S+B Exercise 23.2

Find n eigenvalues + related eigenvectors:

(a) $M_a = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

eigenvalues

$$\text{DeT}\left(M_a - \lambda I_2\right) = 0$$

$$\text{DeT}\left(M_a - \lambda I\right) = (3-\lambda)(5-\lambda) = 0$$

$$15 - 8\lambda + \lambda^2 = 0$$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{64 - 4(1)15}}{2} = 4 \pm 1$$

$$\lambda_1 = 3 \quad \lambda_2 = 5$$

Note: M_a is a lower triangular matrix and it's diagonal elements are equal to it's eigenvalues.

~~eigen~~

eigenvectors: For $\lambda_1 = 3$

$$M_a V = 3V$$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 3V_{11} \\ 3V_{21} \end{bmatrix}$$

$$3V_{11} = 3V_{11}$$

$$4V_{11} + 5V_{21} = 3V_{21}$$

$$2V_{21} = -4V_{11}$$

$$V_{21} = -2V_{11}$$

So the eigenvector associated w/ $\lambda_1 = 3$ is

$\begin{bmatrix} V_{11} \\ -2V_{11} \end{bmatrix}$. ~~Normalize By Dividing~~

$$\frac{1}{V_{11}} \begin{bmatrix} V_{11} \\ -2V_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Normalize

The eigenvector associated w/ $\lambda_1 = 3$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

~~For $\lambda_2 = 5$~~

$$\cancel{\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = 5 \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}}$$

For $\lambda_2 = 5$

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = 5 \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$$

$$3V_{12} = 5V_{12}$$
$$4V_{12} + 5V_{22} = 5V_{22}$$

From $3V_{12} = 5V_{12}$ $\quad V_{12} = 0$

This gives $5V_{22} = 5V_{22}$

So eigenvector is $\begin{bmatrix} 0 \\ V_{22} \end{bmatrix}$

Normalize $\quad \dfrac{1}{V_{22}} \begin{bmatrix} 0 \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The eigenvector associated with $\lambda_2 = 5$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note Well: ~~For~~ The elements of an eigenvector are unique only up to a multiplicative constant, so we are free to normalize.

23.2 (b) $M_b = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$

eigenvalues $\det(M_b - r I_2) = 0$

$$\det(M_b - rI) = (-1-r)(4-r) + 6 = 0$$

$$-4 - 3r + r^2 + 6 = 0$$

$$r^2 - 3r + 2 = 0$$

$$r_{1,2} = \frac{3 \pm \sqrt{(3)^2 - 4(1)2}}{2} = \frac{3}{2} \pm \frac{1}{2}$$

$$r_1 = 1, \quad r_2 = 2$$

eigenvectors

For $r_2 = 2$

$M_b V = r \cdot V$, $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 2v_{11} \\ 2v_{12} \end{bmatrix}$

$-v_{11} + 3v_{12} = 2v_{11}$

$-2v_{11} + 4v_{12} = 2v_{12}$

From the first Eqn $v_{12} = v_{11}$

From the Second $-2v_{11} = -2v_{12}$, so eigenvector $\begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix}$.

Normalize $\frac{1}{v_{11}} \begin{bmatrix} v_{11} \\ v_{11} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The eigenvector Associated with $r_2 = 2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $r_1 = 1$

$M_b \cdot V = 1 \cdot V.$ $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$ $\quad -V_{12} + 3V_{22} = V_{12}$
$-2V_{12} + 4V_{22} = V_{22}$

so $V_{12} = \frac{3}{2} V_{22}$. eigen vector is $\begin{bmatrix} \frac{3}{2} V_{22} \\ V_{22} \end{bmatrix}$

Normalize $\frac{2}{3} V_{22}^{-1} \begin{bmatrix} \frac{3}{2} V_{22} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$ (or $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$).

The eigen vector Associated w/ $r_1 = 1$ is $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

23.2 (c) $M_c = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

eigen values $\mathrm{Det}(M_c - rI) = 0$

$$\mathrm{Det}\begin{bmatrix} -r & -2 \\ 1 & -3-r \end{bmatrix} = -r(-3-r) + 2 = 3r + r^2 + 2 = 0$$

$$r^2 + 3r + 2 = 0 \qquad r_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4(1)2}}{2} = \frac{-3}{2} \pm \frac{1}{2}$$

$r_1 = -1 \qquad r_2 = -2$

↘

Scanned with CamScanner

eigen vectors   for $r_1 = -1$

$$M_c \cdot V = -1 V. \quad \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} -V_{11} \\ -V_{12} \end{bmatrix}$$

$$-2V_{12} = -V_{11}$$
$$V_{11} - 3V_{12} = -V_{12}$$

so $V_{12} = \frac{1}{2} V_{11}$. $V = \begin{bmatrix} V_{11} \\ \frac{1}{2} V_{11} \end{bmatrix}$

Normalize: eigenvector $= \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ (or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$).

For $r_2 = -2$

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} -2V_{12} \\ -2V_{22} \end{bmatrix}$$

$$-2V_{22} = -2V_{12}$$
$$V_{12} - 3V_{22} = -2V_{22}$$

so $V_{12} = V_{22}$.

eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

23.2 ⓓ $Md = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ $\text{Det}\left(Md - r I_3\right) = 0$

$$\text{Det}\begin{bmatrix} -r & 0 & -2 \\ 0 & (7-r) & 0 \\ 1 & 0 & (-3-r) \end{bmatrix} = -r(7-r)(-3-r) + 0 + 0 + 2(7-r) - 0 - 0$$

$$= -r(-21 - 4r + r^2) + 14 - 2r$$

$$= -r^3 + 21r + 4r^2 + 14 - 2r = -r^3 + 4r^2 - 19r + 14 = 0$$

$\cancel{= -r^3 + 22r}$

$$-r^3+4r^2-19r+14=0$$

$$(r+1)\left[-r^2+14+5r\right]=0$$

$$(r+1)(-r+7)(r+2)=0$$

$r_1=-1 \quad r_2=-2 \;.\; r_3=7$ are the eigenvalues

eigenvectors

For $r_1=-1$

$$Md\cdot v=-1v. \begin{bmatrix}0 & 0 & -2\\ 0 & 7 & 0\\ 1 & 0 & -3\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\\ v_{13}\end{bmatrix}=\begin{bmatrix}-v_{11}\\ -v_{12}\\ -v_{13}\end{bmatrix}$$

~~So v_{13}~~ So $-2v_{13}=-v_{11}$ or $v_{13}=\frac{1}{2}v_{11}$

and $7v_{12}=-v_{12}$ so $v_{12}=0$

and $v_{11}-3v_{13}=-v_{13}$ so $v_{13}=\frac{1}{2}v_{11}$

~~So~~ eigenvector $=\begin{bmatrix}v_{11}\\ 0\\ \frac{1}{2}v_{11}\end{bmatrix}$ Normalize $\begin{bmatrix}1\\ 0\\ \frac{1}{2}\end{bmatrix}$ (or $\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$).

