L1. Introduction to Macroeconomics

These notes introduce the notions of a social planner maximizing economywide welfare subject to resource feasibility, how market incompleteness can arise endogenously, decentralization as a competitive equilibrium, and a role for government policy with and without commitment.

1 Environment

- An *environment* is a statement of population, preferences, and technologies (production, matching, information, commitment).
- Time: Discrete time $t = 1, 2, 3, ...$
- Population: Each period t a new generation of 2 period lived agents are born. Thus in any given period t, there is a youthful generation born in period t and an old generation born in period $t-1$. Each generation has a large number of identical agents. This implies that no single agent has any power to influence aggregates (no market power).
- Production Technology: Each generation is endowed with w_1 units of nonstorable consumption goods in youth and 0 in old age. One can think of this as income in working age years and no income in retirement.[1]
- Preferences: The utility function of an agent of generation t is

$$U\left(c_t^t, c_{t+1}^t\right) = \ln\left(c_t^t\right) + \ln\left(c_{t+1}^t\right)$$

  where $\left(c_t^t, c_{t+1}^t\right)$ is per capita consumption of generation t in period t (youth) and period $t+1$ (old age). Note that $\ln(\cdot)$ is an increasing, concave, and differentiable function.
- Matching Technology: In any period t, only young agents of generation t and old agents of generation $t-1$ are matched.

generation\time	...	t	$t+1$	$t+2$
...		...		
$t-1$	...	$(c_t^{t-1}, 0)$	dead	
t	...	(c_t^t, w_1)	$(c_{t+1}^t, 0)$	dead
$t+1$	...	unborn	(c_{t+1}^{t+1}, w_1)	$(c_{t+2}^{t+1}, 0)$

[1] Instead of endowments of nonstorable goods, one can think of this as a production economy where each agent has an endowment of one unit of time in youth and there is a home production technology $Y_t = w_1 \cdot n_t$. Since there is no disutility in preferences to supplying labor, their income will be w_1 which is identical to what I have described.

- Demographics induces a natural matching friction in this model: Generation t (the young) is matched with Generation $t-1$ (the old) but not with Generation $t+1$ (the unborn).

- Notes:

  - Preferences and Technologies are time independent, so one natural candidate allocation is time independent.
  - This is a heterogeneous agent model. At any time t, there are two types of agents in the economy with observable differences based on age and earnings.
  - There is an intertemporal consumption smoothing problem for a young agent who has income now but wants to save for old age when she has no income:

    * she can't lend to someone of her own generation since they are all identical
    * she can't lend to an old agent since they will be dead next period (and hence cannot repay the loan)
    * she can't lend to the next period's young since they aren't even born yet.
    * This is a source of endogenous market incompleteness due to missing markets (i.e. the environment rules out a private loan market)

2 Planner's problem

- In any period t, given preferences and technologies, if a planner weights each generation alive equally and can allocate resources between them, what is the optimal allocation?

- Since goods are nonstorable, resource feasibility in any period t implies

$$c_t^t + c_t^{t-1} \leq w_1. \tag{1}$$

- The planning problem is then to choose $(c_t^t, c_t^{t-1}) \in \mathbb{R}_+^2$ to solve:[2]

$$\begin{aligned} \max_{(c_t^t, c_t^{t-1}) \in \mathbb{R}_+^2} \ln\left(c_t^t\right) \quad &+ \quad \ln\left(c_t^{t-1}\right) \\ \text{s.t. } c_t^t + c_t^{t-1} \quad &\leq \quad w_1 \end{aligned}$$

  or (given that preferences are strictly increasing in consumption so that the resource constraint will be binding):

$$\max_{c_t^t} \ln\left(c_t^t\right) + \ln\left(w_1 - c_t^t\right) \tag{2}$$

[2] Note that the objective function of the planner is different from $U(c_t^t, c_{t+1}^t)$.

- The first order necessary and sufficient condition (given strictly concave preferences and a linear constraint) is given by

$$\frac{1}{c_t^t} = \frac{1}{w_1 - c_t^t} \Longleftrightarrow w_1 - c_t^t = c_t^t \Longleftrightarrow c_t^t = \frac{w_1}{2} \tag{3}$$

which implies from equality in (1)

$$c_t^{t-1} = w_1 - \frac{w_1}{2} = \frac{w_1}{2}. \tag{4}$$

- In order to implement this allocation, the planner needed to take $\frac{w_1}{2}$ away from the young and give it to the old.

3 Decentralized problem

- How can this allocation be implemented in a decentralized economy?

3.1 Social Security Taxes

- The government can simply tax young agents $\frac{w_1}{2}$ and transfer it to the old.
- This is similar to the current pay-as-you-go social security system in the U.S.
- With a constant or growing population it is feasible to transfer $\frac{w_1}{2}$. However, in a shrinking population like Japan, it is not feasible to make constant transfers through time.
  - To see this, let N_t be the size of the generation t population and n be the growth rate (which could be negative) so that $N_t = (1+n)N_{t-1}$. In that case the resource constraint is

    $$N_t c_t^t + N_{t-1} c_t^{t-1} \leq N_t w_1 \Longleftrightarrow c_t^t + \frac{1}{1+n} c_t^{t-1} \leq w_1$$

  - If $c_t^t = c_t^{t-1} = w_1/2$, then when $n < 0$ we have

    $$\frac{w_1}{2}\left(1 + \frac{1}{1+n}\right) > w_1$$

    so the proposed allocation is not resource feasible.
  - You should formulate and solve the planner's problem and associated tax/transfer policy when population size shrinks or grows (remember to get the weights on the population correct in the objective function).

3.2 Government Liabilities

- Since all young are alike and there is no storage technology, how can they save for old age? They could lend the goods to an old person alive in period t, but the old person cannot commit to repaying the loan in period $t+1$ since they will be dead. Thus a private loan market cannot be implemented.

- Consequence of the missing private loan market:

$$U(w_1, 0) = -\infty \tag{5}$$

- Can a government improve upon this outcome with government issued assets? Since the government will be around next period they could issue government debt to the young of generation t and promise to pay back in $t+1$ when the generation t becomes old by issuing new debt to generation $t+1$ young in exchange for their goods. Thus, there is a potential role for government debt.

- However, if the government cannot *commit* to pay back next period (e.g. Greece or Argentina), what else can be done?

- Fiat (unbacked paper) currency carries no commitment on the part of the government (when currency was backed by gold, you could get gold for it from the government which has fundamental value since it is a good conductor of electricity, doesn't tarnish, etc.).

- Thus, money is accepted today in exchange for goods only if there is the *belief* that it will be accepted in the future. If people do not believe it will be accepted in the future, then they will not accept it today. In that case we are back to autarky as in (5).

- What if people believe that other people will accept their money at $t+1$ in exchange for consumption goods at price p_{t+1} (that is, the relative price of goods in terms of money[3])? Can we achieve the same allocations as the planner problem?

- Suppose the government issues M units of currency to the initial old and maintains that level through time. Taking price p_1 as given (since agents have no market power), the initial old person's problem is

$$\max_{c_1^0} \ln(c_1^0)$$
$$\text{s.t. } p_1 c_1^0 \leq M.$$

  Since the utility function is strictly increasing, the solution is

$$c_1^0 = M/p_1. \tag{6}$$

[3] That is, p_t =(#\$/1 consumption good). Alternatively, the relative price of money in terms of consumption goods is $1/p_t$ =(# consumption goods/1\$).

But the old can't eat fiat money, so they must trade their money to someone for goods? Who wants it?

- Taking prices p_t for all t as given, the agent's problem is then to choose $(c_t^t, c_{t+1}^t, M_{t+1}^t) \in \mathbb{R}_+^3$ to solve

$$\begin{aligned} \max_{(c_t^t, c_{t+1}^t, M_{t+1}^t)\in\mathbb{R}_+^3} \ln\left(c_t^t\right) &+ \ln\left(c_{t+1}^t\right) \\ \text{s.t. } p_t c_t^t + M_{t+1}^t &= p_t w_1 \qquad (7) \\ p_{t+1} c_{t+1}^t &= M_{t+1}^t \qquad (8) \end{aligned}$$

where $p_t c_t^t$ and $p_t w_1$ are nominal expenditure and nominal income and M_{t+1}^t is nominal savings. This probem can be written simply as[4]

$$\max_{M_{t+1}^t} \ln\left(w_1 - \frac{M_{t+1}^t}{p_t}\right) + \ln\left(\frac{M_{t+1}^t}{p_{t+1}}\right). \qquad (9)$$

- The first order necessary condition (fonc) is[5]

$$\frac{1}{p_t}\cdot\left(\frac{1}{w_1 - \frac{M_{t+1}^t}{p_t}}\right) = \frac{1}{p_{t+1}}\cdot\frac{1}{\frac{M_{t+1}^t}{p_{t+1}}} \Longleftrightarrow \left(\frac{1}{p_t w_1 - M_{t+1}^t}\right) = \frac{1}{M_{t+1}^t} \Longleftrightarrow M_{t+1}^t = \frac{p_t w_1}{2}$$

(i.e. the agent saves half of her nominal income). This implies

$$\begin{aligned} p_t c_t^t + \frac{p_t w_1}{2} &= p_t w_1 \Longleftrightarrow c_t^t = \frac{w_1}{2} \qquad (10) \\ p_{t+1} c_{t+1}^t &= \frac{p_t w_1}{2} \Longleftrightarrow c_{t+1}^t = \frac{p_t}{p_{t+1}}\cdot\frac{w_1}{2} \qquad (11) \end{aligned}$$

Note that (10) is identical to the planner's allocation of consumption to the young (3) while (11) is only identical to (4) if $\frac{p_t}{p_{t+1}} = 1$.

- The goods market (12) and money market (13) clearing conditions (i.e. demand equals supply) are:

$$\begin{aligned} c_t^t + c_t^{t-1} &= w_1 \qquad (12) \\ M_{t+1}^t &= M \qquad (13) \end{aligned}$$

- Definition. A *competitive monetary equilibrium* is an allocation $\left\{(c_t^t, c_t^{t-1})\right\}_{\forall t}$ and prices $\{p_t\}_{\forall t}$ such that agents optimize (9) and markets clear (12)-(13).

- So does this CME implement the planner's solution?

[4] Note that $\frac{M_{t+1}^t}{p_{t+1}} \equiv \left(\frac{\#\text{t+1 goods}}{1\$}\right)\cdot(\#\text{t+1 units of money})$ is the number of goods your money can buy in $t+1$. This is called one's "real money holdings" at $t+1$.

[5] Linear constraints and a concave objective imply the fonc are also sufficient for a maximum.

- How do we solve for $\left\{(c_t^t, c_t^{t-1})\right\}_{\forall t}$ and $\{p_t\}_{\forall t}$? Substituting household optimization conditions (6), (10) and (11) into market clearing (12) and (13) gives us:

$$c_t^t + c_t^{t-1} = w_1 \Longleftrightarrow \frac{w_1}{2} + \frac{p_{t-1}}{p_t} \cdot \frac{w_1}{2} = w_1 \Longleftrightarrow \frac{p_{t-1}}{p_t} = 1, \quad (14)$$

$$M_{t+1}^t = M \Longleftrightarrow \frac{p_t w_1}{2} = M \Longleftrightarrow p_t = \frac{2M}{w_1}. \quad (15)$$

- Notes:
  - (14) is a linear first order difference equation in p_t. Linear and nonlinear difference equations are important in macroeconomics since it is inherently dynamic.[6]
  - As can be seen in (15), the monetary equilibrium has the property that prices are high if there is a lot of money and prices are low if goods are plentiful. Further, since the rhs of (15) is independent of t, then prices must be independent of time consistent with (14).

- Besides the monetary equilibrium, there is also an autarkic equilibrium. To see this, suppose everyone believes that no one will accept their money in the next period in exchange for goods. In that case, no one will accept money for goods in the current period.
  - If no one will accept your money next period, then the price of goods in terms of money $p_{t+1} = \infty$ (remember $p_t = \#\$/good$). From (8), that means $c_{t+1}^t = 0$ and if a young person accepts money in period t then $c_t^t = w_1 - M_{t+1}^t/p_t > 0$ in (7) is lower than if they do not accept money (i.e. $U\left(w_1 - M_{t+1}^t/p_t, 0\right) < U(w_1, 0)$).[7]
  - Since agents are assumed atomless, a unilateral deviation to accepting money cannot be optimal since it does not change other's beliefs/actions.
  - Thus, there are multiple pareto (welfare) ranked steady state equilibria.

- Notes:
  - From Walras Law, we could have solved for the monetary equilibrium by just using the money market clearing condition (13) and budget constraints noting that (15) at t and $t+1$ implies (14). That is, the goods market clearing condition (12) is trivially satisfied.[8]

[6] This explains why the first TA sessions involve how to solve difference equations.

[7] Of course, with log preferences since $\ln(0) = \infty$ there is not a technical difference between $U\left(w_1 - M_{t+1}^t/p_t, 0\right)$ and $(w_1, 0)$ so, strictly speaking, the inequality is not defined. However, $U\left(w_1 - M_{t+1}^t/p_t, w_2\right) < U(w_1, w_2)$ holds for all w_2 approaching zero from above.

[8] Walras Law says that if there are N markets, then we need only clear $N-1$ and the budget set will imply the N^{th} market is also satisfied.

- The planner's problem is easier to solve since there are no prices to solve for (this was the idea behind how to solve RBC models).
- A fiat monetary equilibrium is your first example of a rational bubble. While money has no intrinsic or fundamental value, it has positive "market" value (i.e. the price of money in terms of consumption goods $(1/p_t)$ is non-zero).
- This is also an example where adding a market (i.e. money) in an incomplete markets model can make agents better off. In fact, the competitive monetary equilibrium can actually implement the planner's solution. That is, with $\frac{p_t}{p_{t+1}} = 1$ from (14), (10) and (11), then $\left(c_t^t, c_{t+1}^t\right) = \left(\frac{w_1}{2}, \frac{w_1}{2}\right)$.

• Missing markets are clear in this two-period example; at time t, generation t young can't lend to generation $t-1$ old since they will be dead and young can't lend to generation $t+1$ young since they are not born yet. What if there are three period lived agents (young, mid, old) with endowment profile $(0, w_2, 0)$?

generation\time	...	t	$t+1$	$t+2$
...		...		
$t-2$	...	$(c_t^{t-2}, 0)$		
$t-1$	...	(c_t^{t-1}, w_2)	$(c_{t+1}^{t-1}, 0)$	
t	...	$(c_t^t, 0)$	(c_{t+1}^t, w_2)	$(c_{t+2}^t, 0)$
$t+1$			$(c_{t+1}^{t+1}, 0)$	(c_{t+2}^{t+1}, w_2)

- at time $t+1$, generation t mid can't lend to generation $t-1$ old since they will be dead but they can lend to generation $t+1$ young since they are around and want to borrow. So with a commitment technology, it seems possible to sustain a private credit market with three period lived agents.
- However, what if there is no commitment technology to enforce paying back private loans (i.e. just like governments may choose to default, private agents can choose to default)? Why would a mid age person who has income choose to pay back a loan to an old person who will be dead in the future? Do we need a government to coordinate that punishment (i.e. bankruptcy rules)?

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L2. The Labor/Leisure Choice in a Production Economy

The last set of notes introduced a dynamic choice (or tradeoff) between current and future consumption of goods in an endowment economy. These notes introduce a labor/leisure static choice (i.e. a tradeoff between current consumption goods and leisure), efficient allocations, implementation as a competitive equilibrium, income and substitution effects, value functions, envelope conditions, Frisch labor supply elasticity, wedges, the Lucas Critique, and the welfare theorems in a static economy.

1 Environment

- Population: Agents (households) are identical. Since we assume there are a large number of them (specifically there is a "unit measure" of agents) no one agent has any influence on allocations, which is sometimes called an "atomless" economy.[1]
- Technology:
  - Households are endowed with a unit of time which they can allocate to leisure and work (the latter is denoted $n \in [0,1]$).
  - Production: There is a technology available for producing the single good (y) from labor input (n). Let $y = f(n)$ where $f(\cdot)$ is strictly increasing (i.e. $f'(\cdot) > 0$) and concave (i.e. $f''(\cdot) \leq 0$), with $f(0) = 0$. The slope of the production function measures the marginal rate of transformation of labor into consumption goods denoted $MRT = f'(n)$. See Figure L2.1 which plots the case for strictly decreasing returns to scale in production.
- Preferences:
$$U(c,n) = u(c) - g(n)$$
over consumption (c) and labor supplied (n). Assume that $u(\cdot)$ is strictly increasing (i.e. $u'(\cdot) > 0$) and concave (i.e. $u''(\cdot) \leq 0$) and that $g(\cdot)$ is strictly increasing (i.e. $g'(\cdot) > 0$) and strictly convex (i.e. $g''(\cdot) > 0$).

[1] Suppose the only difference between agents is their "name", indexed by $i \in [0,1]$, distributed uniformly. If each agent i takes the same action then $x_i = X$, where X can be thought of as the per capita action. Then the economywide action is given by $\int_0^1 x_i di = X \cdot \int_0^1 di = X \cdot i|_0^1 = X$. Note that even if one agent does something different from X, the economywide action is still X since that agent is of measure zero.

– An indifference curve is the locus of points of (c, n) for which the household is indifferent. Since $\overline{U} = u(c) - g(n)$, then the slope of an indifference curve is given by

$$d\overline{U} = 0 \Longleftrightarrow u'(c)dc - g'(n)dn = 0 \Longleftrightarrow \frac{dc}{dn}|_{d\overline{U}=0} = \frac{g'(n)}{u'(c)} \equiv MRS.$$

where MRS denotes marginal rate of substitution between consumption and leisure/labor. See Figure L2.1.

- Assume

$$\lim_{c\to 0, n\to 0} u'(c)f'(n) > g'(n) \text{ and } u'(f(1))f'(1) < g'(1). \tag{1}$$

These assumptions hold if the marginal utility of consumption is very high at zero consumption and the marginal disutility of work is very high when leisure is zero.

2 Planner's Problem

- Given symmetry, the planner chooses $\{c, n\}$ to maximize the utility of the "representative agent" (named Robinson Crusoe) subject to resource feasibility

$$\max_{c\geq 0, n\in[0,1]} u(c) - g(n)$$
$$\text{s.t. } c \leq y = f(n)$$

or given that utility is increasing in consumption we simplify the problem to

$$\max_{n\in[0,1]} u(f(n)) - g(n). \tag{2}$$

- Given the assumptions on u, f, g, the objective function is concave and the constraint set is compact and convex.

- Given the assumptions in (1), the first order condition is given by

$$u'(f(n^*)) \cdot f'(n^*) = g'(n^*) \tag{3}$$

where n^* denotes the planner's solution. Note that the planner sets the marginal utility benefit from work (the left hand side of (3)) equal to the marginal utility cost of work (the right hand side of (3))

- Equation (3) can be re-written such that at an optimum the $MRT \equiv f'(n) = \frac{g'(n)}{u'(c)} \equiv MRS$ so that the tangency of the indifference curve and the production function selects the optimal allocation. See Figure L2.1.

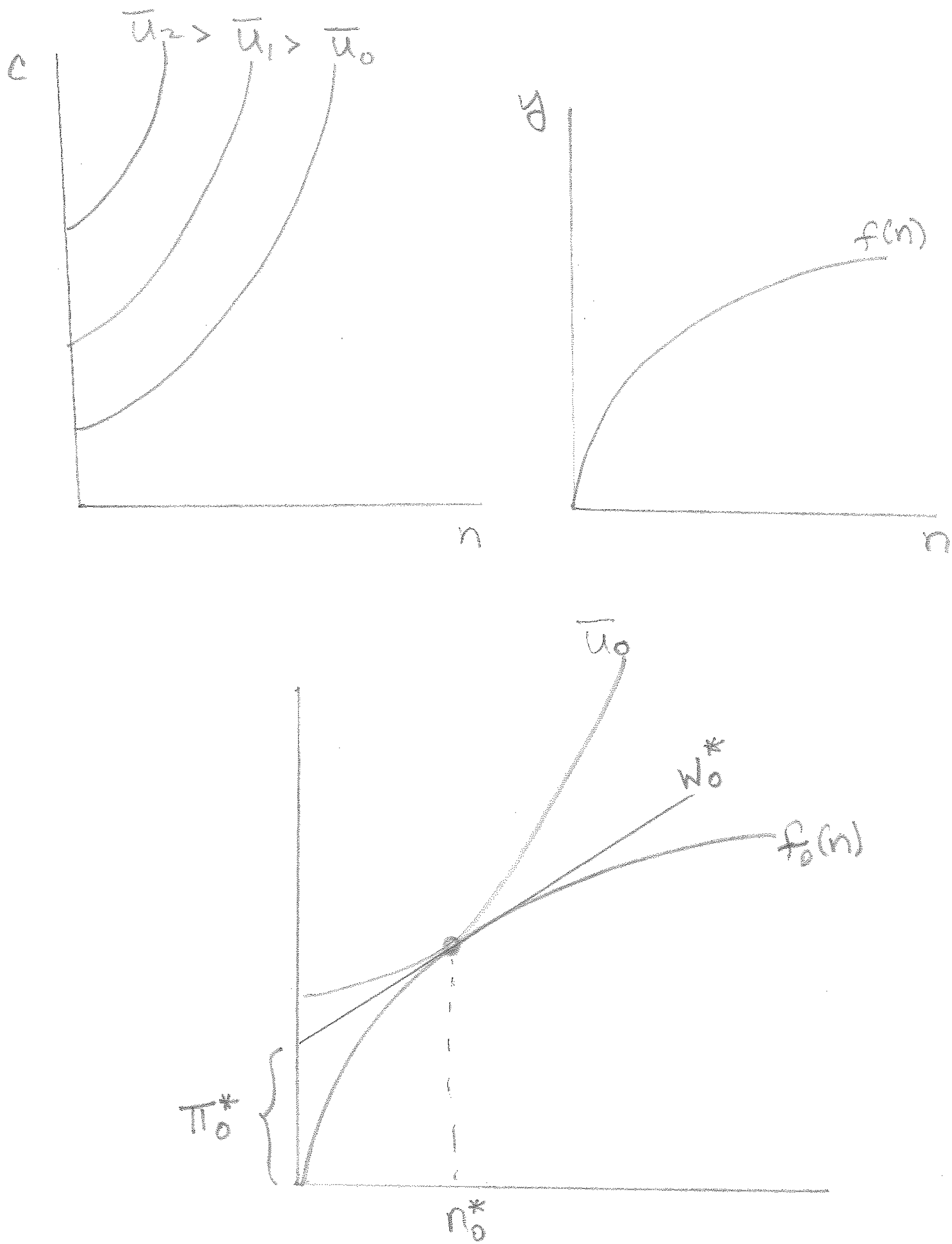
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Figure L2.1

3 Competitive Equilibrium

- We assume competitive markets for goods and labor where agents and firms interact.
- Firms are owned by households. Firm profits π are returned to the households. Households supply labor to firms at relative price of labor in terms of consumption goods w (i.e. the real wage given by $w =$#cons.goods/unit of labor).
- The competitive assumption implies that household and firm choices do not influence profitability and prices.

3.1 Household Problem

- Households choose how much labor to supply and consumption to solve

$$\max_{c\geq 0, n\in[0,1]} u(c) - g(n) \tag{4}$$

$$\text{s.t. } c = wn + \pi \tag{5}$$

Notice the price normalization (i.e. the price of consumption goods in terms of consumption goods is one).

- Forming a lagrangian for (4)-(5)

$$£ = u(c) - g(n) + \lambda \cdot [wn + \pi - c],$$

the first order conditions are given by

$$c \;:\; u'(c) = \lambda \tag{6}$$

$$n \;:\; g'(n) = \lambda \cdot w \tag{7}$$

- Substituting λ from (6) and c from (5) into (7) yields n^s which satisfies the first order condition or

$$u'(wn^s + \pi) \cdot w = g'(n^s) \tag{8}$$

  - The household sets the marginal utility benefit from work (the left hand side of (8)) equal to the marginal utility cost of work (the right hand side of (8)).
  - (8) is one equation in one unknown n^s which implicitly defines labor supply in terms of wages and other income (profits). We call this a decision rule or policy function $n^s(w, \pi)$.

- How does labor supply respond to changes in wages? To understand this we can use the implicit function theorem to differentiate (8) with respect to w and n^s yielding[2]

$$u''(wn^s + \pi) \cdot [w \cdot dn^s + n^s \cdot dw] \cdot w + u'(wn^s + \pi) \cdot dw = g''(n^s) \cdot dn^s$$

[2] By the competitive assumption (i.e. households and firms are small), they take profits as given and we don't differentiate π.

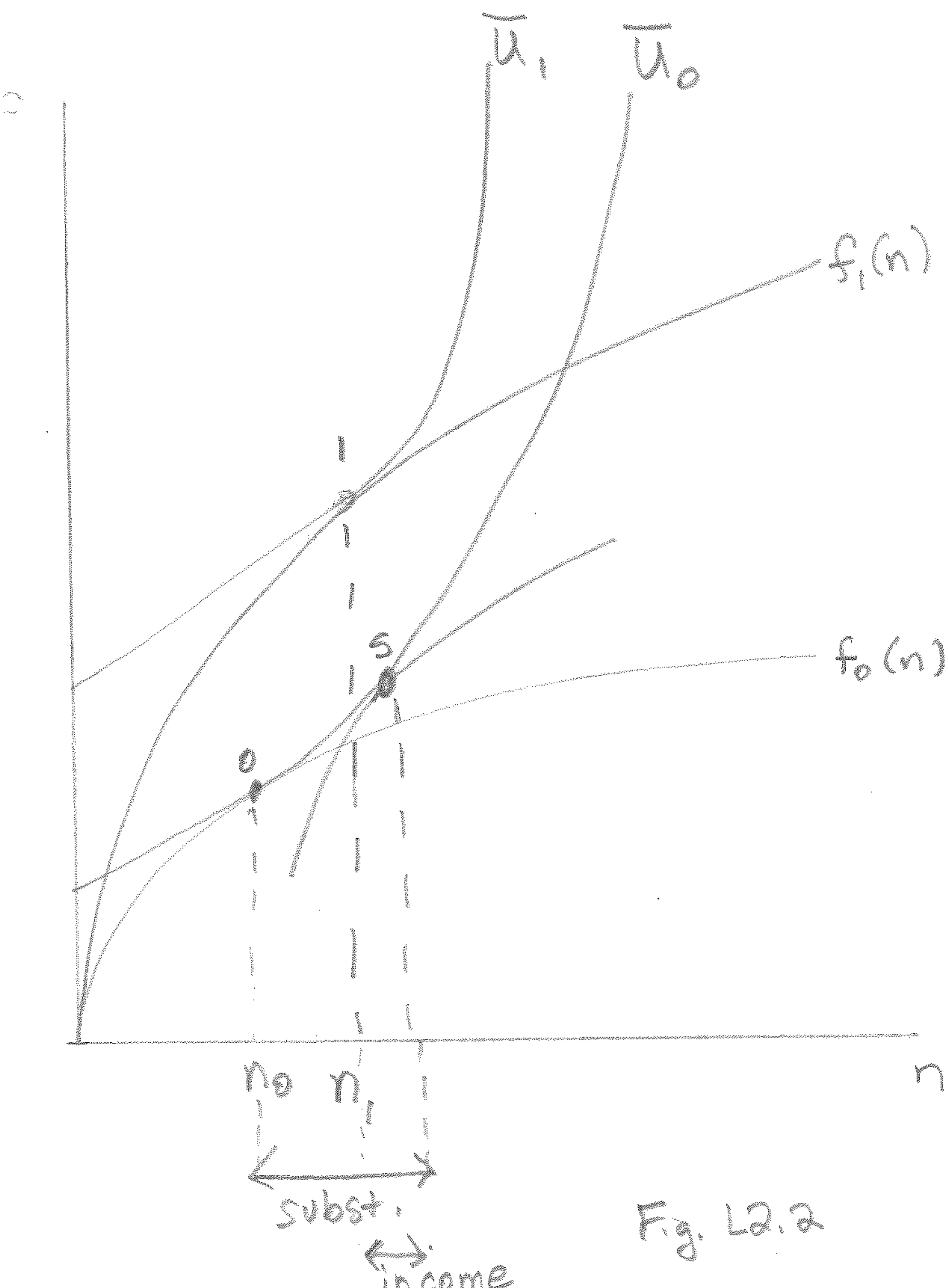

Fig. L2.2

or

$$\frac{dn^s}{dw} = \frac{[u''(wn^s + \pi) \cdot w \cdot n^s + u'(wn^s + \pi)]}{[g''(n^s) - u''(wn^s + \pi) \cdot w^2]} \tag{9}$$

– The denominator of (9) is positive since $g'' > 0$ and $u'' \leq 0$.

– The first term in the numerator is negative; as wages increase, households have more income and thus want to consume more leisure (this is known as the income effect).

– The second term is positive; as wages increase, households substitute away from leisure towards work (this is known as the substitution effect).

– Which effect dominates depends on the curvature of the utility function. Figure L2.2 illustrates the Hicksian case where substitution effects (from 0 to s) outweigh income effects (from s to 1) so that labor supplied rises as real wages (or marginal product of labor) rise (from $f_0(n)$ to $f_1(n)$). These opposing effects are one reason why labor supply does not respond much to wage changes and can even result in a backward bending labor supply curve at high wages.

- Once we know the decision rule $n^s(w, \pi)$, then we know the value (or indirect utility) function associated with a solution to (4). That is,

$$V(w, \pi) = u(w \cdot n^s(w, \pi) + \pi) - g(n^s(w, \pi)) \tag{10}$$

- To understand how wage changes, for example, affect the household's utility (or value function) differentiate (10) with respect to w to yield

$$\begin{aligned}
\frac{dV(w,\pi)}{dw} &= u'(w \cdot n^s(w,\pi) + \pi) \cdot \left[n^s + w \cdot \frac{dn^s}{dw}\right] - g'(n^s(w,\pi)) \cdot \frac{dn^s}{dw} \\
&= u'(w \cdot n^s(w,\pi) + \pi) \cdot n^s + [u'(w \cdot n^s(w,\pi) + \pi) \cdot w - g'(n^s(w,\pi))] \cdot \frac{dn^s}{dw} \\
&= u'(w \cdot n^s(w,\pi) + \pi) \cdot n^s > 0 \qquad (11)
\end{aligned}$$

where the final line is due to the "envelope condition". That is, by the first order condition (8), the term in $[\cdot]$ multiplied by $\frac{dn^s}{dw}$ is zero. Thus, utility increases from a wage increase whether or not labor supply increases.

3.2 Firm Problem

- Firms choose labor to maximize profits given by the following problem:

$$\pi = \max_n f(n) - w \cdot n \tag{12}$$

- The solution to this problem is given by n^d which satisfies the first order condition

$$f'(n^d) = w \tag{13}$$

- This says that the firm chooses the fraction of hours such that the marginal benefit (the marginal product of labor) equals the marginal cost (the wage).
- This defines the firm decision rule $n^d(w)$. By the implicit function theorem
$$f''(n^d)dn^d = dw \Leftrightarrow \frac{dn^d}{dw} = \frac{1}{f''(n^d)} \leq 0$$
since $f''(\cdot) \leq 0$. Thus, the labor demand curve is downward sloping.
- The special case of constant returns to scale where $f(n) = A \cdot n$, so $f''(\cdot) = 0$, implies a perfectly elastic (i.e. horizontal) labor demand curve at $w = A$. That is, the firm is willing to demand any amount of labor supplied at real wage A.

3.3 Market Clearing

- Labor and goods market clearing are given by

$$\begin{aligned} n^d &= n^s \equiv n^{**} && (14) \\ c^{**} &= f(n^{**}) && (15) \end{aligned}$$

- Using Walras Law, we can solve for an equilibrium by using only one of the two market clearing conditions. We will use (14).

3.4 Equilibrium

- A symmetric *competitive equilibrium* is an allocation (n^{**}, c^{**}) and prices w^{**} such that households optimize (8), firms optimize (13), and markets clear (14)-(15).
- Using (8), (12), (13), and (14) we have

$$\begin{aligned} u'\left(w^{**} \cdot n^{**} + [f(n^{**}) - w^{**} \cdot n^{**}]\right) \cdot f^{'}(n^{**}) &= g'(n^{**}) \Longleftrightarrow \\ u'\left(f(n^{**})\right) \cdot f^{'}(n^{**}) &= g'(n^{**}) && (16) \end{aligned}$$

Notice that the competitive outcome (16) is identical to the planner's outcome (3). Further, since $n^* = n^{**}$ we know $c^* = c^{**}$.

- Since in a decentralized equilibrium, $f'(n) = w$, the tangency defines the wage and the intercept defines profits. This allocation is consistent with the separating hyperplane theorem. See Figure L2.1 for the case of strictly decreasing returns to scale in productivity (which generates a positive intercept). This is an important result which has to do with the Welfare Theorems in Section 5.

4 Wedges: The Distortionary Effects of Taxes

- Assume that proportional labor taxes $\tau \in [0, 1]$ are placed on earnings used to finance lump sum transfers back to the household (i.e. $T = \tau \cdot w \cdot n$).[3] In that case, the household budget constraint is written

$$c = (1 - \tau)wn + T + \pi.$$

- In this case, (16) can be written

$$\begin{aligned} g'(\widehat{n}) &= u'\left((1-\tau)\widehat{w} \cdot \widehat{n} + T + [f(\widehat{n}) - \widehat{w} \cdot \widehat{n}]\right) \cdot (1-\tau) \cdot f'(\widehat{n}) \Longleftrightarrow \\ MC &\equiv g'(\widehat{n}) = u'\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot f'(\widehat{n}) \equiv MB \end{aligned} \tag{17}$$

  Note that I used "$\widehat{x}$" to potentially differentiate the competitive choice with taxes from the one without taxes.

- Notice that the only difference between the first best (16) and (17) is the marginal benefit with taxes is now lower than the MC (which is the same in both the first best and the competitive equilibrium with taxes). Since the benefit is less than the cost, households will supply less labor and in turn consume less as well (i.e. $\widehat{n} < n^{**}$ and $\widehat{c} < c^{**}$). This is the distortionary effect of taxation.[4] Chari, et. al. (2007) call $(1 - \tau)$ the "labor wedge".

4.1 Parametric Example

- Suppose we want to answer the question "How much do tax revenues respond to changes in proportional income taxes?" This quantitative question ("How much") depends on the responsiveness of labor supply to changes in real after-tax wages (basically the slope of the labor supply curve).

- There are many ways to decompose how changes in real wages (Marshallian labor supply holds income constant, Hicksian holds utility constant,

[3] Of course generally such income taxes fund government expenditure, which will be coming later in class.

[4] We can formalize this by using the implicit function theorem on (17). In particular,

$$\begin{aligned} g''(\widehat{n})d\widehat{n} &= u''\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot \left[f'(\widehat{n})\right]^2 d\widehat{n} \\ &\quad -u'\left(f(\widehat{n})\right) \cdot f'(\widehat{n})d\tau + u'\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot f''(\widehat{n})d\widehat{n} \\ \Longleftrightarrow &\quad \left\{g''(\widehat{n}) - u''\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot \left[f'(\widehat{n})\right]^2 - u'\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot f''(\widehat{n})\right\} d\widehat{n} \\ &= -u'\left(f(\widehat{n})\right) \cdot f'(\widehat{n})d\tau \end{aligned}$$

or

$$\frac{d\widehat{n}}{d\tau} = \frac{-u'\left(f(\widehat{n})\right) \cdot f'(\widehat{n})}{\left\{g''(\widehat{n}) - u''\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot [f'(\widehat{n})]^2 - u'\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot f''(\widehat{n})\right\}} < 0$$

since $-u'\left(f(\widehat{n})\right) \cdot f'(\widehat{n}) < 0$ and $g''(\widehat{n}) - u''\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot [f'(\widehat{n})]^2 - u'\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot f''(\widehat{n}) > 0$ since $g''(\widehat{n}) > 0$, $-u''\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot [f'(\widehat{n})]^2 > 0$, and $-u'\left(f(\widehat{n})\right) \cdot (1-\tau) \cdot f''(\widehat{n}) > 0$.

Frisch holds the marginal utility of wealth - λ in (7) - constant) affect labor supply.

