"ANS TO PROB SET 3"

ECON 6020

MACRO Theory I

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Wickens. Prob 4.1 The Household Budget constraint may be expressed in different ways from eqn (4.2),

$$\Delta a_{t+1} + c_t = x_t + r_t a_t \quad \text{[Text (4.2)]},$$

Derive the Euler eqn for consumption for each of the following ways of writing the budget constraint.

(a) $\boxed{a_{t+1} = (1+r)\left[a_t + x_t - c_t\right]}$ (1.1)

Note that this is [Text (4.2)] with constant r.

(b) $\boxed{\Delta a_t + c_t = x_t + r a_{t-1}}$ (1.2)

(c) $\boxed{W_t = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s c_{t+s} = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s x_{t+s} + (1+r)a_t}$ (1.3)

Where W_t is household wealth.

Solution: In all three cases we seek to

$$\max_{\{c_{t+s},\, a_{t+s}\}} \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) \quad \text{[Text (4.1)]}$$

Subject to the relevant constraint

(a) For the Constraint (1.1) the Lagrangian is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \Big\{ \beta^s u(c_{t+s}) + \lambda_{t+s}\big[(1+r)(a_{t+s} + x_{t+s} - c_{t+s}) - a_{t+s+1})\big]\Big\} \quad (1.4)$$

The FOC are

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u'(c_{t+s}) - \lambda_{t+s}(1+r) = 0, \; s \geq 0, \quad (1.5)$$

and

$$\frac{\partial \mathcal{L}_t}{\partial a_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}(1+r) = 0, \; s \geq 0, \quad (1.6)$$

From (1.5): $\lambda_{t+s} = \left(\frac{1}{1+r}\right)\beta^s u'(c_{t+s}) \quad (1.7)$

Use (1.7) to Eliminate λ_{t+s} and λ_{t+s+1} in (1.6)

to get

$$-\left(\frac{1}{1+r}\right)\beta^s u'(c_{t+s}) + \left(\frac{1}{1+r}\right)\beta^{s+1} u'(c_{t+s+1})(1+r) = 0 \quad \text{or}$$

$$\left(\frac{1}{1+r}\right)\beta^s u'(c_{t+s}) = \beta^{s+1} u'(c_{t+s+1}) \quad \text{or}$$

$$\boxed{\beta \frac{u'(c_{t+s+1})}{u'(c_{t+s})}(1+r) = 1 \quad (1.8)}$$

Eqn (1.8) is the Euler Eqn for Constraint (a).

(b) For constraint (1.2) the Lagrangian is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{ \beta^s u(c_{t+s}) + \lambda_{t+s} [x_{t+s} + (1+r) a_{t+s-1} - c_{t+s} - a_{t+s}] \} \quad (1.9)$$

The $\underline{\text{FOC}}$ are

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = \beta^s u'(c_{t+s}) - \lambda_{t+s} = 0, \; s \geq 0, \quad (1.10)$$

and

$$\frac{\partial \mathcal{L}_t}{\partial a_{t+s}} = -\lambda_{t+s} + \lambda_{t+s+1}(1+r) = 0, \; s \geq 0. \quad (1.11).$$

From (1.10): $\lambda_{t+s} = \beta^s u'(c_{t+s})$ (1.12)

Use (1.12) in (1.11) to eliminate λ_{t+s} and λ_{t+s+1} to get

$$-\beta^s u'(c_{t+s}) + \beta^{s+1} u'(c_{t+s+1})(1+r) = 0 \quad \text{or}$$

$$\beta^s u'(c_{t+s}) = \beta^{s+1} u'(c_{t+s+1})(1+r) \quad \text{or}$$

$$\boxed{\beta \frac{u'(c_{t+s+1})}{u'(c_{t+s})}(1+r) = 1 \qquad (1.13)}$$

Eqn (1.13), which is the Euler Eqn for constraint (b) is the same as (1.8).

(c) The Lagrangian for constraint (1.3) is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) + \lambda_t \left[\sum_{s=0}^{\infty} \frac{x_{t+s} - c_{t+s}}{(1+r)^s} + (1+r) a_t \right] \quad (1.14)$$

Note, a single constraint

As a_t is given we require only the FOC for c_{t+s}.

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u'(c_{t+s}) - \lambda_t \frac{1}{(1+r)^s} = 0 \text{ or}$$

$$\beta^s u'(c_{t+s}) = \lambda_t \frac{1}{(1+r)^s} \quad (1.15)$$

Leading (1.15) one period (i.e. evaluating $\frac{\partial \mathcal{L}_t}{\partial c_{t+s+1}} = 0$)

gives

$$\beta^{s+1} u'(c_{t+s+1}) = \lambda_t \frac{1}{(1+r)^{s+1}} \quad (1.16)$$

Combining (1.15) and (1.16) gives

$$(1+r)^s \beta^s u'(c_{t+s}) = (1+r)^{s+1} \beta^{s+1} u'(c_{t+s+1}) \text{ or}$$

$$\boxed{\beta \frac{u'(c_{t+s+1})}{u'(c_{t+s})} (1+r) = 1 \quad (1.17)}$$

Eqn (1.17), which is the Euler Eqn for constraint (c) is the same as (1.13) and (1.8).
For all three note the lack of a time subscript on r, that is, the assumption of a constant real interest rate.