For $r_2 = -2$

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} -2v_{21} \\ -2v_{22} \\ -2v_{23} \end{bmatrix}$$

$-2V_{23} = -2V_{21}$ so $V_{23} = V_{21}$

$7V_{22} = -2V_{22}$ so $V_{22} = 0$

$V_{21} - 3V_{23} = -2V_{23}$ So $V_{23} = V_{21}$

eigenvector $\begin{bmatrix} V_{21} \\ 0 \\ V_{21} \end{bmatrix}$ Normalized $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For $r_3 = 7$

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 7v_{31} \\ 7v_{32} \\ 7v_{33} \end{bmatrix}$$

$-2V_{33} = 7V_{31}$

$7V_{32} = 7V_{32}$

$V_{31} - 3V_{33} = 7V_{33}$

use $\boxed{V_{33} = \frac{-7}{2}V_{31}}$ to get $V_{31} = 10\left(\frac{-7}{3}\right)V_{33}$ or $\boxed{V_{33} = \frac{-3}{70}V_{31}}$

Thus $V_{33} = V_{31} = 0$

eigen vector is $\begin{bmatrix} 0 \\ V_{32} \\ 0 \end{bmatrix}$. Normalized $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

S+B Exercise 23.5 Begin from $(A-rI)v=\vec{0}$

Premultiply both sides by A^{-1}

$$A^{-1}(A-rI)v = A^{-1}\vec{0}$$

$(A-\gamma I)V = 0$

or $(I - rA^{-1})v = \vec{0}$ Multiply through by $\frac{-1}{r}$

$AV = \gamma V$

$$\left(\frac{-1}{r}I + A^{-1}\right)V = \vec{0}$$

$V = A^{-1}\gamma V$

$A^{-1}\gamma V - V = 0$

or $\left(A^{-1} - \frac{1}{r}I\right)v = \vec{0}$

$A^{-1}V - \frac{1}{\gamma}V = 0$

$\left(A^{-1} - \frac{1}{\gamma}I\right)V = 0$

If r is eigenvalue of A, $\frac{1}{r}$ is ~~eiege~~ an eigenvalue of A^{-1}.

QED

S+B Exercise 23.7 For each matrix A find P and $D \ni P^{-1}AP = D$.

(a) $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ The eigenvalues of A are $r_1 = 3$ and $r_2 = 2$.

Eigenvector Associated w/ $r_1 = 3$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 3v_{11} \\ 3v_{21} \end{bmatrix}$$

(12)

$3v_{11} = 3v_{11}$

$v_{11} + 2v_{21} = 3v_{21}$ so $v_{21} = +v_{11}$

(12)

and eigenvector is $\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$

(12)

↓

Eigenvector Associated w/ $r_2 = 2$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \underset{21}{v_{12}} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 2\underset{21}{v_{12}} \\ 2v_{22} \end{bmatrix} \qquad 3\underset{21}{v_{12}} = 2\underset{21}{v_{12}} \qquad v_{12} = 0$$

$$\underset{21}{v_{12}} + 2v_{22} = 2v_{22} \qquad v_{22} = v_{22}$$

So eigenvector is $\begin{bmatrix} 0 \\ v_{22} \end{bmatrix}$. Normalized $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$D = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \quad \text{So} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} v_{11} & \underset{21}{v_{12}} \\ v_{21}^{12} & v_{22} \end{bmatrix} \quad \text{so} \quad P = \begin{bmatrix} 1 & 0 \\ +1 & 1 \end{bmatrix}$$

Note $P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Can easily verify $P^{-1}AP = D$.

Exercise 23.7 (b) $\qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$

eigenvalues Solve $\quad -r(5-r) + 1 = 0 \quad$ or $\boxed{r^2 - 5r + 1 = 0}$ (1)

$$r_{1,2} = \frac{5 \pm \sqrt{25-4}}{2} \qquad r_1 = \frac{5 - \sqrt{21}}{2}, \quad r_2 = \frac{5 + \sqrt{21}}{2}$$

eigenvectors: For $r_1 = \dfrac{5-\sqrt{21}}{2}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = r_1 \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$$

$v_{21} = r_1 v_{11}$ (2)

$-v_{11} + 5v_{21} = r_1 v_{21}$

or $v_{11} = (5 - r_1) v_{21}$ (3)

Use (2) in (3) to get $v_{11} = (5r_1 - r_1^2) v_{11}$ and note from (1) that $(5r_1 - r_1^2) = 1$. So (3) just says $v_{11} = v_{11}$.

Using (2) we have the eigenvector $\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5-\sqrt{21}}{2} \end{bmatrix}$

For $r_2 = \dfrac{5+\sqrt{21}}{2}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} r_2 v_{12} \\ r_2 v_{22} \end{bmatrix}$$

$v_{22} = r_2 v_{12}$ (4)

$-v_{12} + 5v_{22} = r_2 v_{22}$

or $v_{12} = (5 - r_2) v_{22}$ (5)

Same argument as above gives that the eigenvector is $\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5+\sqrt{21}}{2} \end{bmatrix}$

So $D = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} \frac{5-\sqrt{21}}{2} & 0 \\ 0 & \frac{5+\sqrt{21}}{2} \end{bmatrix}$

AND $P = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{5-\sqrt{21}}{2} & \frac{5+\sqrt{21}}{2} \end{bmatrix}$

Note Also $P^{-1} = \left(\frac{1}{r_2 - r_1}\right) \begin{bmatrix} r_2 & -1 \\ -r_1 & 1 \end{bmatrix}$

<u>S+B Exercise 23.13</u> Prove Theorem 23.4 (page 592)

Note That The Theorem HAS 2 PARTS; The Proposition + its Converse.

Proposition: if $P = [v_1 \; v_2 \cdots v_k]$ Then $P^{-1}AP = D$.

All notation As in Text.