- Consider the following parametric example which is a special case of Keane and Rogerson (2012, equation 1, with $\eta = \infty$):

$$\begin{aligned} u(c) &= c \\ g(n) &= \frac{\alpha}{1+\frac{1}{\gamma}} \cdot n^{1+\frac{1}{\gamma}} \\ f(n) &= A \cdot n^{\theta} \end{aligned} \tag{18}$$

where $\theta \leq 1$, $\gamma > 0$, and to be consistent with Assumption (1) we need $A\theta < \alpha$.[5]

- As discussed in Keane and Rogerson (2012, equation 4), to uncover the labor supply elasticity parameter γ, MaCurdy (1981) ran a panel regression after taking logs of (7) augmented with proportional taxes to yield

$$\begin{aligned} \ln\left(\alpha \cdot n^{\frac{1}{\gamma}}\right) &= \ln\left(\lambda \cdot w \cdot (1-\tau)\right) \Longleftrightarrow \\ \ln(n) &= b + \gamma \cdot \ln(w) \end{aligned} \tag{19}$$

where $b = \gamma \cdot [\ln(\lambda) + \ln(1-\tau) - \ln(\alpha)]$. That is, the coefficient on the regression of log hours on log wages uncovers the Frisch labor supply elasticity γ.[6] This is an example of how we can estimate parameters of a macro model.

- Keane and Rogerson make clear that micro estimates of γ which come from cross-sectional data (somewhere between 0 to 0.5) are much smaller than the parameterization used in macro models which come from aggregated time series data (somewhere between 2 to 4). Intuitively, to get enough variation in aggregate hours in response to labor demand shocks (e.g. technology shocks which alter the real wage), we need an elastic labor supply curve. Since (19) implies that the slope of the labor supply curve is inversely related to γ,a large γ means a moderate slope which in turn

[5] It is also simple to see that under the parameterization in (18), the indifference curves associated with these preferences in Figure L2.1 are given by

$$\begin{aligned} \overline{U} &= c - \frac{\alpha}{1+\frac{1}{\gamma}} \cdot n^{1+\frac{1}{\gamma}} \Longleftrightarrow \\ c &= \overline{U} + \frac{\alpha}{1+\frac{1}{\gamma}} \cdot n^{1+\frac{1}{\gamma}} \end{aligned}$$

Hence the intersection of indifference curves with the vertical axis (where $n = 0$) is increasing in $\overline{U}$, increasing in n (since $\frac{dc}{dn} = \alpha \cdot n^{\frac{1}{\gamma}} > 0$) at an increasing rate (since $\frac{d^2 c}{dn^2} = \frac{\alpha}{\gamma} \cdot n^{\frac{1-\gamma}{\gamma}} > 0$).

[6] Recall that the elasticity of y with respect to x is defined to be $\frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{d\ln(y)}{d\ln(x)}$.

means more variation (i.e. a higher γ implies a more elastic supply curve).[7]

- Note that profits are given by

$$\pi = f(n) - f'(n) \cdot n = An^{\theta} - A\theta n^{\theta-1} n = (1-\theta) \cdot f(n)$$

With constant returns to scale (i.e. $\theta = 1$) there are no profits while with decreasing returns (i.e. $\theta < 1$) there are profits. Unlike the case plotted in Figure L2.1 for $\theta < 1$, when $\theta = 1$, the production function is linear and the hyperplane would lie on the production function yielding a zero intercept for profits.

- Using the parameterization in (18), we can substitute into (17) to yield:

$$\begin{aligned} \alpha \cdot (\widehat{n})^{\frac{1}{\gamma}} &= (1-\tau) \cdot \frac{A\theta}{(\widehat{n})^{1-\theta}} \Longleftrightarrow \\ \widehat{n} &= \left(\frac{(1-\tau)A\theta}{\alpha} \right)^{1/\left(1+\frac{1}{\gamma}-\theta\right)} \qquad (20) \end{aligned}$$

which makes clear that as labor taxes τ rise, labor supply (and hence the equilibrium level of employment) $\widehat{n}$ falls. That is, as the distortionary wedge rises, the economy moves further from the efficient allocation of labor.

4.2 Lucas Critique

- With constant returns to scale $w = f'(n) = A$, so that (20) implies the labor supply decision rule is a nonlinear function of technology A (taken as exogenous), policy τ, and "deep" preference parameters α and γ.

- Since productivity moves through time, often we make it an explicit function of time A_t. We typically assume that preferences don't change through time. Since the frequency of legislative changes in tax policy τ is very infrequent, a researcher might assume it is not a function of time.

[7] In the data (e.g. Table 1, p. 321 in Hansen (1985)) the standard deviation of aggregate hours is 1.66 and the correlation with output is 0.76. At reasonable parameterizations, the real business cycle model predicts a standard deviation of hours of 0.70 and a correlation of hours with output of 0.98. Hence a model economy with divisible labor is less than half as volatile as in the data.

To bring the model closer to the data, Hansen (1985) and Rogerson (1988) introduced "indivisible labor". In their papers, variation in hours comes about by variation in employment (the extensive margin) rather than variation in hours per worker (the intensive margin). That is, aggregate hours $H_t = h_t \cdot N_t$ where h_t is hours per worker and N_t is per capita # workers. In that case, $var(\log H_t) = var(\log h_t) + var(\log N_t) + 2cov(\log h_t, \log N_t)$. In the data, $\frac{var(\log h_t)}{var(\log H_t)} = 0.2$, $\frac{var(\log N_t)}{var(\log H_t)} = 0.55$, $\frac{2cov(\log h_t, \log N_t)}{var(\log H_t)} = 0.25$.

So indivisibilities in the workday (i.e. hours per worker) can help explain the observed behavior of variability of aggregate hours over the business cycle.

These papers also provided a methodological innovation in real business cycle models since they used lotteries to convexify a nonconvex set.

- Suppose then, we want to find the linear relation between $\widehat{n}_t$ and A_t assuming policy τ is constant.To do so, we can take a linearize (20) around a steady state where productivity (and hence wages) is normalized $A^{ss} = 1$ using a first order Taylor series approximation.[8] In that case, with constant returns to scale $\theta = 1$, we have

$$\begin{aligned}\widehat{n}_t &= \left(\frac{(1-\tau)}{\alpha}\right)^{\gamma} + \gamma\left(\frac{(1-\tau)}{\alpha}\right)^{\gamma-1}(A_t - 1) \Longleftrightarrow \\ \widehat{n}_t - b &= m \cdot (A_t - A^{ss}) \qquad (21)\end{aligned}$$

  where $b = \gamma\left(\frac{(1-\tau)}{\alpha}\right)^{\gamma}$ and $m = \gamma\left(\frac{(1-\tau)}{\alpha}\right)^{\gamma-1}$ (just like the equation of a line $y = b + mx$).

- Equation (21) states that deviations of employment from its steady state value (i.e. long run average) are a linear function of deviations of productivity from its steady state value. One could in fact run a linear regression of demeaned employment on demeaned productivity to find m.

- The Lucas (1976) critique argues that relationships observed in historical data (e.g. like those estimated by large scale econometric models used in the 1970s) may not be invariant to changes in government policy. In the example above, the coefficient m of a regression of employment fluctuations on productivity fluctuations depends on τ. In summary, reduced form linear regressions like (21) are not necessarily stable and should not be used to predict the effect of policy experiments.[9]

4.3 Policy

- A fundamental result in public finance is to tax the least sensitive (most inelastic) good if one wants to maximize tax revenues. To see this, the optimal labor tax rate to maximize total revenue is given by:

$$\begin{aligned}\widehat{\tau} &= \arg\max_{\tau} \tau \cdot f'(\widehat{n}) \cdot \widehat{n} \\ &= \arg\max_{\tau} \tau \cdot A \cdot \theta \cdot \left(\frac{(1-\tau)A\theta}{\alpha}\right)^{\frac{\theta}{\left(1+\frac{1}{\gamma}-\theta\right)}}\end{aligned}$$

[8] Recall the first order approximation of a function $f(x)$ around a given point x_0 is

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0).$$

[9] Parameter instability is also evident in the constant of the log-linear regression (19).

which implies

$$\begin{aligned}
0 &= A\cdot\theta\cdot\left(\frac{(1-\widehat{\tau})A\theta}{\alpha}\right)^{\frac{\theta}{\left(1+\frac{1}{\gamma}-\theta\right)}} \\
&+ \widehat{\tau}\cdot A\cdot\theta\cdot\left(\frac{\theta}{\left(1+\frac{1}{\gamma}-\theta\right)}\right)\cdot\left(\frac{(1-\widehat{\tau})A\theta}{\alpha}\right)^{\frac{\theta}{\left(1+\frac{1}{\gamma}-\theta\right)}-1}\cdot\left(\frac{-A\theta}{\alpha}\right) \\
\Longleftrightarrow \widehat{\tau} = & \quad \frac{\left(1+\frac{1}{\gamma}-\theta\right)}{\left(1+\frac{1}{\gamma}\right)}\in(0,1)
\end{aligned}$$

Note that

$$\frac{d\widehat{\tau}}{d\gamma}=\frac{\left(1+\frac{1}{\gamma}\right)\left(-\gamma^{-2}\right)-\left(1+\frac{1}{\gamma}-\theta\right)\left(-\gamma^{-2}\right)}{\left(1+\frac{1}{\gamma}\right)^2}<0.$$

Hence, the optimal labor tax decreases in the Frisch labor supply elasticity γ. That is, the more sensitive labor supply is to tax changes, the lower taxes should be.

5 Welfare Theorems and Equivalence of Planner and Market Outcomes

- In Section 3.4 we saw that the competitive outcome (16) is identical to the planner's outcome (3) under appropriate assumptions (like no distortionary taxation or "wedges", etc.). Thus there is no way to make anyone better off without making someone else worse off in the competitive equilibrium. This is the idea behind the *First Welfare Theorem*; under appropriate assumptions, a competitive equilibrium is pareto optimal.

- The converse is given by the *Second Welfare Theorem*; under appropriate assumptions, any pareto optimal allocation can be acheived as a competitive equilibrium with appropriate transfers (typically implemented through lump sum taxes and transfers). The second welfare theorem is actually useful computationally in macro; we can implement the allocation that solves the planner's problem (which doesn't involve solving for prices) as a competitive equilibrium (which is harder to solve for).

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L3. Overlapping Generations in an Endowment Economy

These notes are based on Kehoe (1989) "Intertertemporal General Equilibrium Models". It focuses on existence and uniqueness of a competitive equilibrium, stationary and nonstationary equilibria, stability of equilibrium, pareto efficiency, and equilibrium selection.[1]

1 OG Environment

- Population: Each period $t = 1, 2, 3, ...$ a new generation of 2 period lived households are born. Further there is an initial old generation alive in period $t = 1$. To keep notation simple, we will consider a "representative" agent of the unit measure of each generation.
- Technology:
  - Each generation is endowed with w_1 in youth and w_2 in old age of nonstorable consumption goods. Note that endowments don't depend explicitly on time.
  - The initial old are also endowed with an asset $\overline{M} \in \mathbb{R}$ as well as w_2. Note here that the asset is not restricted to be positive (i.e. the initial old can be saddled with "debt").[2] We often call the case where $M \geq 0$ "outside money" and the case where $M < 0$ "inside money".
  - There is a commitment technology which enforces feasible trades. For example, if $w_1 = 0$ and $w_2 > 0$, a generation t household can borrow against its future income but is committed to repay at $t + 1$. For the above example, the young get goods from a bank and give the bank an IOU which the agent must repay to the bank when old. In that case, in every period, the bank has some goods coming in (from the old) and going out (to the young).[3]
- Preferences: The utility function of a household of generation $t \geq 1$ is $U\left(c_t^t, c_{t+1}^t\right)$ where $\left(c_t^t, c_{t+1}^t\right)$ is consumption of generation t in youth (i.e. in period t) and old age (i.e in period $t + 1$). Note that $U(\cdot, \cdot)$ does not depend explicitly on time. The preferences of the initial old are given by $U(c_1^0)$.

[1] Note, I will try to adhere to Kehoe's notation, but in some places I have changed it to be consistent with other notes.

[2] An example where $\overline{M} < 0$ is a guaranteed student loan which the government makes to its citizens when young which is repayed when old.

[3] Note that this example is different from the case in the first set of notes where $w_1 > 0$ and $w_2 = 0$. In that case, the old wanted to borrow but it was not feasible for them to pay back the next period since they would be dead.

– As a parametric example, we will take

$$U\left(c_t^t, c_{t+1}^t\right) = \ln\left(c_t^t\right) + \beta \ln\left(c_{t+1}^t\right)$$

where the household's discount factor is given by $\beta \in [0,1]$. Here we change notation (i.e. Kehoe denotes $\gamma \equiv \beta$).

- We can state the following definitions for this environment.

Definition. An *allocation* is a sequence $\{c_t^{t-1}, c_t^t\}_{t=1}^{\infty}$.
Definition. An allocation is *resource feasible* if $(c_t^{t-1}, c_t^t) \in \mathbb{R}_+^2$ for all $t \geq 1$ and

$$c_t^t + c_t^{t-1} \leq w_1 + w_2, \forall t \geq 1. \tag{1}$$

- Since both technology and preferences don't depend explicitly on time, it is sensible to look for stationary equilibria of this economy. There may also be nonstationary equilibria as we will see.

2 Competitive Equilibrium

- We assume a sequence of competitive markets for goods and assets (e.g. inside or outside money). Let p_t be the relative price of goods in terms of the asset (i.e. the number of units of the asset per 1 unit of consumption good at t).

2.1 Decision Problems

- A generation t household's *sequence* of budget constraints in periods t and $t+1$ are given by

$$\begin{aligned} p_t c_t^t + M_{t+1}^t &= p_t w_1, && (2) \\ p_{t+1} c_{t+1}^t &= p_{t+1} w_2 + M_{t+1}^t. && (3) \end{aligned}$$

where $M_{t+1}^t \in \mathbb{R}$ is the asset holdings of a generation t household chosen in period t to begin period $t+1$.[4][5]

[4] I understand this notation for asset holding is a little confusing but it will prove useful when you start doing dynamic programming.

[5] Another way to write the constraint set is in "real" terms where the real holdings of money at time t is $m_{t+1}^t \equiv \frac{M_{t+1}^t}{p_t}$:

$$\begin{aligned} c_t^t + m_{t+1}^t &= w_1, \\ c_{t+1}^t &= w_2 + \frac{p_t}{p_{t+1}} \cdot m_{t+1}^t. \end{aligned}$$

This form makes clear that in autarky (where money has no value so $m_{t+1}^t = 0$), we have $c_t^t = w_1$ and $c_{t+1}^t = w_2$.

- These constraints can be consolidated by substituting out M_{t+1}^t to yield the consolidated constraint:

$$p_t c_t^t + p_{t+1} c_{t+1}^t = p_t w_1 + p_{t+1} w_2 \tag{4}$$

- It is important to note that if there was a further constraint that $M_{t+1}^t \geq 0$, then it would not be possible to characterize the constraint set by (4) only without reference to M_{t+1}^t.

- This is a critical distinction for when sequential markets economies are equivalent to economies with initial trades conducted through futures markets. While more will be said about this in a later part of the first year, it is the subject of Chapter 8 - Equilibrium with Complete Markets - of Ljundqvist and Sargent (LS). In particular, LS would term (2)-(3) part of a sequential trading structure or "one-period Arrow securities" economy while they would term (4) part of a "time 0" Arrow-Debreu market structure. In the context of (4) for our overlapping generations model, time 0 is when each generation is born (say t), there is a claim for next period's consumption c_{t+1}^tavailable in period t at price p_{t+1}.

- In this case, the household's problem is

$$\begin{aligned} & \max_{c_t^t, c_{t+1}^t} U\left(c_t^t, c_{t+1}^t\right) \\ \text{s.t. } p_t c_t^t + p_{t+1} c_{t+1}^t \quad = \quad & p_t w_1 + p_{t+1} w_2 \end{aligned}$$

2.2 Optimization

- Forming a lagrangian we have

$$\pounds = U\left(c_t^t, c_{t+1}^t\right) + \lambda_t \cdot \left[p_t w_1 + p_{t+1} w_2 - p_t c_t^t - p_{t+1} c_{t+1}^t\right]$$

  Note that the multiplier λ_t is time contingent. The marginal utility from relaxing the intertemporal constraint may vary over time because the relative price ratio (or real interest rate) may be varying over time in non-stationary equilibria.

- The first order conditions are

$$\frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_t^t} = \lambda_t \cdot p_t, \frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_{t+1}^t} = \lambda_t \cdot p_{t+1}, \tag{5}$$

  plus the budget constraint (4).

- Taking the ratio in (5) we have

$$MRS\left(c_t^t, c_{t+1}^t\right) \equiv \frac{\frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_t^t}}{\frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_{t+1}^t}} = \frac{p_t}{p_{t+1}} \equiv MRT_{t,t+1} \tag{6}$$

where as is clear from footnote 4 that the marginal rate of transformation of period t goods into period t+1 goods is not technological (as typical) but through the market.

- Equation (6) makes clear that the real return to substituting consumption goods across time is given by $\left(\frac{p_t}{p_{t+1}}\right)$ which is sometimes written as $(1 + r_{t,t+1})$ where $r_{t,t+1}$ denotes the net real return between t and $t+1$.[6] For example, if there is inflation (i.e. $p_{t+1} > p_t$, so that the ratio is less than one) then the net real return is negative (i.e. $r_{t,t+1} < 0$) and if there is deflation (i.e. $p_{t+1} < p_t$ so that the ratio is greater than one) then the net real return is positive (i.e. $r_{t,t+1} > 0$).

- For a given price ratio $\frac{p_t}{p_{t+1}}$, the solution (4) and (5) can be seen graphically in Figure L3.1 for a case where $\beta w_1 > w_2$ (i.e. where people want to consumption smooth by saving since they have little future consumption).[7]

  - The budget constraint (4) can be written

$$c_{t+1}^t = \frac{p_t}{p_{t+1}} w_1 + w_2 - \frac{p_t}{p_{t+1}} c_t^t \tag{7}$$

    so that the intercept of the budget constraint is $\frac{p_t}{p_{t+1}} w_1 + w_2$ and the slope is $-\frac{p_t}{p_{t+1}}$. One can interpret the intercept as the agent's "permanent income".[8]

  - The first order conditions imply a tangency of the slope of an indifference curve ($-MRS$) with the slope of the budget constraint.[9]

  - In Figure L3.1, the price ratio $\left(\frac{p_t}{p_{t+1}}\right)_A$ makes the household happy to consume its endowment (point A).

  - When the real return to holding the asset rises - that is, when $\left(\frac{p_t}{p_{t+1}}\right)_1 > \left(\frac{p_t}{p_{t+1}}\right)_A$ - households decide to save in order to smooth consumption (point 1).

---

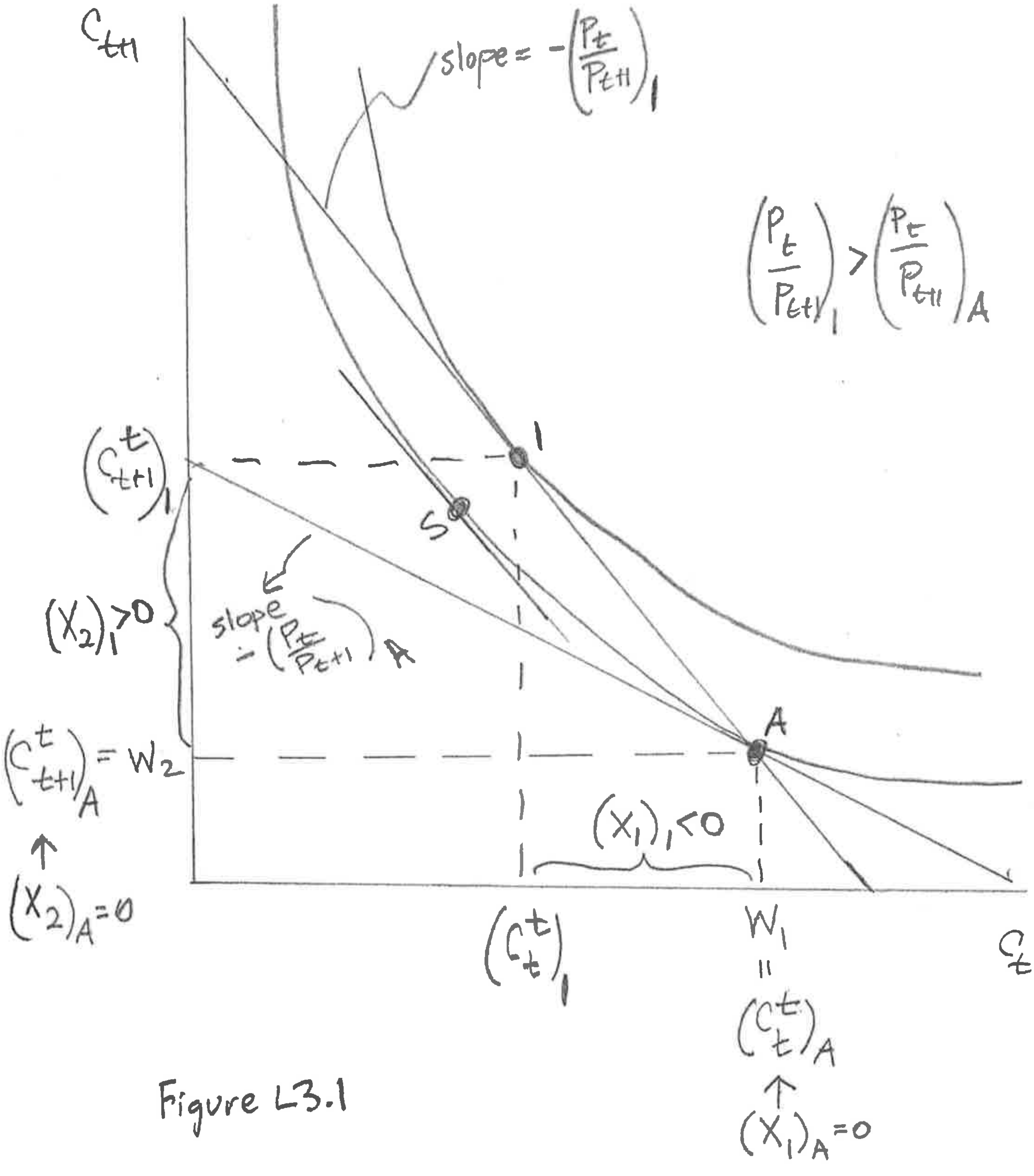
[6] Why is this a real return? Consider the case where $\overline{M}$ is outside money (e.g. dollars). Then $p_t = \#\$/good\_t$. In that case,

$$\frac{p_t}{p_{t+1}} = \frac{\frac{\#\$}{good\_t}}{\frac{\#\$}{good\_t+1}} = \frac{\#good\_t+1}{good\_t}.$$

[7] You should draw this picture for the case where $\beta w_1 < w_2$ to see a case where $x_2 < 0$ and $x_1 > 0$.

[8] This is related to Milton Friedman's permanent income hypothesis (PIH), used to describe how agents spread consumption over their lifetimes. A person's consumption at a point in time is determined not just by their current income but also by their expected income in future years â€' their "permanent income". Its predictions of consumption smoothing, where people spread out transitory changes in income over time, depart from the traditional Keynesian emphasis on excess sensitivity of consumption to current income changes. It provides an explanation for why Keynesian demand management policies might not yield big effects.

[9] Recall an indifference curve is the locus of combinations of (c_t^t, c_{t+1}^t) for which utility is



C_{t+1}
slope = $-\left(\frac{P_t}{P_{t+1}}\right)_1$
$\left(\frac{P_t}{P_{t+1}}\right)_1 > \left(\frac{P_t}{P_{t+1}}\right)_A$
$\left(C^t_{t+1}\right)_1$
1
S
slope $-\left(\frac{P_t}{P_{t+1}}\right)_A$
$(X_2)_1 > 0$
A
$\left(C^t_{t+1}\right)_A = W_2$
$(X_2)_A = 0$
$(X_1)_1 < 0$
$\left(C^t_t\right)_1$
W_1
=
$\left(C^t_t\right)_A$
$(X_1)_A = 0$
C_t

Figure L3.1

– The higher cost to consuming today leads the household to substitute away from current consumption into the less costly future consumption (i.e. substitution effect is from A to S) while the income effect leads the household to consume more of both goods (i.e. income effect is from S to 1).

– The OG literature calls the case where substitution effects outweigh income effects and savings rise as the rate of return rises "gross substitutability".

- *Characterization via Excess Demand Functions*: Define the excess demand function in youth (i.e. period 1 of life) as $x_1(p_t, p_{t+1}) = (c_t^t - w_1)$ and in old age (i.e. period 2 of life) as $x_2(p_t, p_{t+1}) = \left(c_{t+1}^t - w_2\right)$.[10]

  – Figure L3.1 makes clear that we can summarize household optimization in terms of excess demand functions and how excess demands (or saving/dissaving) changes as intertemporal relative prices $\frac{p_t}{p_{t+1}}$ change.

  – Given the strict concavity of $U(\cdot)$, x_1 and x_2 are continuous functions of prices.

  – x_1 and x_2 depend only on $\frac{p_t}{p_{t+1}}$ not on p_t and p_{t+1} separately. This is because preferences don't explicitly depend on prices and prices enter the budget constraint (7) only as a ratio and not on prices separately. This implies that they are homogeneous of degree 0 in prices (p_t, p_{t+1}); that is, if each price is multiplied by α, there is no change in excess demand (i.e. $\frac{\alpha p_t}{\alpha p_{t+1}} = \frac{p_t}{p_{t+1}}$).

  – This is another example of where notation differs from Kehoe where in his notation $y \equiv x_1$ while $z \equiv x_2$.

- In addition to the consumers born in periods $t = 1, 2, ...,$there is a consumer alive only in period 1 who solves the problem

$$\max_{c_1^0} U\left(c_1^0\right)$$
$$\text{s.t. } p_1 c_1^0 = p_1 w_2 + \overline{M}$$

kept constant or

$$\begin{aligned} 0 &= dU = \frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_t^t} dc_t^t + \frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_{t+1}^t} dc_{t+1}^t \iff \\ \frac{dc_{t+1}^t}{dc_t^t} &= -\frac{\frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_t^t}}{\frac{\partial U\left(c_t^t, c_{t+1}^t\right)}{\partial c_{t+1}^t}} = -MRS \end{aligned}$$

[10] The excess demand functions depend on p_t and p_{t+1} since the solution to the household's optimization problem yields decision rules $c_t^t(p_t, p_{t+1})$ and $c_t^t(p_t, p_{t+1})$.

– The excess demand function of this initial old household is given simply by

$$x_0(M, p_1) = c_1^0 - w_2 = \frac{\overline{M}}{p_1} \tag{8}$$

– Notice x_0 is also homogeneous of degree 0 in p_1 and $\overline{M}$ from (8).

– Further, notice from (8) that If money has no value (i.e. $\frac{1}{p_1} = 0$), then even if $\overline{M} \neq 0$ we have $x_0 = 0$ (or autarky).

- The homogeneity results form the basis for a result known as the neutrality of money; if the money supply is doubled and all prices double, there is no change in real allocations.

2.3 Market Clearing

- Goods market clearing (the resource constraint (1) with equality) requires aggregate excess demand is zero in every period $t = 1, 2, ...$ or

$$\begin{aligned} x_1(p_1, p_2) + x_0(\overline{M}, p_1) &= 0, \quad (9) \\ x_1(p_t, p_{t+1}) + x_2(p_{t-1}, p_t) &= 0. \quad (10) \end{aligned}$$

Equation (10) makes clear that equilibrium must satisfy a second order difference equation in prices.

- There are many examples in macroeconomics where an equilibrium must satisfy a second order difference equation. For example, an RBC model equilibrium requires an expectational nonlinear second order difference equation. To simplify the analysis, that nonlinear equation is often linearized around a steady state.

2.4 Equilibrium - Characterization via offer curves:

- We will study the existence and uniqueness of equilibrium by use of an *offer curve*.[11] The offer curve is the locus of optimal excess demands as we vary the price ratio between 0 and ∞ (not inclusive). In particular, from the household budget constraint (4) and optimization we have

$$p_t \cdot x_1(p_t, p_{t+1}) + p_{t+1} \cdot x_2(p_t, p_{t+1}) = 0 \tag{11}$$

which can be written

$$\frac{x_2(p_t, p_{t+1})}{x_1(p_t, p_{t+1})} = -\frac{p_t}{p_{t+1}}. \tag{12}$$

Since $c \geq 0$, we know $x_1(p_t, p_{t+1}) \geq -w_1$ and $x_2(p_t, p_{t+1}) \geq -w_2$. It should be clear that the offer curve can be thought of simply as a translation of Figure L3.1.

[11] Existence and Uniqueness of General Equilibrium will be addressed in your micro class. Chapter 20H in Mas-Colell, Whinston, and Green (1995) describes equilibrium for overlapping generations models in terms of offer curves.

- Since Figure L3.1 was drawn for the case where substitution effects outweigh income effects for all price ratios, it generates offer curve functions. However, there are preferences where the offer curve is backward bending (i.e. a correspondence) due to income effects eventually outweighing substitution effects.

- Notice that market clearing (10) can be written

$$\frac{x_2(p_{t-1}, p_t)}{x_1(p_t, p_{t+1})} = -1 \tag{13}$$

so that the offer curve (12) and market clearing (13) are not necessarily identical.

- Note that the slope of the offer curve is not necessarily equal to the MRS at an optimum given in (12). You will see why in Figure L3.3.

Definition. Given $\overline{M}$, a *perfect foresight competitive equilibrium* is an allocation $\{c_t^{t-1}, c_t^t\}_{t=1}^{\infty}$ and prices $\{p_t\}_{t=1}^{\infty}$ such that:

1. Households optimize (8), (12), and
2. Markets clear (13).

- Why are offer curves a useful construct to characterize a competitive equilibrium in an infinite horizon framework? It is possible to characterize stationary and nonstationary equilibria in one diagram using only 2 equations: household optimization in (12) and market clearing in (13).

- In Figure L3.2 we plot the offer curve and market clearing conditions for two different cases: one where endowments are high in youth and low in old age so that households want to save and one where endowments are low in youth and hight in old age so households want to borrow.

3 Existence and Uniqueness in a Parametric Example

- For our parametric example, the foc (5) we have are

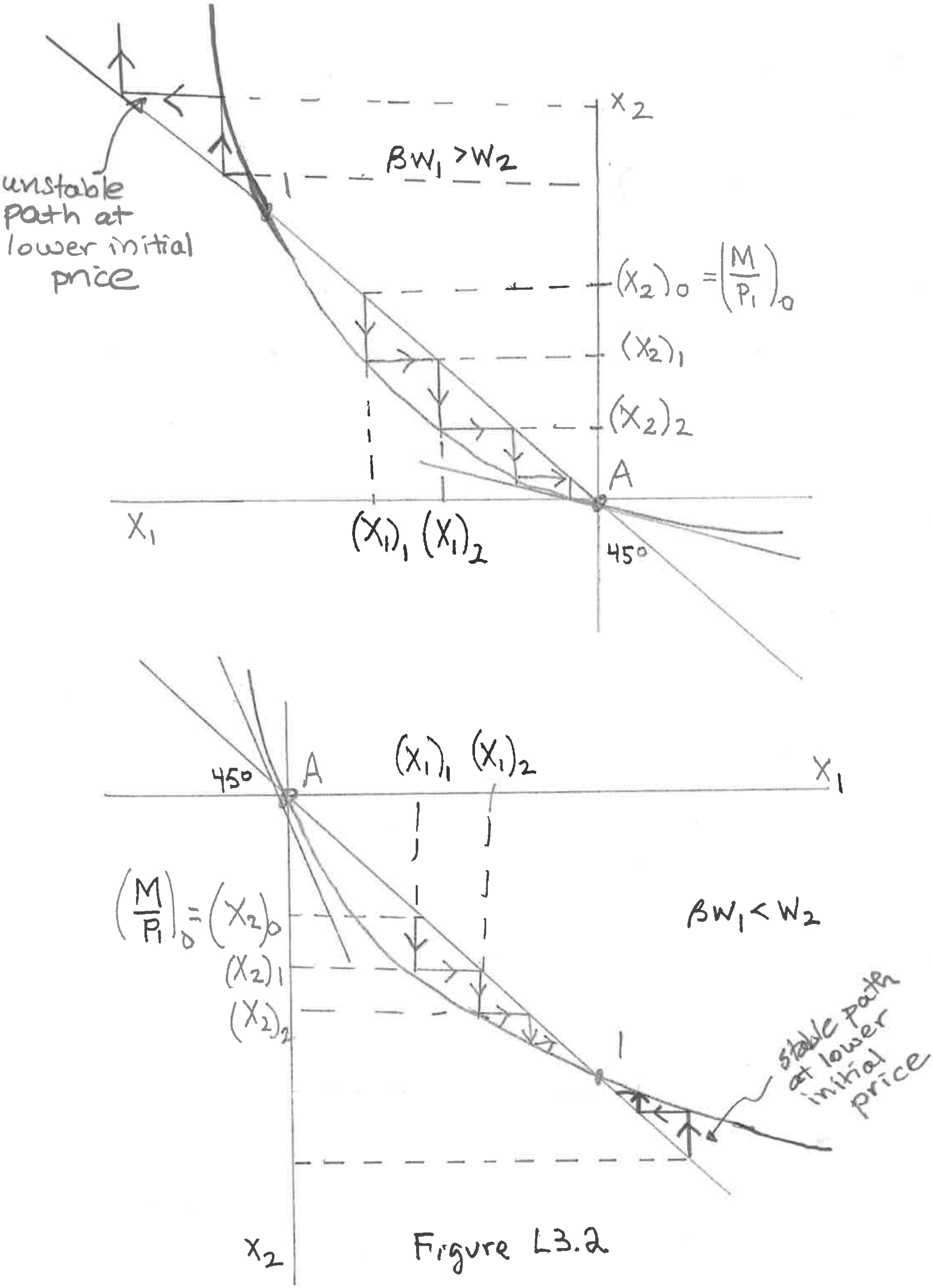
$$\frac{1}{c_t^t} = \lambda_t \cdot p_t, \frac{\beta}{c_{t+1}^t} = \lambda_t \cdot p_{t+1},$$

or taking their ratio

$$p_{t+1} c_{t+1}^t = p_t \beta c_t^t \tag{14}$$

into the budget constraint:

$$p_t \cdot c_t^t + p_t \cdot \beta c_t^t = p_t \cdot w_1 + p_{t+1} \cdot w_2 \Longleftrightarrow c_t^t = \frac{p_t \cdot w_1 + p_{t+1} \cdot w_2}{p_t \cdot (1 + \beta)} \tag{15}$$



X2
BW1 > W2
I
unstable path at lower initial price
(X2)0 = (M/P1)0
(X2)1
(X2)2
A
X1
(X1)1 (X1)2
45°
(X1)1 (X1)2
X1
45°
A
(M/P1)0 = (X2)0
(X2)1
(X2)2
BW1 < W2
I
stable path at lower initial price
X2

Figure L3.2

so that

$$c_{t+1}^t = \frac{\beta \cdot [p_t \cdot w_1 + p_{t+1} \cdot w_2]}{p_{t+1} \cdot (1+\beta)} \tag{16}$$

As stated earlier, equations (15) and (16) make clear that as p_{t+1} rises relative to p_t, current consumption rises and future consumption falls.

- Notice that at an optimum (14) implies the analogue of (6) is given by:

$$MRS\left(c_t^t, c_{t+1}^t\right) \equiv \frac{c_{t+1}^t}{\beta c_t^t} = \frac{p_t}{p_{t+1}}. \tag{17}$$

- The MRS evaluated *at autarky* (i.e. when $(x_1, x_2) = (0, 0)$) is given by

$$\frac{w_2}{\beta w_1} = \frac{p_t}{p_{t+1}} \equiv \frac{1}{\mu}. \tag{18}$$

where it should be noted that since the lhs is independent of time, the price ratio must be independent of time and by implication $p_{t+1} = \mu \cdot p_t$. In that case,

  - if $\beta w_1 > w_2$, then the slope of the budget constraint that is consistent with autarky is $1/\mu < 1 \Longrightarrow \mu > 1$,
  - if $\beta w_1 < w_2$, then the slope of the budget constraint that is consistent with autarky is $1/\mu > 1 \Longrightarrow \mu < 1$.
  - This is another case where I am diverging from Kehoe's notation (what I am calling μ, Kehoe calls β).

