Risk Aversion Measures Continued

$y+h$
$y-h$

$$\pi(y,h) = \frac{1}{2} + \frac{1}{4}h\left[R_A(y)\right]$$

$$R_A(y) = \frac{-u''(y)}{u'(y)}$$

$\theta \in [0,1]$

$$\pi(y,\theta) = \frac{1}{2} + \frac{1}{4}\theta\left[R_R(y)\right]$$

$$R_R(y) = -y\,\frac{u''(y)}{u'(y)}$$

CRRA = Constant relative nsh aversion utility function

(same as CES)

$$u(y) \begin{cases} \dfrac{y^{1-\gamma}}{1-\gamma} & \gamma \geq 0,\ \gamma \neq 1 \\ \ln(y) & \text{if } \gamma = 1 \end{cases}$$

If $u(y) = \ln(y)$

$$u'(y) = 1/y \qquad u''(y) = -\frac{1}{y^2}$$

$$R_R(y) = -y\,\frac{-\frac{1}{y^2}}{\frac{1}{y}} = \boxed{1}$$

$$\pi(y,\theta) = \frac{1}{2} + \frac{\theta}{4} \rightarrow \max \theta$$

$$1 = \frac{1}{2} + \frac{\theta}{4} \rightarrow \theta = 2$$

If $u(y) = \dfrac{y^{1-\gamma}}{1-\gamma}$

$$u'(y) = \frac{\cancel{(1-\gamma)}\, y^{1-\gamma-1}}{\cancel{1-\gamma}} = y^{-\gamma}$$

$$u''(y) = -\gamma\, y^{-\gamma-1} = \frac{-\gamma}{y^{\gamma-1}}$$

$$R_R(y) = -y\,\frac{\frac{-\gamma}{y^{\gamma-1}}}{\frac{1}{y^{-\gamma}}} = \boxed{\gamma}$$

coeff. of risk aversion

In both cases, R_R is constant as it no longer changes as y changes. Persons with a higher γ are more risk averse. $\gamma = 0$ indicates a risk neutral person.

Examples

Bonus
50,000
100,000

$$\pi_1 = \pi_2 = 0.5$$

CRRA: $u(CE)$: $$\frac{CE^{1-\gamma}}{1-\gamma} = \frac{0.5(50000)^{1-\gamma}}{1-\gamma} + \frac{0.5(100000)^{1-\gamma}}{1-\gamma}$$

$$\gamma = 0 \rightarrow \quad CE = 0.5(50000) + 0.5(100000)$$
$$CE = 75,000$$

Note that this is just the Expected Value of the lottery. We saw before that for a risk neutral person ($\gamma = 0$), $CE = E(y)$ and $RP = 0$.

$$RP = E(y) - CE$$

$\gamma = 1 \rightarrow$ We use ln(y) here.

$$u(CE) = \ln(CE) = 0.5\ln(50000) + 0.5\ln(100000)$$

$$\ln(CE) = 111{,}654$$

$$CE = 70{,}969.11$$

$\gamma = 2 \rightarrow$

$$\frac{CE^{1-\gamma}}{1-\gamma} = \frac{0.5(50000)^{1-2}}{1-2} + \frac{0.5(100000)^{1-2}}{1-2}$$

$$CE = 66{,}666$$

$\gamma = 4 \rightarrow$

$$\frac{CE^{1-4}}{1-4} = \frac{0.5(50000)^{1-4}}{1-4} + \frac{0.5(10000)^{1-4}}{1-4}$$

$$CE = 60{,}570.68$$

The lower the CE, the higher the RP, the more risk averse a person is.

Most person show a γ around 4.
From now on we will label γ as A.

Suppose γ was a negative number, then the person is risk-loving, $CE > E(y)$, and RP is negative.

