Final Solution

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Part 1.

1. (a) Take Partial Derivatives

$$\nabla f_1(x_1, x_2) = \left(-2x_1 - \tfrac{5}{2}x_2,\ -\tfrac{5}{2}x_1 - 2x_2\right)$$

$$\nabla f_2(x_1, x_2) = \left(4x_1 + 5x_2,\ 5x_1 + 4x_2\right)$$

(b) $$|J| = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} -2x_1 - \frac{5}{2}x_2 & -\frac{5}{2}x_1 - 2x_2 \\ 4x_1 + 5x_2 & 5x_1 + 4x_2 \end{vmatrix} = 0$$

$\therefore$ f_1, f_2 are dependent

2. $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= -7x^2 + 3y^2 - 25xy + \tfrac{1}{2} = -\tfrac{99}{4}t^2 + 78t + \tfrac{7}{2}$$

3. By Lagrange multiplier, find $\frac{\partial \mathcal{L}}{\partial c_1}$ $\frac{\partial \mathcal{L}}{\partial c_2}$ to equalize λ, and bring the equation of c_1, c_2 back to $\frac{\partial \mathcal{L}}{\partial \lambda}$

$\Rightarrow (c_1^*, c_2^*) = (10, 6)$

Then Compute $D = f_{xx}f_{yy} - f_{xy} = 7 > 0$

$f_{xx} = -2 < 0$

$\Rightarrow (c_1^*, c_2^*)$ is local maximum

4. $f'(x) = 6(1-x)^{-3}$

$f''(x) = 18(1-x)^{-4}$

$f'''(x) = 72(1-x)^{-5}$

3rd order around 0 : $f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$

$= 3 + 6x + 9x^2 + 12x^3$

5.

(a) $\frac{e^{30}}{3} + \frac{599}{3}$

(b) integral by substitution : $2e - 2$

(c) FTC $g(x) = \int_a^{u(x)} f(t)\,dt$

$= g'(x) = \frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right)$

$= f(u) \cdot u'(x)$

$= 9x^2 e^{3x^3} \Rightarrow g(x) = e^{3x^3} + C$

This is already good. though we need to find $g(x)$ based on $g'(x)$

6. $\lim_{x\to 0}\left(\frac{e^x - 1}{x^3 + 3x^2}\right)\sin x$

$= \lim\left(\frac{e^x - 1}{x^2 + 3x}\right)\left(\frac{\sin x}{x}\right)$

$= \left(\lim_{x\to 0}\frac{e^x - 1}{x^2 + 3x}\right)\underbrace{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}_{=1} \overset{L'H}{=} \lim_{x\to 0}\frac{e^x}{2x+3} = \frac{1}{3}$

7. $6x^2 - 2xy - x^2y' + \frac{1}{y} \cdot y' = 0$

$$y' = \frac{6x^2 - 2xy}{x^2 - \frac{1}{y}}$$

8. Ratio Test $\lim \frac{a^{n+1}}{a^n} = \lim_{n\to\infty} \frac{1}{2}(1+\frac{1}{n})^2 = \frac{1}{2} < 1$

$\therefore \sum_{n=1}^{\infty} (\frac{1}{2})^{n+1} n^2$ converges

9. $f'(x) = 9x^2 - 4x + 1$

$f''(x) = 18x - 4 = 0 \Rightarrow x^* = \frac{2}{9}$ is the inflection point

x	$(-\infty, \frac{2}{9})$	$\frac{2}{9}$	$(\frac{2}{9}, \infty)$
$f(x)$	concave	inf.	convex

Part II.

10. (i) $d_1(x, y) = \min\{1, |x-y|\} \geq 0$

(ii) $d_1(x, y) = d_1(y, x)$

(iii) $d_1(x, z) = \min\{1, |x-z|\} \leq \min\{1, |x-y| + |y-z|\}$

since $|x-z| \leq |x-y| + |y-z|$

$d_1(x, y) + d_1(y, z) = \min\{1, |x-y|\} + \min\{1, |y-z|\}$

By the property that $\min\{a, b+c\} \leq \min\{a, b\} + \min\{a, c\}$

$d_1(x, z) \leq d_1(x, y) + d_1(y, z)$

(i) (ii) (iii) $\Rightarrow$ $(\mathbb{R}, d_1)$ is metric space.