- With regard to offer curves, the excess demand functions are

$$x_1(p_t, p_{t+1}) = \frac{p_t \cdot w_1 + p_{t+1} \cdot w_2}{p_t \cdot (1+\beta)} - w_1 = \frac{p_{t+1} \cdot w_2 - p_t \cdot \beta \cdot w_1}{p_t \cdot (1+\beta)} \tag{19}$$

$$x_2(p_t, p_{t+1}) = \frac{\beta \cdot [p_t \cdot w_1 + p_{t+1} \cdot w_2]}{p_{t+1} \cdot (1+\beta)} - w_2 = \frac{\beta \cdot p_t \cdot w_1 - p_{t+1} \cdot w_2}{p_{t+1} \cdot (1+\beta)} \tag{20}$$

- Notice that for strictly positive prices, the excess demand functions are continuous in prices and homogeneous of degree 0. To see the latter,

$$x_1(\alpha p_t, \alpha p_{t+1}) = \left(\frac{\alpha p_{t+1} \cdot w_2 - \alpha p_t \cdot \beta \cdot w_1}{\alpha p_t \cdot (1+\beta)}\right) = \frac{\alpha}{\alpha} \cdot x_1(p_t, p_{t+1}).$$

- For our parametric example, solving (19) for $\frac{p_t}{p_{t+1}}$ in terms of x_1 yields:

$$\begin{aligned} x_1 &= \frac{p_{t+1} \cdot w_2}{p_t \cdot (1+\beta)} - \frac{\beta \cdot w_1}{(1+\beta)} \Longleftrightarrow \\ x_1 + \frac{\beta \cdot w_1}{(1+\beta)} &= \frac{p_{t+1}}{p_t} \cdot \frac{w_2}{(1+\beta)} \Longleftrightarrow \\ \frac{p_{t+1}}{p_t} &= \left[x_1 \cdot \frac{(1+\beta)}{w_2} + \frac{\beta \cdot w_1}{w_2}\right]. \end{aligned} \tag{21}$$

We can then substitute $\frac{p_{t+1}}{p_t}$ from (21) into (20)

$$x_2 = \frac{p_t}{p_{t+1}} \cdot \frac{\beta \cdot w_1}{(1+\beta)} - \frac{w_2}{(1+\beta)}$$

to yield

$$\begin{aligned} x_2 &= \frac{1}{\left[x_1 \cdot \frac{(1+\beta)}{w_2} + \frac{\beta \cdot w_1}{w_2}\right]} \cdot \left[\frac{\beta \cdot w_1}{(1+\beta)}\right] - \frac{w_2}{(1+\beta)} \\ &= \frac{\beta \cdot w_1 \cdot w_2}{[x_1 \cdot (1+\beta)^2 + \beta \cdot (1+\beta) \cdot w_1]} - \frac{w_2}{(1+\beta)} \end{aligned} \tag{22}$$

which gives us x_2 as a function of x_1 (i.e. the offer curve).

3.1 Steady State Equilibria:

- When $x_1 = 0$ in (22), it is clear that $x_2 = 0$ since

$$x_2 = \frac{\beta \cdot w_1 \cdot w_2}{[\beta \cdot (1+\beta) \cdot w_1]} - \frac{w_2}{(1+\beta)} = 0. \tag{23}$$

  Hence $(x_1, x_2) = (0, 0)$ is a steady state autarkic equilibrium.

- There is another solution when $x_1 = -x_2 \neq 0$ (which is also consistent with market clearing (13)) or

$$\begin{aligned} x_2 &= \frac{1}{\left[-x_2 \cdot \frac{(1+\beta)}{w_2} + \frac{\beta \cdot w_1}{w_2}\right]} \cdot \left[\frac{\beta \cdot w_1}{(1+\beta)}\right] - \frac{w_2}{(1+\beta)} \\ \Longleftrightarrow & \quad \frac{(1+\beta)x_2 + w_2}{(1+\beta)} = \frac{\beta \cdot w_1 \cdot w_2}{[\beta \cdot (1+\beta) \cdot w_1 - x_2 \cdot (1+\beta)^2]} \\ \Longleftrightarrow & \quad [(1+\beta)x_2 + w_2] \cdot \left[\beta \cdot (1+\beta) \cdot w_1 - x_2 \cdot (1+\beta)^2\right] = \beta \cdot w_1 \cdot w_2 \cdot (1+\beta) \\ \Longleftrightarrow & \quad 0 = x_2 \cdot \beta \cdot (1+\beta)^2 \cdot w_1 - (1+\beta)x_2^2 \cdot (1+\beta)^2 - w_2 \cdot x_2 \cdot (1+\beta)^2 \\ \Longleftrightarrow & \quad x_2 = \frac{\beta \cdot w_1 - w_2}{(1+\beta)} \end{aligned} \tag{24}$$

- Hence, whether or not the non-autarkic steady state equilibrium occurs with saving (i.e. when $x_2 > 0$) or borrowing (i.e. when $x_2 < 0$) depends on relative endowments and time preference.

  – In particular, if households are relatively highly endowed in youth and are sufficiently patient (i.e. $\beta w_1 > w_2$), then they will have positive excess demand in old age (i.e. $x_2 > 0$) which is financed by negative excess demand (i.e. saving) in youth (i.e. $x_1 = -x_2 < 0$). The top graph in Figure L3.2 plots the offer curve (12) and the market clearing condition (13) for this case.

- On the other hand, if households are relatively highly endowed in old age or are insufficiently patient (i.e. $\beta w_1 < w_2$), then they will have negative excess demand in old age (i.e. $x_2 < 0$) in order to finance positive excess demand (i.e. borrowing) in youth (i.e. $x_1 = -x_2 > 0$). The bottom graph in Figure L3.2 plots the offer curve (12) and the market clearing condition (13) for this case.

- By (12), we know the price ratio at point 1 in each figure has constant prices since there $p_t/p_{t+1} = 1$ (i.e. $p_{t+1} = p_t$) but at point A in the top panel of the figure (where $\beta w_1 > w_2$) we know $p_t/p_{t+1} < 1$ (i.e. $p_{t+1} > p_t$) while at the point A in the bottom figure (where $\beta w_1 < w_2$) we know $p_t/p_{t+1} > 1$ (i.e. $p_{t+1} < p_t$).[12]

- Given that preferences and endowments are independent of time (i.e. the left hand side of (12)), a steady state must imply that p_t/p_{t+1} must be independent of time.

  - The only way for that is if p_t/p_{t+1} is a constant. This doesn't mean prices are necessarily constant in the steady state, simply the ratio is. In particular, $p_t/p_{t+1} = 1/\mu$ or $p_{t+1} = \mu \cdot p_t$. An alternative way to write this is $p_{t+1} = \mu^t p_1$.
  - If $\mu > 1$, then the gross real rate of return on the asset $1 + r_{t.t+1} = p_t/p_{t+1} = 1/\mu < 1$ which implies a negative real rate of return (e.g. there is inflation) in the steady state. This corresponds to point A (where $\beta w_1 > w_2$) in the top panel. In order for households to choose autarky (or at least be indifferent between autarky and the asset) when they want to save, it must be that saving bears a negative rate of return.
  - If $\mu < 1$, then the asset has a positive real return in the steady state. This corresponds to point A (where $\beta w_1 < w_2$) in the bottom panel. In order for households to choose autarky (or at least be indifferent between autarky and the asset) when they want to borrow, it must be that borrowing is costly (i.e. inside money bears a positive rate of return).
  - Recall this is also where notation differs from Kehoe (what he denotes β, we denote μ).

- Notice that a stationary equilibrium also doesn't necessarily mean that an individual household's consumption path is constant over its lifetime, only that aggregate consumption of the young and old is constant.

  - In particular, from (14), we know

$$c_{t+1}^t = \frac{p_t}{p_{t+1}} \beta c_t^t = \frac{\beta}{\mu} c_t^t$$

[12] Note that point 1 in Figure L3.1 is not necessarily point 1 in Figure L3.2.

so that in steady state 1 (where $\beta w_1 > w_2$) in the top panel, consumption declines over the life cycle and if $\mu << 1$, consumption rises over the life cycle. Even if $\mu < 1$, consumption can decline over the life cycle provided $\beta < \mu$.

3.2 Nonstationary Equilibria

- There are also a continuum of non-stationary equilibria (depending on the initial price level) that converge in the limit to a steady state.
- To see this, consider the following algorithm for constructing a nonstationary equilibrium.[13]

1. Pick an initial price level p_1, say $\widehat{p}_1$. This determines the real amount of money $\left(\frac{\overline{M}}{\widehat{p}_1}\right)$ the initial old start with. The higher is $\widehat{p}_1$, the less real money the initial old have (which means the less they can consume in the case of outside money or the more they can consume with inside money). This price determines $x_0(\overline{M}, \widehat{p}_1)$ on the vertical axis.
2. Given $x_0(\overline{M}, \widehat{p}_1)$ market clearing at $t = 1$ requires $x_1(\widehat{p}_1, \widehat{p}_2)$ is read off the 45^o line (see horizontal axis).
3. Given $x_1(\widehat{p}_1, \widehat{p}_2)$ optimization requires $x_2(\widehat{p}_1, \widehat{p}_2)$ is read off the offer curve (see vertical axis). The price ratio associated with this point (i.e $\widehat{p}_1/\widehat{p}_2$) is given by the slope of the line from the origin to that point on the offer curve. Using the earlier nomenclature, this slope is $\frac{-1}{\mu_{1,2}}$.
4. Given $x_2(\widehat{p}_1, \widehat{p}_2)$ market clearing at $t = 2$ requires $x_1(\widehat{p}_2, \widehat{p}_3)$ is read off the 45^o line.
5. Given $x_1(\widehat{p}_2, \widehat{p}_3)$ optimization requires $x_2(\widehat{p}_2, \widehat{p}_3)$ is read off the offer curve. Again, the price ratio associated with this point (i.e $\widehat{p}_2/\widehat{p}_3$) is given by the slope of the line from the origin to that point on the offer curve. Using the earlier nomenclature, this slope is $\frac{-1}{\mu_{2,3}}$.
6. Continuing, given $x_2(p_{t-1}, p_t)$ market clearing at t requires $x_1(p_t, p_{t+1})$ is read off the 45^o line.
7. Given $x_1(p_t, p_{t+1})$ optimization requires $x_2(p_t, p_{t+1})$ is read off the offer curve. Again, the price ratio associated with this point (i.e p_t/p_{t+1}) is given by the slope of the line from the origin to that point on the offer curve. Using the earlier nomenclature, this slope is $\frac{-1}{\mu_{t,t+1}}$.
8. In the limit, this sequence converges to a steady state depending on $\beta w_1 \gtrless w_2$. Along the path, inflation is growing since $\mu_{t-1,t} < \mu_{t,t+1}$.

[13] In Figure L3.2, the subscript "n" outside the parentheses of $(x_1)_n$ on the horizontal axis and outside the parentheses of $(x_2)_n$ on the vertical axis refers to the excess demand of the generation.

- If $\widehat{p}_1 \in \{p_1^1, p_1^A\}$ which are associated with steady states at point 1 or A, respectively, then we are already at the steady state.

- If $\widehat{p}_1 \in (p_1^1, p_1^A)$, then

  - if $\beta w_1 > w_2$ the rising price sequence converges to a steady state where the gross real rate of return on the asset $1/\mu < 1$. Along the path, returns are becoming more and more negative until households are just indifferent between autarky and foregoing current consumption (i.e. saving). This is an example where if beliefs about the initial price level are too high, it could set off an "inflationary" downspin.

  - if $\beta w_1 < w_2$ the declining price sequence converges to a steady state where the gross real rate of return on the asset equals 1. Along the path, borrowing costs are dropping. Only in the autarkic steady state are costs high enough to make households just indifferent between autarky and borrowing to finance current consumption.

- Notice that there is nothing which "pins down" $\widehat{p}_1 \in [p_1^1, p_1^A]$. That is, any $\widehat{p}_1 \in [p_1^1, p_1^A]$ is consistent with a competitive equilibrium. Next we will see that different $\widehat{p}_1$ correspond to different levels of welfare for households.

- Note that besides steady state and nonstationary equilibria as above, if there are preferences such that there is a backward bending offer curve (e.g. Grandmont, J.M. (1985) "On Endogenous Competitive Business Cycles", *Econometrica,* 53, pp. 995-1045), there can be deterministic cycles which are another form of nonstationary equilibria.

- In summary, Gale (1973) termed the first case (where $w_2 < \beta w_1$ or equivalently where the autarkic real rate of return on the asset is negative), the Samuelson case. He termed the second case (where $w_2 > \beta w_1$ or equivalently where the autarkic real rate of return on the asset is positive), the Classical case.

3.3 Stability of non-autarkic steady states

- When $\widehat{p}_1 \notin [p_1^1, p_1^A]$ with $\beta w_1 > w_2$ (i.e. for $\widehat{p}_1 < p_1^1$), the price ratio diverges and the allocation eventually violates feasibility (i.e. $c_t^t, c_{t+1}^t < 0$). That is, those initial prices are inconsistent with equlibrium. Thus the non-autarkic (monetary) steady state is unstable in this case.

- When $\widehat{p}_1 \notin [p_1^1, p_1^A]$ with $\beta w_1 < w_2$ (i.e. for $\widehat{p}_1 < p_1^1$), the price ratio also converges. Thus, the non-autarkic steady state is stable in this case.

3.4 Second order difference equations

- The market clearing condition (10) in this parametric example is given by:

$$\begin{aligned} x_1(p_t, p_{t+1}) + x_2(p_{t-1}, p_t) &= 0 \Longleftrightarrow \\ \frac{p_{t+1} \cdot w_2 - p_t \cdot \beta \cdot w_1}{p_t \cdot (1+\beta)} + \frac{\beta \cdot p_{t-1} \cdot w_1 - p_t \cdot w_2}{p_t \cdot (1+\beta)} &= 0 \Longleftrightarrow \\ \frac{p_{t+1} \cdot w_2}{p_t} + \frac{\beta \cdot p_{t-1} \cdot w_1}{p_t} - (\beta \cdot w_1 + w_2) &= 0 \qquad (25) \end{aligned}$$

- This particular second order difference equation has a special form since it is written in terms of ratios. In particular, let $q_{t+1} \equiv \frac{p_t}{p_{t+1}}$, then (25) can be written as a nonlinear first order difference equation

$$\begin{aligned} \frac{w_2}{q_{t+1}} + \beta \cdot q_t \cdot w_1 - (\beta \cdot w_1 + w_2) &= 0 \Longleftrightarrow \\ \beta \cdot w_1 \cdot q_t \cdot q_{t+1} - (\beta \cdot w_1 + w_2) \cdot q_{t+1} + w_2 &= 0 \Longleftrightarrow \\ a \cdot q_t \cdot q_{t+1} - b \cdot q_{t+1} + c &= 0 \qquad (26) \end{aligned}$$

  where $a \equiv \beta w_1, b \equiv \beta \cdot w_1 + w_2$, and $c = w_2$.

  – This first order difference equation has 2 steady state solutions. To see this, let $q_t = \overline{q}$ in which case (26) can be written

  $$a \cdot \overline{q}^2 - b \cdot \overline{q} + c = 0$$

  which from the quadratic equation has solution

  $$\begin{aligned} \frac{\beta \cdot w_1 + w_2 \pm \sqrt{(\beta \cdot w_1 + w_2)^2 - 4\beta w_1 w_2}}{2\beta w_1} &\Longleftrightarrow \\ \frac{\beta \cdot w_1 + w_2 \pm (\beta \cdot w_1 - w_2)}{2\beta w_1} &\Longleftrightarrow \\ \overline{q} \quad \in \quad \{1, \frac{w_2}{\beta w_1}\} & \end{aligned}$$

  which should look familiar.

  – Stability of the equilibria and paths of this equation can be found by the methods taught in the review session.

4 On Pareto Optimality of Competitive Equilibrium

Definition. An allocation $\{c_t^{t-1}, c_t^t\}_{t=1}^{\infty}$ is *Pareto Optimal (or Pareto Efficient)* if it is resource feasible and if there is no other feasible allocation $\{\widehat{c}_t^{t-1}, \widehat{c}_t^t\}_{t=1}^{\infty}$ such that

$$U\left(\widehat{c}_1^0\right) \geq U\left(c_1^0\right) \text{ and } U\left(\widehat{c}_t^t, \widehat{c}_{t+1}^t\right) \geq U\left(c_t^t, c_{t+1}^t\right)$$

with strict inequality for at least one $t \geq 1$.

4.1 On Efficiency when $\beta w_1 > w_2$

- If $\widehat{p}_1 \in \{p_1^1, p_1^A\}$, we are in a steady state. We know the non-autarkic steady state pareto dominates the autarkic steady state for all generations born in $t \geq 1$ since a household can always choose autarky. To see this, just look at the indifference curves in Figure L3.3. Further, the initial old (born at 0) are also better off at $p_1^1 < p_1^A$ since $\overline{M}/p_1^1 > \overline{M}/p_1^A$.

- If $\widehat{p}_1 \in (p_1^1, p_1^A)$, any $\widehat{p}_1 < \widetilde{p}_1 \in (p_1^1, p_1^A)$ leads to higher utility not only for the initial old since $\overline{M}/\widehat{p}_1 > \overline{M}/\widetilde{p}_1$, but also all subsequent generations since the real benefit to holding the asset is falling over time (alternatively, the slope of the price ratio ray from 0 to the offer curve is becoming flatter, which by tangency with the indifference curve implies utility for the $\widetilde{p}_1$ sequence is below that for the $\widehat{p}_1$ sequence evident in Figure L3.3). However, by the same reasoning, the p_1^1 equilibrium pareto dominates the $\widehat{p}_1$ equilibrium. That is, there is a continuum of pareto ranked equilibria with p_1^1 highest, $\widehat{p}_1$ (say p_1^0 in Figure L3.3) next highest, and p_1^A lowest.

- Note that along the nonstationary path, inequality in consumption is growing over time as consumption fans out across generations until it converges on the autarkic allocation which has the most variation (since there is no smoothing occurring).

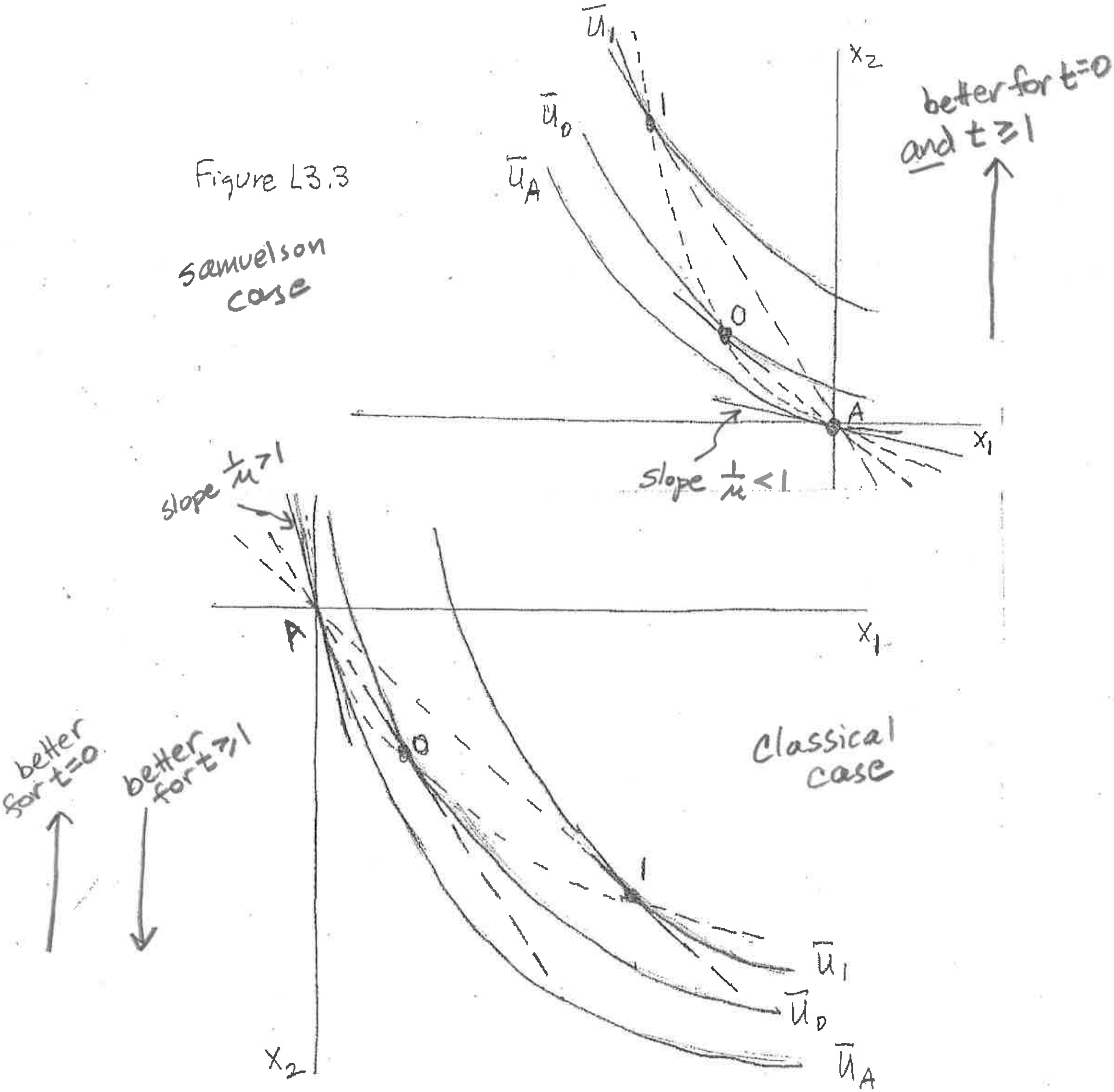
- In summary, $\widehat{p}_1 \in (p_1^1, p_1^A]$ is a competitive equilibrium which is not pareto optimal since it is dominated by a feasible allocation at p_1^1 which makes all generations better off. This appears to violate the First Welfare Theorem (i.e. competitive equilibria which are not pareto optimal). In Section 4.3 we will discuss why the first welfare theorem does not hold in this case.

4.2 On Efficiency when $\beta w_1 < w_2$

- If $\widehat{p}_1 \in \{p_1^1, p_1^A\}$, we are in a steady state. We know the non-autarkic steady state pareto dominates the autarkic steady state for all generations born in $t \geq 1$ since a household can always choose autarky. To see this, just look at the indifference curves in Figure L3.3. However, unlike the previous case, the initial old (born at 0) are not better off at $p_1^1 < p_1^A$ since with $\overline{M} < 0$, consumption is less than their income by $\left|\overline{M}/p_1^1\right| > \left|\overline{M}/p_1^A\right|$. Thus steady state 1 *does not* pareto dominate steady state A since while generations $t \geq 1$ are better off the initial old are worse off.

- If $\widehat{p}_1 \in (p_1^1, p_1^A)$, any $\widehat{p}_1 < \widetilde{p}_1 \in (p_1^1, p_1^A)$ leads to higher utility for all subsequent generations since the real cost of borrowing is falling over time (alternatively, the slope of the price ratio ray from 0 to the offer curve is becoming flatter, which by tangency with the indifference curve implies utility is rising). However, the initial old are not better off at $\widehat{p}_1 < \widetilde{p}_1$ since with $\overline{M} < 0$, consumption is less than their income by $\left|\overline{M}/\widehat{p}_1\right| > \left|\overline{M}/\widetilde{p}_1\right|$. In this case, there are a continuum of pareto non-comparable equilibria

Figure L3.3


Samuelson case
$\bar{u}_1$
$\bar{u}_0$
$\bar{u}_A$
x_2
x_1
1
0
A
slope $\frac{1}{\mu} < 1$
better for t=0 and t ≥ 1
slope $\frac{1}{\mu} > 1$
A
x_1
0
1
Classical case
$\bar{u}_1$
$\bar{u}_0$
$\bar{u}_A$
x_2
better for t=0
better for t ≥ 1

since generations $t \geq 1$ are made better off with lower prices but generation 0 is made worse off.

- In summary, any $\widehat{p}_1 \in [p_1^1, p_1^A]$ is consistent with a competitive equilibrium which is pareto optimal. This seems consistent with the First Welfare Theorem.

- Thus, the Samuelson case where the autarkic real rate of return on the asset is negative, corresponds to dynamic inefficiency while the classical case where the autarkic real rate of return on the asset is positive corresponds to dynamically efficient economies.

- Our interest in normative economics provides a way to think about political economy issues. For instance, in the classical case where $\beta w_1 < w_2$, the initial old and subsequent generations have a welfare conflict over intial prices. A planner who cares about the old might try to coordinate population price beliefs on high prices while a planner who cares about future generations might try to coordinate population price beliefs on low prices.

4.3 On the First Welfare Theorem

The following proposition is typically associated with the first welfare theorem.

Proposition (First Welfare Theorem). Assume that $\{p_t\}_{t=1}^T$ and $\{c_t^t, c_t^{t-1}\}_{t=1}^T$ is a Competitive Equilibrium. Then $\{c_t^t, c_t^{t-1}\}_{t=1}^T$ is Pareto Optimal.

- For the case where $T = \infty$, $\beta w_1 > w_2$, and $M \geq 0$ we saw in Subsection 4.1 that there are many competitive equilibria which are not Pareto Optimal. This implies that the first welfare theorem does not hold for this particular environment. Where does it break down?

- Standard steps in the proof (by contradiction):

  - Assume that $\{c_t^t, c_t^{t-1}\}_{t=1}^T$ is not Pareto Optimal and let $\{\widehat{c}_t^t, \widehat{c}_t^{t-1}\}_{t=1}^T$ be a resource feasible allocation that dominates it.

  - Then because preferences are strictly increasing in consumption (i.e. local nonsatiation), $\{\widehat{c}_t^t, \widehat{c}_t^{t-1}\}_{t=1}^T$ must not be budget feasible (i.e. not affordable for at least one household). That is,

    $$p_1\widehat{c}_1^0 \geq p_1 w_2 + \overline{M} \text{ and } p_t\widehat{c}_t^t + p_{t+1}\widehat{c}_{t+1}^t \geq p_t w_1 + p_{t+1} w_2, \forall t = 1, 2, ...T. \tag{27}$$

    with at least one strict inequality.

  - The next step in the proof would be to sum (27) for all t (typically in static models this is a sum over all individuals, but because here we have one representative household in each generation, it is over generations (indexed by t)). This sum is given by

$$
\begin{aligned}
p_1\widehat{c}_1^0 + \sum_{t=1}^{T} p_t\widehat{c}_t^t + p_{t+1}\widehat{c}_{t+1}^t &> p_1 w_2 + \overline{M} + \sum_{t=1}^{T} p_t w_1 + p_{t+1} w_2 \Longleftrightarrow \\
\sum_{t=1}^{T} p_t\left(\widehat{c}_t^t + \widehat{c}_t^{t-1}\right) &> \sum_{t=1}^{T} p_t\left(w_1 + w_2\right) + \overline{M} + p_{T+1}\left(w_2 - \widehat{c}_{T+1}^T\right) \\
&\geq \sum_{t=1}^{T} p_t\left(w_1 + w_2\right)
\end{aligned} \tag{28}
$$

if $\overline{M} + p_{T+1}\left(w_2 - \widehat{c}_{T+1}^T\right) \geq 0$ (which we will assume).

- But feasibility of the alternative allocation requires

$$
\widehat{c}_t^t + \widehat{c}_t^{t-1} \leq w_1 + w_2, \forall t = 1, 2, ...T. \tag{29}
$$

- With a finite number of periods $T < \infty$, the one strict inequality in (27) generates the strict inequality in (28), which would yield the desired contradiction with a finite number of summations in (29).
- The problem with establishing the contradiction is that with an infinite number of periods, there is no inconsistency between (28) and an infinite summation in (29) since it is possible that both sides of the inequalities are infinity (i.e. undefined). Heuristically, infinity is not greater than infinity.

• The problem is one of "infinity". Balasko and Shell (1980) provide a sufficient condition for Pareto Optimality of competitive equilibrium is given by[14]

$$
\sum_{t=1}^{\infty} \frac{1}{p_t} = \infty \tag{30}
$$

since prices must be strictly positive. This ensures that the value of the endowment on the right hand side of (28) is finite. To see this, consider a steady state where $p_{t+1} = \mu^t p_1$. Then

$$
\sum_{t=1}^{\infty} \mu^{t-1} p_1\left(w_1 + w_2\right) = p_1\left(w_1 + w_2\right) \cdot \sum_{t=1}^{\infty} \mu^{t-1}.
$$

If $\mu < 1$, then $\sum_{t=1}^{\infty} \mu^{t-1} = 1/(1-\mu)$. Otherwise, it is infinity.

• This reconciles why the classical case where $\beta w_1 < w_2$ which yields $\mu < 1$ works. Recall, those are the cases where the competitive equilibrium is pareto optimal. The standard proof for this case, however, needs to be modified since $\overline{M} < 0$ so that the second inequality in (28) needn't hold.

[14] Balasko, Y. and Shell, K., (1980) "The Overlapping-Generations Model I: The Case of Pure Exchange without Money", *Journal of Economic Theory*, 23, 281-306.

- In general however, finiteness is not necessary for pareto optimality.

- To see the relation between planner's problems and pareto optimality, the next result is helpful. We will prove it simply for the "steady state" case where there are current old and current young in any given period t.

Proposition (PO). An allocation $\{\bar{c}_t^t, \bar{c}_t^{t-1}\}$ which satisfies

$$\begin{aligned} \{\bar{c}_t^t, \bar{c}_t^{t-1}\} &= \arg\max \alpha_1 u(c_t^t) + \alpha_2 u(c_t^{t-1}) \\ s.t.\ c_t^t + c_t^{t-1} &= w_1 + w_2 \end{aligned} \tag{PP}$$

with $\{\alpha_1, \alpha_2\} \in [0,1]^2$ such that $\alpha_1 + \alpha_2 = 1$ is Pareto Optimal.

Proof. Assume that $\{\bar{c}_t^t, \bar{c}_t^{t-1}\}$ solves (PP) and by way of contradiction that $\{\bar{c}_t^t, \bar{c}_t^{t-1}\}$ is not Pareto Optimal. Then another allocation exists that raises everybody's utility, strictly for one agent.[15] We have then raised $\alpha_1 u(c_t^t) + \alpha_2 u(c_t^{t-1})$ strictly which contradicts the fact that $\{\bar{c}_t^t, \bar{c}_t^{t-1}\}$ solves (PP).

- Planner's problems in overlapping generations models can be tricky since the objective function may not be defined for all weighting functions. In particular, if the planner weighs all generations equally, then the objective is

$$U(c_1^0) + \sum_{t=1}^{\infty} U\left(c_t^t, c_{t+1}^t\right).$$

  Since this objective may not be well defined (i.e. add up to infinity), we can apply the "overtaking" criterion to determine optimality.

- The "overtaking criterion" states that an allocation $\{c_t^{t-1}, c_t^t\}_{t=1}^T$ overtakes $\{\widehat{c}_t^{t-1}, \widehat{c}_t^t\}_{t=1}^T$ if

$$\lim \inf_{T\to\infty} \left[U(c_1^0) + \sum_{t=1}^{T} U(c_t^t, c_{t+1}^t) - U(\widehat{c}_1^0) + \sum_{t=1}^{T} U(\widehat{c}_t^t, \widehat{c}_{t+1}^t) \right] > 0. \tag{31}$$

  Since the finite sum is well defined, this sequence is well defined. [16] Intuitively, the lhs of (31) is like a derivative yielding a first order condition in the limit.

4.4 Equilibrium Selection

- Since there are multiple equilibria, including pareto ranked equilibria depending on the initial price level $\widehat{p}_1$in the case where $\beta w_1 > w_2$, how do we "select" equilibrium prices?

- Can a government coordinate beliefs on p_1^1 to select the pareto optimal allocation when $\beta w_1 > w_2$?

[15] If that agent has zero weight, then redirect resources to a consumer with positive weight.

[16] For an application of the overtaking criterion see Green and Zhou (2002) "Dynamic Monetary Equilibrium in a Random Matching Economy", *Econometrica*, 70, p.929-969.

- If there is population growth, would a government coordinate on $p_1^1$when $\beta w_1 < w_2$? That is, if there are more young who benefit from point 1 than old would benefit from $\widehat{p}_1 > p_1^1$, would they constitute the majority vote?

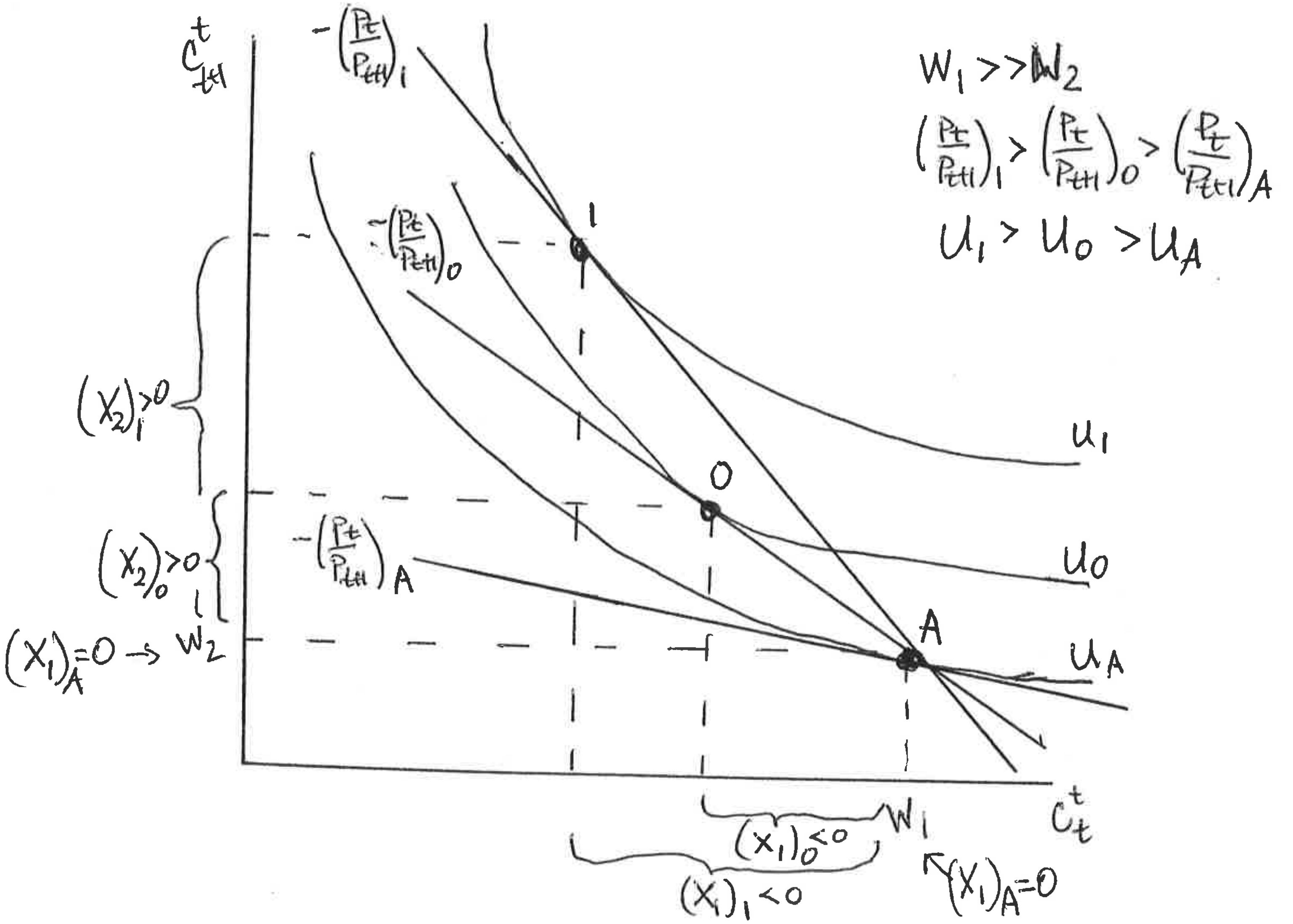
5 Summary

For the given log preferences and endowments we have specified, in the

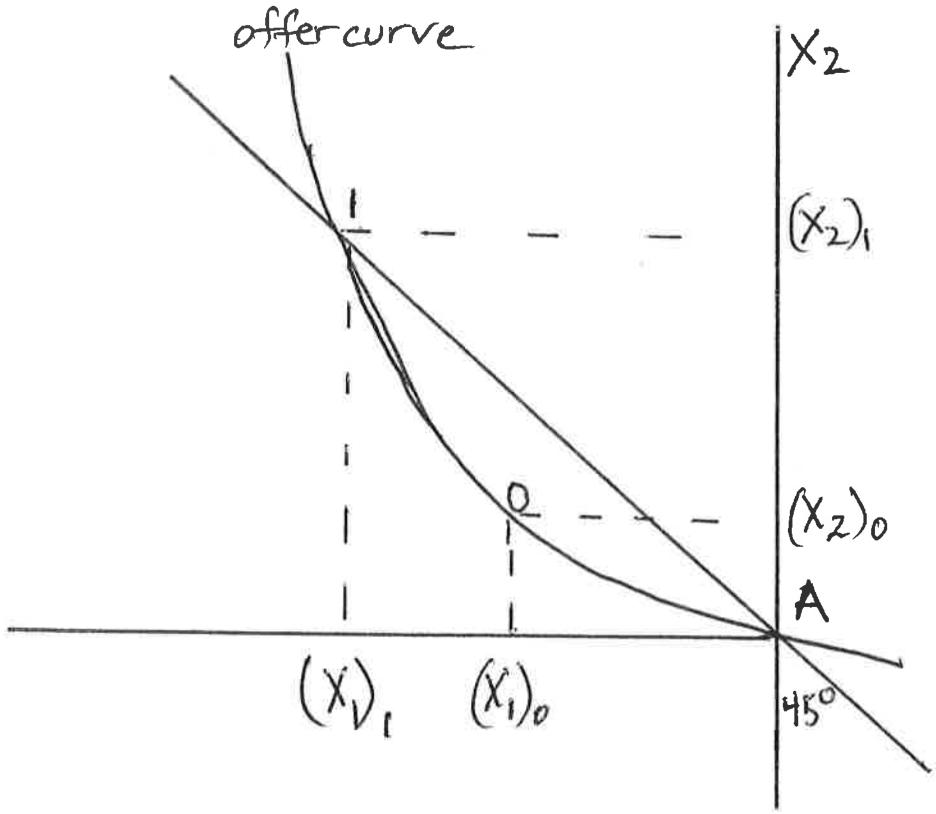
- Samuelson case where $\beta w_1 > w_2$ (so agents want to save):
  - Autarkic steady state with $p_1 = p_A$ such that $\overline{M}/p_A = 0$ and $p_t/p_{t+1} \equiv 1/\mu_A < 1$ (i.e. net real return is negative or $\mu_A > 1$ in order to get people to not want to save).[17]
  - Monetary Steady state with $p_1 = \overline{p}$ such that $x_1 = -\overline{M}/\overline{p} < 0$ where $p_t/p_{t+1} = 1$ (i.e. net real return is zero).[18]
  - Non-stationary equilibria with $p_1 \in (\overline{p}, p_A)$ where $p_t/p_{t+1} \equiv 1/\mu_t < 1$ where $\mu_t > \mu_s > 1$ for $t > s$ (i.e net real return is negative and falling over time (μ_t rising to μ_A) due to "inflation" since $p_{t+1} = \mu_t p_t$ with $\mu_t > 1$).
  - There is a continuum of pareto ranked competitive equilibria with the monetary steady state ranked highest (i.e. there are an infinite number of competitive equilibria which are pareto dominated by the monetary steady state).
  - Can the government implement the monetary steady state by coordinating beliefs such that $p_1 = \overline{p}$?

