CHAPTER

1

Preference and Choice

1.A Introduction

In this chapter, we begin our study of the theory of individual decision making by considering it in a completely abstract setting. The remaining chapters in Part I develop the analysis in the context of explicitly economic decisions.

The starting point for any individual decision problem is a *set of possible (mutually exclusive) alternatives* from which the individual must choose. In the discussion that follows, we denote this set of alternatives abstractly by X. For the moment, this set can be anything. For example, when an individual confronts a decision of what career path to follow, the alternatives in X might be: {go to law school, go to graduate school and study economics, go to business school, . . . , become a rock star}. In Chapters 2 and 3, when we consider the consumer's decision problem, the elements of the set X are the possible consumption choices.

There are two distinct approaches to modeling individual choice behavior. The first, which we introduce in Section 1.B, treats the decision maker's tastes, as summarized in her *preference relation*, as the primitive characteristic of the individual. The theory is developed by first imposing rationality axioms on the decision maker's preferences and then analyzing the consequences of these preferences for her choice behavior (i.e., on decisions made). This preference-based approach is the more traditional of the two, and it is the one that we emphasize throughout the book.

The second approach, which we develop in Section 1.C, treats the individual's choice behavior as the primitive feature and proceeds by making assumptions directly concerning this behavior. A central assumption in this approach, the *weak axiom of revealed preference*, imposes an element of consistency on choice behavior, in a sense paralleling the rationality assumptions of the preference-based approach. This choice-based approach has several attractive features. It leaves room, in principle, for more general forms of individual behavior than is possible with the preference-based approach. It also makes assumptions about objects that are directly observable (choice behavior), rather than about things that are not (preferences). Perhaps most importantly, it makes clear that the theory of individual decision making need not be based on a process of introspection but can be given an entirely behavioral foundation.

Understanding the relationship between these two different approaches to modeling individual behavior is of considerable interest. Section 1.D investigates this question, examining first the implications of the preference-based approach for choice behavior and then the conditions under which choice behavior is compatible with the existence of underlying preferences. (This is an issue that also comes up in Chapters 2 and 3 for the more restricted setting of consumer demand.)

For an in-depth, advanced treatment of the material of this chapter, see Richter (1971).

1.B Preference Relations

In the preference-based approach, the objectives of the decision maker are summarized in a *preference relation*, which we denote by $\succsim$. Technically, $\succsim$ is a binary relation on the set of alternatives X, allowing the comparison of pairs of alternatives $x, y \in X$. We read $x \succsim y$ as "x is at least as good as y." From $\succsim$, we can derive two other important relations on X:

(i) The *strict preference* relation, $\succ$, defined by

$$x \succ y \iff x \succsim y \text{ but not } y \succsim x$$

and read "x is preferred to y."[1]

(ii) The *indifference* relation, $\sim$, defined by

$$x \sim y \iff x \succsim y \text{ and } y \succsim x$$

and read "x is indifferent to y."

In much of microeconomic theory, individual preferences are assumed to be *rational*. The hypothesis of rationality is embodied in two basic assumptions about the preference relation $\succsim$: *completeness* and *transitivity*.[2]

Definition 1.B.1: The preference relation $\succsim$ is *rational* if it possesses the following two properties:

(i) *Completeness:* for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both).

(ii) *Transitivity:* For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

The assumption that $\succsim$ is complete says that the individual has a well-defined preference between any two possible alternatives. The strength of the completeness assumption should not be underestimated. Introspection quickly reveals how hard it is to evaluate alternatives that are far from the realm of common experience. It takes work and serious reflection to find out one's own preferences. The completeness axiom says that this task has taken place: our decision makers make only meditated choices.

Transitivity is also a strong assumption, and it goes to the heart of the concept of

1. The symbol $\iff$ is read as "if and only if." The literature sometimes speaks of $x \succsim y$ as "x is weakly preferred to y" and $x \succ y$ as "x is strictly preferred to y." We shall adhere to the terminology introduced above.

