Exercise #4

3.E.8 For the Cobb-Douglas utility function, verify that the realtionships in

$$e(p, v(p, w)) = w \qquad \text{and} \qquad v(p, e(p, u) = u \tag{3.E.1}$$

and

$$h(p, u) = x(p, e(p, u)) \qquad \text{and} \qquad x(p, w) = h(p, v(p, w)) \tag{3.E.4}$$

hold. Note that the expenditure function can be derived by simply inverting the indirect utility function, and vice versa.

3.E.8 We use the utility function $u(x) = x_1^{\alpha} x_2^{(1-\alpha)}$. To prove (3.E.1),

$$e(p,v(p,w)) = \alpha^{-\alpha}(1-\alpha)^{\alpha-1} p_1^{\alpha} p_2^{1-\alpha}(\alpha^{\alpha}(1-\alpha)^{1-\alpha} p_1^{-\alpha} p_2^{\alpha-1} w) = w,$$

$$v(p,e(p,u)) = \alpha^{\alpha}(1-\alpha)^{1-\alpha} p_1^{-\alpha} p_2^{\alpha-1}(\alpha^{-\alpha}(1-\alpha)^{\alpha-1} p_1^{\alpha} p_2^{1-\alpha} u) = u.$$

To prove (3.E.3),

$$x(p,e(p,u)) = (\alpha^{-\alpha}(1-\alpha)^{\alpha-1} p_1^{\alpha} p_2^{1-\alpha} u)(\alpha/p_1,\ (1-\alpha)/p_2)$$

$$= \left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} u,\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} u\right) = h(p,u),$$

$$h(p,v(p,w)) = \alpha^{\alpha}(1-\alpha)^{1-\alpha} p_1^{-\alpha} p_2^{\alpha-1} w \left(\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha},\ \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\right)$$

$$= w(\alpha/p_1,\ (1-\alpha)/p_2) = x(p,w).$$

1) Walrasian demand - Indirect utility function

$$\max_{\{x_1, x_2\}} u(x) = x_1^{\alpha} x_2^{1-\alpha}$$

$$\text{s.t.:} \quad p_1 x_1 + p_2 x_2 = w$$

$$\mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} + \lambda (w - p_1 x_1 - p_2 x_2)$$

FOC: (x_1): $\alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \iff \lambda = \dfrac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1}$

(x_2): $(1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0 \iff \lambda = \dfrac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2}$

(λ): $w - p_1 x_1 - p_2 x_2 = 0$

From $(x_1) + (x_2)$: $\dfrac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{p_1} = \dfrac{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}{p_2} \iff x_1 = \left(\dfrac{\alpha}{1-\alpha}\right)\left(\dfrac{p_2}{p_1}\right) x_2$

Plug into (5): $\cancel{p_1}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_2}{\cancel{p_1}}\right)x_2 + p_2 x_2 = w \Leftrightarrow \left(\frac{1-\alpha+\alpha}{1-\alpha}\right)p_2 x_2 = w$

$$\Leftrightarrow x_2 = \frac{(1-\alpha)w}{p_2}$$

and: $x_1 = \left(\frac{\alpha}{\cancel{1-\alpha}}\right)\left(\frac{\cancel{p_2}}{p_1}\right)\frac{(\cancel{1-\alpha})w}{\cancel{p_2}} = \frac{\alpha w}{p_1}$

Walrasian demand:

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \end{bmatrix} = \begin{bmatrix} \frac{\alpha w}{p_1} \\ \frac{(1-\alpha)w}{p_2} \end{bmatrix}$$

Plug into utility function to obtain Indirect Utility Function,

$$v(p,w) = u(x(p,w)) = \left(\frac{\alpha w}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w$$

2) Hicksian demand - Expenditure Function

$$\min_{\{x_1,x_2\}} \quad p_1 x_1 + p_2 x_2$$

$$\text{s.t.:} \quad x_1^{\alpha} x_2^{1-\alpha} = u$$

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda\left(u - x_1^{\alpha} x_2^{1-\alpha}\right)$$

FOC: (x_1): $p_1 - \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha} \Leftrightarrow \lambda = \frac{p_1}{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}$

(x_2): $p_2 - \lambda(1-\alpha) x_1^{\alpha} x_2^{-\alpha} \Leftrightarrow \lambda = \frac{p_2}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}}$

$(\lambda): \quad u - x_1^{\alpha} x_2^{1-\alpha} = 0$

From $(x_1) + (x_2)$: $\quad \dfrac{p_1}{\alpha x_1^{\alpha-1} x_2^{1-\alpha}} = \dfrac{p_2}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} \Leftrightarrow x_1 = \left(\dfrac{\alpha}{1-\alpha}\right)\left(\dfrac{p_2}{p_1}\right) x_2$

Plug into (λ): $\quad u = \left(\dfrac{\alpha}{1-\alpha}\right)^{\alpha}\left(\dfrac{p_2}{p_1}\right)^{\alpha} x_2^{\alpha}\, x_2^{1-\alpha} = 0 \Leftrightarrow x_2 = \left(\dfrac{1-\alpha}{\alpha}\right)^{\alpha}\left(\dfrac{p_1}{p_2}\right)^{\alpha} u$

and: $\quad x_1 = \left(\dfrac{\alpha}{1-\alpha}\right)\left(\dfrac{p_2}{p_1}\right)\left(\dfrac{1-\alpha}{\alpha}\right)^{\alpha}\left(\dfrac{p_1}{p_2}\right)^{\alpha} u = \left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\dfrac{p_2}{p_1}\right)^{1-\alpha} u$

