Homework 5

Wei Ye*
QF8915 - Stochastic Calculus

Due on Dec 17, 2022

Problem1

Calculate the partial derivatives of the Black-Scholes formula and demonstrate mathematically or demonstrate using programming computattion.

Solution:

Apply Feyman-Kac Theorem,

$$c(x,t) = E^{x,t}[e^{-r(T-t)}(x(T)-K)^+]$$

Where X is satisfies $dX_t = rX_t dt + \sigma X_t dW_t$. Let $Y_t = \log X_t$. Thus, $dY_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t$

$$Y_T - Y_t = (r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$$

Thus,

$$Y_T \sim \log X + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}z$$

Where z is standard normal distribution.

$$\begin{aligned} E^{x,t}[e^{-r(T-t)}(S_T - k)^+] &= e^{-r(T-t)}E^{\log X,t}[\exp(Y_T - k)^+] \\ &= e^{-r(T-t)}\int_{z<d^-} e^{\log X + (r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}z}\sqrt{2\pi}e^{-\frac{z^2}{2}}dz - Ke{-}r(T-t)\int_{z,d_-}\sqrt{2\pi}e^{-\frac{z^2}{2}}d \\ &= XN(d^+) - Ke^{-r(T-t)}N(d^-) \end{aligned}$$

Now justify the solution: In the PDE, we know:

$$C_t(t,x) + rXC_x(x,t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x,t) = rC(x,t)$$

Put the solution into PDE:

$$-Kre^{-r(T-t)}N(d^-) + rXN(d^+) + \frac{1}{2}\sigma^2 x^2 \cdot 0 = r(XN(d^+) - Ke^{-r(T-t)}N(d^-)) = rC(x,t)$$

They are equal, so we are done.

*2nd year PhD student in Economics Department at Fordham University. Email: wye22@fordham.edu

(Bonus)Problem2

Find the pricing formula for the European style derivative whose payoff is given by function $\log(S_T)$.

Solution:

As before, by Famen-Kac formula:

$$C(S,t) = E^{\log S,t}[e^{-r(T-t)}(\log S_T - K)^+]$$

$$d\log S_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

And:

$$\log S_T \sim \log S_t + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}z$$

$$\begin{aligned} C(S,t) &= E^{\log S,t}[e^{-r(T-t)}(\log S_T - K)^+] \\ &= e^{=r(T-t)}\int_{z<d_1}(\log S + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}z)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz - Ke^{-r(T-t)}N(d_-) \end{aligned}$$

I'm still thinking how to simplify the first part, Maybe wrong or not. Check how to simplify once getting the solution.

(Bonus) Problem3

Solution:

Same with question1 but with variance.

By Feyman-Kac Formula:

$$c(S,t) = E^{S,t}[e^{-r(T-t)}((S(T) - K)^+)^2]$$

Set $Y_t = \log S_t$, $dY_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t$, then $Y_t = (r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)$

$$\begin{aligned} E^{S,t}[e^{-r(T-t)}((S(T) - K)^+)^2] &= e^{-r(T-t)}ES,t((S(T) - K)^+)^2 \\ &= e^{-r(T-t)}\int_{z<d_-}(e^{\log S + (r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}z} - K)^2\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \end{aligned}$$

It's even harder to simplify the quadratic terms in integral...