Chapter 3

Problem: 3.B.3

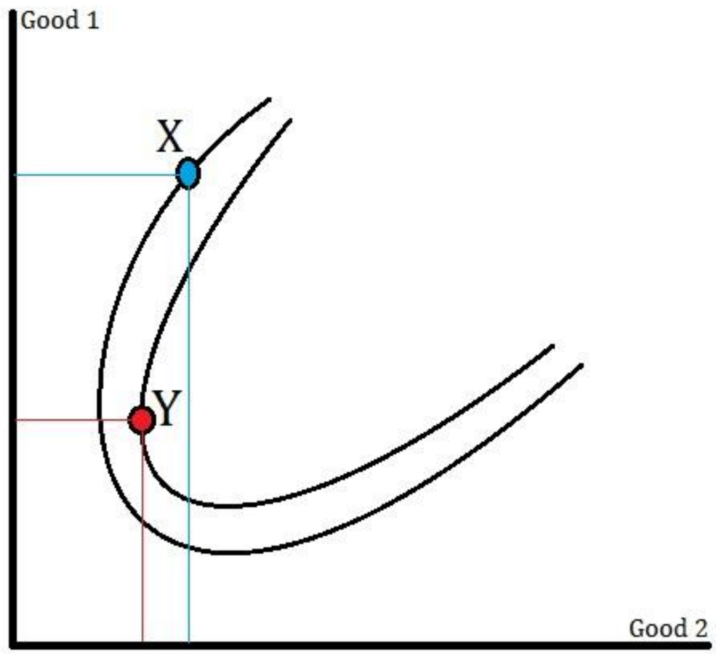
Draw a convex preference relation that is locally non-satiated but is not monotone.

Answer

For the preference relation to be monotone, when $x \gg y$ then it implies that $x \succ y$.

However, we need a preference relation that is not monotone or in other words would contradict monotonicity. Such would be that $x \gg y$ and $y \succ x$.

As we can see in the graph bellow, X has more of both goods compared to Y, but Y is preferred to X.



Problem: 3.C.2

Show that if $u(\cdot)$ is a continuous utility function representing a preference relation $\succsim$, then the preference relation $\succsim$ is continuous.

Answer

Take a sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ such that $x^n \succsim y^n$ for all n.

And in the limit we have $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$.

Then it implies that $u(x^n) \geq u(y^n)$ for all n and continuity of $u(\cdot)$ implies that $u(x) \geq u(y)$.

Hence, $x \succsim y$. Thus, $\succsim$ is continuous.

Problem: 3.C.6

Suppose that in a two commodity world, the consumer's utility function takes the form as follows.

$$u(x) = \left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{1/\rho}$$

This utility function is known as the constant elasticity of substitution (CES) utility function.

a) Show that when $\rho = 1$, indifference curve becomes linear.
b) Show that when $\rho \to 0$, this utility function comes to represent the same preferences as the generalized Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.
c) Show that as $\rho \to -\infty$, indifference curve become "right angles"; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = Min\{x_1, x_2\}$.

Answer

a) For part a), lets plug in $\rho = 1$ into the utility function.

$$u(x) = \left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{1/\rho} = [\alpha_1 x_1^{1} + \alpha_2 x_2^{1}]^{1/1} = \alpha_1 x_1 + \alpha_2 x_2$$

We see that this utility function is a combination of two linear parts and therefore is linear (no powers other than 1).

b) For part b), let's use a monotonic transformation of a utility function. Since any monotonic transformation of a utility function represents the same preference, we are not losing any generality. Let's try using log version of our utility function.

$$\tilde{u}(x) = \ln\big(u(x)\big) = \ln\left[\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right]^{1/\rho} = \frac{1}{\rho}\ln\left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)$$

Now using L'Hopital's rule, let's find a limit of our newly transformed utility function $\tilde{u}(x)$.

$$L'Hopital's\ rule: \lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to 0}\left(\frac{f'(x)}{g'(x)}\right)$$

$$\lim_{\rho\to 0}[\tilde{u}(x)] = \lim_{\rho\to 0}\left[\frac{ln\left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)}{\rho}\right] = \lim_{\rho\to 0}\left[\frac{\frac{\partial}{\partial\rho}\left(ln\left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)\right)}{\frac{\partial}{\partial\rho}(\rho)}\right] = \lim_{\rho\to 0}\left[\frac{\partial}{\partial\rho}\left(ln\left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)\right)\right]$$

$$= \lim_{\rho\to 0}\left\{\frac{1}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}}\left[\alpha_1 x_1^{\rho}\ln(x_1) + \alpha_2 x_2^{\rho}\ln(x_2)\right]\right\}$$

$$= \lim_{\rho\to 0}\left\{\frac{\alpha_1 x_1^{\rho}\ln(x_1) + \alpha_2 x_2^{\rho}\ln(x_2)}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}}\right\} = \frac{\alpha_1\ln(x_1) + \alpha_2\ln(x_2)}{\alpha_1 + \alpha_2}$$