Hicksian demand:

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\dfrac{p_2}{p_1}\right)^{1-\alpha} u \\ \left(\dfrac{1-\alpha}{\alpha}\right)^{\alpha}\left(\dfrac{p_1}{p_2}\right)^{\alpha} u \end{bmatrix}$$

Expenditure function:

$$e(p,u) = p \cdot h(p,u) = p_1 \cdot h_1(p,u) + p_2 \cdot h_2(p,u) = p_1\left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\dfrac{p_2}{p_1}\right)^{1-\alpha} u \,+$$

$$+\, p_2\left(\dfrac{1-\alpha}{\alpha}\right)^{\alpha}\left(\dfrac{p_1}{p_2}\right)^{\alpha} u = \left[p_1^{\alpha}\left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha} p_2^{1-\alpha} + p_2^{1-\alpha}\left(\dfrac{1-\alpha}{\alpha}\right)^{\alpha} p_1^{\alpha}\right] u$$

$$= p_1^{\alpha} p_2^{1-\alpha}\left[\left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\dfrac{1-\alpha}{\alpha}\right)^{\alpha}\right] u = p_1^{\alpha} p_2^{1-\alpha}\left[\left(\dfrac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\dfrac{\alpha}{1-\alpha}\right)^{-\alpha}\right] u$$

$$= p_1^{\alpha} p_2^{1-\alpha}\left[\left(\dfrac{\alpha}{1-\alpha}\right)^{-\alpha}\left(\dfrac{\alpha}{1-\alpha} + 1\right)\right] u = p_1^{\alpha} p_2^{1-\alpha}\left[\left(\dfrac{\alpha}{1-\alpha}\right)^{\alpha}\left(\dfrac{\cancel{\alpha}+1-\cancel{\alpha}}{1-\alpha}\right)\right] u$$

$$= p_1^{\alpha} p_2^{1-\alpha}\, \alpha^{\alpha}\, (1-\alpha)^{-\alpha+1}\, u = p_1^{\alpha} p_2^{1-\alpha}\, \alpha^{\alpha}\, (1-\alpha)^{1-\alpha}\, u$$

Indirect utility function:

$$v(p,w) = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w$$

Expenditure function:

$$e(p,u) = p_1^{\alpha} p_2^{1-\alpha} \alpha^{\alpha} (1-\alpha)^{1-\alpha} u$$

Invert the utility function:

Substitute $v(p,w)$ for u and w for $e(p,u)$ and solve for $e(p,u)$

$$u = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} e(p,u) \iff u = \frac{\alpha^{\alpha}}{p_1^{\alpha}} \frac{(1-\alpha)^{1-\alpha}}{p_2^{1-\alpha}} e(p,u)$$

$$\iff e(p,u) = p_1^{\alpha} p_2^{1-\alpha} \alpha^{\alpha} (1-\alpha)^{1-\alpha} u$$

3.G.3 Consider again the three-good setting of Exercise 3.D.6 in which the consumer has utility function $u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma$. Assume that $\alpha + \beta + \gamma = 1$ and that $b_1 \geq 0$, $b_2 \geq 0$, and $b_3 \geq 0$.

(a) Derive the Hicksian demand and expenditure functions. Check that $e(p, u)$ is (i) homogenous of degree one in p, (ii) strictly increasing in u and nondecreasing in p_ℓ for any ℓ; and that $h(p, u)$ is homogeneous of degree zero in p, and (ii) provides no excess utility.

(b) Show that the derivates of the expenditure function are the Hicksian demand function you derived in (a).

(c) Verify that the Slutsky equation holds.

(d) Verify that the own-substitution terms are negative and that the compensated cross-price effects are symmetric.

3.G.3 (a) Suppose that $\alpha + \beta + \gamma = 1$. Note that

$$\ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3).$$

By the first-order condition of the EMP,

$$h(p,u) = (b_1, b_2, b_3) + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\alpha/p_1, \beta/p_2, \gamma/p_3).$$

Plug this into $p \cdot h(p,u)$, then we obtain the expenditure function

$$e(p,u) = p \cdot b + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma,$$

where $p \cdot b = p_1 b_1 + p_2 b_2 + p_3 b_3$.

To check the homogeneity of the expenditure function,

$$e(\lambda p, u) = \lambda p \cdot b + u(\lambda p_1/\alpha)^\alpha (\lambda p_2/\beta)^\beta (\lambda p_3/\gamma)^\gamma$$
$$= \lambda p \cdot b + \lambda u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma = \lambda e(p,u).$$

To check the monotonicity, assume $b_1 \geq 0$, $b_2 \geq 0$, and $b_3 \geq 0$. Then

$$\partial e(p,u)/\partial u = (p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma > 0,$$
$$\partial e(p,u)/\partial p_1 = b_1 + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\alpha/p_1) > 0,$$
$$\partial e(p,u)/\partial p_2 = b_2 + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\beta/p_2) > 0,$$
$$\partial e(p,u)/\partial p_3 = b_3 + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\gamma/p_3) > 0.$$

To check the concavity, we can show that $D_p^2 e(p,u)$ is equal to

$$u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma \begin{bmatrix} -\alpha(1-\alpha)/p_1^2 & \alpha\beta/p_1p_2 & \alpha\gamma/p_1p_3 \\ \alpha\beta/p_1p_2 & -\beta(1-\beta)/p_2^2 & \beta\gamma/p_2p_3 \\ \alpha\gamma/p_1p_3 & \beta\gamma/p_2p_3 & -\gamma(1-\gamma)/p_3^2 \end{bmatrix}.$$

and then apply the condition in Exercise 2.F.9 to show that $D_p^2 e(p,u)$ is negative semidefinite. An alternative way is to only calculate the 2×2 submatrix obtained from $D_p^2 e(p,u)$ by deleting the last row and the last column and apply the condition in Exercise 2.F.9 to show that it is negative definite. Then note that the homogeneity implies that $D_p^2 e(p,u)p = 0$. Hence we can apply Theorem M.D.4(iii) to conclude that $D_p^2 e(p,u)$ is negative semidefinite. The continuity follows from the functional form.