- Classical case where $\beta w_1 < w_2$ (so agents want to borrow):
  - Autarkic steady state with $p_1 = p_A$ such that $\overline{M}/p_A = 0$ and $p_t/p_{t+1} \equiv 1/\mu_A > 1$ (i.e. net real return is positive or $\mu_A < 1$ in order to get people to not want to borrow).
  - Debt steady state with $p_1 = \overline{p}$ such that $x_1 = -\overline{M}/\overline{p} > 0$ where $p_t/p_{t+1} = 1$ (i.e. net real return is zero).
  - Non-stationary equilibria with $p_1 \in (\overline{p}, p_A)$ where $p_t/p_{t+1} \equiv 1/\mu_t > 1$ where $\mu_s < \mu_t < 1$ for $t > s$ (i.e net real return is positive and falling over time (μ_t rising to 1) since $p_{t+1} = \mu_t p_t$ with $\mu_t < 1$).
  - There are an infinite number of competitive equilibria which are pareto non-comparable since the initial old are worse off the lower is p_1 but later generations are better off.

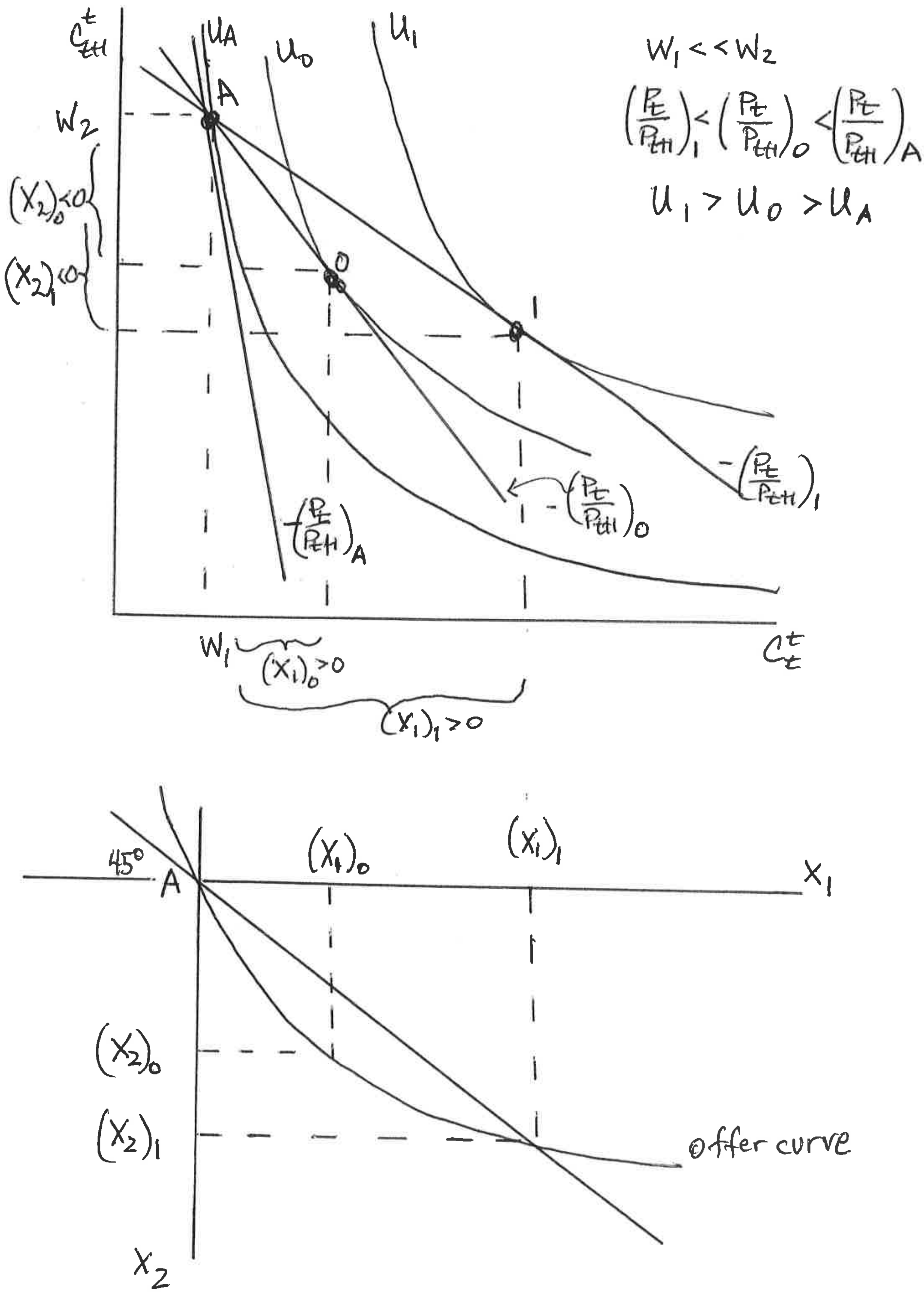
[17]Note that to formalize $\overline{M}/p_A = 0$, we must consider a sequence of finite but large p_A approaching infinity.
[18]Note that if $\beta < 1$, then even in the monetary steady state $c_t^t > c_{t+1}^t$ for all $t \geq 1$.



$W_1 >> W_2$
$\left(\frac{P_t}{P_{t+1}}\right)_1 > \left(\frac{P_t}{P_{t+1}}\right)_0 > \left(\frac{P_t}{P_{t+1}}\right)_A$
$U_1 > U_0 > U_A$
C_{t+1}^t
$-\left(\frac{P_t}{P_{t+1}}\right)_1$
$-\left(\frac{P_t}{P_{t+1}}\right)_0$
$-\left(\frac{P_t}{P_{t+1}}\right)_A$
1
0
A
U_1
U_0
U_A
$(X_2)_1 > 0$
$(X_2)_0 > 0$
$(X_1)_A = 0 \rightarrow W_2$
W_1
C_t^t
$(X_1)_0 < 0$
$(X_1)_1 < 0$
$(X_1)_A = 0$



offer curve
X_2
1
0
A
$(X_2)_1$
$(X_2)_0$
$(X_1)_1$
$(X_1)_0$
45^0

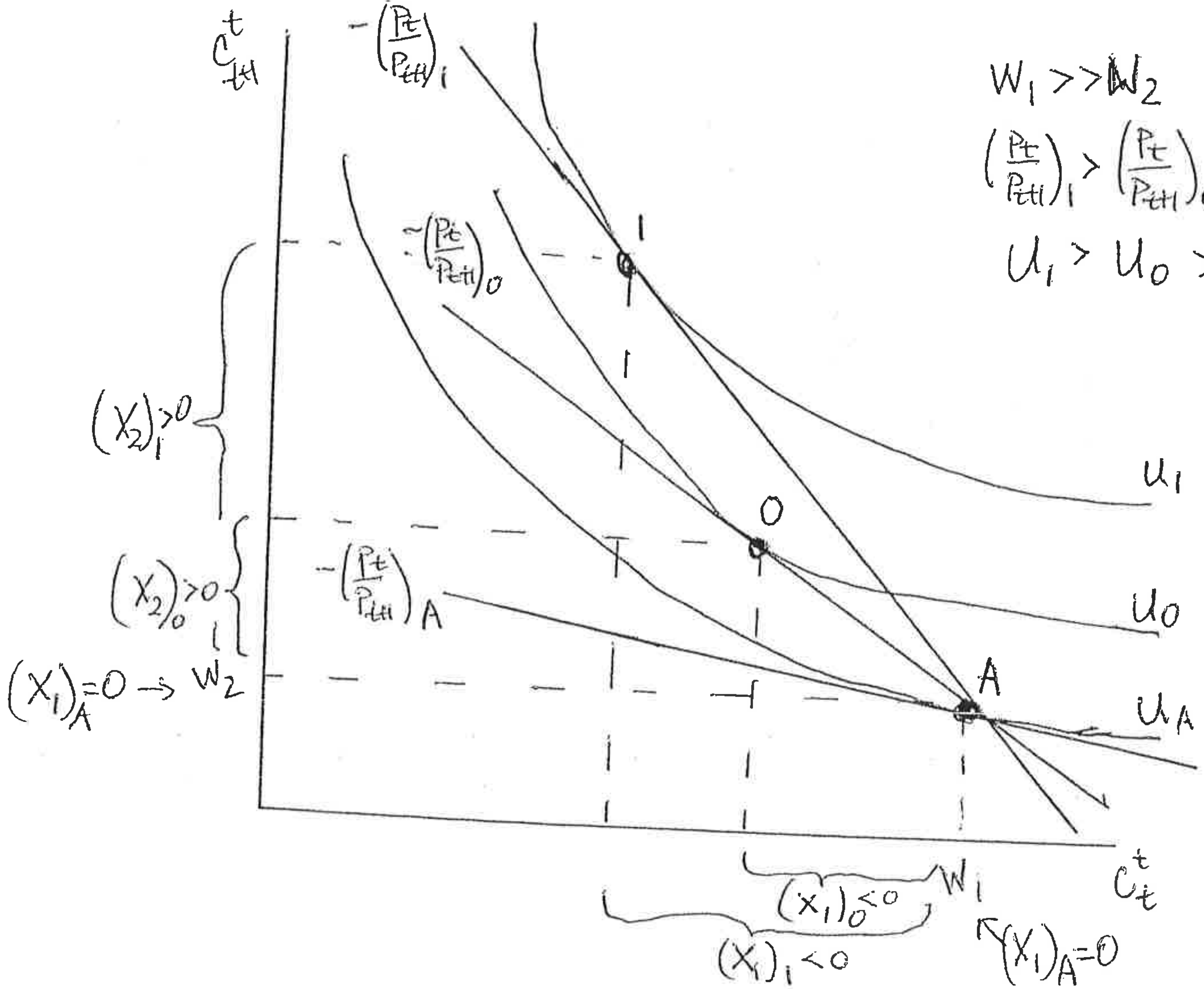


C_{t+1}^t
U_A
U_0
U_1
$W_1 << W_2$
$\left(\frac{P_t}{P_{t+1}}\right)_1 < \left(\frac{P_t}{P_{t+1}}\right)_0 < \left(\frac{P_t}{P_{t+1}}\right)_A$
$U_1 > U_0 > U_A$
A
W_2
$(X_2)_0 < 0$
$(X_2)_1 < 0$
O
$-\left(\frac{P_t}{P_{t+1}}\right)_1$
$-\left(\frac{P_t}{P_{t+1}}\right)_0$
$-\left(\frac{P_t}{P_{t+1}}\right)_A$
W_1
$(X_1)_0 > 0$
$(X_1)_1 > 0$
C_t^t
45°
A
$(X_1)_0$
$(X_1)_1$
X_1
$(X_2)_0$
$(X_2)_1$
offer curve
X_2

$W_1 >> W_2$

$\left(\frac{P_t}{P_{t+1}}\right)_1 > \left(\frac{P_t}{P_{t+1}}\right)_0 > \left(\frac{P_t}{P_{t+1}}\right)_A$

$U_1 > U_0 > U_A$

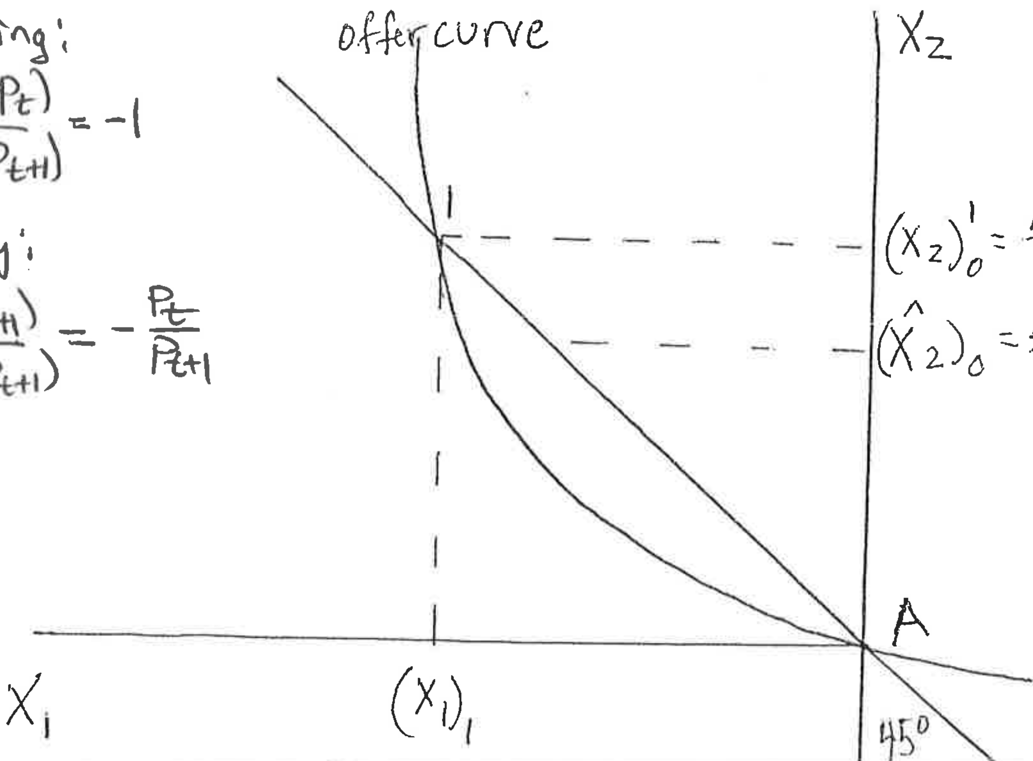


mkt clearing:

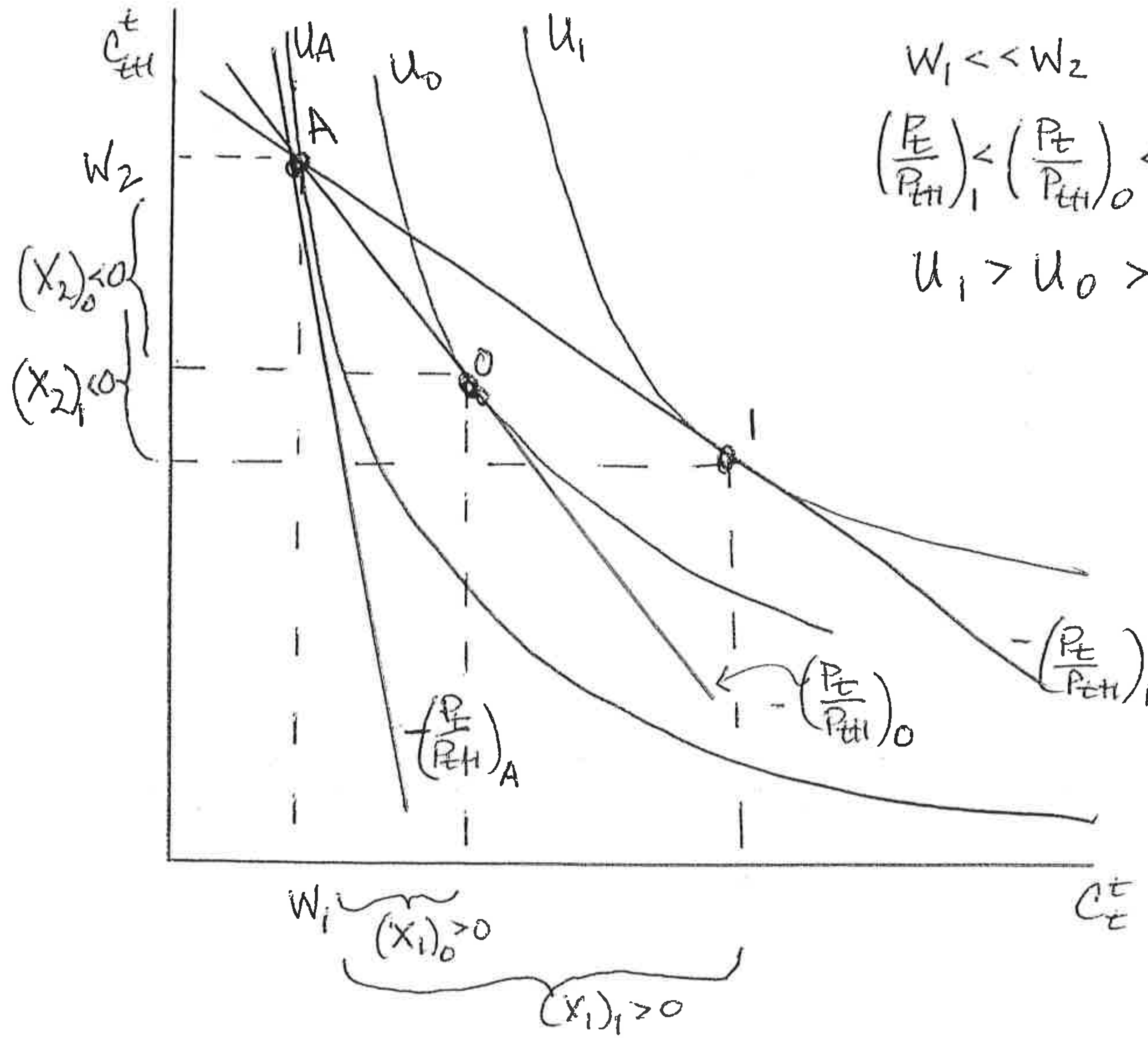
$$\frac{X_2(P_{t-1}, P_t)}{X_1(P_t, P_{t+1})} = -1$$

optimality:

$$\frac{X_2(P_t, P_{t+1})}{X_1(P_t, P_{t+1})} = -\frac{P_t}{P_{t+1}}$$



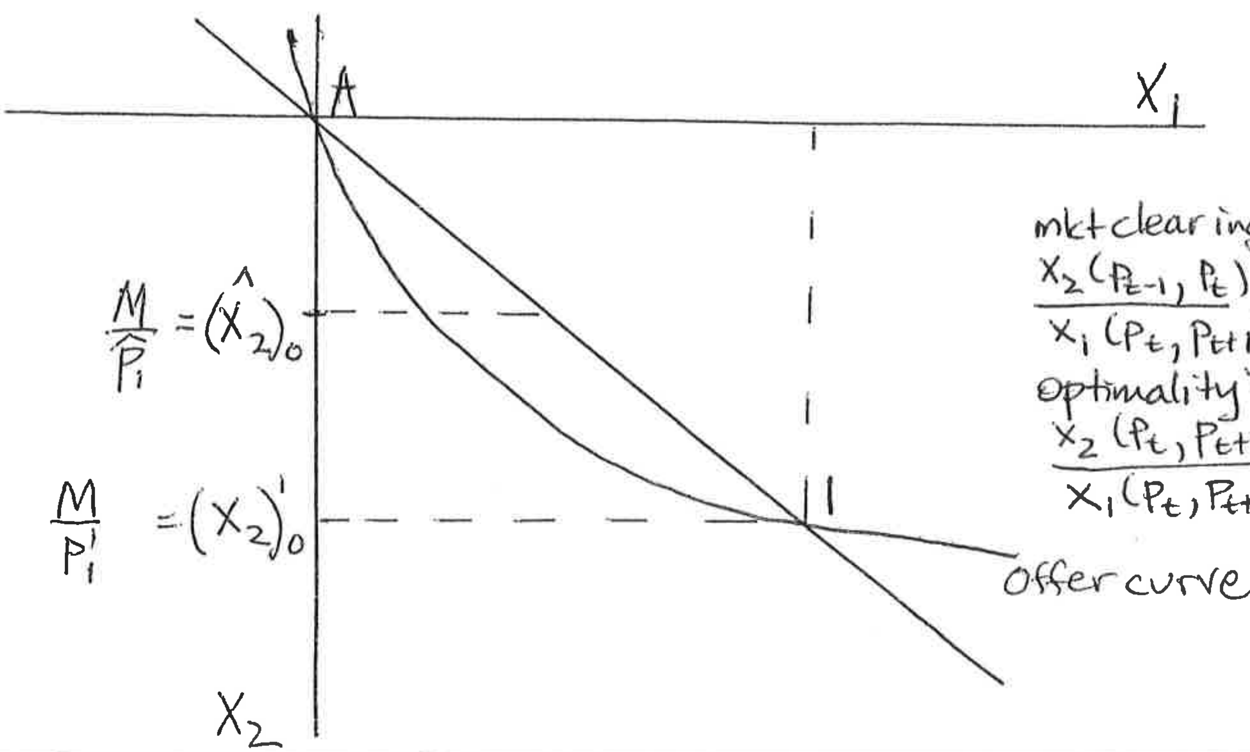
$\hat{P}_1 > P^1_1$



$W_1 << W_2$

$$\left(\frac{P_t}{P_{t+1}}\right)_1 < \left(\frac{P_t}{P_{t+1}}\right)_0 < \left(\frac{P_t}{P_{t+1}}\right)_A$$

$$U_1 > U_0 > U_A$$



$\hat{P}_1 > P'_1$

mkt clearing:

$$\frac{X_2(P_{t-1}, P_t)}{X_1(P_t, P_{t+1})} = -1$$

optimality:

$$\frac{X_2(P_t, P_{t+1})}{X_1(P_t, P_{t+1})} = -\frac{P_t}{P_{t+1}}$$

Econ 712, Corbae
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L4. Government Policy in an Endowment Economy

These notes focus on government policy with commitment. Besides the welfare improving role for government when there are missing markets, it focuses on conditions for the irrelevance of financing policies. Ricardian Equivalence states conditions under which the timing of taxation (to finance a given sequence of government expenditure) does not matter in a frictionless world (i.e. where there are nondistortionary taxes and perfect financial markets). Some people call this the Modigliani-Miller theorem for government taxation.[1]

1 Competitive Equilibrium with Social Security

- Population: Each period $t = 1, 2, 3, ...$ a new generation of 2 period lived households are born. The measure of identical households born in any period grows by $1 + n$. That is, we assume population growth of rate n.
- Technology:
  - Each generation is endowed with w_1 in youth and w_2 in old age of nonstorable consumption goods. Note that endowments don't depend explicitly on time.
  - There is a commitment technology which enables feasible trades to take place.
- Preferences: The strictly increasing, concave and differentiable utility function of a household of generation $t \geq 1$ is

$$U\left(c_t^t, c_{t+1}^t\right) = u(c_t^t) + \beta u(c_{t+1}^t)$$

  where $\left(c_t^t, c_{t+1}^t\right)$ is consumption of generation t in youth (i.e. in period t) and old age (i.e in period $t+1$). Note that $U(\cdot, \cdot)$ does not depend explicitly on time. The preferences of the initial old are given by $U(c_1^0)$. To avoid a binding non-negativity constraint on consumption, assume that $\lim_{c \to 0} u'(c) = \infty$ (known as an Inada condition).
- We assume a sequence of competitive goods markets. Let p_t be the price of period t goods in terms of period 1 goods (so that the normalization is $p_1 = 1$) in that case.

[1] The Modiagliani-Miller theorem in corporate finance stated conditions under which it doesn't matter how (through debt or equity) a firm finances its investment.

- Goods market clearing (or resource feasibility) now requires

$$c_t^{t-1} + (1+n)c_t^t = w_2 + (1+n)w_1, \forall t \geq 1. \tag{1}$$

- Consider the following social security system. The young pay lump sum social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old.[2] The government can provide consumption smoothing rather than an asset market.

- The government budget constraint requires

$$b \leq \tau(1+n). \tag{2}$$

- The household optimization problem is now

$$\max_{c_t^t, c_{t+1}^t} U(c_t^t, c_{t+1}^t) \tag{3}$$

$$s.t.\ p_t c_t^t + p_{t+1}c_{t+1}^t = p_t(w_1 - \tau) + p_{t+1}(w_2 + b) \tag{4}$$

- The first order conditions imply

$$\frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = \frac{p_t}{p_{t+1}} \tag{5}$$

  as well as the budget constraint (4).

- **Definition.** A *Competitive Equilibrium with Social Security* is an allocation $\{c_t^{t-1}, c_t^t\}_{t=1}^{\infty}$, prices $\{p_t\}_{t=1}^{\infty}$, and policy $\{\tau, b\}$ such that:
  - Taking prices and policy as given, households optimize by solving (3)-(4),
  - The goods markets clears (i.e. (1) is satisfied),
  - The government budget constraint (2) is satisfied.

- Evaluating (5) at the equilibrium after-tax/transfer autarchy point is given by

$$\frac{u'(w_1 - \tau)}{\beta u'(w_2 + \tau(1+n))} = \frac{p_t}{p_{t+1}} \equiv 1 + r_{t,t+1} = 1 + r \tag{6}$$

  which makes clear that the real interest rate is independent of time since the left hand side is independent of time.

- Can the introduction of a pay-as-you-go system lead to a pareto improvement over autarky?

[2] Note that because preferences do not depend on leisure, even with distortionary taxes househoulds would supply their unit of time to labor and in fact proportional taxes are equivalent to these lump sum ones. Your problem set will consider preferences which do depend on leisure.

- For any $\tau \in (0, w_1)$, the initial old are strictly better off since they receive $\tau(1+n)$.
- For all other generations born in $t \geq 1$, their equilibrium lifetime utility is given by

$$V(\tau) = u(w_1 - \tau) + \beta u(w_2 + \tau(1+n))$$

- Those generations are better off with a marginal rise in taxes if $V'(\tau)|_{\tau=0} > 0$ or[3]

$$\begin{aligned} V'(\tau)|_{\tau=0} &= -u'(w_1) + \beta u'(w_2)(1+n) > 0 \Longleftrightarrow \\ (1+n) &> \frac{u'(w_1)}{\beta u'(w_2)} = 1 + r^A \qquad (7) \end{aligned}$$

- Thus, social security can lead to a pareto improvement over autarky when the social intertemporal rate of transformation $(1+n)$ exceeds the private intertemporal rate of transformation (equal to the marginal rate of substitution) given by $(1+r^A)$.
- Inequality (7) is the population growth analogue to the Samuelson case (described in the previous notes where the autarkic net interest rate is negative) where equilibria are dynamically inefficient so that social security can improve on allocations (recall the offer curve diagram for $\beta w_1 > w_2$ where $(1+r^A) < 1$).
- The optimal size of $\tau^* \leq w_1$ satisfies

$$(1+n) = \frac{u'(w_1 - \tau^*)}{\beta u'(w_2 + \tau^*(1+n))}$$

  or in the case of log preferences

$$\begin{aligned} \beta(1+n)(w_1 - \tau^*) &= (w_2 + \tau^*(1+n)) \Longleftrightarrow \\ \tau^* &= \frac{\beta w_1 - w_2/(1+n)}{(1+\beta)} \end{aligned}$$

  which should look familiar.

- It is important to note that these social security taxes and transfers are of the lump sum variety. In the real world, the taxes are proportional to one's earnings. In the problem set associated with Conesa and Krueger (1998), we introduce proportional taxes which can distort the labor supply decision. In that case, there is a tradeoff between efficiency losses when young and consumption smoothing when old.

[3] Of course it is possible that if the government over taxes (e.g. $\tau = w_1 - \varepsilon$ where ε is arbitrarily close to zero), then social security can be dominated by autarky for $t \geq 1$ generations.

2 Ricardian Equivalence

- How should a government finance a given stream of expenditures (like wars)?
  - Tax current generations
  - Issue debt now and tax future generations
- In "Essay on the Funding System" (1820) Ricardo said it shouldn't matter for the population's welfare. The idea is similar to a result in corporate finance called the "Modigliani-Miller" theorem which states that how a corporation finances physical investment (through debt or equity) doesn't matter.
- To see the issues, consider the government's budget constraint in an economy with overlapping generations with no population growth. In particular, assume the government must finance a given sequence of government spending $\{g_t\}_{t=1}^{\infty}$ through taxes on the young and old in any generation $\{\tau_t^t, \tau_t^{t-1}\}_{t=1}^{\infty}$ to satisfy the **sequence of constraints**

$$g_t + B_t = \Upsilon_t + q_t B_{t+1}, \forall t \geq 1 \tag{8}$$

  where $\Upsilon_t = \left(\tau_t^t + \tau_t^{t-1}\right)$ are taxes paid by young and old, $B_t > 0$ is one period "real" government debt (discount bonds) and we assume the initial condition $B_1 > 0$. Each discount bond costs q_t goods at time t and pays off 1 good at $t+1$.That is, (8) is the GBC in "real" terms.
- There really is "real" government debt. In the U.S. it is called TIPS (Treasury Inflation Protected Securities). In this case $q_t \equiv \#goods\_t/1$ *Tip bond.*
- Note that this sequence of budget constraints in (8) can be written as a consolidated government budget constraint through recursive substitution. To see this, writing out the first 3 terms in (8)

$$\begin{aligned} g_1 + B_1 &= \Upsilon_1 + q_1 B_2 \\ B_2 &= \Upsilon_2 + q_2 B_3 - g_2 \\ B_3 &= \Upsilon_3 + q_3 B_4 - g_3 \end{aligned}$$

  Then substituting B_3 into the $t = 2$ constraint and B_2 into the $t = 1$ constraint yields

$$\begin{aligned} g_1 + B_1 &= \Upsilon_1 + q_1 \left[\Upsilon_2 + q_2 \left\{\Upsilon_3 + q_3 B_4 - g_3\right\} - g_2\right] \Longleftrightarrow \\ g_1 + q_1 g_2 + q_1 q_2 g_3 &= \Upsilon_1 + q_1 \Upsilon_2 + q_1 q_2 \Upsilon_3 + q_1 q_2 q_3 B_4 - B_1 \end{aligned}$$

  so that recursively substituting for all future periods we have

$$\sum_{t=1}^{T} \left(\prod_{s=1}^{t-1} q_s\right) g_t = \sum_{t=1}^{T} \left(\prod_{s=1}^{t-1} q_s\right) \Upsilon_t - B_1 + \left(\prod_{s=1}^{T} q_s\right) B_{T+1} \tag{9}$$

where we take $\left(\prod_{s=1}^{0} q_s\right) = 1$.

- If $q_t < 1$ (which is consistent with the fact that households discount the future ($\beta < 1$)), then when $T \to \infty$, we can re-write (9) as the **consolidated government budget constraint**

$$\sum_{t=1}^{\infty}\left(\prod_{s=1}^{t-1} q_s\right) g_t = \sum_{t=1}^{\infty}\left(\prod_{s=1}^{t-1} q_s\right) \Upsilon_t - B_1 \tag{10}$$

since

$$\lim_{T\to\infty}\left(\prod_{s=1}^{T} q_s\right) B_{T+1} = 0 \tag{11}$$

provided B_∞ is finite (or growing at a sufficiently slow rate relative to q_t). Condition (11) is called a "No Ponzi Games" condition which rules out excessive future borrowing to fund current borrowing. Note that (11) does not require that $q_t < 1$ for all t (i.e. that real interest rates are always positive); any finite number of negative real rates can be dominated by an infinite number of positive real interest rates periods and still satisfy condition (11).

- From the standpoint of the government's consolidated budget constraint, for a given $\{g_t\}_{t=1}^{\infty}$ and B_1, then any two arbitrary sequences of taxes, say $\{\Upsilon_t\}_{t=1}^{\infty}$ and $\{\widehat{\Upsilon}_t\}_{t=1}^{\infty}$ which satisfies

$$\sum_{t=1}^{\infty}\left(\prod_{s=1}^{t-1} q_s\right) \Upsilon_t = \sum_{t=1}^{\infty}\left(\prod_{s=1}^{t-1} q_s\right) \widehat{\Upsilon}_t,$$

then the timing of taxes makes no difference to the consolidated government budget constraint (10). That is one important part of the proof of the Ricardian Equivalence Theorem. Note that it is important that the alternative tax sequence $\{\widehat{\Upsilon}_t\}_{t=1}^{\infty}$ is evaluated at the "original" price sequence $\{q_t\}_{t=1}^{\infty}$.

- However, in an overlapping generations model with the above preferences, the timing of taxes obviously matters.

  - To see why it is sufficient to consider the effect on an old agent. Their consumption is given by

$$c_1^0 = \left(w_2 - \tau_1^0\right) + B_1.$$

  Hence if instead of $\tau_1^0 > 0$, the initial old do not incur a tax (i.e. $\widehat{\tau}_1^0 = 0$) while the initial young pay that amount next period (i.e. $\widehat{\tau}_2^1 = \tau_1^0/q_1 + \tau_2^1$, there is no difference in government revenue but initial old don't have to pay (at the expense of the old of generation 1).[4]

- Barro (1974) suggested that if, unlike the model above, agents have a bequest motive, the timing of taxes may not matter.
- If there is a bequest motive, the utility function of a household of generation $t \geq 1$ is

$$U_t(c_t^t, c_{t+1}^t, b_{t+2}^t) = u(c_t^t) + \beta u(c_{t+1}^t) + \alpha V_{t+1}(b_{t+2}^t)$$

with $(\alpha, \beta) \in [0,1)^2$. Here V_{t+1} is the maximal utility that a generation $t+1$ household (my child) can achieve given

$$b_{t+2}^t \geq 0 \tag{12}$$

is a bequest from generation t made in old age to its children born at $t+1$ which enters their $t+2$ wealth.[5] Note by definition, a bequest is non-negative.

  - In this two period OG framework, note that children are born to mature (i.e. old) parents.
  - We assume that V is strictly increasing, concave, and differentiable.
  - Note that now U_t in principle depends on time as long as $\alpha > 0$ and these preferences "nest" the previous ones if $\alpha = 0$. As we will see (after applying a recursion) it need not.

- Similarly, the preferences of the initial old are given by

$$U_0(c_1^0, b_2^0) = \beta u(c_1^0) + \alpha V_1(b_2^0)$$

where c_1^0 and b_2^0 are choice variables.

- We will make three assumptions:
  - $w_2 < \min\{\alpha, \beta\} w_1$ (similar to Samuelson case)
  - $\tau_t^t < w_1, \tau_t^{t-1} < w_2$ (for feasibility)
  - $g_t = g$ (for stationarity)

[4] Specifically, writing out the first few terms of the consolidated government budget constraint we have

$$\tau_1^0 + \tau_1^1 + q_1 \left(\tau_2^1 + \tau_2^2\right) + ...$$

versus

$$0 + \tau_1^1 + q_1 \left(\frac{\tau_1^0}{q_1} + \tau_2^1 + \tau_2^2\right) + ...$$

which are identical.

[5] This implicitly assumes bequests are storable. If the transfer had taken place between the old and the young while the old was alive, we call that an "inter vivos" transfer.