Now monotonically transforming this utility function again to get rid of the bottom part and get rid of log part we see that the utility function does indeed reflect Cobb-Douglas utility function.

$$\hat{u}(x) = (\alpha_1 + \alpha_2)\tilde{u}(x) = \alpha_1\ln(x_1) + \alpha_2\ln(x_2)$$

$$\breve{u}(x) = e^{\hat{u}(x)} = e^{\alpha_1\ln(x_1) + \alpha_2\ln(x_2)} = e^{\alpha_1\ln(x_1)}e^{\alpha_2\ln(x_2)} = x_1^{\alpha_1}x_2^{\alpha_2}$$

c) For part c) of the problem, let's suppose that $x_1 \leq x_2$.

We want to show that $x_1 = \lim_{\rho \to -\infty} \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}$. This is equivalent to the Leontief preferences.

Let $\rho < 0$.

Since $x_1 \geq 0$ and $x_2 \geq 0$, we have

$$\alpha_1 x_1^\rho \leq \alpha_1 x_1^\rho + \alpha_2 x_2^\rho$$

Raising to the power of $1/\rho$, we have this.

$$\alpha_1^{1/\rho} x_1 \leq \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}$$

On the other hand, since $x_1 \leq x_2$, we have

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \geq \alpha_1 x_1^\rho + \alpha_2 x_1^\rho$$

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \geq (\alpha_1 + \alpha_2) x_1^\rho$$

Now raising to the power of $1/\rho$, we have this.

$$\left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho} \geq (\alpha_1 + \alpha_2)^{1/\rho} x_1$$

Now, combining the two inequalities that we raised to the power of $1/\rho$, we have the following.

$$(\alpha_1 + \alpha_2)^{1/\rho} x_1 \leq \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho} \leq \alpha_1^{1/\rho} x_1$$

Now taking limits of the two functions around $\left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}$, we have this.

$$\lim_{\rho \to -\infty} \left((\alpha_1 + \alpha_2)^{1/\rho} x_1\right) = \lim_{\rho \to -\infty} \left(\frac{x_1}{(\alpha_1 + \alpha_2)^{-1/\rho}}\right) = x_1$$

$$\lim_{\rho \to -\infty} \left(\alpha_1^{1/\rho} x_1\right) = \lim_{\rho \to -\infty} \left(\frac{x_1}{\alpha_1^{-1/\rho}}\right) = x_1$$

Since $\left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}$ is between the two functions and both of those two functions around converge to x_1 as $\rho \to -\infty$. This means, that $\left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}$ also converges to x_1 as $\rho \to -\infty$.

$$\lim_{\rho \to -\infty} \left(\left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho}\right) = x_1$$

This implies that in the limit of negative infinity, this utility function will equal to either x_1 (if we assume that $x_1 \leq x_2$) or x_2 if we assume $x_2 \leq x_1$. This can be represented as we usually represent Leontief utility function.

$$u(x_1, x_2) = Min\{x_1, x_2\}$$

Problem: 3.D.5 (let u(x)=p*u(x)^p)

Consider the CES utility function

$$u(x) = \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{1/\rho}$$

And assume that $\alpha_1 = \alpha_2 = 1$.

$$u(x) = \left[x_1^\rho + x_2^\rho\right]^{1/\rho}$$

a) Compute Walrasian demand and indirect utility functions.
b) Verify that these two functions satisfy the properties discussed in the textbook. Demand function - homogeneity, Walras' law, uniqueness. Indirect utility function – homogeneity, monotonicity, continuity, quasi-convexity.
c) Derive Walrasian demand for the case of linear utility and the case of Leontief utility. Show that CES Walrasian demand and indirect utility functions approach these as $\rho \to 1$ and as $\rho \to -\infty$, respectively.
d) Elasticity of Substitution (EOS) between good 1 and good 2 is defined bellow. Show that for the CES utility function EOS $EOS = \frac{1}{1-\rho}$, thus justifying its name. What is EOS for the linear, Leontief, and Cobb-Douglas utility functions?

$$EOS_{1,2}(p,w) = -\frac{\partial[x_1(p,w)/x_2(p,w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p,w)/x_2(p,w)}$$

Answer

a) For part a), we will use a monotonic transformation of the utility function and then solve the UMP problem.

$$\tilde{u}(x) = \rho[u(x)]^\rho = \rho\left[x_1^\rho + x_2^\rho\right]$$

$$\mathcal{L} = \rho\left[x_1^\rho + x_2^\rho\right] + \lambda(w - p_1x_1 - p_2x_2)$$

We know from Micro Theory 1, that after solving FOC's for the UMP we get the following result. Substituting in marginal utilities, we solve for either x_1 or x_2.

