5.B.2 The Cobb-Douglas production function with two inputs is given by $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, where $\alpha \geq 0$ and $\beta \geq 0$. Verify that the marginal technical rate of substitution between the two inputs at $z = (z_1, z_2)$ is $MRTS_{12}(z) = \cancel{\frac{\alpha z_1}{\beta z_2}}$. $\frac{\alpha z_2}{\beta z_1}$

Example 5.B.2: *The Cobb–Douglas Production Function* The *Cobb–Douglas* production function with two inputs is given by $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, where $\alpha \geq 0$ and $\beta \geq 0$. The marginal rate of technical substitution between the two inputs at $z = (z_1, z_2)$ is $MRTS_{12}(z) = \underline{\alpha z_2 / \beta z_1}$. ■

We have: $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$ $\quad \alpha \geq 0, \ \beta \geq 0$

Show: $MRTS_{12}(z) = \frac{\alpha z_1}{\beta z_2}$

Definition: Marginal Rate of Technical Substitution (MRTS) of input l for input k at $\bar{z}$:

$$MRTS_{lk}(\bar{z}) = \frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z})}{\partial z_k}}$$

$\downarrow$

measures the additional amount of input k that must be used to keep output at level $\bar{q} = f(\bar{z})$ when the amount of input l is decreased marginally

Answer: $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$

$$\frac{\partial f}{\partial z_1} = \alpha z_1^{\alpha-1} z_2^{\beta} \qquad \frac{\partial f}{\partial z_2} = \beta z_1^{\alpha} z_2^{\beta-1}$$

$$MRTS_{12}(z) = \frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} = \frac{\alpha z_1^{\alpha-1} z_2^{\beta}}{\beta z_1^{\alpha} z_2^{\beta-1}}$$

$$= \frac{\alpha}{\beta} \frac{z_2^{\beta-\beta+1}}{z_1^{\alpha-\alpha+1}}$$

$$= \frac{\alpha z_2}{\beta z_1}$$

5.B.3 Show that for a single-output technology, Y is convex if and only if the production function $f(z)$ is concave.

5.B.3 Suppose first that Y is convex. Let $z, z' \in \mathbb{R}^{L-1}_{+}$ and $\alpha \in [0,1]$, then $(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$. By the convexity,

$$(-(\alpha z + (1-\alpha)z), \alpha f(z) + (1-\alpha)f(z)) \in Y.$$

Thus, $\alpha f(z) + (1-\alpha)f(z) \le f(\alpha z + (1-\alpha)z)$. Hence $f(z)$ is concave.

Suppose conversely that $f(z)$ is concave. Let $(q, -z) \in Y$, $(q', -z') \in Y$, and $\alpha \in [0,1]$, then $q \le f(z)$ and $q' \le f(z')$. Hence

$$\alpha q + (1-\alpha)q' \le \alpha f(z) + (1-\alpha)f(z').$$

By the concavity,

$$\alpha f(z) + (1-\alpha)f(z') \le f(\alpha z + (1-\alpha)z').$$

Thus

$$\alpha q + (1-\alpha)q' \le f(\alpha z + (1-\alpha)z').$$

Hence

$$(-(\alpha z + (1-\alpha)z'), \alpha q + (1-\alpha)q') = \alpha(-z, q) + (1-\alpha)(-z', q') \in Y.$$

Therefore Y is convex.

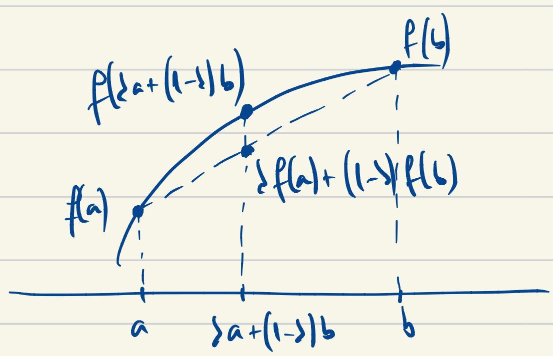
→ production set

→ production function

Show: Y is convex iff $f(z)$ is concave.

Definitions: Convexity: Y is convex if for $y \in Y$, $y' \in Y$ and $\alpha \in [0,1]$ we have $\alpha y + (1-\alpha) y' \in Y$.

Concavity: $f(x)$ is concave over an interval $[x_1, x_2]$ if for any two points a, b in $[x_1, x_2]$ and $0 < \lambda < 1$:
$f(\lambda a + (1-\lambda) b) \geq \lambda f(a) + (1-\lambda) f(b)$.



