[For Glaeser Midterm : Not helpful for Final or Generals]

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Chapter 2

What happens when we differentiate Walras' law $p \cdot x(p,w) = w$ with respect to p?

What is the intuition?

Proposition 2.E.2: If the Walrasian demand function $x(p,w)$ satisfies Walras' law, then for all p and w:

$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0$ for $k = 1, ..., L$

In matrix notation:

$p \cdot D_p x(p,w) + x(p,w)^T = 0^T$

This is the result of differentiating $p \cdot x(p,w) = w$ with respect to p. (product rule)

Intuition: total expenditure cannot change in response to a change in prices.

What happens when we differentiate Walras' law $p \cdot x(p,w) = w$ with respect to w?

What is the intuition?

Proposition 2.E.3: If the Walrasian demand function $x(p,w)$ satisfies Walras' law, then for all p and w:

$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$

In matrix notation:

$p \cdot D_w x(p,w) = 1$

Intuition: Total expenditure must change by an amount equal to any change in wealth

What is WARP for the Walrasian demand function?

Definition 2.F.1: The Walrasian demand function $x(p,w)$ satisfies the weak axiom of revealed preference if the following property holds for any two price-wealth situations (p,w) and (p',w'):

If $p \cdot x(p',w') \leq w$ and $x(p,w) \neq x(p',w')$ then:

$p' \cdot x(p,w) > w'$

Intuition: If the new bundle was affordable at the old prices, and the new and old bundles aren't equal, then the old bundle must not be affordable at the new prices.

What are the two ways in which price changes affect consumers?

1. They alter the relative cost of different commodities

2. They change the consumer's real wealth
[Slutsky anaylsis attempts to isolate first effect]

What is Slutsky wealth compensation?

Verbally: A change in prices is accompanied by a change in the consumer's wealth that makes his initial consumption bundle affordable at new prices.

Formally: If the consumer is originally facing prices p and wealth w and chooses the consumption bundle $x(p, w)$, then when prices change to p', the consumer's wealth is adjusted to $w' = p' \cdot x(p, w)$.

Thus the wealth adjustment is $\Delta w = \Delta p \cdot x(p, w)$. [where $\Delta w = w' - w$ and $\Delta p = p' - p$]

What are compensated price changes?

Also known as "Slutsky compensated price changes"

Price changes that are accompanied by a Slutsky wealth compensation. That is, $\Delta w = \Delta p \cdot x(p, w)$

How can WARP be stated in terms of the demand response to compensated price changes?
What is the name of this property?

The "Compensated Law of Demand"

Proposition 2.F.1 (MEM): Suppose that the Walrasian demand function $x(p, w)$ is homogenous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom $\Longleftrightarrow$the following property holds:

For any compensated price change from an initial situation (p, w) to a new price-wealth pair $(p', w') = (p', p' \cdot x(p, w))$

Then $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$

[with strict inequality whenever $x(p', w') \neq x(p, w)$]

In shorthand: $\Delta p \cdot \Delta x \leq 0$ where $\Delta p = p' - p$ and $\Delta x = [x(p', w') - x(p, w)]$

Intuition: It can be interpreted as a form of the law of demand: price and demand move in opposite directions for *compensated* price changes.

Is WARP enough to yield the law of demand?

No: it is not sufficient to yield the law of demand for price changes that are not compensated.

(law of demand = price and demand move in opposite directions)

If x(p,w) is a differentiable function of p and w, what is the "compensated law of demand" for differential changes in price and wealth dp and dx?

$dp \cdot dx \leq 0$
$dp \cdot [D_p x(p,w) + D_w x(p,w) x(p,w)^T] dp \leq 0$

If we have $x(p,w)$ as a differentiable function of p and w, what is dx?

$dx = D_p x(p,w) dp + D_w x(p,w) dw$
$dx = D_p x(p,w) dp + D_w x(p,w)[x(p,w) \cdot dp]$
$dx = [D_p x(p,w) + D_w x(p,w) x(p,w,)^T] dp$

What is another expression for $S(p,w)$?

The expression in brackets from the compensated law of demand for differential changes, $[D_p x(p,w) + D_w x(p,w) x(p,w)^T]$
is an $L \times L$ matrix denoted by $S(p,w)$.
Formally

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

What is the (l,k)th entry of $S(p,w)$?

$S_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$
First term: unconpensated change in demand of good l due to a change in p_k (when multiplied by dp_k)
Second term: effect of wealth change on demand of good l [see below] (when multiplied by dp_k)

What is $S(p,w)$ known as?
What are its elements called?

The matrix $S(p,w)$ is known as the "substitution," or "Slutsky" matrix
Its elemtns are known as "substitution effects."

#Explanation of Slutsky matrix (p.34)
The matrix $S(p,w)$ is known as the substitution, or Slutsky, matrix, and its elements are known as substitution effects.

The substitution terminology is apt because the term $s_{lk}(p,w)$ measures the differential change in the consumption of commodity l (i.e. the substitution to or from other commodities) due to a differential change in the price of commodity k when wealth is adjusted so that the consumer can still just afford his original consumption bundle (i.e. due solely to a change in relative prices).

To see this, note that the change in demand for good l if wealth is left unchanged is $\frac{\partial x_l(p,w)}{\partial p_k}dp_k$.

For the consumer to be able to "just afford" his original consumption bundle, his wealth must vary by the amount $x_k(p,w)dp_k$.

The effect of this wealth change on the demand for good l is then $\frac{\partial x_l(p,w)}{\partial w}[x_k(p,w)dp_k]$.

The sum of these two effects is therefore exactly $s_{lk}(p,w)dp_k$.

What is a key matrix property that $S(p,w)$ satisfies?

Proposition 2.F.2: If a differentiable Walrasian demand function $x(p,w)$ satisfies homogeneity of degree zero, Walras' law and WARP, then at any (p,w) the Slutsky matrix $S(p,w)$ satisfies

$v \cdot S(p,w)v \leq 0$ for any $v \epsilon \mathbb{R}^L$

What does $v \cdot S(p,w)v \leq 0$ for any $v \epsilon \mathbb{R}^L$imply for $S(p,w)$?

A matrix satisfying this property is called negative semidefinite.

What does negative semidefiniteness imply about diagonal entries?

$S(p,w)$ being negative semidefinite implies that $s_{ll}(p,w) \leq 0$

That is, the substitution effect of good l with respect to a change in its own price is always positive.

Is $S(p,w)$ symmetric?

For $L = 2$, $S(p,w)$ is necessarily symmetric.

For $L > 2$, $S(p,w)$ is **not** necessarily symmetric under assumptions made in Chapter 2:

(homogeneity of degree zero, Walras' law, WARP)

What is necessary for demand to be generated from preferences?

Symmetric slutsky matrix at all (p,w)

Suppose that the Walrasian demand function $x(p,w)$ is differentiable, homogenous of degree zero, and satisfies Walras' law.

What is $p \cdot S(p,w)$?

Proposition 2.F.3:
$p \cdot S(p,w) = 0$ and
$S(p,w)p = 0$ for any (p,w)

Chapter 3

What in addition to rationality, what are the two other sets of assumptions about preferences that are used?

What are the elements of these sets?

1. Desirability assumptions
Capturing the idea that more goods are generally preferred to less
Includes locally non-satiated, monotone, strongly monotone
2. Convexity assumptions
Concerns trade-offs that consumer is willing to make among different goods
Includes convexity, stricy convexity

When is $\succeq$ locally nonsatiated?

Definition 3.B.3: The preference relation $\succeq$is locally nonsatiated if for every $x \epsilon X$ and every $\varepsilon > 0$, there exists $y \epsilon X$ such that $||y - x|| < \varepsilon$ and $y \succ x$.

Says that there is a preferred bundle in an arbitrary small distance from x.

Note that this bundle y could have less of every commodity than x. However, this does rule out the extreme case that all goods are bads, in which case 0 would be a satiation point.

Also, local nonsatiation (and hence monotonicity) rules out "thick" indifffer-ence sets.

When is $\succeq$ monotone?

Definition 3.B.1: The preference relation $\succeq$on X is monotone if $x \epsilon X$ and $y \gg x$ implies $y \succeq x$.

Intuition: The assumption that preferences are monotone holds as long as commodities are "goods" rather than "bads." [Even if some commodity is a bad, can redefine consumption activity as the "absence of" that bad to recover monotonicity."]

When is $\succeq$ strongly monotone?

Definition 3.B.1: The preference relation $\succeq$on X is strongly monotone if $x \epsilon X$ and $y \geq x$, where $y \neq x$, implies $y \succ x$.

Monotonicity: we may have indifference when a new bundle has more of some, but not all, goods.

Strong monotonicity: if a new bundle has a larger amount of one good (and no less of any other), then the new bundle is strictly preferred to the old.

For a $\succeq$ and a consumption bundle x, what are the indifference set, the lower contour set, and the upper contour set?
What is an implication of local nonsatiation on these sets?

Indifference set containing point x is the set of all bundles that are indifferent to x: formally $\{y \epsilon X : \ y \sim x\}$.

Upper contour set of bundle x is the set of all bundles that are as least as good as x: $\{y \epsilon X : \ y \succeq x\}$.

Lower contour set of bundle x is the set of all bundles that x is at least as good as: $\{y \epsilon X : \ x \succeq y\}$.

Local nonsatiation rules out "thick" indifference sets.

When is $\succeq$ convex?

Definition 3.B.4: The preference relation $\succeq$ on X is convex if for every $x \epsilon X$, the upper contour set $\{y \epsilon X : \ y \succeq x\}$is convex; that is, if $y \succeq x$ and $x \succeq z$, then $\alpha y + (1-\alpha) z \succeq x$ for any $\alpha \epsilon [0,1]$.

What does convexity mean intuitively?

Convexity can be interpreted in terms of diminishing marginal rates of substitution. That is, it takes increasingly larger amounts of one commodity to compensate for successive losses of the other.

Can also be viewed as a formal expression of a basic inclination of economic agents for diversification.

When is $\succeq$ strictly convex?