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2}; \qquad MU_1 = \rho^2 x_1^{\rho-1}; \qquad MU_2 = \rho^2 x_2^{\rho-1}$$

$$\frac{\cancel{\rho}^2 x_1^{\rho-1}}{p_1} = \frac{\cancel{\rho}^2 x_2^{\rho-1}}{p_2} \rightarrow x_1^{\rho-1}p_2 = x_2^{\rho-1}p_1 \rightarrow x_2 = x_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}} \; and \; x_1 = x_2\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

Plugging x_2 value into the budget constraint, we can solve the demand function for x_1.

$$w = p_1x_1 + p_2x_2 = p_1x_2\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2x_2 \rightarrow x_2\left(p_1\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2\right) = w$$

$$x_2=\frac{w}{\left(p_1^{\frac{\rho-1+1}{\rho-1}}p_2^{\frac{-1}{\rho-1}}+p_2\right)}=\frac{w}{\left[p_1^{\frac{\rho}{\rho-1}}p_2^{\frac{-1}{\rho-1}}+p_2\right]}*\frac{\left(p_2^{\frac{1}{\rho-1}}\right)}{\left(p_2^{\frac{1}{\rho-1}}\right)}=\frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho-1+1}{\rho-1}}}=\frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}$$

Letting $\delta=\frac{\rho}{\rho-1}$, we can simplify the expression of x_2 to the following.

$$x_2=\frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}=\frac{wp_2^{\frac{\rho}{\rho-1}-\frac{\rho-1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}=\frac{wp_2^{\delta-1}}{p_1^{\delta}+p_2^{\delta}}$$

Same way solving for the value x_1 we get the following.

$$w=p_1x_1+p_2x_2=p_1x_1+p_2x_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}}$$

$$x_1=\frac{w}{p_1+p_2^{\frac{\rho}{\rho-1}}p_1^{\frac{-1}{\rho-1}}}*\frac{\left(p_1^{\frac{1}{\rho-1}}\right)}{\left(p_1^{\frac{1}{\rho-1}}\right)}=\frac{wp_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}=\frac{wp_1^{\delta-1}}{p_1^{\delta}+p_2^{\delta}}$$

Therefore, the Walrasian demand function is as follows.

$$x^*(p,w)=\begin{bmatrix}\left(\frac{wp_1^{\delta-1}}{p_1^{\delta}+p_2^{\delta}}\right)\\\left(\frac{wp_2^{\delta-1}}{p_1^{\delta}+p_2^{\delta}}\right)\end{bmatrix}$$

The indirect utility function is derived by plugging in the demand function into the original utility. I will use the form with powers in ρ instead of the later one with powers in δ.

$$v(p,w)=u\big(x^*(p,w)\big)=\left[\left(\frac{wp_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}\right)^{\rho}+\left(\frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}\right)^{\rho}\right]^{1/\rho}=\left[\frac{w^{\rho}\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)}{\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\rho}}\right]^{1/\rho}$$

$$=w\left[\frac{\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)}{\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\rho}}\right]^{1/\rho}=w\left[\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{1-\rho}\right]^{1/\rho}=w\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}$$

This can be expressed with powers in δ for simplification of the function.

$$v(p,w)=w\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}=\frac{w}{\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}}=\frac{w}{\left(p_1^{\delta}+p_2^{\delta}\right)^{1/\delta}}$$

b) For part b), we will show that the functions we derived in the previous part hold all the necessary properties.

Starting with the demand function, lets show that it is homogeneous of degree zero, satisfies Walras' law and is unique.

$$x(\alpha p, \alpha w) = \begin{bmatrix} \left(\dfrac{\alpha w \alpha^{\delta-1} p_1^{\delta-1}}{\alpha^\delta p_1^\delta + \alpha^\delta p_2^\delta}\right) \\ \left(\dfrac{\alpha w \alpha^{\delta-1} p_2^{\delta-1}}{\alpha^\delta p_1^\delta + \alpha^\delta p_2^\delta}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha^\delta}{\alpha^\delta}\left(\dfrac{w p_1^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \\ \dfrac{\alpha^\delta}{\alpha^\delta}\left(\dfrac{w p_2^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{w p_1^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \\ \left(\dfrac{w p_2^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \end{bmatrix} = x(p, w)$$

$$p \cdot x(p, w) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \cdot \begin{bmatrix} \left(\dfrac{w p_1^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \\ \left(\dfrac{w p_2^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \end{bmatrix} = \frac{w p_1^{\delta-1} p_1^1 + w p_2^{\delta-1} p_2^1}{p_1^\delta + p_2^\delta} = w\left(\frac{p_1^\delta + p_2^\delta}{p_1^\delta + p_2^\delta}\right) = w$$

It is also a unique function, because for every choice of p and w there exists a unique value of x.