Additional Problem 1

A. Lagrangian, etc.

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \left\{ \beta^s U(C_{t+s}) + \lambda_{t+s} \left[(1+r(W_{t+s})) W_{t+s} - C_{t+s} - W_{t+s+1} \right] \right\} \quad (5)$$

FOC

$$\frac{\partial \mathcal{L}_t}{\partial C_t} = \beta^s U'(C_{t+s}) - \lambda_{t+s} = 0 \qquad \text{or}$$

$$\lambda_{t+s} = \beta^s U'(C_{t+s}) \quad (6)$$

$$\frac{\partial \mathcal{L}_t}{\partial W_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1} \left[1 + r(W_{t+s+1}) + r'(W_{t+s+1}) \cdot W_{t+s+1} \right] = 0$$

OR NOTING FROM (3) THAT $r'(W_{t+s+1}) = -\xi$
AND USING (3) FOR $r(W_{t+s+1})$

$$\lambda_{t+s} = \lambda_{t+s+1} \left[1 + \rho - \xi W_{t+s+1} - \xi W_{t+s+1} \right]$$

OR USING (6)

$$U'(C_{t+s}) = \beta U'(C_{t+s+1}) \left[1 + \rho - 2\xi W_{t+s+1} \right] \quad (7)$$

EQN (7) IS THE INTERTEMPORAL OPTIMALITY CONDITION.

B. Growth Rate of Consumption

A 1st order Taylor Series approx of $u'(C_{t+1})$ around C_t gives

$$u'(C_{t+1}) \cong u'(C_t) + u''(C_t)(C_{t+1} - C_t) \qquad (8)$$

Divide (8) by $u'(C_t)$ to get

$$\frac{u'(C_{t+1})}{u'(C_t)} = 1 + \frac{u''(C_t)}{u'(C_t)} \Delta C_{t+1} \qquad (9)$$

Set $S=0$ in (7) and use the result in (9) to write

$$\frac{1}{\beta[1+\rho-2\xi W_{t+1}]} = 1 + \frac{u''(C_t)}{u'(C_t)} \Delta C_{t+1} \quad \text{or}$$

$$\frac{\Delta C_{t+1}}{C_t} = -\left[\frac{u'(C_t)}{u''(C_t) \cdot C_t}\right]\left[1 - \frac{1}{\beta[1+\rho-2\xi W_{t+1}]}\right] \qquad (10)$$

Note from (2) that

$$-\left[\frac{u'(C)}{u''(C) \cdot C}\right] = \frac{1}{\gamma} \quad \text{So (10) Becomes}$$

$$\frac{\Delta C_{t+1}}{C_t} = \frac{1}{\gamma}\left[1 - \frac{1}{\beta[1+\rho-2\xi W_{t+1}]}\right] \qquad (12)$$

C.) Steady-State W

From (12) it follows that $\frac{\Delta C_{t+1}}{C_t} = 0$ where

$$\frac{1}{\beta[1+\rho-2\xi W]} = 1 \quad \text{or}$$

$$1+\theta = 1+\rho - 2\xi W_s \quad \text{or}$$

$$\theta - \rho = -2\xi W_s \quad \text{or}$$

$$\boxed{W_s = \frac{\rho-\theta}{2\xi} > 0 \qquad (13)}$$

looking ~~ahead~~ ahead, when $W_t = W_s$ we are on the $\Delta C_{t+1} = 0$ locus.

D. Phase Diagram

1.) $\Delta W_{t+1} = 0$. From (4) with (3), $\Delta W_{t+1} = 0$ where

$$0 = (\rho - \xi W_t) W_t - C_t \quad \text{or where}$$

$$\boxed{C_t \Big|_{\Delta W = 0} = \rho W_t - \xi W_t^2 \qquad (14)}$$

This eqn describes the $\Delta W_{t+1} = 0$ locus. Note that it is a quadratic function.