Note: As usual to establish equivalence: $A \Leftrightarrow B$ we need to prove $A \rightarrow B$ and $B \rightarrow A$.

Answer: Two parts: (1) Convexity of $Y \Rightarrow$ Concavity of $f(z)$.

(2) Concavity of $f(z) \Rightarrow$ Convexity of Y.

(1) Convexity of $Y \Rightarrow$ Concavity of $f(z)$.

Suppose Y is convex.

Let $z, z' \in \mathbb{R}^{L-1}_{+}$ and $\alpha \in [0,1]$. Then $(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$.

↳ production plans using input vectors z and z' are feasible

By convexity of Y:

$$\left(-(\alpha z + (1-\alpha) z'),\ \alpha f(z) + (1-\alpha) f(z')\right) \in Y$$

↳ if each of the productions plans is individually feasible then a combination of the two is also feasible.

Thus: $\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z')$

↳ the combination of the two production plans should yield at most as much as the output from the combination of the input vectors

Hence, $f(z)$ is concave.

(2) Concavity of $f(z) \Rightarrow$ Convexity of Y.

Conversely, suppose that $f(z)$ is concave.

Let $(q, -z) \in Y$, $(q, -z') \in Y$ and $\alpha \in [0,1]$.

↳ consider two feasible production plans with input vectors z and z'.

Then $q \leq f(z)$ and $q' \leq f(z')$.

↳ by free disposal, production function $f(z)$ gives the max amount q of output that can be produced given inputs z

It follows that:

$$q \leq f(z) \Rightarrow \alpha q \leq \alpha f(z)$$

and $q' \leq f(z') \Rightarrow (1-\alpha) q \leq (1-\alpha) f(z')$

for $\alpha \in [0,1]$.

Summing up the inequalities:

$$\alpha q + (1-\alpha) q' \leq \alpha f(z) + (1-\alpha) f(z')$$

By concavity of $f(z)$:

$$\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z').$$

Substituting into the previous inequality, it will also hold that:

$$\alpha q + (1-\alpha) q' \leq f(\alpha z + (1-\alpha) z').$$

↳ the idea is that if we have $a \leq b$ and $b \leq c$ then for sure $a \leq c$.

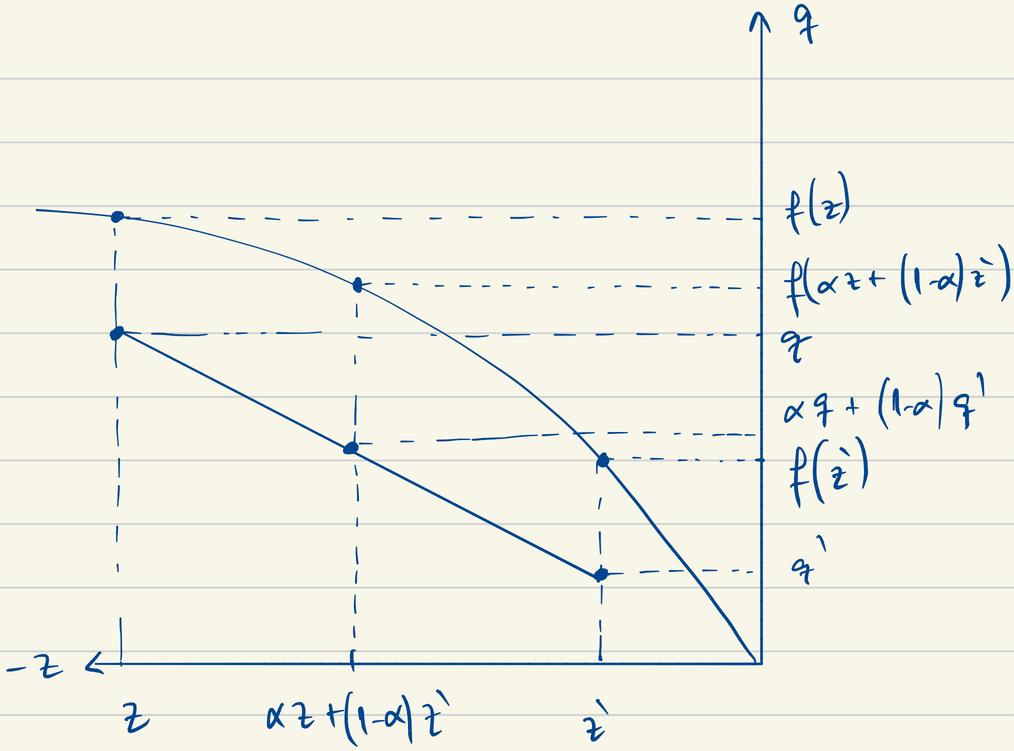
the economic intuition is that if we have two production plans that yield the same output but use different input combinations, then a production vector that uses an average of the inputs used in those two plans will do at least as either one of the two.