Now we will show that the indirect utility function satisfies homogeneity of degree zero, monotonicity and continuity.

$$v(bp, bw) = \frac{bw}{\left(b^\delta p_1^\delta + b^\delta p_2^\delta\right)^{1/\delta}} = \frac{bw}{\left[b^\delta\left(p_1^\delta + p_2^\delta\right)\right]^{1/\delta}} = \frac{bw}{b\left(p_1^\delta + p_2^\delta\right)^{1/\delta}} = \frac{w}{\left(p_1^\delta + p_2^\delta\right)^{1/\delta}} = v(p, w)$$

To show monotonicity, we will show the signs of the first derivatives.

$$\frac{\partial v(p, w)}{\partial w} = \frac{1}{\left(p_1^\delta + p_2^\delta\right)^{1/\delta}} > 0\ (always\ positive)$$

$$\frac{\partial v(p, w)}{\partial p_1} = \frac{-w\dfrac{1}{\delta}\delta p_1^{\delta-1}}{\left(p_1^\delta + p_2^\delta\right)^{\left(\frac{1}{\delta}\right)+1}} = -\frac{w p_1^{\delta-1}}{\left(p_1^\delta + p_2^\delta\right)^{\left(\frac{\delta+1}{\delta}\right)}} < 0\ (always\ negative)$$

$$\frac{\partial v(p, w)}{\partial p_2} = \frac{-w\dfrac{1}{\delta}\delta p_2^{\delta-1}}{\left(p_1^\delta + p_2^\delta\right)^{\left(\frac{1}{\delta}\right)+1}} = -\frac{w p_2^{\delta-1}}{\left(p_1^\delta + p_2^\delta\right)^{\left(\frac{\delta+1}{\delta}\right)}} < 0\ (always\ negative)$$

Also, the indirect utility function is continuous because it is differentiable at any point.

To prove quasi-convexity by the property of homogeneity, it is sufficient to prove that for any $v \in R$ and $w > 0$, the set $(p \in R^2: v(p, w) \leq v)$ is convex. Define $f(p) = \left(p_1^\delta + p_2^\delta\right)^{1/\delta}$ and $f(p)^\delta = p_1^\delta + p_2^\delta$.

If $\delta \in (0,1)$, then $f(p)^\delta = p_1^\delta + p_2^\delta$ is a concave function. Hence, $\left(p \in R^2: \left(f(p)^\delta\right)^{\frac{1}{\delta}} \geq v\right)$ is convex for every v. Since, $v(p, w) = \frac{w}{f(p)}$, this implies that $(p \in R^2: v(p, w) \leq v)$ is convex for every v and w.

If $\delta < 0$, then $f(p)^\delta = p_1^\delta + p_2^\delta$ is a convex function. Hence, $\left(p \in R^2: \frac{1}{f(p)} = \left(f(p)^\delta\right)^{-\frac{1}{\delta}} \leq v\right)$ is convex for every v. Since, $v(p, w) = \frac{w}{f(p)}$, this implies that $(p \in R^2: v(p, w) \leq v)$ is convex for every v and w.

c) For part c) of the problem we will derive Walrasian demand functions for linear, Leontief preferences. From the previous part, we already have demand function for the general case of the CES utility function.

Linear utility function with $\alpha_1 = \alpha_2 = 1$ can be stated as the following.

$$u(x) = x_1 + x_2$$

Thus, solving the UMP problem we get everything as follows.

$$\mathcal{L} = x_1 + x_2 + \lambda(w - p_1x_1 - p_2x_2)$$

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2}; \qquad MU_1 = 1; \qquad MU_2 = 1$$

$$\lambda = \frac{1}{p_1} = \frac{1}{p_2}$$

We know that λ shows marginal utility per dollar. If marginal utility per dollar is higher in good 1, individual should consume more of good 1 until marginal utilities per dollar are equal. Except, in this case, they will not converge – the indifference curve is a straight line.