Definition 3.B.5: The preference relation $\succeq$ is strictly convex if for every $x \epsilon X$, we have that $y \succeq x$, $z \succeq x$ and $y \neq z$ implies $\alpha y + (1-\alpha)z \succ x$ for all $\alpha \epsilon (0,1)$.

What are two classes of preference relations for which it is possible to deduce the consumer's entire preference relation from a single indifference set?

Homothetic and quasilinear preferences.

When is $\succeq$ homothetic?

A *monotone* preference relation $\succeq$ is homothetic if all indifference sets are related by proportional expansion along rays; that is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.

When is $\succeq$quasilinear?

Definition 3.B.7: The preference relation $\succeq$on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to commodity 1 (called in this case, the numeraire commodity) if:

i. All the indifference sets are parallel displacements of each other along the axis of commodity 1.

That is, if $x \sim y$, then $x + \alpha e_1 \sim y + \alpha e_1$ for $e_1 = (1, 0, ..., 0)$ and any $\alpha \epsilon \mathbb{R}$

ii. Good 1 is desirable; that is $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Note that in definition there is no lower bound on the possible consumption of the first commodity.

3.C Preference and Utility

What is a lexicographic preference relation?
Why is it important? What features do they have and what do they lack?

Lexicographic preference relations are named for the way a dictionary is organized: commodity 1 has the highest priority in preference ordering (above all else), then commodity 2, and so on.

For $X = \mathbb{R}_+^2$, define $x \succeq y$ if $x_1 > y_1$; or "$x_1 = y_1$and $x_2 > y_2$."

A lexicographic preference relation is complete, transitive, monotone, strictly convex. It is not continuous.

Yet no utility function exists that represents this preference ordering.

This is because no two distinct bundles are indifferent.

What are the assumptions needed to ensure the existance of a utility function?

Continuity and rationality.

When is $\succeq$continuous?

What does this imply about the upper contour set and lower contour set?

Definition 3.C.1: The preference relation $\succeq$on X is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$with $x^n \succeq y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succ y$.

Contunuity says that the consumer's preferences cannot exhibit "jumps." (cannot reverse her preference at the limiting points of these sequences).

How can continuity be stated in terms of upper contour sets and lower contour sets?

An equivalent way of stating the notion of continuity is to say that for all x, the upper contour set $\{y \epsilon X : \ y \succeq x\}$and the lower contour set $\{y \epsilon X : \ x \succeq y\}$are both closed, that is, they include their boundaries.

Suppose that the rational preference relation $\succeq$on X is continuous. What can you get?

Proposition 3.C.1: Then there is a *continuous* utility function $u(x)$ that represents $\succeq$.

[Not only are rationality and continuity sufficient for the existance of a utility function representation, they guarantee the existance of a continuous utility function]

Is the utility function $u(\cdot)$ that represents a preference relation $\succeq$unique?

No, any strictly increasing transformation of $u(\cdot)$, say $v(x) = f(u(\cdot))$, where $f(\cdot)$ is a strictly increasing function, also represents $\succeq$.

Are all utility functions representing $\succeq$continuous?

No; any strictly increasing but discontinuous transformation of a continuous utility function also represents $\succeq$.

What is another common convenience assumption about utility functions?

What is an important counter-example?

For analytical purposes, it is convenient if $u(\cdot)$ is assumed to be differentiable.

It is possible, however, for continuous preferences not to be representable by a differentiable utility function.

The simplest counter-example is the case of Leontief preferences.

MWG assumes utility functions to be twice continuously differentiable.

What are Leontief preferences?

Leontief preferences: $x'' \succeq x' \iff \min\{x_1^{''}, x_2^{''}\} \geq \min\{x_1^{'}, x_2^{'}\}$

The nondifferentiability arises because of the kink in the indifference curves when $x_1 = x_2$.

Restrictions on preferences translate into restrictions on the form of utility functions.

What does the property of monotonicity of preferences imply about the utility function?

Monotonicty of preferences implies that the utility function is increasing: $u(x) > u(y)$ if $x \gg y$.

What does the property of convexity of preferences imply about the utility function?

The property of convexity of preferences implies that $u(\cdot)$ is quasiconcave.

Note, however, that the convexity of $\succeq$ of does **not** imply the stronger property that $u(\cdot)$ is concave.

What is the formal definition of quasiconcavity for $u(\cdot)$?

The utility function $u(\cdot)$ is quasiconcave if the set $\left\{y\epsilon\mathbb{R}_+^L : \ u(y) \geq u(x)\right\}$is convex for all x.

Or, equivalently, if $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), \ u(y)\}$ for any x, y and all $\alpha\epsilon[0,1]$.

[Convexity of $\succeq$ of does **not** imply the stronger property that $u(\cdot)$ is concave, which is

that $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$ for any x,y and all $\alpha\epsilon[0,1]$.

When is a continuous $\succeq$ homothetic?

A continuous $\succeq$on $X \epsilon \mathbb{R}_+^L$is homothetic $\iff$ it admits a utility function $u(x)$ that is homogenous of degree one [i.e. such that $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$].

When is a continuous $\succeq$ quasilinear?

A continuous $\succeq$on $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$is quasilinear with respect to the first commodity $\iff$ it admits a rational utility function of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.

Which of the following properties are ordinal and which are cardinal? Which are preserved under arbitrary increasing transformations?
increasing; quasiconcave; homothetic; quasilinear

1&2: Monotonicity and convexity of $\succeq$imply increasing and quasiconcave. These are *ordinal* properties of $u(\cdot)$ that are preserved under any arbitrary increasing transformation of the utility index.
3&4: $\succeq$ that fulfill criteria for homothetic and quasilinear utility representations merely say that there is at least one utility function that has the specified form. These are *cardinal* properties that are not preserved.

3.D Utility Maximization

#Assume throughout that consumer has a rational, continuous and locally nonsatiated preference relation.
Take $u(x)$ to be a continuous utility function representing these preferences.
Assume consumption set is $X = \mathbb{R}_+^L$ for concreteness.

What is the utility maximization problem (UMP)?

The consumer's problem of choosing her most preferred consumption bundle given prices $p \gg 0$ and wealth level $w > 0$ can be stated as the UMP:
$\max_{x \geq 0} \ u(x)$
$s.t. \ p \cdot x \leq w$
The consumer chooses a consumption bundle in the Walrasian budget set $B_{p,w} = \{x \epsilon \mathbb{R}^L : \ p \cdot x \leq w\}$ to maximize her utility level.

What are two conditions which guarantee a solution to the UMP?

If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

What is the rule that assigns the set of optimal consumption vectors in the UMP?

The rule that assigns the optimal consumption vectors in the UMP to each price-waelth situation $(p, w) \gg x$ is denoted by $x(p, w)$ and is known as the Walrasian (or ordinary or market) demand correspondance. As a general matter, for a given $(p, w) \gg 0$ the optimal set $x(p, w)$ may have more than one element.

When $x(p, w)$ is single valued for all (p, w), we refer to it as the Walrasian (or market or ordinary) demand function (or Marshallian demand function).

Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. What properties will the Walrasian demand correspondance $x(p, w)$ possess?

Proposition 3.G.2:

i. Homogeneity of degree zero in $(p, w) : x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.

ii. Walras' law: $p \cdot x = w$ for all $x \epsilon x(p, w)$.

iii. Convexity and uniqueness, with conditions (see below)

When is $x(p, w)$ a convex set?

If $\succeq$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set.

When does $x(p, w)$ consist of a single element?

If $\succeq$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

If $u(\cdot)$ is continuously differentiable, what is a useful manner in which the optimal consumption bundle $x^* \epsilon x(p, w)$ be characterized?

By means of first-order conditions. The Kuhn-Tucker conditions say that if $x^* \epsilon x(p, w)$ is a solution to the UMP, then there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $l = 1, \ldots, L$:

$\frac{\partial u(x^*)}{\partial c_l} \leq \lambda p_l$

[if $x^*_l > 0$, then we have $\frac{\partial u(x^*)}{\partial c_l} = \lambda p_l$]

An equivalent formulation:

If we denote the gradient vector of $u(\cdot)$ at x as $\nabla u(x) = \begin{bmatrix} \frac{\partial u(x)}{\partial x_1} \\ \vdots \\ \frac{\partial u(x)}{\partial x_L} \end{bmatrix}$
we can write the above notation in matrix form: $\nabla u(x) \leq \lambda p$
and $x^*[\nabla u(x) - \lambda p] = 0$.
At an interior optimum (i.e. if $x^* \gg 0$), we have
$\nabla u(x) = \lambda p$.

What does the above discussion imply about the relationship between two important vectors at an interior optimum?

At an interior optimum, the gradient vector of a consumer's utility function $\nabla u(x^*)$ must be proportional to the price vector p.

What is the relationship for two goods l and k involving the gradient vector of $u(\cdot)$ and the price vector, formally and verbally?

If $u(x^*) \gg 0$, then for any two goods l and k we have:
$\frac{\frac{\partial u(x*)}{\partial x_l}}{\frac{\partial u(x*)}{\partial x_k}} = \frac{p_l}{p_k}$
The expression on the left $MRS_{lk}(x^*)$**,** the marginal rate of substitution of good l for good k at x^*.
Verbally, this condition tells us:
At an interior optimum, the marginal rate of substitution between two goods must be equal to their price ratio.
[Were this not the case, the consumer could do better by marginally changing her consumption]

Where is there an exception to the formula $\frac{\frac{\partial u(x*)}{\partial x_l}}{\frac{\partial u(x*)}{\partial x_k}} = \frac{p_l}{p_k}$?

If the consumer's optimization bundle lies on the boundary of the consumption set (some $x_i^* = 0$), the gradient vector will not be proportional to the price vector.

What is the marginal utility of wealth in the UMP?

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#Will generally assume Walrasian demand to be suitably continuous and differentiable.

[These properties hold under general conditions – if preferences are continuous, strictly convex, locally nonsatiated]

What is the indirect utility function?

The indirect utility function $v(p,w)$ is an analytic tool for measuring the utility value at any $x^* \epsilon\ x(p,w)$. The utility value of the UMP is denoted $v(p,w)\epsilon\mathbb{R}$ for each $(p,w) \gg 0$.

