Exercises #9

11.B.5 Suppose that at fixed input prices of $\overline{w}$ a firm produces output with differentiable and strictly convex cost function $c(q,h)$, where $q \geq 0$ is its output level (whose price is $p > 0$) and h is the level of a negative externality generated by the firm. The externality affects a single consumer, whose derived utility function takes the form $\phi(h) + w$. The actions of the firm and consumer do not affect any market prices.

(a) Derive the first-order condition for the firm's choice of q and h.

(b) Derive the first-order conditions characterizing the Pareto optimal levels of q and h. (Since the consumer's utility function is quasilinear in wealth, the utility possibilities frontier is a linear line with slope -1; therefore the Pareto optimal level of q and h can be found by the sum of the profit function and utility function omitting wealth.)

(c) Suppose that the government taxes the firm's output level. Show that this cannot restore efficiency, unlike a direct tax on the externality.

(d) Show, however, that in the limiting case where h is necessarily produced in fixed proportions with q, so that $h(q) = \alpha q$ for some $\alpha > 0$, a tax on the firm's output *can* restore efficiency. What is the efficiency-restoring tax?

11.B.5 **(a)** The firm solves $\max_{q,h} p\cdot q - c(q,h)$, and the FOCs are:

$$(1)\quad p \leq \frac{\partial c(q^*,h^*)}{\partial q}, \text{ with equality if } q^* > 0,$$

$$(2)\quad 0 \leq \frac{\partial c(q^*,h^*)}{\partial h}, \text{ with equality if } h^* > 0.$$

(b) Since the consumer's utility function is quasilinear with respect to money, the utility posibility frontier is a linear line with slope -1, and therefore we can find the Pareto optimal level of h and q by maximizing the sum of the profit function and the utility function without wealth, i.e.,

$$\max_{q,h} \; p\cdot q - c(q,h) + \phi(h),$$

which yields the FOCs,

$$(3)\quad p \leq \frac{\partial c(q^o,h^o)}{\partial q}, \text{ with equality if } q^o > 0,$$

$$(4)\quad \phi'(h) \leq \frac{\partial c(q^o,h^o)}{\partial h}, \text{ with equality if } h^o > 0.$$

(c) Let t denote the tax rate on output, the firm solves:

$$\max_{q,h} \; p\cdot q - c(q,h) - t\cdot q,$$

which yields the FOCs,

$$(5)\quad p \leq \frac{\partial c(q,h)}{\partial q} + t, \text{ with equality if } q > 0,$$

$$(6)\quad 0 \leq \frac{\partial c(q,h)}{\partial h}, \text{ with equality if } h > 0.$$

This gives the same level of $h = h^*$ as in part (a) above. If, however, a tax is imposed on h, say τ, the firm solves:

$$\max_{q,h} \; p\cdot q - c(q,h) - \tau\cdot h,$$

which yields the FOCs,

$$(7)\quad p \leq \frac{\partial c(q,h)}{\partial q}, \text{ with equality if } q > 0,$$

$$(8)\quad -\tau \leq \frac{\partial c(q,h)}{\partial h}, \text{ with equality if } h > 0,$$

and if we set $\tau = -\phi_1'(h^o)$ then this gives the level $h = h^o$ as in part (b) above, and efficiency is restored.

(d) Let t denote the tax rate on output, the firm solves:

$$\max_{q} \; p\cdot q - c(q,\alpha q) - t\cdot q,$$

which yields the FOC,

$$(9)\quad p \leq \frac{\partial c(q,\alpha q)}{\partial q} + \frac{\partial c(q,\alpha q)}{\partial q} + t, \text{ with equality if } q > 0,$$

and if we set $t = -\alpha\cdot\phi_1'(h^o)$ then the pair (q^o,h^o) will solve this FOC as in part (b) above, and efficiency is restored.