- Now consider the budget constraints of agents. Let $a_{t+1}^t \in \mathbb{R}$ be the asset holdings of the young of generation t chosen in period t for use in period $t+1$ (notice that we do not require a_{t+1}^t to be non-negative). For simplicity we assume that private debt and public debt are perfect substitutes so that their price is the same.

  – In general we now have

$$\begin{aligned} c_t^t + q_t a_{t+1}^t &= w_1 - \tau_t^t & (13)\\ c_{t+1}^t + q_{t+1} b_{t+2}^t &= w_2 - \tau_{t+1}^t + a_{t+1}^t + b_{t+1}^{t-1} & (14) \end{aligned}$$

  with $b_{t+1}^t \geq 0$. Hence, when agents are old, they have two sources of funds: their own assets from the previous period and bequests from their dead parents (scary).

  – Since until now agents were selfish (i.e. $\alpha = 0$), we knew $b_{t+2}^t = 0$.

  – Since a_{t+1}^t is not required to be non-negative, substituting a_{t+1}^t from (14) into (13) yields the **consolidated lifetime budget constraint**

$$\begin{aligned} c_t^t + q_t\left[c_{t+1}^t + q_{t+1} b_{t+2}^t - \left(w_2 - \tau_{t+1}^t\right) - b_{t+1}^{t-1}\right] &= w_1 - \tau_t^t \Longleftrightarrow & (15)\\ c_t^t + q_t c_{t+1}^t + q_t q_{t+1} b_{t+2}^t &= w_1 - \tau_t^t + q_t (w_2 - \tau_{t+1}^t + b_{t+1}^{t-1}). \end{aligned}$$

  – Note that we can define total lifetime resources available to a generation t agent (the right hand side of (15)) by

$$e^t = w_1 - \tau_t^t + q_t(w_2 - \tau_{t+1}^t) + q_t b_{t+1}^{t-1} = W^t - \Psi^t + q_t b_{t+1}^{t-1} \quad (16)$$

  where lifetime (or permanent) income $W^t = w_1 + q_t w_2$ and lifetime taxes $\Psi^t = \tau_t^t + q_t \tau_{t+1}^t$.

  – The initial old budget constraint is given by

$$c_1^0 + q_1 b_2^0 = w_2 - \tau_1^0 + B_1 \quad (17)$$

- Goods and asset market clearing requires that $\forall t \geq 1$

$$\begin{aligned} c_t^{t-1} + c_t^t + g &= w_1 + w_2 & (18)\\ b_{t+1}^{t-1} + a_{t+1}^t &= B_{t+1} & (19) \end{aligned}$$

  where one can think of bequeathing T-bills to one's children. The rhs of the asset market clearing condition (19) is supply of government debt and the lhs is the demand for net debt.

- Suppose for the moment that the non-negativity constraints on bequests never bind. In that case we can neglect the constraint that $b_{t+2}^t \geq 0$. Taking (17) and the first several periods of (15) we have

$$\begin{aligned} q_1 b_2^0 &= w_2 - \tau_1^0 + B_1 - c_1^0 & (20)\\ q_1 b_2^0 &= c_1^1 + q_1 c_2^1 + q_1 q_2 b_3^1 - \left(w_1 - \tau_1^1\right) - q_1\left(w_2 - \tau_2^1\right) & (21)\\ q_2 b_3^1 &= c_2^2 + q_2 c_3^2 + q_2 q_3 b_4^2 - \left(w_1 - \tau_2^2\right) - q_2\left(w_2 - \tau_3^2\right) & (22) \end{aligned}$$

Substituting (22) into (21) into (20) we have

$$\begin{aligned} & w_2 - \tau_1^0 + B_1 - c_1^0 \\ = \quad & c_1^1 + q_1 c_2^1 + q_1 \left\{ c_2^2 + q_2 c_3^2 + q_2 q_3 b_4^2 - \left(w_1 - \tau_2^2\right) - q_2 \left(w_2 - \tau_3^2\right) \right\} - \left(w_1 - \tau_1^1\right) - q_1 \left(w_2 - \tau_2^1\right) \end{aligned}$$

or

$$\begin{aligned} c_1^0 + c_1^1 + q_1 \left(c_2^1 + c_2^2\right) + q_1 q_2 c_3^2 + q_1 q_2 q_3 b_4^2 \quad = & \\ & \left[(w_1 + w_2) - \left(\tau_1^1 + \tau_1^0\right)\right] + q_1 \left[(w_1 + w_2) - \left(\tau_2^2 + \tau_2^1\right)\right] \\ & + q_1 q_2 \left(w_2 - \tau_3^2\right) + B_1. \end{aligned}$$

Continuing in the same way that we did in (9) - that is, recursively substituting for all future periods - we have

$$\sum_{t=1}^{T} \left(\prod_{s=1}^{t-1} q_s \right) \left(c_t^{t-1} + c_t^t\right) = \sum_{t=1}^{T} \left(\prod_{s=1}^{t-1} q_s \right) (w_1 + w_2) - \sum_{t=1}^{T} \left(\prod_{s=1}^{t-1} q_s \right) \Upsilon_t + B_1 - \prod_{s=1}^{T} q_s \left(a_{T+1}^T + b_{T+1}^{T-1}\right) \tag{23}$$

where $\left(\prod_{s=1}^{0} q_s \right) = 1$.

- As in (10), if $q_t < 1$, then when $T \to \infty$, we can re-write (23) as the **consolidated dynastic household budget constraint**

$$\sum_{t=1}^{\infty} \left(\prod_{s=1}^{t-1} q_s \right) \left(c_t^{t-1} + c_t^t\right) = \sum_{t=1}^{\infty} \left(\prod_{s=1}^{t-1} q_s \right) (w_1 + w_2) - \sum_{t=1}^{\infty} \left(\prod_{s=1}^{t-1} q_s \right) \Upsilon_t + B_1 \tag{24}$$

provided the household debt holdings are finite.

- But then, as long as any two arbitrary sequences of taxes, say $\{\Upsilon_t\}_{t=1}^{\infty}$ and $\{\widehat{\Upsilon}_t\}_{t=1}^{\infty}$ which satisfies

$$\sum_{t=1}^{\infty} \left(\prod_{s=1}^{t-1} q_s \right) \Upsilon_t = \sum_{t=1}^{\infty} \left(\prod_{s=1}^{t-1} q_s \right) \widehat{\Upsilon}_t,$$

then the timing of taxes makes no difference to the consolidated household budget (24). Unlike the case without bequests, the timing of taxes doesn't matter for a given generation's budget constraint because if they do not get taxed now, they will leave a bequest to their child (who may leave it to his child...) to pay for the future tax. Again, it is important that the alternative tax sequence $\{\widehat{\Upsilon}_t\}_{t=1}^{\infty}$ is evaluated at the "original" price sequence $\{q_t\}_{t=1}^{\infty}$.

- This is all true provided that the constraint (12) (i.e $b_{t+2}^t \geq 0$) does not bind, since if it binds, we cannot arbitrarily consolidate the budget constraint.

- The initial old problem is given by

$$\begin{aligned} V_0(B_1) &= \max_{c_1^0 \geq 0, b_2^0 \geq 0} \beta u(c_1^0) + \alpha V_1(b_2^0) \qquad (25) \\ s.t.\ c_1^0 + q_1 b_2^0 &= w_2 - \tau_1^0 + B_1 \end{aligned}$$

- In general, given any bequest b, generation t's problem is given by

$$V_t(b) = \max_{c_t^t, c_{t+1}^t, b_{t+2}^t \geq 0} u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right) + \alpha V_{t+1}\left(b_{t+2}^t\right) \qquad (26)$$

$$s.t.\ c_t^t + q_t c_{t+1}^t + q_t q_{t+1} b_{t+2}^t = w_1 - \tau_t^t + q_t(w_2 - \tau_{t+1}^t + b) \qquad (27)$$

- Thus, for generation $t = 1$ we have

$$\begin{aligned} V_1(b) &= \max_{c_1^1, c_2^1, b_3^1 \geq 0} u\left(c_1^1\right) + \beta u\left(c_2^1\right) + \alpha V_2\left(b_3^1\right) \qquad (28) \\ s.t.\ c_1^1 + q_1 c_2^1 + q_1 q_2 b_3^1 &= w_1 - \tau_1^1 + q_1(w_2 - \tau_2^1) + q_1 b \end{aligned}$$

- (28) into (25) implies the initial old problem is given by

$$\begin{aligned} V_0(B_1) &= \max_{c_1^0 \geq 0, c_1^1, c_2^1, b_2^0 \geq 0, b_3^1 \geq 0} \beta u(c_1^0) + \alpha\left\{u\left(c_1^1\right) + \beta u\left(c_2^1\right) + \alpha V_2\left(b_3^1\right)\right\} \\ s.t.\ c_1^0 + q_1 b_2^0 &= w_2 - \tau_1^0 + B_1 \\ c_1^1 + q_1 c_2^1 + q_1 q_2 b_3^1 &= w_1 - \tau_1^1 + q_1(w_2 - \tau_2^1) + q_1 b_2^0 \end{aligned}$$

- Recursively substituting an infinite number of times using (26) we have

$$V_0(B_1) = \max_{\left\{c_t^{t-1}, c_t^t, b_{t+1}^{t-1}\right\}_{t=1}^{\infty} \geq 0} \beta u(c_1^0) + \sum_{t=1}^{\infty} \alpha^t \left\{u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right)\right\} \qquad (29)$$

$$s.t.\ c_1^0 + q_1 b_2^0 = w_2 - \tau_1^0 + B_1 \qquad (30)$$

$$c_t^t + q_t c_{t+1}^t + q_t q_{t+1} b_{t+2}^t = w_1 - \tau_t^t + q_t(w_2 - \tau_{t+1}^t + b_{t+1}^{t-1}), \forall t \geq 1 \qquad (31)$$

- Problem (29)-(31) is the "infinite sequence" formulation of the "recursive" formulation (25)-(27) of the overlapping generations model with a bequest motive. It means the overlapping generations model with bequests, under certain assumptions, is equivalent to an infinite horizon model with borrowing constraints (i.e. $\left\{b_{t+1}^{t-1} \geq 0\right\}_{t=1}^{\infty}$) with an interpretation that there are two types of "goods" in every period for the infinitely lived consumer.

- Stokey and Lucas Recursive Methods in Economic Dynamics establishes (in Theorems 4.2 and 4.3) the conditions under which the infinite sequence formulation (they call it SP for sequence problem) and the recursive formulation (they call it FE for functional equation) are equivalent. Note that while there is effectively an infinite number of first order conditions we must take to solve the infinite sequence problem while there are only 3 first order conditions to solve in the recursive problem, we have to solve for the function $V_t(b)$ (an infinite dimensional object) to solve the recursive (dynamic programming problem).

- Forming a Lagrangian for this problem we have

$$\begin{aligned}\pounds &= \beta u(c_1^0) + \sum_{t=1}^{\infty} \alpha^t \left\{u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right)\right\} + \lambda_0\left[w_2 - \tau_1^0 + B_1 - c_1^0 - q_1 b_2^0\right] + \mu_0 b_2^0 \\ &\quad + \sum_{t=1}^{\infty} \lambda_t \left\{w_1 - \tau_t^t + q_t(w_2 - \tau_{t+1}^t + b_{t+1}^{t-1}) - c_t^t - q_t c_{t+1}^t - q_t q_{t+1} b_{t+2}^t\right\} + \mu_t \left\{b_{t+2}^t\right\}\end{aligned}$$

  where λ_t denotes the multiplier on the budget constraints (30)-(31) and μ_t denotes the multiplier on the bequest (no borrowing) constraints $\left\{b_{t+1}^{t-1} \geq 0\right\}_{t=1}^{\infty}$ for a generation t household.

- The first order conditions are given by

$$\begin{aligned} c_1^0 &: \beta u'(c_1^0) = \lambda_0 & (32)\\ b_2^0 &: -q_1\lambda_0 + \mu_0 + \lambda_1 q_1 = 0 & (33)\\ c_t^t &: \alpha^t u'(c_t^t) = \lambda_t & (34)\\ c_{t+1}^t &: \alpha^t \beta u'(c_{t+1}^t) = q_t \lambda_t & (35)\\ b_{t+2}^t &: -\lambda_t q_t q_{t+1} + \mu_t + \lambda_{t+1} q_{t+1} = 0 & (36)\end{aligned}$$

  plus the budget constraints.

- Conditions (34) and (36) imply

$$\mu_t = \alpha^t u'(c_t^t) q_t q_{t+1} - \alpha^{t+1} u'(c_{t+1}^{t+1}) q_{t+1}$$

  in which case the "borrowing" constraint is not binding (i.e. $\mu_t = 0$) if

$$\begin{aligned} u'(c_t^t) q_t - \alpha u'(c_{t+1}^{t+1}) &= 0 \Longleftrightarrow \\ \frac{u'(c_t^t)}{\alpha u'(c_{t+1}^{t+1})} &= \frac{1}{q_t}. \end{aligned} \tag{37}$$

  But the lhs of (37) is just the dynastic marginal rate of substitution (between the young of any two generations t and $t+1$) and the right hand is the real rate of return on bequests (marginal rate of transformation of goods by the bond market).

- In a steady state, $c_t^t = c_{t+1}^{t+1}$ and $q_t = \overline{q}$ so that from (37), a necessary condition for a nonbinding constraint is

$$\frac{1}{\alpha} = \frac{1}{\overline{q}}$$

  or $1 = \frac{\alpha}{\overline{q}}$. Since this is a discount bond, $\overline{q} = 1/(1+r)$ so since $\alpha < 1$,the steady state real rate of return $r > 0$.[6]

[6] Note that (34) and (35) in a steady state where $c_t^t = c^y$ and $c_{t+1}^t = c^o$, their ratio

$$\frac{u'(c^y)}{\beta u'(c^o)} = \frac{1}{\overline{q}}.$$

Thus it is not necessary that $\beta = \alpha$. In fact, if $\beta \neq \alpha$, then $c^y \neq c^o$ in a steady state.

- Note further that if $\beta = 0$, then this model is isomorphic to a standard representative agent infinite horizon (RAIH) decision problem with discount factor α. To see this, the foc in (35) implies that the marginal benefit of increasing old age consumption c_{t+1}^t is given by $\alpha^t \beta u'(c_{t+1}^t)$ while the marginal cost is $q_t \lambda_t > 0$. When $\beta = 0$, there is no benefit to old age consumption and only a cost; hence the optimal choice is $c_{t+1}^t = 0$, in which case the lifetime budget constraint (24) and objective (29) look like their RAIH counterparts.

- In summary, we have constructed a competitive equilibrium with $q_t = \alpha$ where the timing of taxes does not affect the right hand side of the dynasty's consolidated budget constraint and since taxes do not explicitly enter preferences implies that the consumption allocation is unchanged with any changes in the timing of taxes.[7]

2.1 On the implications of imperfect financial markets

- The above analysis assumed that $a_{t+1}^t \in \mathbb{R}$ so that there was a consolidated constraint for each generation which was then used to consolidate the dynasty's budget constraint.

- To see why borrowing constraints can play an important role in the timing of taxes, we will present this "financial friction" in the context of the two period household problem without bequests (for simplicity we drop the generational superscript t).

- The borrowing constraint is given by $a_{t+1} \geq \underline{a}$ where $\underline{a} \leq 0$. A strict no borrowing constraint is given by $\underline{a} = 0$.

- The household problem is

$$
\begin{aligned}
& \max_{(c_t, c_{t+1}) \in \mathbb{R}_+, a_{t+1} \geq \underline{a}} U(c_t, c_{t+1}) \\
s.t.\ c_t + q_t a_{t+1} &= w_1 - \tau_t \\
c_{t+1} &= w_2 - \tau_{t+1} + a_{t+1}
\end{aligned}
$$

- Letting $\Psi_t = \tau_t + q_t \tau_{t+1}$, the consolidated budget constraint in (c_t, c_{t+1}) is given by

$$
\begin{aligned}
c_{t+1} &= \frac{w_1 - \tau_t}{q_t} + w_2 - \tau_{t+1} - \frac{1}{q_t} c_t \\
\iff & \ c_{t+1} = \frac{W_t - \Psi_t}{q_t} - \frac{1}{q_t} c_t
\end{aligned}
$$

[7] Note that even though the construction with $q = \alpha$ has $c_t^t = c^y$ and $c_{t+1}^t = c^o$ independent of time, that doesn't mean that τ_t^t and τ_t^{t-1} have to be independent of time (just must respect feasibility condition listed above).

in which case the intercept is given by $\frac{W_t - \Psi_t}{q_t}$ and the slope of the intertemporal budget constraint is given by $-\frac{1}{q_t}$ (with the discount bond interpretation this is just $-(1+r_t)$).

- However, that is not the only constraint. It must also be the case that $a_{t+1} \geq \underline{a}$ or $\frac{w_1 - \tau_t - c_t}{q_t} \geq \underline{a}$.

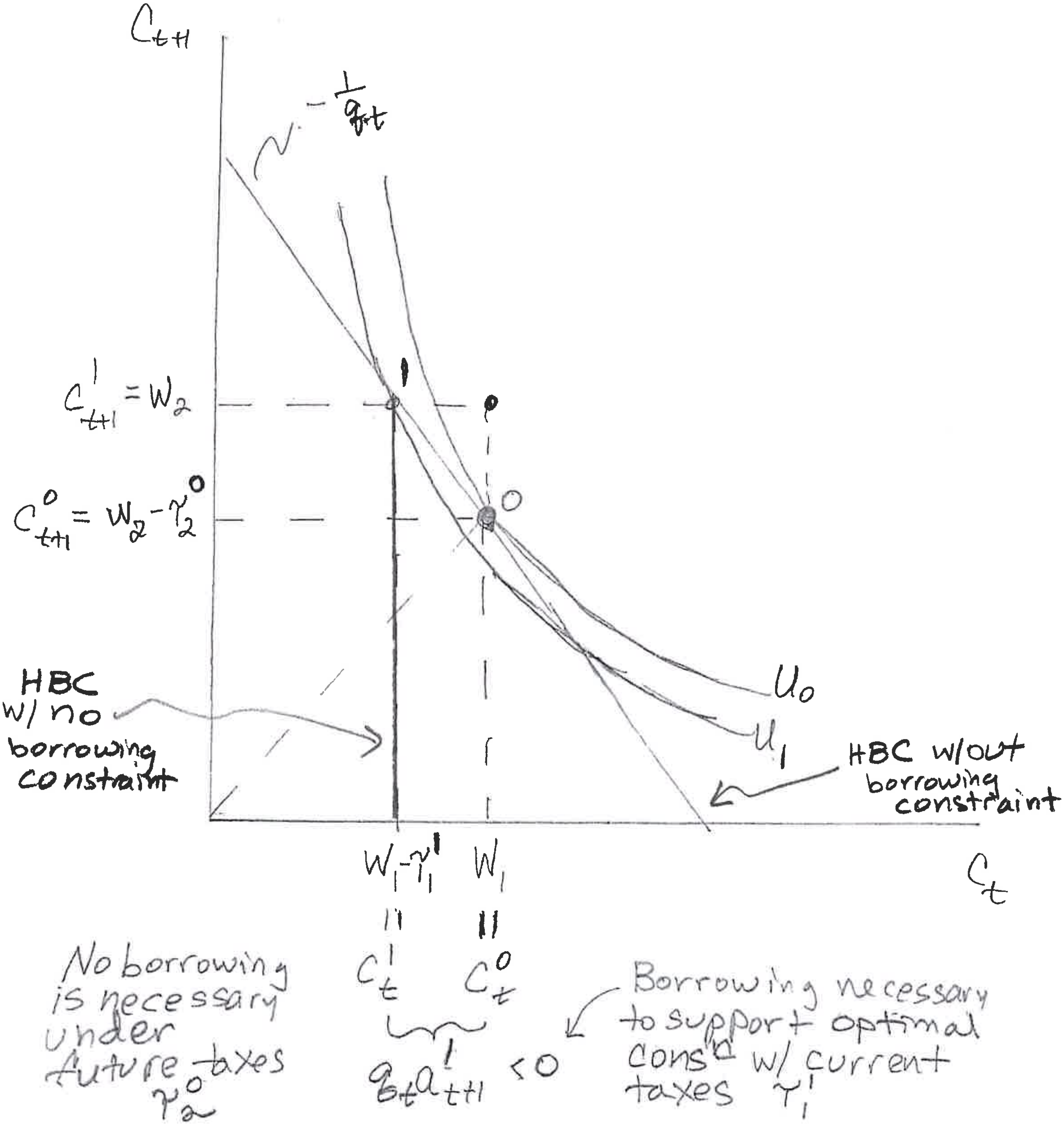
- Figure L4.1 graphs a case where the timing of taxes (τ_1, τ_2) matter if there are borrowing constraints.

  – If current taxes are zero while future taxes are $\tau_2 > 0$, it is possible to support the best outcome (point 0).

  – If future taxes are zero while current taxes are $\tau_1 > 0$, then if it is possible to borrow $q_t a_{t+1}$ (along HBC without a borrowing constraint) it is still possible to support the best outcome. That is, if $q_t a_{t+1} = -\tau_1 < 0$, then the household can borrow against future income in order to pay currently high taxes.

  – However, in this latter case, if there is a no borrowing constraint (i.e. $a_{t+1} \geq 0$) so that the HBC has a "kink" in it, then it is not possible to support the best outcome (point 1).

Figure L41



C_{t+1}
$-\frac{1}{q_t}$
$C^1_{t+1} = W_2$
$C^0_{t+1} = W_2 - \tau^0_2$
1
0
HBC w/ no borrowing constraint
U_0
U_1
HBC w/out borrowing constraint
$W_1 - \tau^1_1$
W_1
C_t
C^1_t
C^0_t
No borrowing is necessary under future taxes τ^0_2
$\underbrace{C^1_t \quad C^0_t}_{q_t a^1_{t+1} < 0}$
Borrowing necessary to support optimal cons'n w/ current taxes τ^1_1

Notes on Social Security Reform in a Life-Cycle Model

ECO 712 Theory: macro sequence

Instructor: Dean Corbae

In these lecture notes we study the macroeconomic consequences of eliminating the Social Security system in the U.S. To do so, we set up a simple general equilibrium overlapping generations model and then show how to solve it.

Reference: J. Conesa, and D. Krueger (1999): *Social Security Reform with Heterogeneous Agents*, Review of Economic Dynamics, 2, p. 757-95.

Environment

Each period a continuum of agents is born. Agents live for J periods after which they die. The population growth rate is n per year (which is the model period length). Thus, the relative size of each cohort of age j, ψ_j, is given by:

$$\psi_{i+1} = \frac{\psi_i}{1+n}, \text{ for } i = 1, ..., J-1$$

with $\psi_1 = \tilde{\psi} > 0$. It is convenient to normalize ψ, so that it sums up to 1 across all age groups.

Newly born agents (i.e. $j = 1$) are endowed with no initial capital (i.e. $k_j = 0$) but can subsequently save in capital which they can rent to firms at rate r. A worker of age j supplies labor $\ell_j \in [0, 1]$ and pays proportional social security taxes on her labor income $\tau w e_j \ell_j$ until she retires at age $J^R < J$, where e_j is the age-efficiency profile shown in figure 1. Upon retirement, agent receives pension benefits b.

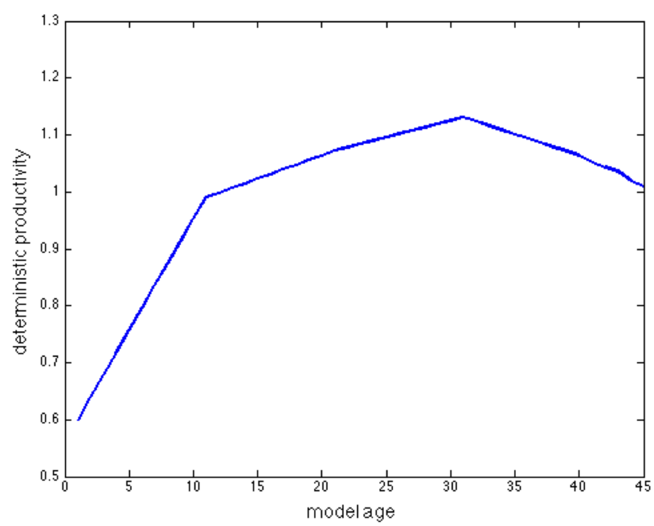


Figure 1: **Age-efficiency profile**

The instantaneous utility function of a worker at age $j = 1, 2, ..., J^R - 1$ is given by:

$$u^W(c_j, \ell_j) = \frac{\left(c_j^{\gamma}(1-\ell_j)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma}$$

with c_j denoting consumption and ℓ_j denoting labor supply at age j. The weight on consumption is γ and the coefficient of relative risk aversion is σ. The instantaneous utility function of a retired agent at age $j = J^R, ..., J$ is given by:

$$u^R(c_j) = \frac{c_j^{1-\sigma}}{1-\sigma}. \tag{1}$$

Preferences are then given by

$$\sum_{j=1}^{J^R-1} \beta^{j-1} u^W(c_j, \ell_j) + \sum_{j=J^R}^{J} \beta^{j-1} u^R(c_j)$$

There is a constant returns to scale production technology $Y = F(K, L) = K^\alpha L^{1-\alpha}$ with α denoting capital share, Y denoting aggregate output, K denoting aggregate capital stock and L denoting aggregate effective labor supply. The capital depreciates at rate δ. The labor and capital markets are perfectly competitive, so that $w = F_2(K, L)$ and $r = F_1(K, L) - \delta$.

Equilibrium

Household's choose $\{c_j, k_{j+1}\}_{j=1}^J \in \mathbb{R}_+^{2J}$ and $\{\ell_j\}_{j=1}^{J^R-1} \in [0,1]^{J^R-1}$ to

$$\max \sum_{j=1}^{J^R-1} \beta^{j-1} u^W(c_j, \ell_j) + \sum_{j=J^R}^{J} \beta^{j-1} u^R(c_j) \tag{2}$$

subject to

$$c_j + k_{j+1} = (1-\tau) w e_j \ell_j + (1+r) k_j, j = 1, ...J^R - 1 \tag{3}$$

$$c_j + k_{j+1} = b + (1+r) k_j, j = J^R, ..., J. \tag{4}$$

Definition. A *stationary competitive equilibrium* is an allocation $\{c_j, k_{j+1}\}_{j=1}^J$ and $\{\ell_j\}_{j=1}^{J^R-1}$, prices (r, w), policy (τ, b) and an age-dependent distribution ψ_j, such that:

- Given prices (r, w) and policy (τ, b), households choose $\{c_j, k_{j+1}\}_{j=1}^J \in \mathbb{R}_+^{2J}$ and $\{\ell_j\}_{j=1}^{J^R-1} \in [0,1]^{J^R-1}$ to solve (2)-(4).
- Firms optimize in competitive input markets:
  - marginal cost of labor equals marginal benefit of labor (i.e. $w = F_2(K, L)$);
  - marginal cost of capital equals marginal benefit of capital (i.e. $r = F_1(K, L) - \delta$);
- Markets clear:
  - goods: $\sum_{j=1}^J \psi_j [c_j + k_{j+1}] = F(K, L) + (1-\delta) K$;
  - rental capital: $K = \sum_{j=1}^J \psi_{j-1} k_j$;
  - rental labor: $\sum_{j=1}^{J^R-1} \psi_j e_j l_j = L$.
- The government budget constraint satisfies:

$$b = \frac{\tau w L}{\sum_{j=J^R}^J \psi_j}.$$

Dynamic programming problem

We will use finite dynamic programming to solve for the household's problem. The dynamic programming problem of a retired agent reads:

$$V_j(k) = \max_{k'} \left\{ u^R((1+r)k + b - k') + \beta V_{j+1}(k') \right\} \tag{5}$$

subject to $k' \geq 0$ and $k' = 0$ if $j = J$. In the expression above, V_j stands for the value function (indirect utility) of agent at age j. A prime denotes tomorrow's variables. The discount factor is β. Observe that we substituted agent's budget constraint into the utility function.

Note that in the very last period, the value function, $V_J(k)$, is known and given by (1) with $c = (1+r)k+b$. Knowing $V_J(k)$, we can proceed backwards to recover the optimal savings of retired agents, which we denote by $k'_j(k)$.

The dynamic programming problem of a worker is given by:

$$V_j(k) = \max_{k',l} \left\{ u^W(w(1-\tau)l + (1+r)k - k', l) + \beta V_{j+1}(k') \right\} \tag{6}$$

subject to $k = 0$ if $j = 1$ and $0 \leq l \leq 1$.

The dynamic programming problem of a worker involves maximization over an additional control, which is individual labor supply, l. We can, however, make use of the household's first-order condition with respect to l combined with the budget constraint (3) to arrive at the following expression for l_j:

$$l_j = \frac{\gamma(1-\tau)e_j w - (1-\gamma)\left[(1+r)k - k'\right]}{(1-\tau)e_j w}$$

Denote the optimal individual labor supply by $l_j(k)$.

Computing a stationary equilibrium

The algorithm to compute the stationary equilibrium consists of the following steps:

1. Make initial guesses of the steady state values of the aggregate capital stock K and aggregate labor L.
2. Compute social security benefits, b, and the prices, w and r, implied by these guesses.
3. Compute the household's decision functions by backward induction.
4. Compute the optimal path for savings and labor for the new born generation by forward induction given that the initial capital stock of newborns is 0.
5. Compute the aggregate capital stock and aggregate labor supply.
6. Update K and L and return to step 2 until convergence.

Parametrization of the model

J	J^R	n	τ	$\{e_j\}$	γ	σ	β	α	δ
66	46	0.011	0.11	Hansen (1993)	0.42	2.0	0.97	0.36	0.06

Table 1: Parametrization of the model

Experiment

There is an unanticipated change in the social security program: in period t the social security tax rate, τ, and the pension benefit, b, are set equal to 0 and stay there forever.

Results

	No heterogeneity	
Var.	In. St.St.	Fi. St.St.
b	50%	0%
r	6.0%	4.9%
w	1.18	1.25
h	32.8%	34.5%
K Y	2.98	3.30
y	1.04	1.17
SS y	38.9%	0
$cv(lab)$	0.52	0.51
$cv(weal)$	0.81	0.93
EV^{SS}	—	12.7%

Figure 2: **Changes in aggregate variables, prices and welfare across steady states** ($In.St.St.$ – initial steady state (with social security), $Fi.St.St.$ – final steady state (without social security), b – replacement rate, r – interest rate, w – wage, h – average hours worked, K/Y – capital to output ratio, y – output, SS/y – share of social security contributions in output, cv – coefficient of variation, EV^{SS} – consumption equivalent variation).

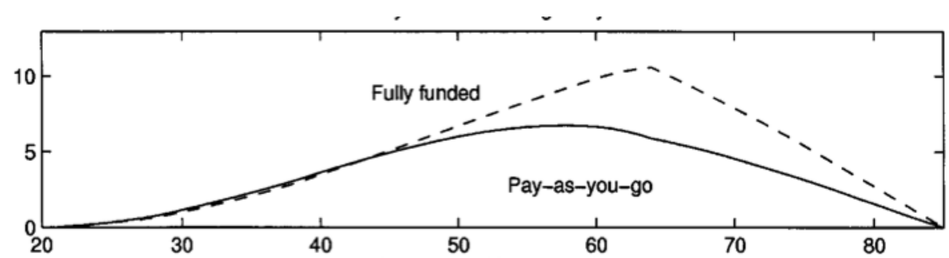


Figure 3: Age-wealth profile (pay-as-you-go – with social security, fully funded – without social security)

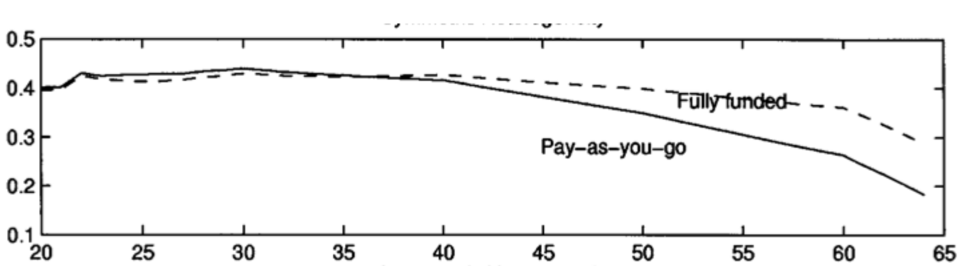


Figure 4: Age-labor profile (pay-as-you-go – with social security, fully funded – without social security)

Econ 712, Corbae

Fall 2021

L5. Intro to Government Commmitment Problems

This draws on the simple framework in Stokey (1989, AER). In a static environment with a possible time inconsistency problem on the part of the government, it uses timing as a way to illustrate commitment or lack thereof. The government chooses policy before households choose their actions in a commitment (or Ramsey) equilibrium while the government chooses policy after households have chosen their action in a no-commitment (or sustainable) equilibrium.[1] In a dynamic environment, however, it is clear that even if the government chooses policy after households in one period, households can react to its choice in the following period. In this way, households may be able to punish a government for bad policy choices in previous periods (this is known as a trigger strategy).

Another method introduced below is what macro people often call "Big K, little k" (here "Big X, little x)". If all other identical agents are taking action X, an atomless individual's optimal decision x does not affect aggregates due to the continuum assumption. Consistency (given symmetry) requires that $x = X$ but we must allow individuals to contemplate choosing an action different from X (rather than impose it).

1 A Simple Environment

- Continuum (unit measure) of identical households and one "big" government
- Each household (hh) chooses $x \in \mathcal{X}$.
- Government (govt.) chooses $y \in Y$ to maximize household utility.
- If X is chosen by all hh, y is feasible for the govt if $(X, y) \in Z \subset \mathcal{X} \times Y$
- Hh preferences represented by $w(x, X, y)$ which is the utility of a hh if it chooses x, all other hhs choose X, and the govt chooses y.

[1] For this taxonomy, see also V.V. Chari (1988) "Time Consistency and Optimal Policy Design", Federal Reserve Bank of Minneaopolis Quarterly Review, Vol. 12, No. 4, p. 17-31.
In particular, when there is no government commitment to its policies, optimal behavior by private agents requires that they forecast future policies as being sequentially rational for the government. A sequence of private outcomes and policy rules satisfying optimality by private agents and sequential rationality in government policy choices is called a "sustainable equilibrium". This concept is closely related to the notion of subgame perfection and sequential equilibrium in game theory. Sustainable equilibrium extends these ideas to environments with anonymous, atomless private agents.

2 Equilibrium

2.1 Static First best

- The planner chooses for all identical hhs, so $x = X$ by definition.
- $(X^f, y^f) = \arg\max_{(X,y)\in Z} w(X, X, y)$.
- Characterization by foc:

$$\begin{aligned} X &: \quad w_1 + w_2 = 0 \\ y &: \quad w_3 = 0 \end{aligned} \tag{1}$$

2.2 Static Commitment Equilibrium (RE)

- Timing: govt chooses $y \in Y$ before hhs choose $x \in \mathcal{X}$.
- Work backwards: Given y, hh chooses x (which yields a decision rule $\eta(y)$). Consistency and optimality requires:

$$\eta(y) = \arg\max_{x\in\mathcal{X}} w(x, \eta(y), y) \tag{2}$$

- Given $\eta(y)$, govt chooses y :

$$\begin{aligned} y^r &= \arg\max_{y\in Y} w(\eta(y), \eta(y), y) \\ s.t.(\eta(y), y) &\in Z \end{aligned} \tag{3}$$

- In general $X^f \neq X^r \equiv \eta(y)$, so Ramsey outcomes are not necessarily first best since

$$\begin{aligned} x &: \quad w_1 = 0 \\ y &: \quad (w_1 + w_2)\, \eta'(y) + w_3 = 0. \end{aligned} \tag{4}$$

2.3 Static No Commitment Equilibrium (SE)

- Timing: hh chooses $x \in \mathcal{X}$ before govt chooses $y \in Y$.
- Work backwards: Given X^n, govt chooses y:

$$\begin{aligned} \varphi(X^n) &= \arg\max_{y\in Y} w(X^n, X^n, y) \\ s.t.(X^n, y) &\in Z \end{aligned} \tag{5}$$

- Since each hh is small, it realizes that its impact on the govt is infinitesimal. It expects that if all other hhs choose X^n, the govt will choose $\varphi(X^n)$. Consistency and optimality requires

$$X^n = \arg\max_{x\in\mathcal{X}} w(x, X^n, \varphi(X^n)) \tag{6}$$

- In general $X^r \neq X^n$ since in the RE, the government takes into account its influence on hh, while in the SE it does not:

$$\begin{aligned} x &: w_1 = 0 \\ y &: w_3 = 0. \end{aligned} \tag{7}$$

- Note that if the govt moves first (i.e. commits), it can always choose $y^n = \varphi(X^n)$. Hh then chooses $x = X^n$, so that it is always possible to implement the NCE as a RE. Hence, the utility of the RE is always at least as high as the utility of a SE.