Portfolio Management

→ mean variance framework

1) Capital Allocation in risky portfolio (P) versus a risk free asset (F) that make up a complete portfolio (C)

Suppose we allocate an investment of \$100,000 as follows:

100,000
- P = 60,000 (60%)
  - D = 20,000 (33.3%)
  - E = 40,000 (66.7%)
- F = 40,000 (40%)

How to reduce risk? Reallocate

- C=100,000 (100%)
  - P = 30,000 (30%)
    - D = 10,017 (33.3%)
    - E = 19,983 (66.7%)
  - F = 70,000 (70%)

2) Asset Allocation

- Risky Portfolio
  - equity
  - debt
  - forex
  - commodities
  - options/derivative

3) Security Allocation

(within each asset class)

Equity → AMZN, Meta, GOOG

Commodities → Gold, Oil, Coffee

Then we select riskier and more risk free options from these.

The allocation percentages depends solely on a persons level of risk aversion.

Back to C, P, and F:

y = % invested in risky portfolio P
r = return

$$r_c = y\, r_p + (1-y)\, r_f$$

$$E(r_c) = y\, E(r_p) + (1-y)\, r_f$$

rearranged: (distribute r_f and factor out y from $yE(r_p) - yr_f$).

$$\underbrace{E(r_c)}_{\text{expected return}} = r_f + y\underbrace{\left(E(r_p) - r_f\right)}_{\substack{\text{risk premium} \\ (E - CE)}} \quad ①$$

$$Var(r_c) = var(yr_p, (1-y)\, r_f)$$

By variance sum law

$$\sigma_c^2 = var(yr_p) + \cancel{var((1-y)r_f)} + \cancel{2y(1-y)\, cov}(r_p, r_f)$$

$$\sigma_c^2 = var(yr_p) = \boxed{y^2 \sigma_p^2} \quad ②$$

The last two terms cancel because the variance of constant r_f is zero, and a covariance involving a constant also equals zero.

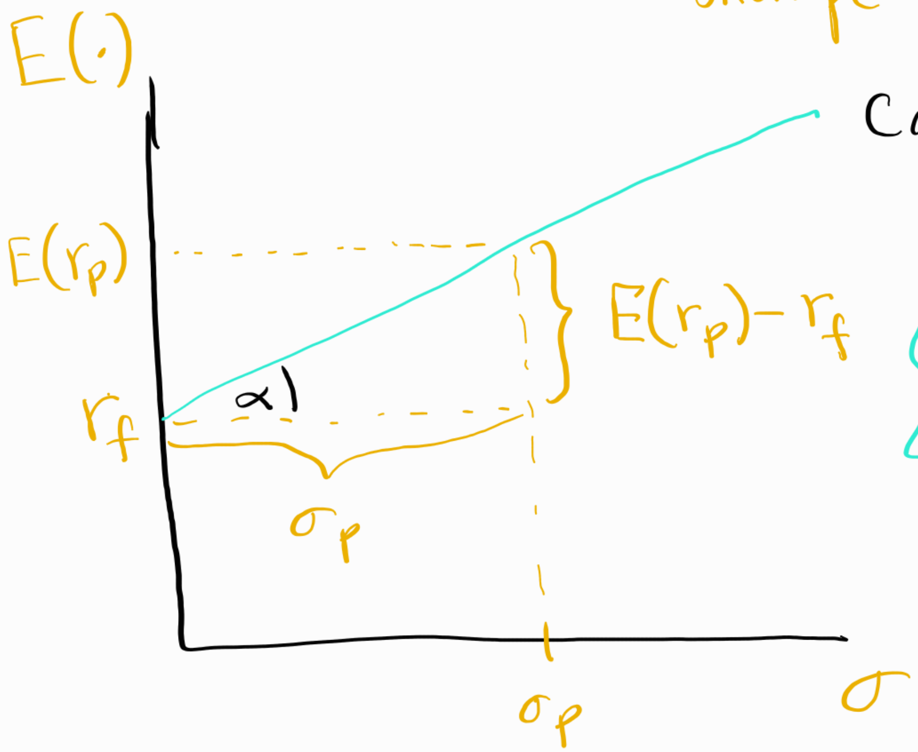
$$\boxed{y = \frac{\sigma_c}{\sigma_p}} \quad (3)$$

