Session 7

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2.4. Theorem $(A^T = A)$

a. If A is a square matrix

$\Rightarrow A + A^T$ is a symmetric matric

$$(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

b. AA^T, A^TA are symmetric matrices.

*c. A, B are symmetric $n \times n$ matrices.

$A + B$ symmetric

*d. A is symmetric, kA is symmetric

3). Matrix Inverse

3.1. Def. $\begin{cases} A^{-1} \ni & AA^{-1} = A^{-1}A = I_{n \times n} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{n \times n} \\ A_{n \times n} & \end{cases}$

3.2. If is invertible, then A^{-1} is unique.

non-invertible $\rightarrow$ singular.

3.3. Theorem. A is $n \times n$ matrix, A is invertible.

Coefficient matrix for system of linear equation $A\vec{x} = b$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Then. $\vec{x} = A^{-1} b$

n obs, k ind. var's (regressors)

$$\underset{\sim}{y} = \mathbb{X} \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}.$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ \vdots & & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}_{n \times (k+1)} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{(k+1) \times 1} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 + \beta_1 x_{11} + \cdots + \beta_k x_{1k} \\ \beta_0 + \beta_1 x_{21} + \cdots + \beta_k x_{2k} \\ \vdots \\ \beta_0 + \beta_1 x_{n1} + \cdots + \beta_k x_{nk} \end{bmatrix}_{n \times 1} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_1 x_{11} + \cdots + \beta_k x_{1k} + \varepsilon_1 \\ \beta_0 + \beta_1 x_{21} + \cdots + \beta_k x_{2k} + \varepsilon_2 \\ \vdots \\ \beta_0 + \cdots + \varepsilon_n \end{bmatrix}$$

$$\hat{\underset{\sim}{\beta}} = \underbrace{(\mathbb{X}^T \mathbb{X})}_{\text{invertible always.}}{}^{-1} \mathbb{X}^T \underset{\sim}{y}$$

$$\hat{\underset{\sim}{y}} = \mathbb{X} \hat{\underset{\sim}{\beta}} = \mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T \underset{\sim}{y} \equiv H \underset{\sim}{y}$$

where $H \equiv \mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T$

(hat matrix

The MLE of σ^2.

$$\hat{\sigma}^2 = \frac{\overbrace{(\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\beta}})^T (\underset{\sim}{y} - \mathbb{X}\hat{\underset{\sim}{\beta}})}^{\square^T \square}}{n} \quad \text{Invertible}$$

$\underset{\sim}{e}$ (errors)

3.4. Remark. The order maters.

$$(AA^T)BC = DC.$$

ex.

$$\boxed{A^{-1}}(AA^T)BC = \boxed{A^{-1}}DC.$$

$$(A^{-1}A)A^TBC = A^{-1}DC$$

$$A^{-1}A = I$$

$$I \cdot A = A$$

$$A^TBC = A^{-1}DC$$

$$A^TBC\boxed{C^{-1}} = A^{-1}DC\boxed{C^{-1}}$$

$$A^TB(\underbrace{CC^{-1}}_{I}) = A^{-1}D(\underbrace{CC^{-1}}_{I}).$$

$$A^TB = A^{-1}D.$$

3.5. Theorem. $\boxed{\text{2x2. Matrix Inverse Calculation.}}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) $ad-bc \neq 0$ $\boxed{A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

(ii) $ad-bc=0$ 无 A^{-1}

3.6.ex. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$ad - bc = 1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$k[a_{ij}] = [ka_{ij}]$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

3.7. Theorem

a. $(A^{-1})^{-1} = A$

b. $(cA)^{-1} = \frac{1}{c} A^{-1}$

c. $(AB)^{-1} = \underbrace{B^{-1}A^{-1}}$, $(AB)^T = B^T A^T$

d. $A^{-n} = (A^{-1})^n = (A^n)^{-1}$

3.8. Theorem. The Fundamental Theorem of Invertible Matrices (FTIM)

Let A be an $n \times n$ matrix. The followings are equivalent

a. A is invertible (A is non-singular) Full rank.

b. $A\vec{x} = \vec{b}$ has a unique solution ($\forall b \in \mathbb{R}^n$) $\Leftrightarrow$ e. $\text{rank}(A) = n$

c. $A\vec{x} = \vec{0}$ has only the trivial solution ($\vec{0}$ - trival)

d. The reduced echelon form of A is I_n. $\Leftrightarrow$ f. row vectors are IND.

$$\begin{cases} 1.\ R_i \pm R_j \\ 2.\ R_i \pm cR_j \end{cases}$$

$\Updownarrow$

ex. Step 1. $R_2 + 2R_1$ Step 2 $R_3 + 3R_1$ Step 3 $R_3 - R_2$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

