Lecture 3

§5. Applications of derivatives

1) Taylor Expansion

① $f(x) \in C[a,b]$ $f(x) \in C^{(n)}[a,b]$

② $f^{(n+1)}(x)$ exists.

Ⅰ Taylor expansion w/ Lagrian remander

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{n}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

① ① ② ⓝ ⓝ+

$f(x) = ① + ①$ → 1st order Taylor expan

$\approx ① + ① + ②$ → 2nd order Taylor expansion

$\approx ① + \dots ⓝ$ → nth order Taylor exp?

ξ is a number between x, x_0

Remark 1: O, o

(Bounded) Big O: $a_n \overset{\text{seq}}{=} O(N^{\lambda})$ ⇔ $\frac{a_N}{N^{\lambda}}$ is bounded. (λ: natural number)

(Convergence) Little o: $a_n = o(N^{\lambda})$ ⇔ $\frac{a_N}{N^{\lambda}} \to 0 \ (N \to \infty)$

Ⅱ Taylor Expansion w/ Peano Remainder.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o(x-x_0)^n$$

Peano Remainder.

Remark 2: set $x_0 = 0$

Taylor series — Maclaurin Series

Ex: Maclaurin Series of e^x

$f(0) = 1 \quad f'(0) = 1 \dots$

Thus $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$

$= \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$

$e^x \approx 1+x$ (when x is very small)

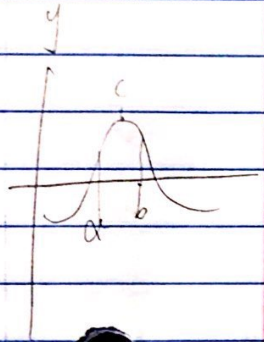
2) mid-value theorem.

① Rolle's Theorem.

(i) $f(x) \in [a,b]$

(ii) $f'(x)$ exist on (a,b)

(iii) $f(a) = f(b)$



$\Rightarrow \exists$ atleast one point $c \in (a,b)$ s.t $f'(c)=0$

weirtrass extreme value theorem:

$f(x) \in [a,b]$, $\exists \sup f(x)$, $\inf f(x)$

Fermat's theorem : $f'(c)=0$

② Lagrange's Mean Value Theorem (1st-MVT)

(i) $f(x) \in [a,b]$

(ii) $f'(x)$ exists on (a,b)

$\Rightarrow \exists$ at least one point $c \in (a,b)$ s.t $f(b)-f(a) = f'(c)(b-a)$

$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\Delta f(x)}{\Delta x}$.

③ Cauchy's MVT (2nd MVT)

(i) $f(x), g(x) \in C[a,b]$

(ii) $f'(x), g'(x)$, exist on $[a,b]$

(iii) $g'(x) \neq 0$

$\Rightarrow \forall x \in (a,b)$ $\exists c \in (a,b)$ s.t $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Ex: Validate 2nd-MVT for $f(x)=x^4$, $g(x)=x^2$ on $[1,2]$

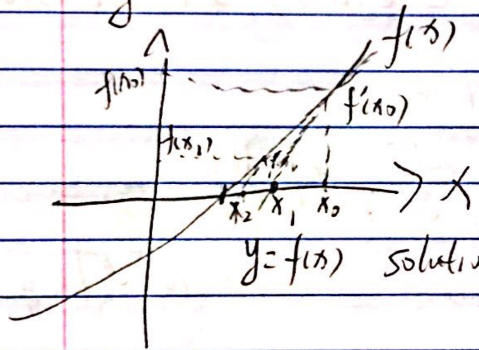
$f(b)-f(a)=2^4-1^4=15$ $\qquad f'(x)=4x^3$

$g(b)-g(a)=2^2-1^2=3$ $\qquad g'(x)=2x$

$$LHS=\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{15}{3}=5 \Leftrightarrow 2x^2 \leftarrow \frac{4x^3}{2x}=RHS$$

$$x=\pm\sqrt{\tfrac{5}{2}} = c_{1,2} \qquad \text{exist!}$$

3) New-Raphson Method



$x_0 \to x_1 \to x_2 \cdots \to x^*$

4) Convex functions

(1) Def: convex (Downward)