3 Distortionary Taxation in a Static Environment

3.1 Environment

- Continuum (unit measure) of identical households and one "big" benevolent government (wishing to maximize the utility of its citizens).
- HH receives nonstorable endowment ω.
- Two storage technologies: "productive" technology yields gross return $R > 1$ (i.e. net return is $r > 0$) and "pillow" yields gross return equal to 1 (i.e. zero net return) at the end of the period from an investment at the beginning of the period. Given $R > 1$, the efficient thing to do is to "invest" in the productive technology rather than "hoard" under the pillow.[2]
- HH prefs: $U(c, g)$ where c is a private good, g is a public good, U is strictly increasing, concave, and differentiable with

$$U_c(\omega, (R-1)\omega) < U_g(\omega, (R-1)\omega). \tag{8}$$

Thus if all households invest, and the principal is used for private consumption while the interest is used for public goods, the marginal benefit of public goods (rhs) exceeds the marginal cost of more public goods (i.e. marginally less private goods on the lhs). As will be seen this assumption means that the government will ultimately need to raise sufficient tax revenue if it wants to implement the first best.

3.2 Planner's solution

- Since $R > 1$, planner invests ω into productive technology. Then problem is

$$\begin{aligned} &\max_{c,g} U(c, g) \\ &s.t. c + g = R\omega \end{aligned} \tag{9}$$

[2] Note that Stokey uses different notation ($R = (1 + i)$).

where the constraint is the resource feasibility constraint.

- FOC

$$\begin{aligned} -U_c(c^f, g^f) + U_g(c^f, g^f) &= 0 \qquad (10) \\ \iff \frac{-U_c^f}{U_g^f} &= -1. \end{aligned}$$

- In Figure 1a, this states that at the first best $MRS_{c,g}$ (slope of the indifference curve) = $MRT_{c,g}$ (slope of resource constraint).

- By assumption (8), we know that $c = \omega$ and $g = (R-1)\omega$ can't be the first best allocation (see Figure L51). In particular, from (8)

$$\frac{-U_c^f(\omega, (R-1)\omega)}{U_g^f(\omega, (R-1)\omega)} > -1.$$

  This is where the indifference curve cuts the resource constraint at the endowment point.

3.3 Decentralization

- Govt taxes $\tau \in [0,1]$ productive storage to pay for public goods (i.e. $g = \tau RX$) where X is aggregate investment in the productive technology.[3] Note that these taxes are proportional to the amount of storage the household undertakes and hence are potentially distortionary. The real world application is a version of the capital gains tax.[4]

- $x \in \mathcal{X} = [0, \omega]$ denotes the amount of the endowment a HH invests and $m = \omega - x$ the amount it hoards, while $(X, M) \in [0, \omega] \times [0, \omega]$ is all other hhs choice.

- Since in a decentralized solution $x + m = \omega$ and $c = m + (1-\tau)Rx$, we now can see why $w(x, X, \tau) = U(\omega - x + (1-\tau)Rx, \tau RX)$ fits into the analysis of Section 2. Notice that even though a hh doesn't care directly about other hhs' choices (i.e. X) or the tax rate (i.e τ), they still show up in w.

3.4 Commitment Equilibrium (RE)

- Timing: Govt commits to a tax rate τ, then HHs move.

[3] Again, I have introduced different notation than Stokey for this section (She uses y, instead of τ).

[4] Strictly speaking the capital gains tax would be $\tau(R-1)X$, but this makes for easier analysis.

- Work backwards. The HH problem is:

$$\max_{x,m,c} U(c,g) \tag{11}$$
$$\begin{aligned} x+m &= \omega \\ c &= m+(1-\tau)Rx \end{aligned}$$

- Problem (11) can be simplified. Taking τ as given (since a hh is small), other hhs decisions $X^r(\tau)$ which also depend on those same taxes, and the government budget constraint, the household's problem amounts to choosing an optimal investment decision rule:

$$x^r(\tau) = \arg\max_x U(\omega - x + (1-\tau)Rx, \tau R X^r(\tau)) \tag{12}$$

- Since all hhs are identical, consistency of an equilibrium implies $x^r(\tau) = X^r(\tau)$.

- Thus, taking household decision rules $X^r(\tau)$ as given, the benevolent govt chooses τ so that:

$$\tau^r = \arg\max_\tau U(\omega - X^r(\tau) + (1-\tau)RX^r(\tau), \tau RX^r(\tau)) \tag{13}$$

Definition 1 *A **Ramsey Equilibrium** (RE) is a household decision rule $x^r(\tau) \in [0,\omega]$ (implying $m^r(\tau)$ and $c^r(\tau)$) which solves (12), consistency of decisions $x^r(\tau) = X^r(\tau)$, goods market clearing $c + g = \omega + (R-1)X^r(\tau, g)$, and a tax rate τ^r which satisfies government budget balance and maximizes household utility (i.e. solves (13)).*

- To solve this optimization problem, notice that once x is chosen, then m and c follow.

- It's actually harder to solve for x via the foc. Instead look at Figure L51.

  - If $(1-\tau)R \geq 1$ (i.e. the after tax gross return to investment in the productive technology exceeds the return to hoarding), then the hh will invest rather than hoard (i.e. $x = \omega, m = 0$). This only happens if taxes are sufficiently low.
  - If $(1-\tau)R < 1$ which happens if taxes are sufficiently high, the hh will hoard (i.e. $x = 0, m = \omega$).
  - Thus the decision rule is given by

$$x^r(\tau) = \begin{cases} \omega & \text{if } \tau \leq (R-1)/R \\ 0 & \text{if } \tau > (R-1)/R \end{cases}$$

- Notice that if all households, which are ex-ante identical make the same choice, there is a Laffer curve; tax revenue (which here just equals g) as a function of the tax rate τ increases at first, then decreases. In particular,

$$g = \begin{cases} \tau R\omega & \text{if } \tau \leq \frac{R-1}{R} \\ 0 & \text{if } \tau > \frac{R-1}{R} \end{cases}.$$

So in $([0,1], g)$ space, g is increasing in τ until $\frac{R-1}{R}$, then 0 thereafter (see Figure L51).

- Given the above cutoff rule, feasibility requires

$$c = \begin{cases} (1-\tau)R\omega & \text{if } \tau \leq \frac{R-1}{R} \\ \omega & \text{if } \tau > \frac{R-1}{R} \end{cases}.$$

So in $([0,1], c)$ space, c is decreasing in τ until $\frac{R-1}{R}$, then ω thereafter.

- Hence, up to $\tau = \frac{R-1}{R}$, there is a tradeoff, utility from consumption of the private good is decreasing but utility from consumption of the public good is increasing. For any $\tau > \frac{R-1}{R}$, c is the same as at $\tau = \frac{R-1}{R}$,but g is discretely lower, so this τ can't be optimal. Furthermore, by assumption (8), we know that any $\tau < \frac{R-1}{R}$ cannot be optimal, since the first best calls for $g^f > (R-1)\omega = g^r\left(\frac{R-1}{R}\right)$ and $c^f < \omega = c^r\left(\frac{R-1}{R}, (R-1)\omega\right)$. Thus, $\tau^r = \frac{R-1}{R}$. Notice that at $\tau^r = \frac{R-1}{R}$, government tax revenues are $\left(\frac{R-1}{R}\right) R\omega = (R-1)\omega$ as in (8). Hence we end up at point $\mathcal{R}$ in Figure L5.1.

- Since the first best could always choose the Ramsey allocation, we know that welfare is lower than the first best. This should not be surprising since the decentralized economy uses distortionary taxes (rather than lump sum ones) to pay for the public good.

3.5 No Commitment Equilibrium (SE)

- Timing: HHs commit to an investment/hoarding choice (since (τ, g) has not been chosen yet, Hh choice cannot be contingent on it). Then govt moves.

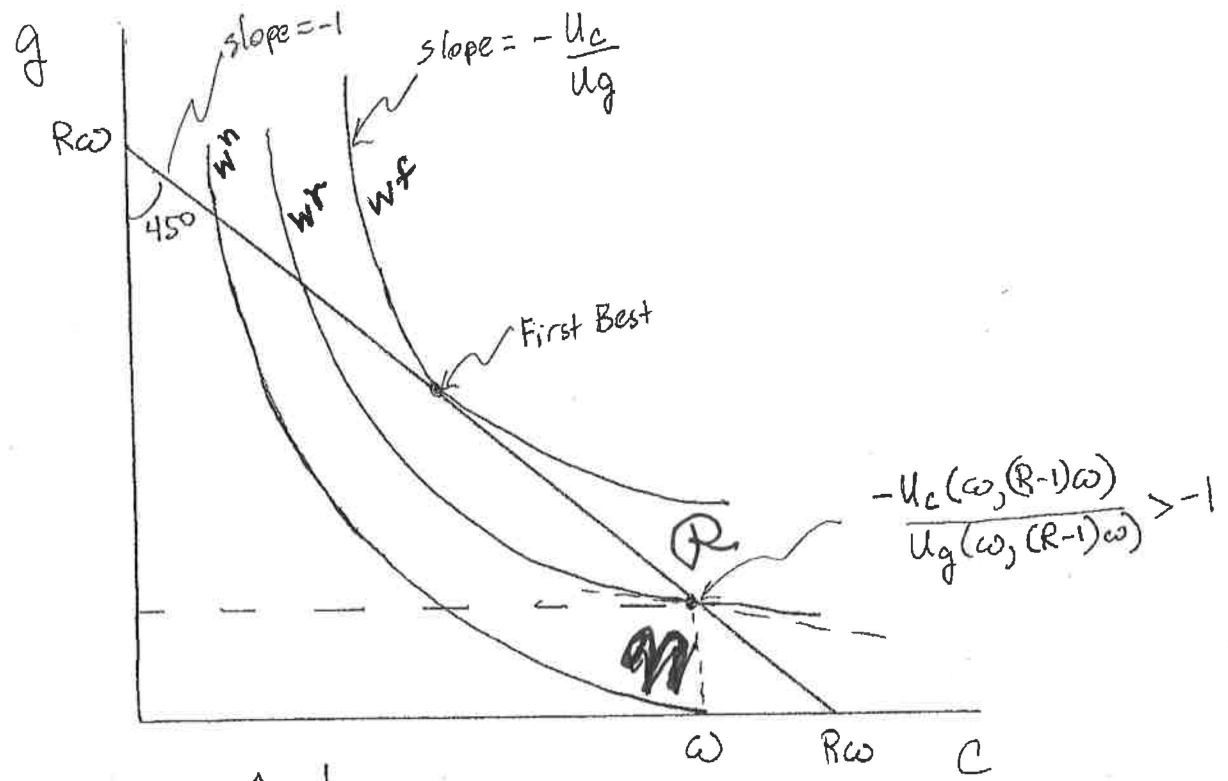
- Work backwards. Taking household investment choices X and the government budget constraint $g = \tau RX$ as given, the govt problem is:

$$\tau^n(X) = \arg\max_{\tau} U(\omega - X + (1-\tau)RX, \tau RX) \tag{14}$$

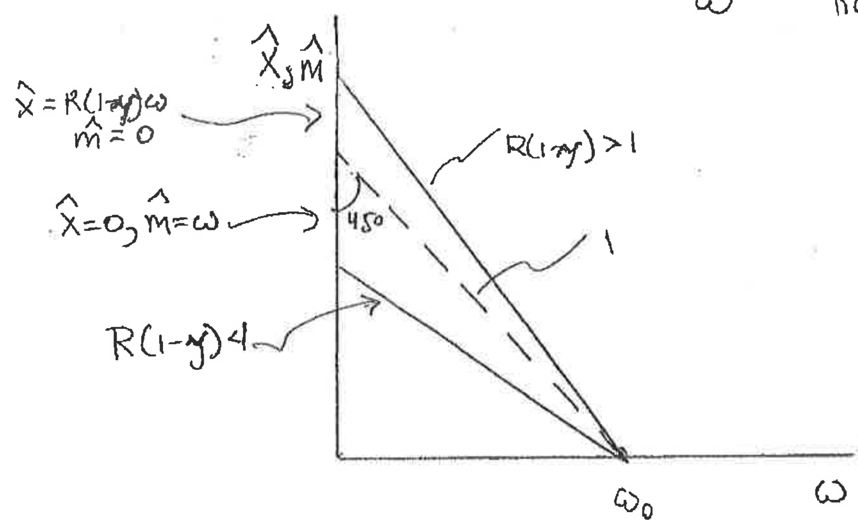
- Since a tiny hh doesn't influence the govt's problem, they take the govt decision rule $\tau^n(X)$ as given. The HH problem is

$$x^n = \arg\max_{x} U(\omega - x + (1-\tau^n(X))Rx, \tau^n(X)RX) \tag{15}$$

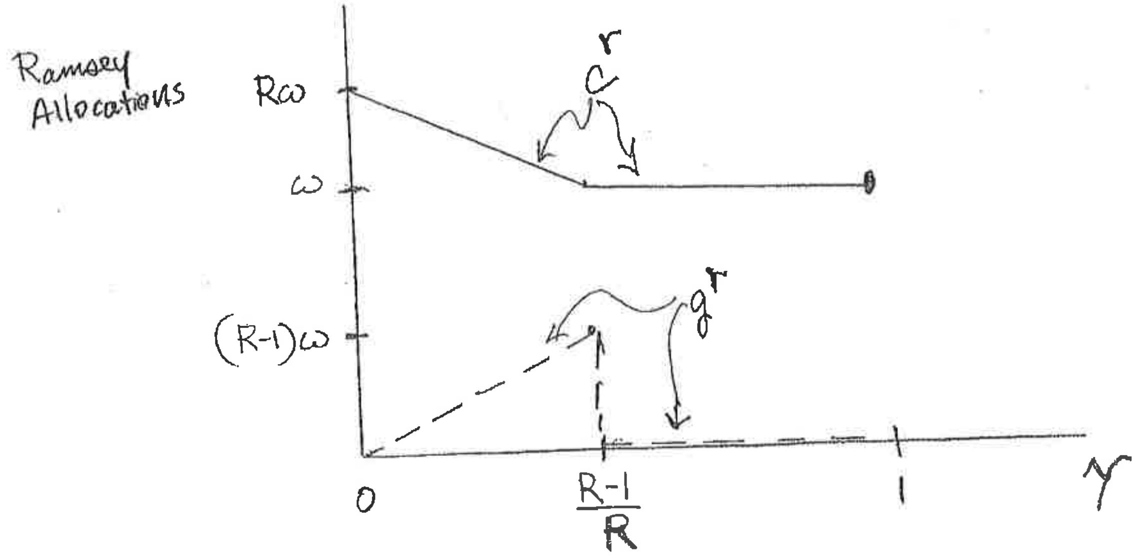
Notice that the household decision x^n depends on the belief of what government tax rule $\tau^n(X)$ will be implemented.



g
slope=-1
slope = $-\frac{U_c}{U_g}$
$R\omega$
w^n
w^r
w^f
45^0
First Best
$\frac{-U_c(\omega,(R-1)\omega)}{U_g(\omega,(R-1)\omega)} > -1$
R
W
ω
$R\omega$
C



$\hat{X}, \hat{M}$
$\hat{x} = R(1-\gamma)\omega$
$\hat{m} = 0$
$R(1-\gamma) > 1$
$\hat{x}=0, \hat{m}=\omega$
45^0
1
$R(1-\gamma) < 1$
ω_0
ω



Ramsey Allocations
$R\omega$
c^r
ω
g^r
$(R-1)\omega$
0
$\frac{R-1}{R}$
1
γ

- Consistency then requires $x^n = X^n$. In the language of game theory, this is like a restriction to symmetric equilibria.

Definition 2 *A* ***(Subgame Perfect) No Commitment Equilibrium*** *(SE) is a household decision x^n (which induces m^n and c^n) that solves (15), consistency of decisions $x^n = X^n$, goods market clearing, and a tax decision rule $\tau^n(X^n)$ which satisfies government budget balance and maximizes household utility given X^n (i.e. solves (14)).*

- The FOC wrt τ in the govt's problem (14) is :

$$-RXU_c(c^n, g^n) + RXU_g(c^n, g^n) = 0$$

- Notice that since both terms are multiplied by RX, this foc is the same as the first best (10). But with only proportional taxes, the only way to implement $(c^f < \omega, g^f > (R-1)\omega)$ is to "overtax" (i.e. $\tau^n > \frac{R-1}{R}$) in order to lower consumption and raise public goods expenditure.

- The incentive for the government to "overtax" is an example of a time inconsistency (or in the language of game theory "dynamic inconsistency") problem.[5]

- Household's understand the government has an incentive to overtax (i.e. choose $\tau^n > \frac{R-1}{R}$). Knowing this, hhs hoard and don't invest (i.e. $X^n = 0$).

- Budget balance then implies $g^n = \tau^n R \cdot 0 = 0$ and $c^n = \omega$. Hence we end up at point $\mathcal{N}$ in Figure L5.1.

- As one can see in Figure L5.1, without commitment, HHs realizing it is subgame perfect for the government to overtax in stage 2, choose a subgame perfect hoarding strategy in stage 1, yielding the worst possible welfare outcome.

- These two results (private underprovision of investment and underprovision of public goods) is the reason why lack of commitment is so bad. In particular, $w^n = U(\omega, 0) < w^r < w^f$.

- The high tax rate after the investment decision has been made (in which case it is perfectly inelastic) should not be surprising. It is a general result that revenue creation from increased taxation is highest facing inelastic demand or supply.

- The important point is that even though the government has aligned preferences (is trying to maximize hh welfare), the state where the government takes actions is different in the two cases. In the SE case, it takes the hh action as given, while in the RE case it takes the decision rule as given.

[5] From wikipedia: In the context of game theory, dynamic inconsistency is a situation in a dynamic game where a player's best plan for some future period will not be optimal when that future period arrives. A dynamically inconsistent game is not subgame perfect.

4 Repeated No-Commitment Equilibrium (RNCE)

- In a static game without commitment, the analysis above shows there are excessively high tax rates and low productive investment. There is little commitment in the real world, but it is also not static.
- Here we will consider whether it is possible to support low tax behavior by the government (e.g. the Ramsey allocation) which welfare (at $\mathcal{R}$) dominates the static no commitment allocation (at $\mathcal{N}$) through what are called "trigger strategies" which punish the government for deviations.
- While goods are nonstorable across periods, we can store (or encapsulate) a history h_t of past government tax behavior.[6]
- Dynamic timing without commitment:

1. beginning of period t
   (a) start in a given history of actions h_t
   (b) hhs choose investment $X_t \in [0, \omega]$
   (c) government chooses taxes $\tau_t \in [0, 1]$
   (d) history is updated $h_{t+1} = f(h_t, X_t, \tau_t)$
2. beginning of period $t + 1$
   (a) start with a given history of actions h_{t+1}
   (b) hhs choose investment X_{t+1}
   (c) etc.

- In the static no commitment economy, the government chose τ_t (in 1c) after hhs chose X_t (in 1b). In the dynamic no commitment economy, households choose X_{t+1}(in 2b) after government chooses τ_t (in 1c).
- Thus, in a repeated game hhs can condition their choice x on the government's past τ choice. If the government overtaxed in the last period (as in the static SE), then the hh can "punish" the govt by hoarding.
- Suppose the no commitment model is repeated infinitely often and that all agents discount the future at rate $\beta \in (0, 1)$. Even though the government moves after households, repetition means that households also move after the government.
- Let the "state space" be denoted H, standing for "history of actions".[7]

[6] This endogenous state variable is not strictly payoff relevant (i.e. it does not enter preferences or technologies).

[7] Again, I changed the notation from S in Stokey to H to make clear the state can encapsulate the history of past actions.

- Each period, hhs observe the current state (i.e. history) before they choose their action. We will restrict attention to a particular type of strategy which does not depend on calendar time (so is stationary), but depends on history. That is, a stationary investment strategy for the hh is a function $\sigma : H \to [0, \omega]$

- Each period, the govt observes both the state and the actions of households before it chooses its policy, so a stationary tax strategy for the govt is a function $\gamma : H \times [0, \omega] \to [0, 1]$ with $(X, \gamma(h, X)) \in [0, \omega] \times [0, 1]$, $\forall X \in [0, \omega], \forall h \in H$.

- The law of motion for the state variable is a function $f : H \times [0, \omega] \times [0, 1] \to H$ describing next period's state as a function of the current state (history) and the current outcome. The function f generates a history. Think of this as a record keeping technology.

- Since each agent takes as given the strategies of all other agents and the law of motion of the state variable, the dynamic problem the agent faces is stationary.

Definition 3 *A* ***sustainable (subgame perfect) equilibrium*** *(SE) is a pair of stationary policy and investment rules (γ, σ) such that given γ, σ solves the hh dynamic decision problem for every state h and given σ, γ solves the government dynamic decision problem for every state $(h, \sigma(h))$.*[8]

4.1 Supporting a superior equilibrium with grim trigger strategies.

- Even if there is no commitment, is it possible to support the "better" Ramsey allocation in a dynamic environment?

- We will focus on a particular stationary strategy called a "grim trigger strategy" on a very simply state space $H = \{0, 1\}$. In this grim trigger strategy the household will punish the government forever if it ever deviates from the Ramsey tax policy.

- Recall from the **static** environment (i.e. which did not depend on history) that $w(x, X, \tau) = U(\omega - x + (1 - \tau)Rx, \tau RX)$ and:

  - optimal investment and consistency requires

$$x^o(\tau) = \arg \max_{x \in [0, \omega]} w(x, X^o(\tau), \tau) \tag{16}$$

    and $x^o(\tau) = X^o(\tau)$.

[8] This equilibrium with trigger strategies is not to be confused with a markov perfect equilibrium.

- optimal tax choice in the no commitment environment

$$\tau^n(X) = \arg\max_{\tau\in[0,1]} w(X, X, \tau) \tag{17}$$

- While in general we could consider any allocation and policy $(X^o, \tau^o) \in [0, \omega] \times [0, 1]$ with $w(x^o, X^o, \tau^o) > w(x^n, X^n, \tau^n)$, here we will focus on the Ramsey outcome where $X^o = X^r = \omega$ and $\tau^o = \tau^r = (R-1)/R$.
- Let $H = \{0, 1\}$. Let the initial state be $h = 0$. Interpret $h = 0$ as the state where no deviation has ever occurred.
- Let the law of motion for the state variable be:

$$f(h, X, \tau) = \begin{cases} 1 & \text{if } h = 0, X = X^r, \tau \neq \tau^r \\ h & \text{otherwise} \end{cases} \tag{18}$$

  - Given we start with $h = 0$, as long as no deviation has occurred, the new value of the state variable is 0.
  - The first time a deviation from τ^r occurs following investment to X^r by hhs, the value of the state variable changes from 0 to 1, and then it remains at 1 forever since $h \neq 0$ invalidates the top branch of (18). This supports a grim trigger strategy.
  - Notice this law of motion does not punish household deviations from X^r and government deviations following household deviations. Just government deviations preceded by household play of X^r.
  - There can be many different types of punishment strategies and many possible equilibria. For instance, there are less harsh punishment strategies which punish for a finite amount of time, etc. The harshest penalty makes it easier to support good outcomes.
- Let the stationary strategy for each hh be

$$\sigma(h) = \begin{cases} x^r & \text{if } h = 0 \\ x^n & \text{if } h = 1 \end{cases}. \tag{19}$$

  That is, each hh chooses the action x^r or x^n depending on whether a deviation has occurred. Notice that the "punishment" should be harsh to support the best behavior.
- Let the strategy for the government be

$$\gamma(h, X) = \begin{cases} \tau^r & \text{if } h = 0, X = X^r \\ \tau^n(X) & \text{otherwise} \end{cases} \tag{20}$$

  That is, govt chooses the policy τ^r if no deviation has ever occurred and if hhs have chosen X^r, otherwise it chooses what is optimal in that particular state (i.e. $\tau^n(X)$ which is the one-shot argmax (17) since in that state, history is no longer relevant).

- Do these strategies constitute a sustainable (subgame perfect) equilibrium that support the Ramsey allocation? Subgame perfection means that we must check that households and government do not have an incentive to deviate in any state (e.g. history), even those that are not "on-the-equilibrium path". Specifically, even though we are trying to find conditions under which the good Ramsey equilibrium can be supported with trigger strategies "on-the-equilibrium path" (where the government doesn't overtax) we must also verify that if the government overtaxes "off-the-equilibrium path", all agents (households and the government) play optimal strategies.

- HH ($\sigma(h)$ needs to be optimal across all possible states/histories):

  - Since no individual can affect the evolution of the state variable (i.e. x does not appear in (18), the policy of the govt in (20), or the actions of other hhs), its future payoffs are independent of its current action. This implies that the individual's problem is a sequence of **static** problems.

  - Hence for the individual, it is sufficient to check that its action maximizes its current utility in all possible histories, assuming all other hhs adopt σ and the govt adopts γ. The strategy is optimal if there are no gains to deviation or

$$\text{For } h = 0,\ w(x^r, X^r, \tau^r) \geq w(x, X^r, \tau^r), \forall x \in [0, \omega] \tag{21}$$

$$\text{For } h = 1,\ w(x^n, X^n, \tau^n) \geq w(x, X^n, \tau^n), \forall x \in [0, \omega] \tag{22}$$

    * It follows from the definition of $x^r = \arg\max_x w(x, X^r, \tau^r)$ in (16) that (21) holds.In what follows, this will be "on-the-equilibrium path" behavior.
    * It follows from the definition of $x^n = \arg\max_x w(x, X^n, \tau^n(X^n))$ in (16) that (22) holds. In what follows this will be "off-the-equilibrium path" behavior.
    * Thus the stationary strategy σ follows the optimal static strategies we learned in Section 3 across all possible histories (both those that happen "on-the-equilibrium path" (which will be $h = 0$ if we can support the Ramsey allocation) and "off-the-equilibrium path" (which will be $h = 1$ if we can support the Ramsey allocation).

- Govt ($\gamma(h, X)$ needs to be optimal across all possible states/histories):

  - If $h = 1$ (there's been a prior deviation) or if $h = 0$ and $X \neq X^r$ (there's a current deviation by hhs), the govt cannot affect the evolution of the state variable in (18) (i.e. it stays at h according to $f(\cdot)$). Thus, it must solve a sequence of **static** problems as in Section 3.

In these cases it is sufficient to check if γ maximizes the govt's current utility. Since $\tau^n(X) = \arg\max_{\tau\in} w(X, X, \tau)$ satisfies (17), the required condition holds.

– If $h = 0$ and $x = X^r$, then the govt can affect the evolution of the state variable by a one shot deviation, so its future utilities depend on its current policy. For the govt not to deviate, we need[9]

$$w(x^r, X^r, \tau^r) + \frac{\beta}{1-\beta} w(x^r, X^r, \tau^r) \geq w(x^r, X^r, \tau) + \frac{\beta}{1-\beta} w(x^n, X^n, \tau^n), \forall \tau \neq \tau^r \tag{23}$$

The lhs is the value if the govt follows the proposed strategy and the rhs is its utility after it deviates and then responds optimally to x^n.

- Rearranging (23) yields

$$\frac{\beta}{1-\beta} \left[w(x^r, X^r, \tau^r) - w(x^n, X^n, \tau^n) \right] \geq \left[w(x^r, X^r, \tau) - w(x^r, X^r, \tau^r) \right] \tag{24}$$

The current gain from deviating (the term on the rhs) must be less than the total discounted future benefit of not deviating (the term on the lhs).

- Since $w(x^r, X^r, \tau^r) > w(x^n, X^n, \tau^n)$ by assumption, the lhs $\to \infty$ as $\beta \to 1$, while the rhs is constant. Hence if hh and govt are sufficiently patient, we can support such outcomes (and in particular, the Ramsey commitment outcome can be supported even when there is not an explicit commitment technology).

- Notice that if $\beta = 0$,then we are back in the static case and (24) is given by $0 \geq \left[w(x^r, X^r, \tau) - w(x^r, X^r, \tau^r) \right]$, a weak contradiction. That is, the r allocation cannot be supported in the myopic case.

- A grim trigger strategy imposes the harshest punishment (permanent autarky), providing the biggest stick to disincentive a deviation to bad behavior. What if the punishment strategy is only for T periods? In that

[9] Under certain conditions, showing that a one-shot deviation does not yield higher utility is sufficient to prove that the candidate policy is optimal. This is the idea behind Bellman's Principle of Optimality that is the basis of dynamic programming.

case (23) would be written[10]

$$
\begin{aligned}
& w(x^r, X^r, \tau^r) + \frac{\beta}{1-\beta} w(x^r, X^r, \tau^r) \\
\geq \quad & w(x^r, X^r, \tau) + \beta(1 + \beta + ... + \beta^T) w(x^n, X^n, \tau^n) + \frac{\beta^{T+1}}{1-\beta} w(x^r, X^r, \tau^r), \forall \tau \neq \tau^r \\
\iff \quad & \frac{\beta}{1-\beta} \left[w(x^r, X^r, \tau^r) - w(x^n, X^n, \tau^n) - \beta^T \left\{ w(x^r, X^r, \tau^r) - \beta w(x^n, X^n, \tau^n) \right\} \right] \\
\geq \quad & [w(x^r, X^r, \tau) - w(x^r, X^r, \tau^r)]
\end{aligned} \tag{25}
$$

Clearly (25) nests (24); just set $T = \infty$. Since $w(x^r, X^r, \tau^r) - \beta w(x^n, X^n, \tau^n) > 0$, the lhs of (25) is smaller than the lhs of (24). This proves it is harder to support good behavior with weaker punishment.

- Note that this is the first case where you have an aggregate endogenous state variable given by the transition function $h_{t+1} = f(h_t, X_t, \tau_t)$. It is endogenous since endogenous decisions like X_t influence the future state variable. In the next set of notes (Diamond's production economy), you will see another endogenous transition function for aggregate capital $K_{t+1} = (1-\delta)K_t + I_t$ where I_t is an endogenous gross investment choice (the analogue to X_t in these notes).

[10] Note that

$$
\begin{aligned}
& (1 + \beta + \beta^2 + ... + \beta^T) \\
\iff \quad & \frac{1 + \beta + \beta^2 + ... + \beta^T - \beta - \beta^2 - ... - \beta^T - \beta^{T+1}}{(1-\beta)} \\
\iff \quad & \frac{(1 - \beta^{T+1})}{(1-\beta)}
\end{aligned}
$$

where the second line follows from the first by multiplying by $\frac{1-\beta}{1-\beta}$.

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L6. Overlapping Generations in a Production Economy

The GDP accounting identity is C+I+G+NX=Y. We have focused only on private and public consumption till now (C+G). Investment fluctuates around 15-20% of US GDP. These notes introduce investment based on Diamond's (1965) overlapping generations model with production. This set of notes forms the basis of the neoclassical growth model without uncertainty that you will see a lot in macro (after adding stochastics).

1 Environment

- Population: Each period a unit measure of identical two period lived agents are born. There is zero population growth.
- Endowments: young agents of generation t are endowed with one unit of labor time.
- Technology:
  - The production technology is given by the Cobb-Douglas function:
$$Y_t \equiv F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$
    where K_t is physical capital, L_t is labor, and $\alpha \in (0,1)$.
  - In macro, α is chosen to match the capital share of income in the data. We will show this later.
  - Capital is storable. The capital stock evolves as $K_{t+1} = I_t + (1-\delta)K_t$ where I_t is gross investment and δ is depreciation.
  - Consumption and capital goods are perfect substitutes (e.g. corn). This implies that their relative price is 1.
  - The initial old hold the capital stock $\overline{K}_1$.
- Preferences: A representative agent of generation t has a utility function given by
$$\ln(c_t^t) + \beta \ln(c_{t+1}^t)$$
  where c_t^t and c_{t+1}^t is consumption of the generation t agent in youth and old age, respectively, and $\beta \in (0,1)$. Since preferences don't depend on leisure, the young agent will supply her unit of labor time inelastically (i.e. $n_t = 1$).

2 Planner's Problem

- The planner's problem is given by[1]:

$$\max_{c_1^0,(c_t^t,c_{t+1}^t,K_{t+1})_{t=1}^\infty} \ln(c_1^0)+\sum_{t=1}^{\infty}\left[\ln(c_t^t)+\beta\ln c_{t+1}^t\right]$$
$$c_t^{t-1}+c_t^t+K_{t+1}=F(K_t,1)+(1-\delta)K_t \tag{1}$$
$$c_t^{t-1},c_t^t,K_{t+1}\geq 0$$

- The Lagrangian associated with this problem is given by:

$$L=\ln(c_1^0)+\sum_{t=1}^{\infty}\left[\ln(c_t^t)+\beta\ln c_{t+1}^t\right]+\sum_{t=1}^{\infty}\theta_t\left[F(K_t,1)+(1-\delta)K_t-c_t^{t-1}-c_t^t-K_{t+1}\right]$$

- The first order conditions with respect to c_1^0, c_t^t, c_{t+1}^t, and K_{t+1} for $t\geq 1$ are:

$$c_t^t:\frac{1}{c_t^t}=\theta_t \tag{2}$$
$$c_{t+1}^t:\frac{\beta}{c_{t+1}^t}=\theta_{t+1} \tag{3}$$
$$K_{t+1}:\theta_t=\theta_{t+1}\left[F_K(K_{t+1},1)+1-\delta\right] \tag{4}$$

  - (2) and (3) imply for $t\geq 1$

$$\frac{1}{c_{t+1}^{t+1}}=\frac{\beta}{c_{t+1}^t}$$
$$\Longrightarrow\quad c_{t+1}^t=\beta c_{t+1}^{t+1} \tag{5}$$

  - (2) and (4) imply for $t\geq 1$

$$\frac{1}{c_t^t}=\frac{1}{c_{t+1}^{t+1}}\left[F_K(K_{t+1},1)+1-\delta\right]$$
$$\Longrightarrow\ \alpha K_{t+1}^{\alpha-1}+1-\delta=\frac{c_{t+1}^{t+1}}{c_t^t} \tag{6}$$

- Optimality (5) and (6) along with the resource constraint (1) is 3 equations in 3 unknowns $(c_t^t,c_{t+1}^t,K_{t+1})$.

- (5) and (1) imply

$$c_t^{t-1}+c_t^t+K_{t+1}=K_t^\alpha+(1-\delta)K_t\Longleftrightarrow c_t^t=\frac{K_t^\alpha+(1-\delta)K_t-K_{t+1}}{(1+\beta)} \tag{7}$$

[1] Here we are invoking the overtaking criterion from previous notes to consider this objective finite and hence well defined.

Then (7) into (6) implies

$$
\begin{aligned}
\alpha K_{t+1}^{(\alpha-1)} + 1 - \delta &= \frac{c_{t+1}^{t+1}}{c_t^t} = \frac{\frac{K_{t+1}^{\alpha}+(1-\delta)K_{t+1}-K_{t+2}}{(1+\beta)}}{\frac{K_t^{\alpha}+(1-\delta)K_t-K_{t+1}}{(1+\beta)}} \\
\Longleftrightarrow \left(\alpha K_{t+1}^{(\alpha-1)} + 1 - \delta\right) &\cdot \left(K_t^{\alpha} + (1-\delta)K_t - K_{t+1}\right) = K_{t+1}^{\alpha} + (1-\delta)K_{t+1} - K_{t+2} \\
&\Longleftrightarrow K_{t+2} + -(1+\alpha)K_{t+1}^{\alpha} + \alpha K_t^{\alpha} K_{t+1}^{(\alpha-1)} \qquad (8) \\
&+ (1-\delta)\left[\alpha K_{t+1}^{(\alpha-1)} K_t - K_{t+1} + K_t^{\alpha} + K_t(1-\delta) - K_{t+1}\right] = 0
\end{aligned}
$$

Hence the first best capital stock evolves as a second order nonlinear difference equation.

- There are two ways to find the steady state optimal capital stock.
  - The hard way; plug $K_t = \overline{K}^{SP}$ into (8).[2]
  - The easy way; since $\frac{c_{t+1}^{t+1}}{c_t^t} = 1$ in a steady state, use (6) to yield

$$\alpha \left(\overline{K}^{SP}\right)^{(\alpha-1)} + 1 - \delta = 1 \Longleftrightarrow \overline{K}^{SP} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}. \qquad (9)$$

3 Decentralized Competitive Economy

- At the beginning of any period t, production takes place by competitive firms which choose how much capital K_t to rent from the old at rate r_t and how much labor L_t to hire from the young at price w_t in order to maximize profits (denoted Π_t). The return on savings by the old is $r_t - \delta$.

