Homework 2

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QF8915 - Stochastic Calculus

Due on Nov 22, 2022

Problem1

Find the conditional density of $W(t)$ given $W(t) = y$

Solution:

First, we know the joint density of $f_{W(s)W(t)}(x, y)$ be:

$$
\begin{aligned}
f_{W(s)W(t)}(x,y) &= \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \cdot \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{y^2}{2(t-s)}} \\
&= \frac{1}{2\pi\sqrt{s(t-s)}} e^{-(\frac{x^2}{2s} + \frac{y^2}{2(t-s)})}
\end{aligned}
$$

Thus, the conditional density would be:

$$
\begin{aligned}
f_{W(s)|W(t)} &= \frac{f_{W(s)W(t)}(x,y)}{f_{W(t)}(y)} \\
&= \frac{\frac{1}{2\pi\sqrt{s(t-s)}} e^{-(\frac{x^2}{2s} + \frac{y^2}{2(t-s)})}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}} \\
&= \frac{\sqrt{t}}{\sqrt{2\pi s(t-s)}} e^{-(\frac{x^2}{2s} + \frac{y^2}{2(t-s)} - \frac{y^2}{2t})}
\end{aligned}
$$

Problem2

Show that $W(t)^3 - 3tW(t)$ is a martingale.

Solution:

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We assume $0 \le s < t$

$$\begin{aligned} E[W(t)^3 - 3tW(t)|\mathcal{F}_s] &= E[(W(t)-W(s)+W(s))^3 - 3tW(t)|\mathcal{F}_s] \\ &= E[(W(t)-W(s))^3 + 3(W(t)-W(s))^2W(s) + 3(W(t)-W(s))W(s)^2 + W(s)^3 - 3 \\ &= 0 + 3(t-s)W(s) + 0 + W(s)^3 - 3tW(s) \\ &= W(s)^3 - 3sW(s) \end{aligned}$$

Thus, It's a martingale.

Problem3

Show that $\int_0^t sdW(t) = tW(t) - \int_0^t W(s)ds$.

Solution:

$$\begin{aligned} \int_0^t sdW(t) &= \lim \sum_{j-1} s_j \Delta W_j \\ &= \lim \sum_j (s_jW_j - s_{j-1}W_{j-1}) - W_j(s_j - s_{j-1}) \\ &= \lim \sum_j \Delta(s_jW_j) - \sum_j W_j \Delta s_j \\ &= tW(t) - \int_0^t W(s)ds \end{aligned}$$

Problem4

Find the distribution of Z, where $Z(\omega) = \int_0^1 W(\omega, t)dt$

Solution:

$\int_0^1 W(\omega,t)dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} W(\frac{i}{n}) \cdot \frac{1}{n}$ Thus,

$$E(\int_0^1 W(\omega,t)dt) = E(\lim_{n\to\infty} \sum_{i=0}^{n-1} W(\frac{i}{n}) \cdot \frac{1}{n}) = 0$$

$Var(Z(\omega)) = E(\lim_{n\to\infty} \sum_{i=0}^{n-1} W(\frac{i}{n}) \cdot \frac{1}{n})^2 = \frac{1}{n^2} \cdot \frac{1}{n}(\frac{(n-1)n}{2}) = \frac{n-1}{2n^2} = 0?$

So the $Z(\omega)$ is normal distribution with mean and variance as above.

Problem5

Find the distribution for $\int_0^T e^t dW(t)$.

Solution:

Since $\int_0^T e^t dW(t) = \sum_{t=0}^T -1 e^t (W(t) - W(t-1))$, and as $W(t)$ is normal distribution, the limit of normal distribution is still normal.

$$
\begin{aligned}
E \lim \sum_{t=0}^{T-1} e^{t-1}(W(t) - W(t-1)) &= \lim \sum E(e^{t-1}) E(W(t) - W(t-1)) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
var(\int_0^T e^t dW(t)) &= E(\int_0^T e^t dW(t))^2 \\
&= \int_0^T e^{t \cdot 2} dt \\
&= \frac{1}{2} e^{2t} |_0^T \\
&= \frac{1}{2}(e^{2T} - 1)
\end{aligned}
$$

Thus, the distribution is $\mathcal{N}(0, \frac{1}{2}(e^{2T} - 1))$

Problem6

Find mean and variance for $\int_0^T W(t)^3 dW(t)$.

Solution:

$$
\begin{aligned}
E(\int_0^T W(t)^3 dW(t)) &= \int_0^T E(W(t)^3) dW(t) \\
&= \lim \sum E(W(t))^3 E(W(t) - W(t-1)) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
var(\int_0^T w(t)^3 dW(t)) &= E(\int_0^T w(t)^3 dW(t))^2 \\
&= \int_0^T W(t)^6 dt \\
&= a
\end{aligned}
$$

a is a number we can get from the last question of HW1, but unfortunately, i didn't derive the result successfuly while doing hw1.

Problem7(Bonus Question)

Show that $\int_0^t W(s)^2 dW(s) = \frac{1}{3} W(t)^3 - \int_0^t W(t) ds$.

Solution:

$$\int_0^t W(s)^2 dW(s) = \lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i)^2[W(t_{i+1}) - W(t_i)]$$

From quesiton3, $\int_0^t W(t)ds = tW(t) - \int_0^t sdW(t)$ So $\frac{1}{3}W(t)^3 - \int_0^t W(t)ds = \frac{1}{3}W(t)^3 - tW(t) + \int_0^t sdW(t)$ Not done yet, my quesiton is how to deal with the LHS, and compute $W(t)^2$ integral. Check with homework solution later.

Problem8(Bonus Question)

Find the distribution of $\int_0^T g(s)dW(t)$.

Solution:

$$\begin{aligned} E(\int_0^T g(s)dW(t)) &= E(\lim \sum g(s)(W(t) - W(t-1)) \\ &= 0 \end{aligned}$$

$$\begin{aligned} Var(\int_0^T g(s)dW(t)) &= E(\int_0^T g(s)dW(t))^2 \\ &= \int_0^T g(s)^2 ds \end{aligned}$$

Thus, the distribution is $\mathcal{N}(0, \int_0^T g(s)^2 ds)$

Note: I have a few questions about this problem. If s is unrelevant to T, then this variance becomes a constant number. But I guess there is hidden infomration $0 \le s \le T$.