2. Note that there is no unified terminology in the literature; *weak order* and *complete preorder* are common alternatives to the term *rational preference relation*. Also, in some presentations, the assumption that $\succsim$ is *reflexive* (defined as $x \succsim x$ for all $x \in X$) is added to the completeness and transitivity assumptions. This property is, in fact, implied by completeness and so is redundant.

rationality. Transitivity implies that it is impossible to face the decision maker with a sequence of pairwise choices in which her preferences appear to cycle: for example, feeling that an apple is at least as good as a banana and that a banana is at least as good as an orange but then also preferring an orange over an apple. Like the completeness property, the transitivity assumption can be hard to satisfy when evaluating alternatives far from common experience. As compared to the completeness property, however, it is also more fundamental in the sense that substantial portions of economic theory would not survive if economic agents could not be assumed to have transitive preferences.

The assumption that the preference relation $\succsim$ is complete and transitive has implications for the strict preference and indifference relations $\succ$ and $\sim$. These are summarized in Proposition 1.B.1, whose proof we forgo. (After completing this section, try to establish these properties yourself in Exercises 1.B.1 and 1.B.2.)

Proposition 1.B.1: If $\succsim$ is rational then:

(i) $\succ$ is both *irreflexive* ($x \succ x$ never holds) and *transitive* (if $x \succ y$ and $y \succ z$, then $x \succ z$).

(ii) $\sim$ is *reflexive* ($x \sim x$ for all x), *transitive* (if $x \sim y$ and $y \sim z$, then $x \sim z$), and *symmetric* (if $x \sim y$, then $y \sim x$).

(iii) if $x \succ y \succsim z$, then $x \succ z$.

The irreflexivity of $\succ$ and the reflexivity and symmetry of $\sim$ are sensible properties for strict preference and indifference relations. A more important point in Proposition 1.B.1 is that rationality of $\succsim$ implies that both $\succ$ and $\sim$ are transitive. In addition, a transitive-like property also holds for $\succ$ when it is combined with an at-least-as-good-as relation, $\succsim$.

An individual's preferences may fail to satisfy the transitivity property for a number of reasons. One difficulty arises because of the problem of *just perceptible differences*. For example, if we ask an individual to choose between two very similar shades of gray for painting her room, she may be unable to tell the difference between the colors and will therefore be indifferent. Suppose now that we offer her a choice between the lighter of the two gray paints and a slightly lighter shade. She may again be unable to tell the difference. If we continue in this fashion, letting the paint colors get progressively lighter with each successive choice experiment, she may express indifference at each step. Yet, if we offer her a choice between the original (darkest) shade of gray and the final (almost white) color, she would be able to distinguish between the colors and is likely to prefer one of them. This, however, violates transitivity.

Another potential problem arises when the manner in which alternatives are presented matters for choice. This is known as the *framing* problem. Consider the following example, paraphrased from Kahneman and Tversky (1984):

> Imagine that you are about to purchase a stereo for 125 dollars and a calculator for 15 dollars. The salesman tells you that the calculator is on sale for 5 dollars less at the other branch of the store, located 20 minutes away. The stereo is the same price there. Would you make the trip to the other store?

It turns out that the fraction of respondents saying that they would travel to the other store for the 5 dollar discount is much higher than the fraction who say they would travel when the question is changed so that the 5 dollar saving is on the stereo. This is so even though the ultimate saving obtained by incurring the inconvenience of travel is the same in both cases.[3] Indeed, we would expect indifference to be the response to the following question:

> Because of a stockout you must travel to the other store to get the two items, but you will receive 5 dollars off on either item as compensation. Do you care on which item this 5 dollar rebate is given?

If so, however, the individual violates transitivity. To see this, denote

x = Travel to the other store and get a 5 dollar discount on the calculator.
y = Travel to the other store and get a 5 dollar discount on the stereo.
z = Buy both items at the first store.

The first two choices say that $x \succ z$ and $z \succ y$, but the last choice reveals $x \sim y$. Many problems of framing arise when individuals are faced with choices between alternatives that have uncertain outcomes (the subject of Chapter 6). Kahneman and Tversky (1984) provide a number of other interesting examples.

At the same time, it is often the case that apparently intransitive behavior can be explained fruitfully as the result of the interaction of several more primitive rational (and thus transitive) preferences. Consider the following two examples

(i) A household formed by Mom (M), Dad (D), and Child (C) makes decisions by majority voting. The alternatives for Friday evening entertainment are attending an opera (O), a rock concert (R), or an ice-skating show (I). The three members of the household have the rational individual preferences: $O \succ_M R \succ_M I$, $I \succ_D O \succ_D R$, $R \succ_C I \succ_C O$, where $\succ_M$, $\succ_D$, $\succ_C$ are the transitive individual strict preference relations. Now imagine three majority-rule votes: O versus R, R versus I, and I versus O. The result of these votes (O will win the first, R the second, and I the third) will make the household's preferences $\succsim$ have the intransitive form: $O \succ R \succ I \succ O$. (The intransitivity illustrated in this example is known as the *Condorcet paradox*, and it is a central difficulty for the theory of group decision making. For further discussion, see Chapter 21.)