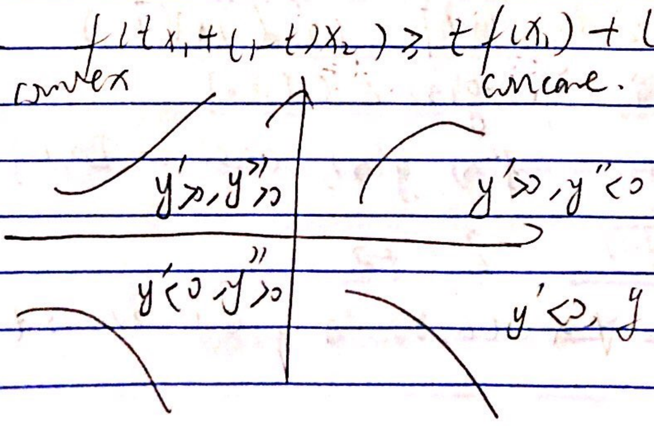
$f(x)\in[a,b]$ $\quad f(tx_1+(1-t)x_2)\le tf(x_1)+(1-t)f(x_2)$

$t\in[0,1)$

Concave $\quad f(tx_1+(1-t)x_2)\ge tf(x_1)+(1-t)f(x_2)$

$y''>0 \Rightarrow$ convex

$y''<0 \Rightarrow$ concave



sufficient conditions for convexity/concavity

SP: $f(x)$ exist on $[a,b]$, $f''(x)$ exist on (a,b)
$g(x)$ $\quad g''(x)$
$\Rightarrow$ two following conditions sufficient for convex/concave.

property:

① f, g are convex, $af+bg$ is convex $a,b \in \mathbb{R}$

② $u=g(x)$ is convex $y=f(u)$ is also convex.
and $y=f(u)$ is convex non-decreasing.

$\Rightarrow f(g(x))$ is also convex

③ (i) $u=g(x)$ concave ; (ii) $y=f(u)$ is convex, and non-increasing

$\Rightarrow y=f(g(x))$ is convex

④ local max/min of convex func on $[a,b]$ is also global max/min.

(4) Inflection point (convexity changes)

critical point $f'(x)=0$
Inflection point $f''(x)=0$

Ex: convex/concave? $f(x)=-x^3+6x^2-2x+1$

step 1: $f'(x)=-3x^2+12x-2$.
$f''(x)=-6x+12$

step 2: $f''(x) \overset{set}{=} 0$ $\quad -6x+12=0 \Rightarrow x=2$.

concave $\quad x=2 \quad$ convex

§6. Infinite series

1) Def. $\sum_{n=1}^{\infty} a_n$ — infinite series.

2) convergence of inf. series

$$S_n = \sum_{i=1}^{n} a_i \quad \text{— partial sum}$$

If the seq $\{S_n\}_{n=1}^{\infty}$ is convergent, $\lim_{n\to\infty} S_n = S$ exists ($S \in \mathbb{R}$)
then we say $\sum_{n=1}^{\infty} a_n$ is convergent.

Ex: Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad \text{convergent?} \Rightarrow \text{divergent.}$$

S_{2^1}: $S_2 = \sum_{n=1}^{2} \frac{1}{n} = 1 + \frac{1}{2} = \frac{3}{2}$

S_{2^2}: $S_4 = \sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$

$$= \frac{3}{2} + \frac{1}{2} = 2.$$

S_{2^3}: $S_8 = \sum_{n=1}^{8} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \overbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}^{\frac{1}{2}}$$

$$= \frac{5}{2}$$

$$S_{16} = \sum_{n=1}^{16} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}} + \underbrace{\left(\frac{1}{16} + \dots + \frac{1}{16}\right)}_{8}$$

$S_{2^k} > 1 + \frac{k}{2}$ $\quad 2^k = n \quad S_n > 1 + \frac{k}{2} = 1 + \frac{\frac{\lg n}{\lg 2}}{2}$ $\quad \frac{1}{2}$

$k \lg_2 = \lg n.$

3) Test of convergence.

