Session 9

Chunyu Qu 08/19/20

§5. Orthogonality.

1). Orthogonal Sets of Vectors

1.1. Def. A set of vetors $\{\vec{v}_1 \cdots \vec{v}_k\}$ in $\mathbb{R}^n$.
It is called $\boxed{\text{orthogonal set}}$ if
$\forall i, j \in \{1, 2 \cdots k\}$ $i \neq j$. $\vec{v}_i \cdot \vec{v}_j = 0$

1.2. Theorem. An orthogonal set of nonzero vectors in $\mathbb{R}^n$ are IND

orthonormal set
$\uparrow$ unit vector
orthogonal set
$\downarrow$ basis (IND)
orthonormal basis

1.3. Def. An $\boxed{\text{orthonormal basis}}$ for subspace $W \subset \mathbb{R}^n$, is a basis that is an orthogonal set
A set of vectors in $\mathbb{R}^n$ is called an $\boxed{\text{orthonormal set}}$ if it is an orthogonal set of unit vectors

(i) orthogonal
(ii) Unit vector

1.4. ex. $\vec{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ $\vec{q}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$S \overset{set}{=} \{\vec{q}_1, \vec{q}_2\}$

Determine if S is an orthonormal set.

step 1. $\vec{q}_1 \cdot \vec{q}_2 = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$ → orthogonality

step 2. $\left.\begin{array}{l} \|q_1\| = \sqrt{q_{11}^2 + q_{12}^2 + q_{13}^2} = \sqrt{q_1 \cdot q_1} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1 \\ \|q_2\| = \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = 1 \end{array}\right\}$ → basis

$\Rightarrow$ S is an orthonormal set.

Determine if $S = \{\vec{q}_1, \vec{q}_2, \vec{q}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1/2 \\ -2 \end{bmatrix} \right\}$.

form orthonormal basis for $\mathbb{R}^3$.

(i) If S is orthogonal set or not.

$$q_i \cdot q_j = 0 \quad (i \neq j)$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1 + (-1) = 0$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1/2 \\ -2 \end{bmatrix} = 2 - 2 = 0$$

$$\begin{bmatrix} 1 \\ 8 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1/2 \\ -2 \end{bmatrix} = 2 + (-4) + 2 = 0$$

S is orthogonal set

(ii) $\vec{q}_1, \vec{q}_2, \vec{q}_3$ are LID ?

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 8 & -1/2 \\ 1 & -1 & -2 \end{bmatrix} \xrightarrow{\text{R.E.}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 8 & -1/2 \\ 0 & -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 15\frac{1}{2} \end{bmatrix} \longleftrightarrow I_3$$

$\vec{q}_1, \vec{q}_2, \vec{q}_3$ form an orthogonormal basis of $\mathbb{R}^3$

$\boxed{1.5}$ Theorem. $Q_{m\times n}$. The column of Q form an orthonormal set iff. $Q^TQ = I_n$.

square!

1.6. Def. $Q_{n\times n}$ whose columns form an orthonormal set, we call Q an orthogonal matrix

1.7. Theorem. $Q_{n\times n}$ is orthognal

$\Leftrightarrow$ $Q^{-1} = Q^T$

1.8. Ex $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Show A is orthognal. ① $Q^TQ = I_3$

② $Q^{-1} = Q^T$.

$$A^TA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I_3.$$

※ 1.9. Theorem. $Q_{n\times n}$ is square matrix

The following are equivalent

a. Q is orthogonal

b. $\|Q\vec{x}\| = \|\vec{x}\|$ $\forall \vec{x} \in \mathbb{R}^n$ invariance of norm in terms of orthogonal matrix

c. $\underset{n\times n}{Q}\underset{n\times 1}{\vec{x}} \cdot Q\vec{y} = \vec{x}\cdot\vec{y}$, $\forall \vec{x}, \vec{y} \in \mathbb{R}$

* Thm. If Q is an orthogonal matrix, then its rows form an orthonormal set.

1.10 Properties Q_1, Q_2 are orthogonal matrices

a. Q_i^{-1} is orthogonal

b. $|Q_i| = \pm 1$

c. λ is eigenvalue of $Q_i \Rightarrow |\lambda| = 1$

d. $Q_1 Q_2$ is orthogonal

2). The Gram-Schimdt Process.