Regarding (14) Note That

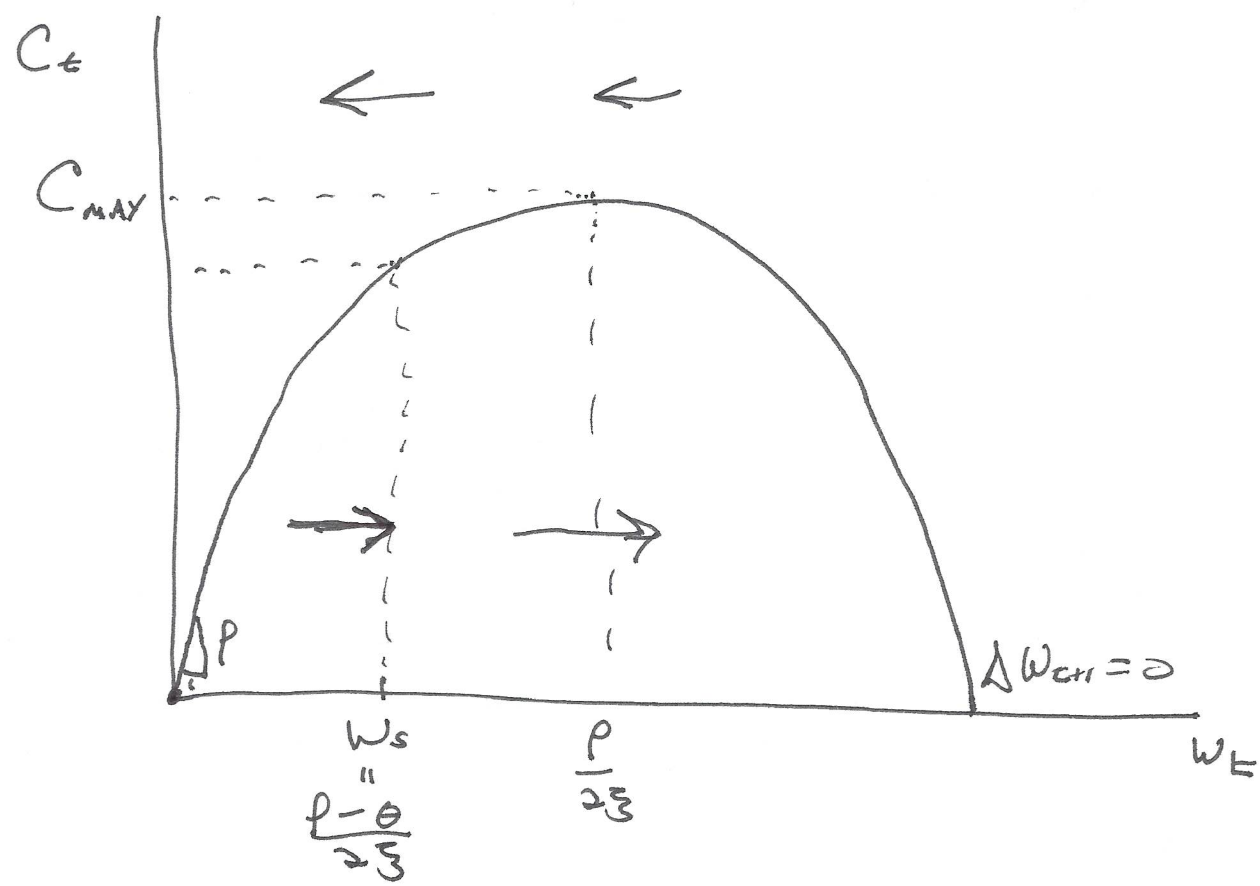
a) At $W_t = 0$, $C_t \big|_{\Delta W = 0} = 0$ So it Passes through origin.

b) $$\frac{d\, C_t\big|_{\Delta W=0}}{d\, W_t} = \rho - 2\xi W_t \quad (15)$$ So Slope is $\rho > 0$ at $W_t = 0$

c.) $$\frac{d^2\, C_t\big|_{\Delta W=0}}{d\, W_t^2} = -2\xi < 0$$ So it is Strictly Concave

d) The function Achieves a maximum value for C_t where $\rho - 2\xi W_t = 0$ or where

$$W_t = \frac{\rho}{2\xi} > W_s = \frac{\rho - \theta}{2\xi} \quad (16)$$



e) Dynamics off of $\Delta W_{t+1} = 0$

From (4) with (3)

$$\Delta W_{t+1} = (\rho - \xi W_t) W_t - C_t$$

$\dfrac{\partial W_{t+1}}{\partial C_t} = -1 < 0$ So above $\Delta W_{t+1} = 0$, W_t is falling through time and below $\Delta W_{t+1} = 0$, W_t is increasing through time.

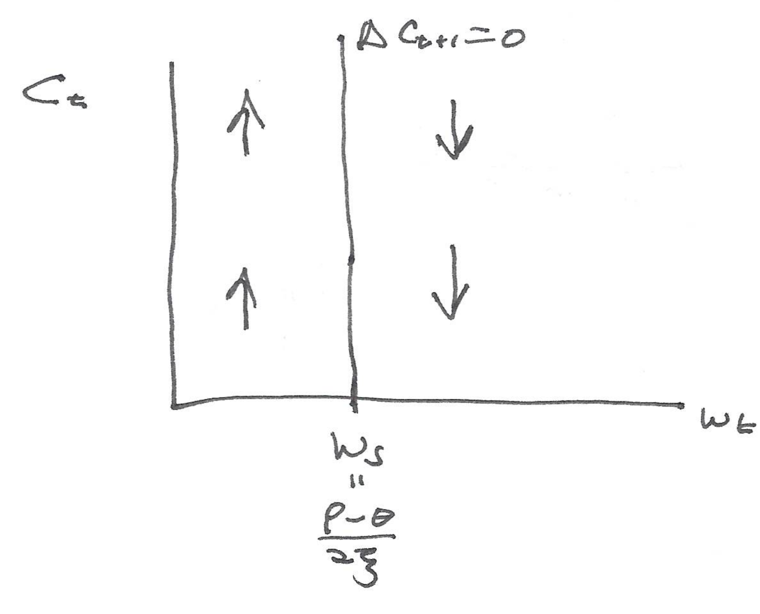
2.) $\Delta C_{t+1} = 0$. We established in part C. that $\Delta C_{t+1} = 0$ when $W_t = W_s = \dfrac{\rho - \theta}{2\xi}$ (13).

