Note about Bordered Var-Cov Matrix

	w_D	w_E
w_D	var	cov
w_E	var	cov

where n is the number of different assets in the risky portfolio, the number of variances = n, and the number of different covariances = $\frac{n(n-1)}{2}$

For each of the n assets, there are $n-1$ covariances, including the repetitions.

covariance $\geqslant 0$

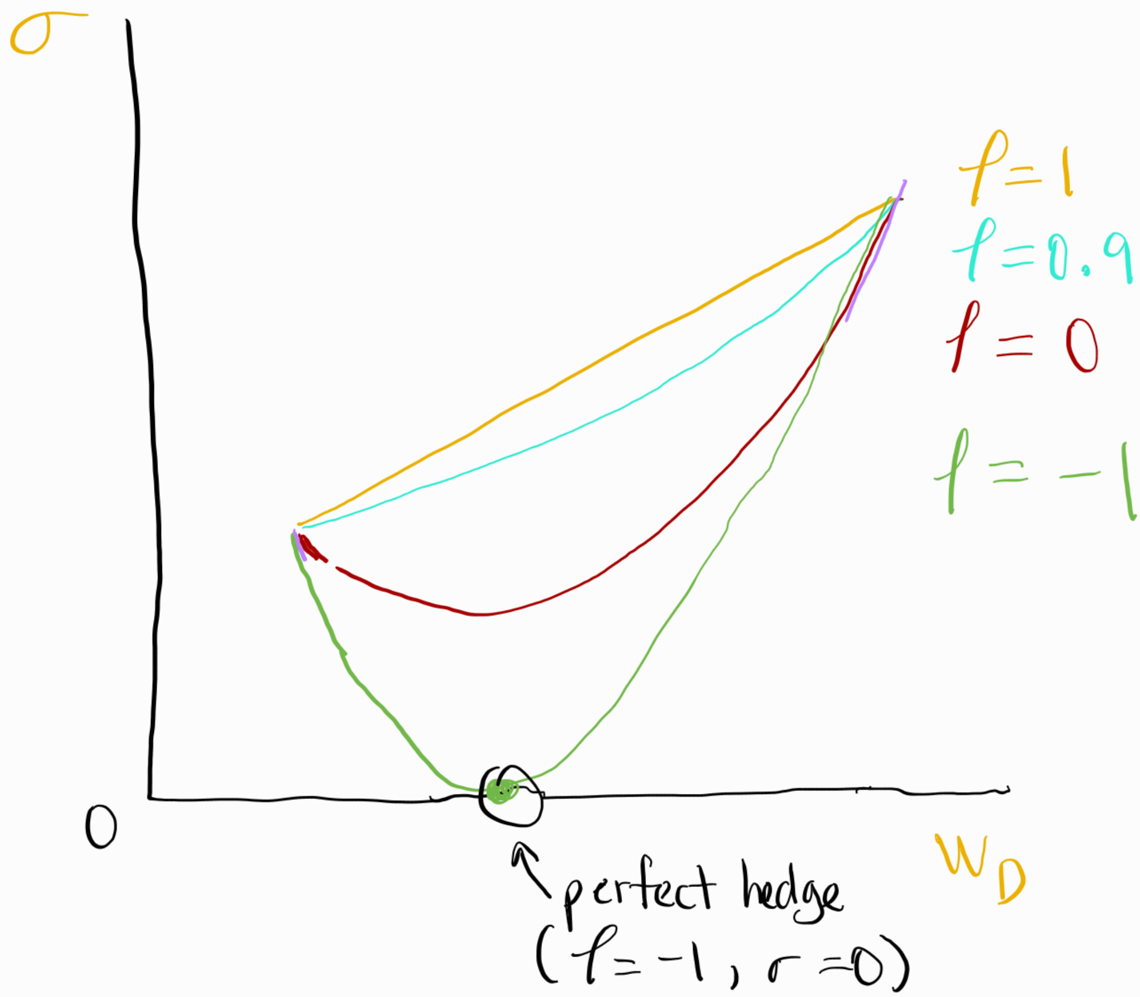
correlation coefficient $= \rho$ where $-1 \leq \rho \leq 1$

$\rho = 0$ indicates linear independence

$$\rho = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$$

When $\rho = 1$, $\sigma_p = w_D \sigma_D + w_E \sigma_E$ and there is no diversification and thus no risk reduction. Diversification and risk reduction happens for values of ρ less than 1.

