Lecture #2

2) unique path of partial differentiation

suppose $f(x,y) \to L_1$ $(x,y) \to (a,b)$

$z = f(x,y)$ i.e., $z \to L_1$

show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ doesn't exist

$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

step1: let's approach $(0,0)$ along x-axis. i.e. $y=0$ — C_1

given $f(x,0) = \frac{x^2}{x^2} = 1 = L_1$

$\boxed{f(x,y) \to 1 \text{ as } (x,y) \to (0,0) \text{ along x-axis.}}$

step2: Approach $(0,0)$ along y-axis. — C_2

i.e., $x=0$ $f(0,y) = -1 = L_2$

$\boxed{f(x,y) \to -1 \text{ as } (x,y) \to (0,0) \text{ along y-axis}}$

3) continuity $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$f(x,y)$ is continuous at (a,b).

4) Partial Derivative

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

partial differentiation

$\frac{df}{dx} \to$ differentiation

If $z = f(x,y)$, we write $f_x(x,y) = f_x = \frac{\partial f}{\partial x}$

$= \frac{\partial}{\partial x} f(x,y) = D_x f$

$f(u(x,y))$

$f_x = \frac{df}{du} \cdot \frac{\partial u}{\partial x}$

Ex1: $f(x,y) = \sin\left(\frac{x}{1+y}\right)$ Find f_x, f_y

$$f_x = \frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \frac{1}{1+y}$$

$$f_y = \frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \frac{-x}{(1+y)^2}$$

5) Higher Derivatives.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

6) Linear approximation

Tangent plane: $z = f(x,y)$

W.t.f a tangent plane at (x_0, y_0, z_0) to this surface.

$z = f(x,y) \to f(x_0,y_0)$

$$\underbrace{z - z_0}_{\Delta z} = \underbrace{f_x(x_0,y_0)}_{\text{component of } f \text{ on } x\text{-axis}}\underbrace{(x-x_0)}_{\Delta x} + \underbrace{f_y(x_0,y_0)}_{\text{component of } f \text{ on } y\text{-axis}}\underbrace{(y-y_0)}_{\Delta y}.$$

$\Rightarrow z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$

$\Rightarrow$ called linear approximation of f at (x_0,y_0)

Ex: $z = 2x^2 + y^2$. Now, we want to find tangent plane to the elipital parobolic at $(1,1,\underset{z_0}{3})$

step 1: $f_x = 4x \quad f_y = 2y$.

step 2: $z - 3 = 4x(x-1) + 2y(y-1)$

$\Rightarrow z = 4x^2 - 4x + 2y^2 - 2y + 3$

Remark:

$\mathbb{R}^2$: tangent line. $\xrightarrow{\text{approximate}}$ tangent point

$\mathbb{R}^3$: tangent plane $\longrightarrow$ point.

7) Partially differentiable

① In $\mathbb{R}$, recap $\Delta y = f(a+\Delta x) - f(a)$

$\Delta y = f'(a)\Delta x + \varepsilon \cdot \Delta x \qquad \varepsilon \to 0$ as $\Delta x \to 0$

we say f is differential at a.

$\Delta x \to 0 \quad \Delta y \to 0$

$\frac{\Delta y}{\Delta x} \to f'(a)$

if only $\Delta x \to 0$

$\frac{\Delta y}{\Delta x} \to f'(a) + \varepsilon$

$f(x,y)-f(a,b)$.

(2) $\mathbb{R}^2$ — Partialy differentiable.

$Z=f(x,y)$ is PD at (a,b) if $\Delta Z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$.

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

(8) Differentials

(1) $\mathbb{R}$ dx — $dy = f'(x)dx$ $\quad \frac{dy}{dx} = f'(x)$

(2) $\mathbb{R}^2$ $Z=f(x,y)$ $dx\,dy$

$$dz \stackrel{def}{=} f_x(a,b)dx + f_y(a,b)dy$$
$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Total diff

Ex 1: $Z=f(x,y) = x^2+3xy-y^2$

$$dZ = f_x(x,y)dx + f_y(x,y)dy$$
$$= (2x+3y)dx + (3x-2y)dy$$

T.D

Ex2: If x changes from $2 \to 2.05$

y $\quad 3 \to 2.96$

compute ΔZ (PD)

dz (total diff, tD)

At $x=2$ $\quad \Delta x = 2.05-2 = 0.05 \approx dx$

$y=3$ $\quad \Delta y = 2.96-3 = -0.04 \approx dy$

$$dz \approx (2\cdot 2+3\cdot 3)\cdot 0.05 + (3\cdot 2-2\cdot 3)(-0.04)$$
$$= 0.65$$

$$\Delta Z = f(x,y) - f(x_0,y_0)$$
$$= 2.05^2 + 3\times 2.05\times 2.96 - 2.96^2 - (2^2+3\cdot 2\cdot 3-3^2)$$

9) Chain Rule.

$z = f(x, y)$
$x = g(t, s)$ $y = (t, s)$ > Differential.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Ex: $z = x^2 y + 3xy^4$ $x = \sin 2t$
$y = \cos t$

$$\frac{dz}{dt} = (2xy + 3y^4) 2\cos 2t + (x^2 + 12xy^3)(-\sin t)$$

10) Directional derivative & gradient.

(1) D.D of f at (x_0, y_0) is the direction of a unit vector $u = (a, b)$, defined as $D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$ If this limit exists

(2) Theorem. If f is differentiable w.r.t x, y, then. we say f has D.D in the direction vector $u = (a, b)$

$\Rightarrow D_u f(x, y) = f_x(x, y) a + f_y(x, y) b$

Ex: find D.D of $f(x, y) = x^3 - 3xy + 4y^2$.

vector ← $\underset{\sim}{u}$ is unit direction vector $= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

step 1 $\underset{\sim}{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $a = \frac{\sqrt{2}}{2} = b$.

step 2 $f_x(x, y) = 3x^2 - 3y$

$f_y(x, y) = -3x + 8y$

D.D $f(x, y) = (3x^2 - 3y) \frac{\sqrt{2}}{2} + (-3x + 8y) \cdot \frac{\sqrt{2}}{2}$

(3) gradient.