We have: $\bar{w}$: fixed inputs (fixed)

$c(q, h)$: cost function (strictly convex)

$q \geq 0$: output level (with price $p > 0$)

h : level of negative externality generated by the firm

$\phi(h) + w$: derived utility function

actions of firms and consumers do not affect market prices

Recall: Derived utility function:

$$v_i(p, w_i, h) = \max_{\{x_i \geq 0\}} u_i(x_i, h)$$

$$\text{s.t.: } p \cdot x_i \leq w_i$$

utility function is quasilinear w.r.t to numeraire commodity:

$$v_i(p, w_i, h) = \phi_i(p, h) + w_i = \phi_i(h) + w_i$$

given that prices are unaffected by consumer's choices

with $\phi_i(\cdot)'' < 0$.

a) FOC for the firm's choices of q and h:

$$\max_{\{q,h\}} \ p \cdot q - c(q,h)$$

note: firm is price-taker

FOC: (q): $p - \dfrac{\partial c(q,h)}{\partial q} \leq 0 \iff p \leq \dfrac{\partial c(q,h)}{\partial q}$

with equality if $q^* > 0$ (interior solution)

(h): $- \dfrac{\partial c(q,h)}{\partial q} \leq 0 \iff \dfrac{\partial c(q,h)}{\partial q} \geq 0$

with equality if $h^* > 0$

b) FOC for the Pareto optimal levels of q and h:

$$\max_{\{q,h\}} \quad \phi(h) + w + p \cdot q - c(q,h)$$ → sum of utility and profit

FOC: (q): $p - \dfrac{\partial c(q,h)}{\partial q} \leq 0 \Leftrightarrow p \leq \dfrac{\partial c(q,h)}{\partial q}$

with equality if $q^{0} > 0$

(h): $\dfrac{\partial \phi(h)}{\partial h} - \dfrac{\partial c(q,h)}{\partial h} \leq 0 \Leftrightarrow \dfrac{\partial \phi(h)}{\partial h} \leq \dfrac{\partial c(q,h)}{\partial h}$

with equality if $h^{0} > 0$

c) Show that a tax on the firm's output level does not restore efficiency but a direct tax on the externality does.

Let t: tax rate on output

$$\max_{\{q,h\}} \; p \cdot q - c(q,h) - t \cdot q$$

FOC: (q): $p - \dfrac{\partial c(q,h)}{\partial q} - t \leq 0 \iff p \leq \dfrac{\partial c(q,h)}{\partial q} + t$

with equality if $q > 0$

(h): $-\dfrac{\partial c(q,h)}{\partial h} \leq 0 \iff \dfrac{\partial c(q,h)}{\partial h} \geq 0$

with equality if $h > 0$

This gives the same level of the externality $h = h^*$ as in part (a).

It is therefore <u>not</u> Pareto optimal.

If a tax is imposed on h, say τ, the firm solves:

$$\max_{\{q,h\}} \ p \cdot q - c(q,h) - \tau \cdot h$$

FOC: (q): $p - \dfrac{\partial c(q,h)}{\partial q} \leq 0 \iff p \leq \dfrac{\partial c(q,h)}{\partial q}$

with equality if $q > 0$

(h): $-\dfrac{\partial c(q,h)}{\partial h} - \tau \leq 0 \iff \dfrac{\partial c(q,h)}{\partial h} \geq -\tau$

with equality if $h > 0$

If we set $\tau = -\dfrac{\partial \phi(h^\theta)}{\partial h^\theta}$ then the FOC for (h) becomes:

$$\frac{\partial c(q,h)}{\partial h} \geq \frac{\partial \phi(h^\theta)}{\partial h^\theta} \qquad \left(\text{or } \frac{\partial \phi(h^\theta)}{\partial h^\theta} \leq \frac{\partial c(q,h)}{\partial h}\right)$$

and this gives $h = h^\theta$ as in part (b) and efficiency is restored.

d) Show that if $h(q) = \alpha q$, $\alpha > 0$, a tax on firm's output can restore efficiency.

