Session 10

Chunyu Qu 08/20/20

5). Positive Definite

5.1. Def. A symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

is called PD if

$\forall \vec{x} \in \mathbb{R}^2, \quad \vec{x}^T A \vec{x} > 0 \quad (\vec{x} \neq \vec{0})$

5.2. Remark. For 2x2 A. A is PD if

(i) $a > 0$

(ii) $ac - b^2 > 0$

5.3. Theorem $A_{n \times n}$.

A symmetric matrix $A_{n \times n}$ is PD iff.

any of the following is true

(i) $\vec{x}^T A \vec{x} > 0 \quad (\forall \vec{x} \in \mathbb{R}^n, \vec{x} \neq 0)$

(ii) All eigenvalues of are positive.

$(\lambda_i > 0, \quad i = 1, 2 \cdots n)$

(iii) All the upper left submatrices has positive determinant.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{15} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{25} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{35} \\ \vdots & \vdots & & \ddots & \\ a_{51} & a_{52} & a_{53} & \cdots & a_{55} \end{bmatrix}$$

$|A_1| > 0$

$|A_2| > 0$

$|A_3| > 0$

$|A_4| > 0$

$|A_5| > 0$

ex. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $\quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Solution 1 $\boxed{\text{by (iii)}}$

$|A_1| = |2| > 0$

$|A_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

$|A_3| = 2 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$

$= 2 \cdot 3 + 1 \cdot (-2) = 4 > 0$

$\therefore A$ is PD

Solution 2. $\boxed{\text{by (ii)}}$ Find λ's.

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)^2-1\right] + (-1)(2-\lambda)$$

$$= (2-\lambda)\left[\lambda^2-4\lambda+4-2\right] = (2-\lambda)\underbrace{(\lambda^2-4\lambda+2)}_{\shortparallel\ 0} = 0$$

$$\lambda_1 = 2, \quad \lambda_{2,3} = \frac{4 \pm \sqrt{16-8}}{2}$$

$$= 2 \pm \sqrt{2} \quad > 0$$

A is PD

5.4. Remark If A is PD, we can decompose A into LU.

$$L = \begin{bmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ \vdots & & & \ddots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

Actually $U = \underbrace{\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}}_{\checkmark\ D} \underbrace{\begin{bmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}}_{S}$

?

The tricky thing is we can verify that $S = L^T$

i.e. $A = LU = LDS \overset{S=L^T}{=} LDL^T$

5.5. LDL^T Decomposition

If A is PD, then A can be represented as $A = L\,D\,L^T$

L ↓ Lower triangular matrix; D ↘ diagonal matrix

5.6 ex.

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

Step 1. LU factorization. Row Echelon Form

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} \xrightarrow[R_3+4R_1]{R_2-3R_1} \begin{bmatrix} 4 & 12 & -16 \\ 0 & 1 & 5 \\ 0 & 5 & 34 \end{bmatrix}$$

$-(-4)$

$$\xrightarrow{R_3-5R_2} \begin{bmatrix} 4 & 12 & -16 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix} \equiv U.$$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix}}$$

Step 2. Solve for S

$$U = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

12, 3, -16, -4, 5, 5

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = LDL^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 1 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = L^T$

5.7. Cholesky Decomposition.

$$A = LDL^T$$

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix} \begin{bmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{bmatrix}$$

$$= \sqrt{D}\sqrt{D}$$

$$A = \underbrace{\boxed{L\sqrt{D}}}_{R^T} \underbrace{\boxed{\sqrt{D}L^T}}_{R}$$

$$R^T = (\sqrt{D}L^T)^T = (L^T)^T(\sqrt{D})^T = L\sqrt{D}$$

$\boxed{A = R^T R}$ is called Cholesky decomposition.

5.6. ex (continued)

$$A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 1 & \\ & & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 3 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & & \\ & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}}_{R^T} \underbrace{\begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}}_{R.}$$

$$= R^T R$$

5.7. Theorem

A is PD $\Leftrightarrow$ $\exists$ R (upper triangular matrix)

$\ni.$ $A = R^T R$

5.8. Positive Semi-definite (PSD)

$$\vec{x}^T A \vec{x} \geq 0 \qquad (\forall \vec{x} \in \mathbb{R}^n,\ \vec{x} \neq \vec{0})$$

5.9.

A is PSD. iff. any of the following holds.

(i) $\vec{x}^T A \vec{x} \geq 0$, $\forall \vec{x} \in \mathbb{R}^n$

(ii) All the eigenvalues $\lambda_i \geq 0$ $i = 1, \cdots n$.

(iii) All the principal submatrices have determinant ≥ 0

(iv) $A = R^T R$ $\quad R =$ ( ) is upper triangular

5.10.

ex. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

use (iii) to determine A is PSD.