Furthermore, note from (12) that an increase in wealth causes a decline in ΔC_{t+1}:

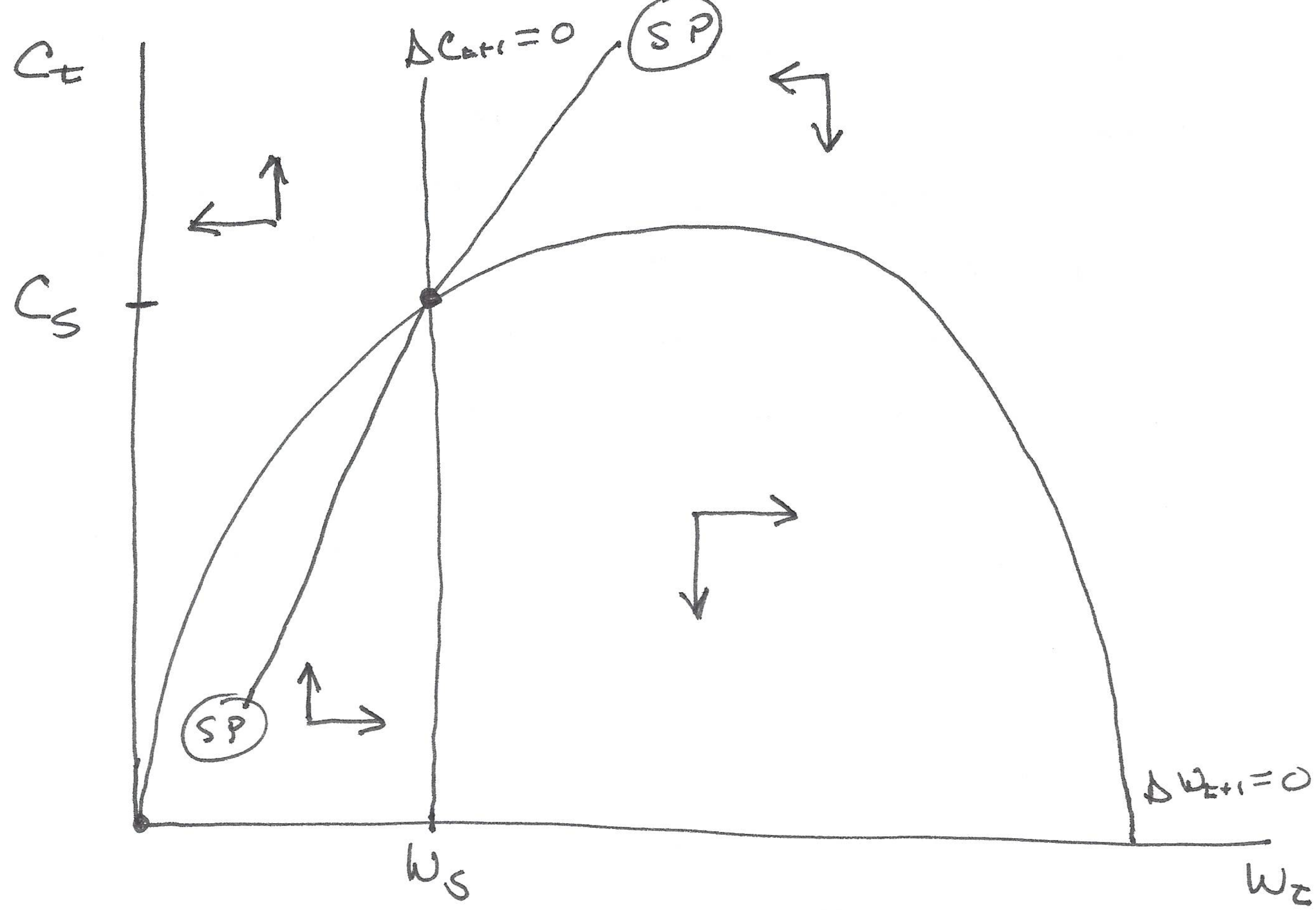
Specifically, $\uparrow W \Rightarrow \downarrow \beta[1+\rho-2\xi W] \Rightarrow \uparrow \dfrac{1}{\beta[\sim]}$

$$\Rightarrow \downarrow \left[1 - \frac{1}{\beta[\sim]}\right] \Rightarrow \downarrow \frac{\Delta C_{t+1}}{C_t} = \downarrow \Delta C_{t+1}$$