It is equal to $u(x^*)$ for any $x^* \epsilon\ x(p,w)$.

What are key properties of the indirect utility function $v(p,w)$? (4) What assumptions need to be made?

Proposition 3.D.3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$defined on the consumption set $X \epsilon \mathbb{R}^L_+$.

The indirect utility function $v(p,w)$ is:

i. Homogenous of degree zero

ii. Strictly increasing in w and nonincreasing in p_l for any l.

iii. Quasiconvex; that is the set $\{(p,w) : v(p,w) \leq \bar{v}\}$is convex for any $\bar{v}$.

iv. Continuous in p and w.

#Adam's summary of indirect utility:

When we plug $x(p,w)$ back into $u(x)$ we get the indirect utility function:

$v(p,w) = u(x(p,w)) = u(\arg\max_x\ \{u(x) - \lambda[p \cdot x - w]\})$

It is homogenous of degree zero, strictly increasing in w and nonincreasing in $p_i \forall i$, quasiconvex, and continuous.

3.E Expenditure Minimization Problem

Assume that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$.

As with the UMP, when $p \gg 0$ a solution to the EMP exists under very general conditions. The constraint set merely needs to be nonempty; that is, $u(\cdot)$ must attain values at least as large as u for some x.

What is the expenditure minimization problem (EMP)? Verbally, how does it differ from the UMP?

The EMP for $p \gg 0$ and $u > u(0)$:

$\min_{x \geq 0}\ p \cdot x\ \ s.t.\ u(x) \geq u.$

Whereas the UMP computes the maximal level of utility that can be obtained given the wealth w, the EMP computes the minimal level of wealth required required to reach utility level u.

It captures the same aim of efficient use of the consumer's purchasing power while reversing the roles of objective function and constraint.

The optimal consumption bundle x^* is the least costly bundle that still allows the consumer to achieve the utility level u.

How is the equivalence between the optimal x^*in the UMP and the optimal u in the EMP stated formally?

Proposition 3.E.1: Suppose that that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$, and that the price vector is $p \gg 0$. We have:

i. If $x^*$is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$.

Moreover, the mimimzed expenditure level in this EMP is exactly w.

ii. If $x^*$is optimal in the EMP when the required utility level is $u > u(0)$, then $x^*$is optimal in the UMP when wealth is $p \cdot x^*$.

Moreover, the maximized utility level in this UMP is exactly u.

What is $e(p, u)$?

Given prices $p \gg 0$ and required utility level $u > u(0)$, the value of the EMP is denoted $e(p, u)$. The function $e(p, u)$ is called the expenditure function.

Its value for any (p, u) is simply $p \cdot x^*$,where x^* is any solution to the EMP.

Proposition 3.E.2's characterization of the basic properties of the expenditure function parallel the properties of the indirect utility function for UMP.

What are the key properties of the expenditure function $e(p, u)$? (4) What assumptions need to be made?

Proposition 3.E.2: Suppose that that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$.

The expenditure function $e(p, u)$ is:

i. Homogenous of degree one in p.

ii. Strictly increasing in u and nondecreasing in p_l for any l.

iii. *Concave* in p.

iv. Continuous in p and u.

What is the formal relationship between the expenditure function and the indirect utility function?

$e(p, v(p, w)) = w$ and $v(p, e(p, u)) = u$

These conditions imply that for a fixed price vector $\bar{p}$, $e(\bar{p}, \cdot)$ and $v(\bar{p}, \cdot)$ are inverses to one another.

[In fact, Proposition 3.E.2 can be directly derived from Proposition 3.D.3, and vice versa.]

What is the set of optimal commodity vectors in the EMP called?

The set of optimal commodity vectors in the EMP is denoted $h(p, u) \subset \mathbb{R}^L_+$, and is known as the Hicksian, or compensated, demand correspondance (or function is single valued).

What are the three basic properties of Hicksian demand?

This parellels proposition 3.D.2 for Walrasian demand.

Proposition 3.E.3:

Suppose that $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$.

Then for any $p \gg 0$,the Hicksian demand correspondance $h(p, u)$ possesses the following properties:

i. Homogeneity of degree zero in p : $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.

ii. No excess utility: For any $x \in h(p, u)$, $u(x) = u$.

iii. Convexity/uniqueness:

If $\succsim$ is convex, then $h(p, u)$ is a convex set

If $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Can the optimal consumption bundle in the EMP be characterized FOCs?

Yes: as in the UMP, when $u(\cdot)$ is differentiable, the optimal conumption bundle can be characterized using FOCs.

[The FOCs bear a close similarity (but not the same as) those of the UMP: $x^*[p - \lambda \nabla u(x^*)] = 0$

What is the formal relationship between Hicksian and Walrasian demand correspondances?

$h(p,u) = x(p, e(p,u))$ and $x(p,w) = h(p, v(p,w))$

Why is $h(p,u)$ called "compensated demand correspondance"? What is Hicksian wealth compensation?

As prices vary, $h(p,u)$ gives precisely the level of demand that would arise if the consumer's wealth were simlataneously adjusted to keep her utility level at u.

This type of of wealth compensation is known as Hicksian wealth compensation.

[As $p \longrightarrow p'$, $\Delta w_{hicks} = e(p',u) - w$. Thus the demand function $h(p,u)$ keeps the consumer's utility level fixed as prices change, in contrast with Walrasian demand which keeps money wealth fixed but allows utility to vary.]

What is the Compensated Law of Demand? How does this relate to Hicksian demand?

An important property of Hicksian demand is that it satisfies the compensated law of demand: demand and price move in opposite directions for price changes that are accompanied by Hicksian wealth compensation.

What is the formal statement that the single-valued Hicksian demand function $h(p,u)$ satisfies the compensated law of demand?

Proposition 3.E.4: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p,u)$ consists of a single element for all $p \gg 0$.

Then, the Hicksian demand function $h(p,u)$ satisfies the compensated law of demand:

For all p' and p",

$$(p" - p') \cdot [h(p", u) - h(p'u)] \leq 0$$

What is an important implication of the Compensated Law of Demand for own-price effects?

For compensated demand, own-price effects are nonpositive.

In particular, if only p_l changes, Proposition 3.E.4 implies that $(p_l" - p_l') \cdot [h_l(p_l", u) - h_l(p_l', u)] \leq 0$.

Is the compensated law of demand also true for Walrasian demand?

No: Walrasian demand need not satisfy the law of demand.
For example, the demand for a good can decrease when its price falls.

3.G Relationships between Demand, Indirect Utility, and Expenditure Functions

Assume that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$.
Restrict attention to cases where $p \gg 0$.
Assume throughout the $\succsim$ is strictly convex, so that Walrasian and Hicksian demands $x(p, w)$ and $h(p, w)$ are single-valued.

How can $e(p, u)$ be written in terms of the Hicksian demand $h(p, u)$?

$e(p, u) = p \cdot h(p, u)$

How can $h(p, u)$ be written in terms of $e(p, u)$?

Proposition 3.G.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$.

For all p and u, the Hicksian demand $h(p, u)$ is the derivative vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u)$$

That is, $h_l(p, u) = \frac{\partial e(p,u)}{\partial p_l}$ for all $l = 1, \ldots, L$.
Thus, given the expenditure function, we can calculate the consumer's Hicksian demand function simply by differentiating.

What are properties of the derivative of the Hicksian demand function that follow from $h(p, u) = \nabla_p e(p, u)$? (4)

Proposition 3.G.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$.

Suppose that $h(\cdot, u)$ is continuously differentiable at (p, u), and denote its $L \times L$ derivative matrix $D_p h(p, u)$. Then:
i. $D_p h(p, u) = D^2_p e(p, u)$
ii. $D_p h(p, u)$ is a negative semidefinite matrix.
iii. $D_p h(p, u)$ is a symmetric matrix.
iv. $D_p h(p, u) p = 0$.

What does the negative semidefiniteness of $D_p h(p,u)$ imply?

Negative semidefiniteness implies that own price effects are non-positive: $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$.

[The negative semidefiniteness of $D_p h(p,u)$ is the differential analog of the compensated law of demand.

We saw (3.E.5) that $dp \cdot dh(p,u) \leq 0$. Now we have $dh(p,u) = D_p h(p,u)dp$; substituting gives:

$dp \cdot D_p h(p,u)dp \leq 0$ for all dp.

$\Longrightarrow D_p h(p,u)$ is negative semidefinite.]

What does the symmetry of $D_p h(p,u)$ imply?

It implies that the cross-derivates between any two goods l and k must satisfy:

$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_l}$

[Not easy to interepret in plain economic terms–says that 'when you clime a mountain, you will cover the same height regardless of route.']

What are substitutes? What are complements?

We define two goods l and k to be:

Substitutes at (p,u) if $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$

Complements at (p,u) if $\frac{\partial h_k(p,u)}{\partial p_l} \leq 0$.

What does $D_p h(p,u)$ imply about substitutes or complements?

Because $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$, property (iv) implies that there must be one good k for which $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$.

Hence every good has at least one substitute.

What is the Slutsky Equation?

Proposition 3.G.3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$.

Then, for all (p,w), and $u = v(p,w)$, we have:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \quad [for\ all\ l,k]$$

Equivalently, in matrix notation:

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$$

How can we get a decomposition of the effect of a price change into income and substitution effects?

We can spin things around to get a decomposition of the effect of a price change on Marshallian demand into income and substitution effects:

$$\frac{\partial x_l(p,w)}{\partial p_k} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{Substitution\ Effect} - \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{Income\ Effect}$$

What is the matrix of price derivatives of the Hicksian demand function?

Proposition 3.G.3 implies that the matrix of price derivatives $D_p h(p,u)$ of the Hicksian demand function is equal to the *Slutsky substitution matrix*

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

Where

$$S_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

First term: unconpensated change in demand of good l due to a change in p_k (when multiplied by dp_k)
Second term: effect of wealth change on demand of good l [see below] (when multiplied by dp_k)

What does this tell us about the Slutsky matrix when demand is generated from preference maximization?
How does this differ from the choice-based theory?