We are given $\{\vec{x}_1, \cdots \vec{x}_k\}$ as a basis for subspace $W \subset \mathbb{R}^n$.

$$\Downarrow$$

$\{\vec{v}_1 \cdots \vec{v}_k\} \rightarrow$ orthonormal basis

2.1. G-S Process Algorithm.

step 1. $\vec{v}_1 = \vec{x}_1$

step 2. $\vec{v}_2 = \vec{x}_2 - \underbrace{\left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right) \cdot \vec{v}_1}$

step 3. $\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\right) \cdot \vec{v}_2$

$\vdots$

step k. $\vec{v}_k = \vec{x}_k - \left(\frac{\vec{v}_1 \cdot \vec{x}_k}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_k}{\vec{v}_2 \cdot \vec{v}_2}\right) \vec{v}_2 - \cdots - \left(\frac{\vec{v}_{k-1} \cdot \vec{x}_k}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}}\right) \vec{v}_{k-1}$

$\{\vec{v}_1 \cdots \vec{v}_k\}$ is an orthonormal basis

2.2. ex. $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

Use G-S process to construct an orthonormal basis for

$W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \subset \mathbb{R}^4$

Step 1 Use G-S to find an orthonormal set.

(i) $\vec{v}_1 = \vec{x}_1$

(ii) $$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2-1+1}{1+1+1+1}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

(iii) $$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{2-2-1+2}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{3+3+\frac{1}{2}+1}{\frac{9}{4}+\frac{9}{4}+\frac{1}{4}+\frac{1}{4}}\begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \overset{3/2}{\frac{\frac{15}{2}}{\frac{20}{4}}}\begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

$$A$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

Step 2. Normalize $\underbrace{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}_{\text{orthonormal set}} \to \underbrace{\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}}_{\text{orthonormal basis}}$.

$$\vec{q}_1 = \frac{1}{\|\vec{v}_1\|} \cdot \vec{v}_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{q}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2 = \frac{1}{\sqrt{\frac{9+9+1+1}{4}}} \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \frac{1}{\frac{2\sqrt{5}}{2}} \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix}$$

$$\vec{q}_3 = \frac{1}{\|\vec{v}_3\|} \vec{v}_3 = \frac{1}{\sqrt{\frac{1+1+4}{4}}} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\frac{1}{\frac{\sqrt{6}}{2}} = \frac{2}{\sqrt{6}}$$

2.3. QR Factorization

Let A be an $m \times n$ matrix w/. IND columns. Then A can be factored as $\underset{m \times n}{A} = \underset{m \times n}{Q}\underset{n \times n}{R}$.

Rectang. A

Rectang. Q — $m \times n$ matrix w/. orthonormal columns.

Square R — invertible upper triangular matrix.

Suppose $\vec{a}_1 \cdots \vec{a}_n$ are IND, which are columns of A.

Let $\vec{q}_1 \cdots \vec{q}_n$ be the orthonormal vectors obtained by G-S process.

Step1. G-S.

Step2. normalization

$\exists\ r_{1i}, r_{2i}, \cdots r_{ii} \quad i = 1, \cdots, n$

a_i can be expressed as LC of $r_{1i}\vec{q}_1 + r_{2i}\vec{q}_2 + \cdots + r_{ii}\vec{q}_i$

$$\vec{a}_i = r_{1i}\vec{q}_1 + r_{2i}\vec{q}_2 + \cdots r_{ii}q_n$$

or.

$$\vec{a}_1 = r_{11}\vec{q}_1 \rightarrow r_{11}$$

$$\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2 \rightarrow [\vec{q}_1 \mid \vec{q}_2 \mid \vec{a}_2] \rightarrow r_{12}, r_{22}$$

$$\vdots$$

$$\vec{a}_n = r_{1n}\vec{q}_1 + r_{2n}\vec{q}_2 + \cdots + r_{nn}\vec{q}_n \rightarrow r_{1n}, \vdots, r_{nn}.$$

$$A = \underbrace{[q_1 \mid q_2 \cdots \mid q_n]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ 0 & & \ddots & \\ & & & r_{nn} \end{bmatrix}}_{R}.$$

2.4. Ex. Find QR Factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Step1. G-S. to have $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ – orthogonal set

$$\vec{V}_1 = \vec{X}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{V}_2 = \vec{X}_2 - \left(\frac{\vec{V}_1 \cdot \vec{X}_2}{\vec{V}_1 \cdot \vec{V}_1}\right)\vec{V}_1 = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad \vec{V}_3 = \vec{X}_3 - \left(\frac{\vec{V}_1 \cdot \vec{X}_3}{\vec{V}_1 \cdot \vec{V}_1}\right)\vec{V}_1 - \left(\frac{\vec{V}_2 \cdot \vec{X}_3}{\vec{V}_2 \cdot \vec{V}_2}\right)\vec{V}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Step2. Normalization to have $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ – orthonormal basis

$$\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{q}_1 \qquad \vec{q}_2 \qquad \vec{q}_3$$

$$\begin{bmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{bmatrix} \quad \begin{bmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

orthonormal basis

Step3. $$Q = \begin{bmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{bmatrix} \quad 4\times 3.$$

$A = QR \Rightarrow Q^T A = \underbrace{Q^T Q}_{I} R = R$

orthogonal matrix

not required

$$R = \underline{Q^T A} = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}_{4\times 3} =$$

Solve OLS. Given $A_{n\times n}$.