To check the homogeneity of the Hicksian demand function,

$$h(\lambda p, u) = (b_1, b_2, b_3) + u(\lambda p_1/\alpha)^\alpha (\lambda p_2/\beta)^\beta (\lambda p_3/\gamma)^\gamma (\alpha/\lambda p_1, \beta/\lambda p_2, \gamma/\lambda p_3).$$
$$= (b_1, b_2, b_3) + u\lambda^{\alpha+\beta+\gamma-1}(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\alpha/p_1, \beta/p_2, \gamma/p_3)$$
$$= h(p,u).$$

To check no excess utility,

$$u(h(p,u)) = u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\alpha/p_1)^\alpha (\beta/p_2)^\beta (\gamma/p_3)^\gamma = u.$$

The uniqueness is obvious.

(b) We calculated the derivatives $\partial e(p,u)/\partial p_\ell$ in (a). If we compare them with $h_\ell(p,u)$, then we can immediately see $\partial e(p,u)/\partial p_\ell = h_\ell(p,u)$.

(c) By (b), $D_p h(p,u) = D_p^2 e(p,u)$. In (a), we calculated $D_p^2 e(p,u)$. In Exercise 3.D.6, we showed

$$x(p,w) = (b_1, b_2, b_3) + (w - p \cdot b)(\alpha/p_1, \beta/p_2, \gamma/p_3)$$

and hence $D_w x(p,w) = (\alpha/p_1, \beta/p_2, \gamma/p_3)$ and

$$D_p x(p,w) = -(w - p \cdot b)\begin{bmatrix} \alpha/p_1^2 & 0 & 0 \\ 0 & \beta/p_2^2 & 0 \\ 0 & 0 & \gamma/p_3^2 \end{bmatrix} - \begin{bmatrix} \alpha/p_1 \\ \beta/p_2 \\ \gamma/p_3 \end{bmatrix}(b_1, b_2, b_3).$$

Using these results, we can verify the Slutsky equation.

(d) Use $D_p h(p,u) = D_p^2 e(p,u)$ and the explicit expression of $D_p^2 e(p,u)$ in (a).

a) $u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma$ with $\alpha + \beta + \gamma = 1$, $b_1, b_2, b_3 \geq 0$.

Monotone transformation:

$$\ln u(x) = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3)$$

Hicksian demand:

$$\min_{\{x_1, x_2, x_3\}} \quad p_1 x_1 + p_2 x_2 + p_3 x_3$$

$$\text{s.t.} \quad \ln u = \alpha \ln(x_1 - b_1) + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3)$$

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + p_3 x_3 + \lambda \left[\ln u - \alpha \ln(x_1 - b_1) - \beta \ln(x_2 - b_2) - \gamma \ln(x_3 - b_3) \right]$$

$$(x_1): \quad p_1 - \lambda \frac{\alpha}{x_1 - b_1} = 0 \iff \lambda = \frac{(x_1 - b_1) p_1}{\alpha}$$

$$(x_2): \quad p_2 - \lambda \frac{\beta}{x_2 - b_2} = 0 \iff \lambda = \frac{(x_2 - b_2) p_2}{\beta}$$

$$(x_3): \quad p_3 - \lambda \frac{\gamma}{x_3 - b_3} = 0 \iff \lambda = \frac{(x_3 - b_3) p_3}{\gamma}$$

$$(\lambda): \quad \ln u - \alpha \ln(x_1 - b_1) - \beta \ln(x_2 - b_2) - \gamma \ln(x_3 - b_3) = 0$$

From $(x_1) + (x_2)$: $\dfrac{(x_1 - b_1) p_1}{\alpha} = \dfrac{(x_2 - b_2) p_2}{\beta} \iff x_2 = \dfrac{\beta}{\alpha} \dfrac{p_1}{p_2} (x_1 - b_1) + b_2$

From $(x_1) + (x_3)$: $\dfrac{(x_1 - b_1) p_1}{\alpha} = \dfrac{(x_3 - b_3) p_3}{\gamma} \iff x_3 = \dfrac{\gamma}{\alpha} \dfrac{p_1}{p_3} (x_1 - b_1) + b_3$

Into (λ): $\ln u = \alpha \ln(x_1 - b_1) + \beta \ln\left(\dfrac{\beta}{\alpha} \dfrac{p_1}{p_2} (x_1 - b_1) + \cancel{b_2 - b_2} \right) + \gamma \ln\left(\dfrac{\gamma}{\alpha} \dfrac{p_1}{p_3} (x_1 - b_1) + \cancel{b_3 - b_3} \right)$

$$\iff \ln u = \alpha \ln(x_1 - b_1) + \beta \ln\left(\frac{\beta}{\alpha}\right) + \beta \ln\left(\frac{p_1}{p_2}\right) + \beta \ln(x_1 - b_1) + \gamma \ln\left(\frac{\gamma}{\alpha}\right) + \gamma \ln\left(\frac{p_1}{p_3}\right) + \gamma \ln(x_1 - b_1)$$

