Name:______________________

Solutions to
The Stochastic Calculus Midterm Exam

Fall 2022

Note 1: The exam time is from 3:30PM to 5:30PM. You will have 15 minutes after the exam (until 5:45PM) to scan/take picture of your answers to email to the dedicated email address (please use copy-paste to insure no typing errors):

`msqfeconometrics2015@gmail.com`

Don't forget to cc yourself and check if the file is attached!!!

Note 2: Although this exam is openbook, you are only allowed to reference the lecture notes, HW, and your notes on your laptop. Nothing else is allowed: No use of the Internet or communicating to any other people or source is allowed.

Note 3: Details for your solutions. Solutions with missing arguments in calculations/derivations will NOT receive full credit.

Note 4: You can write your answers to this PDF le as markers/comments. Please provide the Necessary

Note: *Please provide the details in all your solutions.*

Problem 1 [Brownian Motion][20 points]. Let W_t be a standard BM with the initial condition $W_0 = 0$ where $t \in [0, T]$. Set

$$Z_t = W_t - \frac{t}{T} W_T.$$

(i). Find the distribution of Z_t $(0 < t < T)$.

(ii). For $0 < s < t < T$, find the covariance $Cov(Z_s, Z_t)$.

(iii). For $0 < s < t < T$, what is the joint distribution for (Z_s, Z_t)? Note: you can give the joint density, or give the name of the distribution and all the necessary parameters. [Note: extra space provided on next page]

Solutions. (i) First note that Z_t is normally distributed for any t, since W_t and $W(T)$ are all normally distributed. So to specify the distribution of Z_t, we only need to find the mean and variance parameter for this normal random variable. This is easy:

$$\begin{aligned}
\mathbb{E}Z_t &= \mathbb{E}\left[W_t - \frac{t}{T}W(T)\right] = \mathbb{E}W_t - \frac{t}{T}\mathbb{E}W(T) = 0 - \frac{t}{T}\cdot 0 = 0 \\
Var\left[Z_t\right] &= \mathbb{E}\left(Z_t\right)^2 = \mathbb{E}\left[W_t - \frac{t}{T}W(T)\right]^2 \\
&= \mathbb{E}\left[W_t^2 - 2\frac{t}{T}W_tW(T) + \frac{t^2}{T^2}W(T)^2\right] \\
&= t - 2\frac{t}{T}\mathbb{E}W_tW(T) + \frac{t^2}{T^2}\mathbb{E}W(T)^2 \\
&= t - 2\frac{t}{T}t + \frac{t^2}{T^2}T \\
&= t - \frac{t^2}{T} = t\left(1 - \frac{t}{T}\right).
\end{aligned}$$

(ii). For $0 < s < t < T$, find the covariance $Cov(Z_s, Z_t)$.

First note that the Z_t process is centered (i.e., it has mean 0). The computation easily follows from the fact that $\mathbb{E}W_sW_t = s$ for $s < t$:

$$\begin{aligned}
Cov(Z_s, Z_t) &= \mathbb{E}\left[Z_sZ_t\right] \\
&= \mathbb{E}\left[W_s - \frac{s}{T}W(T)\right]\left[W_t - \frac{t}{T}W(T)\right] \\
&= \mathbb{E}\left[W_sW_t - \frac{t}{T}W_sW(T) - \frac{s}{T}W_tW(T) + \frac{s}{T}\frac{t}{T}W(T)^2\right] \\
&= s - \frac{st}{T} - \frac{st}{T} + \frac{st}{T^2}T \\
&= s - \frac{st}{T}
\end{aligned}$$

(iii). For $0 < s < t < T$, what is the joint distribution for (Z_s, Z_t)? Note: you can give the joint density, or give the name of the distribution and all the necessary parameters.

It is easy to see that Z_s and Z_t have a joint normal distribution. We have shown in (i) that the Z_t (and hence Z_s) has mean 0. So all we need to do is to find the covariance matrix. This can be done using the covariance calculation (note that the result in (ii) also holds for $s = t$)

$$\begin{aligned} Var(Z_s) &= Cov(Z_s, Z_s) = s - \frac{s^2}{T} \text{ [setting } s = t \text{ in (ii)]} \\ Var(Z_t) &= t - \frac{t^2}{T}. \end{aligned}$$

Hence, the joint distribution for (Z_s, Z_t) is

$$N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s - \frac{s^2}{T} & s - \frac{st}{T} \\ s - \frac{st}{T} & t - \frac{t^2}{T} \end{pmatrix}\right).$$

You can write down the density if you want from here.