(i) 1×1 submatrices $\quad 2 \geq 0,\ 2 \geq 0,\ 2 \geq 0$

(ii) 2×2 submatrices $\quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 \geq 0$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 \geq 0$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 \geq 0$

(iii) 3×3 submatrix $\quad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{R_3/3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix}$

$\xrightarrow{R_2 \to 2R_2 + R_1} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0$

$\therefore$ A is PSD. not PD

6) Trace & Kernel, Range.

6.1. Def. $trA = a_{11} + \cdots a_{nn}$, $A_{n\times n}$

6.2. Theorem $A_{n\times n}$ has n eigenvalues $\lambda_1 \cdots \lambda_n$.

$\Rightarrow$ (i) $\sum_{i=1}^{n} \lambda_i = trA$

(ii) $\prod_{i=1}^{n} \lambda_i = |A|$

6.3. Kernel & Range.

Let $T: V \to W$ V, W are subspaces of $\mathbb{R}^n$.

- kernel of T, denoted $ker(T)$, is defined

$$Ker(T) = \{ \vec{v} \in V : T(\vec{v}) = \vec{0} \}.$$

T_A: $A\vec{v} = 0 \Rightarrow \vec{v}$ solution to homogeneous system, $\xrightarrow{\text{Def of}}$ $null(T_A)$ null(space)

- Range of, $range(T) \stackrel{def}{=} \{ \vec{w} \in W : \vec{w} = T(\vec{v})$ for some $\vec{v} \in V \}$.

6.4. ex. $A_{m\times n}$, and $T = T_A$ is the corresponding matrix transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

Find $ker(T)$

$$ker(T) = \{ \vec{v} \text{ in } \mathbb{R}^n : T(\vec{v}) = \vec{0} \}.$$

$$= \{ \vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = 0 \}.$$

$$= null(A)$$

7). Least square solution
Generalised matrix inverse

7.1. Def 1 $A_{m\times n}$, $\vec{b} \in \mathbb{R}^n$ $A\vec{x} = \vec{b}$ is system of linear equations.

Least square solution of $A\vec{x} = \vec{b}$ is a vector $\overline{\vec{x}} \in \mathbb{R}^n$ $\ni$.

$$\|\vec{b} - A\overline{\vec{x}}\| \le \|b - A\vec{x}\|. \ (\forall \vec{x} \in \mathbb{R}^n)$$

7.2. Theorem. $\bar{x} = (A^T A)^{-1} A^T \vec{b}$

recap $\hat{\beta} = (X^T X)^{-1} X^T \vec{b}$ — OLS solution to coefficients.

is the solution to the coefficients estimators for

$Y = X\beta + \varepsilon$

7.3. Theorem. LS $\leftrightarrow$ QR.

Let $A_{m\times n}$ w/. IND. columns in $\mathbb{R}^n$.

$\vec{b} \in \mathbb{R}^n$

If $A = QR$

$\Rightarrow$ the unique least square solution for $A\vec{x} = b$

$$\boxed{\overline{\vec{x}} = R^{-1} Q^T \vec{b}}$$

7.4. Pseudo inverse (Moore–Penrose Inverse)

$A^T A$

For a non-square matrix $A_{m\times n}$

If. A has IND. columns, then

we can define P-inverse A^+

$$A^+ = (A^T A)^{-1} A^T$$

7.5. Properties of pseudoinverse

a. $AA^+A = A$ $\quad AA^{-1}A = A$

b. $A^+AA^+ = A^+$ $\quad A^{-1}AA^{-1} = A^{-1}$

c. AA^+, A^+A are symmetric $\quad AA^{-1} = I$

d. If A is invertible, then $A^{-1} = A^+$

e. $(A^+)^+ = A$

f. $(\alpha A)^+ = \frac{1}{\alpha} A^+$ $\quad (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

7.6 ex.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}_{3\times 2} \quad \text{Find } A^+$$

$$A^+ = \underbrace{\underbrace{(\underset{2\times 3}{A^T}\ \underset{3\times 2}{A})^{-1}}_{2\times 2}\ \underset{2\times 3}{A^T}}_{2\times 3}$$

$$= \left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{42-36} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

8). Singular Value Decomposition.