(ii) Intransitive decisions may also sometimes be viewed as a manifestation of a change of tastes. For example, a potential cigarette smoker may prefer smoking one cigarette a day to not smoking and may prefer not smoking to smoking heavily. But once she is smoking one cigarette a day, her tastes may change, and she may wish to increase the amount that she smokes. Formally, letting y be abstinence, x be smoking one cigarette a day, and z be heavy smoking, her initial situation is y, and her preferences in that initial situation are $x \succ y \succ z$. But once x is chosen over y and z, and there is a change of the individual's current situation from y to x, her tastes change to $z \succ x \succ y$. Thus, we apparently have an intransitivity: $z \succ x \succ z$. This *change-of-tastes* model has an important theoretical bearing on the analysis of addictive behavior. It also raises interesting issues related to commitment in decision making [see Schelling (1979)]. A rational decision maker will anticipate the induced change of tastes and will therefore attempt to tie her hand to her initial decision (Ulysses had himself tied to the mast when approaching the island of the Sirens).

It often happens that this change-of-tastes point of view gives us a well-structured way to think about *nonrational* decisions. See Elster (1979) for philosophical discussions of this and similar points.

Utility Functions

In economics, we often describe preference relations by means of a *utility function*. A utility function $u(x)$ assigns a numerical value to each element in X, ranking the

3. Kahneman and Tversky attribute this finding to individuals keeping "mental accounts" in which the savings are compared to the price of the item on which they are received.

elements of X in accordance with the individual's preferences. This is stated more precisely in Definition 1.B.2.

Definition 1.B.2: A function $u: X \to \mathbb{R}$ is a *utility function representing preference relation* $\succsim$ if, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Note that a utility function that represents a preference relation $\succsim$ is not unique. For any strictly increasing function $f: \mathbb{R} \to \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function representing the same preferences as $u(\cdot)$; see Exercise 1.B.3. It is only the ranking of alternatives that matters. Properties of utility functions that are invariant for any strictly increasing transformation are called *ordinal*. *Cardinal* properties are those not preserved under all such transformations. Thus, the preference relation associated with a utility function is an ordinal property. On the other hand, the numerical values associated with the alternatives in X, and hence the magnitude of any differences in the utility measure between alternatives, are cardinal properties.

The ability to represent preferences by a utility function is closely linked to the assumption of rationality. In particular, we have the result shown in Proposition 1.B.2.

Proposition 1.B.2: A preference relation $\succsim$ can be represented by a utility function only if it is rational.

Proof: To prove this proposition, we show that if there is a utility function that represents preferences $\succsim$, then $\succsim$ must be complete and transitive.

Completeness. Because $u(\cdot)$ is a real-valued function defined on X, it must be that for any $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. But because $u(\cdot)$ is a utility function representing $\succsim$, this implies either that $x \succsim y$ or that $y \succsim x$ (recall Definition 1.B.2). Hence, $\succsim$ must be complete.

Transitivity. Suppose that $x \succsim y$ and $y \succsim z$. Because $u(\cdot)$ represents $\succsim$, we must have $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Therefore, $u(x) \geq u(z)$. Because $u(\cdot)$ represents $\succsim$, this implies $x \succsim z$. Thus, we have shown that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$, and so transitivity is established. ■

At the same time, one might wonder, can *any* rational preference relation $\succsim$ be described by some utility function? It turns out that, in general, the answer is no. An example where it is not possible to do so will be discussed in Section 3.G. One case in which we can always represent a rational preference relation with a utility function arises when X is finite (see Exercise 1.B.5). More interesting utility representation results (e.g., for sets of alternatives that are not finite) will be presented in later chapters.