The graph below shows how σ on w_D changes as the value of ρ changes:



As mentioned in previous notes, when $\rho = -1$ it is possible to choose a w_D such that your composite risk / volatility for your entire risky portfolio $= 0$.

Capital Asset Pricing Model (CAPM)

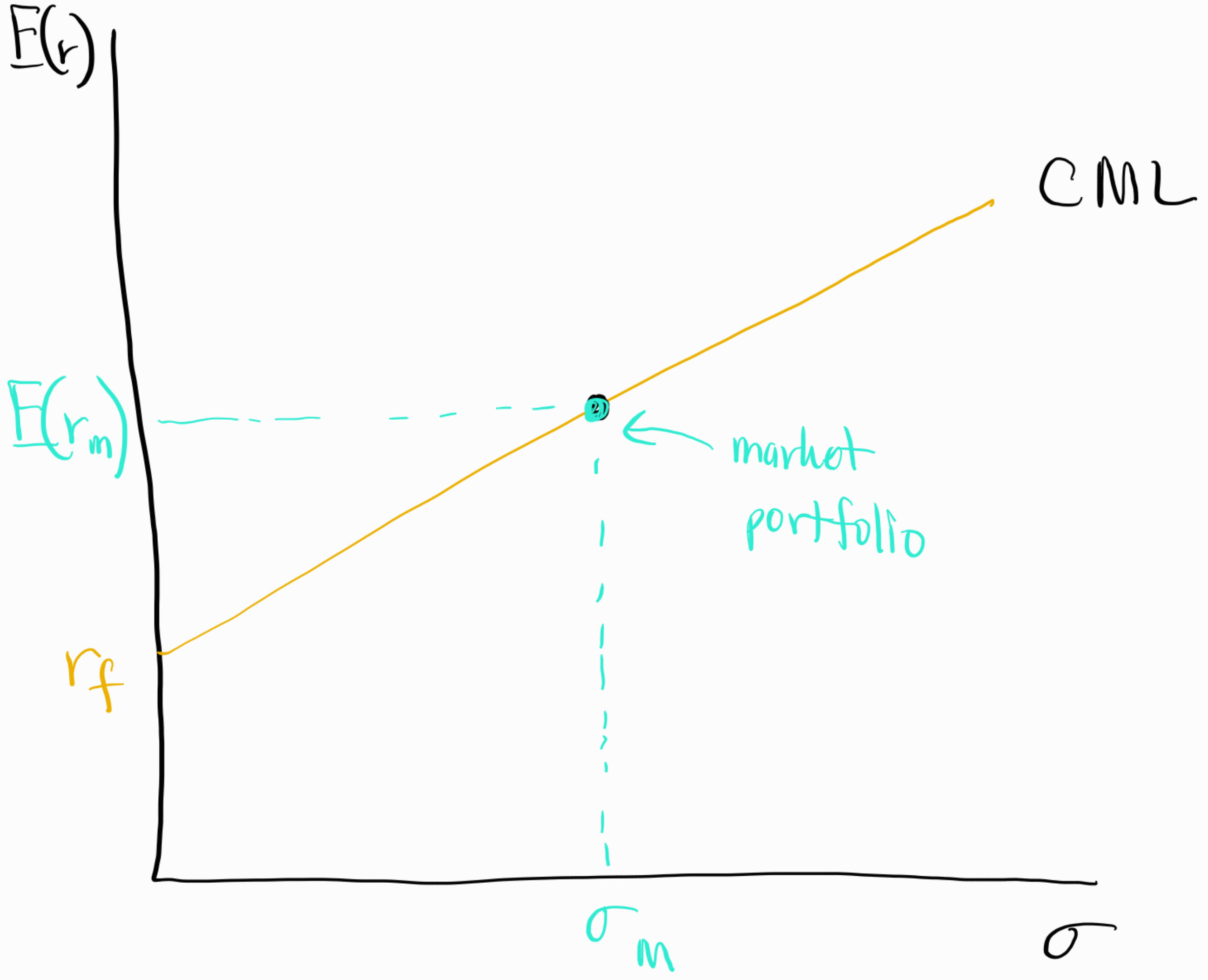
→ equilibrium model

→ Assumptions

1. perfect competition (everyone is a price taker)
2. everyone is equal except in there initial endownments
3. no distortional taxes or transaction costs
4. everyone has same investment time horizon (myopia)
5. everyone invests in publicly traded companies (so in theory all information is the same across agents)
6. everyone is a mean-variance optimizer
7. everyone reacts in the same way to public news (so everyone has the same probability distribution).