8.1 A is symmetric we have the spectrum decomposition $Q^TAQ = D \Leftrightarrow A = Q^{-1}DQ^T$, $Q^T = Q^{-1}$.

But A is $m\times n$ $\Rightarrow$ do not have such a orthogonal matrix Q

8.2. Algorithm for SVD

"diagonal" matrix

$$\underset{m\times n}{A} = U \Sigma V^T$$

U: $m\times m$, square orthogonal

Σ: $m\times n$

V^T: $n\times n$, square orthogonal

$$\boxed{\Sigma} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$$

square

$$D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

singular values of A $\rightarrow$ eigenvalues of A^TA

Find the eigenvectors of A, G-S & Normalization to convert the eigenvectors to $\vec{v}_1 \cdots \vec{v}_n$

$$V = [\vec{v}_1 \cdots \vec{v}_n]$$

$$U = [\vec{u}_1 \cdots \vec{u}_m]$$

$$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i$$

8.3. ex. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times 3}$

Step 1. $\underbrace{\underset{3\times 2}{A^T}\ \underset{2\times 3}{A}}_{3\times 3} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 2. Find singular values.

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0.$

$(1-\lambda) \cdot \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^3 - (1-\lambda) = 0$

$(1-\lambda)(\lambda^2 - 2\lambda \cancel{+1} \cancel{-1}) = 0$

$(1-\lambda)\lambda(\lambda - 2) = 0$

$\Rightarrow \begin{cases} \lambda_1 = 2, & \sigma_1 = \sqrt{\lambda_1} = \sqrt{2} \\ \lambda_2 = 1, & \sigma_2 = \sqrt{\lambda_2} = 1 \\ \lambda_3 = 0, & \sigma_3 = 0 \end{cases}$

Step 3. Find eigenvectors.

solve $[A - \lambda I \,|\, \vec{0}]$

$\vec{x}_1, \vec{x}_2, \vec{x}_3$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Step 4. Form $V = [\vec{v}_1 \cdots \vec{v}_n] = \left[\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right]$

orthonormal basis

$\begin{cases} \text{G-S} \\ \text{Normalization} \end{cases}$

$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$

Step 5. Form $\Sigma = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]_{2\times 3}.$

Step 6. Form $U = [\vec{u}_1, \vec{u}_2]$ with $\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i$

$\vec{u}_1 = \frac{1}{\sqrt{2}} \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}$

$\vec{u}_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$> U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Finally $A = U \Sigma V^T$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\left[\begin{array}{cc|c} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}}_{V^T}$$

Lagrange Multiplier – constrained optimization

(1 constraint)

$$\boxed{\begin{array}{ll} \max & F(x,y) \\ \text{s.t.} & g(x,y)=0 \end{array}}$$

1. Step 1. Write Lagrange function

$$\mathcal{L}(x,y;\lambda) = F(x,y) + \lambda g(x,y)$$

λ → Lagrange multiplier.

Step 2. Give F.O.C.'s for each variables in $\mathcal{L}(x,y;\lambda)$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} \overset{\text{set}}{=} 0$$

Step 3. Solve for x^*, y^* as the extremes

2. Ex. $F(x,y) = 2y + x$

s.t. $g(x,y) = y^2 + xy - 1 = 0$

Find the extrema of $F(x,y)$ given the condition $g(x,y)=0$

Step 1. $\mathcal{L}(x, y, \lambda) = 2y + x + \lambda(y^2 + xy - 1)$

$= x + \lambda xy + 2y + \lambda y^2 - \lambda.$

Step 2. $\frac{\partial \mathcal{L}}{\partial x} = 1 + \lambda y \overset{set}{=} 0$ ①

$\frac{\partial \mathcal{L}}{\partial y} = 2 + 2\lambda y + \lambda x \overset{set}{=} 0$ ②

$\frac{\partial \mathcal{L}}{\partial \lambda} = y^2 + xy - 1 = 0$ ③

Step 3. ① $\Rightarrow \lambda = -\frac{1}{y}$

② $\Rightarrow 2 + \lambda(2y + x) = 0$

$\Rightarrow \lambda = -\frac{2}{2y+x}$

① ② $\Rightarrow +\frac{1}{y} = +\frac{2}{2y+x}$ ③ $\Rightarrow y \neq 0$

$2y = 2y + x \Rightarrow x = 0$

put x back to ③

$y^2 - 1 = 0 \Rightarrow y = \pm 1$

Finally, we have $(0, 1)$ $(0, -1)$ as our extrema.

when $(x, y) = (0, 1)$ $F(0, 1) = 2$ is maximum

$(x, y) = (0, -1)$ $F(0, -1) = -2$ is minimum.

3. ex. $f(x,y) = x^2 + 3xy + y^2 - x + 3y$

s.t. $g(x,y) = x^2 - y^2 - \frac{1}{3} = 0.$

Find the extrema.

Step 1. $\mathcal{L}(x, y; \lambda) = x^2 + 3xy + y^2 - x + 3y$

$$+ \lambda\left(x^2 - y^2 - \frac{1}{3}\right)$$

Step 2. F.O.C.