1.C Choice Rules

In the second approach to the theory of decision making, choice behavior itself is taken to be the primitive object of the theory. Formally, choice behavior is represented by means of a *choice structure*. A choice structure $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family (a set) of nonempty subsets of X; that is, every element of $\mathscr{B}$ is a set $B \subset X$. By analogy with the consumer theory to be developed in Chapters 2 and 3, we call the elements $B \in \mathscr{B}$ *budget sets*. The budget sets in $\mathscr{B}$ should be thought of as an exhaustive listing of all the choice experiments that the institutionally, physically, or otherwise restricted social situation can conceivably pose to the decision maker. It need not, however, include all possible subsets of X. Indeed, in the case of consumer demand studied in later chapters, it will not.

(ii) $C(\cdot)$ is a *choice rule* (technically, it is a correspondence) that assigns a nonempty set of chosen elements $C(B) \subset B$ for every budget set $B \in \mathscr{B}$. When $C(B)$ contains a single element, that element is the individual's choice from among the alternatives in B. The set $C(B)$ may, however, contain more than one element. When it does, the elements of $C(B)$ are the alternatives in B that the decision maker *might* choose; that is, they are her *acceptable alternatives* in B. In this case, the set $C(B)$ can be thought of as containing those alternatives that we would actually see chosen if the decision maker were repeatedly to face the problem of choosing an alternative from set B.

Example 1.C.1: Suppose that $X = \{x, y, z\}$ and $\mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$. One possible choice structure is $(\mathscr{B}, C_1(\cdot))$, where the choice rule $C_1(\cdot)$ is: $C_1(\{x, y\}) = \{x\}$ and $C_1(\{x, y, z\}) = \{x\}$. In this case, we see x chosen no matter what budget the decision maker faces.

Another possible choice structure is $(\mathscr{B}, C_2(\cdot))$, where the choice rule $C_2(\cdot)$ is: $C_2(\{x, y\}) = \{x\}$ and $C_2(\{x, y, z\}) = \{x, y\}$. In this case, we see x chosen whenever the decision maker faces budget $\{x, y\}$, but we may see either x or y chosen when she faces budget $\{x, y, z\}$. ■

When using choice structures to model individual behavior, we may want to impose some "reasonable" restrictions regarding an individual's choice behavior. An important assumption, the weak axiom of revealed preference [first suggested by Samuelson; see Chapter 5 in Samuelson (1947)], reflects the expectation that an individual's observed choices will display a certain amount of consistency. For example, if an individual chooses alternative x (and only that) when faced with a choice between x and y, we would be surprised to see her choose y when faced with a decision among x, y, and a third alterative z. The idea is that the choice of x when facing the alternatives $\{x, y\}$ reveals a proclivity for choosing x over y that we should expect to see reflected in the individual's behavior when faced with the alternatives $\{x, y, z\}$.[4]

The weak axiom is stated formally in Definition 1.C.1.

Definition 1.C.1: The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* if the following property holds:

If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

In words, the weak axiom says that if x is ever chosen when y is available, then there can be no budget set containing both alternatives for which y is chosen and x is not.

4. This proclivity might reflect some underlying "preference" for x over y but might also arise in other ways. It could, for example, be the result of some evolutionary process.

Note how the assumption that choice behavior satisfies the weak axiom captures the consistency idea: If $C(\{x, y\}) = \{x\}$, then the weak axiom says that we cannot have $C(\{x, y, z\}) = \{y\}$.[5]

A somewhat simpler statement of the weak axiom can be obtained by defining a *revealed preference relation* $\succsim^*$ from the observed choice behavior in $C(\cdot)$.

Definition 1.C.2: Given a choice structure $(\mathscr{B}, C(\cdot))$ the *revealed preference relation* $\succsim^*$ is defined by

$$x \succsim^* y \Leftrightarrow \text{there is some } B \in \mathscr{B} \text{ such that } x, y \in B \text{ and } x \in C(B).$$

We read $x \succsim^* y$ as "x is revealed at least as good as y." Note that the revealed preference relation $\succsim^*$ need not be either complete or transitive. In particular, for any pair of alternatives x and y to be comparable, it is necessary that, for some $B \in \mathscr{B}$, we have $x, y \in B$ and either $x \in C(B)$ or $y \in C(B)$, or both.

We might also informally say that "x is revealed preferred to y" if there is some $B \in \mathscr{B}$ such that $x, y \in B$, $x \in C(B)$, and $y \notin C(B)$, that is, if x is ever chosen over y when both are feasible.