$$\iff \ln u = \underbrace{(\alpha + \beta + \gamma)}_{1} \ln(x_1 - b_1) + \beta \ln\left(\frac{\beta}{\alpha}\right) + \gamma \ln\left(\frac{\gamma}{\alpha}\right) + \beta \ln\left(\frac{p_1}{p_2}\right) + \gamma \ln\left(\frac{p_1}{p_3}\right)$$

$$\Leftrightarrow \ln(x_1 - b_1) = \ln u - \beta \ln\left(\frac{\beta}{\alpha}\right) - \gamma \ln\left(\frac{\gamma}{\alpha}\right) - \beta \ln\left(\frac{p_1}{p_2}\right) - \gamma \ln\left(\frac{p_1}{p_3}\right)$$

$$\Leftrightarrow x_1 - b_1 = u \left[\frac{\beta}{\alpha}\right]^{\beta} \left(\frac{\gamma}{\alpha}\right)^{-\gamma} \left(\frac{p_1}{p_2}\right)^{-\beta} \left(\frac{p_1}{p_3}\right)^{-\gamma} = u \left(\frac{\alpha}{\beta}\right)^{\beta} \left(\frac{\alpha}{\gamma}\right)^{\gamma} \left(\frac{p_2}{p_1}\right)^{\beta} \left(\frac{p_3}{p_1}\right)^{\gamma}$$

$$\Leftrightarrow x_1 = b_1 + u \, \frac{p_1^{\alpha+\beta+\gamma}}{p_1^{\alpha+\beta+\gamma}} \, \frac{\alpha^{\alpha+\beta+\gamma}}{\alpha^{\alpha+\beta+\gamma}} \left(\frac{\cancel{\alpha}}{\beta}\right)^{\beta} \left(\frac{\cancel{\alpha}}{\gamma}\right)^{\gamma} \left(\frac{p_2}{\cancel{p_1}}\right)^{\beta} \left(\frac{p_3}{\cancel{p_1}}\right)^{\gamma}$$

$$\Leftrightarrow x_1 = b_1 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1}$$

and $$x_2 = \frac{\beta}{\cancel{\alpha}} \frac{\cancel{p_1}}{p_2} \left[\cancel{b_1} + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\cancel{\alpha}}{\cancel{p_1}} - \cancel{b_1}\right] + b_2$$

$$\Leftrightarrow x_2 = b_2 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\beta}{p_2}$$

and $$x_3 = \frac{\gamma}{\cancel{\alpha}} \frac{\cancel{p_1}}{p_3} \left[\cancel{b_1} + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\cancel{\alpha}}{\cancel{p_1}} - \cancel{b_1}\right] + b_3$$

$$\Leftrightarrow x_3 = b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\gamma}{p_3}$$

Hicksian demand:

$$h(p, u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \\ h_3(p,u) \end{bmatrix} = \begin{bmatrix} b_1 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1} \\ b_2 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\beta}{p_2} \\ b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\gamma}{p_3} \end{bmatrix}$$

Expenditure function:

$$e(p,u) = p \cdot h(p,u) = p_1 \cdot h_1(p,u) + p_2 \cdot h_2(p,u) + p_3 \cdot h_3(p,u)$$

$$= p_1 \left[b_1 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1} \right]$$

$$+ p_2 \left[b_2 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\beta}{p_2} \right]$$

$$+ p_3 \left[b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\gamma}{p_3} \right]$$

$$= p_1 b_1 + p_2 b_2 + p_3 b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} (\underbrace{\alpha+\beta+\gamma}_{1})$$

$$= p_1 b_1 + p_2 b_2 + p_3 b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma}$$

(i) Check: $e(p,u)$ homogeneous of degree one: $e(\lambda p, u) = \lambda e(p,u)$

$$e(\lambda p, u) = \lambda p_1 b_1 + \lambda p_2 b_2 + \lambda p_3 b_3 + u \left(\frac{\lambda p_1}{\alpha}\right)^{\alpha} \left(\frac{\lambda p_2}{\beta}\right)^{\beta} \left(\frac{\lambda p_3}{\gamma}\right)^{\gamma}$$

$$= \lambda (p_1 b_1 + p_2 b_2 + p_3 b_3) + \lambda^{\alpha+\beta+\gamma} u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma}$$

$$= \lambda e(p,u)$$

(ii) Check: $e(p,u)$ strictly increasing in u and nondecreasing in p_ℓ, $\forall \ell$

and $h(p,u)$ homogeneous of degree two.

$$e(p,u) = p_1 b_1 + p_2 b_2 + p_3 b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma}$$

$$\frac{\partial e(p,u)}{\partial u} = \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} > 0$$

$$\frac{\partial e(p,u)}{p_1} = b_1 + u \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \cancel{\alpha} \left(\frac{p_1}{\alpha}\right)^{\alpha-1} \frac{1}{\cancel{\alpha}}$$

$$= b_1 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1} > 0 \quad (\text{given } b_1, b_2, b_3 \geq 0)$$

$$= h_1(p,u) \quad (\text{for (b)})$$

$$\frac{\partial e(p,u)}{\partial p_2} = b_2 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_3}{\gamma}\right)^{\gamma} \cancel{\beta} \left(\frac{p_2}{\beta}\right)^{\beta-1} \frac{1}{\cancel{\beta}}$$

$$= b_2 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\beta}{p_2} > 0$$

$$= h_2(p,u) \quad (\text{for (b)})$$

$$\frac{\partial e(p,u)}{\partial p_3} = b_3 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\cancel{\gamma}\left(\frac{p_3}{\gamma}\right)^{\gamma-1}\frac{1}{\cancel{\gamma}}$$