$\text{Rank}(A) = n$

Find $\vec{x} \in \mathbb{R}^n \ni. \min \|A\vec{x} - \vec{b}\|$

If we can QR factorize A i.e. $A = QR$.

$$\|A\vec{x} - \vec{b}\| \underset{\substack{Q^T \text{ is orthogonal} \\ \text{matrix.}}}{\overset{\|Q\cdot\vec{x}\| = \|\vec{x}\|}{=\!=}} \|Q^T(A\vec{x} - \vec{b})\|$$

$$= \|Q^TA\vec{x} - Q^T\vec{b}\|$$

$Q^T = Q^{-1}$

$$= \|Q^{-1}A\vec{x} - Q^{-1}\vec{b}\|$$

$A = QR$

$$= \|R\vec{x} - Q^{-1}\vec{b}\|$$

$\underline{Q^{-1}A} = Q^{-1}QR$

$= \underline{R}$

3) Spectrum decomposition

3.1. Def. A is orthogonally diagonalizable if $\exists$ an orthogonal matrix Q and a diagonal matrix D

$\ni$ $Q^T A Q = D$

$\Updownarrow$ $A = Q D Q^T$ $\quad Q^T = Q^{-1}$

Recap. $A = P D P^{-1}$

3.2. Algorithm for Spectrum Decomposition.

$A_{n\times n} = Q D Q^T$

Step 1. Find eigenvalues $\lambda_1 \cdots \lambda_n$

$\downarrow \quad \downarrow$

eigenvectors $\vec{x}_1 \cdots \vec{x}_n$

Step 2. G-S $\{\vec{x}_1 \cdots \vec{x}_n\} \to \{\vec{v}_1 \cdots \vec{v}_n\} \Rightarrow$ orthogonal set

Step 3. Normalize $\{\vec{v}_1 \cdots \vec{v}_n\} \to \{\vec{q}_1 \cdots \vec{q}_n\} \to$ orthonormal basis

Step 4. $$A = \underbrace{\begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix}}_{Q^T}$$

$$= \lambda_1 \underbrace{\vec{q}_1 \vec{q}_1^{\,T}}_{n\times n} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \cdots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$$

3.3.ex. Spectrum decompose $A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Step 1. Find $\lambda_1, \lambda_2, \lambda_3$

$\vec{x}_1, \vec{x}_2, \vec{x}_3$

$\lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = 1$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Step 2. G-S $\quad \boxed{\vec{v}_1, \vec{v}_2, \vec{v}_3}$ orthogonal set

Step 3. normalize $\vec{v}_1, \vec{v}_2, \vec{v}_3$

orthonormal basis

$$\vec{q}_1 = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \quad \vec{q}_2 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \quad \vec{q}_3 = \begin{bmatrix} -\sqrt{6}/6 \\ -\sqrt{6}/3 \\ -\sqrt{6}/6 \end{bmatrix}$$

Step 4. $$A = \begin{bmatrix} \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & -\sqrt{6}/3 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -\sqrt{6}/6 & -\sqrt{6}/3 & -\sqrt{6}/6 \end{bmatrix}$$

$$= \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$$

$$= 4 \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix}^T + \cdots$$

4). LU factorisation.

4.1. Def. $A_{n\times n}$. $A = LU$

L → Lower triangular, U → upper triangular.

4.2. Ex $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$

Step 1. Reduce A to row echelon form.

$\boxed{R_2 = R_2 - 2R_1}$ set.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow[E_2:\ \boxed{R_3+R_1}]{E_1:\ \boxed{R_2-2R_1}} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{E_3:\ \boxed{R_3+2R_2}} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = U$$

Step 2. Determine L

E_1 – elementary matrix express a row operation on a matrix A.

Row operation on A ⟺ EA

$E_1 A$, $\boxed{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ a_{21}

$(B+C)A = BA + CA$

$$E_1 A = \left(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C\ 3\times 3} \right) A \quad \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}_{3\times 3} \begin{matrix} R_1 \\ \\ \\ \end{matrix}$$

$$= BA + CA = \boxed{A} + \begin{bmatrix} 0 & 0 & 0 \\ (-2)\times 2 & (-2)\times 1 & (-2)\times 3 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \begin{matrix} \\ -2R_1 \\ \\ \end{matrix}$$

$$= A + \begin{bmatrix} 0 \\ -2R_1 \\ 0 \end{bmatrix} \iff R_2 - 2R_1 \qquad R_2 = R_2 - 2R_1$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ (-2)a_{11} & (-2)a_{12} & (-2)a_{13} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-2a_{11} & a_{22}-2a_{12} & a_{23}-2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$A = LU$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$$