<u>Proof</u>: By Definition of eigenvalues + eigenvectors

$$\underset{(k\times k)}{A} \; \underset{(k\times 1)}{V_i} = \underset{(1\times 1)}{r_i} \; \underset{(k\times 1)}{V_i} \qquad \text{for } i = 1, 2, \ldots k.$$

$\searrow$

That is
$$\underset{(K\times K)}{A}\ \underset{(K\times 1)}{V_1} = \underset{(1\times 1)}{\Gamma_1}\ \underset{(K\times 1)}{V_1}$$
$$A V_2 = \Gamma_2 V_2$$
$$\vdots$$
$$A V_K = \Gamma_K V_K$$

HORIZONTAL CONCATENATION gives

$$\underset{(K\times K)}{\begin{bmatrix} \underset{(K\times 1)}{Av_1} & \underset{(K\times 1)}{Av_2} & \cdots & \underset{(K\times 1)}{Av_K} \end{bmatrix}} = \begin{bmatrix} \underset{(K\times 1)}{V_1\Gamma_1} & \underset{(K\times 1)}{V_2\Gamma_2} \cdots & \underset{(K\times 1)}{V_K\Gamma_K} \end{bmatrix} \text{ or}$$

$$A\begin{bmatrix} V_1 & V_2 & \cdots & V_K \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & \cdots & V_K \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \Gamma_K \end{bmatrix}$$

Thus

$A P = P D.$ PREMULTIPLY BOTH SIDES BY P^{-1} TO get (NOTE P^{-1} exists By Assumption)

$$P^{-1} A P = D$$ QED PROPOSITION

Conversely if $P^{-1}AP = D$ the columns of P must be eigenvectors and diagonal entries of D must be eigenvalues

Let $P \equiv \begin{bmatrix} \underset{(K\times 1)}{\phi_1} & \underset{(K\times 1)}{\phi_2} & \cdots & \underset{(K\times 1)}{\phi_K} \end{bmatrix}$ and $D \equiv \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_K \end{bmatrix}$

Premultiply both sides of $P^{-1}AP = D$ by P to get

$AP = PD$ or

$$A\begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_K \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_K \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & d_K \\ 0 & \cdots & & \end{bmatrix}$$

or column by column

$$A\phi_1 = d_1\phi_1$$
$$A\phi_2 = d_2\phi_2$$
$$\vdots$$
$$A\phi_K = d_3\phi_K$$

✓ since these satisfy the defn of eigenvectors and eigenvalues we have that the d_i are the eigenvalues of A and the ϕ_i are the associated eigenvectors.

QED Converse

S+B Exercise 23.29 Verify eqn (39)

Text Page 611

$$P^{-1}AP = \begin{bmatrix} 1+3i & 0 \\ 0 & 1-3i \end{bmatrix} \qquad (39)$$

where $P = \begin{bmatrix} 1 & 1 \\ 3i & -3i \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6}i \\ \frac{1}{2} & \frac{1}{6}i \end{bmatrix}$

AND $A = \begin{bmatrix} 1 & 1 \\ -9 & 1 \end{bmatrix}$.

To verify

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{6}i \\ \frac{1}{2} & \frac{1}{6}i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -9 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3i & -3i \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2} + \frac{9}{6}i & \frac{1}{2} - \frac{1}{6}i \\ \frac{1}{2} - \frac{9}{6}i & \frac{1}{2} + \frac{1}{6}i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3i & -3i \end{bmatrix} =$$

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{9}{6}i\right)+\left(\frac{1}{2}-\frac{1}{6}i\right)3i & \left(\frac{1}{2}+\frac{9}{6}i\right)-\left(\frac{1}{2}-\frac{1}{6}i\right)3i \\ \left(\frac{1}{2}-\frac{9}{6}i\right)+\left(\frac{1}{2}+\frac{1}{6}i\right)3i & \left(\frac{1}{2}-\frac{9}{6}i\right)-\left(\frac{1}{2}+\frac{1}{6}i\right)3i \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}+\frac{9}{6}i+\frac{3}{2}i-\frac{3}{6}i^2\right) & \left(\frac{1}{2}+\frac{9}{6}i-\frac{3}{2}i+\frac{3}{6}i^2\right) \\ \left(\frac{1}{2}-\frac{9}{6}i+\frac{3}{2}i+\frac{3}{6}i^2\right) & \left(\frac{1}{2}-\frac{9}{6}i-\frac{3}{2}i-\frac{3}{6}i^2\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1+3i & 0 \\ 0 & 1-3i \end{bmatrix}$$ This verifies (39)

S+B Exercise 23.48: If A and B are p.d. symmetric matrices Prove That $A+B$ is p.d. symmetric.

Clearly A and B are conformable (They have the same Dimension) Let Both Be $K \times K$.

Since A is p.d. Then for ANY $x \in \mathbb{R}^K \ni x \neq 0$, $x'Ax > 0$. The Same Argument gives $x'Bx > 0$. Thus $x'Ax + x'Bx > 0$. Since $x'Ax + x'Bx = x'(A+B)x$, it follows That $x'(A+B)x > 0 \;\; \forall \; x \in \mathbb{R}^K \ni x \neq 0$ and Therefore $(A+B)_{(K \times K)}$ is p.d.

To Show symmetry Let $C_{(K \times K)} = A + B$ and let a_{ij}, b_{ij}, and c_{ij} denote The i,j th element of A, B, and C, respectively. Symmetry of A and B implies that $a_{ij} = a_{ji}$ $\forall i \neq j$ and that $b_{ij} = b_{ji}$ $\forall i \neq j$. Since $c_{ij} = a_{ij} + b_{ij}$ it follows that $c_{ij} = c_{ji}$ $\forall i \neq j$ and thus $C = A+B$ is symmetric.

S+B Exercise 16.1 (a-f)

(a) $M = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Det $M = 2-1 = 1 > 0$
So both roots same sign

Trace $M = 3 > 0$ so at least one root is positive.

Therefore both roots are positive. M is p.d.

(b) $M = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

det $M_1 = -3$
det $M = 15 - 16 = -1$

So M is indefinite.

Alternatively, Det $M < 0$ so the roots are of opposite sign.
M is indefinite.

(c) $M = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

Det $M = 2 > 0$
So both roots same sign

Trace $M = -9$ so at least one root is negative

Therefore both roots are negative. M is N.d.

(d) $M = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

Det $M = 0$ so at least one root is zero
Trace $M = 10$ so at least one root is positive.

$r_1 = 0$ $\quad r_2 = 10$ $\qquad$ M is pos. semi definite

(e) $M = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

Det $M = -25 < 0$

So no roots are zero and since the product of the roots is negative either all three are negative or two are positive and one is negative.

Trace $= 11 > 0$ So at least one root is positive.