Proposition 3.G.2 tells us that when demand is generated from preference maximization, $S(p,w)$ must possess the following three properties: it must be negative semidefinite, symmetric, and satisfy $S(p,w)p = 0$.

This is because $D_p h(p,u) = S(p,w)$

The difference is *symmetry*: In the choice-based approach, discussed in section 2.F, saw that if $x(p,w)$ satisfies WARP, Walras' law, and homogeneity of degree zero, then $S(p,w)$ is negative semidefinite with $S(p,w)p = 0$. However, we saw that unless $L = 2$, the Slutsky substitution matrix need not be symmetric.

$\Longrightarrow$Restrictions imposed on demand in the preference-based approach are stronger than those arising in the choice-based theory built on WARP

$\Longrightarrow$In 3.I, see that it is impossible to find preferences that rationalize demand when Slutsky matrix is not symmetric.

So $e(p,u)$ is the value function of the EMP, and we saw that the minimizing vector of the EMP,$h(p,u)$, is equivalent to the derivative of $e(p,u)$ with respect to $p : h(p,u) = \nabla_p e(p,u)$.

Why can't we say that Walrasian demand $x(p,w)$ is equivalent to the derivative with respect to p of its value function $v(p,w)$?

The Walrasian demand, an ordinal concept, cannot equal the price derivative of the indirect utility function, which can vary with increasing transformations of utility.

However, with a small correction in which we normalize the derivatives of $v(p,u)$ with respect to p by the marginal utility of wealth, it holds true.

What is Roy's Identity?

Proposition 3.G.4: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$.

Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})} \nabla_p v(\bar{p},\bar{w})$$

That is, for every $l = 1, \ldots, L$:

$$x_l(\bar{p},\bar{w}) = -\frac{\frac{\partial v(\bar{p},\bar{w})}{\partial p_l}}{\frac{\partial v(\bar{p},\bar{w})}{\partial w}}$$

Why is Roy's identity important?

Roy's identity provides a substantial payoff: Walrasian demand is much easier to compute from indirect than from direct utility.

To derive $x(p, w)$ from the indirect utility function, no more than the calculation of derivatives is involved; no system of first-order equations needs to be solved.

Thus, it may often be more conveenient to express tastes in indirect utility form.

3.I Welfare Evaluation

#Normative side of consumer theory, called *welfare analysis.*

#Focus on the welfare effect of a *price change*

Assume consumer has fixed wealth $w > 0$ and that the price vector is initially p^0. Look to evaluate consumer's welfare of a change from p^0 to a new price vector p^1.

What is a *money metric* utility function? How can this function be used to provide a measure of welfare change in dollars?

This class of functions is used to express a measurement of the welfare change in dollars.

We have the function $e(\bar{p}, v(p, w))$, which serves as an indirect utility function:

For any utility function $v(p, w)$ and any price vector $\bar{p} \gg 0$, $e(\bar{p}, v(p, w))$ gives the wealth required to reach utility level of $v(p, w)$ when prices are $\bar{p}$.

$e(\bar{p}, v(p, w))$ is therefore a strictly increasing function of (p, w).

We have $e(\bar{p}, v(p, w))$ as an indirect utility function of $\succsim$ in regards to (p, w).

Hence:

$$e(\bar{p}, v(p^0, w)) - e(\bar{p}, v(p^1, w))$$

provides a measure of welfare that can be expressed in dollars.

How can EV and CV be defined?

Although money metric utility function (above) can be constructed for any $\bar{p} \gg 0$, two natural choices for $\bar{p}$ are p^0 [EV] and p^1 [CV].

Let $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$.

We also note that $e(p^0, u^0) = e(p^1, u^1) = w$. Then define:

$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$

$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$

Verbally:

EV: The amount of money the consumer would be indifferent to accepting in lieu of a price change. [Before price change] Another phrasing: the change in her wealth that woudl be equivalent to the price change in terms of its welfare impact.

CV: Net revenue of a planner who compensates a consumer for the price change after it occurs, bringing her back to her original utility level.

The specific dollar amounts calculated using the EV and CV will differ because of the differing price vectors at which compensation is assumed to occur.

What are the graphical representations of EV and CV?

[EV left, CV right]

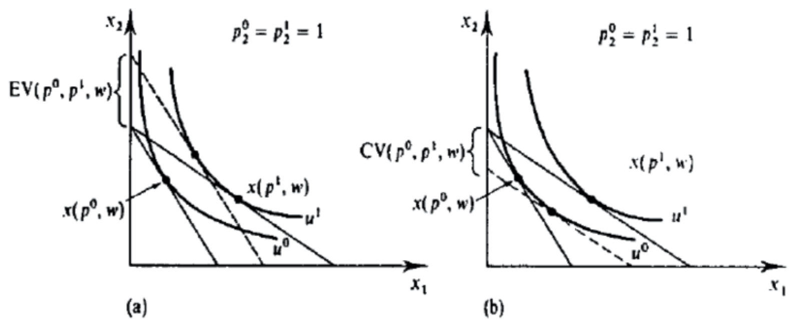


Figure 3.1.2
The equivalent (a) and compensating (b) variation measures of welfare change.

How can the EV and CV be expressed using the Hicksian?

For simplicity, assume only the price of good 1 changes, so that $p_1^0 \neq p_1^1$ and $p_l^0 = p_l^1$ for all $l \neq 1$.

We have the Hicksian demand: $h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1}$

We also note that $e(p^0, u^0) = e(p^1, u^1) = w$.

For EV:

$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w =$

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1) dp_1$$

For CV:

$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0) =$

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0) dp_1$$

What is the graphical depiction of EV and CV using the Hicksian?

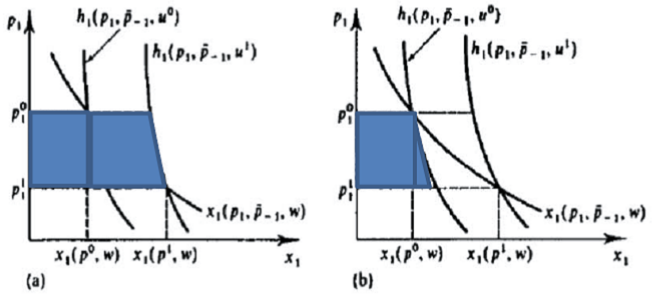


Figure 3.1.3
(a) The equivalent variation.
(b) The compensating variation.

[EV left, CV right]

From the equation, $EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)dp_1$, we see that the EV can be represented as the area lying between p_1^1 and p_1^0 to the left of the Hicksian demand curve for good 1 associated with utility level u^1.

The above graph depicts the situation where good 1 is normal.

If good 1 is normal, $EV(p^0, p^1, w) > CV(p^0, p^1, w)$.

If good 1 is inferior, $EV(p^0, p^1, w) < CV(p^0, p^1, w)$.

When are EV and CV the same for a single good price change?

If there is no wealth effect for good 1, the EV measures are the same because we have

$h_1(p_1, \bar{p}_{-1}, u^1) = h_1(p_1, \bar{p}_{-1}, u^0) = x_1(p_1, \bar{p}_{-1}, w)$

In the case of no wealth effects, we call the common value of CV and EV, which is also the value of the area lying between p_1^0 and p_1^1 and to the left of the market (Walrasian) demand curve for good 1, the change in *Marshallian consumer surplus.*

[One example of this could arise if the underlying preferences are quasilinear with respect to some good $l \neq 1$]

How can the *deadweight loss from commodity taxation* be calculated using the Hicksian demand curve at utility level u^1?

The deadweight loss measure can be represented in terms of the Hicksian demand curve at utility level u^1:

We have tax $= T = tx_1(p^1, w) = th_1(p^1, u^1)$ we can write the deadweight loss as follows:

$(-T) - EV(p_0, \bar{p}_{-1}, u^1) = e(p^0, u^1) - e(p^1, u^1) - T$

$$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^1)dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^1)$$

$$= \int_{p_1^0}^{p_1^0+t} \left[h_1(p_1, \bar{p}_{-1}, u^1) - h_1(p_1^0 + t, \bar{p}_{-1}, u^1) \right] dp_1$$

Can a measure of deadweight loss be calculated using the Hicksian demand curve at utility level u^0?

A similar deadweight loss measure can be represented in terms of the Hicksian demand curve at utility level u^0. It measures the loss from commodity taxation, but in a different way.

In particular, the deadweight loss can be measured as a deficit the government would run to compensate the taxed consumer to return her welfare to original levels (u^0).

The government would run a deficit if the tax collected $th_1(p^1, u^0)$ is less than $-CV(p^0, p^1, w)$.

The deficit can be written as:

$-CV(p^0, p^1, w) - th_1(p^1, u^0) = e(p^1, u^0) - e(p^0, p^0) - th_1(p^1, u^0)$

$$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0) dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^0)$$

$$= \int_{p_1^0}^{p_1^0+t} \left[h_1(p_1, \bar{p}_{-1}, u^0) - h_1(p_1^0 + t, \bar{p}_{-1}, u^0) \right] dp_1$$

Graphically, how does the *deadweight loss from commodity taxation* calculated at u^1compare to the deadweight loss calculated at u^0?

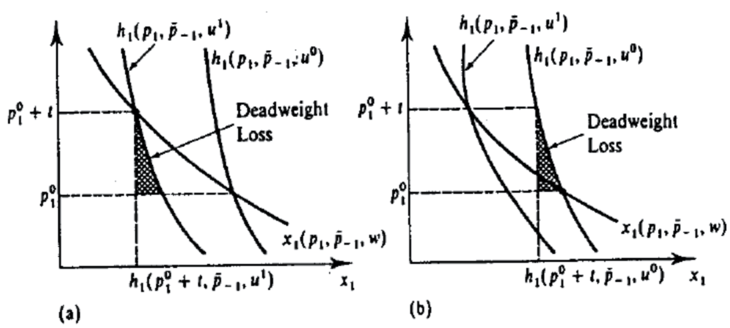


Figure 3.1.4
The deadweight loss from commodity taxation.
(a) Measure based at u^1.
(b) Measure based at u^0.