*3.9. Theorem. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}

General Algorithm to find A^{-1}

$$[A \mid I] \xrightarrow{\text{seq. of op.}} [\, I \mid \boxed{A^{-1}} \,]$$

Remark: if you cannot reduce A to I, $\nexists A^{-1}$

3.10. ex. Find A^{-1} where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 - R_1}{\overset{R_2 - 2R_1}{=}} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{(-\frac{1}{2})R_2}{=} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 - R_2}{=} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

$$\overset{R_1 + R_3,\ R_2 + 3R_3}{=} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

$$\overset{R_1 - 2R_2}{=} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

I ✓. $\quad \| \quad A^{-1}$

3.11.ex. Find A^{-1}, where $A = \begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & -4 & 1 & 0 & 0 \\ -4 & -1 & 6 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3+R_1]{R_2+2R_1 \mid R_1/2} \left[\begin{array}{ccc|ccc} 1 & 1/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|ccc} 1 & 1/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & -5 & - & \end{array}\right]$$

4). Subspaces, Basis, Dimension, Rank.

4.1. Def. A subspace of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ $\ni$

(i) $\vec{0} \in S$

(ii) If $\vec{u}, \vec{v} \in S$, then $\vec{u}+\vec{v} \in S$. (S is closed under addition)

(iii). If $\vec{u} \in S$, $c \in \mathbb{R}$, $c\vec{u} \in S$ (S is closed under scalar multiplication)

4.2. Def. Let A be a $m \times n$ matrix.

(i) Row space $-$ row(A): subspace of $\mathbb{R}^n$ spanned by the rows of A.

(ii) Column space $-$ col(A): . . . . columns of A.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{row}(A) = \text{span}\{[1\ 0\ 2], [0\ 1\ 1], [0\ 0\ 1]\}$$

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Def: equivalent matrix $A \overset{r}{\sim} B$

Use row elementary operations $\left(\begin{array}{l} R_i + cR_j \\ R_i \leftrightarrow R_j \end{array}\right)$

Switch between A and B

4.3. Theorem. B is row equivalent to A

$B \overset{r}{\sim} A$

then row(B) = row(A)

4.4. Def. Let A be a $m \times n$ matrix.

Null space $-$ null(A): subspaces of $\mathbb{R}^n$ consisting (spanned) of solutions to $A\vec{x} = \vec{0}$

4.5. Def. A basis of subspace S of $\mathbb{R}^n$ is a set of vectors in S $\ni$

(i) span S

(ii) IND.

4.6. ex. (i) $\vec{e}_1, \vec{e}_2 \cdots \vec{e}_n$ — Standard basis. any pair of standard basis are orthogonal to each o-

(ii) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & 7 \end{bmatrix}$

$\text{row}(A) = \text{span}\{[1\ \ 2\ \ 1], [0\ \ 1\ \ 3], [1\ \ 4\ \ 7]\}$

$[1\ \ 4\ \ 7] = [1\ \ 2\ \ 1] + 2[0\ \ 1\ \ 3]$

$\vec{r}_3 = \vec{r}_1 + 2\cdot\vec{r}_2$

$\text{row}(A) = \text{span}\{[1\ \ 2\ \ 1], [0\ \ 1\ \ 3]\}$.

$\|[1\ \ 2\ \ 1]\| \neq 1 \Rightarrow$ not standard basis. X

$\sqrt{1^2+2^2+1} = \sqrt{6}$ $\quad \frac{1}{\sqrt{6}}[1\ \ 2\ \ 1] = \left[\frac{\sqrt{6}}{6}\ \ \frac{\sqrt{6}}{3}\ \ \frac{\sqrt{6}}{6}\right]$

$\Rightarrow$ Standard basis ✓

ex. 4.7. Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

Step 1. $\longrightarrow$ Row Echelon

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2. Take out the row vectors.

$[1 \quad 0 \quad 1 \quad 0 \quad -1]$

$[0 \quad 1 \quad 2 \quad 0 \quad 3]$

$[0 \quad 0 \quad 0 \quad 1 \quad 4]$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_2 \times 2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3/6} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

4.8. Theorem $\boxed{\text{Basis Theorem}}$ S is a subspace of $\mathbb{R}^n$. Then any two bases have the same number of vectors.

Bases are not unique

4.9. Def. S is subspace of $\mathbb{R}^n$. # of vectors in bases is called the dimension of S. $\dim S$.

4.10. Theorem $\dim \text{row}(A) = \dim \text{col}(A)$

4.11. Def. (Rank II) $\text{rank}(A) = \dim \text{row}(A) = \dim \text{col}(A)$

4.12. Theorem $\text{rank}(A^T) = \text{rank}(A)$

4.13. Def (Nullity) $\text{nullity}(A) - \dim \text{null}(A)$

spanned by ↓
solutions $Ax = 0$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

4.14. Rank Theorem.

If A is $n \times n$ $\text{rank}(A) + \text{nullity}(A) = n$.

