Chapter 15

Handout Question 8

Consider a 2 good, 2 consumer economy.
Consumer A has preferences represented by $u_A(x) = 4x_1 + 3x_2$.
Consumer B has preferences represented by $u_B(x) = 3\ln(x_1) + 4\ln(x_2)$.
Suppose consumer A's endowment is $\omega_A = (12{,}9)$.
Suppose consumer B's endowment is $\omega_B = (8{,}11)$.

What is the competitive equilibrium of this economy?

Answer

First, we solve the individual Walrasian demand functions for each consumer.

For consumer A:

$$\mathcal{L} = u_A + \lambda(w - p_1x_1 - p_2x_2) \rightarrow \lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \rightarrow \frac{4}{p_1} = \frac{3}{p_2} \rightarrow \frac{p_1}{p_2} = \frac{4}{3}$$

$$If\ \frac{p_1}{p_2} > \frac{4}{3},\ then\ \frac{MU_1}{p_1} < \frac{MU_2}{p_2}\ and\ w = p_1 0 + p_2x_2$$

$$If\ \frac{p_1}{p_2} > \frac{4}{3},\ then\ x_A(p,w) = \left(0, \frac{w}{p_2}\right)$$

$$If\ \frac{p_1}{p_2} < \frac{4}{3},\ then\ x_A(p,w) = \left(\frac{w}{p_1}, 0\right)$$

$$If\ \frac{p_1}{p_2} = \frac{4}{3},\ then\ x_A(p,w) = (\{x_1, x_2\} \in R_+^2 : p_1x_1 + p_2x_2 = w_A)$$

For consumer B:

$$\mathcal{L} = u_B + \lambda(w - p_1x_1 - p_2x_2) \rightarrow \lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \rightarrow \frac{3}{x_1p_1} = \frac{4}{x_2p_2}$$

$$4x_1p_1 = 3x_2p_2 \rightarrow x_2 = \left(\frac{4}{3}\right)\left(\frac{p_1}{p_2}\right)x_1 \rightarrow w = p_1x_1 + p_2\left(\frac{4}{3}\right)\left(\frac{p_1}{p_2}\right)x_1$$

$$w = x_1\left(p_1 + \left(\frac{4}{3}\right)p_1\right) \rightarrow w = x_1\left(\frac{7}{3}\right)p_1 \rightarrow x_1 = \left(\frac{3}{7}\right)\left(\frac{w}{p_1}\right)$$

$$x_1 = \left(\frac{3}{4}\right)\left(\frac{p_2}{p_1}\right)x_2 \rightarrow w = \frac{3\cancel{p_1}p_2}{4\cancel{p_1}}x_2 + p_2x_2 \rightarrow w = \frac{7}{4}p_2x_2 \rightarrow x_2 = \left(\frac{4}{7}\right)\left(\frac{w}{p_2}\right)$$

$$x_B(p,w) = \left\{\left(\frac{3w}{7p_1}\right), \left(\frac{4w}{7p_2}\right)\right\}$$

Total Endowment in the economy of the good 1 and good 2 is

$$\omega_1 = 12 + 8 = 20 \qquad and \qquad \omega_2 = 11 + 9 = 20$$

The wealth of consumer A and consumer B is

$$w_A = p \cdot \omega_A = 12p_1 + 9p_2$$

$$w_B = p \cdot \omega_B = 8p_1 + 11p_2$$

Plugging these in to the Walrasian demand function gives us the offer curves of the two consumers

$$If\ \frac{p_1}{p_2} > \frac{4}{3},\ then\ \ OC_A(p) = \left(0, \frac{12p_1 + 9p_2}{p_2}\right)$$

$$If\ \frac{p_1}{p_2} < \frac{4}{3},\ then\ \ x_A(p,w) = \left(\frac{12p_1 + 9p_2}{p_1}, 0\right)$$

$$OC_B(p) = \left\{\left(\frac{(24p_1 + 33p_2)}{7p_1}\right), \left(\frac{(32p_1 + 44p_2)}{7p_2}\right)\right\}$$