$$= b_3 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\frac{\gamma}{p_3} > 0$$

$$= h_3(p,u) \quad \text{(for (b))}$$

$h(p,u)$ homogeneous of degree two: $h(\lambda p, \lambda u) = h(p,u)$

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \\ h_3(p,u) \end{bmatrix} = \begin{bmatrix} b_1 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\frac{\alpha}{p_1} \\ b_2 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\frac{\beta}{p_2} \\ b_3 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\frac{\gamma}{p_3} \end{bmatrix}$$

$$h_1(\lambda p, \lambda u) = b_1 + u\left(\frac{\lambda p_1}{\alpha}\right)^{\alpha}\left(\frac{\lambda p_2}{\beta}\right)^{\beta}\left(\frac{\lambda p_3}{\gamma}\right)^{\gamma}\frac{\alpha}{\lambda p_1}$$

$$= b_1 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\frac{\alpha}{p_1}\frac{\cancel{\lambda^{\alpha+\beta+\gamma}}}{\cancel{\lambda}} = h_1(p,u)$$

$$h_2(\lambda p, \lambda u) = b_2 + u\left(\frac{\lambda p_1}{\alpha}\right)^{\alpha}\left(\frac{\lambda p_2}{\beta}\right)^{\beta}\left(\frac{\lambda p_3}{\gamma}\right)^{\gamma}\frac{\beta}{\lambda p_2}$$

$$= b_2 + u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_3}{\gamma}\right)^{\gamma}\frac{\beta}{p_2}\frac{\cancel{\lambda^{\alpha+\beta+\gamma}}}{\cancel{\lambda}} = h_2(p,u)$$

$$h_3(\lambda p, \lambda u) = b_3 + u\left(\frac{\lambda p_1}{\alpha}\right)^\alpha \left(\frac{\lambda p_2}{\beta}\right)^\beta \left(\frac{\lambda p_3}{\gamma}\right)^\gamma \frac{\gamma}{\lambda p_3}$$

$$= b_3 + u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\gamma}{p_3} \frac{\cancel{\lambda^{\alpha+\beta+\gamma}}}{\cancel{\lambda}} = (p, u)$$

(iii) check: $h(p,u)$ provides no excess utility

Plug $h(p,u)$ into $u(x)$ and show that $u(h(p,u)) = u$.

$$u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma$$

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \\ h_3(p,u) \end{bmatrix} = \begin{bmatrix} b_1 + u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\alpha}{p_1} \\ b_2 + u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\beta}{p_2} \\ b_3 + u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\gamma}{p_3} \end{bmatrix}$$

$$u(h(p,u)) = \left[u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\alpha}{p_1}\right]^\alpha \times \left[u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\beta}{p_2}\right]^\beta \times$$

$$\times \left[u\left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\gamma}{p_3}\right]^\gamma = u^{\alpha+\beta+\gamma} \left[\cancel{\left(\frac{p_1}{\alpha}\right)^\alpha} \cancel{\left(\frac{p_2}{\beta}\right)^\beta} \cancel{\left(\frac{p_3}{\gamma}\right)^\gamma}\right]^{\alpha+\beta+\gamma} \times$$

$$\times \left[\cancel{\left(\frac{\alpha}{p_1}\right)^\alpha} \cancel{\left(\frac{\beta}{p_2}\right)^\beta} \cancel{\left(\frac{\gamma}{p_3}\right)^\gamma}\right] = u$$

b) Check: $\dfrac{\partial e(p,u)}{\partial p_\ell} = h_\ell(p,u)$, $\forall \ell$ (Shephard's Lemma)

see (a, ii)

c) Check: Slutsky equation:

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w) \quad \forall \ell, k$$

Walrasian demand:

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ x_3(p,w) \end{bmatrix} = \begin{bmatrix} b_1 + \left(\frac{\alpha}{p_1}\right)\left(w - p_1 b_1 - p_2 b_2 - p_3 b_3\right) \\ b_2 + \left(\frac{\beta}{p_2}\right)\left(w - p_1 b_1 - p_2 b_2 - p_3 b_3\right) \\ b_3 + \left(\frac{\gamma}{p_3}\right)\left(w - p_1 b_1 - p_2 b_2 - p_3 b_3\right) \end{bmatrix}$$

From 3.D.6.(b)

Example: $\dfrac{\partial h_1(p,u)}{\partial p_2} = \dfrac{\partial x_1(p,w)}{\partial p_2} + \dfrac{\partial x_1(p,w)}{\partial w} x_2(p,w)$

$$\frac{\partial x_1(p,w)}{\partial p_2} = -\frac{\alpha}{p_1} b_2 \qquad \frac{\partial x_1(p,w)}{\partial w} = \frac{\alpha}{p_1}$$

Hicksian demand:

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \\ h_3(p,u) \end{bmatrix} = \begin{bmatrix} b_1 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1} \\ b_2 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\beta}{p_2} \\ b_3 + u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\gamma}{p_3} \end{bmatrix}$$

$$\frac{\partial h_1(p,u)}{\partial p_2} = u \left(\frac{p_1}{\alpha}\right) \cancel{\beta} \left(\frac{p_2}{\beta}\right)^{\beta-1} \frac{1}{\cancel{\beta}} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1} = u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p}^{-1} ,$$

$$= u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \frac{\alpha}{p_1} \frac{\beta}{p_2}$$

$$\frac{\partial x_1(p,w)}{\partial p_2} + \frac{\partial x_1(p,w)}{\partial w} x_2(p,w) = \cancel{-\frac{\alpha}{p_1} b_2} + \frac{\alpha}{p_1} \left[\cancel{b_2} + \left(\frac{\beta}{p_2}\right)\left(w - p_1 b_1 - p_2 b_2 - p_3 b_3\right)\right]$$

$$= \frac{\alpha}{p_1} \frac{\beta}{p_2} \left[w - p_1 b_1 - p_2 b_2 - p_3 b_3 \right] = \left[e(p,u) - p_1 b_1 - p_2 b_2 - p_3 b_3 \right]$$

$$= \frac{\alpha}{p_1} \frac{\beta}{p_2} \left[u \left(\frac{p_1}{\alpha}\right)^{\alpha} \left(\frac{p_2}{\beta}\right)^{\beta} \left(\frac{p_3}{\gamma}\right)^{\gamma} \right] = \frac{\partial h_1(p,u)}{\partial p_2}$$

d) Check: own substitution terms are negative and compensated cross-price effects are symmetric.

Own Substitution terms: $\frac{\partial h_\ell(p,u)}{\partial p_\ell} \leq 0$ $\left(D_p h(p,u) \text{ negative semi definite} \right)$

Compensated cross-price effects: $\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_\ell}$ $\left(D_p h(p,u) \text{ symmetric} \right)$

Hicksian demand:

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \\ h_3(p,u) \end{bmatrix} = \begin{bmatrix} b_1 + u \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\alpha}{p_1} \\ b_2 + u \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\beta}{p_2} \\ b_3 + u \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\gamma}{p_3} \end{bmatrix}$$

$$\frac{\partial h_1(p,u)}{\partial p_2} = u \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\alpha}{p_1} \frac{\beta}{p_2}$$

Example: $\frac{\partial h_1(p,u)}{\partial p_1} < 0$ and $\frac{\partial h_1(p,u)}{\partial p_2} = \frac{\partial h_2(p,u)}{\partial p_1}$

$$h_1(p,u) = b_1 + u \left(\frac{1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \alpha \, p_1^{\alpha-1}$$

$$\frac{\partial h_1(p,u)}{\partial p_1} = u \left(\frac{1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \alpha \underbrace{(\alpha-1)}_{<0} p_1^{\alpha-2} < 0$$

$$\frac{\partial h_2(p,u)}{\partial p_1} = u \, \cancel{\alpha} \left(\frac{p_1}{\alpha}\right)^{\alpha-1} \frac{1}{\cancel{\alpha}} \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\beta}{p_2} = u \left(\frac{p_1}{\alpha}\right)^\alpha \left(\frac{p_2}{\beta}\right)^\beta \left(\frac{p_3}{\gamma}\right)^\gamma \frac{\alpha}{p_1} \frac{\beta}{p_2}$$

$$= \frac{\partial h_1(p,u)}{\partial p_2}$$

3.G.15 Consider the utility function

$$u = 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}}.$$

(a) Find the demand functions for goods 1 and 2 as they depend on prices and wealth.
(b) Find the compensated demand function $h(\cdot)$.
(c) Find the expenditure function, and verify that $h(p,u) = \nabla_p e(p,u)$.
(d) Find the indirect utility function and verify Roy's identity.

3.G.15 (a) $x(p_1,p_2,w) = \left(\dfrac{p_2 w}{p_1p_2 + 4p_1^2}, \dfrac{4p_1 w}{4p_1p_2 + p_2^2}\right)$.

(b) $h(p_1,p_2,u) = \left(\left(\dfrac{p_2 u}{2(4p_1 + p_2)}\right)^2, \left(\dfrac{p_1 u}{4p_1 + p_2}\right)^2\right)$.

(c) $e(p_1,p_2,u) = \dfrac{p_1p_2u^2}{4(4p_1 + p_2)}$. It is then easy to show that $\nabla_p e(p_1,p_2,u) = h(p_1,p_2,u)$.

(d) $v(p_1,p_2,w) = 2(w/p_1 + 4w/p_2)^{1/2}$. To verify Roy's identity, use

$\partial v(p_1,p_2,w)/\partial p_1 = (w/p_1 + 4w/p_2)^{-1/2}(-w/p_1^2)$,

$\partial v(p_1,p_2,w)/\partial p_2 = (w/p_1 + 4w/p_2)^{-1/2}(-4w/p_2^2)$,

$\partial v(p_1,p_2,w)/\partial w = (w/p_1 + 4w/p_2)^{-1/2}(1/p_1 + 4/p_2)$.

a) Walrasian demand : $x(p, w)$

$$\max_{\{x_1,x_2\}} u(x) = 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}}$$

s.t.: $p_1x_1 + p_2x_2 = w$

$$\mathcal{L} = 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}} + \lambda\left[w - p_1x_1 - p_2x_2\right]$$

FOC: (x_1): $x_1^{-\frac{1}{2}} - \lambda p_1 = 0 \Leftrightarrow \lambda = \dfrac{x_1^{-\frac{1}{2}}}{p_1}$

(x_2): $2x_2^{-\frac{1}{2}} - \lambda p_2 = 0 \Leftrightarrow \lambda = \dfrac{2x_2^{-\frac{1}{2}}}{p_2}$

(λ): $w - p_1x_1 - p_2x_2 = 0$

From $(x_1) + (x_2)$: $\dfrac{x_1^{-\frac{1}{2}}}{p_1} = \dfrac{2x_2^{-\frac{1}{2}}}{p_2} \Leftrightarrow x_1 = \left[2\dfrac{p_1}{p_2}x_2^{-\frac{1}{2}}\right]^{-2}$

$$\Leftrightarrow \quad x_1 = \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2 x_2$$

