Homework #1 Solution

Problem 1. Let $W(t)$ be the standard Brownian Motion on the interval $[0,T]$.

(a) Write down the density of $W(T)$.

(b) What is the joint density for $W(s)$ and $W(t)$ where $0 \le s < t \le T$.

Answer:

(a). The definition of BM ensures that $W(T)$ (which is just $W(T) - W(0) = W(T)$, since $W(0) = 0$) is a normal random variable with mean 0 and variance T. So it has density

$$f(x) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}.$$

(b). The definition of BM requires that $W(s)$ and $W(T) - W(s)$ to be independent normal random variables. Hence $W(s)$ and $W(t) - W(s)$ are joinly normally distributed. If follows that $W(s)$ and $W(t)$ are joint normally distributed. To derive the joint density of $W(s)$ and $W(t)$, we only need to find the mean, variance, and the correlation (or covariance) of them. These parameters are easily computed using properties of BM:

$$\begin{aligned}
\mathbb{E}W(s) &= 0, \mathbb{E}W(t) = 0 \\
Var\left[W(s)\right] &= \mathbb{E}\left[W(s)\right]^2 = s \\
Var\left[W(t)\right] &= \mathbb{E}\left[W(t)\right]^2 = t \\
Cov(W(s), W(s)) &= \mathbb{E}\left[W(s)W(t)\right] \\
&= \mathbb{E}\mathbb{E}\left[W(s)\left[W(t) - W(s) + W(s)\right]|F_s\right] \\
&= \mathbb{E}\mathbb{E}\left[W(s)\left[W(t) - W(s)\right]|F_s\right] + \mathbb{E}\mathbb{E}\left[W(s)W(s)|F_s\right] \\
&= \mathbb{E}\left[W(s)\mathbb{E}\left[W(t) - W(s)\right]|F_s\right] + \mathbb{E}\mathbb{E}\left[W(s)^2|F_s\right] \\
&= 0 + \mathbb{E}\left[W(s)^2\right] = s.
\end{aligned}$$

It follows that the correlation coefficient between $W(s)$ and $W(t)$ is

$$\rho = \frac{Cov(W(s), W(s))}{\sqrt{Var\left[W(s)\right] Var\left[W(t)\right]}} = \frac{s}{\sqrt{st}} = \sqrt{\frac{s}{t}}.$$

Recalling the form of bi-variate normal density (with mean vector (μ_x, μ_y), and covariance matrix

$$\begin{pmatrix} \sigma_x^2 & \rho\sigma_x\rho_y \\ \rho\sigma_x\rho_y & \sigma_y^2 \end{pmatrix}$$

is given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right).$$

Substitution yields the joint density of $W(s)$ and $W(t)$ is just

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{s}\sqrt{t}\sqrt{1-\frac{s}{t}}} \exp\left(-\frac{1}{2(1-\frac{s}{t})}\left[\frac{x^2}{s} + \frac{y^2}{t} - \frac{2\sqrt{\frac{s}{t}}xy}{\sqrt{s}\sqrt{t}}\right]\right) \\ &= \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{1}{2(1-s/t)}\left[\frac{x^2}{s} + \frac{y^2}{t} - \frac{2xy}{s/t}\right]\right). \end{aligned}$$

Problem 2. Let $W(t)$ be the standard Brownian Motion on the interval $[0, T]$.

(a) Compute the conditional expectation of $\mathbb{E}[W(t)|W(s) = c]$, where $0 < s < t < T$ and c is a fixed constant.

(b) Compute the expectation $\mathbb{E}[W(t)^2]$

(c) (Bonus question) Compute $\mathbb{E}[W(t)^6]$

(c) (Bonus question) Compute Expectation $\mathbb{E}[e^{1+2W(t)}]$.

Answer:

(a).

$$\begin{aligned} \mathbb{E}[W(t)|W(s) &= c] = \mathbb{E}[W(s) + (W(t) - W(s))|W(s) = c] \\ &= \mathbb{E}[W(s)|W(s) = c] + \mathbb{E}(W(t) - W(s))|W(s) = c] \\ &= c + 0 = c. \end{aligned}$$

(b). From the definition of BM, $W(t)$ is a normal random variable with mean 0 and variance t. So

$$\mathbb{E}[W(t)^2] = Var(W(t)) = t.$$

(c). Note that $W(t)$ is distributed as $N(0,t)$. To save notation, set $X = W(t)$, then $Y = X/\sqrt{t}$ is a standard normal random variable. Hence

$$\begin{aligned}\mathbb{E}[W(t)^6] &= \mathbb{E}\left[\sqrt{t}Y\right]^6 \\ &= t^3\mathbb{E}\left[Y\right]^6,\end{aligned}$$

so as long as we can evaluate the 6-th moment of a standard normal random variable we are done. But this is easy:

$$\mathbb{E}\left[Y\right]^6 = \int y^6\phi(y)dy$$

where

$$\phi(y) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{y^2}{2}\right].$$

This can be done by integration by parts. Note that ϕ satisfies

$$d\phi(y) = -y\phi(y),$$

we have

$$\begin{aligned}\mathbb{E}\left[Y\right]^6 &= \int y^6\phi(y)dy \\ &= \int -y^5 d\phi(y) \\ &= -y^5 d\phi(y)|_{-\infty}^{+\infty} - \int(-5y^4)\phi(y)dy \text{ Note the first term is 0 (why?)} \\ &= 5\int y^4\phi(y)dy \\ &\equiv 5\mathbb{E}\left[Y\right]^4.\end{aligned}$$

We can use the same trick to evaluate $\mathbb{E}\left[Y\right]^4$

$$\begin{aligned}\mathbb{E}\left[Y\right]^4 &= \int y^4\phi(y)dy \\ &= \int -y^3 d\phi(y) \\ &= \left[-y\phi(y)\right]_{-\infty}^{+\infty} - \int(-3y^2)\phi(y)dy \text{ Note the first term is 0} \\ &= 3\int y^2\phi(y)dy \\ &= 3Var\left[N(0,1)\right] \\ &= 3\cdot 1.\end{aligned}$$

Puting these together, we get

$$\mathbb{E}\left[W(t)\right]^6 = 5\cdot 3\cdot 1\cdot t^3 = 15t^3.$$

(c).

$$
\begin{aligned}
\mathbb{E}\left[e^{1+2W(t)}\right] &= \int e^{1+2x}\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{x^2}{2t}\right]dx \\
&= \int \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{x^2}{2t}+2x+1\right]dx \\
&= \int \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(x-2t)^2}{2t}+2t+1\right]dx \text{ Note: complete the square} \\
&= \exp(1+2t)\int \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(x-2t)^2}{2t}\right]dx \text{ the integrand is a normal density} \\
&= \exp(1+2t)\cdot 1 \\
&= \exp(1+2t).
\end{aligned}
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