Homework 1

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QF8915 - Stochastic Calculus

Due on Nov 15, 2022

Problem1

(a) Write down the density of $W(T)$.

Solution:

Since $W(T)$ is Standard Brownian motion, it's density is like normal distribution via CLT. Thus,

$$f_{W(T)}(x) = \frac{1}{\sqrt{2\pi T}} e^{\frac{-x^2}{2T}}$$

(b) Joint density of $W(s)$ and $W(t)$ for $0 \leq s < t \leq T$

Solution:

$f_{W(s)W(t)}(x, y)$ can be written[1] in $f_{W(s)} f_{W(t)-W(s)}(x, y)$, thus:

$$\begin{aligned} f_{W(s)W(t)}(x, y) &= f_{W(s)} f_{W(t)-W(s)}(x, y) \\ &= \frac{1}{\sqrt{2\pi s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{\frac{-x^2}{2s}} e^{\frac{-y^2}{t-s}} \\ &= \frac{1}{2\pi\sqrt{s(t-s)}} e^{-\frac{1}{2}(\frac{x^2}{s} + \frac{y^2}{t-w})} \end{aligned}$$

Problem 2

(a) Compute the conditional expectation of $E[W(t)|W(s) = c]$

Solution:

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[1]It can be seen at `https://www2.isye.gatech.edu/~sman/courses/6759/6759-5-BrownianMotion.pdf` page 9.

Since from the property that Brownian Motion is martingale. Thus:

$$
\begin{aligned}
E[W(t)|W(s) = c] &= W(s)(by\ martingale) \\
&= c
\end{aligned}
$$

(b) Compute the expectation $E[W(t)^2]$

Solution:

$$
\begin{aligned}
E[W(t)^2] &= var(W(t)) + E[W(t)]^2 \\
&= t - 0 \\
&= t
\end{aligned}
$$

(c) (Bonus)Compute $E[W(t)^6]$

Solution:

We use MGF to solve this question[2]. $M_{W(t)}(t) = E(e^{tW(t)})$. So $E[W(t)^6] = \frac{d^6 M(t)}{dt^6}|_{t=0} = 15t^3$

(d) (Bonus) Compute Expectation $E(e^{1+2^{W(t)}})$

Solution:

Since we know the density of $W(t)$, so it's easy to get the expected valuse.

$$
E(e^{1+2^{W(t)}}) = \int_0^T e^{1+2^{W(t)}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{W(t)^2}{2t}} dt
$$

Not done yet, question is how to compute the integral with respect to $W(t)$ above

[2]My reference for this question is : `https://math.stackexchange.com/questions/2135702/expectation-of-standard-brownian-motion?rq=1`