Therefore, there are 2 positive roots and one negative root

M is indefinite

(f) $M = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Eigenvalues solve $\text{Det}(M - rI) = 0$ or

$$\text{Det}\begin{bmatrix} (-1-r) & 1 & 0 \\ 1 & (-1-r) & 0 \\ 0 & 0 & (-2-r) \end{bmatrix} = -\left[(1+r)(1+r)(2+r)\right] + (2+r) = 0$$

or $(2+r) - \left[(r^2+2r+1)(2+r)\right] = (2+r) - \left[2+r+2r^2+r^3+4r+2r^2\right] = 0$

or $(-1)\left[r^3+4r^2+4r\right] = (-r)\left[r^2+4r+4\right] = 0$

or $(-r)(r+2)(r+2) = 0$. Roots: $r_1 = 0, r_2 = -2, r_3 = -2$

So M is Neg. Semi. Definite

Problem 1

C+W 17.2, #4:

(a) $y_{t+1} + 3y_t = 4$ $\qquad y_0 = 4$

(i) $y_s = -3y_s + 4$

$4y_s = 4$

$y_s = 1$

(ii) $y_{t+1} = -3y_t + 4$

$(y_{t+1} - 1) = -3y_t - 3 = -3(y_t - 1)$

$Z_{t+1} = -3(Z_t)$

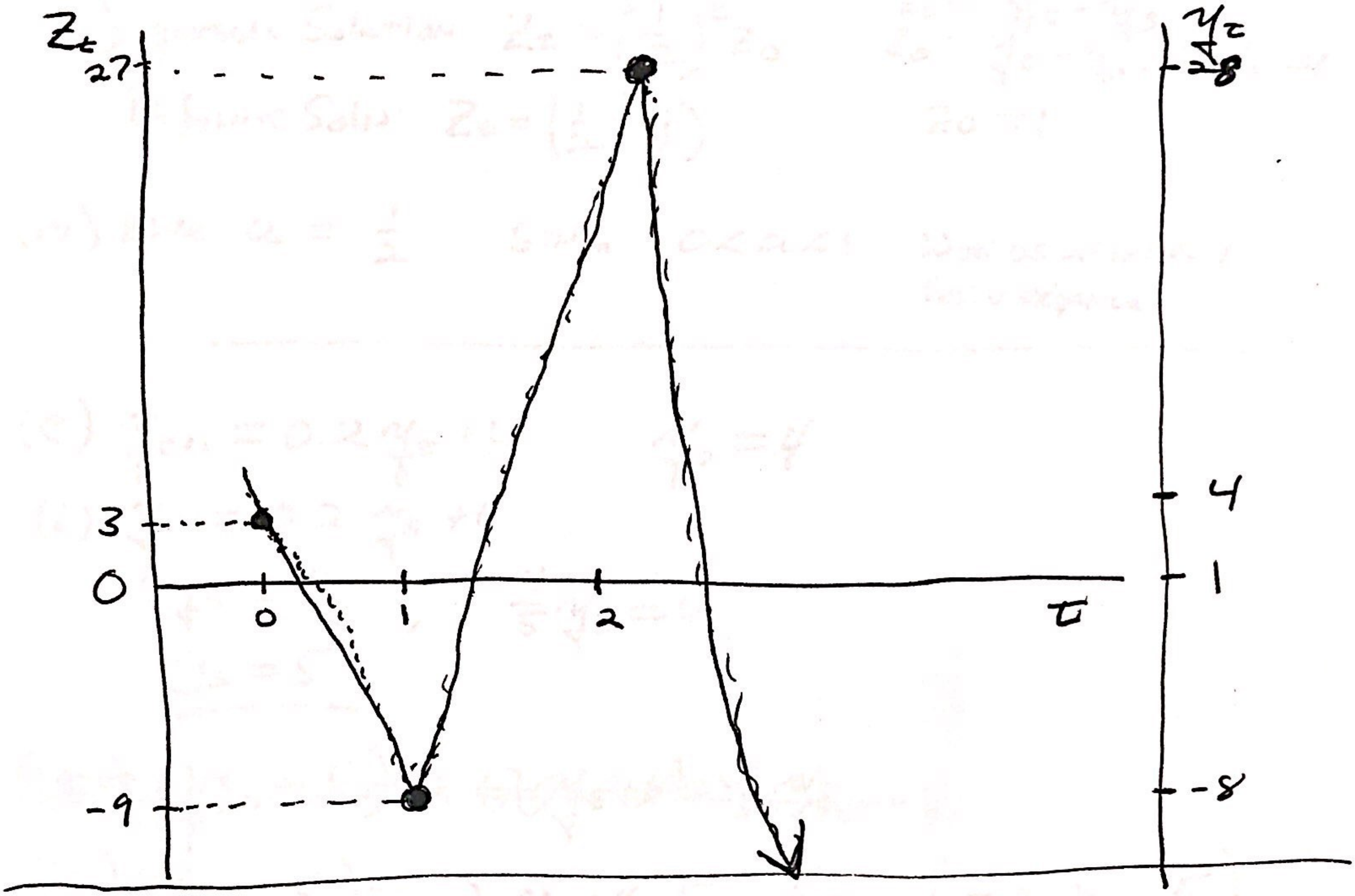
(iii) genl Soln $\quad Z_t = (-3)^t Z_0$ $\qquad Z_0 = y_0 - y_s$

def. Soln $\quad Z_t = (-3)^t(3)$ $\qquad Z_0 = (4-1) = 3$

(iv) Here $a = -3$

Since $a < 0$ $\quad Z_t$ oscillates (Thus y_t oscillates)

Since $|a| > 1$ $\quad Z_t$ diverges (Thus y_t diverges)



(b) $\underline{2y_{t+1} - y_t = 6}$ $\qquad y_0 = 7$

(i) $y_{t+1} = \frac{1}{2}y_t + 3$

$y_s = \frac{1}{2}y_s + 3$

$\frac{1}{2}y_s = 3$

$\underline{y_s = 6}$

(ii) $(y_{t+1} - 6) = \frac{1}{2}y_t - 3 = \frac{1}{2}(y_t - 6)$

$z_{t+1} = \frac{1}{2}z_t$

(iii) general Solution: $Z_t = \left(\frac{1}{2}\right)^t Z_0$ $\quad$ $Z_t = y_t - y_s$
$Z_0 = y_0 - y_s = 7 - 6 = 1$

Definite Soln: $Z_t = \left(\frac{1}{2}\right)^t (1)$ $\quad$ $Z_0 = 1$

(iv) Here $a = \frac{1}{2}$ Since $0 < a < 1$ Nonoscillatory Convergence

---

(e) $y_{t+1} = 0.2\, y_t + 4$ $\quad$ $y_0 = 4$

(i) $y_s = 0.2\, y_s + 4$

$.8 y_s = 4$, $\frac{4}{5} y_s = 4$,

$\underline{y_s = 5}$

~~(ii) $(y_{t+1} - 5)$ ... $\frac{1}{5} y_t$~~

(ii) $(y_{t+1} - 5) = .2\, y_t + 4 - 5 = \frac{1}{5} y_t - 1 = \frac{1}{5}(y_t - 5)$