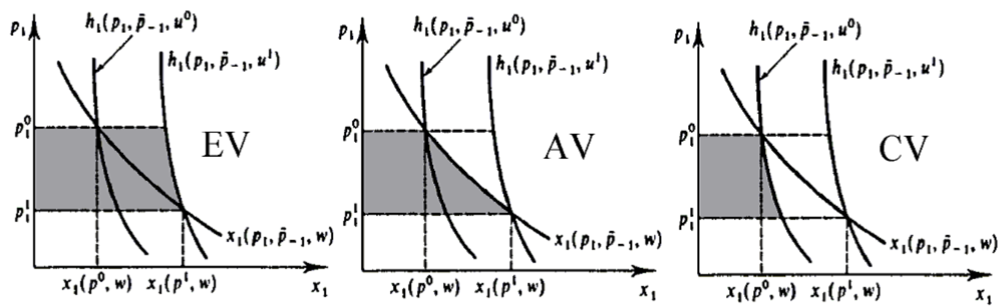
What is the Average Variation?

The Marshallian surplus (or average variation), which is what you see in many undergraduate textbooks, is used frequently as an approximation to EV and CV. It is equal to the area under the Marshallian demand curve:

$$AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x_1(p_1, \bar{p}_{-1}, w) dp_1$$

Only correct (and same to EV and CV) when there are no wealth effects.

For a normal good and a price decline, compare the graphs of EV, CV and AV. What is their order from biggest to smallest?



For a normal good and a price decline, $EV > AV > CV$. This will be reversed for a price rise or an inferior good.

[Welfare Analysis with Partial Information]

If we only know p^0, p^1 and $x^0 = x(p^0, w)$, how can we tell if this price change has made the consumer better off?

Proposition 3.I.1: Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$.

If $(p^1 - p^0) \cdot x^0 < 0$, then the consumer is strictly better off under price-wealth situation (p^1, w) than under (p^0, w).

[If $(p^1 - p^0) \cdot x^0 > 0$, what can we say about the change in consumer's welfare?]

The short answer is that you can't say anything: it could go up or down.

This is because of the convexity of the indifference curve–the new price vector may still be outside the indifference curve. However, if this difference is scaled by some $\alpha \epsilon (0, 1)$, then it will make the consumer worse off:

Proposition 3.I.2: Suppose that the consumer has a differentiable expenditure function.

Then if $(p^1 - p^0) \cdot x^0 > 0$, there is a sufficiently small $\bar{\alpha} \epsilon (0, 1)$ such that for all $\alpha < \bar{\alpha}$, we have $e(\alpha p^1 + (1 - \alpha) p^0, u^0) > w$, and so the consumer is strictly better off under price-wealth situation (p^0, w) than under $(\alpha p^1 + (1 - \alpha) p^0, w)$.

Chapter 4

#AG: Chapter 4 could be renamed "reasons applied general equilibrium theorists like identical homothetic preferences."

What are the two margins on which demand can adjust?

The intensive margin (how much each person buys)
The extensive margin (whether each person buys)

What two key issues are we interested in regarding aggregate demand?

1. To what extent does aggregate demand display similar properties (e.g. law of demand) to individual demand?
2. To what extent can we use a representative consumer framework to capture aggregate outcomes?

What is the definition of aggregate demand?

Let $x^i(p_1, p_2, \ldots, p_L, w)$ denote the demand for the i^{th}consumer out of a total of I consumers.

Aggregate demand is

$$x(p_1, p_2, \ldots, p_L, w) = \sum_{i=1}^{I} x^i(p_1, p_2, \ldots, p_L, w)$$

When can demand be aggregated as above?

The necessary and sufficient condition for this to be the case is for preferences to admit an indirect utility function of the *Gorman polar form*.

That is, it can be written as:

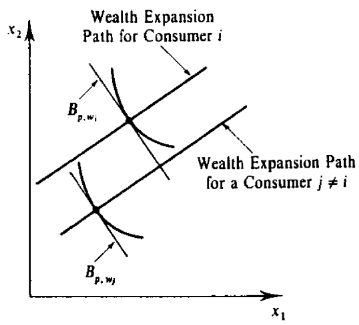
$$v_i(p, w) = a_i(p) + b(p)w_i$$

[Note that $b(p)$, the coefficient on the w_i, is the same for everyone, but the a_i can change]

#This must hold because aggregate demand must be the same for two different distributions of wealth with the same aggregate wealth.

#In particular, starting from any wealth distribution, the aggregate response better be the same if I increase individual $i's$ wealth by dw_i and keep everyone else's wealth constant, or I increase individual $j \neq i$'s wealth by dw_j.

#In other words, the wealth expansion paths (Engel curves) must be straight parellel lines.



What are two cases of the Gorman form that are commonly seen?

1. Quasilinear preferences with respect to the same numeraire good
2. When all preferences are identical and homothetic (but wealth can vary)

When does the aggregate demand function satisfy the weak axiom? Does this always hold?

Definition 4.C.1: The aggregate demand function $x(p, w)$ satisfies the weak axiom if

$$p \cdot x(p', w') \leq x \wedge x(p'w') \neq x(p, w)$$

$$\Longrightarrow p' \cdot x(p, w) > w' \text{ for any } (p, w) \text{ and } (p', w')$$

[$\wedge$is logical conjunction. Think "and". Results in true if both operands are true; otherwise false]

$\Longrightarrow$ This may not hold:

Individual demand functions always satisfy the weak axiom, but aggregation may not.

Why are we interested in whether aggregate demand satisfies the weak axiom?

Because this translates into the law of compensated demand.

What is the definition of the Uncompensated Law of Demand (ULD) for an individual demand function $x^i(p,w)$?

Definition 4.C.2: The individual demand function $x^i(p,w)$ satisfies the Uncompensated Law of Demand if:

$$(p'-p)\cdot\left[x^i(p',w_i)-x^i(p,w_i)\right]\leq 0$$

[Note that the law of *compensated* demand is a similar expression, except of course that wealth is adjusted for the new bundle of goods. The difference is that $w'=p'\cdot x(p,w)$ and we have $(p'-p)\cdot[x(p',\mathbf{w}')-x(p,w)]\leq 0$]

for any p,p' and w_i, with strict equality if $x^i(p',w_i)\neq x^i(p,w_i)$

What is the definition of the Uncompensated Law of Demand for an aggregate demand function $x(p,w)$?

(Very similar)

The aggregate demand function $x(p,w)$ satisfies the Law of Uncompensated Demand if:

$$(p'-p)\cdot(x(p',w)-x(p,w))\leq 0$$

for any p,p' and w, with strict equality if $x(p',w)\neq x(p,w)$

What is one property of individual Walrasian demand functions that will lead to the Law of Uncompensated Demand holding for aggregate demand?

Proposition 4.C.1: If every consumer's Walrasian demand function satisfies the Law of Uncompensated Demand, so does the aggregate demand $x(p_1,p_2,\ldots,p_L,w)=\sum_{i=1}^{I}x^i(p_1,p_2,\ldots,p_L,w)$

What is the other consequence of having every consumer's Walrasian demand function satisfy the Uncompensated Law of Demand (ULD)?

As a consequence, aggregate demand $x(p,w)$ satisfies the weak axiom.

What is a property of the individual preference relationship that will lead to that individual's consumer demand function $x^i(p, w_i)$ satisfying the Law of Uncompensated Demand?

Proposition 4.C.2: If $\succsim^i$ is homothetic, then $x^i(p, w_i)$ satisfies the Law of Uncompensated Demand.

[Note that if $\succsim^i$ is homothetic, then $x^i(p, w_i)$ is homogenous of degree one in w_i, i.e. $\alpha x^i(p, w_i) = x^i(p, \alpha w_i)$]

4.D Aggregate Demand and the Existance of a Representative Consumer

#The best-case scenario for aggregation is that we can relate aggregate demand to a rational preference relation and then use all the tools of chapter 3 for this "representative consumer." Then we can use the Slutsky equation, etc.

#A normative representative consumer requires more assumptions than a positive representative consumer.

#$(w_i(p, w))$ is a distribution rule

When does a positive representative consumer exist?

Definition 4.D.1: A positive representative consumer exists if there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p, w)$ is precisely the Walrasian demand function generated by this preference relation.

That is, $x(p, w) \succ x$ whenever $x \neq x(p, w)$ and $p \cdot x \leq w$.

What is a Bergson-Sameulson social welfare function?

Definition 4.D.2: A Bergson-Sameulson social welfare function is a function $W : \mathbb{R}^L \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, u_2, \ldots, u_L) \epsilon \mathbb{R}^L$ of utility levels for the I consumers in the economy.

What property of $W(\cdot)$ implies redistributive taste?

Concavity of $W(\cdot)$ implies redistributive taste.

What is the "benevolent social planner's problem"?

The benevolent social planner's problem:

$$\max_{\{u\}_1^I} W\left(v_1(p,w_1), v_2(p,w_2), \ldots, v_I(p,w_I)\right)$$

$$s.t. \quad \sum_{i=1}^{I} w_i \leq w$$

where $(w_i(p,w))$ is the distribution rule that is a solution to the social welfare maximization problem.

How can the solution to the benevolent social planner's problem be represented?

Proposition 4.D.1: Suppose that for each level of prices p and aggregate wealth w the wealth distribution $(w_1(p,w), w_2(p,w), \ldots, w_I(p,w))$ solves the benevolent social planner's problem.

Then the value function $v(p,w)$ is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p,w) = \sum_{i=1}^{I} x_i(p, w_i)$.

What is a normative representative consumer?

Definition 4.D.3: The positive representative consumer $\succsim$ for the aggregate demand $x(p,w) = \sum_{i=1}^{L} x_i(p, w_i(p,w))$ is a *normative representative consumer* relative to the social welfare function $W(\cdot)$ if for every (p,w) the distribution of wealth $(w_1(p,w), w_2(p,w), \ldots, w_I(p,w))$ solves the benevolent social planner's problem, and therefore the value function of the problem is an indirect utility function for $\succsim$.

In other words, a positive representative consumer allows us to infer aggregate demand but not social welfare. A normative representative consumer allows social welfare analysis. This requires stronger assumptions.