So to the right of $\Delta C = 0$ $\Delta C_{t+1} < 0$ and to the left of $\Delta C = 0$ $\Delta C_{t+1} > 0$



3.) Combining Results we have The phase Diagram Below



we have Steady-State wealth is $W_s = \dfrac{\rho - \theta}{2\xi}$ (13)
Substituting from (13) into (14) gives

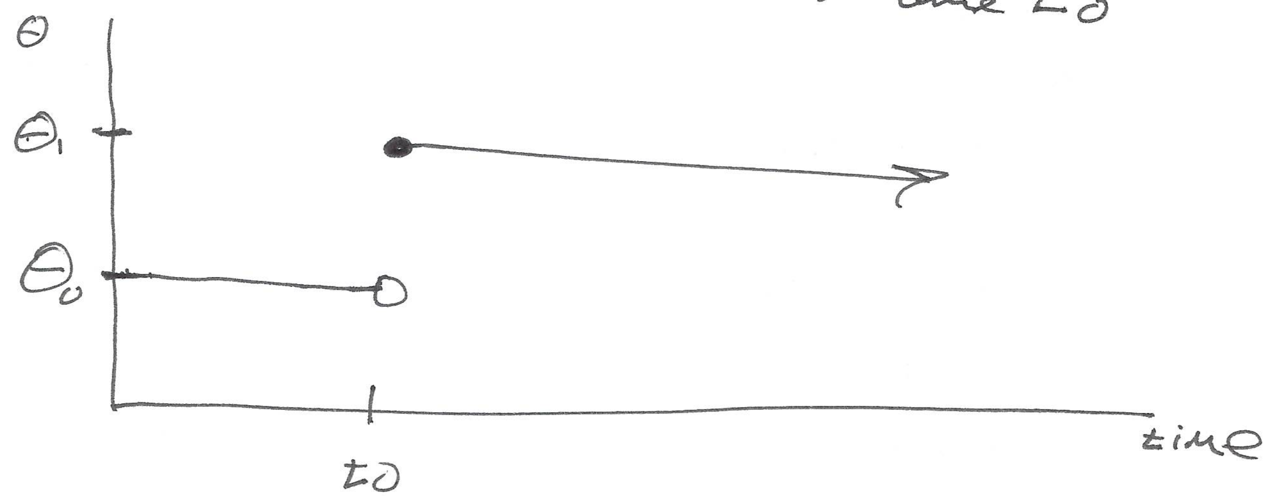
$$C_s = \rho\left(\frac{\rho-\theta}{2\xi}\right) - \xi\left(\frac{\rho-\theta}{2\xi}\right)^2$$

(which may admit a Simplification.)