4.15. FTIM II.

The followings are equivalent

(a). A is invertible

(b). $A\vec{x} = \vec{b}$ has a unique solution ($\forall \vec{b} \in \mathbb{R}^n$)

(c) $A\vec{x} = \vec{0}$ has only trivial solution

(d) Reduced echelon form of A is I_n

(e) $\text{rank}(A) = n$

(f) $\text{nullity}(A) = 0$

(g) The column/row vectors of A are IND

(h) The column/row vectors of A span $\mathbb{R}^n$.

(i) The column/row vectors form a basis for $\mathbb{R}^n$.

16. ex. Find bases for row(A), col(A), null(A)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$$

(i). $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$[1 \ \ 0 \ \ -1]$, $[0 \ \ 1 \ \ 2]$

(ii) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Step 1. we transpose A. and RE.(A^T)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

step 2. Extract the row vector and transpose them.

$[1 \ \ 1]^T$ $[0 \ \ 1]^T$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ bases for col(A)

$\begin{bmatrix} \cancel{1} & 0 & -1 & 0 \\ \cancel{0} & 0 & 0 & 0 \end{bmatrix}$

$x_1 = x_3$

(iii) Step 1. Find RE (A)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$x_1 - (-1)x_3 = 0$

$x_3 = t \in \mathbb{R}$

Step 2. Solve $A\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$x_2 = s \in \mathbb{R}$

$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \quad x_1 = x_3, \quad x_2 = -2x_3. \quad x_3 \in \mathbb{R}^1$$

dim nu

set $x_3 = t$, $t \in \mathbb{R} \Rightarrow x_1 = t$, $x_2 = -2t$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, basis for null(A) is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

5). Linear Transformations.

5.1. Def: $T: \mathbb{R}^{n} \to \mathbb{R}^{m}$ is LT:

(1) $T(\vec{u}+\vec{v}) = T(\vec{v})+T(\vec{u})$ $(\forall \vec{u}, \vec{v} \in \mathbb{R}^n)$

(2) $T(c\vec{v}) = c\,T(\vec{v})$ $(\forall \vec{v} \in \mathbb{R}^n,\ c \in \mathbb{R})$

Condensed version.

$\boxed{T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)}$

$\forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$.

$c_1, c_2 \in \mathbb{R}$

5.2. Ex. Check if $T: \mathbb{R}^2 \to \mathbb{R}^3$ as

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \boxed{2x-y} \\ \boxed{3x+4y} \end{bmatrix}$$

is LT or not.

(i) Suppose we have $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

$$\boxed{T(\vec{u}+\vec{v})} = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1+x_2 \\ 2(x_1+x_2) - (y_1+y_2) \\ 3(x_1+x_2) - 4(y_1+y_2) \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ (2x_1-y_1)+(2x_2-y_2) \\ (3x_1+4y_1)+(3x_2+4y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ 2x_1-y_1 \\ 3x_1+4y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2-y_2 \\ 3x_2+4y_2 \end{bmatrix} = T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$= \boxed{T(\vec{u}) + T(\vec{v})}$$

(ii) $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$\boxed{T(c\vec{u})} = T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right)$$

$$= \begin{bmatrix} cx \\ 2cx - cy \\ 3cx + 4cy \end{bmatrix} = c\begin{bmatrix} x \\ 2x - y \\ 3x + 4y \end{bmatrix}$$

$$= c\,T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

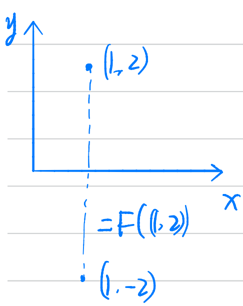
$$= \boxed{c\,T(\vec{u})}$$

(i)(ii) ✓ $\Rightarrow$ T is LT.

5.3. Theorem. For $A_{m\times n}$. $T_A: \mathbb{R}^n \to \mathbb{R}^m$ defined as $T_A(\vec{x}) = A\vec{x}$ ($\forall \vec{x}$ in $\mathbb{R}$) is a LT.

$$T_A\vec{x} = A\vec{x}$$

5.4 ex. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$, that sends each point to its reflection in the x-axis. Show F is LT. $F\begin{bmatrix} x \\ y \end{bmatrix}$



$$\forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^n, \quad \boxed{F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$$\begin{bmatrix} x \\ -y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

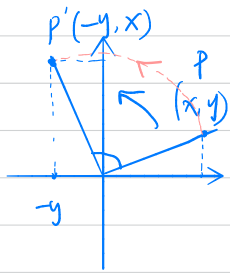
$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{2\times 1}$$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

F is LT.