Chapter 5

Describe the set theoretic approach.

1. *The production vector y* [note the lowercase] describes the net output of L commodities from a production process.

#Negative values correspond to inputs, while positive values correspond to outputs

2. *The production set* Y [note the uppercase] is the set of all production vectors that are feasible.

#Can be described by a transformation function $F(\cdot)$, which is defined so that

$$Y = \left\{ y \epsilon \mathbb{R}^L : F(y) \leq 0 \right\}$$

With this definition, $F(y) = 0$ defines the production possibilities set.

What is is the production possibilities set? Why is does it equal zero?

We have $F(y) = 0$ defines the production possibilities set [also known as the transformation frontier, and $F(\cdot)$ is called the transformation function].

You can think of $F(\cdot)$ as telling you how much you get out for what you put in.

$F(\cdot)$ cannot be positive because you can't get out more than what you put in.

$F(\cdot)$ could be negative, but you can do better.

What is the optimization question using this (set theoretic) approach?

The profit maximization problem is:

$$\max_{y|\ F(y) \leq 0} p \cdot y$$

If $F(\cdot)$ is differentiable, have FOC

$$\max_y p \cdot y - \lambda F(y) \implies$$

$$p - \lambda \frac{\partial F(y)}{\partial y_l} = 0 \ \ \forall\, l \implies$$

$$p = \lambda \frac{\partial F(y)}{\partial y_l} \ \ \forall\, l$$

What is the MRT?

MRT is the marginal rate of transformation. It is produced by dividing the FOCs of two different goods, i and j.

$$MRT_{ij}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial p_i}}{\frac{\partial F(\bar{y})}{\partial p_j}} = \frac{p_i}{p_j}$$

What else is the MRT equal to?

In the UMP, we had that $MRS_{ij} = \frac{\frac{\partial u(x*)}{\partial x_i}}{\frac{\partial u(x*)}{\partial x_j}} = \frac{p_i}{p_j}$

So we have: $MRT_{ij} = MRS_{ij} = \frac{p_i}{p_j}$ in competitive equilibrium, a classic result.

What are the properties of production sets? (9)

i. Y is non-empty
ii. Y is closed.
iii. No free lunch. $[Y \cap \mathbb{R}^L_+ \subset \{0\}]$
iv. Possilibility of inaction $[0 \in Y]$
v. Free disposal $[Y - \mathbb{R}^L_+ \subset Y]$
vi. Irreversibility $[y \in Y,\ y \neq 0 \implies -y \notin Y]$
vii. Non-increasing returns to scale [any feasible input-output vector can be scaled down, that is: $y \in Y \implies \alpha y \in Y \ \ \forall \alpha \in [0,1]$]
viii. Non-decreasing returns to scale [any feasible input-output vector can be scaled up, that is: $y \in Y \implies \alpha y \in Y \ \ \forall \alpha \geq 1$]
ix. Constant returns to scale [conjunction of properties vii and viii. Production Set exhibits constant returns to scale if $y \in Y \implies \alpha y \in Y \forall \alpha \geq 0$. Geometrically, Y is a cone.]
x. Addivity (or free entry) $[y \in Y, y' \in Y \implies y + y' \in Y]$
xi. Convexity $[y, y' \in Y, \alpha \in [0,1] \implies \alpha y + (1-\alpha)y' \in Y]$
xii. Y is a convex cone. [conjunction of ix and xi. $y, y' \in Y$,$\alpha \geq 0, \beta \geq 0 \implies \alpha y + \beta y' \in Y$

Describe the production function approach.

[Usually done in the case of a single output]
Input vector $= z$.
Prodction function $f(z)$ describes maximum number of outputs $f(z)$ produced with input vector z.

What is the optimization question using this (production function) approach?

Where p is the final output price
w is the vector of factor prices
The profit maximization problem is:

$$\max_z pf(z) - w \cdot z$$

The nice thing about this approach is that it defines optimal factor demand functions $z^*(p, w)$.

The FOC is:

$$p\frac{\partial f(z)}{\partial z_l} \leq w_l \quad \forall l \text{ with equality if } z_l > 0$$

What is the MRTS?

The MRTS is the marginal rate of technical substitution, which is achieved by dividing the ratio of factor prices:

$$MRTS_{ij} = \frac{\frac{\partial f(z)}{\partial z_i}}{\frac{\partial f(z)}{\partial z_j}} = \frac{w_i}{w_j}$$

What is the cost minimization problem?

Asks how to produce at the lowest cost while achieving a minimum amount of output.

$$\min_{z \geq 0 | f(z) \geq q} \quad w \cdot z$$

The factor demands it defines are $z(w, q)$ rather than $z(w, p)$.

Chapter 6

What is a lottery?

A lottery is a formal device that is used to represent risky alternatives. The outcomes that may result are certain. Formally:

Definition 6.B.1: A simple lottery L is a list $L = (p_1, p_2, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\sum_{n=1}^{N} p_n = 1$, where p_n is the probability of outcome n occurring.

How can a lottery be represented geometrically?

A simple lottery can be represented geometrically as a point in the $(N-1)$ dimensional simplex, $\Delta = \left\{p\epsilon\mathbb{R}_+^N : \ p_1 + p_2 + \ldots + p_N = 1\right\}$.

When $N = 3$, the simplex can be depicted in two dimensions as an equilateral triangle. Each vertex represents the degenerate lottery where one outcome is certain and the other two outcomes have probability zero.

What is a compound lottery?

A compound lottery allows the outcomes of a lottery themselves to be simple lotteries.

Definition 6.B.2: Given K simple lotteries $L_k = \left(p_1^k, \ldots, p_N^k\right), \ k = 1, \ldots, K$, and probabilities $\alpha_k \geq 0$ with $\sum_{k=1}^K \alpha_k = 1$, the compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

What is a reduced lottery?
How can it be calculated?

For any compound lottery $L_k = \left(p_1^k, \ldots, p_N^k\right), \ k = 1, \ldots, K$, we can calculate a corresponding reduced lottery $L = (p_1, \ldots, p_N)$ that generates the same ultimate distribution over outcomes.

The probability of outcome n in the reduced lottery is: $p_n = \alpha_1 p_n^1 + \cdots + \alpha_K p_n^K$ for $n = 1, \ldots, N$.

Therefore, the reduced lottery L of any compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ can be obtained by vector addition:

$L = \alpha_1 L_1 + \cdots + \alpha_K L_K \epsilon \Delta$.

#Theoretical framework rests on the "consequentialist premise": assume that for any risky alternative, only the reduced lottery over final outcomes is of relevance to the decision-maker.

#Putting decision-maker's problem in general framework of Ch 1:

-Take set of alternatives, denoted here by ℓ, to be the set of all simple lotteries over the set of outcomes C.

-Assume that the decision maker has a rational preference relation $\succsim$ on ℓ, a complete and transitive relation allowing comparison of any pair of lotteries.

-[If anything, rationality assumption is stronger here than Ch 1]

-Two additional assumptions: continuity and independence (most contraversial); below.

What is the continuity axiom for a preference relation $\succsim$ on the space of simple lotteries ℓ?

What is the consequence of the continuity axiom?

Definition 6.B.3: The preference relation $\succsim$ on the space of simple lotteries ℓ is continuous if for any $L, L', L'' \epsilon \ell$, the two sets:

$$\{\alpha \epsilon [0,1] : \ \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$$

$$\{\alpha \epsilon [0,1] : \ L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

In words, continuity means that small changes in probabilities do not change the nature of the ordering between two lotteries. Rules out the case where the decision maker has lexicographic preferences for alternatives with a zero probability of some outcome.

As in Chapter 3, the continuity axiom implies the existance of a utility function representing , a function $U : \ \ell \to \mathbb{R}$ such that $L \succsim L' \iff U(L) \geq U(L')$.

What is the independence axiom?

Definition 6.B.4: The preference relation $\succsim$ on the space of simple lotteries ℓ satisfies the independence axiom if for all $L, L', L'' \epsilon \ell$ and $\alpha \epsilon (0,1)$ we have:

$$L \succsim L' \iff \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

In other words, if we mix each of the two lotteries with a third one, then the preference ordering of the two resulting mixtures does not depend on (is independent of) the particular third lottery used.

What is the expected utility form of a utility function U?

What is this form called?

Definition 6.B.5: The utility function $U : \ \ell \to \mathbb{R}$ has an *expected utility form* if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \epsilon \ell$ we have:

$$U(L) = u_1 p_1 + \cdots + u_N p_N$$

A utility function $U : \ \ell \to \mathbb{R}$ with expected utility form is called a *von Neumann-Morgensterm (v.N-M) expected utility function.*

What functional form must an expected utility function take?

Proposition 6.B.1: A utility function $U : \ell \to \mathbb{R}$ has an expected utility form $\Longleftrightarrow$ it is *linear.*

That is, $\Longleftrightarrow$ it satisfies the property that:

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \epsilon \ell$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0$, $\sum_k \alpha_k = 1$.

What type of property is the expected utility property? What kind of transformations are allowed? (verbal and formal statement)

The expected utility property is a *cardinal* property of utility functions defined on the space of lotteries.

The expected utility form is preserved only by increasing linear transformations:

Proposition 6.B.2: Suppose that $U : \ell \to \mathbb{R}$ is a v.N-M expected utility function for the preference relation $\succsim$on ℓ. Then $\widetilde{U} : \ell \to \mathbb{R}$ is another v.N-M utility function for $\succsim$ $\Longleftrightarrow$ there are scalars $\beta > 0$ and γ such that

$$\widetilde{U}(L) = \beta U(L) + \gamma \text{ for every } L \epsilon \ell$$

What is the expected utility theorem?

Proposition 6.B.3: Suppose that the rational preference relation $\succsim$ on the space of loteries ℓ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form.

That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$, we have

$$L \succsim L' \iff \sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p'_n u_n$$

Expected Utility Theorem:

Proof:

For simplicity, we assume that there are best and worst lotteries in ℓ, $\bar{L}$ and $\underline{L}$ (so, $\overline{L} \succsim L \succsim \underline{L}$ for all $L \epsilon \ell$).

