AE HW2

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Applied Econometrics
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Problem Set 2
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Chapter 3 Problems:
3.1, 3.3, 3.5, 3.7, 3.8

I. Background Knowledge

1. Different Types of Convergence

(1) Basic Convergence

$x_N \to a \ (n \to \infty)$: $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \ni \forall N > N_\varepsilon \ |x_N - a| < \varepsilon$.

(2) Convergence in Probability (P-Convergence)

$x_N \xrightarrow{P} a$: $\forall \varepsilon > 0, \ \lim\limits_{N \to \infty} P(|x_N - a| > \varepsilon) = 0 \ (N \to \infty)$

which is equivalent to

$\forall \varepsilon > 0, \ \lim\limits_{N \to \infty} (|x_N - a| < \varepsilon) = 1 \quad (N \to \infty)$

(3) Covergence in Distribution (D-Convergence)

$x_N \xrightarrow{d} x$: $F_N(\xi) \to F(\xi) \quad \forall \xi \in \mathbb{R} \ (N \to \infty)$

Alternatively, $\underset{\sim}{x}_N \xrightarrow{d} \underset{\sim}{x}$ iff. $\forall \underset{\sim}{c} \ni \underset{\sim}{c}'\underset{\sim}{c} = 1$, $\underset{\sim}{c}'\underset{\sim}{x}_N \xrightarrow{d} \underset{\sim}{c}'\underset{\sim}{x}$ (Df 3.7)

(4) Convergence almost Surely (A.S.-convergence)

$x_N \xrightarrow{a.s.} a$: $\forall \varepsilon > 0, \ P\left(\lim\limits_{N \to \infty} |x_N - a| < \varepsilon\right) = 1$

$\boxed{\text{Remark 2}}$

A.S.-Convergence $\Rightarrow$ P-convergence $\Rightarrow$ D-convergence

2. Big O and little o

$\boxed{\text{Df3.2}}$ (Basic O & o) $\quad \frac{a_N}{N^\lambda}$ bounded: $a_N = O(N^\lambda)$

$\frac{a_n}{N^\lambda} \to 0 \ (n \to \infty)$: $\quad a_N = o(N^\lambda)$

$\boxed{\text{prop 1}}$ $\quad o(N^\lambda) \Rightarrow O(N^\lambda)$

This property is a fundamental property for convergent sequence.

* $\boxed{\text{Remark 1}}$ Key properties of convergent sequence (CS)

If $x_n \to a$, $y_n \to b$ $\quad x_n \le z_n \le y_n$. $(n \to \infty)$

CS1. a is unique

CS2. $a \le b$

CS3 If $a = b$, then $z_n \to a$ (squeese)

CS4. $\{x_n\}$ is bounded

CS5. If $a > 0$, then $\exists N_0 \in \mathbb{N}$, $\ni \forall N > N_0$ $\quad x_n > 0$

$\boxed{\text{Df3.3}}$ $(O_p(1), o_p(1))$

P-bounded: $\forall \varepsilon > 0, \exists b_\varepsilon < \infty, \exists N_\varepsilon \ni \forall N > N_\varepsilon, \ P(|x_N| \ge b_\varepsilon) < \varepsilon$

$\Rightarrow x_N \equiv O_p(1)$

P-convergence: $\quad x_N \xrightarrow{P} 0 \Rightarrow x_N \equiv o_p(1)$

$\boxed{\text{Df 3.4}}$ ($O_p(a_N)$, $o_p(a_N)$)

$$X_N = o_p(a_N) \quad \text{if} \quad \frac{X_N}{a_N} = o_p(1)$$

$$X_N = O_p(a_n) \quad \text{if} \quad \frac{X_N}{a_N} = O_p(1)$$

$\boxed{\text{Lemma 3.1}}$ $\quad X_n \xrightarrow{p} a \Rightarrow O_p(1)$

This is a generalized version of convergence $\Rightarrow$ bounded as prop 1.

$\boxed{\text{prop 2}}$ $\quad C_N = O_p(1) \Leftrightarrow C_N = O(1)$

$$C_N = o_p(1) \Leftrightarrow C_N = o(1)$$

This prop is important as it says in real space, convergence in probability is equivalent to convergence.