Step 1. R.E.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \xrightarrow[\textcircled{2}\ R_3 + R_1]{\textcircled{1}\ R_2 - 2 \cdot R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{\textcircled{3}\ R_3 + 2 \cdot R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \equiv U$$

$$\textcircled{1}\ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\textcircled{2}\ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\textcircled{3}\ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_2 - 2R_1$

$$A \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ -2 & 5 & 5 \end{bmatrix} = EA$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EA = \left(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) A$$

$$= A + \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}_{3\times 3}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2\cdot 2 & -2\cdot 1 & -2\cdot 3 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ -2 & 5 & 5 \end{bmatrix}$$

c_{12}, c_{21}, c_{22}, c_{23}, c_{31}

properties of elementary matrix

(1) Operation $R_j + c_{ji} \cdot R_i \Leftrightarrow E = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & c_{ji} & \ddots & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

(2) $E_1 = \begin{bmatrix} 1 & & & & \\ c_{21} & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & & 1 \end{bmatrix}$

$E_k \cdots E_2 E_1 = \begin{bmatrix} 1 & & & & \\ c_{21} & 1 & & & \\ c_{31} & c_{32} & 1 & & \\ \vdots & & & \ddots & \\ c_{n1} & \cdots & \cdots & c_{n(m-1)} & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ c_{31} & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & & 1 \end{bmatrix}$

$E_k = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & c_{n(m-1)} & 1 \end{bmatrix}$

(3) $E = \begin{bmatrix} 1 & & \\ & \ddots & \\ c_{ij} & & \ddots \\ & & & 1 \end{bmatrix}$ $\quad E^{-1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ -c_{ij} & & \ddots \\ & & & 1 \end{bmatrix}$

$E_2 \qquad R_3 = R_3 + 1 R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

a_{31}

$$E_2 A = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) A$$

$$\underset{I}{\parallel}$$

$$= A + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+a_{11} & a_{32}+a_{12} & a_{33}+a_{13} \end{bmatrix}$$

$\Leftrightarrow \qquad R_3 = R_3 + R_1$

$E_2 : \quad R_3 + 2 R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

a_{32}

$$
\boxed{
\begin{aligned}
E_1 &: R_2 - 2R_1 \to R_2^{\text{New}} \\
E_2 &: R_3 + R_1 = R_3 - (-1)R_1 \\
E_3 &: R_3 + 2R_2 = R_3 - (-2)R_2
\end{aligned}
}
\Rightarrow
$$

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}
$$

$$L_{21} = 2$$

$$L_{31} = -1$$

$$L_{32} = -2$$

$$
\begin{array}{ccc}
E_1 & E_2 & E_3 \\
\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}
\end{array}
$$

$$\boxed{E_3(E_2(E_1A)) = U}$$

$$E_3E_2E_1A = U$$

$$E_1^{-1}E_2^{-1}E_3^{-1}E_3E_2E_1A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

$$
= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{L_2}
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L_3}
U
$$

$$L$$

- Theorem: Elementary matrix is invertible.

- Corollary: Elementary matrix is $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$$

4.3.

Algorithm for LU.

Step 1. (i) Reduce A to its row echelon form U → upper triangular matrix

(ii) Record all your operations in format $R_i - c_{ij} R_j$

Step 2. set up L based on I w/ replacing $l_{ij} = c_{ij}$

4.4. ex. LU. factorize $A = \begin{bmatrix} 3 & 1 & 3 & -4 \\ 6 & 4 & 8 & -10 \\ 3 & 2 & 5 & -1 \\ -9 & 5 & -2 & 4 \end{bmatrix}$

Step 1.

$$A \xrightarrow{\substack{R_2 - (2)R_1 \\ R_3 - (1)R_1 \\ R_4 + 3R_1 \\ -(-3)R_1}} \begin{bmatrix} 3 & 1 & 3 & -4 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 8 & 7 & -16 \end{bmatrix} \xrightarrow{\substack{R_3 - (\frac{1}{2})R_2 \\ R_4 - (4)R_2}} \begin{bmatrix} 3 & 1 & 3 & -4 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & 8 \end{bmatrix}$$

$$\xrightarrow{\substack{-(-1)R_3 \\ R_4 + R_3}} \begin{bmatrix} 3 & 1 & 3 & -4 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$

Step 2.

$$L = \begin{array}{c} \quad l_{21} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{2} & 1 & 0 & 0 \\ \boxed{1} & \boxed{\frac{1}{2}} & 1 & 0 \\ \boxed{-3} & \boxed{4} & \boxed{-1} & 1 \end{bmatrix} \end{array}$$

Remark: the LU fact'n we discuss in class work only on matrices that can be reduced to their row echelon form w.o. interchanging rows.