Homework #4 Stochastic Calculus Solutions

Ito Formula for Ito Processes

Note: *I have not double checked the answers, so there could be computational errors here. Use these solutions as a hint for how to solve the problems.*

Problem 1. Let X_t be Ito process

$$dX_t = W(t)dt + [W(t)]^2 dW(t).$$

Apply Ito's formula to express $Y_t = \log(1 + (X_t)^2)$ as an Ito process (i.e., as a stochastic integral plus a Riemann integral).

Answer: Here $Y_t = f(X_t)$ for

$$f(x) = \log(1 + x^2)$$

and

$$dX_t = W(t)dt + [W(t)]^2 dW(t).$$

The Ito's formula gives

$$\begin{aligned}
dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t) \cdot (dX_t)^2 \\
&= \frac{1}{1+(X_t)^2}dX_t + \frac{1}{2}\left[-\frac{2X_t}{[1+(X_t)^2]^2}\right] \cdot [W(t)]^4\,dt \\
&= \frac{1}{1+(X_t)^2}W(t)^2 dW(t) + \frac{1}{1+(X_t)^2}W(t)dt - \left[\frac{X_t}{[1+(X_t)^2]^2}\right] \cdot [W(t)]^4\,dt \\
&= \frac{1}{1+(X_t)^2}W(t)^2 dW(t) + \left[\frac{\left[1+(X_t)^2\right]W(t) - X_t[W(t)]^4}{[1+(X_t)^2]^2}\right] dt.
\end{aligned}$$

The equalevant integral form

$$Y_t = \int_0^t \frac{1}{1+(X_s)^2}W(s)^2 dW(s) + \int_0^t \left[\frac{\left[1+(X_s)^2\right]W(s) - X_s[W(s)]^4}{[1+(X_s)^2]^2}\right] ds. \text{ [Note: } Y_0 = 0]$$

The first is term is a stochastic integral, and the second term is a Riemann integral.

Problem 2. Suppose that X_t an Ito process that satisfies

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dW(t);\ X(0) = 0.$$

Let $Y_t = (1+t)X_t$. Use Ito's formula to find out what stochastic differential equation Y_t satisfies? Identify Y_t as a Brownian Motion.

Answer: Here $Y_t = f(t, X_t)$ for $f(t,x) = (1+t)x$. The Ito's formula for Ito processes gives us

$$\begin{aligned} dY_t &= f_t(t,X_t)dt + f_x(t,X_t)dX_t \\ &= X_t dt + (1+t)dX_t \\ &= X_t dt + (1+t)\left[-\frac{1}{1+t}X_t dt + \frac{1}{1+t}dW(t)\right] \\ &= dW(t). \end{aligned}$$

The equivalent integral form is

$$Y_t = W(t),$$

since $Y(0) = X(0) = 0$ and $W(0) = 0$.

Problem 3. Solve $dX_t = X_t dt + dW(t)$. **Hint:** This is a special case of the O-U process used in the Vasicek model.

Answer: Without the $dW(t)$ part, the ordinary differential equation $dx = xdt$ has solution

$$x = e^t.$$

This leads us to try solutions of the type $X_t = e^t Y_t$ for some process Y_t. Such Y_t satisfy

$$Y_t = e^{-t} X_t$$

and Ito's formula gives

$$\begin{aligned} dY_t &= -e^{-t}X_t dt + e^{-t}dX_t \text{ [Why the Ito term vanishes?]} \\ &= -e^{-t}X_t dt + e^{-t}\left[X_t dt + dW(t)\right] \\ &= e^{-t}dW(t). \end{aligned}$$

The integral form gives

$$Y_t = Y_0 + \int_0^t e^{-s}dW(s).$$

Convert back to X_t yields

$$X_t = X_0 + e^t \int_0^t e^{-s}dW(s).$$

Note that although we can not further obtain an explicit expression for the stochastic integral, it is easy to see that the stochastic integral is normally distributed and we can easily calculate its distributional parameters.

Problem 4. Solve $dX_t = -X_t dt + e^{-t} dW(t)$.

Answer: Similiar to the argument in Problem 3, we are lead to try solutions of the type $X_t = e^{-t} Y_t$ for some process Y_t.[here we have the negative sign instead of the positive sign]. The rest of the solution is also similar: such Y_t satisfy

$$Y_t = e^t X_t$$

and Ito's formula gives

$$\begin{aligned} dY_t &= e^t X_t dt + e^t dX_t \text{ [Why the Ito term vanishes?]} \\ &= e^t X_t dt + e^t \left[-X_t dt + e^{-t} dW(t) \right] \\ &= dW(t). \end{aligned}$$

Hence $Y_t = W(t)$, and it follows that $X_t = e^{-t} W(t)$, which is a scaled BM.