$Z_{t+1} = \frac{1}{5} Z_t$

$\underline{Z_{t+1} = 0.2\, Z_t}$

(iii) gen'l Soln: $Z_t = (0.2)^t Z_0$ $\quad$ $Z_t = y_t - y_s$

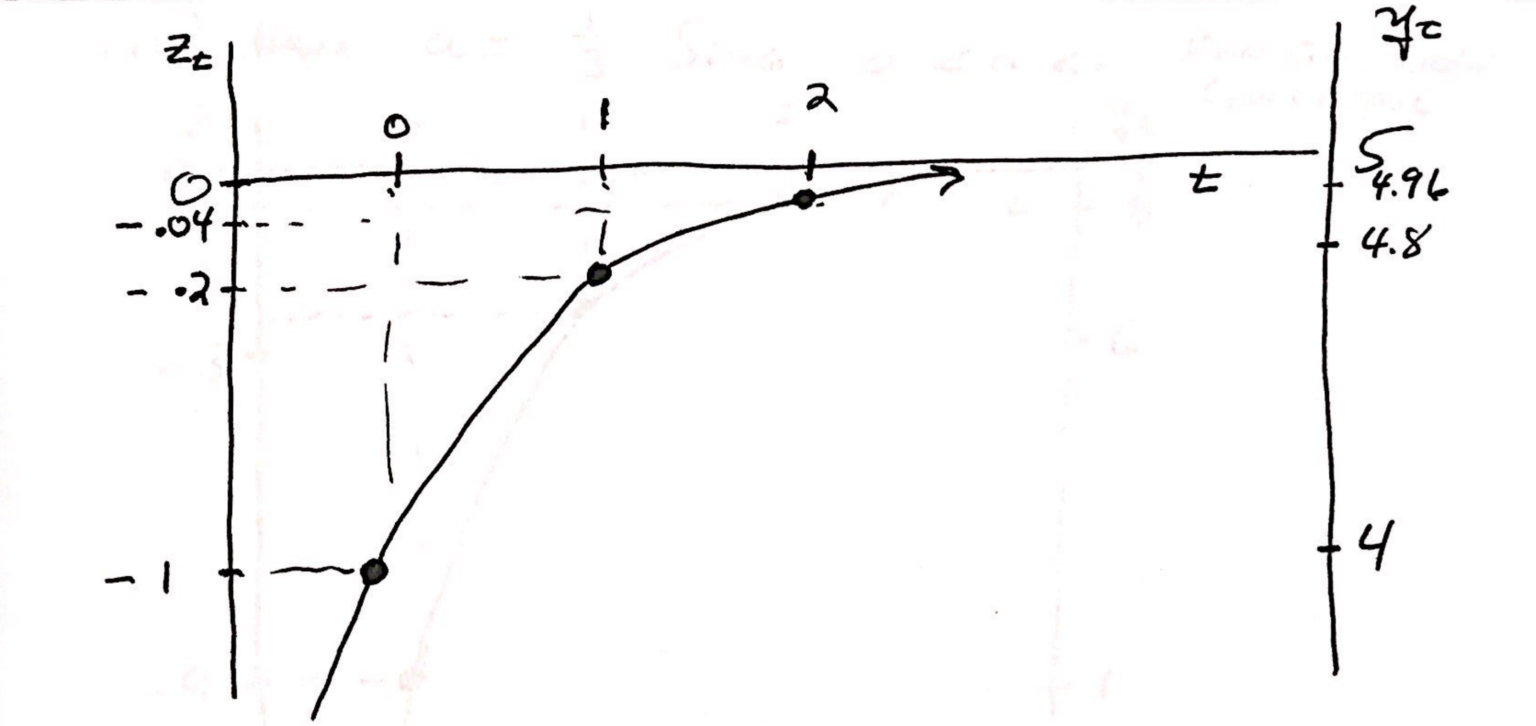
definite Soln: $Z_t = (0.2)^t (-1)$ $\quad$ $Z_0 = y_0 - y_s$

