Homework 4

Wei Ye*
QF8915 - Stochastic Calculus

Due on Dec 16, 2022

Problem 1

Apply Ito's formula to express $Y_t = \log(1 + (X_t)^2)$ as an Ito process.

Solution:

First, we assume $Y_t = f(X_t) = \log(1 + (X_t)^2)$. Thus, $f'(X_t) = \frac{2X_t}{1+X_t^2}$, and $f''(X_t) = \frac{2(1+X_t^2)-2X_t(2X_t)}{(1+X_t^2)^2} = \frac{2(1-X_t^2)}{(1+X_t^2)^2}$

$$\begin{aligned}
dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)dX_t dX_t \\
&= \frac{2X_t}{1+X_t^2}[W(t)dt + W(t)^2 dW_t] + \frac{1}{2} \cdot \frac{2(1-X_t^2)}{(1+X_t^2)^2}W(t)^4 dt \\
&= [\frac{2X_t^2}{1+X_t^2}W(t) + \frac{1-X_t^2}{(1+X_t^2)^2}W(t)^4]dt + \frac{2X_t}{1+X_t^2}[W(t)]^2 dW(t)
\end{aligned}$$

Thus:

$$Y_t = Y_0 + \int_0^t \frac{2X_s}{1+X_s}W(s) + \frac{1-X_s^2}{(1+X_s^2)^2}W(s)^4 ds + \int_0^t \frac{2X_s}{1+X_s^2}W(s)^2 dW(s)$$

Problem 2

Let $Y_t = (1+t)X_t$. Use Ito's formula to find out what stochastic differential equation Y_t satisfies? Identify Y_t as a Brownian Motion.

Solution:

*2nd year PhD student in Economics Department at Fordham University. Email: wye22@fordham.edu

Let $Y_t = f(t, X_t) = (1+t)X_t$. Therefore, $f_t(t, X_t) = X_t$, $f_x(t, X_t) = 1 + t$, and $f_{xx} = 0$.

$$\begin{aligned} dY_t &= f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}dX_t dX_t \\ &= X_t dt + (1+t)dX_t \\ &= X_t dt + (1+t)[-\frac{1}{1+t}X_t d_t + \frac{1}{1+t}dW(t)] \\ &= X_t dt - X_t dt + dW_t \\ &= dW_t \end{aligned}$$

Thus, we can easily see Y_t is brownian motion.

Problem 3

Solve $dX_t = X_t dt + dW(t)$.

Solution:

Set an equation[1] $f(t, X_t) = e^{-t}X_t$

$$\begin{aligned} df(t, X_t) &= -e^{-t}X_t dt + e^{-t}[X_t dt + dW_t] \\ &= -e^{-t}X_t dt + e^{-t}X_t dt + e^{-t}dW_t \\ &= e^{-t}dW(t) \end{aligned}$$

$$e^{-t}X_t = X_0 + \int_0^t e^{-u}dW(u)$$

Therefore:

$$X_t = e^t X_0 + e^t \int_0^t e^{-u}dW(u)$$
$$E(X_t) = e^t X_0$$

Problem 4

Solve $dX_t = -X_t d_t + e^{-t}dW(t)$

Solution:

Let $f(t, X_t) = e^t X_t$

$$\begin{aligned} df(t, X_t) &= e^t X_t dt + e^t[-X_t dt + e^{-t}dW(t)] \\ &= dW(t) \end{aligned}$$

Hence,

$$e^t X_t = x_0 + \int_0^t dW_t$$

[1]For the convenience of our computation, I let β in the lecture is equal to -1, only in this way, we can cancel out X_t in later steps.

$$X_t = e^{-t} X_0 + e^{-t} \int_0^t dW_t$$

$$\begin{aligned} E(X_t) &= e^{-t} X_0 + e^{-t} \int_0^t dW_u \\ &= e^{-t} X_0 \end{aligned}$$