Answer Key

Econ 6700: Math Methods II, Midterm Exam, Fall 2020.

You may consult the book, your notebook, or any other materials, but you may not work with another student or consult any other individual; all of the work must be yours and yours alone.

Answer any three of the following four questions. All questions receive equal weight.

Question 1. Let **A** be a (2x2) matrix whose elements are labelled as follows,

$\mathbf{A}_{(2\times2)} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Let r_1 and r_2 denote the eigenvalues of $\mathbf{A}$. Consider the eigenvectors of **A** normalized so that the first element of each is 1. Thus, denote the eigenvectors of **A** that are associated with r_1 and r_2, respectively, by

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ v_{12} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ v_{22} \end{bmatrix}$.

A.) Derive expressions for v_{12} and v_{22}, each in terms of $a_{11}, a_{12}, a_{21}, a_{22}, r_1$, and r_2. Derive expressions for r_1 and r_2, each in terms of a_{11}, a_{12}, a_{21}, and a_{22}.

B.) Prove that the sum of the eigenvalues of $\mathbf{A}_{(2\times2)}$ equals the sum of its diagonal elements. Prove that the product of the eigenvalues of $\mathbf{A}_{(2\times2)}$ is equal to its determinant.

C.) Suppose that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ where $\mathbf{D}_{(k\times k)}$ is a diagonal matrix. Prove that the diagonal elements of $\mathbf{D}_{(k\times k)}$ are the eigenvalues of $\mathbf{A}_{(k\times k)}$ and that the columns of $\mathbf{P}_{(k\times k)}$ are the corresponding eigenvectors.

(Exam continued next page.)

Question 2. Consider the following model:

$$y_t = y_{t+1} - (i_t - \pi_{t+1}) \qquad (1)$$

$$\pi_t = \beta\pi_{t+1} + \alpha y_t + v_t \qquad (2)$$

$$i_t = \gamma\pi_t + \varepsilon_t \qquad (3)$$

where y_t, i_t, and π_t denote output, the nominal interest rate, and inflation, respectively. The parameters β, α, and γ are positive. The shocks v_t and ε_t are both i.i.d. $(0, \sigma^2)$.

A.) Rewrite the system described by equations (1)-(3) as $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t$ where $\mathbf{x}_t = \begin{pmatrix} y_t \\ \pi_t \end{pmatrix}$. Give each of the elements of the matrices $\mathbf{A}$ and $\mathbf{B}$ in terms of the parameters, β, α, and γ. Specify the elements of $\mathbf{u}_t$ in terms of the variables and parameters of the original system, equations (1)-(3).

B.) Now, instead, use the system described by equations (1)-(3) to derive a 2nd order difference equation in π_t. Show how the coefficients and shocks in this 2nd order difference equation depend on the parameters and variables in the original system, equations (1)-(3).

C.) Suppose now that, instead of both shocks being i.i.d. $(0, \sigma^2)$, each is a consant. That is, let $v_t = v_s$ and $\varepsilon_t = \varepsilon_s$ $\forall t$ where v_s and ε_s are nonzero constants. Using the 2nd order difference equation that you derived in part B, derive the stationary-state value of π_t. Give the stationary state value of π_t as a function of β, α, γ, v_s and ε_s.

Question 3. Using Hamilton's notation, consider a Markov chain. The transition probability matrix is $\mathbf{P}_{(N\times N)}$ and ξ_t is an $(N\times 1)$ column vector whose ith element is 1 when $s_t = i$ and whose other elements are zero.

You may take the following propositions as given

(i) A matrix and its transpose have the same eigenvalues.

(ii) $E(\xi_{t+M} \mid \xi_t) = \mathbf{P}^M \xi_t$

(iii) For the decomposition $\mathbf{P} = \mathbf{T}\Lambda\mathbf{T}^{-1}$, the first row of $\mathbf{T}^{-1}$ is the (transpose of) the eigenvector of $\mathbf{P}'$ associated with the $(1,1)$ element of Λ.

A.) Prove that 1 is an eigenvalue of $\mathbf{P}$. Show that there exists a probability vector π such that $\pi = \mathbf{P}\pi$.