$Z_0 = 4 - 5 = -1$

(iv) Here $a = 0.2$

Since $0 < a < 1$

Nonoscillatory (monotonic) convergence



C+W 17.3, #3

(a) $y_{t+1} - \frac{1}{3}y_t = 6$ $\qquad y_0 = 1$

(i) $y_s = \frac{1}{3}y_s + 6$

$\frac{2}{3}y_s = 6$

$y_s = 9$

(ii) $y_{t+1} = \frac{1}{3}y_t + 6$

$(y_{t+1} - 9) = \frac{1}{3}y_t - 3 = \frac{1}{3}(y_t - 9)$

$z_{t+1} = \frac{1}{3}(z_t)$

(iii) gen'l Soln: $z_t = \left(\frac{1}{3}\right)^t z_0$

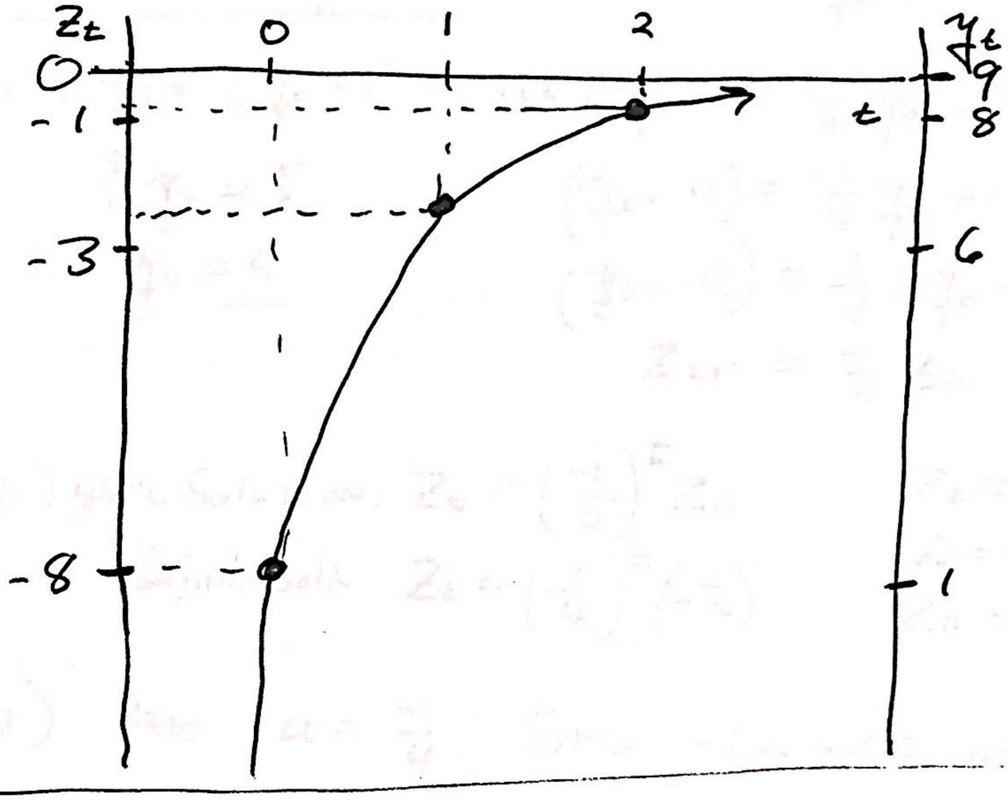
$z_t = y_t - y_s$

$z_0 = y_0 - y_s$

$z_0 = 1 - 9 = -8$

Definite Soln: $z_t = \left(\frac{1}{3}\right)^t (-8)$

(iv) Here $a = \frac{1}{3}$. Since $0 < a < 1$ Nonoscillatory Convergence



(b) $y_{t+1} + 2y_t = 9$ $\qquad$ $y_0 = 4$

(i) $y_s = -2y_s + 9$

$3y_s = 9$

$y_s = 3$

(ii) $y_{t+1} = -2y_t + 9$

$y_{t+1} - 3 = -2y_t + 6 = -2(y_t - 3)$

$z_{t+1} = -2z_t$

(iii) Gen'l Soln: $z_t = (-2)^t z_0$

Def Soln: $z_t = (-2)^t (1)$

$z_t = y_t - y_s$

$z_0 = y_0 - y_s$

$z_0 = 4 - 3 = 1$

(iv) Here $a = -2$

Since $a < -1$ Oscillatory Divergence

(c) $y_{t+1} + \frac{1}{4} y_t = 5$ $\qquad y_0 = 2$

(i) $y_s = \frac{-1}{4} y_s + 5$

$\frac{5}{4} y_s = 5$

$y_s = 4$

(ii) $y_{t+1} = \frac{-1}{4} y_t + 5$

$(y_{t+1} - 4) = \frac{-1}{4} y_t + 1$

$(y_{t+1} - 4) = \frac{-1}{4} (y_t - 4)$

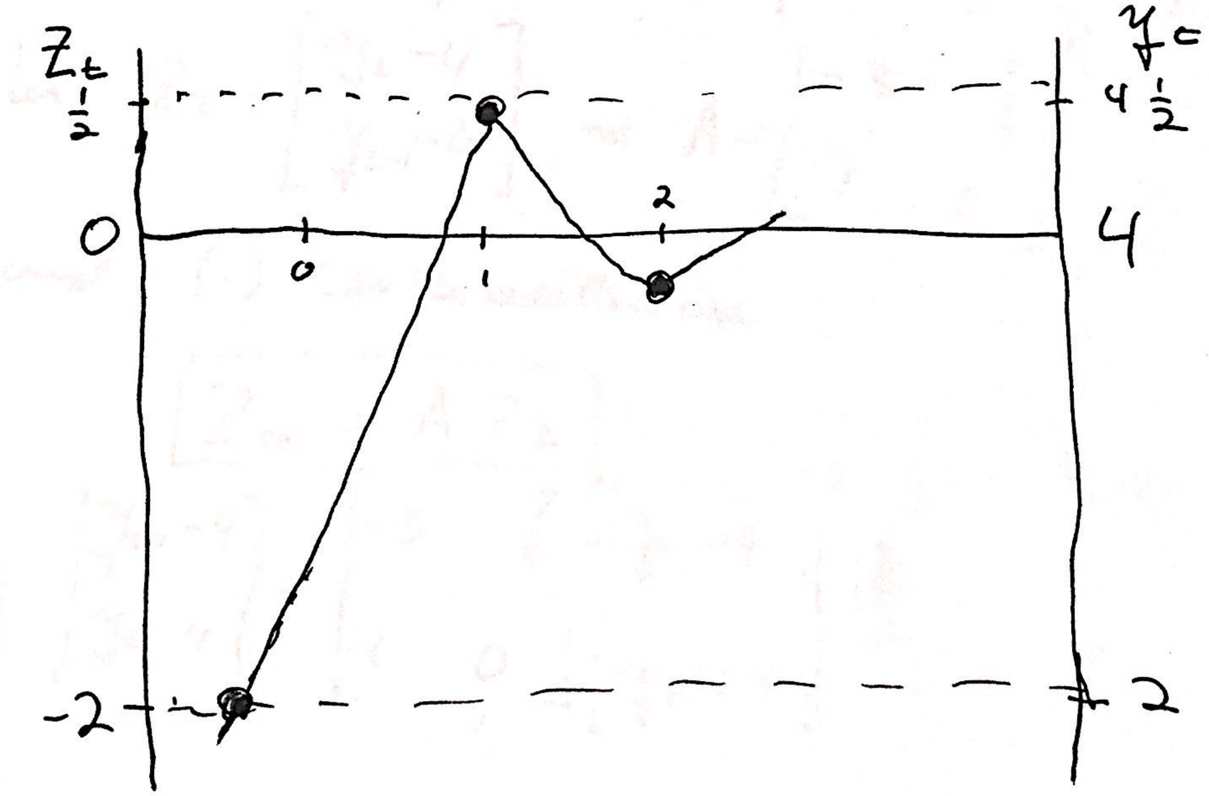
$Z_{t+1} = \frac{-1}{4} Z_t$

(iii) gen'l Solution; $Z_t = \left(\frac{-1}{4}\right)^t Z_0$ $\qquad Z_t = y_t - y_s$

Definite SolN: $Z_t = \left(\frac{-1}{4}\right)^t (-2)$ $\qquad Z_0 = y_0 - y_s$

$Z_0 = 2 - 4 = -2$

(iv) Here $a = \frac{-1}{4}$. Since $-1 < a < 0$ oscillatory converge



PROBLEM 2 C+W 18.1, # 4

(a) $y_{t+1} + 3y_t - \frac{7}{4}y_{t-1} = 9$ $y_0 = 3,\ y_{-1} = 1$

(i) $y_s + 3y_s - \frac{7}{4}y_s = 9$

$\frac{9}{4}y_s = 9$

$\underline{\underline{y_s = 4}}$

(ii) $y_{t+1} = -3y_t + \frac{7}{4}y_{t-1} + 9$

$(y_{t+1} - 4) = -3y_t + \frac{7}{4}y_{t-1} + 5$

$(y_{t+1} - 4) = -3y_t + \frac{7}{4}y_{t-1} + 12 - 7$

$(y_{t+1} - 4) = -3y_t + 12 + \frac{7}{4}y_{t-1} - 7$

$(y_{t+1} - 4) = -3(y_t - 4) + \frac{7}{4}(y_{t-1} - 4)$ (1)

Let $Z_t = \begin{bmatrix} y_t - 4 \\ y_{t-1} - 4 \end{bmatrix}$ and $A = \begin{bmatrix} -3 & \frac{7}{4} \\ 1 & 0 \end{bmatrix}$

and (1) can be written as

$$\boxed{Z_{t+1} = A Z_t}$$

$$\begin{bmatrix} y_{t+1} - 4 \\ y_t - 4 \end{bmatrix} = \begin{bmatrix} -3 & \frac{7}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - 4 \\ y_{t-1} - 4 \end{bmatrix}$$

(iii) The general Solution is $Z_t = A^t Z_0$

$$Z_0 = \begin{bmatrix} y_0 - y_s \\ y_{-1} - y_s \end{bmatrix} = \begin{bmatrix} 3-4 \\ 1-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