$\boxed{\text{Lemma 3.2}}$ $\quad W_N = o_p(1), \quad X_N = o_p(1)$

$$Y_N = O_p(1), \quad Z_N = O_p(1)$$

$\Rightarrow$ $X_N + W_N = o_p(1)$ Additivity

$Y_N +/\cdot\ Z_N = O_p(1)$ Add/productivity

$X_N \cdot Y_N = o_p(1)$ cross product

o is always dominant with O.

$\boxed{\text{prop3}}$ $x_N = O_p(1) \Rightarrow x_N \pm a = O_p(1)$

$\boxed{\text{lemma3.4}}$ (Slutsky) $g \in C(\mathbb{R}^k)$, $g: \mathbb{R}^k \to \mathbb{R}^J$. $\text{plim}\, g(x_N) = g(\text{plim}\, x_N)$

$\boxed{\text{Lemma3.5}}$ $\underset{\sim}{x}_N \xrightarrow{d} \underset{\sim}{x} \Rightarrow x_N = O_p(1)$

$\boxed{\text{Lemma3.6}}$ (CMT). $\underset{\sim}{x}_N \xrightarrow{d} \underset{\sim}{x} \Rightarrow g(\underset{\sim}{x}_N) \xrightarrow{d} g(\underset{\sim}{x})$, $g \in C(\mathbb{R}^k)$

3. Asymptotic Normal Distributed

$\boxed{\text{Df 3.6}}$ If $x_N \xrightarrow{N} x$, $x \sim N(\mu, \sigma^2)$, then

$$x_N \xrightarrow{d} N(\mu, \sigma^2) \quad \text{or} \quad x_N \overset{a}{\sim} N(\mu, \sigma^2)$$

$\boxed{\text{Cor3.2}}$ $z_N \xrightarrow{d} N(0, V)$, z_n is $k \times 1$

(1) $A' z_N \xrightarrow{d} N(0, A'VA)$

(2) $z_N' V^{-1} z_N \xrightarrow{d} \chi^2_k$

$\boxed{\text{Df3.8}}$ $\hat{\theta}_N$ is consistent estimator of θ if $\hat{\theta}_N \xrightarrow{P} \theta$

$\boxed{\text{Df 3.9}}$ ($\sqrt{N}$ - Asymptotically Normally Distributed)

(1) $\sqrt{N}(\hat{\theta}_N - \theta) \overset{a}{\sim} N(0, V)$ (2) $V_{p \times p}$ is PSD

Then we say $\hat{\theta}_N$ is $\sqrt{N}$-AND

V is the asymptotic variance of $\sqrt{N}(\hat{\theta}_N - \theta)$, i.e.

$$V \equiv \text{Avar}(\sqrt{N}(\hat{\theta}_N - \theta))$$

$\boxed{\text{Theorem 3.1}}$ $\boxed{\text{WLLN}}$ (Weak law of large numbers)

(1) $W_i \overset{iid}{\sim} D.$ $(i=1,2,\cdots,N)$, (2) $E[W_i] < \infty$

(here each W_i is a $g \times 1$ vector, i.e. a variable of g observations)

$$\Rightarrow \overline{W}_N \equiv \frac{\sum_{i=1}^{N} W_i}{N} \xrightarrow{P} \mu_W \equiv E(W)$$

Note $\underset{\sim}{W} = [\underset{\sim}{W_1}, \cdots W_N] = \left[\begin{bmatrix} W_{11} \\ \vdots \\ W_{1g} \end{bmatrix}, \cdots \begin{bmatrix} W_{N1} \\ \vdots \\ W_{Ng} \end{bmatrix}\right]$

* $\boxed{\text{SLLN}}$ (Strong law of large numbers)

(1) $W_i \overset{iid}{\sim} D$ (2) $E(W_i) = \mu_i < \infty$

$$\Rightarrow \overline{W}_N \equiv \frac{\sum^{N} W_i}{N} \xrightarrow{a.s} \mu_W \equiv E(\underset{\sim}{W})$$