B.) Assuming that all other eigenvalues are real and less than 1 in absolute value, prove that $\lim_{M\to\infty} \mathbf{P}^M = \pi\psi'$ where, here, $\psi_{(N\times 1)}$ is a vector of ones. That is, $\psi_{(N\times 1)} = [1, 1, \cdots 1]'$.

(Exam continued next page.)

Question 4. Throughout this question you should assume that y_t is covariance stationary. You should also assume that $|\phi| < 1$ and that ε_t is white noise.

A.) Derive the MA(∞) representation of the following two stochastic processes:

$$y_t = c + \phi y_{t-1} + \varepsilon_t \qquad (1)$$

and

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}. \qquad (2)$$

B.) Using the lag operator, derive the autoregressive, AR, representation of the following MA(1) process:

$$y_t = \mu + \varepsilon_t - \theta \varepsilon_{t-1}, \quad |\theta| < 1. \qquad (3)$$

What is the order of the AR representation?

C.) Let γ_j denote the jth autocovariance of y_t and let ρ_j denote its jth autocorrelation. Derive γ_j and ρ_j, $j = 0, 1$, for the case where y_t is MA(1).

D.) Let P_t denote the expected present discounted value of y_t. Thus $P_t = E_t \sum_{i=0}^{\infty} \beta^i y_{t+i}$ where $0 < \beta < 1$. Here $E_t[\bullet]$ denotes $E[\bullet | I_t]$ where I_t, the current information set, is $I_t = \{y_t, y_{t-1}, y_{t-2}, \ldots; P_{t-1}, P_{t-2}, P_{t-3}, \ldots; \Theta\}$, and where Θ denotes the structure of the relevant stochastic process including its parameters (and including β).
Derive the value of P_t (the REE solution) for the ARMA(1,1) process in equation (2) above, where $c = 0$.

A.) If V_j is the e-vector of A associated w/ the e-value r_j then

$$(A - r_j I) V_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ or

$$\begin{bmatrix} a_{11} - r_j & a_{12} \\ a_{21} & a_{22} - r_j \end{bmatrix} \begin{bmatrix} 1 \\ v_{j2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First row: $(a_{11} - r_j) + a_{12} v_{j2} = 0$ or

$$\boxed{v_{j2} = \frac{r_j - a_{11}}{a_{12}} \qquad (1)}$$ for $j = 1, 2$.

Second row: $a_{21} + (a_{22} - r_j) v_{j2} = 0$ or

$$\boxed{v_{j2} = \frac{-a_{21}}{a_{22} - r_j} \qquad (2)}$$ for $j = 1, 2$.

Either (1) or (2) answers this part of part A.

I show below that v_{j2} from (1) is the same as v_{j2} from (2) but such a proof is NOT part of the answer.

A.) CONTINUED: If r is an e-value of A Then

$$\mathrm{Det}\,(A - rI) = 0 \quad \text{or}$$

$$\mathrm{Det}\begin{bmatrix} (a_{11}-r) & a_{12} \\ a_{21} & (a_{22}-r) \end{bmatrix} = 0 \quad \text{or}$$

$$(a_{11}-r)(a_{22}-r) - a_{21}a_{12} = 0 \quad \text{or}$$

$$a_{11}a_{22} - (a_{11}+a_{22})r + r^2 - a_{21}a_{22} = 0 \quad \text{or}$$

$$r^2 - (a_{11}+a_{22})r + (a_{11}a_{22} - a_{21}a_{12}) = 0 \qquad (3)$$

Thus

$$r_{1,2} = \frac{1}{2}\left\{(a_{11}+a_{22}) \pm \left[(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})\right]^{1/2}\right\} \qquad (4)$$

Eqn (4) Answers The Second part of PART A.

<u>Digress</u> To Show That (1) and (2) give The Same v_{j2}.