If $\overline{L} \sim \underline{L}$, then all lotteries in ℓ are indifferent and the conclusion of the proposition holds trivially.

Hence, from now on we assume $\overline{L} \succ \underline{L}$,

Step 1:
If $L \succ L'$ and $\alpha\epsilon(0,1)$, then $L \succ \alpha L + (1-\alpha)L' \succ L'$.
This follows from the independence axiom, which implies that:
$L = \alpha L + (1-\alpha)L \succ \alpha L + (1-\alpha)L' \succ \alpha L' + (1-\alpha)L' = L'$

Step 2:
Let $\alpha, \beta\epsilon[0,1]$. Then $\beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L} \iff \beta > \alpha$.
Suppose that $\beta > \alpha$. Note first that we can write
$\beta\overline{L} + (1-\beta)\underline{L} = \gamma\overline{L} + (1-\gamma)[\alpha\overline{L} + (1-\alpha)\underline{L}]$ where $\gamma = \frac{\beta-\alpha}{1-\alpha}\epsilon(0,1]$.
Forward:
A. By step 1, we know that $\overline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L}$
B. By step 1, we know that $\gamma\overline{L} + (1-\gamma)[\alpha\overline{L} + (1-\alpha)\underline{L}] \succ \alpha\overline{L} + (1-\alpha)\underline{L}$
So now we conclude that $\beta\overline{L} + (1-\beta)\underline{L} \succ \alpha\overline{L} + (1-\alpha)\underline{L}$
Converse:
Suppose that $\beta \leq \alpha$.
1. If $\beta = \alpha$, we must have $\beta\overline{L} + (1-\beta)\underline{L} \sim \alpha\overline{L} + (1-\alpha)\underline{L}$
2. Suppose that $\beta < \alpha$. By the argument from the previous paragraph (reversing the roles of β and α), we must have
$\alpha\overline{L} + (1-\alpha)\underline{L} \succ \beta\overline{L} + (1-\beta)\underline{L}$.

Step 3:
For any $L\epsilon\ell$, there is a unique α_L such that $[\alpha_L\overline{L} + (1-\alpha_L)\underline{L}] \sim L$.
Existance of such a $\alpha_L$is implied by continuity of $\succsim$ and the fact that
.....

6.C: Money Lotteries and Risk Aversion

How can a lottery over monetary outcomes be described by a CDF?

Denote amounts of money by the continuous variable x.
We can describe a monetary lottery by means of a *cumulative distribution function (CDF)* $F: \mathbb{R} \to [0,1]$.
That is, for any x, $F(x)$ is the probability that the realized payoff is less than or equal to x. [Think percentile]
Now, we can take the lottery space ℓ to be the set of all distribution functions over nonnegative amounts of money, or, more generally, over an interval $[a, \infty)$.
[Note: formalism with distribution functions as opposed to density functions ($F(x) = \int_{-\infty}^{x} f(t)dt$ for all x) is the advantage of generality; it does not exclude a priori the possibility of a discrete set of outcomes]

How can a compound lottery be expressed through a CDF?

The final distribution of money, $F(\cdot)$, induced by a compound lottery $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ is just the weighted average of the distributions induced by each of the lotteries that constitute it: $F(\cdot) = \sum_{k=1}^{K} \alpha_k F_k(x)$, where $F_k(\cdot)$ is the distribution of the payoff under lottery L_k.

[The CDF of the total lottery is the sum of the CDFs of the individual lotteries, weighted by their probabilities (α)]

This is an example of the general property that distribution functions preserve the linear structure of lotteries.

How can the expected utility theorem be applied to outcomes defined by a continuous variable?

Each (nonnegative) amount of money x can be assigned a utility value $u(x)$, with the property that any $F(\cdot)$ distribution function can be evaluated by a utility function $U(\cdot)$ of the form

$$U(F) = \int u(x) dF(x)$$

What are the two types of utility functions considered in money lotteries, $u(\cdot)$ and $U(\cdot)$?

The *von-Neumann-Morgenstern (v.N-M)* expected utility function $U(\cdot)$ is defined on lotteries.

The *Bernoulli utility function* $u(\cdot)$ is defined on sure amounts of money.

Assume (makes sense in the current monetary context) that the Bernoulli utility function is *increasing* and *continuous.*

In large part, the analytical power of the expected utility formulation hinges on specifying the Bernoulli utility function in such a manner that it captures interesting economic attributes of choice behavior.

What is Jensen's Inequality?
What property is this expression used for?

(If preferences admit an expected utility representation with Bernoulli utility function $u(x)$):

The decision maker is risk averse $\iff$

$$\int u(x) dF(x) \leq u\left(\int x dF(x) \right)$$

for all $F(\cdot)$.

What are the definitions of risk aversion, risk neutral, and strictly risk averse?

Definition 6.C.1: A decision maker is a risk averter (or exhibits risk aversion) if for any lottery $F(\cdot)$, the degenerate lottery $\int xdF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself.

Risk neutral: decision maker is indifferent between $F(\cdot)$ and $\int xdF(x)$

Strictly risk averse: indifference holds only when two lotteries $\left(F(\cdot) and \int xdF(x)\right)$ are the same [i.e. when $F(\cdot)$ is degenerate].

What properties of the utility function are equivalent to risk aversion, strict risk aversion and risk neutrality?

Risk aversion: *Risk aversion is equivalent to the concavity of* $u(\cdot)$, the Bernoulli utility function of money.

[This makes sense since the marginal utility of money is decreasing]

Strict risk aversion: Equivalent to strict concavity of $u(\cdot)$.

Risk neutrality: Equivalent to linear $u(\cdot)$. Jensen's must hold with equality.

What is the certainty equivalent?

The *certainty equivalent* of $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$. That is:

$$u\left(c(F, u)\right) = \int u(x)dF(x)$$

What is the probability premium?

For any fixed amount of money x and positive number ε, the *probability premium* denoted by $\pi(x, \varepsilon, u)$ is the excess in winning probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x - \varepsilon$ and $x + \varepsilon$. That is:

$$u(x) = \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u(x + \varepsilon) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u(x - \varepsilon).$$

What 3 properties are equivalent to a decision-maker being risk averse (who has a Bernoulli utility function)?

i. $u(\cdot)$ is concave

ii. $c(F,u) \leq \int x dF(x)$ for all $F(\cdot)$
iii. $\pi(x,\varepsilon,u) \geq 0$ for all x, ε.

Representing Assets:

If we assume N assets (one of which may be the safe asset) with asset n giving a return of $z_n$per unit of money invested. These returns are jointly distributed according to a distribution function $F(z_1,\ldots,z_N)$. The utility of a holding *portfolio* of assets $(\alpha_1,\ldots,\alpha_L)$ is then:

$$U(\alpha_1,\ldots,\alpha_N) = \int u(\alpha_1 z_1 + \cdots + \alpha_N z_N) dF(z_1,\ldots,z_N)$$

This utility function for portfolios is also increasing, continuous and concave.

What is the *Arrow-Pratt coefficient of absolute risk aversion*?

Definition 6.C.3: Given a (twice differentiable) Bernoulli utility function $u(\cdot)$ for money, the *Arrow-Pratt coefficient of absolute risk aversion* at x is defined as

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

This is the amount of money you would pay to avoid an infinitesimally small risk.

It makes sense to relate it to the curvature of the utility function, $u''(x)$, and to make it independent of transformations, scale by $u'(x)$. The negative sign allows more curvature in the concave state to simply more risk aversion.

What is the *coefficient of relative risk aversion*?

Definition 6.C.5: Given a Bernoulli utility function $u(\cdot)$, the *coefficient of relative risk aversion* at x is

$$r_R(x,u) = -\frac{xu''(x)}{u'(x)}$$

This is the fraction of your wealth you would give up to avoid an infinitesimally small risk.

6.D Comparison of Payoff Distributions in Terms of Return and Risk

#If risk aversion allows us to compare risk preferences, stochastic dominance allows us to compare lotteries.

What is first-order stochastic dominance?

"The distribution $F(\cdot)$ yields unambiguously higher returns than the distribution $G(\cdot)$"

Proposition 6.D.1: The distribution of monetary payoffs $F(\cdot)$ *first-order stochastically dominates the distribution* $G(\cdot)$ $\iff$

$$F(x) \leq G(x) \quad \text{for every } x$$

[Here, the functions are CDFs, hence $F(x) \leq G(x)$ $\forall x$ implies that there is more density at higher probabilities in the CDF $F(x)$]

We also can think about it:

[AG notes] $F(\cdot)$ stochastically dominates $G(\cdot)$ if it is a rightward shift of $G(\cdot)$ at all points. In other words, it is a stronger version of saying its mean is higher [which does not imply stochastic dominance]. It gives a better payoff with higher probability.

This also gives us the result:

Definition 6.D.1: The distribution $F(\cdot)$ first order stochastically dominates $G(\cdot)$ if, for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$, we have:

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

What is second-order stochastic dominance? What is another way it can be characterized?

$F(\cdot)$ *second-order stochastically dominates* $G(\cdot)$ if they have the same mean and $F(\cdot)$ is less risky than $G(\cdot)$, e.g. $G(\cdot)$ has a higher variance.

Definition 6.D.2: For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates* (or *is less risky than*) $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R}_+ \to \mathbb{R}$, we have:

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

Another way of saying this is that $G(\cdot)$ is a *mean-preserving spread* of $F(\cdot)$. [For example, if $G(\cdot)$ is a lottery of each of the outcomes of $F(\cdot)$ (with the same mean as that outcome)]

6.E State-Dependent Utility

#Consider the possibility that the decision maker may care not only about his montary returns but also about the underlying events, or states of nature, that cause them.

#Denote set of sates by S and an individual state by $s \epsilon S$. For simplicity, assume set of states is finite and that each state s has a well-defined, objective probability $\pi_s > 0$ that it occurs.

#Because we take S to be finite, we can represent a random variable with monetary payoffs by the vector $(x_1, \ldots, x_S)$ where x_s is the nonnegative *payoff vector* in state s.