Using (3) in (1)

$$E(r_c) = r_f + \frac{\sigma_c}{\sigma_p}\left[E(r_p) - r_f\right]$$

$$E(r_c) = r_f + \underbrace{\frac{\left[E(r_p) - r_f\right]}{\sigma_p}}_{\text{Sharpe Ratio}} \sigma_c$$

Equation for capital allocation line



Capital Allocation Line (CAL)

$\tan \alpha$ = sharpe ratio (slope)

$$= \frac{E(r_p) - r_f}{\sigma_p}$$

Example :

$E(r_p) = 15 \quad \sigma_p = 22 \quad r_f = 7$

assume $y = 0.7$

① $E(r_c) = 7 + 0.7(15-7) = 12.6$

② $\sigma_c = 0.7(22) = 15.4$

$$S\,(\text{Sharpe Ratio}) = \frac{12.6-7}{15.4} = \frac{15-7}{22} = \underline{0.36}$$

$$\left[\frac{E(r_c) - r_f}{\sigma_c}\right] \quad \left[\frac{E(r_p) - r_f}{\sigma_p}\right]$$

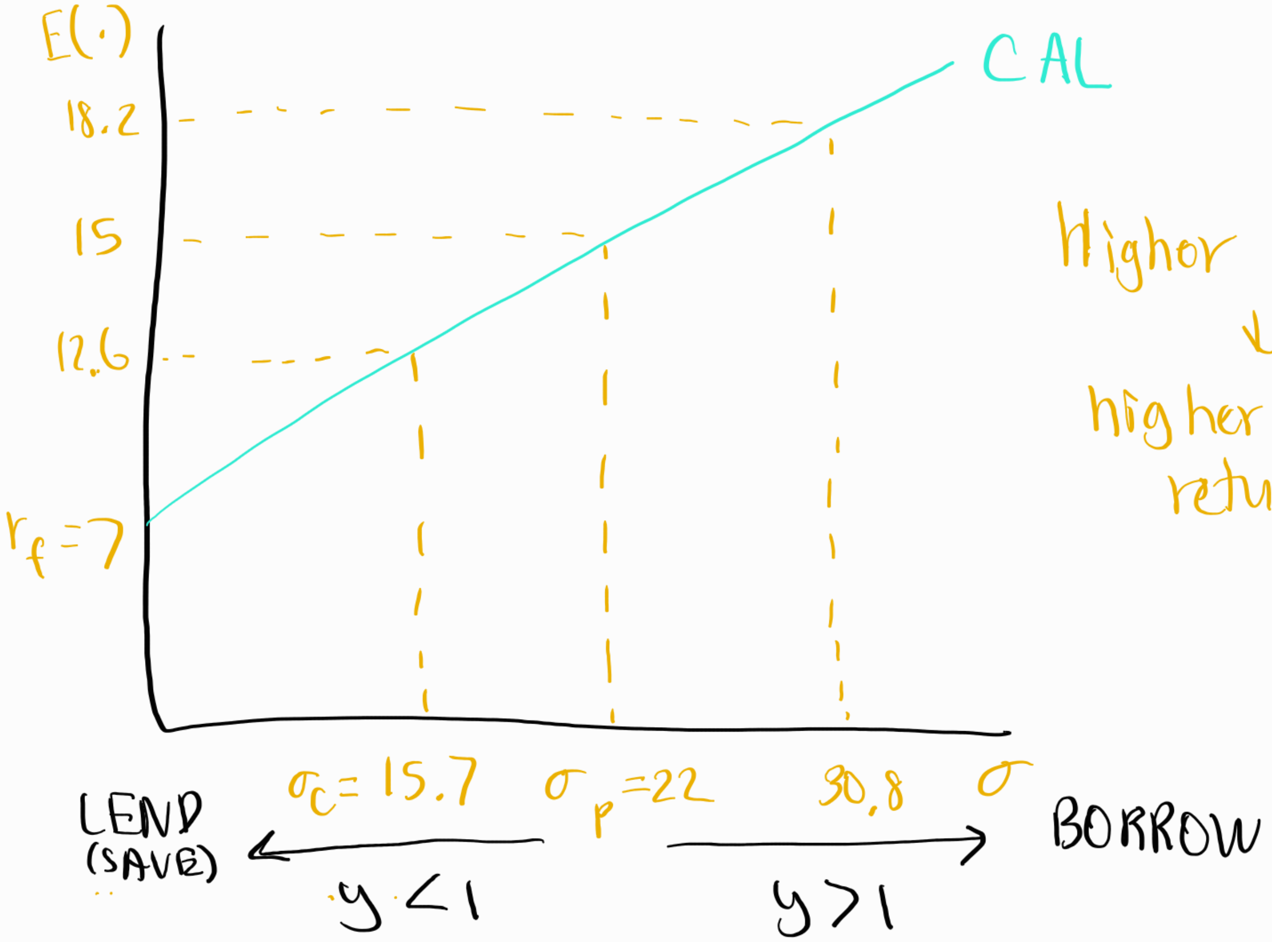
assume $y = 1.4$ (so we have a short of -0.4)

