ECON 7020
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Problem Set 6

Problems from McCandless and Wallace:
Chapter 9 Exercises:
9.1-9.6

Problem 1. Take a two-period OLG model with production. Agents maximize their discounted stream of utility over the two periods of their life. The budget constraint of the young is given by: $c_t^h(t) = w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1)$ and the budget constraint of the old is given by: $c_t^h(t+1) = w(t+1)\Delta_t^h(t+1) + r^l(t)l^h(t) + r^k(t+1)k^h(t+1)$. Suppose utility is given by: $u_t^h = c_t^h(t)[c_t^h(t+1)]^\beta$. Also assume that factor markets are prefectly competitive and the production function is given by $Y(t) = \gamma(t)L(t)^{1-\alpha}K(t)^\alpha$. In addition, let the population be constant so that $N(t)+N(t-1) = 1$ for all t and $\gamma(t+1) = (1+g)\gamma(t)$.

a. Derive the lifetime budget constraint (LBC) and state the no arbitrage condition. Why must the arbitrage condition hold in equilibrium?

Answer: To derive the LBC we solve for $l^h(t)$ in the budget contraint when young and when old. Setting them equal to each other and after some rearrangement we arive at:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r^l(t)} = w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r^l(t)} - k^h(t+1)[1 - \frac{r^k(t+1)}{r^l(t)}]$$

To see why the no arbitrage condition must hold in equilibrium, consider the term: $[1 - \frac{r^k(t+1)}{r^l(t)}]$. If we have the case in which $r^k(t+1) > r^l(t)$ then the term $[1 - \frac{r^k(t+1)}{r^l(t)}]$ is negative which means agents want to borrow and buy an infinite amount of capital. Since there are no lenders in the case of homogeneous agents and there are only a finite number of resources this cannot be an equilibrium. Now consider the case in which $r^k(t+1) < r^l(t)$: this would then make agents want to hold the minimum amount of capital which is zero by constraint. As long as we have production that results in diminishing marginal product of capital, this would result in $r^k(t+1)$ being driven to infinity which would be a contradiction of the condition $r^k(t+1) < r^l(t)$. So this cannot be an equilibrium. We must have $r^k(t+1) = r^l(t)$ in equilibrium to rule out the cases discussed above.

b. Derive the individual savings function for an arbitrary agent h.

Answer: In this economy, the savings for agent h of generation t is the difference between time t labor income and time t consumption:

$$s_t^h(r^l(t)) = w(t)\Delta_t^h(t) - c_t^h(t) \tag{1}$$

from the textbook on page 235, you can then follow his steps to arrive at the savings function for an arbitrary agent h:

$$s_t^h(r^l(t)) = \frac{\beta w(t)\Delta_t^h(t)}{1+\beta} - \frac{w(t+1)\Delta_t^h(t+1)}{(1+\beta)r^l(t)} \tag{2}$$

c. Define a perfect foresight competitive equilibrium.

Answer: **Definition** A perfect foresight competitive equilibrium for an economy with labor endowments and a production function of $\gamma(t)L(t)^{1-\alpha}K(t)^\alpha$ is a sequence of $K(t)$, $r^l(t)$, $w(t)$, and $r^k(t+1)$ for $t \geq 1$ such that, given an initial capital stock $K(1) > 0$,

$S_t(r^l(t)) = K(t+1)$,

$r^l(t) = r^k(t+1)$,

$w(t) = (1-\alpha)\gamma(t)L(t)^{-\alpha}K(t)^\alpha$,

and $r^k(t) = \alpha\gamma(t)L(t)^{1-\alpha}K(t)^{\alpha-1}$,

hold for all $t >= 1$.

d. Using the fact that $L(t) = N(t)\Delta_t^h(t) + N(t-1)\Delta_{t-1}^h(t)$ solve for the steady state capital stock.

Answer: Please see page 239 for the textbook derivation.

e. Assuming $g > 0$ find the steady state growth rate of the capital stock. What is the growth rate of output?

Answer: Please see page 248 for the textbook derivation.