$\nabla f(x, y) = (f_x(x, y), f_y(x, y))$

$= \frac{\partial f}{\partial x} \cdot \underset{\sim}{i} + \frac{\partial f}{\partial y} \cdot j$

$D_u f(x,y) = \nabla f(x,y) \cdot \underset{\sim}{u}$ where $\underset{\sim}{u} = (a,b)$.

$= \left(\frac{\partial f}{\partial x}\cdot \underset{\sim}{i}, \frac{\partial f}{\partial y}\cdot \underset{\sim}{j}\right)(a,b) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$

inner prod $\underset{\sim}{h} \cdot \underset{\sim}{k} = (h_1,h_2)\cdot(k_1,k_2) = h_1k_1 + h_2k_2$

11) Finding Extremams.

(1) local max, min.

Th: f has a local max/min at (a,b) and the 1-order P.D of f exist at (a,b)

$\Rightarrow f_x(a,b) = f_y(a,b) = 0$

Ex 1 $f(x,y) = x^2+y^2-2x-6y+14$ $\Rightarrow (x-1)^2+(y-3)^2+4$

max, min Find the extremeums

step 1: $f_x(x,y) = 2x-2 = 0$ $\Rightarrow x^* = 1$ → critical points.

$f_y(x,y) = 2y-6 = 0$ $y^* = 3$

step 2: $f(x,y) = 1+9-2-18+14 = 4$

$= (x-1)^2+(y-3)^2+4$

Ex 2 $f(x,y) = y^2 - x^2$.

$f_x = -2x$ $f_y = 2y$.

$\quad \| \qquad\qquad \|$

$\quad 0 \qquad\qquad 0$

$(x^*, y^*) = (0,0)$.

Along x-axis $(y=0)$

$f(x,y) = 0 - x^2 = -x^2 \le 0$

$f(x,y) < 0$ if $x \neq 0$

Along y-axis $(x=0)$

$f(x,y) = -y^2 \le 0$

$f(x,y) < 0$ if $y \neq 0$

$(0,0)$ is neither min nor max. Saddle point

(2) Second Derivative test

(i) SD of f is continuous on disk w.(a,b)

(ii) $f_x(a,b) = f_y(a,b) = 0$

$$\boxed{D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2}$$

Criterium.

(a) if $D > 0$, $f_{xx}(a,b) > 0 \Rightarrow$ local min

(b) if $D > 0$ $f_{xx}(a,b) < 0 \Rightarrow$ local max

(c) if < 0, don't have max/min $\rightarrow$ saddle

~~*~~ (d) if $D = 0$ $\Rightarrow$ no info

Ex: Find local max/min of $f(x,y) = x^4 + y^4 - 4xy + 1$

step1 : critical values:

$$f_x = 4x^3 - 4y \stackrel{set}{=} 0$$

$$f_y = 4y^3 - 4x \stackrel{set}{=} 0$$

$$\begin{cases} x^3 - y = 0 \rightarrow y = x^3 \\ y^3 - x = 0 \end{cases} \Rightarrow x^9 - x = 0 \Rightarrow$$

$x_1 = 0$, $x_2 = -1$, $x_3 = 1$ $\Rightarrow (0,0)$ $(-1,-1)$,

$y_1 = 0$ $y_2 = -1$ $y_3 = 1$ $(1,1)$

step2: Compute f_{xx}, f_{yy}, f_{xy}.

$f_{xx} = 12x^2$ $f_{xy} = -4$ $f_{yy} = 12y^2$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$= 144x^2y^2 - 16$

step 3 calculate.

$D(0,0) < 0 \Rightarrow$ saddle.

$D(1,1) > 0$, $f_{xx}(1,1) > 0 \Rightarrow$ local min

$D(-1,-1) > 0$, $f_{xx}(-1,-1) < 0 \Rightarrow$ max

(3) Find absolute min, max

~~step 1: find the values of f at critical points of D, D is bdd.~~

step 2. Find the extreme values of f on the boundary of D

~~step 3: Compare step 1 and step 2.~~

(2) ~~Jacobian Matrix~~

① $\mathbb{R}^2$ $y_1 = f_1(x_1, x_2)$ $y_2 = f_2(x_1, x_2)$

$$|J| = \left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

Jacobian Determinant

2x2 square

$y_1 = \sin x_1 + \frac{x_2}{3}$ $y_2 = (\ln x_1) \cdot x_2$.

$$|J| = \begin{vmatrix} \cos x_1 & \frac{1}{3} \\ \frac{x_2}{x_1} & \ln x_1 \end{vmatrix}$$

If $|J|$ is not square.

② $f_1, f_2, \cdots f_n$ of $x_1, x_2 \ldots$

$$|J| = \left| \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right| = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

ex: $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$.

Transformation of volume elements between cartesian ad spherical polar coordinate system

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

(3) Determine function dependence using Jacobian Determinat Test

$|J|$ will be identically 0 for all $x_1 \dots x_n$ iff $f_1, \cdots f_n$ are functionally dependent.

Ex $g_1 = 2u+3v$

$g_2 = 4u^2 + 12uv + 9v^2$.

$$|J| = \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_1}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 8u+12v & 12u+18v \end{vmatrix}$$

$$= 24u+36v - 24u - 36v = 0$$

$\Rightarrow g_1$ ad g_2 are functionally dependent.