$\boxed{\text{Theorem 3.2}}$ $\boxed{\text{CLT}}$ (Central limit Theorem)

(1) $W_i \overset{iid}{\sim} D$ $(i=1,2,\cdots N)$ (2) $E(W_i) = 0$ (3) $E(W_i^2) < \infty$

$$\Rightarrow \frac{\sum^{N} W_i}{N^{\frac{1}{2}}} \xrightarrow{d} N(0, B)$$

where $B = Var[\underset{\sim}{W_i}] = E[\underset{\sim}{W}\, \underset{\sim}{W_i}']$ is PSD

$\boxed{\text{prop 4}}$ From Df 3.9, $\sqrt{N}(\hat{\theta}_N - \theta) \overset{a}{\sim} N(0, V)$

$$\Rightarrow \hat{\theta}_N \overset{a}{\sim} N(\theta, \frac{V}{N})$$

i.e. $AVar(\hat{\theta}_N) = \frac{V}{N}$

$\boxed{\text{Df 3.10}}$ $\sqrt{N}(\hat{\theta}-\theta) \overset{a}{\sim} N(0, V)$, $V_{p\times p}$ is PD with V_{jj},

$\hat{V}_N \to V$. the asymptotic standard error of $\hat{\theta}_{Nj}$ is

$$se(\hat{\theta}_{Nj}) = \left(\frac{V_{Njj}}{N}\right)^{\frac{1}{2}}, \quad se(\hat{\theta}) = V^{\frac{1}{2}}$$

$\boxed{\text{prop 5}}$ (1) $\sqrt{N}$ - consistent : $\sqrt{N}(\hat{\theta}_N-\theta) \equiv O_p(1)$

or $\hat{\theta}_N-\theta \equiv O_p(N^{-\frac{1}{2}})$

We say $\hat{\theta}_N$ is $\sqrt{N}$ - consistent estimator of θ

i.e. $\hat{\theta}_N \xrightarrow{p} \theta \quad (N\to\infty)$

From Df 3.9, with lemma 3.5, we naturally have prop 5. i.e, $\sqrt{N}$ - AND $\Rightarrow$ $\sqrt{N}$ → consistent

$\Rightarrow$ consistent

4. Asymptotic Properties of test stats.

$\boxed{\text{Df 3.13}}$ $\boxed{\text{Asymptotic size}}$ of a testing procedure

$$\lim_{N\to\infty} P_N(\text{Reject } H_0 \mid H_0 \text{ is true})$$

$\boxed{\text{Consistent test}}$ $\lim_{N\to\infty} P_N(\text{reject } H_0 \mid H_1 \text{ is true}) = 1$

Asymptotic size is the limit of Type I error.

$\sqrt{N}(\hat{\theta}-\theta) \sim N(0, V)$

$\boxed{\text{Lemma 3.8}}$ $\hat{\theta}$ is $\sqrt{N}$-AND. V is PD. Then for any matrix $R_{Q\times P}$, $Q \le P$ w/. $\text{rank}(R)=Q$

(1) $\sqrt{N}R(\hat{\theta}_N-\theta) \overset{a}{\sim} \text{Normal}(0, RVR')$

(2) $[\sqrt{N}R(\hat{\theta}_N-\theta)]'(RVR')[\sqrt{N}R(\hat{\theta}_N-\theta)] \overset{a}{\sim} \chi^2_Q$

To test: $H_0: R\underset{\sim}{\theta}=\underset{\sim}{r}$ vs $H_1: R\underset{\sim}{\theta}\neq\underset{\sim}{r}$

$\underset{\sim}{r}$ is an arbitrary vector to be tested.

We conduct χ^2 test as follows.

Wald Test.

$$W_N \equiv (R\hat{\theta}_N - r)'\left(R\frac{\hat{V}_N}{N}R'\right)^{-1}(R\hat{\theta}_N - r) \sim \chi^2_Q$$

Lemma 3.8 expands the property of normal distn to the p-convergence scale.

5. Delta Method.

The previous sections give conditions under which a standardized random variables has a limit normal distribution. There are some times, we are interested in the functions of the random variable instead of itself, in which cases we adopt delta method.