Hence, $\left(-(\alpha z + (1-\alpha) z'), \alpha q + (1-\alpha) q'\right)$

$$= \alpha(-z, q) + (1-\alpha)(-z', q') \in Y$$

↳ the production plan that uses an average of the inputs in the two starting plans is also feasible

Therefore, Y is convex.



q
f(z)
f(αz + (1-α)z')
q
αq + (1-α)q'
f(z')
q'
-z
z
αz + (1-α)z'
z'

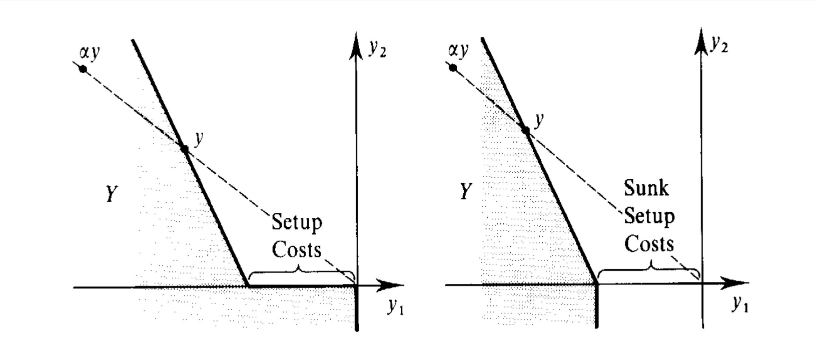
5.C.1 Prove that, in general, if the production set Y exhibits non-decreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$.

5.C.1 If there is a production plan $y \in Y$ with $p \cdot y > 0$, then, by using $\alpha y \in Y$ with a large $\alpha > 1$, it is possible to attain any sufficiently large profit level. Hence $\pi(p) = \infty$. If, on the contrary, $p \cdot y \leq 0$ for all $y \in Y$, then $\pi(p) \leq 0$. Thus we have either $\pi(p) = +\infty$ or $\pi(p) \leq 0$.

Definition: Non decreasing returns to scale: $\forall y \in Y$ we have $\alpha y \in Y$ for any scale $\alpha \geq 1$.

↳ any feasible input-output vector can be scaled up.

can occur if there are fixed setup costs to start producing.



Answer: If there is a production plan $y \in Y$ with $p \cdot y > 0$ (positive profit)
then by using $\alpha y \in Y$ with large $\alpha > 1$
(i.e. from non decreasing returns to scale) it is possible to attain any sufficiently large profit level:

Hence, $\pi(p) = +\infty$.

If $p \cdot y \leq 0$ for all $y \in Y$, then:

$$\pi(p) \leq 0.$$

Thus, we can either have $\pi(p) = +\infty$ or $\pi(p) \leq 0$.

5.C.8 Alpha Incorporated (AI) produces a single output q from two inputs z_1 and z_2. You are assigned to determine AI's technology. You are given 100 monthly observations. Two of these monthly observations are shown in the following table.

	Input prices		Input levels		Output price	Output level
Month	w_1	w_2	z_1	z_2	p	q
3	3	1	40	50	4	60
95	2	2	55	40	4	60

In light of these two monthly observations, what problem will you encounter in trying to accomplish your task?

5.C.8 The cost that AI incurred in month 95 is $2 \cdot 55 + 2 \cdot 40 = 190$, but it could attain the same output level with a lower cost by using the input combination of month 3: $2 \cdot 40 + 2 \cdot 50 = 180$. Thus the problem we will encounter is that, perhaps due to mis-observation and/or some restrictions that AI faced outside the market, the profit-maximizing production plans are not observed to have been used and it is impossible to use those observations in order to recover its technology based on Proposition 5.C.2(iii) or 5.C.1(iii).