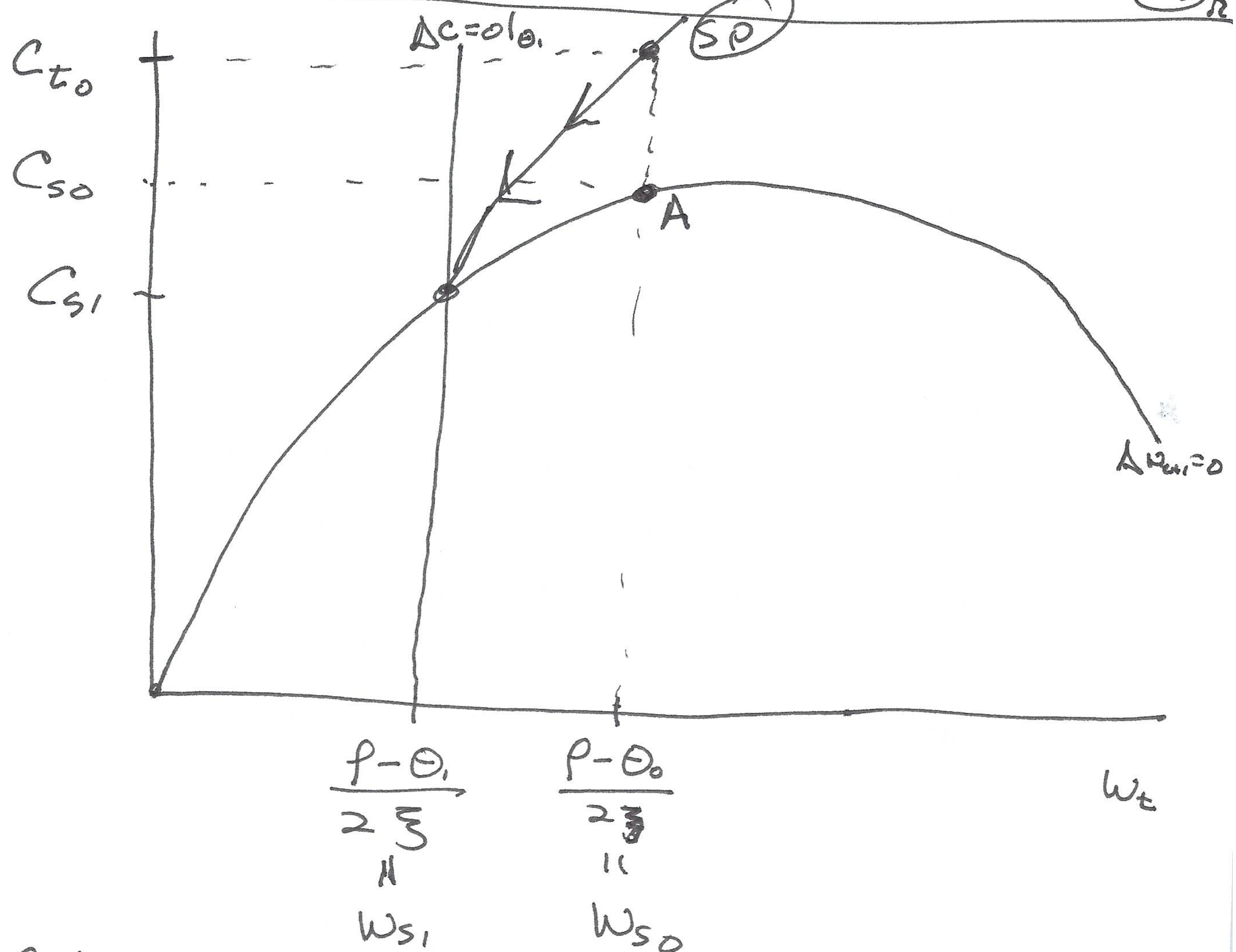
E.) Effects of increase in Θ

Looking at (14) and (13) we see that an increase in Θ will reduce W_S (shift $\Delta C=0$ to the right) but leave the $\Delta W=0$ locus unchanged.

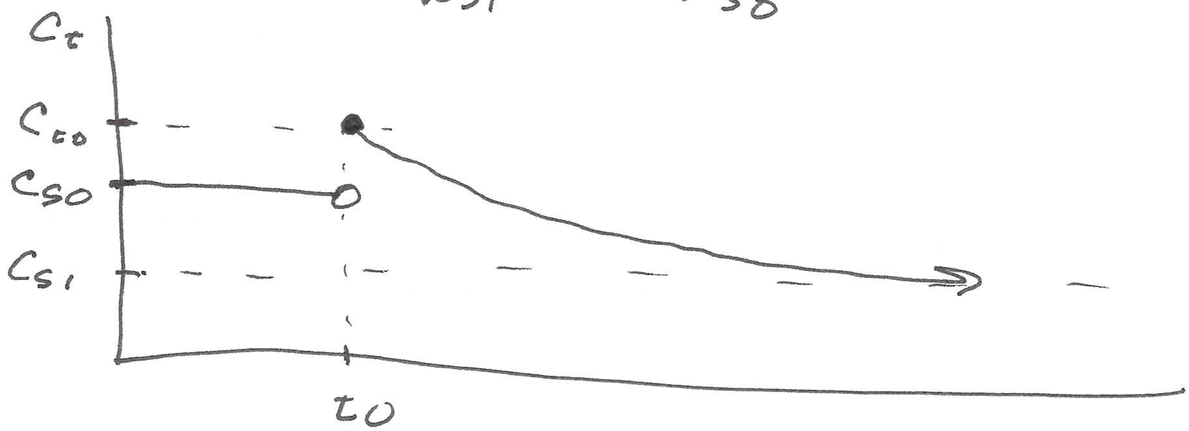
Let Θ_0 denote the original value and let the unanticipated increase in Θ to $\Theta_1 > \Theta_0$ occur at time t_0



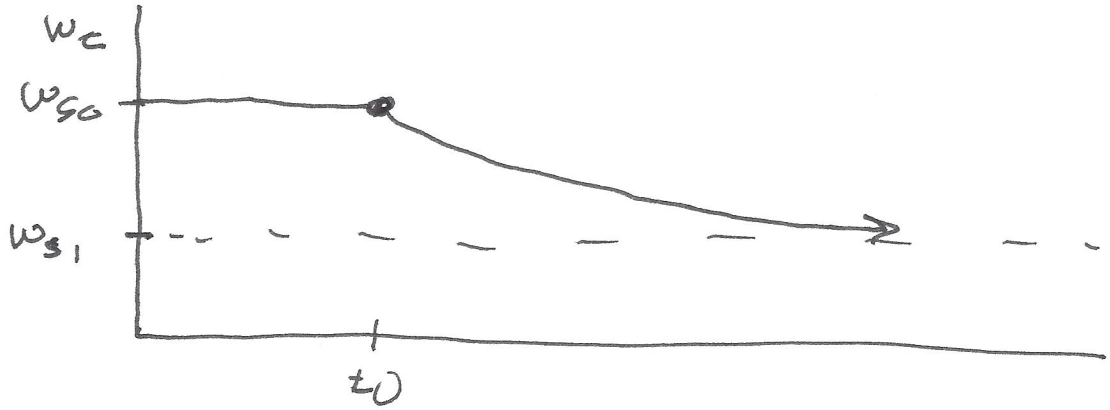
So in the phase diagram below, before time t_0 the economy is at point A. At t_0 the economy jumps to point B (there is a jump up in C_t, above C_{S0}). The economy then proceeds along the new saddle path, to the southwest with W_t and C_t both falling through time, converging to lower steady-state values, C_{S1} and W_{S1}, respectively.