[2] Plugging $K_t = \overline{K}^{SP}$ into (8) yields:

$$
\begin{aligned}
\overline{K}^{SP} - (1+\alpha)\left(\overline{K}^{SP}\right)^{\alpha} + \alpha\left(\overline{K}^{SP}\right)^{\alpha}\left(\overline{K}^{SP}\right)^{(\alpha-1)} & \\
+(1-\delta)\left[\alpha\left(\overline{K}^{SP}\right)^{(\alpha-1)}\overline{K}^{SP} - \overline{K}^{SP} + \left(\overline{K}^{SP}\right)^{\alpha} + \overline{K}^{SP}(1-\delta) - \overline{K}^{SP}\right] &= 0 \Longleftrightarrow \\
1 - (1+\alpha)\left(\overline{K}^{SP}\right)^{(\alpha-1)} + \alpha\left(\overline{K}^{SP}\right)^{2(\alpha-1)} + (1-\delta)\left[(\alpha+1)\left(\overline{K}^{SP}\right)^{(\alpha-1)} - 1 - \delta\right] &= 0 \Longleftrightarrow \\
\delta^2 - \delta(\alpha+1)\left(\overline{K}^{SP}\right)^{(\alpha-1)} + \alpha\left(\overline{K}^{SP}\right)^{2(\alpha-1)} &= 0 \Longleftrightarrow \\
\left(\alpha\left(\overline{K}^{SP}\right)^{\alpha-1} - \delta\right)\left(\left(\overline{K}^{SP}\right)^{\alpha-1} - \delta\right) &= 0 \Longrightarrow \\
\overline{K}_1^{SP} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}, \overline{K}_2^{SP} = \left(\frac{1}{\delta}\right)^{\frac{1}{1-\alpha}} &
\end{aligned}
$$

The resource constraint (1) implies zero aggregate consumption if $\overline{K}^{SP} = \left(\frac{1}{\delta}\right)^{\frac{1}{1-\alpha}}$. Therefore, the only steady state optimal capital stock is $\overline{K}^{SP} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}$.

- At the end of any period t, the young choose how much of their wage income to consume c_t^t and how much capital to save k_{t+1}^t. Profits, if there are any, are transferred to the old after which they consume their savings plus interest.

3.1 Firm Problem

- Firms choose K_t and L_t to maximize profits given by

$$\max_{K_t, L_t} \Pi_t = F(K_t, L_t) - w_t L_t - r_t K_t$$

- The first order conditions with respect to labor and capital yield labor demand and capital demand

$$w_t = F_L(K_t^d, L_t^d) = (1-\alpha)(K_t^d)^\alpha (L_t^d)^{-\alpha} \tag{10}$$

$$r_t = F_K(K_t^d, L_t^d) = \alpha (K_t^d)^{\alpha-1}(L_t^d)^{1-\alpha} \tag{11}$$

- Note that with constant returns to scale in production, firms earn zero profits in equilibrium (i.e. $\Pi_t = F(K_t^d, L_t^d) - w_t L_t^d - r_t K_t^d = 0$).

- We can estimate the parameter α off of the capital share in income. Specifically

$$\begin{aligned} \text{Capital Share} &= \frac{r_t K_t}{Y_t} = \frac{\alpha (K_t)^{\alpha-1}(L_t)^{1-\alpha} K_t}{(K_t)^\alpha (L_t)^{1-\alpha}} = \alpha, \\ \text{Labor Share} &= \frac{w_t L_t}{Y_t} = \frac{(1-\alpha)(K_t)^\alpha (L_t)^{-\alpha} L_t}{(K_t)^\alpha (L_t)^{1-\alpha}} = 1-\alpha. \end{aligned}$$

3.2 Agents Problem

- Again noting that preferences don't depend on leisure so agents inelasitically supply their unit of labor to firms at a given wage rate w_t, generation $t \geq 1$ agents choose how much to save k_{t+1}^t in youth, rent their capital and receive any profits from firms as well as sell the remainder of their undepreciated capital in old age in order to solve

$$\max_{c_t^t \geq 0, c_{t+1}^t \geq 0, k_{t+1}^t \geq 0} \ln c_t^t + \beta \ln c_{t+1}^t$$

$$c_t^t + k_{t+1}^t = w_t \tag{12}$$

$$c_{t+1}^t = (1 + r_{t+1} - \delta) k_{t+1}^t + \Pi_{t+1} \tag{13}$$

- Substituting for c_t^t and c_{t+1}^t from the budget constraints directly into the objective and neglecting inequality constraints, we have

$$\max_{k_{t+1}^t} \ln\left(w_t - k_{t+1}^t\right) + \beta \ln\left((1 + r_{t+1} - \delta) k_{t+1}^t + \Pi_{t+1}\right).$$

- The first order condition in k_{t+1}^t yields the "Euler equation":

$$u'(c_t^t) = \beta \cdot (1 + r_{t+1} - \delta)\, u'(c_{t+1}^t) \Longleftrightarrow \frac{1}{c_t^t} = \frac{\beta \cdot (1 + r_{t+1} - \delta)}{c_{t+1}^t} \tag{14}$$

  The left hand side is the marginal cost of saving (lost current utility) and the right hand side is the marginal benefit (PDV of future utility). (14) can also be rearranged to yield the MRS=MRT expression you've seen before.

- Neglecting the $k_{t+1}^t \geq 0$ constraint (to be verified), consolidating (12) and (13) gives you the life-time budget constraint:

$$c_t^t + \frac{c_{t+1}^t}{(1 + r_{t+1} - \delta)} = w_t + \frac{\Pi_{t+1}}{(1 + r_{t+1} - \delta)} \tag{15}$$

- Substituting c_{t+1}^t from (14) into (15) yields

$$\begin{aligned} c_t^t &= \frac{w_t + \frac{\Pi_{t+1}}{(1+r_{t+1}-\delta)}}{(1+\beta)}, \\ c_{t+1}^t &= \frac{\beta\left((1 + r_{t+1} - \delta)\, w_t + \Pi_{t+1}\right)}{(1+\beta)} \end{aligned}$$

- Savings which support this consumption profile are given by

$$k_{t+1}^t = \frac{\beta w_t - \frac{\Pi_{t+1}}{(1+r_{t+1}-\delta)}}{(1+\beta)}. \tag{16}$$

- Notice that if $\Pi_{t+1} = 0$ (which is consistent with a competitive equilibrium), then savings are simply a constant fraction $\beta/(1+\beta)$ of current earnings w_t.
  - Thus the constraint $k_{t+1}^t \geq 0$ is not binding.
  - This provides a microfoundation for the exogenous constant savings rate used in the Nobel prize winning Solow (1956) model under a set of (strong) assumptions. It also establishes that the neglecting the non-negativity constraint on capital is valid in a competitive equilibrium.

3.3 Market Clearing

- Goods, Capital, and Labor market clearing are given by

$$\begin{aligned} &\text{Demand} = \text{Supply} \\ c_t^t + c_t^{t-1} + K_{t+1} - (1-\delta)K_t &= K_t^{\alpha} L_t^{1-\alpha} \qquad (17) \\ K_{t+1} &= k_{t+1}^t \qquad (18) \\ L_t &= 1 \qquad (19) \end{aligned}$$

Note that the goods market clearing condition (17) is like the national accounting identity $C + I = Y$. The asset market clearing condition (18) is consistent with the flow of savings by all households being equal to flow of investment demanded by firms.[3]

3.4 Equilibrium Definition

- A competitive equilibrium is a sequence

$\{c_1^{0*}, (c_t^{t*}, c_{t+1}^{t*}, k_{t+1}^{t*})_{t=1}^{\infty}, (K_t^*, L_t^*, Y_t^*, \Pi_t^*)_{t=1}^{\infty}, (r_t^*, w_t^*)_{t=1}^{\infty}\}$ such that:
(i) Households optimize,
(ii) Firms maximize profits,
(iii) Markets clear.

3.5 Equilibrium Dynamics

- Using (10), (18), (19) with $\Pi_t = 0$ for all t into (16) we have

$$K_{t+1}^* = \frac{\beta(1-\alpha)(K_t^*)^{\alpha}}{(1+\beta)} \tag{20}$$

Thus, the decentralized law of motion for capital follows a first order nonlinear difference equation.[4]

- There are 2 solutions to (20).
  - One is $K_t^* = 0$,
  - In (K_t^*, K_{t+1}^*) space, it is simple to see that $\frac{dK_{t+1}^*}{dK_t^*} > 0$ and $\frac{d^2K_{t+1}^*}{dK_t^{*2}} < 0$. Further, $\lim_{K_t^* \longrightarrow 0} \frac{dK_{t+1}^*}{dK_t^*} > 1$ and $\lim_{K_t^* \longrightarrow \infty} \frac{dK_{t+1}^*}{dK_t^*} = 0$.
  - The dynamics are given in Figure L6.1. For any $\overline{K}_1 > 0$, the unique path of capital converges to a steady state $\overline{K}^*$ given by

$$\overline{K}^* = \frac{\beta(1-\alpha)(\overline{K}^*)^{\alpha}}{(1+\beta)} \Longleftrightarrow \overline{K}^* = \left(\frac{\beta(1-\alpha)}{(1+\beta)}\right)^{\frac{1}{1-\alpha}}. \tag{21}$$

[3] That is, the flow of investment is given by

$$K_{t+1} - (1-\delta)K_t = K_t^{\alpha} L_t^{1-\alpha} - c_t^t - c_t^{t-1}.$$

The flow of savings is

$$k_{t+1}^t - k_t^{t-1}$$

where

$$k_t^{t-1} = (1-\delta)K_t.$$

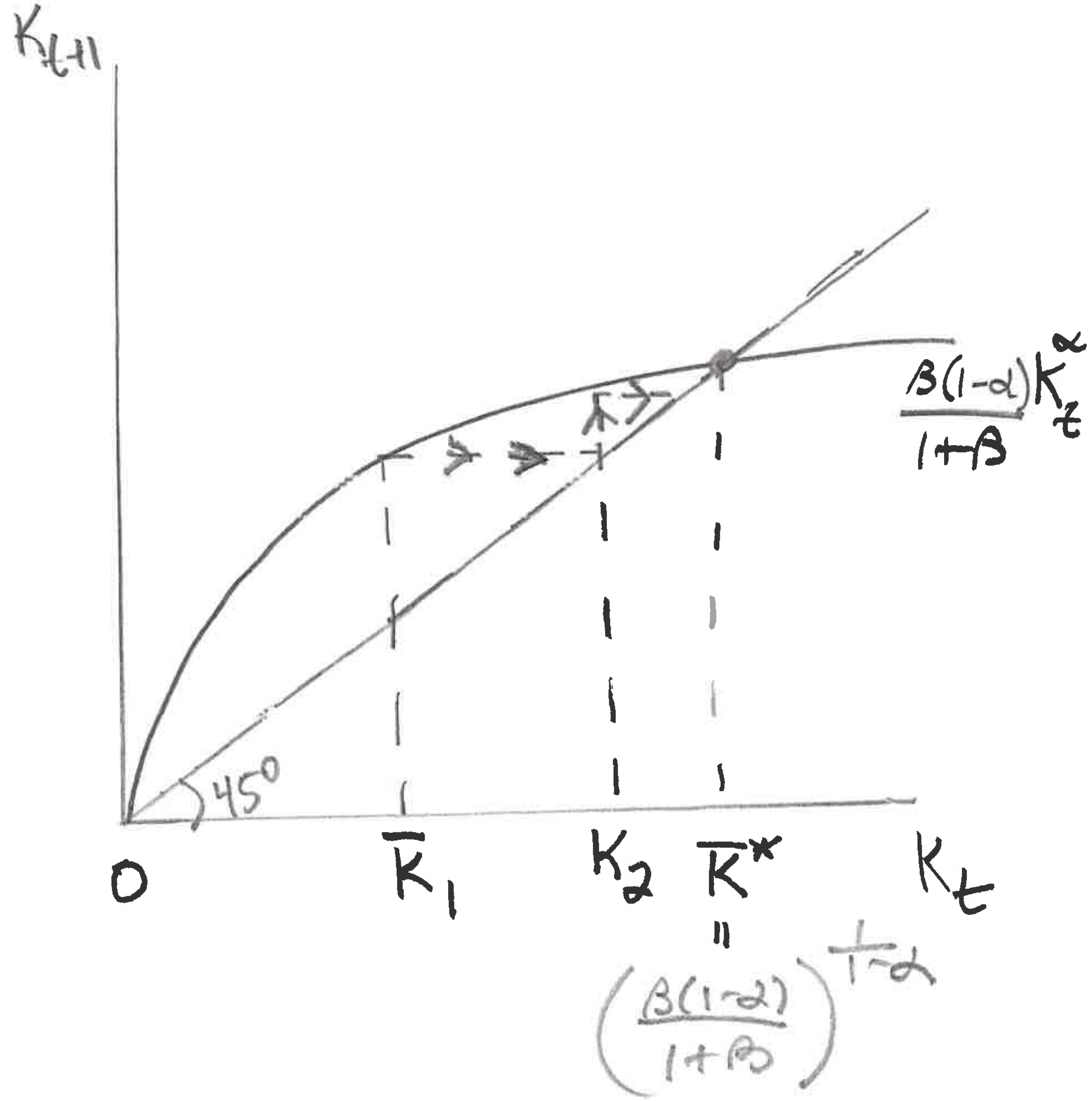
Hence

$$K_{t+1} - (1-\delta)K_t = k_{t+1}^t - (1-\delta)K_t$$

which is (18).

[4] While the competitive equilibrium law of motion in this OG economy is a first order difference equation, the competitive equilibrium law of motion in a dynastic economy (or nonstochastic growth model) is a second order equation.

Figure L6.1



K_{t+1}
$\frac{\beta(1-\alpha)}{1+\beta}K_t^{\alpha}$
45°
0
$\bar{K}_1$
K_2
$\bar{K}^*$
K_t
$= \left(\frac{\beta(1-\alpha)}{1+\beta}\right)^{\frac{1}{1-\alpha}}$

- It is simple to establish that the steady state is stable (the figure does it for $\overline{K}_1 < \overline{K}^*$ and you should do it for $\overline{K}_1 > \overline{K}^*$).
- This provides another example of convergence of an endogenous variable starting at an initial condition different from the steady state (recall Kehoe's notes where $p_1 \neq p^{ss}$).

3.6 Comparing the First Best and Competitive Steady State Allocations

- Comparing (9) and (21), there is over-accumulation of capital in the long run in a competitive equilibrium if

$$\overline{K}^{SP} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} < \left(\frac{\beta(1-\alpha)}{(1+\beta)}\right)^{\frac{1}{1-\alpha}} = \overline{K}^* \tag{22}$$

or $\alpha/\delta < \beta(1-\alpha)/(1+\beta)$. An alternative way to think about this condition, since the marginal product of capital is decreasing in capital, is that interest rates in the competitive equlibrium are lower than what a social planner would choose for the economy.

- What would explain why the long run capital stock is higher in the competitive equilibrium than the planner's problem (i.e. a violation of the first welfare theorem)? Recall that in a competitive equilibrium where $\Pi_t = 0$, the fact that wages are paid only in the first period means we are in the Samuelson case of dynamic inefficiency where the real return was negative.

- How can we think about this inefficiency?

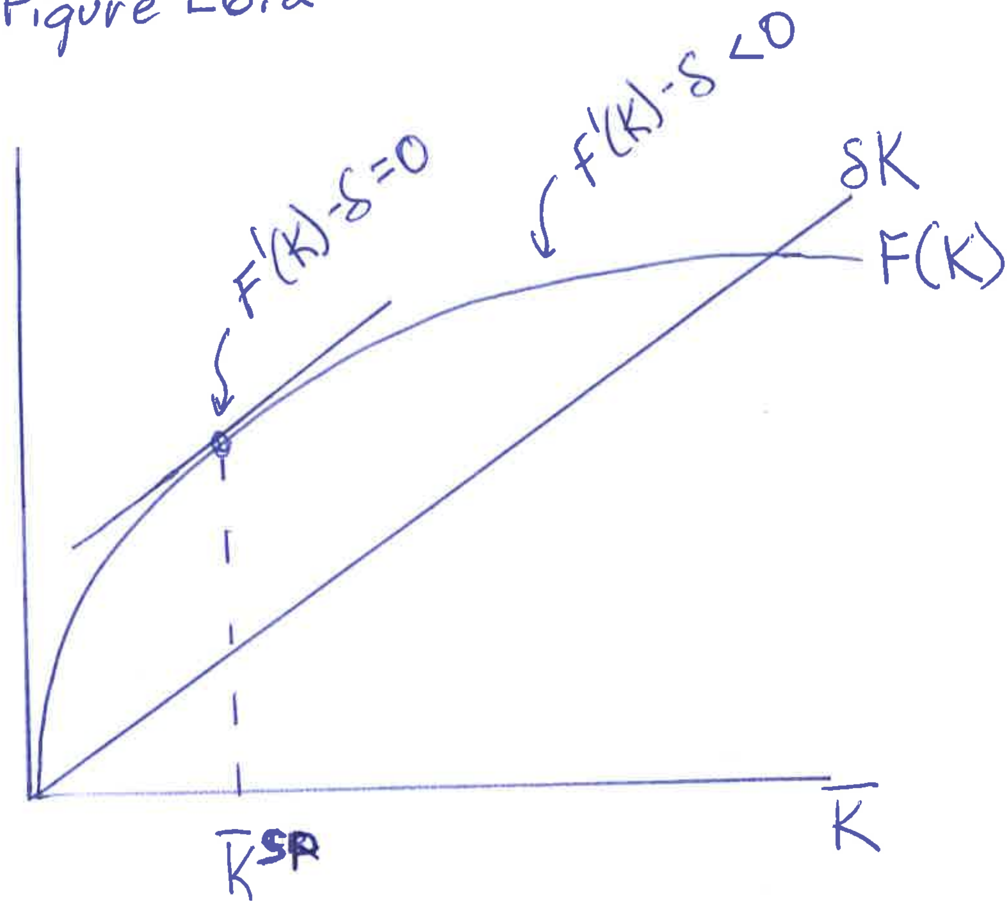
  - In a steady state, resource feasibility (or goods market clearing) implies

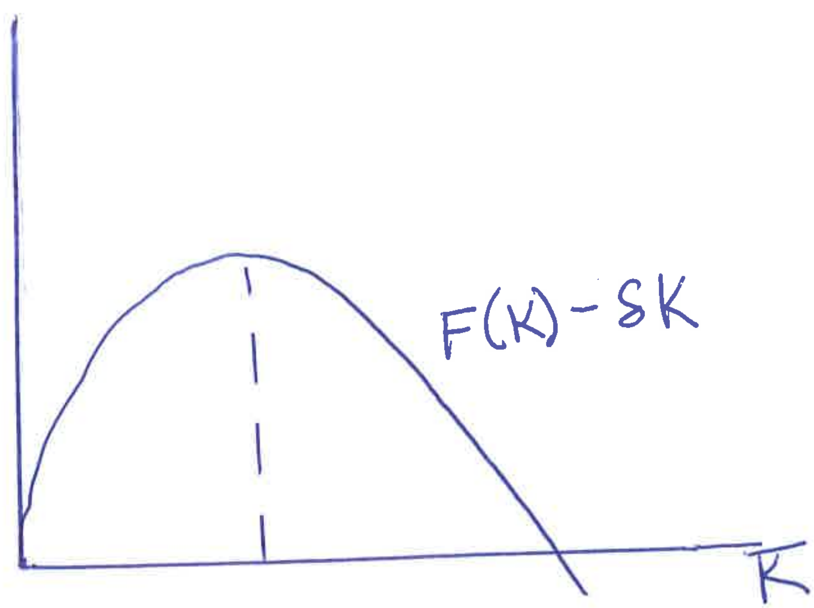
$$C + \widehat{K} - (1-\delta)\widehat{K} = F\left(\widehat{K}\right) \Longleftrightarrow C = F\left(\widehat{K}\right) - \delta\widehat{K}$$

  where C is aggregate consumption.
  - Suppose the return to saving is negative (i.e. capital is such that $F'\left(\widehat{K}\right) - \delta < 0$ which implies $\frac{dC}{d\widehat{K}} < 0$), then it is possible to decrease the capital stock and increase aggregate consumption.[5] See Figure L6.2.
  - Agents from generation t face wages determined by capital stock decisions of agents from generation $t-1$. If generation $t-1$ "over-saves" then wages for generation t will be higher, inducing higher

[5] Note that at $F'\left(\widehat{K}\right) - \delta = 0$, $\widehat{K} = \left(\frac{\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} = K^{SP}$. With zero population growth, this level of capital is sometimes called the Golden rule.

Figure L6.2





savings for generation t. The savings decisions of each generation create pecuniary externalities on future generations. These pecuniary externalities are second order in an infinitely lived agent model.

- This provides a possible explanation for why capital taxes, which make the marginal benefit of saving in the form of capital lower, may be used to implement an optimal allocation.
- However, in U.S. data, the capital share (α) is roughly $0.333-0.4$, the annual depreciation rate around 0.10, and the discount factor (measured off the real T-bill rate) is roughly $0.96-0.99$. Then the lhs of (22) is about 3 while the rhs is less than 1. This suggests we consider whether the log preference specification is appropriate since with more risk aversion, agents in the competitive equilibrium may overaccumulate to ensure sufficient consumption smoothing over their lifetime.[6]

4 Alternative Decentralization (Claims to Firm Profits)

- One way to justify the assumption that profits are distributed to the old is to assume there is a stock market for claims to the profits of firms after renting capital. Normalize the initial supply of stocks $\overline{S}=1$ by each firm and assume the share of stock is held by the initial old and does not change over time (i.e. there are no "seasoned equity issuances"). Stocks are a claim to firm profits (paid as dividends) to their shareholders when they are old, i.e. $d_t=\Pi_t$ per share at time t. Generation t' agents choose how many shares to buy $s^t_{t+1}\in[0,1]$ at the share price of p_t. When they grow old, they receive dividend $d_{t+1}s^t_{t+1}$ and sell their shares (i.e. at p_{t+1}).
- The agent's budget constraints are then given by

$$c^t_t+k^t_{t+1}+p_t s^t_{t+1}=w_t$$
$$c^t_{t+1}=\left(d_{t+1}+p_{t+1}\right)s^t_{t+1}+(1+r_{t+1}-\delta)k^t_{t+1}$$

- Attaching multiplier λ_t to the first period budget constraint and μ_{t+1} to the second period budget constraint, the first order conditions with respect to savings in the form of capital and stocks (neglecting corners) are:

$$k^t_{t+1} \ :\ \lambda_t=\mu_{t+1}\left(1+r_{t+1}-\delta\right)$$
$$s^t_{t+1} \ :\ \lambda_t p_t=\mu_{t+1}\left(d_{t+1}+p_{t+1}\right)$$

[6] That is, with more general preferences the interest rate and depreciation enter the household's first order conditions. In the CRRA case where $u(c_t)=\frac{(c_t)^{1-\gamma}}{1-\gamma}$ with $\gamma>1$, household savings in (16) are given by

$$k^t_{t+1}=\frac{\beta^{\frac{1}{\gamma}}\left(1+r_{t+1}-\delta\right)^{\frac{1-\gamma}{\gamma}}w_t}{1+\beta^{\frac{1}{\gamma}}\left(1+r_{t+1}-\delta\right)^{\frac{1-\gamma}{\gamma}}}.$$

- The only way these two conditions can be satisfied is if

$$1 + r_{t+1} - \delta = \frac{d_{t+1} + p_{t+1}}{p_t} \tag{23}$$

which states the return on capital should be the same as the return on stock (in this nonstochastic environment). This is a no arbitrage condition between assets.

- There is another market clearing condition

$$s_{t+1}^t = 1. \tag{24}$$

- In an equilibrium where $\Pi_t = 0$, then $d_t = 0$. But with a positive marginal product of capital, $r_{t+1} > 0$, the only way to get people to hold the stock is for them to expect stock prices to keep rising to generate positive capital gains. This would imply a stock market bubble since prices are rising despite claims to profits which are zero. It is similar to how beliefs can sustain a monetary equilibrium.

- To bring fundamentals back to the stock market, we could get rid of the capital rental market and let the profits be defined by $\Pi_t = F(K_t, L_t) - w_t L_t$ which are non-zero.

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L7. Idiosyncratic Uncertainty in an Overlapping Generations Economy

These notes introduce idiosyncratic uncertainty with a unit measure of agents in a simple OG model. It focuses on how asset market structure (fully state contingent claims versus noncontingent claims) can or cannot achieve allocational efficiency.

Since this is the first time I introduce uncertainty, let me be more specific. There are two primary types of randomness we study in macro:

- Idiosyncratic uncertainty: each agent gets an independent draw.
- Aggregate uncertainty: each agent gets the same draw (like perfectly correlated idiosyncratic shocks).

With a unit measure of agents (like an infinite number of draws in statistics where the law of large numbers applies), idiosyncratic uncertainty does not generate aggregate uncertainty, which makes it easy to analyze.

With uncertainty, there arises questions about whether asset markets can provide insurance to smooth consumption against all possible realizations of the shocks. The idea of complete markets is that there are enough (state contingent) assets to provide such insurance whereas with incomplete markets assets are not fully contingent and hence may not provide perfect consumption insurance.

1 Environment

- Population: Two period OG model without population growth. Each generation has a unit measure of ex-ante identical agents.
- Technology:
  - The time endowment is one for both periods.
  - When an agent of generation t is young, they choose the fraction of time $H_t^t \in [0,1]$ to be invested in human capital accumulation and the fraction of time to work $n_t^t \in [0,1]$ where $1 = n_t^t + H_t^t$.
  - When old, an agent can be in one of two states $s \in \{e, u\}$ which determines their employment status. Specifically, agents are employed (i.e. $s = e$) with probability $1 - \upsilon$ and are unemployed (i.e. $s = u$) with probability υ.
    - Note that while agents are ex-ante identical, they are ex-post heterogeneous within a generation.

- Since there is a unit measure of agents and these shocks are identical and independently drawn across agents, this implies the unemployment rate is υ for old people of any generation (by the law of large numbers the probabilities imply nonrandom population fractions).[1]
    - Note that while this assumption implies that every single person is uncertain about their future state (of employment), there is no aggregate uncertainty (about the fractions of employed and unemployed).
- Productivity (units of nonstorable goods produced per unit of time) for young agents is 1, whereas productivity for old agents is the amount of human capital accumulated H_t^t multiplied by a constant A and 0 productivity if they are unemployed. Thus output for a young person is $y_t^t = n_t^t$ but for an old person it is state contingent $y_{t+1}^{s,t}$;that is, for an employed old person it is $y_{t+1}^{e,t} = AH_t^t n_{t+1}^{e,t}$ and for an unemployed old person it is $y_{t+1}^{u,t} = 0$.
- We will assume that $A(1-\upsilon) > 1$ so that the expected net return to acquiring human capital is positive. This is consistent with an education wage premium we see in the data.

- Preferences: For simplicity, we assume there is no utility from consumption in youth and no utility from leisure in any period. The (expected) utility of agents of generation t is given by

$$[(1-\upsilon)\log c_{t+1}^{e,t} + \upsilon \log c_{t+1}^{u,t}]$$

  where $c_{t+1}^{e,t}$ and $c_{t+1}^{u,t}$ is state contingent consumption of the old when employed and unemployed, respectively. To be clear about the notation, consumption depends on age and state (i.e. $c_{t+1}^{s,t}$) just like individual output.

- Timing:
    - When young:
        - Choose how much to work n_t^t and accumulate human capital H_t^t.
        - In the decentralized version, agents choose assets.
    - When old:
        - Agents receive their employment shock s and choose how much to work $n_{t+1}^{s,t}$.
        - In the decentralized version, asset transactions are completed
        - Agents consume.

- Note that neither preferences nor technologies depend explicitly on time.

[1] This is not formally true, but here we will assume it. See Al-Najjar, N. (2004) "Aggregation and the law of large numbers in large economies", *Games and Economic Behavior*, 47, p. 1-35.

2 Planner's Problem

- The social planner chooses $\{c_t^{e,t-1}, c_t^{u,t-1}, n_t^t, H_t^t\}_{t=1}^{\infty}$ to solve

$$\max \sum_{t=1}^{\infty} (1-\upsilon)\log c_t^{e,t-1} + \upsilon \log c_t^{u,t-1} \tag{1}$$

subject to

$$\begin{aligned} (1-\upsilon)c_t^{e,t-1} + \upsilon c_t^{u,t-1} &= n_t^t + (1-\upsilon)AH_{t-1}^{t-1} && (2) \\ 1 &= n_t^t + H_t^t && (3) \end{aligned}$$

- Note first that while $1-\upsilon$ and υ are probabilities faced by individuals in their objective function, they are population fractions in the planner's problem by the law of large numbers. Further, note that by the law of large numbers, there is no (aggregate) uncertainty facing the planner since everything is a known fraction even though there is uncertainty facing individuals.

- Given logarithmic utility when old, the nonnegativity constraints on $c_{t+1}^{e,t}$ and $c_{t+1}^{u,t}$ can be neglected. Then, after substituting $n_t^t = 1 - H_t^t$, the Lagrangian for this problem is

$$\begin{aligned} &\sum_{t=1}^{\infty} (1-\upsilon)\log c_t^{e,t-1} + \upsilon \log c_t^{u,t-1} \\ &+\lambda_t^1 \left[(1-H_t^t) + (1-\upsilon)AH_{t-1}^{t-1} - (1-\upsilon)c_t^{e,t-1} - \upsilon c_t^{e,t-1}\right] \\ &+\lambda_t^2 H_t^t + \lambda_t^3(1-H_t^t) \end{aligned}$$

- The Kuhn-Tucker first-order conditions are given by

$$\begin{aligned} H_t^t &: \quad -\lambda_t^1 + \lambda_{t+1}^1(1-\upsilon)A + \lambda_t^2 - \lambda_t^3 = 0 && (4) \\ c_t^{e,t-1} &: \quad \frac{(1-\upsilon)}{c_t^{e,t-1}} = \lambda_t^1(1-\upsilon) && (5) \\ c_t^{u,t-1} &: \quad \frac{\upsilon}{c_t^{u,t-1}} = \lambda_t^1 \upsilon && (6) \end{aligned}$$

plus complementary slackness conditions

$$\lambda_t^2 H_t^t = 0, \qquad \lambda_t^3(1-H_t^t) = 0$$

- What is the optimal stationary allocation?

- In a stationary solution to the planner's problem, $\lambda_t^1 = \lambda_{t+1}^1 > 0$ (since utility is increasing in consumption the constraints bind). Therefore, from condition (4), we have

$$\lambda_t^3 - \lambda_t^2 = [A(1-\upsilon) - 1]\lambda_t^1 > 0 \tag{7}$$

Since by definition, $\lambda_t^3, \lambda_t^2 \geq 0$ and they cannot both be positive at same time (H_t^t cannot be both 0 and 1), condition (7) implies that $\lambda_t^3 > 0, \lambda_t^2 = 0$. Therefore, $\overline{H}_t^t = 1$, which implies $\overline{n}_t^t = 0$. Notice this is where the assumption $A(1-\upsilon) > 1$ is important. Since human capital accumulation is so productive (i.e. A is sufficiently high), the efficient thing is to go to school in order to maximize resources next period rather than to use young people's time to provide resources for this period.

- From conditions (5) and (6), we have $\overline{c}_t^{e,t-1} = \overline{c}_t^{u,t-1}$. Thus, there is perfect consumption insurance; the planner insures against unemployment risk by "taxing" those who are employed (the "lucky" employed are cross-subsidizing the "unlucky" unemployed). Plug in the solution $\overline{H}_t^t = 1$ into the feasibility constraint, we can derive the solution of the social planner's problem in the steady state as

$$\overline{c}_t^{e,t-1} = \overline{c}_t^{u,t-1} = (1-\upsilon)A, \quad \overline{H}_t^t = 1 \tag{8}$$

3 Competitive Equilibrium with State Contingent Claims

- Now consider an environment with the same preferences and technologies but is decentralized as a competitive equilibrium with *Arrow-Debreu securities*. The securities (claims to consumption in period $t+1$ in the employed state ($a_{t+1}^{e,t}$) or unemployed state ($a_{t+1}^{u,t}$)) are traded when the agent of generation t is born at prices q_t^e and q_t^u respectively. These securities allow the young to also sell claims to their future income and their current income in the claims market. Specifically:
  - If $a_{t+1}^{s,t} > 0$, the agent gives up $q_t^s a_{t+1}^{s,t}$ units of the good at time t in exchange for receiving $a_{t+1}^{s,t}$ goods at $t+1$.
  - If $a_{t+1}^{s,t} < 0$, the agent receives $q_t^s \left|a_{t+1}^{s,t}\right|$ units of the good at time t in exchange for giving up $\left|a_{t+1}^{s,t}\right|$ goods at $t+1$.
- This is known as a "complete markets" economy since there is a claim for every possible random state of the world $s \in \{e, u\}$.
- For generation t, facing prices q_t^e and q_t^u, the problem is to choose $a_{t+1}^{e,t}, a_{t+1}^{u,t}, H_t^t \in [0,1]$ to solve

$$\max (1-\upsilon)\log c_{t+1}^{e,t} + \upsilon \log c_{t+1}^{u,t}$$

subject to

$$\begin{aligned} q_t^e a_{t+1}^{e,t} + q_t^u a_{t+1}^{u,t} &= n_t^t & (9)\\ c_{t+1}^{e,t} &= AH_t^t + a_{t+1}^{e,t} & (10)\\ c_{t+1}^{u,t} &= a_{t+1}^{u,t} & (11)\\ n_t^t + H_t^t &= 1 & (12) \end{aligned}$$

as well as $c_{t+1}^{e,t} \geq 0, c_{t+1}^{u,t} \geq 0, H_t^t \in [0,1]$.

- Since there are no constraints on $a_{t+1}^{e,t}$ and $a_{t+1}^{u,t}$, we can re-write the constraint set (9)-(12) as the consolidated (or in the language of Ljungqvist and Sargent the "time 0 Arrow-Debreu") budget constraint:

$$q_t^e c_{t+1}^{e,t} + q_t^u c_{t+1}^{u,t} = 1 - H_t^t + q_t^e A H_t^t \tag{13}$$

- Definition. A **Competitive Arrow-Debreu (Complete Markets) Equilibrium** is given by

$$\{\{q_t^e, q_t^u\}_{t=1}^{\infty}, \{c_t^{e,t-1}, c_t^{u,t-1}, n_t^t, H_t^t\}_{t=1}^{\infty}\}$$

  such that

  1. Given $\{q_t^e, q_t^u\}_{t=1}^{\infty}$, the allocation solves the utility maximization problem of each generation
  2. Markets clear every period, i.e.

$$\begin{aligned}(1-\upsilon)c_t^{e,t-1} + \upsilon c_t^{u,t-1} &= n_t^t + (1-\upsilon)AH_{t-1}^{t-1} && (14)\\ (1-\upsilon)a_t^{e,t-1} + \upsilon a_t^{u,t-1} &= 0 && (15)\end{aligned}$$

- Does the competitive equilibrium implement the planner's solution? Since there is a complete set of asset markets which spans the idiosyncratic uncertainty and there are no other frictions, we know by an application of the Second Welfare theorem we should be able to implement the planner's solution as a competitive equilibrium.

- We begin by substituting the budget constraints, then the Lagrangian of the problem is given by

$$\begin{aligned}\mathcal{L} &= (1-\upsilon)\log c_{t+1}^{e,t} + \upsilon \log c_{t+1}^{u,t}\\ &+ \lambda_t^1[(1-H_t^t) - q_t^e(c_{t+1}^{e,t} - AH_t^t) - q_t^u c_{t+1}^{u,t}]\\ &+ \lambda_t^2 H_t^t + \lambda_t^3(1-H_t^t)\end{aligned}$$

- The first-order conditions are

$$\begin{aligned}H_t^t &: \lambda_t^1(-1+q_t^e A) + \lambda_t^2 - \lambda_t^3 = 0 && (16)\\ c_{t+1}^{e,t} &: \frac{1-\upsilon}{c_{t+1}^{e,t}} = q_t^e \lambda_t^1 && (17)\\ c_{t+1}^{u,t} &: \frac{\upsilon}{c_{t+1}^{u,t}} = q_t^u \lambda_t^1 && (18)\end{aligned}$$

- To determine if the planner's solution satisfies these equations and market clearing, we ask if the allocation in (8) satisfies (14), (16)-(18).

- If one examines the first order conditions for the planners problem (4-6) and the first order conditions for the AD equilibrium, they are identical if $q_t^e = 1 - \upsilon$ and $q_t^u = \upsilon$.

- So let's assume $\widehat{q}_t^e = (1 - \upsilon)$, $\widehat{q}_t^u = \upsilon$ and verify these prices are consistent with a competitive equilibrium. Similar to the procedure of solving the social planner's problem, from (16) and the assumption $A(1 - u) > 1$, we have $\lambda_t^3 > 0$, which implies $\widehat{H}_t^t = 1$ and $\widehat{n}_t^t = 0$.