Must Show
$$\left(\frac{r_j - a_{11}}{a_{12}}\right) = \left(\frac{-a_{21}}{a_{22} - r_j}\right) \quad \text{or}$$

$$(r_j - a_{11})(a_{22} - r_j) = -a_{21}a_{12} \quad \text{or}$$

$$-a_{11}a_{22} - r_j^2 + (a_{11}+a_{22})r_j = -a_{21}a_{12} \quad \text{or}$$

$$0 = r_j^2 - (a_{11}+a_{22})r_j + a_{11}a_{22} - a_{21}a_{12} \quad \text{which is (3)}$$

<u>QED</u>

B.) Let $Q_1 \equiv (a_{11} + a_{22})$ and $Q_2 \equiv [(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})]$

So that (4) gives

$$r_1 = \frac{1}{2}\left[Q_1 - Q_2^{1/2}\right] \text{ and } r_2 = \frac{1}{2}\left[Q_1 + Q_2^{1/2}\right] \quad (5)$$

Thus $r_1 + r_2 = \frac{1}{2}\left[Q_1 - Q_2^{1/2}\right] + \frac{1}{2}\left[Q_1 + Q_2^{1/2}\right]$

$r_1 + r_2 = Q_1$ so

$r_1 + r_2 = a_{11} + a_{22}$ <u>QED</u> Sum of Diag elements = Sum of Roots

Also $r_1 \cdot r_2 = \frac{1}{2}\left[Q_1 - Q_2^{1/2}\right]\frac{1}{2}\left[Q_1 + Q_2^{1/2}\right]$

$$= \frac{1}{4}\left[Q_1^2 - Q_2^{1/2}Q_1 + Q_1Q_2^{1/2} - (Q_2^{1/2})^2\right]$$

$$= \frac{1}{4}\left[(a_{11}+a_{22})^2 - (a_{11}+a_{22})^2 + 4(a_{11}a_{22} - a_{21}a_{12})\right]$$

Thus $r_1 \cdot r_2 = a_{11}a_{22} - a_{21}a_{12} = \text{Det}(A)$. <u>QED</u> Prod of Roots = Det(A)

There is An Alternate Proof on P 599 of Simon + Blume.

C.) We have $P^{-1}AP = D$ where D is a diagonal matrix.

Let $\underset{(K\times K)}{P} \equiv \left[\underset{(K\times 1)}{V_1} \quad \underset{(K\times 1)}{V_2} \quad \cdots \quad \underset{(K\times 1)}{V_K}\right]_{(K\times K)}$ and

let $D \equiv \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_K \end{bmatrix}_{(K\times K)}$

Premultiply both sides of $P^{-1}AP = D$ by P to get

$$AP = PD \qquad \text{or}$$

$$A\left[V_1, V_2, \cdots V_K\right] = \left[V_1 \; V_2 \cdots V_K\right] \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_K \end{bmatrix}$$

Equating, column by column we have

$$\left.\begin{aligned} Av_1 &= d_1 V_1 \text{ or } (A - d_1 I)V_1 = 0_{(K\times 1)} \\ Av_2 &= d_2 V_2 \text{ or } (A - d_2 I)V_2 = 0_{(K\times 1)} \\ &\vdots \\ Av_K &= d_K V_K \text{ or } (A - d_K I)V_K = 0_{(K\times 1)} \end{aligned}\right]$$

So the d_j and V_j satisfy the defn of an eigenvalue and of an eigenvector, respectively, for $j = 1, 2, \ldots K$.

<u>QED</u>

Question 2

Begin From

$$y_t = y_{t+1} - (i_t - \pi_{t+1}) \quad (1)$$

$$\pi_t = \beta \pi_{t+1} + \alpha y_t + v_t \quad (2)$$

$$i_t = \gamma \pi_t + \varepsilon_t \quad (3)$$

A.) Use (3) in (1) to get

$$y_t = y_{t+1} - \gamma \pi_t + \pi_{t+1} - \varepsilon_t \quad \text{or}$$

$$y_{t+1} = y_t + \gamma \pi_t - \pi_{t+1} + \varepsilon_t \quad (4)$$

Re-write (2) as $\beta \pi_{t+1} = \pi_t - \alpha y_t - v_t$ or

$$\boxed{\pi_{t+1} = \frac{1}{\beta} \pi_t - \frac{\alpha}{\beta} y_t - \frac{1}{\beta} v_t \quad (5)}$$

use (5) in (4) to get

$$y_{t+1} = y_t + \gamma \pi_t - \left[\frac{1}{\beta} \pi_t - \frac{\alpha}{\beta} y_t - \frac{1}{\beta} v_t\right] + \varepsilon_t \quad \text{or}$$

$$\boxed{y_{t+1} = \left(1 + \frac{\alpha}{\beta}\right) y_t + \left(\gamma - \frac{1}{\beta}\right) \pi_t + \left(\varepsilon_t + \frac{1}{\beta} v_t\right) (6)}$$