We know that the contract curve is where the MRS of both consumers' are equal (where the slopes of their indifference curves match).

make the MRS of both of the consumer's equal to find the contract curve.

$$MRS_A = \frac{\partial u_A/\partial x_1}{\partial u_A/\partial x_2} = \frac{4}{3} \qquad\qquad MRS_B = \frac{\partial u_B/\partial x_1}{\partial u_B/\partial x_2} = \frac{3x_1^{-1}}{4x_2^{-1}}$$

$$MRS_A = MRS_B \qquad \frac{4}{3} = \frac{3x_2^B}{4x_1^B} \qquad \frac{x_2^B}{x_1^B} = \frac{16}{9}$$

Thus the contract curve is a straight line from point 0_B.

Also, we can draw the indifference curve of consumer A in red which is $-MRS_A = -\frac{\partial u_A/\partial x_1}{\partial u_A/\partial x_2} = -\frac{4}{3}$ and goes through the initial endowment point ω.

At the point where the contract curve and the MRS of consumer A crosses, we know that is where consumer B's indifference curve is also tangent because at that point $-MRS_B = -\frac{\partial u_B/\partial x_1}{\partial u_B/\partial x_2} = -\frac{3x_2}{4x_1}$ has the same slope. Indifference curve of consumer B is in blue.

Thus, at x^*, we have $MRS_A = MRS_B$ and our equilibrium allocation of goods. The prices thus must be $\frac{p_1}{p_2} = \frac{4}{3}$.

$$Assuming\ \frac{p_1}{p_2} = \frac{4}{3},\quad OC_{B,1} = \left(\frac{(24(4) + 33(3))}{7(4)}\right) = \frac{195}{28} = 6.964$$

$$Assuming\ \frac{p_1}{p_2} = \frac{4}{3},\quad OC_{B,2} = \left(\frac{(32(4) + 44(3))}{7(3)}\right) = \frac{128 + 132}{21} = 12.381$$

Thus, the rest is taken by consumer A.

$$OC_{A,1} = 20 - \left(\frac{(24(4) + 33(3))}{7(4)}\right) = 20 - \frac{195}{28} = 13.036$$

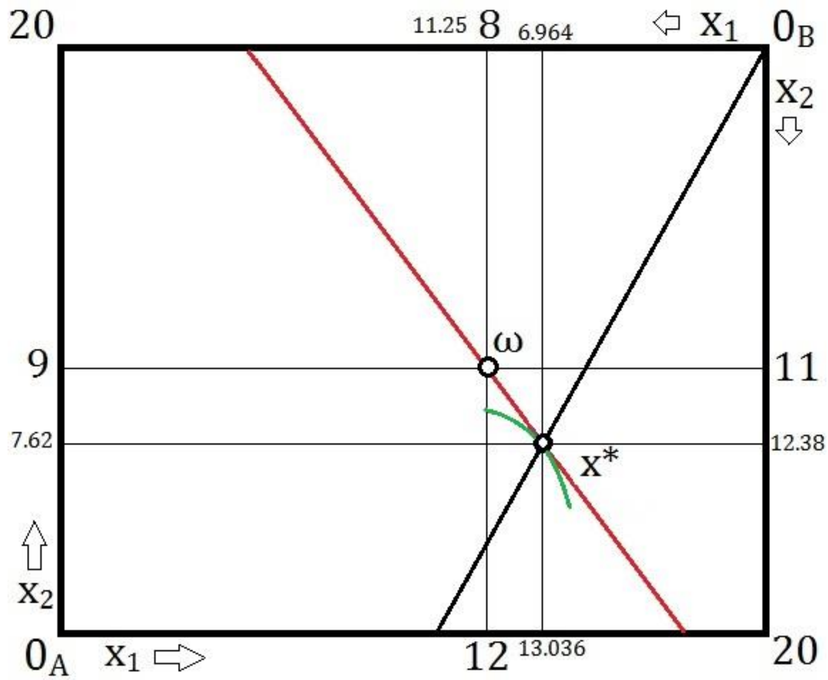
$$OC_{A,2} = 20 - \left(\frac{(32(4) + 44(3))}{7(3)}\right) = 20 - \frac{128 + 132}{21} = 7.619$$