2. T T T T F (all Cauchy seq's should be convergent)

Part III

11. (a) $\begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

(b) $|A| = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -1-2=-3$

(c) $|A| \underset{R_3 - 2R_1}{\overset{R_2 + R_1}{=}} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 0 \end{vmatrix} \overset{R_3 - R_2}{=} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{vmatrix} = 1 \cdot 1 \cdot (-3) = (-3)$

(d) $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ -2 & 0 & 1 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

(e) By FTMI.

(f) Find $\dfrac{\det A_1(b) \quad \det A_2(b) \quad \det A_3(b)}{\det A}$

$\Rightarrow \quad x_1 = 2 \quad x_2 = 1 \quad x_3 = -1$

12.

(a) Solve $B\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \text{null}(B) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$

(b) $|A_1| = 3$, $|A| = 8 > 0 \Rightarrow$ PD

(c) $T_A^{-1} = A^{-1} = \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix}$

(d) $\ker(B) = \text{null}(B) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$

e. Solve $|A-\lambda I| = 0$

$\lambda_1 = 2 \quad \lambda_2 = 4$

$\lambda_1 = 2$ solve $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 4$ solve $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

f. Diagonalize $A = PDP^{-1}$

$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

g. $A^k = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & \\ & 4^k \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

h. G-S $\underset{\sim}{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\underset{\sim}{v_2} = \underset{\sim}{x_2} - \left(\frac{\underset{\sim}{v_1} \cdot \underset{\sim}{x_2}}{\underset{\sim}{v_1} \cdot \underset{\sim}{v_1}} \right) \cdot \underset{\sim}{v_1} = \begin{bmatrix} -4/5 \\ 12/5 \end{bmatrix}$

normalise $\underset{\sim}{q_1} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

$\underset{\sim}{q_2} = \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

i $Q = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

$R = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{10} & 6/\sqrt{10} \\ 0 & 8/\sqrt{10} \end{bmatrix}$

j. $A^{+} = (A^T A)^{-1} A^T = \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix}$

13.

(i) T (ii) T (iii) T (iv) T

(v) T (vi) T (vii) F

14.

(1.) $A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix} \xrightarrow{R_3 - \frac{1}{3}R_1} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & \frac{10}{3} & 1 \end{pmatrix}$

$\xrightarrow{R_3 - \frac{10}{3}R_1} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{pmatrix}$

$U = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{10}{3} & 1 \end{pmatrix}$

(2) To determine if A has LDL^T, we need to check if A is PD.

$3 > 0$

$\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3 > 0$

$\begin{vmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 4 & 1 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 3 \cdot (1-12) - 2 \cdot (-3)$

$= -33 + 6 = -27 < 0$

Thus A does not have LDL^T

(3) NO Cholesky decomp.

Bonus

1. $\operatorname{adj} B = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}$

2. $QDQ^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \cdots \end{pmatrix}^T$

3. $A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

(a) solve for eigenvalues & eigenvectors

$$\det(A^TA - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 4-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda)(1-\lambda) + -(4-\lambda) = (4-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (4-\lambda)(\lambda^2 - 2\lambda) = \lambda(4-\lambda)(\lambda-2) \overset{\text{set}}{=} 0$$

$\lambda_1 = 4 \quad \lambda_2 = 0 \quad \lambda_3 = 2$

$\sigma_1 = 2 \quad \sigma_2 = 0 \quad \sigma_3 = \sqrt{2} \quad \Rightarrow \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

(b) $\lambda = 4 \Rightarrow \vec{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \lambda = 2 \Rightarrow \vec{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \lambda = 0 \Rightarrow \vec{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\xrightarrow{G-S} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{\text{Norm.}} V = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

(c) $\vec{u}_1 = \frac{1}{2} A \vec{q}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{u}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$A = U \Sigma V^T$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$