C_{t_0}
C_{s_0}
C_{s_1}
$\Delta c = 0|_{\Theta_1}$
SP
A
$\Delta \dot{R}_{w} = 0$
$\frac{\rho - \Theta_1}{2\xi} = w_{s_1}$
$\frac{\rho - \Theta_0}{2\xi} = w_{s_0}$
w_t



C_t
C_{t_0}
C_{s_0}
C_{s_1}
t_0



w_t
w_{s_0}
w_{s_1}
t_0

Economically, note that ~~A~~ $\uparrow \theta = \downarrow \beta$ so this is a change in preferences such that agents value the future less (discount the future more). So they increase ~~~~ current consumption and draw down their current wealth. As wealth, and therefore (interest) income decline they must consume less. The economy converges to a new steady state with less wealth and lower consumption.

Setup the Lagrangian:

$$\mathcal{L}_t = \sum_{s=0}^{\infty}\left\{\beta^s u(c_{t+s}) + \lambda_{t+s}\left[(1+r)W_{t+s} - c_{t+s} - W_{t+s+1}\right]\right\}$$

F.O.C.

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u'(c_{t+s}) - \lambda_{t+s} = 0 \qquad (4)$$

$$\frac{\partial \mathcal{L}_t}{\partial W_{t+1+s}} = -\lambda_{t+s} + \lambda_{t+s+1}\left[(1+r)\right] = 0 \qquad (5)$$

From (4)

$$\lambda_{t+s} = \beta^s u'(c_{t+s}) \qquad (6)$$

using (6) in (5) gives

$$\beta^s u'(c_{t+s}) = \beta^{s+1} u'(c_{t+s+1})(1+r) \qquad \text{or}$$

$$\boxed{u'(c_{t+s}) = \beta(1+r)u'(c_{t+s+1}) \qquad (7)}$$

Using $\beta = \frac{1}{1+\theta}$, $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, and evaluating at $s=0$

$$\boxed{C_t^{-\gamma} = \left[\frac{1+r}{1+\theta}\right] C_{t+1}^{-\gamma} \qquad (8)}$$

Eqns (7) and (8) are two versions of the intertemporal optimality condition.

To Derive the growth rate of consumption use the linearization procedure in Wickens.

A 1st order Taylor Series approx of $u'(C_{t+1})$ around C_t yields

$$u'(C_{t+1}) \cong u'(C_t) + u''(C_t)\left(C_{t+1} - C_t\right) \qquad (9)$$

Divide (9) by $u'(C_t)$ to get

$$\frac{u'(C_{t+1})}{u'(C_t)} = 1 + \frac{u''(C_t)}{u'(C_t)} \Delta C_{t+1} \qquad (10)$$

Set $s=0$ in (7) and use (10) in the Result to get

$$\Delta C_{t+1} = \frac{u'(C_t)}{u''(C_t)}\left[\frac{1}{\beta(1+r)} - 1\right] \qquad (11)$$

Divide Through by C_t to get

$$\frac{\Delta C_{t+1}}{C_t} = -\left[\frac{u'(C_t)}{u''(C_t)\cdot C_t}\right]\left[1 - \frac{1}{\beta(1+r)}\right] \qquad (12)$$

Note that $u'(C_t) = C_t^{-\gamma}$

$$u''(C_t) = -\gamma\, C_t^{-(\gamma+1)}$$

so that the coefficient of relative risk aversion is

$$-\left[\frac{u''(C_t)}{u'(C_t)}\right]\cdot C_t = -\left[\frac{-\gamma\, C_t^{-\gamma} C_t^{-1}}{C_t^{-\gamma}}\right]\cdot C_t = \gamma \qquad (13)$$

Use (13) in (12) to get

$$\frac{\Delta C_{t+1}}{C_t} = \left(\frac{1}{\gamma}\right)\left[1 - \frac{1}{\beta(1+r)}\right] \qquad (14)$$

Note that $1 - \frac{1}{\beta(1+r)} = 1 - \frac{1}{\frac{1+r}{1+\theta}} = 1 - \frac{1+\theta}{1+r}$

$$= \frac{1+r-(1+\theta)}{1+r} = \frac{r-\theta}{1+r}$$

This in (14) gives

$$\boxed{\frac{\Delta C_{t+1}}{C_t} = \left(\frac{1}{\gamma}\right)\left[\frac{r-\theta}{1+r}\right] \qquad (15)}$$

Equation (15) describes the rate of growth of consumption, $\frac{\Delta C_{t+1}}{C_t}$.