Problem 2. [Total/QuadraticVariation] [20]. What are the Total Variation and Quadratic Variation for deterministic function $f(x) = -x^4$ over the interval $[-3, 3]$? **Hint:** You can write down the answer with brief reasons by graphing the function (no rigorus derivations are required). We talked about this in class so DO NOT spend more than 10 minutes on this problem.

Solution:

The total variation is $3^4 + 3^4 = 2 \times 81 = 162$. [Why? see my discussions in the lecture]

The quadratic variation is 0. [Quadratic variaion is always 0 when the total variation is finite.]

Problem 3 [Stochastic Integrals] [20points]. Let W_t be a standard BM starting from $W_0 = 0$. Using Ito's formula to compute the following stochastic integrals/Expectations (Riemann integrals are considered done):

(1).

$$\int_0^t [W_s]^2 \, dW_s,$$

(2).

$$\int_0^t (s^2 + e^{W_s}) dW_s$$

Solutions

(1). Use function $f(x) = \frac{1}{3}x^3$ [why working with this function? see my lecture] and apply Ito's formula to $f(W_t)$ to get

$$\begin{aligned} f(W_t) - f(W_0) &= \int_0^t f'(W_s) \, dW_s + \frac{1}{2}\int_0^t f''(W_s) \, ds \\ \frac{1}{3}(W_t)^3 - \frac{1}{3}(W_0)^3 &= \left[\int_0^t [W_s]^2 \, dW_s\right] + \frac{1}{2}\int_0^t 2[W_s] \, ds \\ \frac{1}{3}(W_t)^3 &= \left[\int_0^t [W_s]^2 \, dW_s\right] + \int_0^t W_s ds \\ \int_0^t [W_s]^2 \, dW_s &= \frac{1}{3}(W_t)^3 - \int_0^t W_s ds. \end{aligned}$$

(2). Use function [why ?] $f(t,x) = t^2x + e^x$ in Ito's formula to get

$$\begin{aligned}
f(W_t) - f(W_0) &= \int_0^t f_t(W_s)\,ds + \int_0^t f_x(W_s)\,dW_s + \frac{1}{2}\int_0^t f_{xx}(W_s)\,ds \\
f(W_t) - f(W_0) &= \int_0^t (2sW_s)ds + \left[\int_0^t \left[s^2 + e^{W_s}\right]dW_s\right] + \frac{1}{2}\int_0^t e^{W_s}ds \\
t^2W_t + e^{W_t} - \left[0^2 \cdot W_0 + e^{W_0}\right] &= \int_0^t (2sW_s)ds + \left[\int_0^t \left[s^2 + e^{W_s}\right]dW_s\right] + \frac{1}{2}\int_0^t e^{W_s}ds \\
t^2W_t + e^{W_t} - 1 &= \int_0^t (2sW_s)ds + \left[\int_0^t \left[s^2 + e^{W_s}\right]dW_s\right] + \frac{1}{2}\int_0^t e^{W_s}ds \quad [W_0 = 0!] \\
\int_0^t \left[s^2 + e^{W_s}\right]dW_s &= t^2W_t + e^{W_t} - 1 - \int_0^t (2sW_s)ds - \frac{1}{2}\int_0^t e^{W_s}ds \\
\int_0^t \left[s^2 + e^{W_s}\right]dW_s &= t^2W_t + e^{W_t} - 1 - \int_0^t (2sW_s + \frac{1}{2}e^{W_s})ds.
\end{aligned}$$

And there is no need to simplify the Riemann intergral on the left-hand side.