With this terminology, we can restate the weak axiom as follows: "*If x is revealed at least as good as y, then y cannot be revealed preferred to x.*"

Example 1.C.2: Do the two choice structures considered in Example 1.C.1 satisfy the weak axiom? Consider choice structure $(\mathscr{B}, C_1(\cdot))$. With this choice structure, we have $x \succsim^* y$ and $x \succsim^* z$, but there is no revealed preference relationship that can be inferred between y and z. This choice structure satisfies the weak axiom because y and z are never chosen.

Now consider choice structure $(\mathscr{B}, C_2(\cdot))$. Because $C_2(\{x, y, z\}) = \{x, y\}$, we have $y \succsim^* x$ (as well as $x \succsim^* y$, $x \succsim^* z$, and $y \succsim^* z$). But because $C_2(\{x, y\}) = \{x\}$, x is revealed preferred to y. Therefore, the choice structure $(\mathscr{B}, C_2)$ violates the weak axiom. ■

We should note that the weak axiom is not the only assumption concerning choice behavior that we may want to impose in any particular setting. For example, in the consumer demand setting discussed in Chapter 2, we impose further conditions that arise naturally in that context.

The weak axiom restricts choice behavior in a manner that parallels the use of the rationality assumption for preference relations. This raises a question: What is the precise relationship between the two approaches? In Section 1.D, we explore this matter.

1.D The Relationship between Preference Relations and Choice Rules

We now address two fundamental questions regarding the relationship between the two approaches discussed so far:

5. In fact, it says more: We must have $C(\{x, y, z\}) = \{x\}$, $=\{z\}$, or $=\{x, z\}$. You are asked to show this in Exercise 1.C.1. See also Exercise 1.C.2.

(i) If a decision maker has a rational preference ordering $\succsim$, do her decisions when facing choices from budget sets in $\mathscr{B}$ necessarily generate a choice structure that satisfies the weak axiom?

(ii) If an individual's choice behavior for a family of budget sets $\mathscr{B}$ is captured by a choice structure $(\mathscr{B}, C(\cdot))$ satisfying the weak axiom, is there necessarily a rational preference relation that is consistent with these choices?

As we shall see, the answers to these two questions are, respectively, "yes" and "maybe".

To answer the first question, suppose that an individual has a rational preference relation $\succsim$ on X. If this individual faces a nonempty subset of alternatives $B \subset X$, her preference-maximizing behavior is to choose any one of the elements in the set:

$$C^*(B, \succsim) = \{x \in B: x \succsim y \text{ for every } y \in B\}$$

The elements of set $C^*(B, \succsim)$ are the decision maker's most preferred alternatives in B. In principle, we could have $C^*(B, \succsim) = \varnothing$ for some B; but if X is finite, or if suitable (continuity) conditions hold, then $C^*(B, \succsim)$ will be nonempty.[6] From now on, we will consider only preferences $\succsim$ and families of budget sets $\mathscr{B}$ such that $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$. We say that the rational preference relation $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

The result in Proposition 1.D.1 tells us that any choice structure generated by rational preferences necessarily satisfies the weak axiom.

Proposition 1.D.1: Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$, satisfies the weak axiom.

Proof: Suppose that for some $B \in \mathscr{B}$, we have $x, y \in B$ and $x \in C^*(B, \succsim)$. By the definition of $C^*(B, \succsim)$, this implies $x \succsim y$. To check whether the weak axiom holds, suppose that for some $B' \in \mathscr{B}$ with $x, y \in B'$, we have $y \in C^*(B', \succsim)$. This implies that $y \succsim z$ for all $z \in B'$. But we already know that $x \succsim y$. Hence, by transitivity, $x \succsim z$ for all $z \in B'$, and so $x \in C^*(B', \succsim)$. This is precisely the conclusion that the weak axiom demands. ■

Proposition 1.D.1 constitutes the "yes" answer to our first question. That is, if behavior is generated by rational preferences then it satisfies the consistency requirements embodied in the weak axiom.

In the other direction (from choice to preferences), the relationship is more subtle. To answer this second question, it is useful to begin with a definition.