$\frac{\Delta C_{t+1}}{C_t} > 0$ if $r > \theta$

$\frac{\Delta C_{t+1}}{C_t} < 0$ if $r < \theta$

$\frac{\Delta C_{t+1}}{C_t} = 0$ if $r = \theta$

Add'l prob (Romer 7.5)

$\boxed{\text{Additional Problem 3}}$

$$\max \underset{t}{E} \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) \qquad (1)$$

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \qquad (2)$$

$$\text{s.t.} \quad A_{t+1} = (1+r)A_t + y_t - c_t \qquad (3)$$

(a)

$$\mathcal{L}_t = \underset{t}{E} \sum_{s=0}^{\infty} \beta^s \left\{ u(c_t) + \lambda_t\left[(1+r)A_t + y_t - c_t - A_{t+1}\right] \right\}$$

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = u'(c_t) - \lambda_t = 0 \qquad (4)$$

$$\frac{\partial \mathcal{L}_t}{\partial A_{t+1}} = -\lambda_t + \beta \underset{t}{E} \lambda_{t+1}(1+r) = 0 \qquad (5)$$

Eqn (4) gives $\lambda_t = u'(c_t)$ (6)

use (6) in (5) to get

$$u'(c_t) = \beta(1+r) \underset{t}{E} u'(c_{t+1}) \qquad (7)$$

on →

$$C_t^{-\theta} = \left(\frac{1+r}{1+\rho}\right) \underset{t}{E}\, C_{t+1}^{-\theta} \qquad (8)$$

Eqn (8) is The Euler Eqn.

(b) $\ln C_{t+1}/t \sim N(\underset{t}{E} \ln C_{t+1}, \sigma^2)$

So $\underset{t}{E}\left[C_{t+1}^{-\theta}\right] = \underset{t}{E}\left[e^{-\theta \ln C_{t+1}}\right]$

$$= e^{-\theta \underset{t}{E} \ln C_{t+1}} \, e^{\theta^2 \sigma^2/2} \qquad (9)$$

(Above b/c $-\theta \ln C_{t+1}/t \sim N(-\theta \underset{t}{E} \ln C_{t+1}, \theta^2\sigma^2)$)

Use (9) in (8) to get

$$\left[e^{-\theta \ln C_t}\right] = \left(\frac{1+r}{1+\rho}\right) e^{-\theta \underset{t}{E} \ln C_{t+1}} \, e^{+\theta^2\sigma^2/2} \qquad (10)$$

or, taking logs,

$$-\theta \ln C_t = \ln\left(\frac{1+r}{1+\rho}\right) - \theta \underset{t}{E} \ln C_{t+1} + \frac{\theta^2\sigma^2}{2}$$

or

$$\boxed{\ln C_t = \underset{t}{E} \ln C_{t+1} + \frac{-1}{\theta} \ln\left(\frac{1+r}{1+\rho}\right) - \frac{\theta\sigma^2}{2} \qquad (11)}$$

Equation (11) or eqN (10) is the answer to part b.

(c) Let $\frac{-\theta\sigma^2}{2} + \frac{-1}{\theta}\ln\left(\frac{1+r}{1+\rho}\right) \equiv -a$, a constant.

Add + Subtract $\ln C_{t+1}$ to RHS (11) to get

$$\ln C_t = \ln C_{t+1} + \left(E_t \ln C_{t+1} - \ln C_{t+1}\right) - a$$

or

$$\boxed{\ln C_{t+1} = a + \ln C_t + u_{t+1} \qquad (13)}$$

where $u_{t+1} = \left(\ln C_{t+1} - E_t \ln C_{t+1}\right)$

u_{t+1} is white noise via Rational Expectations.

~~(d) Re-write (11) as~~

$$\cancel{E_t\left[\ln C_{t+1} - \ln C_t\right] = \frac{\theta\sigma^2}{2}}$$

(d) Re-write (11) as

$$\left[E_t \ln C_{t+1} - \ln C_t\right] = \frac{1}{\theta}\ln\left(\frac{1+r}{1+\rho}\right) + \frac{\theta\sigma^2}{2}$$

$$\uparrow r \Rightarrow \uparrow \ln\left(\frac{1+r}{1+\rho}\right) \Rightarrow \uparrow E_t\left[\ln C_{t+1} - \ln C_t\right]$$

$$\uparrow \sigma^2 \Rightarrow \uparrow \frac{\theta\sigma^2}{2} \Rightarrow \uparrow E_t\left[\ln C_{t+1} - \ln C_t\right]$$

$$\frac{\partial\left[E_t \ln C_{t+1} - \ln C_t\right]}{\partial\theta} = \frac{-1}{\theta^2}\ln\left(\frac{1+r}{1+\rho}\right) + \frac{\sigma^2}{2} \lesseqgtr 0\ ?$$

So $\uparrow r$ or $\uparrow \sigma^2$ Increases Expected Cons growth.
but $\uparrow \theta$ has an ambiguous effect.