Problem 4 [Ito Isometry][20 points] Find the variance of Y_t defined by

$$Y_t = \int_0^t (s + [W_s]^2)dW_s.$$

Solution:

Recall that stochastic integrals are margingales and have 0 expections (I discussed this in the lecture. So we have

$$\begin{aligned}
Var(Y_t) &= \mathbb{E}\,(Y_t)^2 \\
&= \mathbb{E}\left(\int_0^t \left[s + [W_s]^2\right]^2 ds\right) \text{ [by Ito Isometry]} \\
&= \int_0^t \mathbb{E}\left\{\left[s + [W_s]^2\right]^2\right\} ds \text{ [change order of integration]} \\
&= \int_0^t \mathbb{E}\left\{s^2 + 2s\,[W_s]^2 + [W_s]^4\right\} ds \\
&= \int_0^t \left[s^2 + 2s\mathbb{E}\,[W_s]^2 + \mathbb{E}\,[W_s]^4\right] ds
\end{aligned}$$

From here, you can recall that W_s has normal distribution $N(0,s)$, so you can compute its second and fourth moment to get

$$\begin{aligned}
\mathbb{E}\,[W_s]^2 &= s \\
\mathbb{E}\,[W_s]^4 &= 3s^2. \text{ [4th moment of Normal distribution].}
\end{aligned}$$

Hence we get

$$\begin{aligned}
Var(Y_t) &= \int_0^t \left[s^2 + 2s^2 + 3s^2\right] ds \\
&= \int_0^t \left[6s^2\right] ds \\
&= 2t^3.
\end{aligned}$$

Note: I have not double checked the numbers, so the above could contain numerical errors. But the method/idea should be correct.

Problem 5 [Ito Formula] [10 points]. Let X_t be the following Ito process

$$dX_t = (W_t - [W_t]^2)dt + 3\sqrt{W_t}dW_t, \text{ with } X_0 = 2$$

Apply Ito's formula to $Y_t = X_t^3$ to express Y_t as a stochastic integral (plus Riemann integral).

Solution. Ito's formula gives

$$\begin{aligned}
dY_t &= 3X_t^2 dX_t + \frac{1}{2}6X_t (dX_t)^2 \\
&= 3X_t^2 \left[(W_t - W_t^2)dt + 3\sqrt{W_t}dW_t\right] + \frac{1}{2}6X_t \cdot \left[3\sqrt{W_t}\right]^2 dt \\
&= \left[3X_t^2(W_t - W_t^2) + 27X_tW_t\right] dt + 9X_t^2\sqrt{W_t}dW_t. \text{ (collect similar terms)}
\end{aligned}$$

The integral form of it is

$$\begin{aligned}
Y_t - Y_0 &= \int_0^t \left(3X_s^2(W_s - W_s^2) + 27X_sW_s\right) ds + \int_0^t 9X_s^2\sqrt{W_s}dW_s \\
&= \int_0^t 9X_s^2\sqrt{W_s}dW_s + \int_0^t \left(3X_s^2(W_s - W_s^2) + 27X_sW_s\right) ds.
\end{aligned}$$

This is what is asked in the problem: the first is stochastic integral, the second is Riemann integeral.

Problem 6 [A Job Interview Question] [10 points]. Compute the covariance of W_t and $\int_0^t W_s ds$:

$$Cov\left(W_t, \int_0^t W_s ds\right).$$

Hint: This integral above is actually a Riemann Integral (integrating with respect to dt and not dW_s), as you have seem in the HW. Use definition of covariance $Cov(X,Y) = \mathbb{E}[XY] - (\mathbb{E}X)(\mathbb{E}Y)$ with $X = W_t$ and $Y = \int_0^t W_s ds$. And you can evaluate the relevant moments for answer this question. The problem is easier than it appears (as usually happens in job interviews).

Solution. Let $X = W_t$, and $Y = \int_0^t W_s ds$. Then we have

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}W_t = 0 \text{ (property of BM)} \\
\mathbb{E}Y &= \mathbb{E}\int_0^t W_s ds = \int_0^t (\mathbb{E}W_s)\, ds = \int_0^t 0 ds = 0.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}\left[W_t \int_0^t W_s ds\right] \\
&= \mathbb{E}\left[\int_0^t W_t W_s ds\right] \\
&= \int_0^t \mathbb{E}[W_t W_s]\, ds \\
&= \int_0^t s ds \quad \text{[homework problem: } \mathbb{E}W_tW_s = \min(s,t)\text{]} \\
&= \frac{t^2}{2},
\end{aligned}$$

which is the answer.