Definition 1.D.1: Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if

$$C(B) = C^*(B, \succsim)$$

for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

In words, the rational preference relation $\succsim$ rationalizes choice rule $C(\cdot)$ on $\mathscr{B}$ if the optimal choices generated by $\succsim$ (captured by $C^*(\cdot, \succsim)$) coincide with $C(\cdot)$ for

6. Exercise 1.D.2 asks you to establish the nonemptiness of $C^*(B, \succsim)$ for the case where X is finite. For general results, See Section M.F of the Mathematical Appendix and Section 3.C for a specific application.

all budget sets in $\mathscr{B}$. In a sense, preferences explain behavior; we can interpret the decision maker's choices as if she were a preference maximizer. Note that in general, there may be more than one rationalizing preference relation $\succsim$ for a given choice structure $(\mathscr{B}, C(\cdot))$ (see Exercise 1.D.1).

Proposition 1.D.1 implies that the weak axiom must be satisfied if there is to be a rationalizing preference relation. In particular, since $C^*(\cdot, \succsim)$ satisfies the weak axiom for any $\succsim$, only a choice rule that satisfies the weak axiom can be rationalized. It turns out, however, that the weak axiom is not sufficient to ensure the existence of a rationalizing preference relation.

Example 1.D.1: Suppose that $X = \{x, y, z\}$, $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$, $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. This choice structure satisfies the weak axiom (you should verify this). Nevertheless, we cannot have rationalizing preferences. To see this, note that to rationalize the choices under $\{x, y\}$ and $\{y, z\}$ it would be necessary for us to have $x \succ y$ and $y \succ z$. But, by transitivity, we would then have $x \succ z$, which contradicts the choice behavior under $\{x, z\}$. Therefore, there can be no rationalizing preference relation. ■

To understand Example 1.D.1, note that the more budget sets there are in $\mathscr{B}$, the more the weak axiom restricts choice behavior; there are simply more opportunities for the decision maker's choices to contradict one another. In Example 1.D.1, the set $\{x, y, z\}$ is not an element of $\mathscr{B}$. As it happens, this is crucial (see Exercises 1.D.3). As we now show in Proposition 1.D.2, if the family of budget sets $\mathscr{B}$ includes enough subsets of X, and if $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, then there exists a rational preference relation that rationalizes $C(\cdot)$ relative to $\mathscr{B}$ [this was first shown by Arrow (1959)].

Proposition 1.D.2: If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) the weak axiom is satisfied,
(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$; that is, $C(B) = C^*(B, \succsim)$ for all $B \in \mathscr{B}$. Furthermore, this rational preference relation is the *only* preference relation that does so.

Proof: The natural candidate for a rationalizing preference relation is the revealed preference relation $\succsim^*$. To prove the result, we must first show two things: (i) that $\succsim^*$ is a rational preference relation, and (ii) that $\succsim^*$ rationalizes $C(\cdot)$ on $\mathscr{B}$. We then argue, as point (iii), that $\succsim^*$ is the unique preference relation that does so.

(i) We first check that $\succsim^*$ is rational (i.e., that it satisfies completeness and transitivity).

Completeness By assumption (ii), $\{x, y\} \in \mathscr{B}$. Since either x or y must be an element of $C(\{x, y\})$, we must have $x \succsim^* y$, or $y \succsim^* x$, or both. Hence $\succsim^*$ is complete.

Transitivity Let $x \succsim^* y$ and $y \succsim^* z$. Consider the budget set $\{x, y, z\} \in \mathscr{B}$. It suffices to prove that $x \in C(\{x, y, z\})$, since this implies by the definition of $\succsim^*$ that $x \succsim^* z$. Because $C(\{x, y, z\}) \neq \varnothing$, at least one of the alternatives x, y, or z must be an element of $C(\{x, y, z\})$. Suppose that $y \in C(\{x, y, z\})$. Since $x \succsim^* y$, the weak axiom then yields $x \in C(\{x, y, z\})$, as we want. Suppose instead that $z \in C(\{x, y, z\})$; since $y \succsim^* z$, the weak axiom yields $y \in C(\{x, y, z\})$, and we are in the previous case.