What is the *extended expected utility representation*?

The generalization allows for a different function $u_s(\cdot)$ in every state. Before, had the same Bernoulli utility function $u(\cdot)$ which is called "state-independent" or "state-uniform" [doesn't vary with different states].

[EG: It is like the independence axiom]

Definition 6.E.2: The preference relation $\succsim$has an *extended expected utility representation* if for every $s \epsilon S$, there is a function $u_s : \mathbb{R}_+ \to \mathbb{R}$ such that $(x_1, \ldots, x_S) \epsilon \mathbb{R}^S_+$ and $(x^{'}_1, \ldots, x^{'}_S) \epsilon \mathbb{R}^S_+$

$$(x_1, \ldots, x_S) \succsim (x^{'}_1, \ldots, x^{'}_S) \iff \sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x^{'}_s)$$

When is there an extended expected utility representation? What is its expression?

Proposition 6.E.2: Suppose that there are at least three states and that the preferences $\succsim$on $\mathbb{R}^S_+$are continuous and satisfy the sure-thing axiom. Then $\succsim$admits and extended utility representation:

$$U(x_1, \ldots, x_s) = \sum_{s=1}^{S} \pi_s u_s(x_s)$$

[If $u_s(x_s) = u(x_s)$, then preferences are not state-contingent; we are in the special case of expected utility]

Compensating Differentials

#Two main types of equilibrium assumptions

First, market clearing – supply equals demand

Second, no arbitrage assumption (focus of this topic)

#Equalizing differences or compensating differentials:

Returns or costs must offset other things.

[Application of the hedonic model–which values something by decomposing it into specific parts, and valuing each component]

What is the core assumption about utility levels in compensating differentials?

#Heart of compensating differentials is that utility must be equal across people.

Let each job have a set of characteristics represented by a vector of n continuous variables z.

What does the no arbitrage assumption tell us? (Verbally and formally)

The no arbitrage assumption is that people move jobs if utility is greater in one job, which equalizes utility at some level $\bar{u}$.

We can write:

$$\bar{\mathbf{u}} = u(w(z_1, \ldots, z_n), z_1, \ldots, z_n)$$

This gives us:

$$0 = \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z_i} + \frac{\partial u}{\partial z_i}$$

$$\implies \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z_i} = -\frac{\partial u}{\partial z_i}$$

In the above set-up, what does $\frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z_i} = -\frac{\partial u}{\partial z_i}$ mean intuitively?

Jobs with unpleasant characteristics [where $\frac{\partial u}{\partial z_i}$ is negative] have higher wages.

How can we model the real estate market with a no arbitrage assumption, assuming each location has different rent and characteristics?

What happens to rent with characteristics?

Each location has different rent and characteristics: characteristics z will be exogenous, rents will be endogenous and will need to clear the market so that utility is equal across space.

$$\bar{u} = u(w - r(z_1, \ldots, z_n), z_1, \ldots, z_n)$$

FOC:
$0 = \frac{\partial u}{\partial x} \cdot \left(-\frac{\partial r}{\partial z_i}\right) + \frac{\partial u}{\partial z_i} \implies$

$$\frac{\partial u}{\partial x} \cdot \frac{\partial r}{\partial z_i} = \frac{\partial u}{\partial z_i}$$

Rents rise exactly to make you pay for characteristics.

3.1 Alonso-Muth-Mills Model

#Goal is to capture relationship between distance from city center and housing prices and housing density.
#Story is going to be all about commuting time

What is the utility function and budget constraint for the AMM Model?
[Need at least three ingredients:
1. housing consumption (assume just talking about land quantity)
2. cost of commuting
3. some alternative use of money]

utility function: $U(C, A)$
such that $C = w - wT(d) - p(d)A$
so we have $u(w - wT(d) - p(d)A, A)$

Step 1: Individual Maximization
Write the maximization equation. What are the choice variables? What are the FOCs?

Choice variables: $A,\ d$

$$\max_{A,d} u(w - wT(d) - p(d)A, A)$$

FOC (A) :

$$\frac{\partial u}{\partial A} + \frac{\partial u}{\partial C}(-p(d)) = 0 \quad \implies \quad \frac{\partial u}{\partial A} = p(d)\frac{\partial u}{\partial C} \quad \text{a.k.a} \quad \implies \quad u_A = p(d)u_C$$

FOC (d) :

$$\frac{\partial u}{\partial C}\left(-w\frac{\partial T}{\partial d}-A\frac{\partial p}{\partial d}\right)=0 \implies w\frac{\partial T}{\partial d}=-A\frac{\partial p}{\partial d} \text{ a.k.a } \implies wT'(d)=-Ap'(d)$$

What further simplifying assumption is made?

Assume $T(d)=td$. Then:
FOC (d) : $\frac{wt}{A}=-p(d)$

Step 2: Impose Spatial Equilibrium
What does imposing spatial equilibrium mean verbally?

Assume that people live at each distance d on some interval $\left[0,\bar{d}\right]$ and are freely mobile.

They must be indifferent between all values of d on $\left[0,\bar{d}\right]$. Hence each consumer has utility value $\bar{u}$.

How do we use this formally?

To take advantage of this, set utility equal to $\bar{u}$ across space.
We then use the implicit function theorem to define $A^*(d)$:

$$\bar{u}=u(A^*(d),w-wT(d)-p(d)A^*(d)) \quad \text{for all } d$$

Differentiating by d, we get:

$$\frac{\partial u}{\partial A^*}\cdot\frac{\partial A^*}{\partial d}+\frac{\partial u}{\partial C}\left(-w\frac{\partial T}{\partial d}-A^*\frac{\partial p}{\partial d}\right)=0$$

We have $\frac{\partial A^*}{\partial d}=0$. Hence:

$$w\frac{\partial T}{\partial d}=-A^*\frac{\partial p}{\partial d}$$

Using the linearized version of $T(d)=td$, we get:

$$\frac{wt}{A^*}=-p'(d)$$

This is the same as the 2nd FOC before, *but now it holds everywhere, not just at the optimum.*

It pins down the price gradient.

How do these two expressions differ:

$$\max_{A,d} u(w-wT(d)-p(d)A,A) \quad \text{and} \quad \bar{u}=u(A^*(d),w-wT(d)-p(d)A^*(d))$$

The first is the individual maximization problem. It yields FOCs that hold at the maximized value for that individual.

The second equation describes the spatial equilibrium for the entire population. Using the implicit function theorem and taking the total derivative lets us derive conditions that hold everywhere, not just as optimum.

Step 3: Comparative Statics

Assume city is populated by homogeneous consumers.

How does lot size A change with distance? (How to take comparative static?)

Take the *FOC of individual's demand curve with respect to A* above, and use the implicit function theorem to define $A^*(d)$:

$$\frac{\partial u}{\partial A} = p(d)\frac{\partial u}{\partial C} \implies u_A(A^*, w-wtd-p(d)A^*) - p(d)u_C(A^*, w-wtd-p(d)A^*) = 0 = G(A^*(d), d)$$

Taking comparative statics gives (using implicit function theorem):

$$\frac{\partial A^*}{\partial d} = -\frac{\frac{\partial G}{\partial d}}{\frac{\partial G}{\partial A}} = -\frac{\frac{\partial u}{\partial A}\left(-wt - A^*\frac{\partial p}{\partial d}\right) - p(d)\frac{\partial u}{\partial C}\left(-wt - A^*\frac{\partial p}{\partial d}\right) - \frac{\partial p}{\partial d}\cdot\frac{\partial u}{\partial C}}{\frac{\partial^2 u}{\partial A^2} - 2p(d)\frac{\partial^2 u}{\partial A \partial C} + \frac{\partial^2 u}{\partial C^2}p(d)^2}$$

$$= \frac{\left(\frac{\partial u}{\partial A} - p(d)\frac{\partial u}{\partial C}\right)\left(-wt - A^*\frac{\partial p}{\partial d}\right) - \frac{\partial p}{\partial d}\cdot\frac{\partial u}{\partial C}}{\frac{\partial^2 u}{\partial A^2} - 2p(d)\frac{\partial^2 u}{\partial A \partial C} + \frac{\partial^2 u}{\partial C^2}p(d)^2}$$

We have from *Spatial Equilibrium* (Step 2) that $\frac{wt}{A^*} = -p(d)$ which gives $-wt - A^*\frac{\partial p}{\partial d} = 0$. Hence the above expression can be simplified to:

$$= \frac{\frac{\partial p}{\partial d}\cdot\frac{\partial u}{\partial C}}{\frac{\partial^2 u}{\partial A^2} - 2p(d)\frac{\partial^2 u}{\partial A \partial C} + \frac{\partial^2 u}{\partial C^2}p(d)^2}$$

$$= \frac{-\frac{\partial p}{\partial d}\cdot\frac{\partial u}{\partial C}}{-\left[\frac{\partial^2 u}{\partial A^2} - 2p(d)\frac{\partial^2 u}{\partial A \partial C} + \frac{\partial^2 u}{\partial C^2}p(d)^2\right]}$$

[In EG's notation]

$$= \frac{-p'(d)u_C}{-\left[u_{AA} - 2p(d)u_{AC} + p(d)^2 u_{CC}\right]}$$

We know the expression is positive (the denominator is positive by SOC $<$ 0; we know $\frac{\partial u}{\partial C} > 0$ can see $\frac{\partial p}{\partial d} < 0$).

Hence lot sizes rise with distrance from city center.

Step 4: Deriving Estimating Equations

#A problem is that we don't have anything we can directly estimate.
For this we need to assume a functional form.
Classic case: $u(A, c) = c + \alpha \ln(A)$

Math

Implicit Function Theorem

Let $G(x, y)$ be a C^1 function on a ball about (x_0, y_0) in $\mathbb{R}^2$.
Suppose $G(x_0, y_0) = c$ and consider the expression

$$G(x, y) = c$$

If $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$, then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that:

1. $G(x, y(x)) = c$ for all x in I
2. $y(x_0) = y_0$
3. We have
$$y'(x_0) = \frac{\partial y^*}{\partial x} = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}$$