Note: to show F is LT, is equivalent to find a matrix $A \ni F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}$

5.5. Ex. Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ $\ni$ rotates each point $90°$ counterclockwise



$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$$= x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$R\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow$ R is LT.

Recipe to determine LT

Step 1. Express the LT. in form of

$$T\vec{u} = u_1\vec{e}_1^* + u_2\vec{e}_2^* + \cdots u_n\vec{e}_n^*$$

where. $\vec{u} \in \mathbb{R}^n$. $\vec{e}_1^* = \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\vec{e}_2^* = \begin{bmatrix} 0 \\ a_2 \\ \vdots \\ 0 \end{bmatrix}$ ...

Step 2. $A = \left[\vec{e}_1^* \,\middle|\, \vec{e}_2^* \,\middle|\, \cdots \; \vec{e}_n^*\right]$

5.6. Remark $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \\ e \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \\ f \end{bmatrix}$

5.7. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be LT. Then T is a matrix transformation based on matrix A. Denoted as T_A, A is $m \times n$.

$$A = [\, T(\vec{e}_1) \mid T(\vec{e}_2) \cdots \mid T(\vec{e}_n) \,].$$

$\to$ stand matrix of T_A

denoted as $A = [T]$

5.8. Composition of transformation.

$$S \circ T(v) \overset{\text{def}}{=} S(T(v))$$

5.9. Theorem. Let $T: \mathbb{R}^m \to \mathbb{R}^n$, $S: \mathbb{R}^n \to \mathbb{R}^p$ > LT.

$$\Rightarrow S \circ T: \mathbb{R}^m \to \mathbb{R}^p \text{ is also a LT.}$$

$$[S \circ T] = [S][T]$$

5.10. EX. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$.

$S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + 3y_2 \\ 2y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$ $\quad S: \mathbb{R}^2 \to \mathbb{R}^3$.

Find the standard matrix of $S \circ T$.

a). Direct substitution.

$$S \circ T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = S\left(T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = S\left(\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}\right) = \begin{bmatrix} x_2 - 3x_1 \\ 2x_2 - x_1 \\ x_2 + x_1 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$[S \circ T] = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

b). matrix multiplication.

W.T.S. $[S]\cdot[T] =$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

$$S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1+3y_2 \\ 2y_1+y_2 \\ y_1-y_2 \end{bmatrix}$$

$[S]$ $$\begin{bmatrix} y_1+3y_2 \\ 2y_1+y_2 \\ y_1-y_2 \end{bmatrix} = y_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y_2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$[S] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$[T]$ $$\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = x_1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\boxed{[S\circ T] = [S][T]}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}_{3\times2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{2\times2}$$

$$= \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}_{3\times2}$$

5.11. Inverse of LT.

Let S, T be LT : $\mathbb{R}^n \to \mathbb{R}^n$.

Then S & T are inverse transformations

if (i) $S \circ T = I_n$ and (ii) $T \circ S = I_n$.

5.12. Theorem $[T^{-1}] = [T]^{-1}$

6) Eigenvalues & Eigenvectors.

6.1 Def. $A_{n\times n}$, λ is called an eigenvalue of A if $\exists\ \vec{x} \neq \vec{0} \ni A\vec{x} = \lambda\vec{x}$

$\vec{x}$ ↓ eigenvector of A corresponding to λ

6.2. Def. All the eigenvetor is called eigenspace E_λ.

6.3 ex. Find all the eigenvalues & eigonvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

(i) Solve $A\vec{x} = \lambda\vec{x}$ $\quad A\vec{x} - \lambda\vec{x} = 0$

$A\vec{x} - \lambda I\vec{x} = 0$

$(A - \lambda I)\vec{x} = 0$

Solve det $|A - \lambda I| = 0$ $\quad \left|\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\right| = 0$

$\left|\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right| = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$

$= (\lambda - 2)(\lambda - 4)$

$\lambda_1 = 2 \qquad \lambda_2 = 4.$

Recap: solve $(A-\lambda I)x=0$

(ii) when $\lambda_1 = 2$.

solve $[A-2I \mid 0] = \left[\begin{array}{cc|c} 3-2 & 1 & 0 \\ 1 & 3-2 & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y_1 + y_2 = 0$

$y_1 = -y_2$

set $y_2 = t \in \mathbb{R}$
$\Rightarrow y_1 = -t$

$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

corresponding eigenvector. $\lambda = 2$.

(iii) when $\lambda_2 = 4$

solve $(A-4I)x=0$

$\Leftrightarrow [A-4I \mid 0] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$y_1 - y_2 = 0 \Rightarrow y_1 = y_2$

let $y_2 = t \in \mathbb{R} \Rightarrow y_1 = t$

$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvector for $\lambda = 4$

(iv) Eigenspace span$\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Remark: an eigenvector does not change direction in LT.