(ii) We now show that $C(B) = C^*(B, \succsim^*)$ for all $B \in \mathscr{B}$; that is, the revealed preference relation $\succsim^*$ inferred from $C(\cdot)$ actually generates $C(\cdot)$. Intuitively, this seems sensible. Formally, we show this in two steps. First, suppose that $x \in C(B)$. Then $x \succsim^* y$ for all $y \in B$; so we have $x \in C^*(B, \succsim^*)$. This means that $C(B) \subset C^*(B, \succsim^*)$. Next, suppose that $x \in C^*(B, \succsim^*)$. This implies that $x \succsim^* y$ for *all* $y \in B$; and so for each $y \in B$, there must exist some set $B_y \in \mathscr{B}$ such that $x, y \in B_y$ and $x \in C(B_y)$. Because $C(B) \neq \varnothing$, the weak axiom then implies that $x \in C(B)$. Hence, $C^*(B, \succsim^*) \subset C(B)$. Together, these inclusion relations imply that $C(B) = C^*(B, \succsim^*)$.

(iii) To establish uniqueness, simply note that because $\mathscr{B}$ includes all two-element subsets of X, the choice behavior in $C(\cdot)$ completely determines the pairwise preference relations over X of any rationalizing preference.

This completes the proof. ■

We can therefore conclude from Proposition 1.D.2 that for the special case in which choice is defined for all subsets of X, a theory based on choice satisfying the weak axiom is completely equivalent to a theory of decision making based on rational preferences. Unfortunately, this special case is too special for economics. For many situations of economic interest, such as the theory of consumer demand, choice is defined only for special kinds of budget sets. In these settings, the weak axiom does not exhaust the choice implications of rational preferences. We shall see in Section 3.J, however, that a strengthening of the weak axiom (which imposes more restrictions on choice behavior) provides a necessary and sufficient condition for behavior to be capable of being rationalized by preferences.

Definition 1.D.1 defines a rationalizing preference as one for which $C(B) = C^*(B, \succsim)$. An alternative notion of a rationalizing preference that appears in the literature requires only that $C(B) \subset C^*(B, \succsim)$; that is, $\succsim$ is said to rationalize $C(\cdot)$ on $\mathscr{B}$ if $C(B)$ is a subset of the most preferred choices generated by $\succsim$, $C^*(B, \succsim)$, for every budget $B \in \mathscr{B}$.

There are two reasons for the possible use of this alternative notion. The first is, in a sense, philosophical. We might want to allow the decision maker to resolve her indifference in some specific manner, rather than insisting that indifference means that anything might be picked. The view embodied in Definition 1.D.1 (and implicitly in the weak axiom as well) is that if she chooses in a specific manner then she is, de facto, not indifferent.

The second reason is empirical. If we are trying to determine from data whether an individual's choice is compatible with rational preference maximization, we will in practice have only a finite number of observations on the choices made from any given budget set B. If $C(B)$ represents the set of choices made with this limited set of observations, then because these limited observations might not reveal all the decision maker's preference maximizing choices, $C(B) \subset C^*(B, \succsim)$ is the natural requirement to impose for a preference relationship to rationalize observed choice data.

Two points are worth noting about the effects of using this alternative notion. First, it is a weaker requirement. Whenever we can find a preference relation that rationalizes choice in the sense of Definition 1.D.1, we have found one that does so in this other sense, too. Second, in the abstract setting studied here, to find a rationalizing preference relation in this latter sense is actually trivial: Preferences that have the individual indifferent among all elements of X will rationalize *any* choice behavior in this sense. When this alternative notion is used in the economics literature, there is always an insistence that the rationalizing preference relation should satisfy some additional properties that are natural restrictions for the specific economic context being studied.

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EXERCISES

1.B.1$^{\mathrm{B}}$ Prove property (iii) of Proposition 1.B.1.

1.B.2$^{\mathrm{A}}$ Prove properties (i) and (ii) of Proposition 1.B.1.

1.B.3$^{\mathrm{B}}$ Show that if $f: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $u: X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$, then the function $v: X \to \mathbb{R}$ defined by $v(x) = f(u(x))$ is also a utility function representing preference relation $\succsim$.

1.B.4$^{\mathrm{A}}$ Consider a rational preference relation $\succsim$. Show that if $u(x) = u(y)$ implies $x \sim y$ and if $u(x) > u(y)$ implies $x \succ y$, then $u(\cdot)$ is a utility function representing $\succsim$.

1.B.5$^{\mathrm{B}}$ Show that if X is finite and $\succsim$ is a rational preference relation on X, then there is a utility function $u: X \to \mathbb{R}$ that represents $\succsim$. [*Hint*: Consider first the case in which the individual's ranking between any two elements of X is strict (i.e., there is never any indifference), and construct a utility function representing these preferences; then extend your argument to the general case.]