Total Cost (month = 3) = $w_1 z_1 + w_2 z_2 = 3 \times 40 + 1 \times 50 = 170$

Total Revenue (month = 3) = $p \cdot q = 4 \times 60 = 240$

Profit (month = 3) = $240 - 170 = 70$

Total Cost (month = 95) = $w_1 z_1 + w_2 z_2 = 2 \times 55 + 2 \times 40 = 190$

Total Revenue (month = 95) = $p \cdot q = 240$

Profit (month = 95) = 50

The cost of AI incurred in month 95 is 190 but it could achieve the same output level with a lower cost of 170 by using the input combination of month 3.

Thus, it seems that in the month 95 the firm did not choose its optimal production point.

It is possible that the firm faced some production restrictions that are not observable just based on this information. This firm's technology is therefore not immediately recoverable from these observations.

5.C.10 Derive the cost function $c(w,q)$ and conditional factor demand functions (or correspondences) $z(w,q)$ for each of the following single-output constant return technologies with production functions with production functions given by

(a) $f(z) = z_1 + z_2$ (perfect substitutable inputs)

(b) $f(z) = \min z_1, z_2$ (Leontief technology)

(c) $f(z) = (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}}$, $rho \leq 1$ (constant elasticity of substitution)

5.C.10

(a) $c(w,q) = \begin{cases} qw_1 & \text{if } w_1 \leq w_2; \\ qw_2 & \text{if } w_1 > w_2. \end{cases}$

$$z(w,q) = \begin{cases} (q,0) & \text{if } w_1 < w_2; \\ \{(z_1,z_2) \in \mathbb{R}^2_+ : z_1 + z_2 = q\} & \text{if } w_1 = w_2; \\ (0,q) & \text{if } w_1 > w_2. \end{cases}$$

(b) $c(w,q) = (w_1 + w_2)q$. $z(w,q) = (q,q)$.

(c) $c(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(1-1/\rho)}$.

$z(w,q) = q(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)})^{(-1/\rho)}(w_1^{1/(\rho-1)}, w_2^{1/(\rho-1)})$.

Derive: $c(w,q) \rightarrow$ cost function

$z(w,q) \rightarrow$ conditional factor demand function

a) $f(z) = z_1 + z_2$ (perfect substitutable inputs)

CMP: $\min_{\{z \geq 0\}} w \cdot z = w_1 z_1 + w_2 z_2$

s.t: $f(z) \geq q$

$$\mathcal{L} = w_1 z_1 + w_2 z_2 + \lambda(q - f(z))$$

$$= w_1 z_1 + w_2 z_2 + \lambda(q - z_1 - z_2)$$

$(z_1): \quad w_1 - \lambda = 0 \iff \lambda = w_1$

$(z_2): \quad w_2 - \lambda = 0 \iff \lambda = w_2$

$(\lambda): \quad q - z_1 - z_2 = 0$

from $(z_1) + (z_2)$: $\quad w_1 = w_2$

<u>case 1</u>: $\quad w_1 > w_2 \qquad$ (input 1 is more expensive)

$$\frac{w_1}{MP_1} > \frac{w_2}{MP_2} \iff \frac{MP_2}{w_2} > \frac{MP_1}{w_1}$$

→ the per dollar marginal product is higher from input 2

since inputs are perfectly substitutable, the producer will choose <u>only</u> input 2 and will not demand input 1.

$$q = z_1 + z_2 = 0 + z_2 = z_2$$

$$z(w, q) = (z_1, z_2) = (0, q)$$

$$c(w, q) = w_1 z_1 + w_2 z_2 = w_1 \times 0 + w_2 z_2 = w_2 q$$

Case 2 : $w_1 < w_2$

$$\frac{w_1}{MP_1} < \frac{w_2}{MP_2} \Leftrightarrow \frac{MP_2}{w_2} < \frac{MP_1}{w_1}$$

→ the per dollar marginal product is higher from input 1

since inputs are perfectly substitutable, the producer will choose only input 1 and will not demand input 2.

$$q = z_1 + z_2 = z_1 + 0$$

$$z(w, q) = (z_1, z_2) = (q, 0)$$

$$c(w, q) = w_1 z_1 + w_2 z_2 = w_1 z_1 + w_2 \times 0 = w_1 q$$

Case 3 : $w_1 = w_2$

$$\frac{w_1}{MP_1} = \frac{w_2}{MP_2} \Leftrightarrow \frac{MP_1}{w_1} = \frac{MP_2}{w_2}$$

→ the producer is indifferent between the two inputs, and any combination of the two inputs that satisfies the production requirement is optimal.