Into (1): $p_1 \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2 x_2 + p_2 x_2 = w \Leftrightarrow \left(\frac{1}{4}\frac{p_2^2}{p_1} + p_2\right) x_2 = w$

$$\Leftrightarrow \left(\frac{p_2^2 + 4p_1p_2}{4p_1}\right) x_2 = w \Leftrightarrow x_2 = \frac{4p_1 w}{p_2^2 + 4p_1p_2}$$

and: $x_1 = \frac{1}{\cancel{4}}\left(\frac{p_2}{p_1}\right)^2 \frac{\cancel{4}p_1 w}{p_2^2 + 4p_1p_2} = \frac{p_2^2 \cancel{p_1} w}{p_1^{\cancel{2}}(p_2^2 + 4p_1p_2)} = \frac{p_2^{\cancel{2}} w}{p_1\cancel{p_2}(p_2 + 4p_1)} = \frac{p_2 w}{p_1p_2 + 4p_1^2}$

Walrasian demand:

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \end{bmatrix} = \begin{bmatrix} \frac{p_2 w}{p_1p_2 + 4p_1^2} \\ \frac{4p_1 w}{p_2^2 + 4p_1p_2} \end{bmatrix}$$

b) Hicksian demand: $h(p,u)$ (Compensated demand function)

$$\min_{\{x_1,x_2\}} p_1 x_1 + p_2 x_2$$

$$\text{s.t.: } 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}} = u$$

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda\left(u - 2x_1^{\frac{1}{2}} - 4x_2^{\frac{1}{2}}\right)$$

FOC: (x_1): $p_1 - \lambda x_1^{-\frac{1}{2}} = 0 \Leftrightarrow \lambda = \dfrac{p_1}{x_1^{-\frac{1}{2}}}$

(x_2): $p_2 - 2\lambda x_2^{-\frac{1}{2}} = 0 \Leftrightarrow \lambda = \dfrac{p_2}{2x_2^{-\frac{1}{2}}}$

(λ): $u - 2x_1^{\frac{1}{2}} - 4x_2^{\frac{1}{2}} = 0$

From (x_1) + (x_2): $\dfrac{p_1}{x_1^{-\frac{1}{2}}} = \dfrac{p_2}{2x_2^{-\frac{1}{2}}} \Leftrightarrow x_1^{-\frac{1}{2}} = \dfrac{p_1}{p_2} 2x_2^{-\frac{1}{2}}$

$\Leftrightarrow x_1 = \dfrac{1}{4}\left(\dfrac{p_2}{p_1}\right)^2 x_2$

Plug into (λ): $u = 2\left[\dfrac{1}{4}\left(\dfrac{p_2}{p_1}\right)^2 x_2\right]^{\frac{1}{2}} + 4x_2^{\frac{1}{2}} \Leftrightarrow u = \dfrac{p_2}{p_1} x_2^{\frac{1}{2}} + 4x_2^{\frac{1}{2}}$

$\Leftrightarrow u = \left(\dfrac{p_2}{p_1} + 4\right) x_2^{\frac{1}{2}} \Leftrightarrow u = \left(\dfrac{p_2 + 4p_1}{p_1}\right) x_2^{\frac{1}{2}} \Leftrightarrow x_2^{\frac{1}{2}} = \left(\dfrac{p_1}{p_2 + 4p_1}\right) u$

$\Leftrightarrow x_2 = \left(\dfrac{p_1}{p_2 + 4p_1}\right)^2 u^2 \Leftrightarrow x_2 = \left(\dfrac{p_1 u}{4p_1 + p_2}\right)^2$

and $x_1 = \dfrac{1}{4}\left(\dfrac{p_2}{\cancel{p_1}}\right)^2 \left(\dfrac{\cancel{p_1} u}{4p_1 + p_2}\right)^2 = \dfrac{1}{4}\left(\dfrac{p_2 u}{4p_1 + p_2}\right)^2 = \left(\dfrac{p_2 u}{2(4p_1 + p_2)}\right)^2$

Hicksian demand:

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{p_2 u}{2(4p_1 + p_2)}\right)^2 \\ \left(\dfrac{p_1 u}{4p_1 + p_2}\right)^2 \end{bmatrix}$$

c) Expenditure Function:

$$e(p,u) = p \cdot h(p,u) = p_1 \cdot h_1(p,u) + p_2 \cdot h_2(p,u) = p_1 \left(\frac{p_2 u}{2(4p_1+p_2)}\right)^2 +$$

$$+ p_2 \left(\frac{p_1 u}{4p_1+p_2}\right)^2 = \frac{p_1 p_2^2 u^2}{4(4p_1+p_2)^2} + \frac{p_1^2 p_2 u^2}{(4p_1+p_2)^2} = \frac{p_1 p_2^2 u^2 + 4p_1^2 p_2 u^2}{4(4p_1+p_2)^2}$$

$$= \frac{p_1 p_2 u^2 \cancel{(p_2+4p_1)}}{4(4p_1+p_2)^{\cancel{2}}} = \frac{p_1 p_2 u^2}{4(4p_1+p_2)}$$