Given the above assumptions, we should end up with a single risky portfolio that we call the market portfolio.

So essentially we end up with a familiar graph:



where CAL is now CML or capital market line.

To obtain the weights for this risky portfolio, we use the following formula:

$$w_i = \frac{\text{market capitalization of asset } i}{\text{total market capitalization}}$$

where market capitalization is the number of outstanding shares of asset i times the price of asset i.

If there is an asset i that no one wants to buy, the price will lower until the it becomes attractive enough to purchase.

$$y^* = \frac{E(r_m) - r_f}{0.01 \bar{A} \sigma_m^2}$$

(portion to invest in the risky portion of portfolio)

market risk

average market risk aversion coefficient

Some agents will borrow and others will lend. Some will short and others will long. But on average these effects should cancel out.

So if borrowing = lending, then $y_m^* = 1$. Plugging this into the above formula:

$$1 = \frac{E(r_m) - r_f}{0.01\,\bar{A}\,\sigma_m^2}$$

Thus $$\underbrace{E(r_m) - r_f}_{\text{market risk premium}} = 0.01\,\bar{A}\,\sigma_m^2$$

market sentiment

market risk / volatility

Suppose we want to choose an additional security to add to our portfolio. We want to measure the marginal benefit of the additional security.

Looking at Bordered Var-Cov Matrix:

$$
\begin{array}{c|ccccc}
 & w_1 & \cdots & w_i & \cdots & w_n \\
\hline
w_1 & var & - & - & - & - \\
\vdots & & & & & \\
w_i & cov(w_i\, w_1) & & \underset{(var)}{cov(w_i\, w_i)} & & cov(w_i\, w_n) \\
\vdots & & & & & \\
w_n & - & - & - & \cdots & var
\end{array}
$$

If we sum across the highlighted row:

$$w_i \sum_{j=1}^{n} w_j \, cov(r_i, r_j)$$

and since we sum across ALL weights $\{ \Sigma w = 1$,

this is just $\boxed{w_i \cdot cov(r_i, r_j)}$

Going back to our additional security example, we essentially want to see what this security will do to overall portfolio returns and volatility.

Contribution of security i to port. risk premium

Contribution of security i to portfolio volatility

or $$\frac{\cancel{w_i}\left[E(r_i) - r_f\right]}{\cancel{w_i} \cdot cov(r_i, r_m)}$$

The market $\Rightarrow \dfrac{E(r_m) - r_f}{\sigma_m^2}$

(since the market has a single point, $cov(r_i, r_m)$ is $cov(r_m, r_m)$ or σ_m^2)

So $$\frac{E(r_i) - r_f}{cov(r_i, r_m)} = \frac{E(r_m) - r_f}{\sigma_m^2}$$

which gives:

$$E(r_i) - r_f = \underbrace{\frac{cov(r_i, r_m)}{\sigma_m^2}}_{\text{Beta coefficient}}\left[E(r_m) - r_f\right]$$

So we ultimately end up with:

$$E(r_i) - r_f = \beta_i\left[E(r_m) - r_f\right]$$

So a unit change in the market risk premium affects the asset i risk premium by β_i.

If $B < 1$, defensive

If $B > 1$, agressive

Note that the market beta, $B^m = 1$.

$$w_1 E(r_1) = w_1 r_f + w_1 B_1 [E(r_m) - r_f]$$

$$w_2 E(r_2) = w_2 r_f + w_2 B_2 [E(r_m) - r_f]$$

$$\vdots$$

$$w_K E(r_K) = w_K r_f + w_K B_K [E(r_m) - r_f]$$

Summing across all these, we get:

$$\underbrace{\sum_{j=1}^{n} w_j \cdot E(r_j)}_{E(r_p)} = r_f + \underbrace{\sum_{j=1} B_j}_{B_p} [E(rm) - r_f]$$

$$B^{\text{MARKET}} = \frac{\operatorname{cov}(r_m, r_m)}{\sigma_m^2} = \frac{\sigma_m^2}{\sigma_m^2} = 1$$

So $B^m = 1$

We want to be market neutral. So if $\beta_1 = 0.6$ and $\beta_2 = 1.4$, we choose our weights such that:

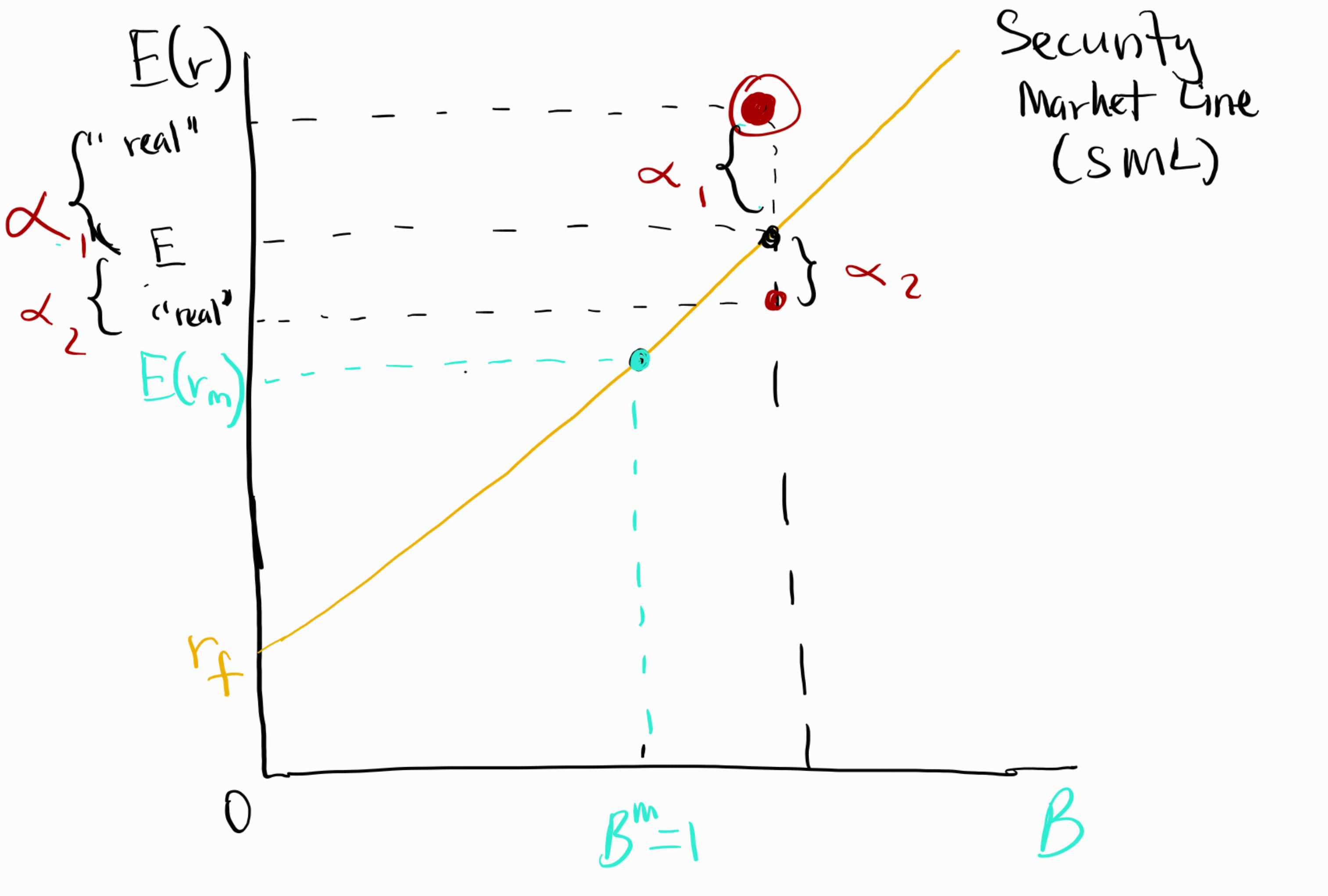
$$w_1 \beta_1 + w_2 \beta_2 = \beta^m = 1$$

$$0.5(0.6) + 0.5(1.4) = 1$$

So the $\sum_{j=1}^{n} w_j \beta_j = 1$

$w_1 = 0.5$ & $w_2 = 0.5$

Going back to our graph (beta instead of sigma)



$\beta = 0$ occurs at r_f. $\beta^m = 1$ gives the market. The α shows the difference between the "real" expected return and the estimated return for a particular beta.

$$E(r_i) - r_f = \beta_i \left[E(r_m) - r_f \right]$$

Test for alpha:

$$E(r_i) - r_f = \alpha + \beta \left[E(r_m) - r_f \right]$$

use a hypothesis test for $\alpha = 0$

If α is positive (higher expected return), then more people will invest in it and the price will increase, reducing returns.

If α is negative (lower expected return), then more people will sell and the price will fall, increasing returns.