$C_t(S_t,t) + \frac{1}{2} C_{xx}(S_t,t)\sigma^2 S_t^2 = r C(S_t,t) - r S_t C_x(S_t,t)$$

$$\Rightarrow \underbrace{C_t(S_t,t) + r S_t C_x(S_t,t) + \frac{1}{2} C_{xx}(S_t,t)\sigma^2 S_t^2 = r C(S_t,t)}_{\text{PDE}} \qquad ①$$

The Boundry condition is

$$\underline{C(S_T,t) = (K-S_T)^2} \qquad ②$$

when if the price S_t goes along the whole path,

①, ② can be written as:

$$C_t(x,t) + r x C_x(x,t) + \frac{1}{2} C_{xx}(x,t)\sigma^2 x^2 = r C(x,t), \qquad ③$$

boundary condition $\quad C(x,t) = (K-X)^2 \qquad ④$

X follows the SDE.

$$dX_t = r X_t dt + \sigma X_t dW_t.$$

2) As X_t is a geometric Brownian motion.

it can be solved.

$$\log(X_T) - \log X_t = (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$$

where $\log X_T$ has the normal distribution w/ mean

$\log x + (r - \frac{1}{2}\sigma^2)(T-t)$. the variance is $\sigma^2(T-t)$,

condition on $X_t = x$

$$\therefore \log(X_T) \sim \log x + (r - \tfrac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}\, Z$$

Z is standard normal variable.

By Feynman-Kac's Theorem.

$$C(x,t) = E^{x,t}\left[e^{-r(T-t)} \cdot (K-X_T)^2\right]$$

$$= e^{-r(T-t)} \int (K-X_T)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= e^{-r(T-t)} \int \left(K - e^{\log x + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}\, z}\right)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= e^{-r(T-t)} K^2 \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz +$$

$$-2Ke^{-r(T-t)} \int e^{\log x + (r-\frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}\, z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$+ e^{-r(T-t)} \int e^{2[\log x + (r-\frac{1}{2}\sigma^2)(T-t)] - 2\sigma\sqrt{T-t}\, z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= e^{-r(T-t)} K^2 - 2Ke^{-r(T-t)} x + x^2 e^{\sigma^2(T-t)}$$

So the pricing function is given by

$$C(x,t) = x^2 e^{\sigma^2(T-t)} - 2Kx + K^2 e^{-r(T-t)}$$

when stock price is S_t, it's $C(S_t, t)$