Shephard's Lemma: $h(p,u) = \nabla_p e(p,u)$

$$h(p,u) = \begin{bmatrix} h_1(p,u) \\ h_2(p,u) \end{bmatrix} = \nabla_p e(p,u) = \begin{bmatrix} \frac{\partial e(p,u)}{p_1} \\ \frac{\partial e(p,u)}{\partial p_2} \end{bmatrix}$$

$$\frac{\partial e(p,u)}{\partial p_1} = \frac{\partial}{\partial p_1}\left(\frac{p_1 p_2 u^2}{4(4p_1+p_2)}\right) = \frac{\partial}{\partial p_1}\left[\frac{1}{4}(p_1 p_2 u^2)(4p_1+p_2)^{-1}\right] =$$

$$= \frac{1}{4}\left[p_2 u^2 (4p_1+p_2)^{-1} - (p_1 p_2 u^2)(4p_1+p_2)^{-2} 4\right] =$$

$$= \frac{1}{4}\left[\frac{p_2 u^2 (4p_1+p_2) - 4(p_1 p_2 u^2)}{(4p_1+p_2)^2}\right] = \frac{1}{4}\left[\frac{\cancel{4p_1p_2u^2} + p_2^2 u^2 - \cancel{4p_1p_2u^2}}{(4p_1+p_2)^2}\right]$$

$$= \frac{p_2^2 u^2}{4(4p_1+p_2)^2} = \left(\frac{p_2 u}{2(4p_1+p_2)}\right)^2 = h_1(p,u)$$

$$\frac{\partial e(p,u)}{\partial p_2} = \frac{1}{4}\left[p_1 u^2 (4p_1 + p_2)^{-1} - (p_1 p_2 u^2)(4p_1+p_2)^{-2}\right]$$

$$= \frac{1}{4}\left[\frac{p_1 u^2 (4p_1 + p_2) - p_1 p_2 u^2}{(4p_1+p_2)^2}\right] = \frac{1}{\cancel{4}}\left[\frac{\cancel{4} p_1^2 u^2 + \cancel{p_1 p_2 u^2} - \cancel{p_1 p_2 u^2}}{(4p_1+p_2)^2}\right]$$

$$= \left(\frac{p_1 u}{4p_1 + p_2}\right)^2 = h_2(p,u)$$

d) Indirect utility function: $v(p,w)$

alternative (i): plug $x(p,w)$ into $u(x)$ (from (a))

alternative (ii): invert $e(p,u)$ (from (c))

$$e(p,u) = \frac{p_1 p_2 u^2}{4(4p_1+p_2)}$$

substitute $e(p,u)$ for w and u for $v(p,w)$ and solve for $v(p,w)$

$$w = \frac{p_1 p_2 (v(p,w))^2}{4(4p_1+p_2)} \Leftrightarrow (v(p,w))^2 = \frac{4w(4p_1+p_2)}{p_1 p_2}$$

$$\Leftrightarrow v(p,w) = \frac{2w^{\frac{1}{2}}(4p_1+p_2)^{\frac{1}{2}}}{(p_1 p_2)^{\frac{1}{2}}} = 2w^{\frac{1}{2}}\left(\frac{4\cancel{p_1}}{\cancel{p_1}p_2} + \frac{\cancel{p_2}}{p_1\cancel{p_2}}\right)^{\frac{1}{2}} = 2w^{\frac{1}{2}}\left(\frac{4}{p_2} + \frac{1}{p_1}\right)^{\frac{1}{2}}$$

$$= 2\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{\frac{1}{2}}$$

Roy's Identity: $$x_\ell(\bar{p},\bar{w}) = -\frac{\dfrac{\partial v_\ell(\bar{p},\bar{w})}{\partial p_\ell}}{\dfrac{\partial v_\ell(\bar{p},\bar{w})}{\partial w}}$$

$$v(p,w) = 2\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{\frac{1}{2}} \qquad x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \end{bmatrix} = \begin{bmatrix} \dfrac{p_2 w}{p_1 p_2 + 4p_1^2} \\ \dfrac{4p_1 w}{p_2^2 + 4p_1 p_2} \end{bmatrix}$$

$$\frac{\partial v(p,w)}{\partial p_1} = \left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(-\frac{w}{p_1^2}\right) \qquad \frac{\partial v(p,w)}{\partial p_2} = \left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(-\frac{4w}{p_2^2}\right)$$

$$\frac{\partial v(p,w)}{\partial w} = \left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)$$

$$-\frac{\dfrac{\partial v(p,w)}{\partial p_1}}{\dfrac{\partial v(p,w)}{\partial w}} = \cancel{-}\frac{\cancel{\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}}\left(\cancel{-}\frac{w}{p_1^2}\right)}{\cancel{\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{\frac{w}{p_1^2}}{\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{w}{p_1^2\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{w}{p_1^2\left(\frac{p_2 + 4p_1}{p_1 p_2}\right)}$$

$$= \frac{w\cancel{p_1}p_2}{p_1^{\cancel{2}}(p_2 + 4p_1)} = \frac{p_2 w}{p_1 p_2 + 4p_1^2} = x_1(p,w)$$

$$-\frac{\dfrac{\partial v(p,w)}{\partial p_2}}{\dfrac{\partial v(p,w)}{\partial w}} = \cancel{-}\frac{\cancel{\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}}\left(\cancel{-}\frac{4w}{p_2^2}\right)}{\cancel{\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{\frac{4w}{p_2^2}}{\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{4w}{p_2^2\left(\frac{1}{p_1} + \frac{4}{p_2}\right)}$$

$$= \frac{4w}{p_2^2\left(\frac{p_2 + 4p_1}{p_1 p_2}\right)} = \frac{4w p_1 \cancel{p_2}}{p_2^{\cancel{2}}(p_2 + 4p_1)} = \frac{4w p_1}{p_2^2 + 4p_1 p_2} = x_2(p,w)$$