Thus, the equilibrium is where

$$p^* = \left(\frac{p_1}{p_2}\right)^* = \frac{4}{3}$$

$$x_A^* = (13.036,\ \ 7.619)$$

$$x_B^* = (6.964,\ \ 13.381)$$



<u>Other calculations that are more or less irrelevant now.</u>

Total demand cannot be above the total endowment of the good.

$$Assuming\ \frac{p_1}{p_2} > \frac{4}{3}, \qquad 0 + \left(\frac{(24p_1 + 33p_2)}{7p_1}\right) = 20\ \rightarrow\ \frac{p_1}{p_2} = \frac{33}{116} \not> \frac{4}{3}\ thus\ not\ solution\ in\ the\ box$$

$$Assuming\ \frac{p_1}{p_2} < \frac{4}{3}, \qquad \frac{12p_1 + 9p_2}{p_1} + \left(\frac{(24p_1 + 33p_2)}{7p_1}\right) = 20\ \rightarrow\ 12 + 9\frac{p_2}{p_1} + \frac{24}{7} + \frac{33}{7}\frac{p_2}{p_1} = 20$$

$$\frac{p_2}{p_1} = \frac{32}{7} * \frac{7}{96} = \frac{1}{3}\ \ \rightarrow\ \frac{p_1}{p_2} = 3 \not< \frac{4}{3}\ thus\ NOT\ solution$$

$$Assuming\ \frac{p_1}{p_2} = \frac{4}{3}, \quad OC_{B,1} = \left(\frac{(24(4) + 33(3))}{7(4)}\right) = \frac{195}{28} = 6.964$$

$$Assuming\ \frac{p_1}{p_2} = \frac{4}{3}, \quad OC_{B,2} = \left(\frac{(32(4) + 44(3))}{7(3)}\right) = \frac{128 + 132}{21} = 12.381$$

If consumer B is better with bundle $(6.964, 12.381)$ than his endowment $(8,11)$ then trade will take place and $\frac{p_1}{p_2} = \frac{4}{3}$ will be the competitive equilibrium. This is only viable if we allow for non-integer quantities of goods.

$$u_B(\omega_B) = 3\ln(8) + 4\ln(11) = 6.238 + 9.592 = 15.829$$

$$u_B\left(\frac{p_1}{p_2} = \frac{4}{3}\right) = 3\ln(6.964) + 4\ln(12.381) = 5.822 + 10.065 = 15.887$$

$$u_B\left(\frac{p_1}{p_2} = \frac{4}{3}\right) > u_B(\omega_B) \ \rightarrow thus, the\ consumer\ B\ will\ want\ to\ trade$$

Consumer A, then, must consume what's left and if that gives more utility than the initial endowment, consumer A will also want to trade.

$$u_A(\omega_A) = 4(12) + 3(9) = 75$$

$$u_A\left(\frac{p_1}{p_2} = \frac{4}{3}\right) = 4(20 - 6.964) + 3(20 - 12.381) = 75$$

$$u_A\left(\frac{p_1}{p_2} = \frac{4}{3}\right) \geq u_A(\omega_A) \ \rightarrow \quad thus, the\ consumer\ A\ will\ want\ to\ trade.$$

Thus, allowing non-integer quantities, shows that $\frac{p_1}{p_2} = \frac{4}{3}$ is the competitive equilibrium relative prices with the quantities showed above, however, if only integer quantities can be traded and consumed, then there is no competitive equilibrium and each consumer consumes his/her endowment.