$\uparrow$ limiting case where h is produced in fixed proportions

$$h(q) = \alpha q \iff q = \frac{h}{\alpha}$$

$$\max_{\{q,h\}} \; p \cdot q - c(q,h) - t q = p \cdot q - c(q,h) - \frac{t h}{\alpha}$$

FOC: (q): $p - \dfrac{\partial c(q,h)}{\partial q} \leq 0 \iff p \leq \dfrac{\partial c(q,h)}{\partial q}$

with equality if $q > 0$

(h): $-\dfrac{\partial c(q,h)}{\partial h} - \dfrac{t}{\alpha} \leq 0 \iff \dfrac{\partial c(q,h)}{\partial h} \geq -\dfrac{t}{\alpha}$

with equality if $h > 0$

If we set $-\dfrac{t}{\alpha} = \dfrac{\partial \phi(h^0)}{\partial h^0} \iff t = -\alpha \dfrac{\partial \phi(h^0)}{\partial h^0}$ then

we obtain $h = h^0$ as in part (b) and efficiency is restored.

11.D.1 First-year graduate students are a hard-working group. Consider a typical class of I students. Suppose that each student i puts in h_i hours of work on his or her classes. This effort involves a disutility of $h_i^2/2$. His or her benefits depend upon how well he or she performs relative to his or her peers and take the form $\phi(h_i/\tilde{h})\ \forall i$, where $\tilde{h} = (1/I)\sum_i h_i$ is the average number of hours put in by all students in the class and $\phi(\cdot)$ is a differentiable concave function, with $\phi'(\cdot) > 0$ and $\lim_{h\to 0}\phi'(h) = \infty$. Characterize the symmetric (Nash) equilibrium. Compare it with the Pareto optimal symmetric outcome. Interpret.

11.D.1 Suppose that $h_1 = h_2 = \cdots h_I \equiv h^*$ is a symmetric Nash equilibrium of the simultaneous move game. The it must be that for every i, h^* solves

$$\underset{h_i}{\text{Max}}\ \phi\left(\frac{h_i}{\frac{1}{I}h_i + \frac{I-1}{I}h^*}\right) - \frac{h_i^2}{2}$$

which yields the FOC

$$\phi'\left(\frac{h_i}{\left(\frac{1}{I}h_i + \frac{I-1}{I}h^*\right)}\right)\cdot\frac{\left(\frac{1}{I}h_i + \frac{I-1}{I}h^*\right) - h_i\cdot\frac{1}{I}}{\left(\frac{1}{I}h_i + \frac{I-1}{I}h^*\right)^2} - h_i = 0\,.$$

For h^* to be a NE this should hold for $h_i = h^*$, i.e., $\phi'(1)\cdot\dfrac{\frac{I-1}{I}\cdot h^*}{(h^*)^2} - h^* = 0$,

which yields $h^* = \left((1 - \frac{1}{I})\cdot\phi'(1)\right)$.

Now, let $h_1 = h_2 = \cdots h_I \equiv h^o$ be a symmetric Pareto optimum. The utility of each student i will be $u_i = \phi(1) - \dfrac{(h^o)^2}{2}$, which immediately implies that $h^o = 0$ is the Pareto optimal outcome. The intuition is simple; by studying, every student imposes a negative externality on others, and the competitive outcome has too much studying.

We have: I students

h_i : hours of work of student i

$\dfrac{h_i^2}{2}$: disutility of work

$\phi\left(\dfrac{h_i}{\tilde{h}}\right)$: benefits with $\phi'(\cdot) > 0$ and $\lim_{h\to 0}\phi'(h) = -\infty$

$\tilde{h} = \dfrac{\sum_i h_i}{I}$: average number of hours of all students

Find: Nash equilibrium and Pareto optimal outcome

Suppose that $h_1 = h_2 = \dots = h^*$ is a symmetric Nash equilibrium.