(1) Here is a 1-D version of delta method.

Let $\hat{\theta}_N$ be a $\sqrt{N}$-AND, i.e. $\sqrt{N}(\hat{\theta}_N-\theta) \overset{a}{\sim} N(0,\sigma^2)$ where V is PD. For a given function g and a specific value θ. Suppose $g'(\theta)$ exist and is not 0. Then,

$$\sqrt{N}\left(g(\hat{\theta}_N) - g(\theta)\right) \overset{a}{\sim} N(0, \sigma^2[g'(\theta)]^2)$$

$$\Rightarrow \text{Avar } g(\hat{\theta}_N) = [g'(\theta)]^2 \text{Avar}(\hat{\theta})$$

(2) Here is a general version.

Let $c: \Theta \to \mathbb{R}^Q$ be a continuously differetiable function on parameter space $\Theta \subset \mathbb{R}^P$ $(Q \leq P)$

Let $C(\theta) \equiv \nabla_\theta c(\theta) = \begin{bmatrix} \frac{\partial c_1}{\partial \theta_1} & \cdots & \frac{\partial c_Q}{\partial \theta_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial c_1}{\partial \theta_P} & \cdots & \frac{\partial c_Q}{\partial \theta_P} \end{bmatrix}$, $\sqrt{N}(\hat{\underset{\sim}{\theta}}_N - \underset{\sim}{\theta}) \overset{a}{\sim} N(0, V)$

Then $\sqrt{N}(c(\hat{\underset{\sim}{\theta}}_N) - c(\underset{\sim}{\theta})) \overset{a}{\sim} N(0, C'(\underset{\sim}{\theta}) V C(\underset{\sim}{\theta}))$

$$\Rightarrow \text{Avar}[c(\hat{\theta}_N)] = C_N' \,\text{Avar}(\hat{\theta}_N)\, C_N.$$

and $\left[\sqrt{N}(c(\hat{\underset{\sim}{\theta}}_N) - c(\underset{\sim}{\theta}))\right]' \left[C'(\underset{\sim}{\theta}) V C(\underset{\sim}{\theta})\right]^{-1} \left[\sqrt{N}(c(\hat{\underset{\sim}{\theta}}_N) - c(\underset{\sim}{\theta}))\right] \overset{a}{\sim} \chi^2_Q$

providing an updated version of Wald Statistics.

i.e. To test $H_0: c(\underset{\sim}{\theta}) = 0$ vs $H_1: c(\underset{\sim}{\theta}) \neq 0$

$$\text{Wald} = c(\hat{\theta}_N)' \left\{\hat{C}_N' \frac{\hat{V}}{N} \hat{C}_N\right\}^{-1} c(\hat{\theta}_N) \sim \chi^2_Q$$

* c is multivariate function, C' is Jacobian.

(3) The updated version of Ase is

(i) 1-D

$$Ase(\hat{\gamma}_N) = [c'(\hat{\theta}_N)] \, Avar(\hat{\theta}_N)$$

(ii) General condition.

$$Ase(\hat{\gamma}_N) = \left[\nabla_\theta c(\hat{\theta}_N)\right] Avar(\hat{\theta}_N) \left[\nabla_\theta c(\hat{\theta}_N)'\right]^{\frac{1}{2}}$$

II. Homework Questions

3.1. Proove lemma 3.1.

If $x_N \xrightarrow{P} a$, then $x_N = O_p(1)$

Pf 1: $x_N \xrightarrow{P} a \Rightarrow x_N - a \xrightarrow{P} 0 \Rightarrow x_N - a = o_p(1)$

$\Leftrightarrow x_N - a = o(1) \Rightarrow x_N - a = O(1)$

$\Leftrightarrow x_N - a = O_p(1)$

Pf 2: First, consider x_N being a sequence of numbers.