Case 1: $p_1 > p_2$	Case 2: $p_1 < p_2$	Case 3: $p_1 = p_2$
$p_1 > p_2$ $\frac{1}{p_1} < \frac{1}{p_2} \quad \rightarrow \quad \frac{MU_1}{p_1} < \frac{MU_2}{p_2}$ MU per dollar is always greater from good 2, therefore, the consumer will spend all of his wealth on good 2. $x(p,w) = \begin{bmatrix} (0) \\ (w/p_2) \end{bmatrix}$ $v_1(p,w) = 0 + \left(\frac{w}{p_2}\right) = \left(\frac{w}{p_2}\right)$	$p_1 < p_2$ $\frac{1}{p_1} > \frac{1}{p_2} \quad \rightarrow \quad \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$ MU per dollar is always greater from good 1, therefore, the consumer will spend all of his wealth on good 1. $x(p,w) = \begin{bmatrix} (w/p_1) \\ (0) \end{bmatrix}$ $v_2(p,w) = \left(\frac{w}{p_1}\right) + 0 = \left(\frac{w}{p_1}\right)$	$p_1 = p_2$ $\frac{1}{p_1} = \frac{1}{p_2} \quad \rightarrow \quad \frac{MU_1}{p_1} = \frac{MU_2}{p_2}$ MU per dollar is always equal from both of the goods. Therefore, the consumer will be equally well of spending his wealth on any (linear) combination of the two goods. $p_1 = p_2 = p$ $\lambda \in [0,1]$ $x(p,w) = \begin{bmatrix} (\lambda)(w/p) \\ (1-\lambda)(w/p) \end{bmatrix}$ $v_3 = \lambda\left(\frac{w}{p_1}\right) + (1-\lambda)\left(\frac{w}{p_2}\right) = \frac{w}{p}$

Indirect utility is obtained by plugging $x(p,w)$ into the utility function but since we have a few cases. Since, we see that depending on the case, we have complete spending on one good or another, or a mix with indirect utility $\frac{w}{p}$ as in the first two cases. Therefore, we can generalize indirect utility function as:

$$v(p,w) = \max\left(\left(\frac{w}{p_1}\right), \left(\frac{w}{p_2}\right)\right)$$

Now, let's show that CES demand function approaches linear demand function as $\rho \to 1$.

$$x(p,w) = \begin{bmatrix} \left(\dfrac{wp_1^{\delta-1}}{p_1^{\delta}+p_2^{\delta}}\right) \\ \left(\dfrac{wp_2^{\delta-1}}{p_1^{\delta}+p_2^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+\left(\frac{p_2}{p_1}\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1+\left(\frac{p_1}{p_2}\right)^{\delta}}\right) \end{bmatrix}, \ \ and \quad \delta = \frac{\rho}{\rho-1}$$

<table>
<tr>
<td rowspan="3">$\rho < 1$
and
$\rho \to 1$

This implies
$\lim_{\rho \to 1^-} \delta = \lim_{\rho \to 1^-} \frac{\rho}{\rho - 1} \to -\infty$</td>
<td>Case 1: $p_2 > p_1$ $\quad \frac{p_2}{p_1} > 1 \ and \ \frac{p_1}{p_2} < 1$
$$\lim_{\delta \to -\infty} \left(\frac{p_2}{p_1}\right)^{\delta} \to 0, \ \ and \quad \lim_{\delta \to -\infty} \left(\frac{p_1}{p_2}\right)^{\delta} \to \infty$$
$$\lim_{\delta \to -\infty} \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+\left(p_2/p_1\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1+\left(p_1/p_2\right)^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+0}\right) \\ \left(\dfrac{wp_2^{-1}}{1+\infty}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{w}{p_1}\right) \\ (0) \end{bmatrix}$$</td>
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<td>Case 2: $p_1 > p_2$ $\quad \frac{p_2}{p_1} < 1 \ and \ \frac{p_1}{p_2} > 1$
$$\lim_{\delta \to -\infty} \left(\frac{p_2}{p_1}\right)^{\delta} \to \infty, \ \ and \quad \lim_{\delta \to -\infty} \left(\frac{p_1}{p_2}\right)^{\delta} \to 0$$
$$\lim_{\delta \to -\infty} \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+\left(p_2/p_1\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1+\left(p_1/p_2\right)^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+\infty}\right) \\ \left(\dfrac{wp_2^{-1}}{1+0}\right) \end{bmatrix} = \begin{bmatrix} (0) \\ \left(\dfrac{w}{p_2}\right) \end{bmatrix}$$</td>
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<td>Case 3: $p_2 = p_1$ $\quad \frac{p_2}{p_1} = 1 \ and \ \frac{p_1}{p_2} = 1$
$$\lim_{\delta \to -\infty} \left(\frac{p_2}{p_1}\right)^{\delta} = 1, \ \ and \quad \lim_{\delta \to -\infty} \left(\frac{p_1}{p_2}\right)^{\delta} = 1$$
$$\lim_{\delta \to -\infty} \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+\left(p_2/p_1\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1+\left(p_1/p_2\right)^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1+1}\right) \\ \left(\dfrac{wp_2^{-1}}{1+1}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{w}{2p_1}\right) \\ \left(\dfrac{w}{2p_1}\right) \end{bmatrix}$$</td>
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$\rho > 1$