Handout Question 9

Consider a 2 good, 2 consumer economy.
Consumer A has preferences represented by $u_A(x) = \min\{4x_1 + 3x_2, 3x_1 + 4x_2\}$.
Consumer B has preferences represented by $u_B(x) = 3x_1 + 4\ln(x_2)$.
Suppose consumer A's endowment is $\omega_A = (12,9)$.
Suppose consumer B's endowment is $\omega_B = (8,11)$.

What is the competitive equilibrium of this economy?

Answer

Total Endowment in the economy of the good 1 and good 2 is

$$\omega_1 = 12 + 8 = 20 \qquad and \qquad \omega_2 = 11 + 9 = 20$$

The wealth of consumer A and consumer B is

$$w_A = p \cdot \omega_A = 12p_1 + 9p_2$$
$$w_B = p \cdot \omega_B = 8p_1 + 11p_2$$

Equating the marginal rate of substitution's in consumption for consumers A and B, we get the following

$MRS_A = \frac{3}{4}$ $MRS_A = \frac{4}{3}$	$MRS_B = \frac{3}{4}x_2^B$

$$MRS_A = \frac{3}{4} = MRS_B = \frac{3}{4}x_2^B \rightarrow \quad x_2^B = 1$$
$$MRS_A = \frac{4}{3} = MRS_B = \frac{3}{4}x_2^B \rightarrow \quad x_2^B = \frac{16}{9}$$

<u>Solving the Walrasian demand function for consumer B</u>

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \rightarrow \quad \frac{3}{p_1} = \frac{4}{x_2 p_2} \quad \rightarrow \quad 3x_2p_2 = 4p_1 \quad \rightarrow \quad x_2^B = \frac{4p_1}{3p_2}$$

Thus, there are two possibilities of prices now

$$x_2^B = \frac{4p_1}{3p_2} = 1 \quad \rightarrow \quad \frac{p_1}{p_2} = \frac{3}{4}$$
$$x_2^B = \frac{4p_1}{3p_2} = \frac{16}{9} \quad \rightarrow \quad \frac{p_1}{p_2} = \frac{4}{3}$$

<u>Continue solving the Walrasian demand function for consumer B</u>

$$w = x_1p_1 + x_2p_2 \quad \rightarrow \quad w = x_1p_1 + p_2\frac{4p_1}{3p_2} \quad \rightarrow \quad x_1 = \frac{w}{p_1} - \frac{4}{3} = \left(\frac{3w - 4p_1}{3p_1}\right)$$

$$x_B(p, w_B) = \begin{bmatrix} \left(\frac{w_B}{p_1} - \frac{4p_1}{3p_1}\right) \\ \left(\frac{4p_1}{3p_2}\right) \end{bmatrix}$$

This means that under the right circumstances (in terms of prices), consumer B will consume some of quantity of good 2 but then will keep consuming good 1 until he depletes his wealth. Quasilinear preferences.

$$OC_B(p) = \begin{bmatrix} \left(\frac{[8p_1 + 11p_2]}{p_1} - \frac{4}{3}\right) \\ \left(\frac{4p_1}{3p_2}\right) \end{bmatrix} = \begin{bmatrix} \left(8 + 11\frac{p_2}{p_1} - \frac{4}{3}\right) \\ \left(\frac{4p_1}{3p_2}\right) \end{bmatrix}$$