$\underbrace{x_N \xrightarrow{P} a}_{①} \Rightarrow$ for $\underbrace{\varepsilon = 1,\ P(|x_N - a| > 1) \to 0\ (N \to \infty)}_{②}$

(Def of limit) $\Leftrightarrow \forall \gamma > 0, \exists N_\gamma \in \mathbb{N}, \ni \forall N > N_\gamma, P(|x_N - a| > 1) < \gamma$ ③

we need $P(|x_N| \geq b_\gamma) \leq P(|x_N - a| > 1)$ so that

$P(|x_N| \geq b_\gamma) < \gamma$

i.e. $\{x_N : |x_N| \geq b_\gamma,\ b_\gamma > 0\} \subset \{x_N : |x_N - a| > 1\}$.

To construct such b_γ, note

$|x_N - a| > 1 \Leftrightarrow x_N > a+1$ or $x_N < a-1$

Since $|a|+1 \geq a+1$ and $-|a|-1 \leq a-1$

we set $b_\gamma \equiv |a| + 1$

Thus by ③, we rewrite $\forall \gamma > 0, \exists b_\varepsilon = |a|+1$,

$\exists N_\gamma \in \mathbb{N}, \ni \forall N > N_\gamma, P(|x_N| \geq b_\gamma) < \gamma$, which is $x_N = O_p(1)$

3.3 Show that, under lemma 3.4 $g(\underset{\sim}{X}_N) = O_p(1)$

Pf: Since given $\underset{\sim}{X}_N \xrightarrow{P} \underset{\sim}{c}$, $g(\underset{\sim}{X}_N) \xrightarrow{P} g(\underset{\sim}{c})$ $(N \to \infty)$

Let $\underset{\sim}{y}_N = g(\underset{\sim}{X}_N)$, $\underset{\sim}{a} = g(\underset{\sim}{c})$

Then $\underset{\sim}{y}_N \xrightarrow{P} \underset{\sim}{a}$

By lemma 3.1. $\underset{\sim}{y}_N = O_p(1)$

i.e. $g(X_N) = O_p(1)$.

3.5 $y_1 \cdots y_N \overset{iid}{\sim} N(\mu, \sigma^2)$

a. Find $Var[\sqrt{N}(\bar{y}_N - \mu)]$

$$\bar{y}_N = \frac{\Sigma y_i}{N} \sim N\left(\frac{\mu \cdot N}{N}, \frac{\sigma^2 \cdot N}{N^2}\right) = N\left(\mu, \frac{\sigma^2}{N}\right)$$

$\therefore W_N = \sqrt{N}(\bar{y}_N - \mu) \sim N(0, \sigma^2)$ i.e. $Var[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$

b. Find $AVar[\sqrt{N}(\bar{y}_N - \mu)]$

By a. $E[W_N] = E[\sqrt{N}(\underset{\sim}{\bar{y}_N} - \mu)] = 0$

By CLT. $\frac{\sum_{i}^{N} W_i}{N^{\frac{1}{2}}} \xrightarrow{d} N(0, B)$

i.e. $\bar{y}_N$ is $\sqrt{N}$ - AND. where

$B = AVar[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$

Thus, we find that $Var[\sqrt{N}(\bar{y}_N - \mu)] = Avar[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$

c. Find $AVar[\bar{y}_N]$

By b, $\sqrt{N}(\bar{y}_N - \mu) \overset{a}{\sim} N(0, \sigma^2)$, $AVar[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$

$\Rightarrow AVar(\bar{y}_N) = Avar(\bar{y}_N - \mu) = Avar\left(\frac{\sqrt{N}(\bar{y}_N - \mu)}{\sqrt{N}}\right) = \frac{Avar(\bar{y}_N - \mu)}{N} = \frac{\sigma^2}{N}$

d. Find $ASD(\bar{y}_N)$

By c. $ASD(\bar{y}_N) = AVar(\bar{y}_N) = \frac{\sigma}{\sqrt{N}}$

e. Find $ase(\bar{y}_N)$

By definition 3.10 if $\sqrt{N}(\bar{y}_N - \mu_y) \sim N(0, V)$,
and $\hat{V}_N \to V$, then ASE of $\bar{y}_{Nj}$ is
$Ase(\bar{y}_{Nj}) = \left(\frac{\hat{V}_{jj}}{N}\right)^{\frac{1}{2}}$

i.e. we obtain se. as follows

(1) Find the unbiased consistent estimator of V. $\hat{V}_N$

By b. $AVar[\sqrt{n}(\bar{y}_N - \mu_y)] = \sigma^2$

The unbiased sample estimator of $V = \sigma^2$ is $\hat{V}_N = \frac{\sum_{i=1}^{N}(y_i - \bar{y}_N)^2}{N-1}$
which is also consistent, i.e. $\hat{V}_N \to V$.