$$q = z_1 + z_2 \Leftrightarrow z_1 = q - z_2$$

$$z(w, q) = (z_1, z_2) = (q - z_2, z_2)$$

or $$z(w, q) = (\{z_1, z_2\} \in \mathbb{R}^2_+ : z_1 + z_2 = q)$$

$$c(w, q) = q w_1 = q w_2$$

Putting together all 3 cases:

$$c(w, q) = \begin{cases} q w_1 & \text{if } w_1 < w_2 \\ q w_1 = q w_2 & \text{if } w_1 = w_2 \\ q w_2 & \text{if } w_1 > w_2 \end{cases}$$

$$z(w, q) = \begin{cases} (q, 0) & \text{if } w_1 < w_2 \\ \{(z_1, z_2) \in \mathbb{R}^2_+ : z_1 + z_2 = q\} & \text{if } w_1 = w_2 \\ (0, q) & \text{if } w_1 > w_2 \end{cases}$$

b) $f(z) = \min\{z_1, z_2\}$ (Leontief technology)

CMP: $\min_{\{z \geq 0\}} \quad wz$

s.t. $f(z) \geq q$ with $q = \min\{z_1, z_2\}$

The CMP problem cannot be solved with the standard Lagrangean because the production function is not differentiable (there is a kink at $z_1 = z_2$).

The producer will use both input equally to minimize costs.

Thus, we must have:

$$z_1 = z_2 = q$$

$$z(w, q) = (z_1, z_2) = (q, q)$$

$$c(w, q) = w_1 z_1 + w_2 z_2 = w_1 q + w_2 q = (w_1 + w_2) q$$

c) $f(z) = \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}}$ with $\rho \leq 1$ (constant elasticity of substitution)

CMP: $\min_{\{z \geq 0\}} \quad w \cdot z = w_1 z_1 + w_2 z_2$

s.t: $f(z) \geq q$ and $f(z) = \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}}$

$$\mathcal{L} = w_1 z_1 + w_2 z_2 + \lambda \left[q - \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho}} \right]$$

(z_1): $w_1 - \lambda \frac{1}{\rho} \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho} - 1} \rho z_1^{\rho - 1} = 0$

$\Leftrightarrow \quad w_1 = \lambda \left(z_1^\rho + z_2^\rho\right)^{\frac{1-\rho}{\rho}} z_1^{\rho-1}$

$\Leftrightarrow \quad \lambda = \dfrac{w_1}{\left(z_1^\rho + z_2^\rho\right)^{\frac{1-\rho}{\rho}} z_1^{\rho-1}}$

(z_2): $w_2 - \lambda \frac{1}{\rho} \left(z_1^\rho + z_2^\rho\right)^{\frac{1}{\rho} - 1} \rho z_2^{\rho - 1} = 0$

$\Leftrightarrow \quad w_2 = \lambda \left(z_1^\rho + z_2^\rho\right)^{\frac{1-\rho}{\rho}} z_2^{\rho-1}$

$\Leftrightarrow \quad \lambda = \dfrac{w_2}{\left(z_1^\rho + z_2^\rho\right)^{\frac{1-\rho}{\rho}} z_2^{\rho-1}}$

(λ): $q - (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}} = 0$

using $(z_1) + (z_2)$:

$$\frac{w_1}{(z_1^\rho + z_2^\rho)^{\frac{1-\rho}{\rho}} z_1^{\rho-1}} = \frac{w_2}{(z_1^\rho + z_2^\rho)^{\frac{1-\rho}{\rho}} z_2^{\rho-1}}$$

$$\Leftrightarrow \frac{w_1}{w_2} = \frac{z_1^{\rho-1}}{z_2^{\rho-1}} \Leftrightarrow \frac{w_1}{w_2} = \left(\frac{z_1}{z_2}\right)^{\rho-1}$$

$$\Leftrightarrow \frac{z_1}{z_2} = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} \Leftrightarrow z_1 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} z_2$$

plug into (λ):

$$q = \left\{ \left[\left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} z_2 \right]^\rho + z_2^\rho \right\}^{\frac{1}{\rho}}$$

$$\Leftrightarrow q^\rho = \left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} z_2^\rho + z_2^\rho \Leftrightarrow q^\rho = \left[\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1 \right] z_2^\rho$$

$$\Leftrightarrow z_2 = q \left[\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1 \right]^{-\frac{1}{\rho}}$$

and $$z_1 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} q \left[\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1 \right]^{-\frac{1}{\rho}}$$