The Definite Soln is $Z_t = A^t \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

(b) $\underline{y_{t+1} - 2y_t + 2y_{t-1} = 1}$ $\quad y_0 = 4,\ y_{-1} = 3$

(i) $y_s - 2y_s + 2y_s = 1$

$\underline{y_s = 1}$

(ii) $y_{t+1} = 2y_t - 2y_{t-1} + 1$

$(y_{t+1} - 1) = 2y_t - 2y_{t-1} - 2 + 2$

$(y_{t+1} - 1) = 2(y_t - 1) - 2(y_{t-1} - 1)$ $\quad (2)$

let $Z_t = \begin{bmatrix} y_t - 1 \\ y_{t-1} - 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$

and (2) can be written as

$$\boxed{Z_{t+1} = A Z_t}$$

$$\begin{bmatrix} y_{t+1} - 1 \\ y_t - 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - 1 \\ y_{t-1} - 1 \end{bmatrix}$$

(iii) The general SolN: $Z_t = A^t (Z_0)$

$$Z_0 = \begin{bmatrix} y_0 - y_s \\ y_{-1} - y_s \end{bmatrix} = \begin{bmatrix} 4 - 1 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Definite SolN: $Z_t = A^t \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(c) $y_{t+1} - y_t + \frac{1}{4} y_{t-1} = 2$ $\qquad y_0 = 7, \; y_{-1} = 4$

(i) $y_s - y_s + \frac{1}{4} y_s = 2$

$$\frac{1}{4} y_s = 2$$

$$\underline{y_s = 8}$$

(ii) $y_{t+1} = y_t - \frac{1}{4} y_{t-1} + 2$

$$(y_{t+1} - 8) = y_t - \frac{1}{4} y_{t-1} - 8 + 2$$

$$(y_{t+1} - 8) = (y_t - 8) - \frac{1}{4}(y_{t-1} - 8) \qquad (3)$$

let $Z_t = \begin{bmatrix} y_t - 8 \\ y_{t-1} - 8 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & \frac{-1}{4} \\ 1 & 0 \end{bmatrix}$ and (3) CAN be

written as $\quad Z_{t+1} = A Z_t$

$$\begin{bmatrix} y_{t+1} - 8 \\ y_t - 8 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-1}{4} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t - 8 \\ y_{t-1} - 8 \end{bmatrix}$$

Problem 3.) C+W 16.2, #6. Use Euler relations to show

(a) $e^{-i\pi} = -1$

Since $e^{-i\theta} = \cos\theta - i\sin\theta$

$$e^{-i\pi} = \cos\pi - i\sin\pi = (-1) - i(0)$$

so $e^{-i\pi} = -1$

(b) $e^{i\pi/3} = \frac{1}{2}(1+\sqrt{3}\,i)$

Since $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$$

so $e^{i\frac{\pi}{3}} = \frac{1}{2}(1+\sqrt{3}\,i)$

(c) $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1+i)$

Since $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

so $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1+i)$

(d) $e^{-3i\left(\frac{\pi}{4}\right)} = -\frac{\sqrt{2}}{2}(1+i)$

Since $e^{-i\theta} = \cos\theta - i\sin\theta$

$$e^{-i\left(\frac{3\pi}{4}\right)} = \cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} - i\left(\frac{1}{\sqrt{2}}\right)$$

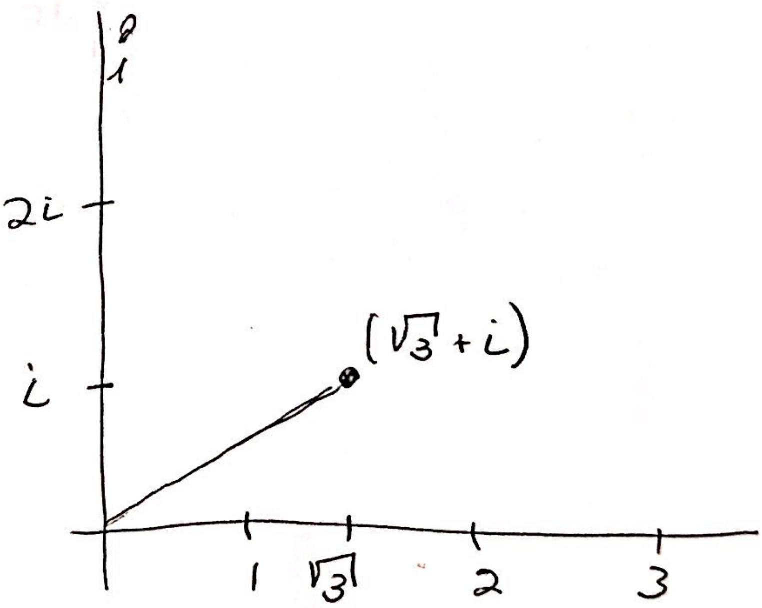
So $e^{-i\left(\frac{3\pi}{4}\right)} = \frac{-\sqrt{2}}{2}(1+i)$

Problem 4. C+W 16.2 #7. Find the Cartesian form of each. Graph (a) and (c).

(a) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\quad\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$$



(b) $4e^{i(\pi/3)}$

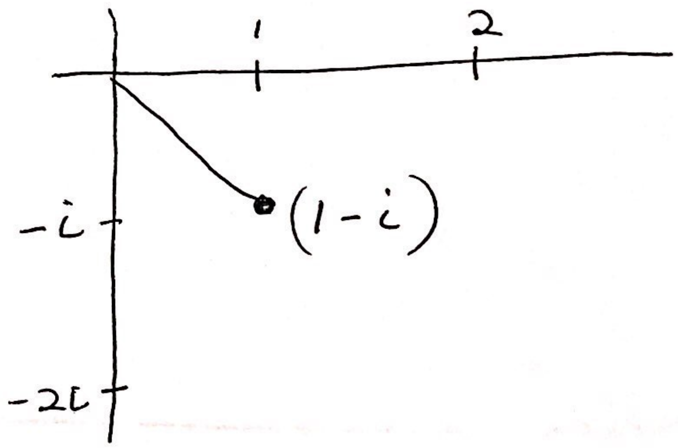
$$e^{i(\pi/3)} = \cos\frac{\pi}{3} + i\sin\pi/3 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