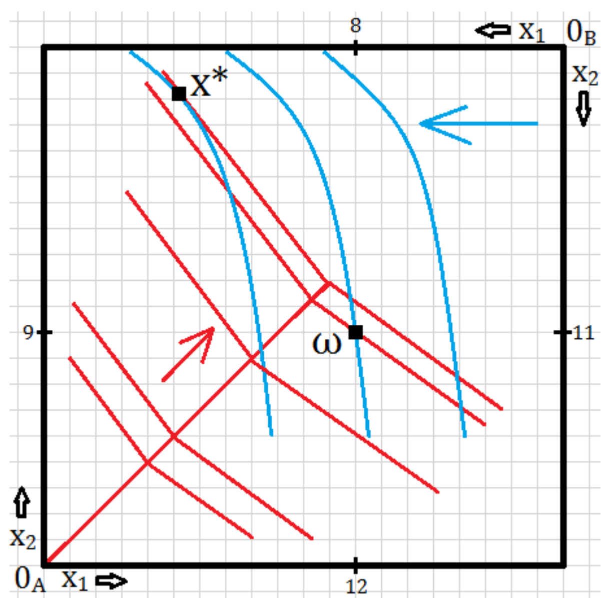
Having two possible equilibrium prices, we can see what the demand of consumer 2 will be.

$$if\ \frac{p_1}{p_2} = \frac{4}{3}, \qquad then\ OC_B(p) = \begin{bmatrix} \left(8 + 11 * \frac{3}{4} - \frac{4}{3}\right) \\ \left(\frac{4 * 4}{3 * 3}\right) \end{bmatrix} = \begin{bmatrix} \left(14\frac{11}{12}\right) \\ \left(\frac{16}{9}\right) \end{bmatrix} \ll 20\ (ok)$$

$$if\ \frac{p_1}{p_2} = \frac{3}{4}, \qquad then\ OC_B(p) = \begin{bmatrix} \left(8 + 11 * \frac{4}{3} - \frac{4}{3}\right) \\ \left(\frac{4 * 3}{3 * 4}\right) \end{bmatrix} = \begin{bmatrix} \left(21\frac{1}{3}\right) \\ (1) \end{bmatrix} \notin \begin{bmatrix} 0{,}20 \\ 0{,}20 \end{bmatrix}\ (not\ ok)$$

Then, consumer A consumes the following amounts

$$x^{A^*} = \begin{bmatrix} 20 - 14\frac{11}{12} \\ 20 - 1\frac{7}{9} \end{bmatrix} = \begin{bmatrix} \left(5\frac{1}{12}\right) \\ \left(18\frac{2}{9}\right) \end{bmatrix}$$



Handout Question 11

Consider a 2 good, 2 consumer economy.
Consumer A has preferences represented by $u_A(x) = \alpha x_1 + x_2$.
Consumer B has preferences represented by $u_B(x) = \beta \ln(x_1) + \ln(x_2)$.

a) Suppose consumer A's endowment is $\omega_A = (10,10)$.
   Suppose consumer B's endowment is $\omega_B = (10,10)$.
   What is the competitive equilibrium of this economy?

b) Suppose $\alpha = 2, \beta = 1$ and fix the total size of the economy at 20 units of each good as above. As you have seen above, there are two kinds of equilibria: interior and boundary. For which endowments will interior equilibria obtain, for which ones' boundary equilibria?

Answer

a) For the first part of the problem, we need to solve for Walrasian demand, then offer curves and finally find the optimal relative prices.

For consumer A

$$u_A(x) = \alpha x_1 + x_2$$

For consumer B

$$u_B(x) = \beta \ln(x_1) + \ln(x_2)$$

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \rightarrow \frac{\beta}{x_1 p_1} = \frac{1}{x_2 p_2} \rightarrow x_1 = \frac{\beta p_2}{p_1} x_2 \rightarrow w = p_1 \frac{\beta p_2}{p_1} x_2 + p_2 x_2 \rightarrow w = (1+\beta) p_2 x_2$$

$$x_2 = \frac{w}{(1+\beta)p_2}$$

$$x_2 = \frac{p_1}{\beta p_2} x_1 \rightarrow w = p_1 x_1 + \frac{1}{\beta} p_1 x_1 \rightarrow w = \left(\frac{\beta}{\beta} + \frac{1}{\beta}\right) p_1 x_1 \rightarrow x_1 = \frac{\beta w}{(1+\beta)p_1}$$

$$x_B(p, w_B) = \left\{\frac{\beta w_B}{(1+\beta)p_1}, \frac{w_B}{(1+\beta)p_2}\right\}$$