1.C.1$^{\mathrm{B}}$ Consider the choice structure $(\mathscr{B}, C(\cdot))$ with $\mathscr{B} = (\{x, y\}, \{x, y, z\})$ and $C(\{x, y\}) = \{x\}$. Show that if $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, then we must have $C(\{x, y, z\}) = \{x\}$, $= \{z\}$, or $= \{x, z\}$.

1.C.2$^{\mathrm{B}}$ Show that the weak axiom (Definition 1.C.1) is equivalent to the following property holding:

> Suppose that $B, B' \in \mathscr{B}$, that $x, y \in B$, and that $x, y \in B'$. Then if $x \in C(B)$ and $y \in C(B')$, we must have $\{x, y\} \subset C(B)$ and $\{x, y\} \subset C(B')$.

1.C.3$^{\mathrm{C}}$ Suppose that choice structure $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom. Consider the following two possible revealed preferred relations, $\succ^*$ and $\succ^{**}$:

$$x \succ^* y \Leftrightarrow \text{there is some } B \in \mathscr{B} \text{ such that } x, y \in B,\ x \in C(B), \text{ and } y \notin C(B)$$

$$x \succ^{**} y \Leftrightarrow x \succsim^* y \text{ but not } y \succsim^* x$$

where $\succsim^*$ is the revealed at-least-as-good-as relation defined in Definition 1.C.2.

(a) Show that $\succ^*$ and $\succ^{**}$ give the same relation over X; that is, for any $x, y \in X$, $x \succ^* y \Leftrightarrow x \succ^{**} y$. Is this still true if $(\mathscr{B}, C(\cdot))$ does not satisfy the weak axiom?

(b) Must $\succ^*$ be transitive?

(c) Show that if $\mathscr{B}$ includes all three-element subsets of X, then $\succ^*$ is transitive.

1.D.1$^{\mathrm{B}}$ Give an example of a choice structure that can be rationalized by several preference relations. Note that if the family of budgets $\mathscr{B}$ includes all the two-element subsets of X, then there can be at most one rationalizing preference relation.

1.D.2[A] Show that if X is finite, then any rational preference relation generates a nonempty choice rule; that is, $C(B) \neq \varnothing$ for any $B \subset X$ with $B \neq \varnothing$.

1.D.3[B] Let $X = \{x, y, z\}$, and consider the choice structure $(\mathscr{B}, C(\cdot))$ with

$$\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}, \{x, y, z\}\}$$

and $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$, as in Example 1.D.1. Show that $(\mathscr{B}, C(\cdot))$ must violate the weak axiom.

1.D.4[B] Show that a choice structure $(\mathscr{B}, C(\cdot))$ for which a rationalizing preference relation $\succsim$ exists satisfies the *path-invariance* property: For every pair $B_1, B_2 \in \mathscr{B}$ such that $B_1 \cup B_2 \in \mathscr{B}$ and $C(B_1) \cup C(B_2) \in \mathscr{B}$, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$, that is, the decision problem can safely be subdivided. See Plott (1973) for further discussion.

1.D.5[C] Let $X = \{x, y, z\}$ and $\mathscr{B} = \{\{x, y\}, \{y, z), \{z, x\}\}$. Suppose that choice is now stochastic in the sense that, for every $B \in \mathscr{B}$, $C(B)$ is a frequency distribution over alternatives in B. For example, if $B = \{x, y\}$, we write $C(B) = (C_x(B), C_y(B))$, where $C_x(B)$ and $C_y(B)$ are nonnegative numbers with $C_x(B) + C_y(B) = 1$. We say that the stochastic choice function $C(\cdot)$ can be *rationalized by preferences* if we can find a probability distribution Pr over the six possible (strict) preference relations on X such that for every $B \in \mathscr{B}$, $C(B)$ is precisely the frequency of choices induced by Pr. For example, if $B = \{x, y\}$, then $C_x(B) = Pr(\{\succ : x \succ y\})$. This concept originates in Thurstone (1927), and it is of considerable econometric interest (indeed, it provides a theory for the error term in observable choice).

(a) Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\frac{1}{2}, \frac{1}{2})$ can be rationalized by preferences.

(b) Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\frac{1}{4}, \frac{3}{4})$ is not rationalizable by preferences.

(c) Determine the $0 < \alpha < 1$ at which $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$ switches from rationalizable to nonrationalizable.