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + 3y - 1 + 2\lambda x \overset{set}{=} 0. \quad ①$$

$$\frac{\partial \mathcal{L}}{\partial y} = 3x + 2y + 3 - 2\lambda y \overset{set}{=} 0. \quad ②$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 - y^2 - \frac{1}{3} = 0 \quad ③.$$

$x \sim y$

Step 3.

$$\boxed{\frac{2x+3y-1}{-\cancel{2}x} = \lambda = \frac{3x+2y+3}{\cancel{2}y}}$$

$\Uparrow$ ① $\Uparrow$ ②.

$$(-x)(3x+2y+3) = y(2x+3y-1).$$

$$-3x^2 - 2xy - 3x = 2xy + 3y^2 - y.$$

$$-3x^2 - 4xy - 3x - 3y^2 + y = 0$$

$$3x^2 + 4xy + 3x + 3y^2 - y = 0. \quad ④$$

$$(ax + by + c)(dx + ey + f) = 0$$

$ad = 3$ $be = 3$ $ae + bd = 4$ $cf = 0$

$bf = -1$

Problems to HW & Some Complement.

1. Repeated eigenvalues.

Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\qquad \lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$

$\lambda_{1,2} = 1 \qquad [A - \lambda I \mid 0] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$x_1 = s$, $x_2 = t$, $x_3 = 0$, $s, t \in \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + t \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$\leftarrow$ two eigenvetors

$\vec{x}_1$, $\vec{x}_2$

$\lambda = 1$ for 2 times $\downarrow$ Algebraic multiplicity

A is not diagonlizable $\Uparrow$ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$<$

$=$

2 eigenvectors $\downarrow$ geometric multiplicity

$\lambda_3 = 2 \Rightarrow \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Downarrow$ A is diagonalizable

Ex $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\qquad \lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = 2$

$\lambda_{1,2} = 1 \qquad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 = t \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

$\leftarrow$ only 1 eigenvetor

$\vec{x}_1$

2. Diagonalizable

$$\boxed{\underset{n\times n}{A} \text{ is diagonalizable iff. } A \text{ has } n \text{ IND eigenvetors.}}$$

3. Complex eigenvalues

ex. $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

A $(a_{ij} \in \mathbb{R}) \to A^T$

H $(h_{ij} \in \mathbb{C}) \to H^*$

$a+bi$ $\downarrow$

conjugate transpose

$a-bi$

Solve $|A - \lambda I| = 0$, $\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 + 4 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 5 = 0$

$\Delta = 4 - 4 \cdot 5 < 0$

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\lambda_1 = 1+2i$ $\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right] \xrightarrow{iR_1} \left[\begin{array}{cc|c} 2 & -2i & 0 \\ 2 & -2i & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 2 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 - 2i x_2 = 0$

$x_1 = i x_2$

Let $x_2 = t \Rightarrow x_1 = it$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\vec{x_1}$

$\lambda_2 = 1 - 2i$ TBD.

HW 6

Q3. Determine $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

are IND.

W.t.S. $c_1 = c_2 = c_3 = 0$

$$\Rightarrow \quad c_1 \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is the only solution

$$\begin{cases} c_1 + 2c_2 + c_3 = 0 \\ 2c_1 + c_2 + c_3 = 0 \\ 4c_1 - c_2 + c_3 = 0 \\ 3c_1 + c_3 = 0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & -1 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 - 4R_1 \\ R_4 - 3R_1}]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -9 & -3 & 0 \\ 0 & -6 & -2 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -9 & -3 & 0 \\ 0 & -6 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 + 2R_2]{R_3 + 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} c_1 - c_2 = 0 \\ 3c_2 + c_3 = 0 \end{cases} \quad c_1 \overset{\text{set}}{=} t \quad c_2 = t \quad c_3 = -3t \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ -3t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$\Rightarrow$ Matrices are not IND.

Q6.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

Row(A), col(A), null(A).

(1) $A \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$Row(A) = span\left([1\ 1\ 0\ 1], [0\ 1\ -1\ 1]\ [0\ 0\ 0\ -2]\right)$

$= span\left([1\ 1\ 0\ 0], [0\ 1\ -1\ 0], [0\ 0\ 0\ 1]\right)$

(2) Col(A) step 1 Reduce A^T to row echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_4-R_1]{R_2-R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_4-R_2]{R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & -2 \end{bmatrix} \rightarrow$$

step 2. $Col(A) = row(A^T)$ w/. transpose each vectors in $row(A^T)$

$$Col(A) = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right)$$

(3) Null(A)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] =$$

Reduced echelon $\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$$\begin{cases} x_4 = 0 \\ x_1 + x_2 = 0 \\ x_2 - x_3 = 0 \end{cases} \xRightarrow[x_3 = t]{\text{set}} \begin{cases} x = -t \\ x_2 = t \\ x_3 = t \\ x_4 = 0 \end{cases} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$null(A) = span\left(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

Final exam 26. (EST. USA).

Wednesday 7:30 am.