(1) Nth-term test.

If the series $\sum a_n \left(\sum_{n=1}^{\infty} a_n\right)$ is convergent,

$\Rightarrow \lim_{n\to\infty} a_n = 0$

note: one direction.

(2) Test for divergence.

If $\lim_{n\to\infty} a_n$ doesn't exist or $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\sum a_n$ $\sum b_n$ are convergent,

$\Rightarrow$ (i) $\sum c a_n = c \sum a_n$

(ii) $\sum (a_n + b_n) = \sum a_n + \sum b_n$.

(iii) $\sum (a_n - b_n) = \sum a_n - \sum b_n$.

(4) Comparison Test.

Remark

$\sum_{n=1}^{\infty} \frac{1}{n^p}$

$p>1$ convergent.

$0<p\leq 1$ divergent

$\sum a_n, \sum b_n$ s.t. $0 \leq a_n \leq b_n \ \forall n$

$\Rightarrow$ (i) If $\sum b_n$ is convergent, $\Rightarrow \sum a_n$ is convergent.

If $\sum a_n$ is divergent $\Rightarrow \sum b_n$ is divergent.

(5) The limit comparison Test. (ratio)

$\sum a_n$ $\sum b_n$. $a_n, b_n > 0$ $(\forall n \in \mathbb{N})$.

i) If $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum a_n, \sum b_n$ are both convergent/divergent

ii) If the ratio goes to zero, then, if $\sum b_n$ is convergent, then $\sum a_n$ is convergent

iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, if $\sum b_n$ is divergent, $\Rightarrow \sum a_n$ divergent

Ex: Determine if $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ convergent?

$e^{\frac{1}{n}} \leq e \quad (n \geq 1)$

$0 \leq \sum_{n=1}^{\infty} \underbrace{\boxed{\frac{e^{\frac{1}{n}}}{n^2}}}_{a_n} \leq \sum_{n=1}^{\infty} \underbrace{\frac{e}{n^2}}_{b_n} = e \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow$ convergent

By comparison Test (i)

$\sum_{n=1}^{\infty} \frac{\frac{1}{e^n}}{n}$ converges to 0

(b) Integral test.

(7) Ratio Test.

$\sum a_n \quad a_n > 0$

(i) $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \implies \sum a_n$ is convergent.

(ii) $> 1 \implies$ divergent.

$= 1 \implies$ converge/diverage.

(8) Root test

$\sum a_n \quad a_n > 0$

(i) $\lim_{n \to \infty} \sqrt[n]{a_n} < 1 \implies \sum a_n$ convergent.

(ii) $> 1 \implies$ divergent.

(iii) $= 1 \implies$ converge/diverge.

Ex: (i) Determine if $\sum \frac{3^n}{n^2}$ is convergent?

Ratio Test

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{3^{n+1}}{(n+1)^2}}{\frac{3^n}{n^2}} = 3 \lim \frac{n^2}{n^2+1}$$

$$= 3 \lim \Big(1 - \underbrace{\frac{1}{n+1}}_{0 \text{ as } n\to\infty}\Big)^2$$

$$\underbrace{\qquad}_{1}$$

$= 3$ Thus this seq is divergent.

(ii) $\sum \frac{n^3}{(\ln 3)^n}$ convergent?

Ratio Test

$$\lim \frac{\frac{(n+1)^3}{(\ln 3)^{n+1}}}{\frac{n^3}{(\ln 3)^n}} = \lim \frac{(\ln 3)^n}{(\ln 3)^{n+1}} \cdot \frac{(n+1)^3}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{\ln 3} \left(1+\frac{1}{n}\right)^3$$

$$= \frac{1}{\ln 3}$$