Taken together ~~eqn(5)~~ eqns (5) and (6) can be written as

$$\begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \left(1+\frac{\alpha}{\beta}\right) & \left(\gamma - \frac{1}{\beta}\right) \\ \frac{-\alpha}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & \frac{1}{\beta} \\ 0 & -\frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \quad (7)$$

c.f.

$$X_{t+1} = A\, X_t + B\, U_t \quad (8)$$

B. Once again, (3) in (1) gives (4)

$$\boxed{y_{t+1} = y_t + \gamma \pi_t - \pi_{t+1} + \varepsilon_t \quad (4)}$$

Invert (2) to write

$$\cancel{\beta \pi_{t+1}}\ \alpha y_t = \pi_t - \beta \pi_{t+1} - v_t \quad \text{or}$$

$$y_t = \frac{1}{\alpha}\pi_t - \left(\frac{\beta}{\alpha}\right)\pi_{t+1} - \frac{1}{\alpha} v_t \quad (9)$$

Thus

$$y_{t+1} = \frac{1}{\alpha}\pi_{t+1} - \left(\frac{\beta}{\alpha}\right)\pi_{t+2} - \frac{1}{\alpha} v_{t+1} \quad (10)$$

Use (9) and (10) in (4) to get

$$\frac{1}{\alpha}\pi_{t+1} - \frac{\beta}{\alpha}\pi_{t+2} - \frac{1}{\alpha}V_{t+1} = \frac{1}{\alpha}\pi_t - \frac{\beta}{\alpha}\pi_{t+1} - \frac{1}{\alpha}V_t + \gamma\pi_t - \pi_{t+1} + \varepsilon_t$$

or

$$\frac{1}{\alpha}\pi_{t+1} - \frac{\beta}{\alpha}\pi_{t+2} - \frac{1}{\alpha}V_{t+1} = \left(\frac{1}{\alpha}+\gamma\right)\pi_t - \left(\frac{\beta}{\alpha}+1\right)\pi_{t+1} + \varepsilon_t - \frac{1}{\alpha}V_t$$

or, collecting

$$-\frac{\beta}{\alpha}\pi_{t+2} + \left(\frac{1}{\alpha}+\frac{\beta}{\alpha}+1\right)\pi_{t+1} - \left(\frac{1}{\alpha}+\gamma\right)\pi_t = \varepsilon_t + \frac{1}{\alpha}(V_{t+1}-V_t)$$

Multiply through by $-\frac{\alpha}{\beta}$ to get

$$\pi_{t+2} - \left(\frac{1}{\beta}+1+\frac{\alpha}{\beta}\right)\pi_{t+1} + \left(\frac{1}{\beta}+\frac{\alpha\gamma}{\beta}\right)\pi_t = \frac{1}{\beta}(V_t - V_{t+1}) - \frac{\alpha}{\beta}\varepsilon_t$$

or

$$\boxed{\pi_{t+2} - \left(1+\frac{1+\alpha}{\beta}\right)\pi_{t+1} + \left(\frac{1+\alpha\gamma}{\beta}\right)\pi_t = \frac{1}{\beta}(V_t - V_{t+1}) - \frac{\alpha}{\beta}\varepsilon_t \qquad (11)}$$

Eqn (11) answers part (B).

c.) In (11) set $Y_t = Y_{t+1} = Y_s$ and $\varepsilon_t = \varepsilon_s$.