$E(r_c) = 7 + 1.4(15-7) = 18.2$

$\sigma_c = 1.4(22) = 30.8$

$$S = \frac{18.2-7}{30.8} = \underline{0.36}$$

Sharpe ratio is the same, reflecting that the slope of CAL is constant. Note that the higher the sharpe ratio, the better (it indicates higher return with lower volatility)



Higher volatility
↓
higher expected returns

Back to utility function specifications:

$$U = E(r_c) - 0.005 A \sigma_c^2$$

This utility function shows that a higher $E(r_c)$ increases U while volatility decreases U dependent on the coeff. of risk aversion A.

$$\max_{y} \; U = \underbrace{r_f + y\left(E(r_p) - r_f\right)}_{1} - 0.005 A \underbrace{y^2 \sigma_p^2}_{2}$$

$$u' = [E(r_p) - r_f] - 0.01 A y \sigma_p^2 = 0$$

$$y^* = \frac{E(r_p) - r_f}{0.01 A \sigma_p^2}$$

Note:

$$\beta = \frac{cov(x,y)}{var(x)}$$

Suppose $A = 4$

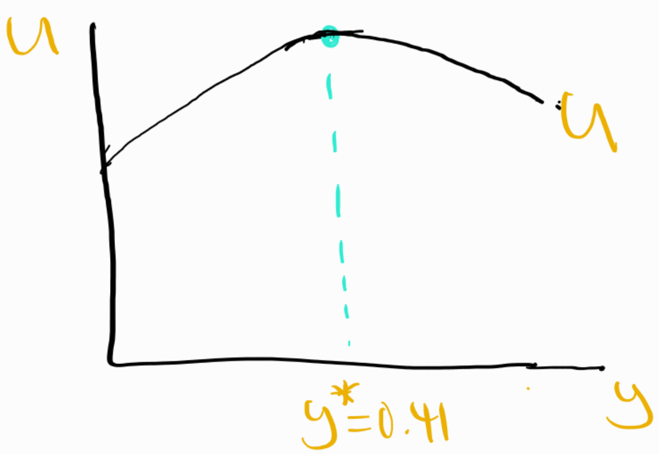
$$y^* = \frac{15 - 7}{0.01(4)(22)^2} = \boxed{0.41}$$

allocation percentage for risky portfolio

Now we can compute expected return & s.d.:

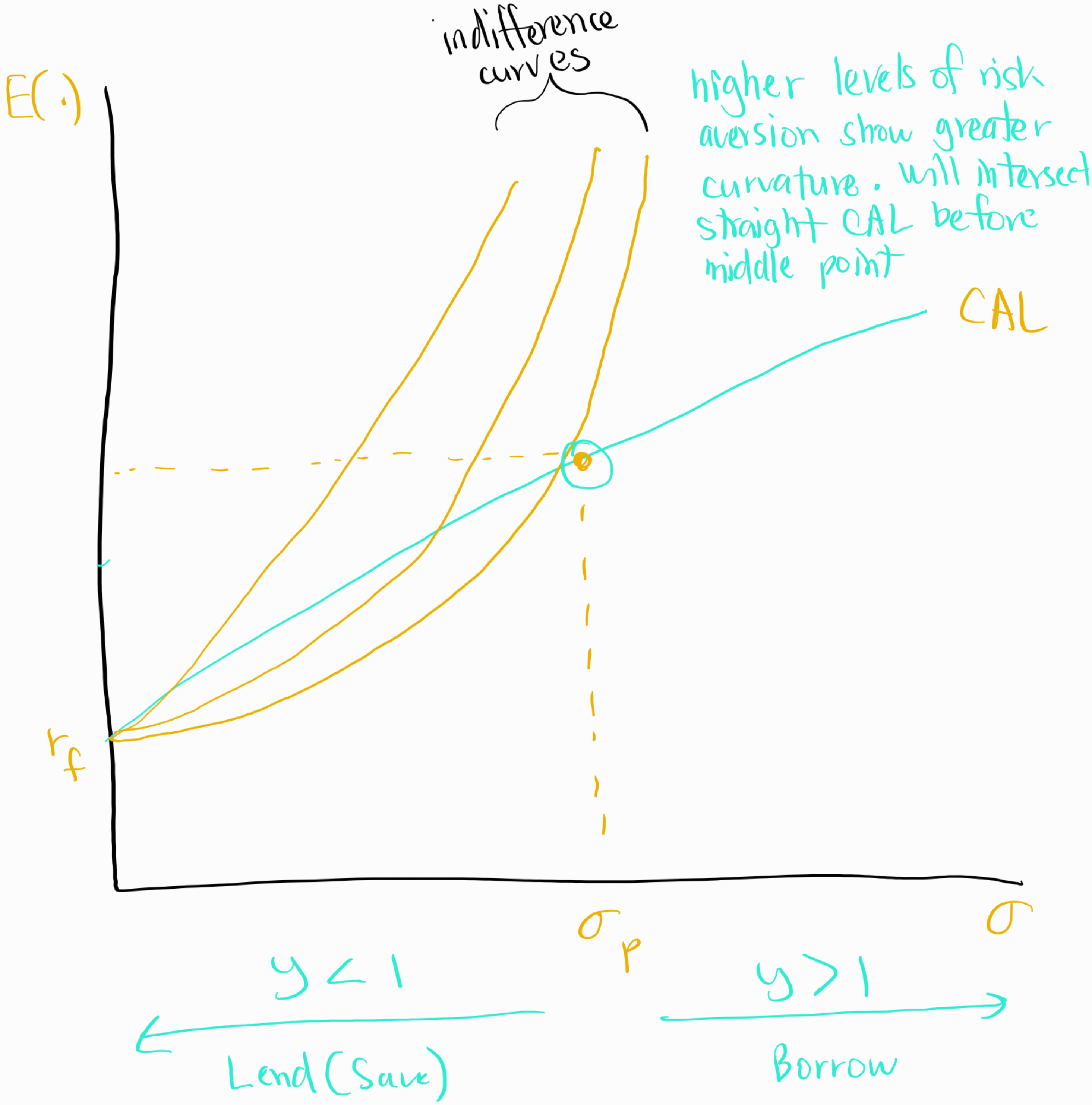
$$E(r_c) = 7 + 0.41(15 - 7) = 10.28$$

$$\sigma_c = 0.41(22) = 9.02$$

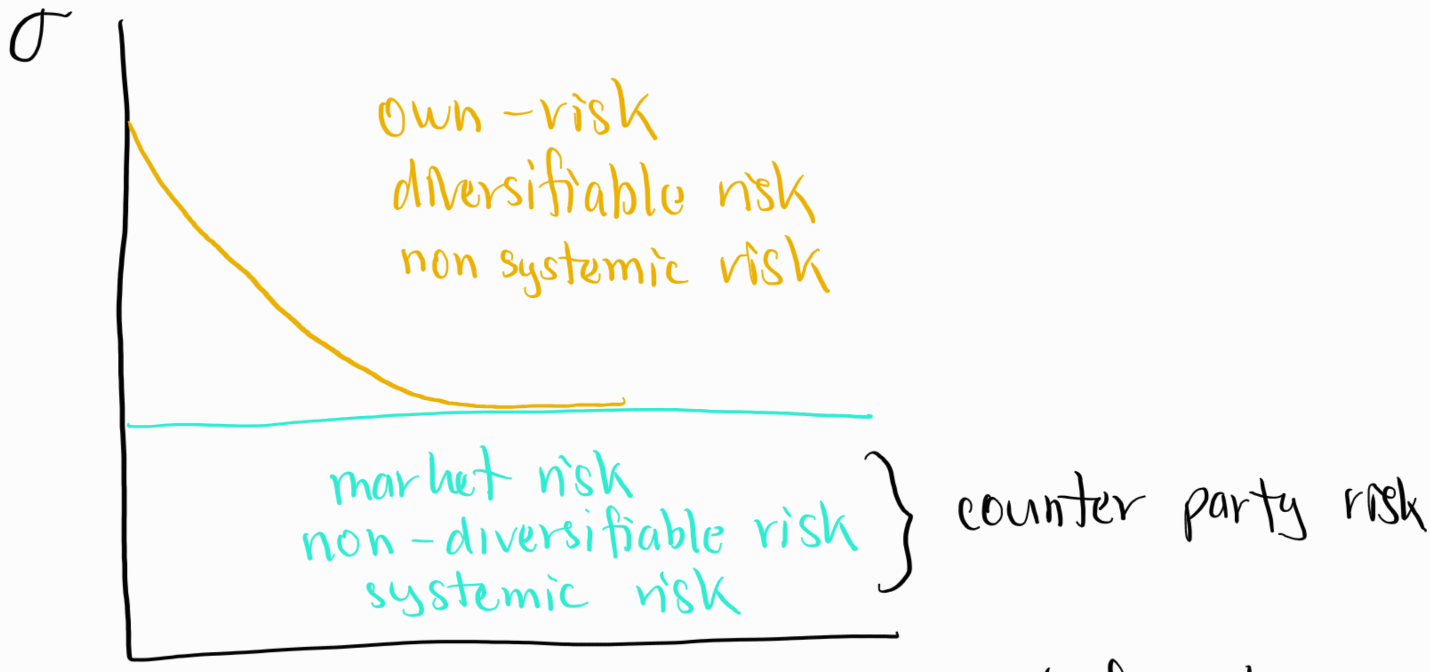


$$u(y^*) = 8.653$$

$$u = 10.28 - 0.005(4)(9.02)^2 = 8.653$$



Portfolio Diversification



$n =$ # of assets
# of securities

For diversification to be useful here, the movement of assets cannot be perfectly positively correlated.

So for assets X & y, and correlation ρ

$-1 \leq \rho \leq 1$ abs value gives magnitude
sign gives direction

$\rho_{x,y} = \underline{1} \rightarrow$ x ↑ 10% y ↑ 10% x ↓ 50% y ↓ 50%