Now, "simplify":

$$z_1 = q \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} \left[\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1\right]^{-\frac{1}{\rho}}$$

$$= q \left[\frac{\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}}}{\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1}\right]^{\frac{1}{\rho}} = q \left[\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\right]^{\frac{1}{\rho}}$$

$$z_2 = q \left[\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1\right]^{-\frac{1}{\rho}}$$

$$= q \left[\frac{1}{\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} + 1}\right]^{\frac{1}{\rho}} = q \left[\frac{1}{\frac{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}}}\right]^{\frac{1}{\rho}}$$

$$= q \left[\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\right]^{\frac{1}{\rho}}$$

$$z(w,q) = \left(q \left[\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}} \right]^{\frac{1}{\rho}} ,\ q \left[\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}} \right]^{\frac{1}{\rho}} \right)$$

$$c(w,q) = w_1 z_1 + w_2 z_2 =$$

$$= w_1 q \left[\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}} \right]^{\frac{1}{\rho}} + w_2 q \left[\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}} \right]^{\frac{1}{\rho}}$$

$$= w_1 q \left[\frac{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}}} \right]^{-\frac{1}{\rho}} + w_2 q \left[\frac{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}} \right]^{-\frac{1}{\rho}}$$

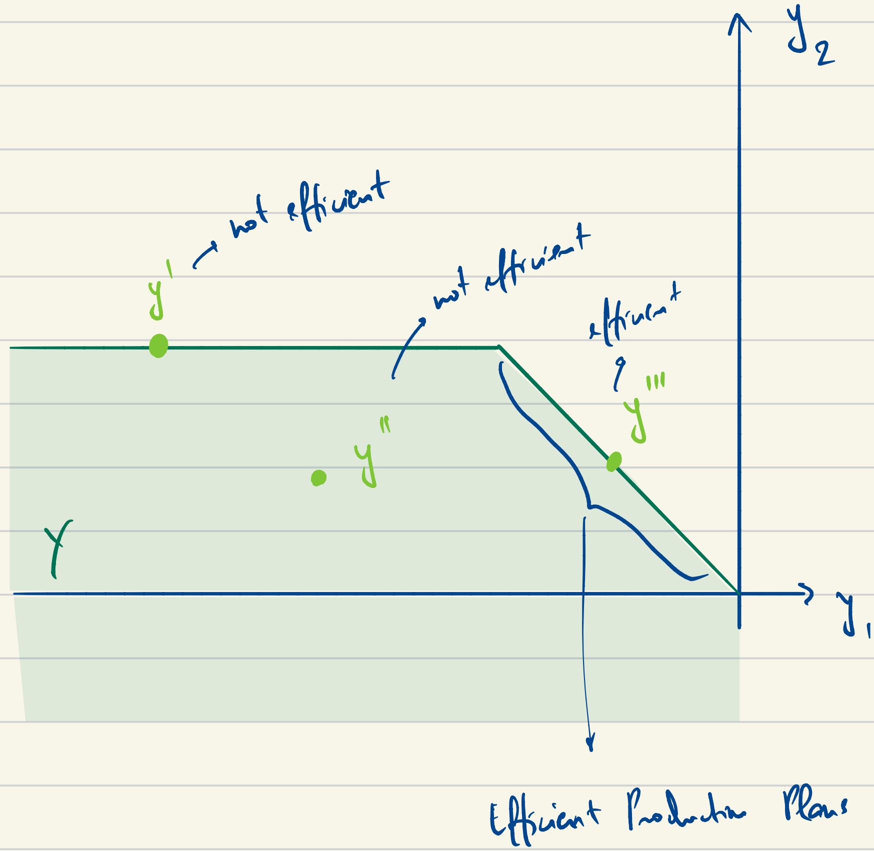
$$= q\, w_1^{1+\frac{1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{-\frac{1}{\rho}} + q\, w_2^{1+\frac{1}{\rho-1}} \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{-\frac{1}{\rho}}$$

$$= q \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right) \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{-\frac{1}{\rho}}$$

$$= q \left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}} \right)^{1-\frac{1}{\rho}}$$

5.F.1 Give an example of a $y \in Y$ that is profit maximizing for some $p \geq 0$ with $p \neq 0$ but that is also inefficient.

5.F.1 The production plan y in Figure 5.F.1(b) is not efficient but it maximizes profit for $p = (0,1)$.



nonnegative prices

Since we assume $p \geq 0$, we can allow some prices to be zero.

Suppose the price of input 1, y_1, is zero. Then being inefficient would not have an impact on profits and the production plan y' would still be profit maximizing.

So y' is profit maximizing even though it is not efficient.