Also set $\pi_{t+2} = \pi_{t+1} = \pi_t$. The ~~Res~~ Result is

$$\cancel{\pi_s - \left(1 + \frac{1+\alpha}{\beta}\right) + \frac{1+\alpha\gamma}{\beta}}$$

$$\pi_s - \left(1 + \frac{1+\alpha}{\beta}\right)\pi_s + \left(\frac{1+\alpha\gamma}{\beta}\right)\pi_s = -\frac{\alpha}{\beta}\varepsilon_s.$$

So

$$\left[1 - 1 - \frac{1+\alpha}{\beta} + \frac{1}{\beta} + \frac{\alpha\gamma}{\beta}\right]\pi_s = \frac{-\alpha}{\beta}\varepsilon_s \quad \text{or}$$

$$\left[\frac{-1}{\beta} - \frac{\alpha}{\beta} + \frac{1}{\beta} + \frac{\alpha\gamma}{\beta}\right]\pi_s = \frac{-\alpha}{\beta}\varepsilon_s \quad \text{or}$$

$$\frac{\alpha(\gamma-1)}{\beta}\pi_s = \frac{-\alpha}{\beta}\varepsilon_s \quad \text{or}$$

$$(\gamma - 1)\pi_s = -\varepsilon_s \quad \text{or}$$

$$\boxed{\pi_s = (1-\gamma)\varepsilon_s \qquad (12)}$$

Eqn (12) Answers Part C

A.) Since P is a transition probability matrix, the elements of each of its columns sum to 1. Thus,

for $\underset{(N\times 1)}{\psi} = (1, 1, \ldots, 1)'$

$$P'\psi = \psi \qquad (1)$$

Rewrite (1) as

$$(P' - 1\cdot I_N)\psi = \vec{0}_{(N\times 1)} \qquad (2)$$

And we see that 1 is an eigenvalue of P', and ψ is its associated eigenvector. Since P and P' have the same eigenvalues it follows that 1 is an eigenvalue of P.

Consider now the eigenvector of P associated with the eigenvalue $\lambda_1 = 1$. Denote this e-vector by $X_{(N\times 1)}$. Thus

$$(P - 1\cdot I_N)X = \vec{0}_{(N\times 1)} \qquad (3)$$

$$X \neq 0$$

And, Thus $$X = PX \qquad (4)$$

If we normalize the elements of X so that they sum to 1, this normalized vector, π, is a probability vector. So

$$\pi = P\pi \qquad (5)$$

and $$\psi'\pi = 1 \qquad (6)$$

B. We seek to establish that

$$\lim_{M \to \infty} P^M = \pi_{(N\times 1)} \psi'_{(1\times N)} \qquad (7)$$

Note that $\pi \psi' = \left[\pi_{(N\times 1)} \; \pi_{(N\times 1)} \cdots \pi_{(N\times 1)} \right]_{(N\times N)}$ (8)

Use the decomposition

$$P = T \Lambda T^{-1} \qquad (9)$$

Where $\Lambda_{(N\times N)}$ is a diagonal matrix with the e-values of P along its principle diagonal and where the columns of $T_{(N\times N)}$ are the corresponding e-vectors of P. Order the e-values in Λ so that $\lambda_1 = 1$ is in the $(1,1)$ position.

~~Since λ_i for $i = 2, 3, \ldots N$ satisfy~~

Since $|\lambda_i| < 1$ for $i = 2, 3, \cdots N$, it follows that

$$\lim_{M\to\infty} \Lambda^M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}_{(N\times N)}$$

and thus

$$\lim_{M\to\infty} P^M = T \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{bmatrix} T^{-1} = X \cdot Y' \quad (9)$$

where $X_{(N\times 1)}$ is the first column of P and $Y'_{(1\times N)}$ is the first row of P.

Since X is 1st Col of T it is the e-vector of P associated with the ~~e-va~~ e-value $\lambda_1 = 1$. It is thus proportional to π:

$$X = \beta \pi \qquad (10)$$

where $\beta \neq 0$ is a scalar.

Since Y' is the first row of T^{-1} it is the (transpose of) e-vector of P' associated w/ the e-value $\lambda_1 = 1$.

Thus

$$P'Y = Y \qquad (11)$$

Now, since

$$P'\Psi = \Psi \qquad (1)$$

Y must be proportional to Ψ:

$$Y = \alpha \Psi \qquad (12)$$

where $\alpha \neq 0$ is a scalar.