and

$\rho \to 1$

This implies

$$\lim_{\rho \to 1^+} \delta = \lim_{\rho \to 1^+} \frac{\rho}{\rho - 1} \to +\infty$$

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Case 1: $p_2 > p_1$ $\quad \frac{p_2}{p_1} > 1\ and\ \frac{p_1}{p_2} < 1$

$$\lim_{\delta \to +\infty} \left(\frac{p_2}{p_1}\right)^{\delta} \to \infty, \quad and \quad \lim_{\delta \to +\infty} \left(\frac{p_1}{p_2}\right)^{\delta} \to 0$$

$$\lim_{\delta \to +\infty} \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1 + \left(p_2/p_1\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1 + \left(p_1/p_2\right)^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1 + \infty}\right) \\ \left(\dfrac{wp_2^{-1}}{1 + 0}\right) \end{bmatrix} = \begin{bmatrix} (0) \\ \left(\dfrac{w}{p_2}\right) \end{bmatrix}$$

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Case 2: $p_1 > p_2$ $\quad \frac{p_2}{p_1} < 1\ and\ \frac{p_1}{p_2} > 1$

$$\lim_{\delta \to -\infty} \left(\frac{p_2}{p_1}\right)^{\delta} \to 0, \quad and \quad \lim_{\delta \to -\infty} \left(\frac{p_1}{p_2}\right)^{\delta} \to \infty$$

$$\lim_{\delta \to -\infty} \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1 + \left(p_2/p_1\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1 + \left(p_1/p_2\right)^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1 + 0}\right) \\ \left(\dfrac{wp_2^{-1}}{1 + \infty}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{w}{p_1}\right) \\ (0) \end{bmatrix}$$

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Case 3: $p_2 = p_1$ $\quad \frac{p_2}{p_1} = 1\ and\ \frac{p_1}{p_2} = 1$

$$\lim_{\delta \to -\infty} \left(\frac{p_2}{p_1}\right)^{\delta} = 1, \quad and \quad \lim_{\delta \to -\infty} \left(\frac{p_1}{p_2}\right)^{\delta} = 1$$

$$\lim_{\delta \to -\infty} \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1 + \left(p_2/p_1\right)^{\delta}}\right) \\ \left(\dfrac{wp_2^{-1}}{1 + \left(p_1/p_2\right)^{\delta}}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{wp_1^{-1}}{1 + 1}\right) \\ \left(\dfrac{wp_2^{-1}}{1 + 1}\right) \end{bmatrix} = \begin{bmatrix} \left(\dfrac{w}{2p_1}\right) \\ \left(\dfrac{w}{2p_1}\right) \end{bmatrix}$$

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$$\lim_{\rho \to 1^-} v_1(p, w) = \lim_{\rho \to 1^+} v_4(p, w) = \left[\left(\frac{w}{p_1}\right)^{\rho} + (0)^{\rho}\right]^{1/\rho} = \frac{w}{p_1}$$

$$\lim_{\rho \to 1^-} v_2(p, w) = \lim_{\rho \to 1^+} v_3(p, w) = \left[(0)^{\rho} + \left(\frac{w}{p_2}\right)^{\rho}\right]^{1/\rho} = \frac{w}{p_2}$$

$$v(p, w) = \max\left\{\frac{w}{p_1}, \frac{w}{p_2}\right\}$$

For the Leontief utility, we have the following.

$$u(x) = \min(x_1, x_2)$$

There is a kink where $x_1 = x_2 \quad \textit{because we have} \quad \alpha_1 = \alpha_2 = 1.$

This implies that the consumer will spend equally on both of the goods.

$$x(p,w) = \begin{bmatrix} \left(\frac{w}{p_1 + p_2}\right) \\ \left(\frac{w}{p_1 + p_2}\right) \end{bmatrix}$$

Now, let's show that CES demand function approaches Leontief demand function as $\rho \to -\infty$.