Then, for every i, h^* solves:

$$\max_{\{h_i\}} \underbrace{\phi\left(\frac{h_i}{\tilde{h}}\right)}_{\text{benefits}} - \underbrace{\frac{h_i^2}{2}}_{\text{disutility}} = \phi\left(\frac{h_i}{\frac{\sum_i h_i}{I}}\right) - \frac{h_i^2}{2}$$

$$= \phi\left(\frac{h_i}{\frac{h_i + (I-1)h^*}{I}}\right) - \frac{h_i^2}{2}$$

FOC: (h_i): $\phi'\left(\frac{h_i}{\frac{h_i}{I} + \frac{(I-1)h^*}{I}}\right)\left[\frac{\left(\frac{I-1}{I}\right)h^*}{\frac{h_i}{I} + \left(\frac{I-1}{I}\right)h^*}\right] - h_i = 0$

$u = h_i$ $\qquad u' = 1$

$v = \frac{h_i}{I} + \frac{(I-1)h^*}{I}$ $\qquad v' = \frac{1}{I}$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} = \frac{\cancel{\frac{h_i}{I}} + \frac{(I-1)h^*}{I} - \cancel{\frac{h_i}{I}}}{\left(\frac{h_i}{I} + \frac{(I-1)h^*}{I}\right)^2}$$

$$= \frac{\frac{(I-1)h^*}{I}}{\left(\frac{h_i}{I} + \frac{(I-1)h^*}{I}\right)^2}$$

$$\Leftrightarrow \quad \phi'\left(\frac{h_i}{\frac{h_i}{I} + \left(\frac{I-1}{I}\right)h^*}\right)\left(\frac{I-1}{I}\right)h^* = h_i\left[\frac{h_i}{I} + \left(\frac{I-1}{I}\right)h^*\right]^2$$

For h^* to be a NE we need: $\quad h_i = h^*$ so:

$$\phi'\Bigg(\underbrace{\frac{h^*}{\frac{h^*}{I} + \left(\frac{I-1}{I}\right)h^*}}_{\frac{h^*}{\frac{\cancel{I}h^*}{\cancel{I}}} = 1}\Bigg)\left(\frac{I-1}{I}\right)\cancel{h^*} = \cancel{h^*}\Bigg[\underbrace{\frac{h^*}{I} + \left(\frac{I-1}{I}\right)h^*}_{\frac{\cancel{I}h^*}{\cancel{I}} = h^*}\Bigg]^2$$

$$\Leftrightarrow \quad \phi'(1)\left(\frac{I-1}{I}\right) = (h^*)^2 \quad \Leftrightarrow \quad h^* = \left[\phi'(1)\left(\frac{I-1}{I}\right)\right]^{\frac{1}{2}}$$

Symmetric Nash Equilibrium:

$$h^* = \left[\phi'(1)\left(\frac{I-1}{I}\right)\right]^{\frac{1}{2}}$$

Let $h_1 = h_2 = \dots = h^\theta$ be the symmetric Pareto optimum:

$$\max_{\{h^\theta\}} \quad u_i = \phi\left(\frac{h^\theta}{\frac{h^\theta}{I} + \left(\frac{I-1}{I}\right)h^\theta}\right) - \frac{(h^\theta)^2}{2}$$

$$= \phi(1) - \frac{(h^\theta)^2}{2}$$

FOC: (h^θ): $\quad -h^\theta \leq 0 \quad$ with equality if $h^\theta > 0$

so $h^\theta = 0$

Pareto optimal symmetric outcome:

$h^\theta = 0$

Comparing NE with Pareto optimal:

$$h^* = \left[\underbrace{\phi'(1)}_{>0} \underbrace{\left(\frac{I-1}{I}\right)}_{>0} \right]^{\frac{1}{2}} > 0 \qquad h^{\Theta} = 0$$

because $\phi'(\cdot) > 0$

so $h^* > h^{\Theta}$

The intuition for this result is as follows:

- The benefits from studying depend on how well each student performs relative to their peers.
- Each student therefore has an incentive to "outperform" their peers.
- By studying, every student imposes a negative externality on others.
- Because each student faces the same incentive, they will all study "too much" relative to what the collective best (which in this case is given by the Pareto optimal) will dictate.
- Therefore, the competitive outcome has too much studying relative to the Pareto optimal.