Use (10) and (12) in RHS(9) to get

↘

$$\lim_{M \to \infty} P^M = (\beta\alpha)\Pi\Psi' \qquad (13)$$

From $E(\xi_{t+M} \mid \xi_t) = P^M \xi_t \qquad (14)$

Note that P^M is itself a transition probability matrix.

For example, if

$$P^M = \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{N1} \\ \phi_{12} & \ddots & & \vdots \\ \vdots & & & \vdots \\ \phi_{1N} & \cdots & & \phi_{NN} \end{bmatrix}$$

$\phi_{12} = \text{Prob}\{S_{t+M}=2 \mid S_t=1\}$

$\phi_{13} = \text{Prob}\{S_{t+M}=3 \mid S_t=1\}$

etc

Thus the elements of each of the columns of P^M must sum to 1.

Note from (8) that the elements of each of the columns of $\Pi\Psi'$ must also sum to 1.

Thus, in (13), $\beta\alpha = 1$ and (7) is established.

<u>QED (7)</u>

Question ~~3~~ 4

4.1 ~~3.1~~

A.)

$$y_t = c + \phi y_{t-1} + \varepsilon_t \qquad (1)$$

$$(1-\phi L) y_t = c + \varepsilon_t$$

$$y_t = \left(\frac{1}{1-\phi L}\right) c + \left(\frac{1}{1-\phi L}\right) \varepsilon_t$$

$$\boxed{y_t = \left(\frac{c}{1-\phi}\right) + \varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots}$$

or $\boxed{y_t = \left(\frac{c}{1-\phi}\right) + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j} \qquad (4)}$

$$y_t = c + \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \qquad (2)$$

$$(1-\phi L) y_t = c + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$y_t = \left(\frac{c}{1-\phi L}\right) + \left(\frac{1}{1-\phi L}\right) \varepsilon_t + \left(\frac{\theta}{1-\phi L}\right) \varepsilon_{t-1}$$

$$y_t = \left(\frac{c}{1-\phi}\right) + \left(\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots\right)$$
$$+ \theta \left(\varepsilon_{t-1} + \phi \varepsilon_{t-2} + \phi^2 \varepsilon_{t-3} + \cdots\right)$$

$$y_t = \left(\frac{c}{1-\phi}\right) + \varepsilon_t + (\phi + \theta) \varepsilon_{t-1} + (\phi^2 + \theta\phi) \varepsilon_{t-2}$$
$$+ (\phi^3 + \phi^2 \theta) \varepsilon_{t-3} + \cdots$$

$$\boxed{y_t = \left(\frac{c}{1-\phi}\right) + \varepsilon_t + (\phi + \theta) \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-1-j} \qquad (5)}$$

B.) $y_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1} \quad |\theta|<1 \quad (3)$

$$y_t = \mu + (1-\theta L)\varepsilon_t$$

$$\left(\frac{1}{1-\theta L}\right)y_t = \left(\frac{\mu}{1-\theta L}\right) + \varepsilon_t \quad \text{or}$$

$$y_t + \theta y_{t-1} + \theta^2 y_{t-2} + \cdots = \left(\frac{\mu}{1-\theta}\right) + \varepsilon_t \quad \text{or}$$

$$\boxed{y_t = -\theta y_{t-1} + -\theta^2 y_{t-2} - \theta^3 y_{t-3} - \cdots + \left(\frac{\mu}{1-\theta}\right) + \varepsilon_t}$$

or

$$\boxed{y_t = \left(\frac{\mu}{1-\theta}\right) - \sum_{j=1}^{\infty}\theta^j y_{t-j} + \varepsilon_t \quad (6)}$$

Eqn (6) is an infinite order AR process, That is AR(∞).