(2) Find Ase

$$Ase(\bar{y}_N) = \left(\frac{\hat{V}}{N}\right)^{\frac{1}{2}} = \left[\frac{\sum_{i=1}^{N}(y_i - \bar{y}_N)^2}{N(N-1)}\right]^{\frac{1}{2}} = \frac{\sum_{i=1}^{N}(y_i - \bar{y}_N)^2}{[N(N-1)]^{\frac{1}{2}}}$$

3.7. $\hat{\theta}$ is $\sqrt{N}$ – as'p normal estimator for θ.
$\hat{\gamma} = \log(\hat{\theta})$ is an estimator of $\gamma = \log\theta$

a. validate $\hat{\gamma}$ is a consistent estimator of γ.

W.t.s. $\hat{\gamma} \xrightarrow{P} \gamma$

Since $\hat{\theta}$ is $\sqrt{N}$ – as'p for θ, i.e.

$\sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, V) \Rightarrow \hat{\theta}_N \xrightarrow{d} N(\theta, \frac{V}{N})$

i.e. $\hat{\theta} \xrightarrow{d} \theta$. where θ follows $N(\theta, \frac{V}{N})$

As $N \to \infty$, the standard error of $\hat{\theta}$ tends to 0 as shown in 3.6. i.e. $\hat{\theta}$ is a consistent estimator of θ. i.e. $\hat{\theta} \xrightarrow{P} \theta$ (P41)

By lemma 3.4. $\log(\hat{\theta}) \xrightarrow{P} \log(\theta)$ where $g(\cdot) = \log(\cdot)$ is continuous i.e. $\hat{\gamma} \xrightarrow{P} \gamma$.

b. Find $Avar[\sqrt{N}(\hat{\gamma}-\gamma)]$ in terms of $Avar[\sqrt{N}(\hat{\theta}-\theta)]$

By , we have $\hat{\gamma}_N \to \gamma$. And log is continuous function.

Suppose $Avar[\sqrt{N}(\hat{\theta}-\theta)] = V$, by delta method. (1)

$$\sqrt{N}\left(g(\hat{\theta}) - g(\theta)\right) \overset{a}{\sim} N(0, V[g'(\theta)]^2)$$

$$\Rightarrow \sqrt{N}(\hat{\gamma}-\gamma) \overset{a}{\sim} N(0, V\frac{1}{\theta^2})$$

Thus $Avar[\sqrt{N}(\hat{\gamma}-\gamma)] = \frac{1}{\theta^2} Avar[\sqrt{N}(\hat{\theta}-\theta)]$

c. Suppose that for a sample of data $\hat{\theta} = 4$, $se(\hat{\theta}) = 2$

What is $\hat{\gamma}$ and its Ase.

(i) By revised Ase (i),

$$Ase(\hat{\gamma}) = [g'(\theta)]^2 Avar[\sqrt{N}(\hat{\theta}-\theta)]$$
$$= \frac{1}{\theta} se(\hat{\theta}) = \frac{2}{4} = \frac{1}{2}.$$

(ii) $\hat{\gamma} = \log \hat{\theta} = \log 4$

d. Use t-test to test $H_0: \theta = 1$ vs $H_1: \theta \neq 1$.

$$t_\theta = \frac{\hat{\theta} - \theta_0}{Ase(\hat{\theta})} = \frac{4-1}{2} = \frac{3}{2} = 1.5, \quad df = N-1$$

e. Use t-test to test $H_0: \gamma = 0$ vs $H_1: \gamma \neq 1$.

$$t_\gamma = \frac{\hat{\gamma} - \gamma_0}{Ase(\hat{\gamma})} = \frac{\log 4}{\frac{1}{2}} = 2\log 4 \approx 2.78, \quad df = N-1$$

For t-value ($\alpha = 0.05$).

df	1	2	3	4	5	10	20	40	1000
t	12.71	4.30	3.2	2.78	2.57	2.22	2.09	2.02	1.96

We have different conclusions from t-test, which is highly sensitive.

