Name:______________________

Stochastic Calculus Midterm Exam

Fall 2022

Important Note: The exam time is from 3:30PM to 5:30PM. You will have 15 minutes after the exam (until 5:45PM) to scan/take picture of your answers to email to the dedicated email address (please use copy-paste to insure no typing errors):

`msqfeconometrics2015@gmail.com`

Don't forget to cc yourself and check if the file is attached!!!

Note: *Please provide the details in all your solutions.*

Problem 1 [Brownian Motion][20 points]. Let W_t be a standard BM with the initial condition $W_0 = 0$ where $t \in [0, T]$. Set

$$Z_t = W_t - \frac{t}{T} W_T.$$

(i). Find the distribution of Z_t $(0 < t < T)$.

(ii). For $0 < s < t < T$, find the covariance $Cov(Z_s, Z_t)$.

(iii). For $0 < s < t < T$, what is the joint distribution for (Z_s, Z_t)? Note: you can give the joint density, or give the name of the distribution and all the necessary parameters. [Note: extra space provided on next page].

[This is extra space for the previous problem]

Problem 2. [Total/QuadraticVariation] [20]. What are the Total Variation and Quadratic Variation for deterministic function $f(x) = -x^4$ over the interval $[-3, 3]$? **Hint:** You can write down the answer with brief reasons by graphing the function (no rigorus derivations are required). We talked about this in class so DO NOT spend more than 10 minutes on this problem.

Problem 3 [Stochastic Integrals] [20points]. Let W_t be a standard BM starting from $W_0 = 0$. Using Ito's formula to compute the following stochastic integrals/Expectations (Riemann integrals are considered done):

(1).

$$\int_0^t [W_s]^2 \, dW_s,$$

(2).

$$\int_0^t (s^2 + e^{W_s}) dW_s$$

Problem 4 [Ito Isometry][20 points] Find the variance of Y_t defined by

$$Y_t = \int_0^t (s + [W_s]^2) dW_s.$$

Problem 5 [Ito Formula] [10 points]. Let X_t be the following Ito process

$$dX_t = (W_t - [W_t]^2)dt + 3\sqrt{W_t}dW_t, \text{ with } X_0 = 2$$

Apply Ito's formula to $Y_t = X_t^3$ to express Y_t as a stochastic integral (plus Riemann integral).

Problem 6 [Job Interview Question] [10 points]. Compute the covariance of W_t and $\int_0^t W_s ds$:

$$Cov\left(W_t, \int_0^t W_s ds\right).$$

Hint: This integral is actually a Riemann Integral, as you have seem in the HW. Use definition of covariance $Cov\left(X,Y\right) = \mathbb{E}\left[XY\right] - \left(\mathbb{E}X\right)\left(\mathbb{E}Y\right)$ with $X = W_t$ and $Y = \int_0^t W_s ds$. And you can evaluate the relevant moments for answer this question. The problem is easier than it appears (as usually happens in job interviews).