$$x(p,w) = \begin{bmatrix} \left(\frac{wp_1^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \\ \left(\frac{wp_2^{\delta-1}}{p_1^\delta + p_2^\delta}\right) \end{bmatrix} = \begin{bmatrix} \left(\frac{wp_1^{-1}}{1 + \left(\frac{p_2}{p_1}\right)^\delta}\right) \\ \left(\frac{wp_2^{-1}}{1 + \left(\frac{p_1}{p_2}\right)^\delta}\right) \end{bmatrix}, \quad and \quad \delta = \frac{\rho}{\rho - 1}$$

$$\rho \to -\infty, \qquad \lim_{\rho \to -\infty} \delta = \lim_{\rho \to -\infty} \frac{\rho}{\rho - 1} \to 1$$

$$\lim_{\delta \to 1} \begin{bmatrix} \left(\frac{wp_1^{-1}}{1 + (p_2/p_1)^\delta}\right) \\ \left(\frac{wp_2^{-1}}{1 + (p_1/p_2)^\delta}\right) \end{bmatrix} = \begin{bmatrix} \left(\frac{wp_1^{-1}}{\frac{p_1 + p_2}{p_1}}\right) \\ \left(\frac{wp_2^{-1}}{\frac{p_1 + p_2}{p_2}}\right) \end{bmatrix} = \begin{bmatrix} \left(\frac{w}{p_1 + p_2}\right) \\ \left(\frac{w}{p_1 + p_2}\right) \end{bmatrix}$$

d) For part d) of the problem, we will compute EOS for CES, linear and Leontief Walrasian demand functions.

$$\frac{x_1(p,w)}{x_2(p,w)} = \frac{\left(\dfrac{wp_1^{\delta-1}}{p_1^\delta + p_2^\delta}\right)}{\left(\dfrac{wp_2^{\delta-1}}{p_1^\delta + p_2^\delta}\right)} = \left(\frac{p_1}{p_2}\right)^{\delta-1}$$

$$\frac{\partial[x_1(p,w)/x_2(p,w)]}{\partial[p_1/p_2]} = \frac{\partial\left[\left(\frac{p_1}{p_2}\right)^{\delta-1}\right]}{\partial[p_1/p_2]} = (\delta-1)\left(\frac{p_1}{p_2}\right)^{\delta-2}$$

$$EOS_{1,2}(p,w) = -\frac{\partial\left[\frac{x_1(p,w)}{x_2(p,w)}\right]}{\partial\left[\frac{p_1}{p_2}\right]}\frac{\frac{p_1}{p_2}}{\frac{x_1(p,w)}{x_2(p,w)}} = -(\delta-1)\left(\frac{p_1}{p_2}\right)^{\delta-2}\frac{\left(\frac{p_1}{p_2}\right)}{\left(\frac{p_1}{p_2}\right)^{\delta-1}} = -(\delta-1) = \frac{1}{1-\rho}$$

For the linear Walrasian demand function, EOS is as follows.

$$\lim_{\rho\to1} EOS_{1,2}(p,w) = \lim_{\rho\to1}\frac{1}{1-\rho} \to \infty$$

For the Leontief Walrasian demand function, EOS is as follows.

$$\lim_{\rho\to-\infty} EOS_{1,2}(p,w) = \lim_{\rho\to1}\frac{1}{1-\rho} \to 0$$

For the Cobb-Douglas Walrasian demand function, EOS is as follows.

$$\lim_{\rho\to0} EOS_{1,2}(p,w) = \lim_{\rho\to0}[-(\delta-1)] = \lim_{\rho\to0}\left[\frac{1}{1-\rho}\right] = 1$$

Problem: 3.D.6 (a, b)

Consider the 3-good setting in which the consumer has utility function:

$$u(x) = (x_1 - b_1)^{\alpha}(x_2 - b_2)^{\beta}(x_3 - b_3)^{\gamma}$$

a) Why can you assume $\alpha + \beta + \gamma = 1$ without loss of generality? Do so for the rest of the problem!
b) Write down the FOCs for the UMP, derive Walrasian demand and indirect utility functions. Use log form of the utility function.

ANSWER

a) Yes. WE can do a utility transformation.

b) First we set up Lagrangian for the UMP and then solve FOCs and solve for x_1, x_2, x_3.

$$\mathcal{L} = \alpha \ln(x_1 - b_1) \ + \beta \ln(x_2 - b_2) + \gamma \ln(x_3 - b_3) + \lambda(w - p_1x_1 - p_2x_2 - p_3x_3)$$