C.) $Y_t \sim MA(1)$ so

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} \qquad (7)$$

$$\varepsilon_t \sim iid(0, \sigma^2)$$

↓

$$\gamma_0 = E\left[(y_t-\mu)^2\right] = E\left[(\varepsilon_t+\theta\varepsilon_{t-1})^2\right]$$

$$= E\left[\varepsilon_t^2 + 2\theta\,\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2\right] = \cancelto{\sigma^2}{E(\varepsilon_t^2)} + 2\theta\,\cancelto{0}{E(\varepsilon_t\varepsilon_{t-1})} + \theta^2\,\cancelto{\sigma^2}{E\varepsilon_{t-1}^2}$$

$$\gamma_0 = \sigma^2 + \theta^2\sigma^2$$

or $\boxed{\gamma_0 = (1+\theta^2)\sigma^2 \qquad (8)}$

$$\gamma_1 = E\left[(y_t-\mu)(y_{t-1}-\mu)\right] = E\left[(\varepsilon_t+\theta\varepsilon_{t-1})(\varepsilon_{t-1}+\theta\varepsilon_{t-2})\right]$$

$$= E\left[\varepsilon_t\varepsilon_{t-1} + \theta\varepsilon_{t-1}^2 + \theta\varepsilon_t\varepsilon_{t-2} + \theta^2\varepsilon_{t-1}\varepsilon_{t-2}\right]$$

$$= \cancelto{0}{E(\varepsilon_t\varepsilon_{t-1})} + \theta\,\cancelto{\sigma^2}{E(\varepsilon_{t-1}^2)} + \theta\,\cancelto{0}{E(\varepsilon_t\varepsilon_{t-2})} + \theta^2\,\cancelto{0}{E(\varepsilon_{t-1}\varepsilon_{t-2})}\,]$$

$\boxed{\gamma_1 = \theta\sigma^2 \qquad (9)}$

Next $\rho_j \equiv \dfrac{\gamma_j}{\gamma_0}$ so

$\boxed{\rho_1 = 1 \quad \text{and} \quad \rho_2 = \dfrac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \left(\dfrac{\theta}{1+\theta^2}\right) \qquad (10)}$

D.) We have

$$P_t = \underset{t}{E} \sum_{i=0}^{\infty} \beta^i y_{t+i} \qquad (11)$$

where $\quad y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \quad (2')$

From (2')

$$\boxed{\underset{t}{E}\, y_t = y_t \qquad (12.1)}$$

$$\underset{t}{E}\, y_{t+1} = E\left[\phi y_t + \varepsilon_{t+1} + \theta \varepsilon_t\right] \text{ so}$$

$$\boxed{\underset{t}{E}\, y_{t+1} = \phi y_t + \theta \varepsilon_t \qquad (12.2)}$$

$$\underset{t}{E}\, y_{t+2} = \underset{t}{E}\left[\phi y_{t+1} + \varepsilon_{t+2} + \theta \varepsilon_{t+1}\right] = \phi \underset{t}{E}\, y_{t+1} \text{ so}$$

$$\boxed{\underset{t}{E}\, y_{t+2} = \phi^2 y_t + \phi \theta \varepsilon_t \qquad (12.3)}$$

$$\underset{t}{E}\, y_{t+3} = \underset{t}{E}\left[\phi y_{t+2} + \varepsilon_{t+3} + \theta \varepsilon_{t+2}\right] = \phi \underset{t}{E}\, y_{t+2} \text{ so}$$

$$\boxed{\underset{t}{E}\, y_{t+3} = \phi^3 y_t + \phi^2 \theta \varepsilon_t \qquad (12.4)}$$

Thus, in general

$$\boxed{\underset{t}{E}\, y_{t+j} = \phi^{j} y_t + \theta \phi^{j-1} \varepsilon_t \quad \text{for } j=1,2,\cdots \quad (12)}$$

So (11) Becomes

$$P_t = y_t + \beta\phi y_t + \beta\phi^2 y_t + \beta\phi^3 y_t + \cdots$$
$$+ \beta\theta\varepsilon_t + \beta^2\theta\phi\varepsilon_t + \beta^3\theta\phi^2\varepsilon_t + \cdots \quad \text{or}$$

$$P_t = \sum_{j=0}^{\infty} (\beta\phi)^j y_t + \beta\theta \sum_{j=0}^{\infty} (\beta\phi)^j \varepsilon_t \quad \text{or}$$

$$\boxed{P_t = \left[\frac{1}{1-\beta\phi}\right] y_t + \left[\frac{\beta\theta}{1-\beta\phi}\right] \varepsilon_t \quad (13)}$$