Knowing the wealth from the initial endowment we can derive the offer curve for consumer B

$$\omega_B = (10,10) \rightarrow w_B = 10p_1 + 10p_2$$

$$OC_B(p) = \left\{\frac{\beta(10p_1 + 10p_2)}{(1+\beta)p_1}, \frac{10p_1 + 10p_2}{(1+\beta)p_2}\right\} = \left\{\frac{10\beta}{(1+\beta)} + \frac{10\beta}{(1+\beta)}\frac{p_2}{p_1}, \frac{10}{(1+\beta)}\frac{p_1}{p_2} + \frac{10}{(1+\beta)}\right\}$$

We know that the contract curve is where MRS of both are equal.

$$MRS_A = \frac{\alpha}{1} = \alpha \qquad MRS_B = \frac{\left(\frac{\beta}{x_1^B}\right)}{\left(\frac{1}{x_2^B}\right)} = \beta \frac{x_2^B}{x_1^B}$$

$$MRS_A = MRS_B \quad \rightarrow \quad \alpha = \beta \frac{x_2^B}{x_1^B} \quad \rightarrow \quad \frac{x_2^B}{x_1^B} = \frac{\alpha}{\beta}$$

We already calculated what x_1^B and x_2^B where thus by plugging into the contract curve equation, we can find equilibrium prices.

$$\frac{\alpha}{\beta} = \frac{x_2^B}{x_1^B} \quad \rightarrow \quad \frac{\alpha}{\beta} = \frac{\left[\frac{w_B}{(1+\beta)p_2}\right]}{\left[\frac{\beta w_B}{(1+\beta)p_1}\right]} \quad \rightarrow \quad \frac{p_1}{p_2\beta} = \frac{\alpha}{\beta} \quad \rightarrow \quad p^* = \left(\frac{p_1}{p_2}\right)^* = \alpha$$

Having the equilibrium prices, we can now solve the equilibrium quantity of consumer B.

$$x_B^* = OC_B(p^*) = \left\{\frac{10\beta}{(1+\beta)} + \frac{10\beta}{(1+\beta)}\frac{1}{\alpha}, \frac{10}{(1+\beta)}\alpha + \frac{10}{(1+\beta)}\right\} = \left\{\frac{10\beta(1+\alpha)}{\alpha(1+\beta)}, \frac{10(1+\alpha)}{1+\beta}\right\}$$

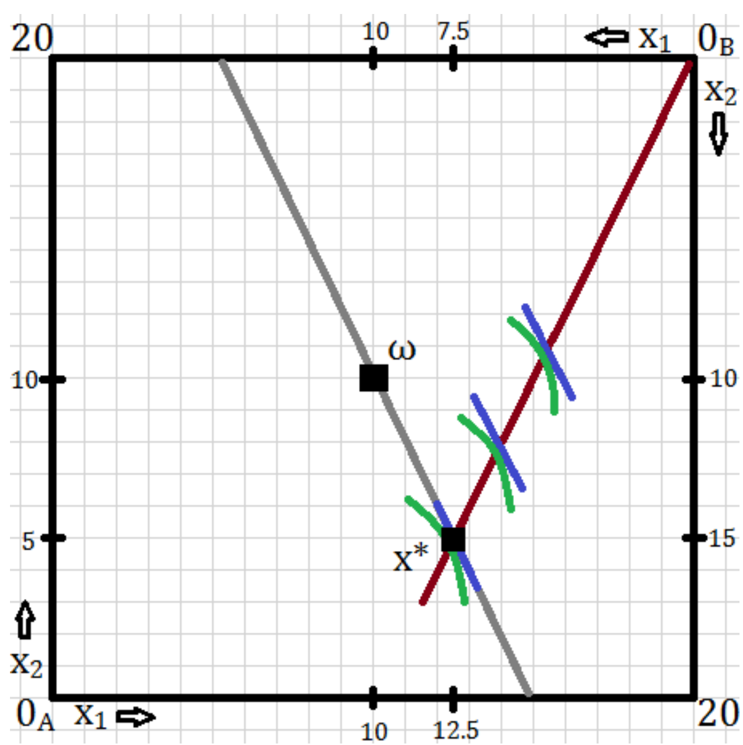
$$x_A^* = OC_A(p^*) = \left\{20 - \frac{10\beta(1+\alpha)}{\alpha(1+\beta)}, 20 - \frac{10(1+\alpha)}{1+\beta}\right\}$$