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_1}{p_1} = \frac{MU_1}{p_1}$$

$$\lambda = \frac{\alpha}{p_1(x_1 - b_1)} = \frac{\beta}{p_2(x_2 - b_2)} = \frac{\gamma}{p_3(x_3 - b_3)}$$

$$\frac{\alpha}{p_1(x_1 - b_1)} = \frac{\beta}{p_2(x_2 - b_2)} \gg x_1 = \frac{\alpha p_2(x_2 - b_2)}{\beta p_1} + b_1; \quad x_2 = \frac{\beta p_1(x_1 - b_1)}{\alpha p_2} + b_2$$

$$\frac{\alpha}{p_1(x_1 - b_1)} = \frac{\gamma}{p_3(x_3 - b_3)} \gg \quad x_1 = \frac{\alpha p_3(x_3 - b_3)}{\gamma p_1} + b_1; \quad x_3 = \frac{\gamma p_1(x_1 - b_1)}{\alpha p_3} + b_3$$

$$\frac{\beta}{p_2(x_2 - b_2)} = \frac{\gamma}{p_3(x_3 - b_3)} \gg x_2 = \frac{\beta p_3(x_3 - b_3)}{\gamma p_2} + b_2; \quad x_3 = \frac{\gamma p_2(x_2 - b_2)}{\beta p_3} + b_3$$

Now plug x_3 and x_2 in terms of x_1 into the constraint w to solve for x_1.

$$w = p_1x_1 + \left[p_2 \frac{\beta p_1(x_1 - b_1)}{\alpha p_2} + b_2\right] + p_3\left[\frac{\gamma p_1(x_1 - b_1)}{\alpha p_3} + b_3\right]$$

$$w = p_1x_1 + \frac{\beta}{\alpha}p_1(x_1 - b_1) + p_2b_2 + \frac{\gamma}{\alpha}p_1(x_1 - b_1) + p_3b_3$$

$$w = p_1x_1 + [p_1(x_1 - b_1)]\left(\frac{\beta + \gamma}{\alpha}\right) + p_2b_2 + p_3b_3$$

$$w = p_1x_1 + [p_1(x_1 - b_1)]\left(\frac{1 - \alpha}{\alpha}\right) + p_2b_2 + p_3b_3$$

$$w - p_2b_2 - p_3b_3 + \left(\frac{1 - \alpha}{\alpha}\right)p_1b_1 = p_1x_1 + p_1x_1\left(\frac{1 - \alpha}{\alpha}\right)$$

$$x_1p_1\left(1 + \frac{1 - \alpha}{\alpha}\right) = w - p_2b_2 - p_3b_3 + \left(\frac{1 - \alpha}{\alpha}\right)p_1b_1$$

$$x_1 = \frac{w - p_2 b_2 - p_3 b_3 + \left(\frac{1}{\alpha}\right) p_1 b_1 - p_1 b_1}{p_1 \left(\frac{1}{\alpha}\right)}$$

$$x_1 = \left(\frac{\alpha}{p_1}\right)(w - p_1 b_1 - p_2 b_2 - p_3 b_3) + b_1 = \left(\frac{\alpha}{p_1}\right)(w - pb) + b_1$$

In the same fashion, we can solve for x_2, x_3.

$$x_2 = \left(\frac{\beta}{p_2}\right)(w - p_1 b_1 - p_2 b_2 - p_3 b_3) + b_2 = \left(\frac{\beta}{p_2}\right)(w - pb) + b_2$$

$$x_3 = \left(\frac{\gamma}{p_3}\right)(w - p_1 b_1 - p_2 b_2 - p_3 b_3) + b_3 = \left(\frac{\gamma}{p_3}\right)(w - pb) + b_3$$

To get the indirect utility function, we just plug the values of x into the original utility function (not the log form we used to get x values).

$$u = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma$$

$$v(p, w) = \left(\left(\frac{\alpha}{p_1}\right)(w - pb) + b_1 - b_1\right)^\alpha \left(\left(\frac{\beta}{p_2}\right)(w - pb) + b_2 - b_2\right)^\beta \left(\left(\frac{\gamma}{p_3}\right)(w - pb) + b_3 - b_3\right)^\gamma$$

$$v(p, w) = \left(\left(\frac{\alpha}{p_1}\right)(w - pb)\right)^\alpha \left(\left(\frac{\beta}{p_2}\right)(w - pb)\right)^\beta \left(\left(\frac{\gamma}{p_3}\right)(w - pb)\right)^\gamma$$

$$v(p, w) = \left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{\beta}{p_2}\right)^\beta \left(\frac{\gamma}{p_3}\right)^\gamma (w - pb)^{\alpha+\beta+\gamma}$$

$$v(p, w) = (w - pb)\left(\frac{\alpha}{p_1}\right)^\alpha \left(\frac{\beta}{p_2}\right)^\beta \left(\frac{\gamma}{p_3}\right)^\gamma$$