3.8. Let $\underset{\sim}{\hat\theta} = (\hat\theta_1, \hat\theta_2)'$ be $\sqrt{N}$-AND for $\underset{\sim}{\theta} = (\theta_1, \theta_2)$

Let $\underset{\sim}{\hat\gamma} = \hat\theta_1 / \hat\theta_2$ be an estimator for $\gamma = \theta_1/\theta_2$

a. Show that $\operatorname{plim} \hat\gamma = \gamma$.

$\because$ $g(\hat\theta_1, \hat\theta_2) = \dfrac{\hat\theta_1}{\hat\theta_2}$ is a continuous function.

$\therefore$ $\operatorname{plim} g(\hat\theta_1, \hat\theta_2) = g[\operatorname{plim}(\hat\theta_1, \hat\theta_2)]$.

$$\operatorname{plim} \frac{\hat\theta_1}{\hat\theta_2} = \frac{\operatorname{plim} \hat\theta_1}{\operatorname{plim} \hat\theta_2}$$

$$\operatorname{plim} \hat\gamma = \frac{\theta_1}{\theta_2} = \gamma.$$

b. Find $\operatorname{Avar}(\hat\gamma)$ in terms of θ and $\operatorname{Avar}(\hat\theta)$ by delta method.

By delta method (2). first, compute $J \equiv \nabla_\theta C(\theta) \equiv C'(\theta)$

$$\nabla_\theta C(\theta) = \begin{pmatrix} \frac{\partial}{\theta_1}\left(\frac{\theta_1}{\theta_2}\right) \\ \frac{\partial}{\theta_2}\left(\frac{\theta_1}{\theta_2}\right) \end{pmatrix}' = \begin{pmatrix} \frac{1}{\theta_2} \\ -\frac{\theta_1}{\theta_2^2} \end{pmatrix}'$$

Then

$$\operatorname{Avar}(\hat\gamma) = C'(\theta) \operatorname{Avar}(\hat\theta)\, C'(\theta)' = \underbrace{\left(\frac{1}{\theta_2}, -\frac{\theta_1}{\theta_2^2}\right)}_{1\times 2} \underbrace{\operatorname{Avar}(\hat\theta)}_{2\times 2} . \underbrace{\begin{pmatrix} \frac{1}{\theta_2} \\ -\frac{\theta_1}{\theta_2^2} \end{pmatrix}}_{2\times 1}$$

c. If $\hat{\theta} = (-1.5, 0.5)'$ $\quad Avar(\hat{\theta}) = \begin{pmatrix} 1 & -0.4 \\ -0.4 & 2 \end{pmatrix}$

Find $Ase(\hat{\gamma})$

$$Avar(\hat{\gamma}) = C(\theta)\, Avar(\hat{\theta})\, C(\theta)'$$

$$= \left(\frac{1}{\hat{\theta}_2}, -\frac{\hat{\theta}_1}{\hat{\theta}_2^2}\right)\Bigg|_{\hat{\theta}=(-1.5,0.5)} Avar(\hat{\theta}) \begin{pmatrix} \frac{1}{\theta_2} \\ -\frac{\theta_1^2}{\hat{\theta}_2} \end{pmatrix}$$

$$= \left(\frac{1}{0.5}, -\frac{-1.5}{0.25^2}\right) \begin{pmatrix} 1 & -0.4 \\ -0.4 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{0.5} \\ 6 \end{pmatrix}$$

$$= (2, 6) \begin{pmatrix} 1 & -0.4 \\ -0.4 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 66.4$$

$$Ase(\hat{\gamma}) = \sqrt{Avar(\hat{\gamma})} = \sqrt{66.4} = 8.15$$