If we assume $\alpha = 2, \beta = 1$, the equilibrium position is

$$\left(\frac{p_1}{p_2}\right)^* = 2$$

$$x_A^* = OC_A(p^*) = \left\{20 - \frac{10(3)}{4}, 20 - \frac{10(3)}{2}\right\} = \{12.5, 5\}$$

$$x_B^* = \{7.5, 15\}$$



b) Suppose $\alpha = 2, \beta = 1$ and fix the total size of the economy at 20 units of each good as above. As you have seen above, there are two kinds of equilibria: interior and boundary. For which endowments will interior equilibria obtain, for which ones' boundary equilibria?

For consumer A

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \rightarrow \frac{2}{p_1} = \frac{1}{p_2} \rightarrow \frac{p_1}{p_2} = 2$$

Thus, the budget line and the indifference curve has the same slope of -2.

For consumer B we calculated the Walrasian demand. Now we just insert $\beta = 1$ to get

$$x_B(p, w_B) = \left\{\frac{(1)w_B}{(1+1)p_1}, \frac{w_B}{(1+1)p_2}\right\} = \left\{\frac{w_B}{2p_1}, \frac{w_B}{2p_2}\right\}$$

Also, since the economy's endowments are limited to 20, the two consumers must share in any way this amount. Let consumer B have a and b amounts. Thus, consumer A, has $(20 - a)$ and $(20 - b)$.

$$w_A = (20 - a)p_1 + (20 - b)p_2$$

$$w_B = ap_1 + bp_2$$

$$OC_A(p) = \left\{20 - \frac{ap_1 + bp_2}{2p_1}, 20 - \frac{ap_1 + bp_2}{2p_2}\right\}$$

$$OC_B(p) = \left\{\frac{ap_1 + bp_2}{2p_1}, \frac{ap_1 + bp_2}{2p_2}\right\}$$

$$MRS_A = MRS_B \rightarrow \frac{x_2^B}{x_1^B} = 2 \rightarrow x_2^B = 2x_1^B \ \ (the\ offer\ curve\ is\ a\ line)$$

$$\frac{x_2^B}{x_1^B} = 2 \rightarrow \frac{\left(\frac{ap_1 + bp_2}{2p_2}\right)}{\left(\frac{ap_1 + bp_2}{2p_1}\right)} = 2 \rightarrow p^* = \left(\frac{p_1}{p_2}\right)^* = 2$$

$$x_B^* = OC_B(p^*) = \left\{\frac{a}{2} + \frac{b}{4}, a + \frac{b}{2}\right\}$$

When both of these quantities of goods 1 and 2 are between 0 and 20, we have an interior solution.

$$0 < \frac{a}{2} + \frac{b}{4} < 20 \rightarrow 0 < 2a + b < 80 \quad (not\ binding)$$

$$0 < a + \frac{b}{2} < 20 \rightarrow a < 20 - \frac{b}{2} \quad where\ a, b\ are\ endowments\ for\ consumer\ B$$

The graph looks as follows.

Any allocation of endowments to the left of the orange line will give a border solution. Any allocation to the right of the orange line will give an interior solution.