so $4e^{i(\pi/3)} = 2\sqrt{3} + 2i$

(c) $\sqrt{2}\, e^{-i(\pi/4)}$

$$e^{-i(\pi/4)} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

so $\sqrt{2}\, e^{-i(\pi/4)} = 1 - i$

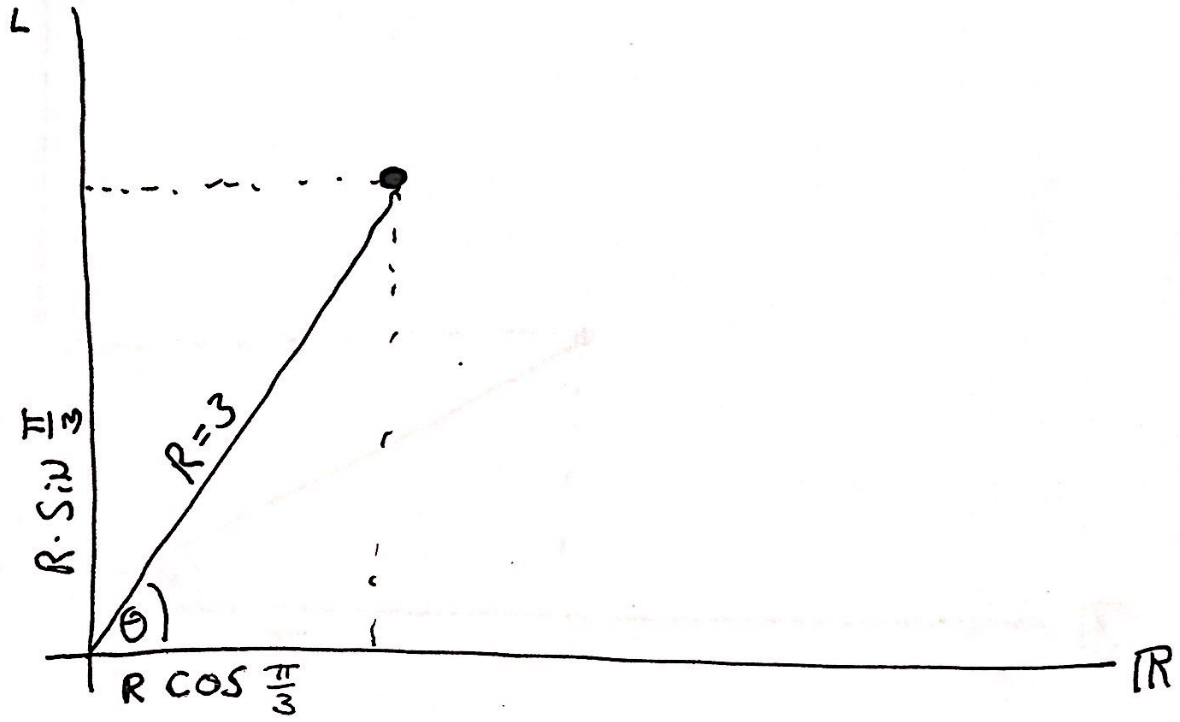


PROBLEM 5 C+W 16.2 #8. FIND THE EXPONENTIAL + POLAR FORM.
GRAPH THE POLAR FORM.

(a) $\frac{3}{2} + \frac{3\sqrt{3}}{2} i = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$= 3\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$ POLAR

$= 3e^{i(\pi/3)}$ EXPONENTIAL

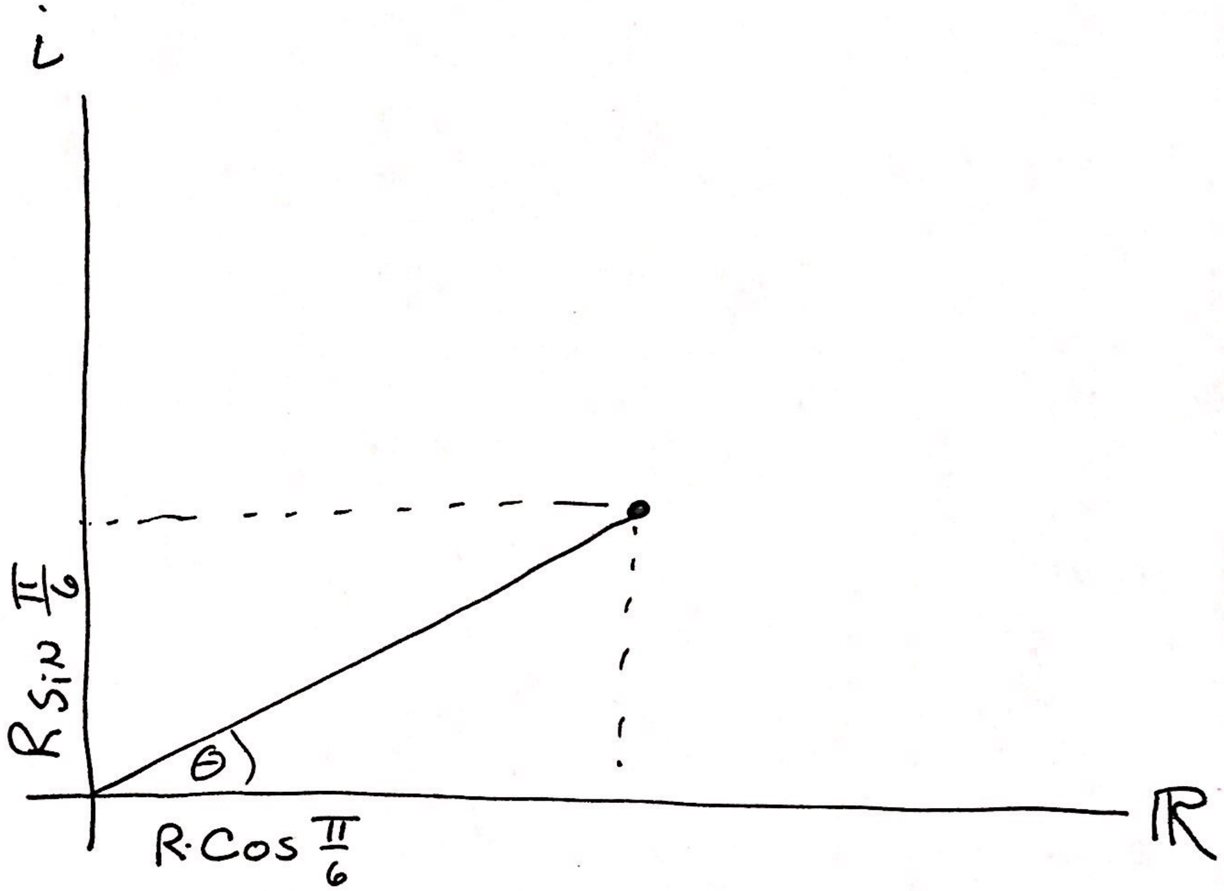


$\Theta = \underset{\text{RADIANS}}{\frac{\pi}{3}} = \underset{\text{Degrees}}{60^\circ}$

(b) $4(\sqrt{3}+i) = 8\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$$= 8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \qquad \text{Polar}$$

$$= 8e^{i\frac{\pi}{6}} \qquad \text{Exponential}$$



	Radians		Degrees
$\theta =$	$\frac{\pi}{6}$	$=$	30°