- Combining (17) and (18) gives

$$\frac{q_t^e c_{t+1}^{e,t}}{q_t^u c_{t+1}^{u,t}} = \frac{1 - \upsilon}{\upsilon}$$

  Which by our assumptions on $\widehat{q}_t^e$ and $\widehat{q}_t^u$ imply $c_{t+1}^{e,t} = c_{t+1}^{u,t}$.

- Inserting into market clearing gives

$$\widehat{c}_{t+1}^{e,t} = \widehat{c}_{t+1}^{u,t} = A(1 - \upsilon) \tag{19}$$

- Therefore, prices $\widehat{q}_t^e = (1 - \upsilon)$, $\widehat{q}_t^u = \upsilon$ and allocations $\widehat{c}_{t+1}^{e,t} = \widehat{c}_{t+1}^{u,t} = A(1 - \upsilon)$, $\widehat{H}_t^t = 1$ and $\widehat{n}_t^t = 0$ are indeed consistent with a competitive equilibrium.

- The key thing to see here is that this competitive Arrow-Debreu equilibrium can implement the social planner's solution.

- As an aside, this example provides a simple way to see the idea behind Walras Law. Specifically, plug budget constraint equations (10) and (11) into the goods market clearing condition (14) with $n_t^t = 0$ and you get the asset market clearing condition (15).

4 Competitive Monetary Equilibrium

- Now we ask if a different asset market structure with a noncontingent asset (as opposed to the state contingent claims studied in Section 3) can implement the planner's solution.

- Consider an environment with the same preferences and technologies but there are competitive markets in goods and money. There is a constant money supply $M_t^s = M^s > 0$. Let $M_{t+1}^t \geq 0$ denote the money chosen by an agent of generation t in period t to be used for consumption in period $t+1$. The price of consumption goods in period t is denoted p_t. Hence any output produced or consumed by an agent is priced at p_t (e.g. a young agent of generation t has value of her production given by $p_t(1 - H_t^t)$).

- Note that money holdings M_{t+1}^t are a "noncontingent" asset; that is, they pay off the same amount in every state in period $t+1$ (there is no s superscript). This is known as an "incomplete markets" economy because this one non-contingent asset does not span the two possible states $s \in \{e, u\}$.

- For generation t facing prices p_t and p_{t+1}, the problem is to choose $c_{t+1}^{e,t}, c_{t+1}^{u,t}, c_t^t, H_t^t \in [0,1], M_{t+1}^t \geq 0$ given by

$$\max(1-\upsilon)\log c_{t+1}^{e,t} + \upsilon \log c_{t+1}^{u,t}$$

subject to

$$\begin{aligned} M_{t+1}^t &= p_t(1-H_t^t) && (20) \\ p_{t+1}c_{t+1}^{e,t} &= p_{t+1}AH_t^t + M_{t+1}^t && (21) \\ p_{t+1}c_{t+1}^{u,t} &= M_{t+1}^t && (22) \end{aligned}$$

and nonnegativity constraints on consumption.

- Definition. A **competitive monetary (incomplete markets) equilibrium** is given by

$$\{\{p_t\}_{t=1}^{\infty}, \{c_t^{e,t-1}, c_t^{u,t-1}, H_t^t, M_{t+1}^t\}_{t=1}^{\infty}\}$$

such that

1. Given $\{p_t\}_{t=1}^{\infty}$, the allocation solves the utility maximization problem of each generation
2. Markets clear every period, i.e.

$$(1-\upsilon)c_t^{e,t-1} + \upsilon c_t^{e,t-1} = (1-H_t^t) + (1-\upsilon)AH_{t-1}^{t-1}$$

$$M_{t+1}^t = \overline{M}$$

- The key observation here is that $H_t^t = 1$ is never optimal. When $H_t^t = 1$, non-negativity of money holdings implies that $M_{t+1}^t = 0$. This leads to $c_{t+1}^{u,t} = 0$ which leads to negative infinity utility. Thus we can ignore the constraints $1 \geq H_t^t$ and $M_{t+1}^t \geq 0$. This also implies that the decentralized allocation of human capital $H_t^t < 1$ will differ from the social planner's allocation $H_t^t = 1$.

- Given the reasoning above and using the notation that $1 + \pi_t = \frac{p_{t+1}}{p_t}$, substituting the budget constraints into the objective function, we can rewrite the utility maximization problem as:

$$\max_{H_t^t}(1-\upsilon)\log\left(AH_t^t + \frac{(1-H_t^t)}{1+\pi_t}\right) + \upsilon\log\left(\frac{1-H_t^t}{1+\pi_t}\right)$$

with first-order condition given by

$$\frac{(1-\upsilon)(A(1+\pi_t)-1)}{A\widetilde{H}_t^t(1+\pi_t)+(1-\widetilde{H}_t^t)} = \frac{\upsilon}{1-\widetilde{H}_t^t} \Longleftrightarrow \widetilde{H}_t^t = \left[\frac{A(1+\pi_t)(1-\upsilon)-1}{A(1+\pi_t)-1}\right] \tag{23}$$

- This gives the allocation

$$\widetilde{c}_{t+1}^{e,t} = A(1-\upsilon), \quad \widetilde{c}_{t+1}^{u,t} = \frac{A\upsilon}{A(1+\pi_t)-1} = \frac{\widetilde{M}_{t+1}^t}{p_{t+1}} \equiv \widetilde{m}_{t+1}^t \tag{24}$$

- Consider a stationary equilibrium with constant prices (i.e. where $\pi_t = 0$). Comparing allocations from the AD equilibrium (19) and the monetary equilibrium (24), we see that while consumption by employed agents are the same in the AD versus monetary equilibrium (i.e. $\widetilde{c}_{t+1}^{e,t} = \widehat{c}_{t+1}^{e,t}$), consumption by the unemployed is lower in the monetary equilibrium than the AD equilibrium (i.e. $\widetilde{c}_{t+1}^{u,t} < \widehat{c}_{t+1}^{u,t}$). This is because aggregate output is lower in the monetary equilibrium than the competitive equilibrium due to lower human capital accumulation (i.e. $\widetilde{H}_t^t < 1 = \widehat{H}_t^t$).

- If one were to use cross-sectional data on consumption of the employed and unemployed as well as college/non-college participation rates, one might say that the data is more likely generated by a model with incomplete (noncontingent) asset markets than one with complete (contingent) asset markets.

- The important takeaway from this analysis is that in the absence of frictions, a complete set of contingent claims in the Arrow-Debreu equilibrium can implement the planner's allocation while it may not be possible for noncontingent assets like money (or bonds) to implement the efficient allocation.

- This can be illustrated simply in Figure 7.1.

  - The steady state allocation from the AD economy implies a budget constraint (13) which satisfies

$$c_{t+1}^{u,t} = \frac{q_t^e}{q_t^u}AH_t^t - \frac{q_t^e}{q_t^u}c_{t+1}^{e,t} = \frac{(1-\upsilon)}{\upsilon}A - \frac{(1-\upsilon)}{\upsilon}c_{t+1}^{e,t}$$

  - The steady state allocation from the monetary economy implies a budget set which satisfies[2]

$$c_{t+1}^{u,t} = \frac{A}{A-1} - \frac{1}{A-1}c_{t+1}^{e,t}$$

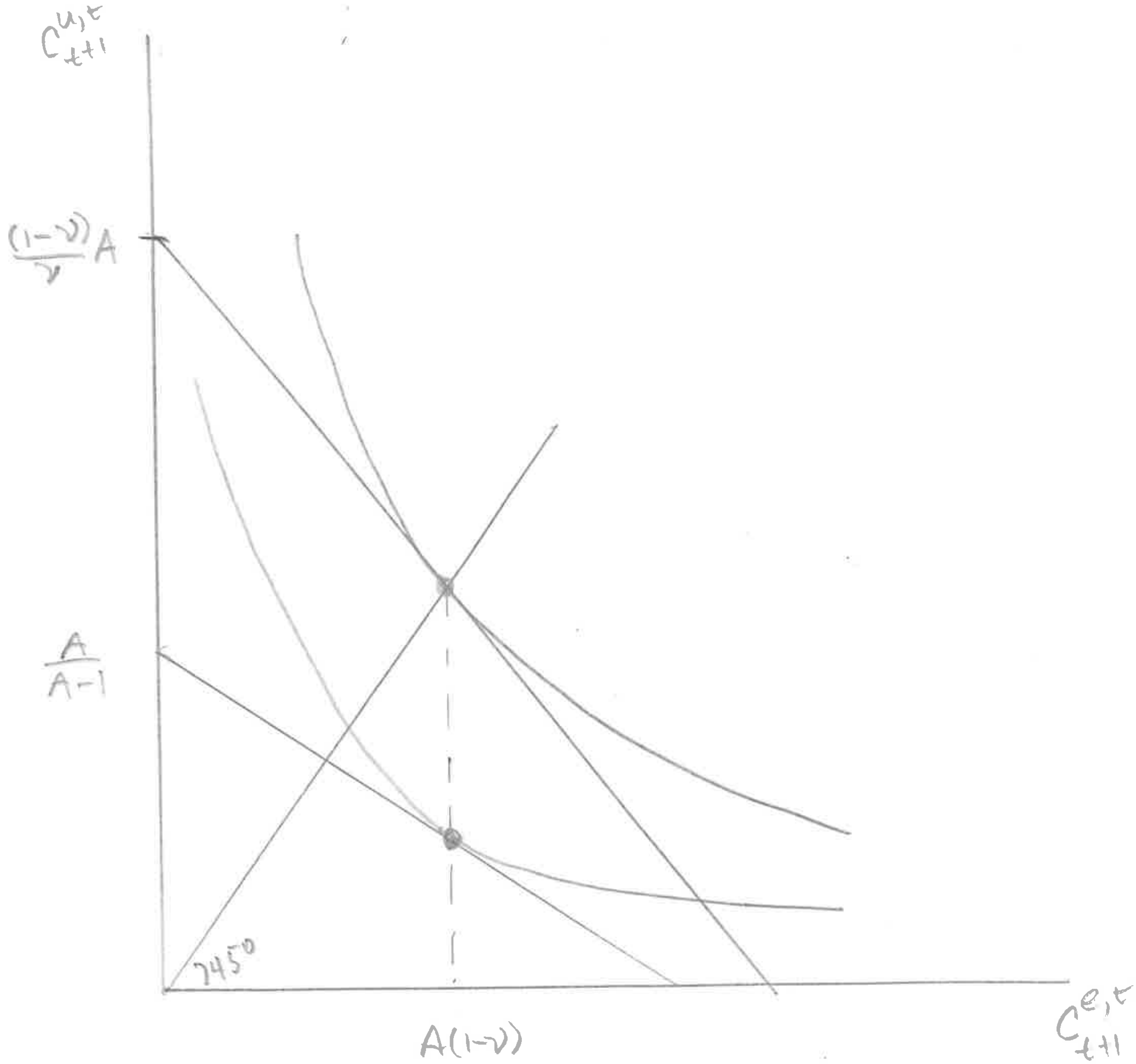
[2]Substituting for H_t^t from (20) into (21) using m_{t+1}^t from (22) yields the consolidated constraint in the text.

- The slopes of the indifference curves are given by

$$\frac{dc_{t+1}^{u,t}}{dc_{t+1}^{e,t}} = \frac{-(1-\upsilon)}{\upsilon} \cdot \frac{u'(c_{t+1}^{e,t})}{u'(c_{t+1}^{u,t})}$$

- Note that $A(1-\upsilon) > 1$ implies $\frac{(1-\upsilon)}{\upsilon}A > \frac{A}{A-1}$ and that $\frac{(1-\upsilon)}{\upsilon} > \frac{1}{A-1}$.

Figure L7.1



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L8. A Simple Model of Private Information Decentralized via a Competitive Banking Industry

These notes introduce idiosyncratic uncertainty with a unit measure of agents in a simple dynamic model with private information about agents' preferences. It is based on Diamond, D. and P. Dybvig (1983) "Bank Runs, Deposit Insurance, and Liquidity", *Journal of Political Economy*, Vol. 91, p. 401-19. The model can (and has) easily be extended to an OG framework. The key new concept it introduces is incentive compatitibility which requires the equilibrium allocation ensures that agents with private information tell the truth about their state or type. The model provides a rationale for why banks exist and how coordination failures can generate runs on financial institutions.

There are three key elements of the model:

1. Individuals are uncertain about when they will want to make expenditures but would like to earn high yields. This idiosyncratic uncertainty produces a demand for liquid assets. We will call these liquidity shocks.

2. Long term high yielding investment projects are costly to interrupt.

3. Expenditure decisions are made sequentially (think of the lines in Mary Poppins). If there is a fractional required reserve banking system (i.e. banks are required to only hold a fraction of their liabilities in liquid cash) as in most of the world, then beliefs matter a lot for whether there will be a bank run.

The first two points highlight a role for an institution (we will call it a bank) to transform illiquid assets into liquid ones in order to provide insurance to people who receive private, random liquidity preference shocks. The third is critical for a belief-based run when there is a fractional required reserve banking system and policy responses like Deposit Insurance.

As simply stated in wiki "The primary motivation for creating the Federal Reserve System (in 1913) was to address banking panics."[1]

1 Environment

- Three periods $t = 0, 1, 2$.
- Population: Unit measure of ex-ante (i.e. $t = 0$) identical agents

[1] http://en.wikipedia.org/wiki/Federal_Reserve_System

- Endowments: All agents have 1 unit of the good at $t = 0$ and prefer to consume either at $t = 1$ or $t = 2$.
- Storage technologies

1. Productive Storage technology: 1 unit of goods invested at $t = 0$ yields $R > 1$ units at $t = 2$. If the storage is interrupted at $t = 1$, the salvage value is the initial investment.

$$\begin{array}{ccc} t=0 & t=1 & t=2 \\ -1 & 1 & R \end{array}$$

2. Pillow Storage technology: 1 unit of goods invested at $t = 1$ yields 1 unit at $t = 2$.

$$\begin{array}{cc} t=1 & t=2 \\ -1 & 1 \end{array}$$

- Preferences: Agents face an iid preference shock (θ) which is realized at $t = 1$ and determines their "type":

  1. Early consumers: $prob(\theta = 1) = \pi$ with preferences $u(c_1)$.That is, they only want to consume at $t = 1$.
  2. Late consumers $prob(\theta = 2) = (1 - \pi)$ with preferences $u(c_2)$.That is, they only want to consume at $t = 2$.

  - Given that these "liquidity" shocks (i.e. early consumers really want liquidity) are iid across agents and there is a unit measure of them, π and $(1 - \pi)$ also denote the population fractions of early and late consumers in the economy.
  - Assume $u'(c) > 0, u'(0) = \infty$, $u'(\infty) = 0$,and CRRA greater than 1.

- Information: Storage in the pillow technology and consumption is unobservable.

2 Autarkic Allocation

- Let W denote the amount withdrawn from the productive technology at $t = 1$ and S denote the amount invested in the pillow storage technology at $t = 1$.
- At $t = 0$ put endowment in productive technology. At $t = 1$, choose $W = 1$ if $\theta = 1$ and $W = 0$ if $\theta = 2$ and $S = 0$. This generates $c_1^A = 1$ and $c_2^A = R$.

3 Planner's problem when type is observable (Efficient or First Best)

$$\max_{(c_1,c_2)\in\mathbb{R}_+,(S,W)\in[0,1]} \pi u(c_1)+(1-\pi)u(c_2) \tag{1}$$

$$\begin{aligned} s.t. S+\pi c_1 &= W \qquad (2)\\ (1-\pi)c_2 &= R(1-W)+S \qquad (3) \end{aligned}$$

where the two constraints are resource feasibility at $t=1$ and $t=2$ respectively and the objective function should not be considered expected utility but the sum of utilities for each agent in the economy.

- Since the short gross return between $t=1$ and $t=2$ (i.e. 1) is dominated by the return to the long asset (i.e. R), it is strictly better not to liquidate more than you need to cover expenditure by early consumers. Hence $S=0$. In this case, problem (1)-(3) can be reduced to

$$\max_W \pi u\left(\frac{W}{\pi}\right)+(1-\pi)u\left(\frac{R(1-W)}{1-\pi}\right)$$

  yields

$$\begin{aligned} u'(c_1^*) &= Ru'(c_2^*) \qquad (4)\\ &\Longrightarrow u'(c_1^*)>u'(c_2^*)\\ &\Longrightarrow c_1^*<c_2^* \end{aligned}$$

  since $R>1$ and $u'(c)>0$.

- With concave preferences, autarky is ex-ante suboptimal relative to the first best (i.e. $c_1^A=1\le c_1^*<c_2^*\le R=c_2^A$).[2]

[2] For example, if $u(c)=c^{1-\alpha}/(1-\alpha)$, then the 2 equations characterizing the first best are given by (4) and the consolidated resource constraint implied by (2)-(3) with $S=0$ substituting for W :

$$\begin{aligned} c_2^* &= R^{1/\alpha}c_1^*\\ \pi c_1^*+\frac{(1-\pi)c_2^*}{R} &= 1 \end{aligned}$$

which yields

$$\begin{aligned} c_1^* &= \frac{1}{\pi+(1-\pi)R^{\frac{1-\alpha}{\alpha}}},\\ c_2^* &= \frac{R^{\frac{1}{\theta}}}{\pi+(1-\pi)R^{\frac{1-\alpha}{\alpha}}}. \end{aligned}$$

But this implies that if $\alpha>1$, then $c_1^*>1$ since $1>\pi+(1-\pi)R^{\frac{1-\alpha}{\alpha}}\iff 1<R^{\frac{\alpha-1}{\alpha}}$. It also implies $c_2^*<R$.

- Hence we see the insurance role that a planner provides; she pools resources in order to insure agents against the "unlucky" event that they have to withdraw early (which generates a low return of 1). Ex-ante agents are willing to pay for that insurance by drawing against the high return $R > 1$ they receive in the event that they withdraw late.

- Note that the planner implements $c_1^* > 1$ by liquidating more of the long technology than $\pi \cdot 1$. This will be important for why there may be bank runs in the decentralized economy.

4 Planner's problem when type is unobservable (Constrained Efficient or Second Best)

- While the planner doesn't know who are early or late consumers (only the fraction of each type), she can design an allocation mechanism such that households send truthful messages about their type (this is known as the revelation principle)

- Truthful revelation requires that:

  1. an early type doesn't report that he is late. That is $u(c_1) \geq u(0)$. The lhs is what the early type gets if he reports early and the rhs is what he gets if he reports late (doesn't get anything at $t = 1$ which is when he wants to eat). Since utility is strictly increasing this requires $c_1 \geq 0$.
  2. a late type doesn't report that he is early. That is $u(c_2) \geq u(c_1)$. The lhs is what the late type gets if he reports late and the rhs is what he gets if he reports early (he stores the c_1 in the short term technology till $t = 2$ then eats). Since utility is strictly increasing, this requires $c_2 \geq c_1$.

- These two constraints are known as incentive compatibility or truth telling constraints[3], which must be added to the planner's problem which is now:

$$\max_{(c_1,c_2)\in\mathbb{R}_+,(S,W)\in[0,1]} \pi u(c_1)+(1-\pi)u(c_2)$$
$$\begin{aligned} s.t\ S+\pi c_1 &= W \\ (1-\pi)c_2 &= R(1-W)+S \\ u(c_1) &\geq u(0) \\ u(c_2) &\geq u(c_1) \end{aligned}$$

But recall the solution to the problem without the incentive compatibility constraint actually satisfies the constraint since $c_2^* > c_1^* > 1$. Hence the first best is actually implementable with private information.

- Note however, it is not typically the case that the private information allocation is identical to the full information allocation; usually incentive compatibility constraints bind and it is not possible to implement the efficient full information allocation when there is private information.
- Note further that after adding incentive compatibility constraints which induce truthtelling, the resource feasibility constraints are unchanged because π early consumers and $1-\pi$ late consumers truthfully report their type.

5 Decentralized Solution: Banks

- Is the planner's solution implementable by a bank offering simple non-contingent deposit contracts which does not know its depositors liquidity needs?
- A deposit contract is specified as one in which agents deposit 1 unit at $t=0$ in return for r_1 units of the good if they withdraw at $t=1$ and r_2 units of the good if they withdraw at $t=2$ **as long as the bank is solvent** at $t=1$: [4]

$t=0$	$t=1$	$t=2$
-1	r_1	r_2

- Agents also have access to the productive and pillow technologies.

[3] The revelation principle assures us that it is sufficient to restrict the allocation to respect simple messages which are incentive compatible. If $\theta \in \{1,2\}$ are the two possible types of agents and $m \in \{1,2\}$ denotes the possible messages that an individual can report, then for a payoff that depends on true type and message $v(\theta,m)$, incentive compatibility requires that the type 1 agent not report he is type 2 and the type 2 agent not report he is type 1 or:

$$\begin{aligned} v(1,1) &\geq v(1,2), \\ v(2,2) &\geq v(2,1). \end{aligned}$$

[4] Solvency means that the bank has resources remaining.

- Withdrawals are served sequentially in random order while the bank is solvent (i.e. until the bank runs out of assets). This is known as the sequential service constraint; a bank's payoff to any agent depends on his place in line due to a solvency constraint and not on future info about agents later in line (effectively this rules out communication between agents).

- To understand the solvency constraint, note that the most that a bank can withdraw from the long run technology at $t = 1$ is 1. Then if f fraction of agents go to the bank to withdraw at $t = 1$ and the bank offers r_1 to each person, then the bank can only meet its obligations if $fr_1 \leq 1$. If $fr_1 > 1$,then the bank is insolvent. Specifically,

  - The payoff associated with withdrawing at $t = 1$ or $t = 2$ per unit of deposit, which depends on one's place in line, is denoted $V_1(f_j, r_1)$ and $V_2(f, r_1)$ respectively where f_j is the fraction of withdrawers serviced before agent j and f is the total fraction of withdrawers serviced at $t = 1$.
  - Type 1 payoffs are then

    $$V_1(f_j, r_1) = \begin{cases} r_1 & \text{if } f_j r_1 \leq 1 \\ 0 & f_j r_1 > 1 \end{cases}$$

    where the first line is the payoff to a type 1 individual if the bank is solvent and the second is if it is insolvent.
  - For type 2 payoffs if solvent and insolvent respectively are

    $$V_2(f, r_1) = \begin{cases} r_2 = \frac{R(1-fr_1)}{1-f} & \text{if } fr_1 \leq 1 \\ 0 & \text{if } fr_1 > 1 \end{cases}.$$

    Note that the payoff if solvent $r_2 = \frac{R(1-fr_1)}{1-f}$ is similar to the planner's resource feasibility constraint at $t = 1$ given by $c_2 = \frac{R(1-W)}{1-\pi}$.

5.1 Can the efficient allocation be implemented by a bank?

- Obviously, $(r_1, r_2) = (c_1^*, c_2^*)$ with $f = \pi$ (i.e. the total fraction served are only early consumers) is feasible and optimal. Competition between banks leads them to offer this deposit contract.

- Since $(r_1, r_2) = (c_1^*, c_2^*)$ dominates autarky, this provides a rationale for why banks exist; they transform illiquid assets into liquid assets (something called maturity transformation).

5.2 Is the decentralized banking equilibrium stable?

- If this is the deposit contract, is the banking system stable? That is, will banks be able to fulfill their contractual obligations in equilibrium? This depends on the behavior of late consumers, which in turn depends on their beliefs.

- Suppose late consumers believe only early consumers will withdraw at $t = 1$. This implies $f = \pi$.
  - Under this belief, a given late consumer believes his utility from withdrawing at $t = 2$ is $u(c_2^*)$.
  - Now consider a deviation given these beliefs. If the late consumer chooses to withdraw at $t = 1$ and use the short term storage technology he will receive utility $u(c_1^*)$ at $t = 2$.
  - Since $c_2^* > c_1^*$, this deviation is suboptimal given his beliefs.
  - In this case we say that the first best banking allocation is a pure strategy Nash equilibrium (i.e. a "good" equilibrium).[5]
- Suppose that late consumers believe that other late consumers will withdraw at $t = 1$ (i.e. they believe there will be a run on the bank). This implies $f = 1$ (remember individual agents are of measure zero).
  - Under this belief, $fr_1 > 1$ since $c_1^* > 1$.
  - In that case, if a given late consumer does what everyone else is doing and withdraws at $t = 1$, he receives c_1^* provided he is early enough in line (i.e. $f_j r_1 \leq 1$) and zero otherwise. Since he can store that under his pillow, he receives either $u(c_1^*)$ or $u(0)$.
  - Now consider a deviation by the agent from what all other late consumers are doing. In particular, if he does not withdraw at $t = 1$, then under these beliefs he receives $u(0)$.
  - If $\overline{f}$ satisfies $\overline{f}r_1 = 1$, then since $u(0)$ is dominated by an $\overline{f}$ chance of $u(c_1^*)$ or a $(1 - \overline{f})$ chance of $u(0)$, it is a best response to withdraw early under these beliefs.
  - Thus there is another "bad" pure strategy Nash equilibrium with bank runs which is pareto dominated by the above "good" equilibrium.[6]

- In the latter case, we see that a bank run can be the result of pessimistic beliefs despite the fact that there was no change in the tastes or tech-

[5] Informally, a set of actions or strategies is a Nash equilibrium if no player can do better by unilaterally changing his or her strategy.

Formally, let

- S_i denote the action or strategy set for player $i \in \{1, ..., n\}$,
- $S = S_1 \times S_1 ... \times S_n$ be the set of possible strategy profiles,
- $x = (x_1, ..., x_n) \in S$ be a given strategy profile,
- x_{-i} be a strategy profile of all players except for player i,
- $f_i(x)$ denote the payoff to agent i given a strategy profile x, and
- $f = (f_1(x), ..., f_n(x))$ be the economywide payoff function.

A strategy profile $x^* \in S$ is a Nash equilibrium if no unilateral deviation in strategy by any single player is profitable for that player, that is

$$\forall i, x_i \in S_i, x_i \neq x_i^* : f_i(x_i^*, x_{-i}^*) \geq f_i(x_i, x_{-i}^*).$$

[6] In this case, ALL agents receive a risky return that has mean 1 (i.e. $\overline{f}r_1 + (1 - \overline{f})0 = 1$). This yields expected value lower than autarky.

nologies of the economy. The multiplicity of pareto ranked equilibria is rampant in coordination games like this.

- Thus can construct sunspot equilibria where some of the time the banking system is stable and other times it is unstable. Under some parameterizations, this sunspot equilibrium might be ex-ante superior to autarky.

6 Optimal Policy Responses to Runs

- If banks can commit to suspend convertibility when withdrawals are too numerous at $t = 1$ (i.e. the first j such that $f_j = \pi$) and late consumers know this, they know the bank will be solvent and there is no need to run. This is shutting the door early enough in Mary Poppins.

- If the government (FDIC) commits to insure depositors such that even if the bank is not able to fulfill its obligations, the depositors receive the full value of their deposits which is funded by a tax system, then this implements the first best without any taxes actually ever being paid.

- Deposit insurance can be shown to be a better option if there is uncertainty about the true number of early consumers π.

- One potential problem is the credibility or time consistency of the commitment to deposit insurance. For instance, when deposit insurance actually has to be paid out, might the choose not to honor that obligation? A paper which has considered this time consistency (no commitment) problem is by Ennis, H. and T. Keister (2009) "Bank Runs and Institutions: The Perils of Intervention", *American Economic Review*, 99, pp. 1588-1607.

- Another problem is that government insurance can mess up the incentives of the bank (if they know that the government will bail them out, then they may take excessive risk, which is known as moral hazard). A paper which considered how incentives are distorted by deposit insurance is by Keeley, M. (1990) "Deposit Insurance, Risk, and Market Power in Banking", *American Economic Review*, 80, pp. 1183-1200.

L9. Private Commitment Frictions

These notes introduce collateral (secured debt) and exclusion from credit markets as a way to mitigate private commitment frictions. Secured debt in the form of mortgages backed by the value of one's house provides one such example (and is typically the largest debt taken out by households). The notes study the decision problem at the heart of Kiyotaki, N. and J. Moore (1997) "Credit Cycles", *Journal of Political Economy*, Vol.105, p. 211-248. Their general equilibrium model illustrates how the dynamic interaction between credit limits and asset prices can generate a powerful transmission mechanism by which exogenous shocks can endogenously persist and become amplified.[1] The notes also apply trigger strategies punishing non-repayment with exclusion from credit markets along the lines of the government problem studied in L5 notes. Exclusion strategies of this sort have been employed in Sovereign Default models like that in Eaton, J. and M. Gersovitz (1981) "Debt with Potential Repudiation: Theoretical and Empirical Analysis", *Review of Economic Studies*, Vol.48, p. 289-309.

1 Environment

- Any given period can be divided into two subperiods, say 0 for the beginning of the period and 1 for the end of the period.
- In any given period, suppose that households (HHs) value services from nonstorable consumption in both subperiods (i.e. c_0 and c_1, respectively) and housing only in subperiod 0 (i.e. h_0 and 0, respectively) according to the utility function

$$u(c_0, h_0, c_1) = \ln(c_0) + \ln(h_0) + \beta c_1.$$

- HHs have zero income in the first subperiod but positive income in the second subperiod (i.e. 0 and y_1, respectively).
- The relative price of housing in terms of consumption in the first and second subperiods are q_0 and q_1, respectively. These prices are endogenous in Kiyotaki and Moore but here we will simply solve the decision problem, not the general equilibrium problem.

[1] A simple GE version is also studied in Kocherlakota, N. (2000) "Creating Business Cycles Through Credit Constraints", *Federal Reserve Bank of Minneapolis Quarterly Review*, Summer, p.2-10.

- HHs can borrow b_1 in subperiod 0 to be repaid with interest (i.e. $(1+r)$) in subperiod 1 using their income y_1. Since HHs do not value housing in subperiod 1, we will assume they sell the house in subperiod 1.
- To summarize, HHs choose $(c_0, c_1, h_0, b_1) \in \mathbb{R}^4_+$ to maximize $u(c_0, h_0, c_1)$ subject to their constraints. We will make the following parametric assumptions:

A1. $q_0(1+r) > q_1$

A2. $\beta y_1 > 2$

2 Commitment Equilibrium

- Suppose the HH can commit to repay their loan.
- The HH problem solves.

$$\max_{(c_0,c_1,h_0,b_1)\in\mathbb{R}^4_+} \ln(c_0) + \ln(h_0) + \beta c_1$$

$$c_0 + q_0 h_0 = b_1 \tag{BC1}$$

$$c_1 = y + q_1 h_0 - (1+r) b_1 \tag{BC2}$$

- Plugging (BC1) and (BC2) into the objective yields lagrangian with multiplier μ on $c_1 \geq 0$ (by linearity we have to worry about this constraint):

$$\pounds = \ln(b_1 - q_0 h_0) + \ln(h_0) + \beta\left[y + q_1 h_0 - (1+r) b_1\right] + \mu\left[y + q_1 h_0 - (1+r) b_1\right]$$

- The foc are

$$h_0 \ : \ \frac{-q_0}{b_1 - q_0 h_0} + \frac{1}{h_0} + \beta q_1 + \mu q_1 = 0 \tag{FOCH}$$

$$b_1 \ : \ \frac{1}{b_1 - q_0 h_0} - \beta(1+r) - \mu(1+r) = 0 \tag{FOCB}$$

- Suppose $c_1 \geq 0$ is nonbinding. Then we have

$$h_0 \ : \ \frac{1}{h_0} + \beta q_1 = \frac{q_0}{b_1 - q_0 h_0} \tag{1}$$

$$b_1 \ : \ \frac{1}{b_1 - q_0 h_0} = \beta(1+r) \tag{2}$$

which is 2 equations in 2 unknowns. (2) into (1)

$$\frac{1}{h_0} = \beta\left[q_0(1+r) - q_1\right] \Longleftrightarrow h_0^* = \frac{1}{\beta\left[q_0(1+r) - q_1\right]} \tag{3}$$

which requires Assumption A.1 $q_0(1+r) > q_1$. (3) into (2) yields

$$\frac{1}{\beta(1+r)} + \frac{q_0}{\beta\left[q_0(1+r) - q_1\right]} = b_1^* \tag{4}$$

(3) and (4) into (BC2) yields

$$\begin{aligned}
c_1 &= y + \frac{q_1}{\beta\left[q_0(1+r) - q_1\right]} - \left[\frac{(1+r)}{\beta(1+r)} + \frac{q_0(1+r)}{\beta\left[q_0(1+r) - q_1\right]}\right] \Longleftrightarrow \\
c_1 &= y + \frac{q_1 - q_0(1+r)}{\beta\left[q_0(1+r) - q_1\right]} - \left[\frac{(1+r)}{\beta(1+r)}\right] \Longleftrightarrow \\
c_1 &= y - \frac{q_0(1+r) - q_1}{\beta\left[q_0(1+r) - q_1\right]} - \left[\frac{1}{\beta}\right] \Longleftrightarrow \\
c_1^* &= y - \left[\frac{1}{\beta}\right] - \frac{1}{\beta} = y - \frac{2}{\beta}
\end{aligned}$$

which requires Assumption A.2 that $\beta y > 2$. Finally, $c_0^* = \frac{1}{\beta(1+r)}$.

3 No Commitment

- Suppose there is no way to enforce repayment of the loan. That is, suppose that if a HH defaults on its loan, they lose nothing.
- What will HHs do? What will lenders do? Since no one will pay back their debt, no one will lend. Hence $(c_0 = 0, c_1 = y, h_0 = 0, b_1 = 0)$.

3.1 Collateral

- Now suppose that while lenders cannot commit the HH to repay, in the event of default (i.e. non-repayment of $(1+r)b_1$ in subperiod 1), lenders can seize κ times the collateral value of the house $q_1 h_0$. We will call

$$b_1 \leq \kappa \cdot \frac{q_1 h_0}{1+r} \tag{CC}$$

  a collateral constraint as in Kiyotaki-Moore. The idea is that the benefit of default (not repaying $(1+r)b_1$) must be less than the cost of the default (losing a multiple of one's collateral $\kappa q_1 h_0$). In that case, the HH will not default. Here, we will allow $\kappa > 1$ so that default can bring an even higher penalty than just the loss of collateral (say lawyers fees, etc.).

- First we can ask if it is possible to implement the full commitment solution with the collateral constraint in (CC). That is, the solution in (3)-(4)

satisfies (CC) if:

$$
\begin{aligned}
b_1^* &\leq \kappa \cdot \frac{q_1 h_0^*}{1+r} \Longleftrightarrow \\
\frac{1}{\beta(1+r)} + \frac{q_0}{\beta\left[q_0(1+r)-q_1\right]} &< \kappa \cdot \frac{q_1}{1+r} \cdot \frac{1}{\beta\left[q_0(1+r)-q_1\right]} \Longleftrightarrow \\
\frac{1}{\beta} + \frac{q_0(1+r)}{\beta\left[q_0(1+r)-q_1\right]} &< \kappa \cdot \frac{q_1}{\beta\left[q_0(1+r)-q_1\right]} \Longleftrightarrow \\
\left[q_0(1+r)-q_1\right] + q_0(1+r) &< \kappa \cdot q_1 \Longleftrightarrow \\
2\frac{q_0(1+r)}{q_1} - 1 &< \kappa. \qquad \text{(BC)}
\end{aligned}
$$

It is unsurprising that high κ (i.e. a high punishment) can deter default. Notice that, consistent with Kiyotaki-Moore, an increase in q_1 can result in HHs moving from a binding constraint to a first best slack constraint.

- Under condition (BC) prices (q_0 and q_1) would not change since preferences and constraints (BC1) and (BC2) did not change and (CC) is non-binding.

- Assume the conditions (BC) for a non-binding constraint do not hold (i.e. $2q_0(1+r) - q_1 > \kappa q_1$ and that $\kappa q_1 > (1+r)q_0$ (note that this need not be inconsistent with assumption A1 above that $(1+r)q_0 > q_1$ for some $\kappa > 1$). I will also assume that prices are the same as in the commitment solution (which is a strong assumption for illustration but would not necessarily hold in general equilibrium). A binding constraint (CC) implies the budget constraints (BC1) and (BC2) are:

$$
\begin{aligned}
c_0 &= \left[\frac{\kappa q_1}{1+r} - q_0\right] h_0, \\
c_1 &= y - (\kappa - 1) q_1 h_0
\end{aligned}
$$

into the objective yields

$$
\begin{aligned}
&\max_{h_0} \ln\left(\left[\frac{\kappa q_1}{1+r} - q_0\right] h_0\right) + \ln h_0 + \beta\left(y - (\kappa-1) q_1 h_0\right) \Longleftrightarrow \\
&\ln\left(\frac{\kappa q_1}{1+r} - q_0\right) + 2 \ln h_0 + \beta\left(y - (\kappa-1) q_1 h_0\right)
\end{aligned}
$$

The foc wrt h_0 is

$$
\frac{2}{h_0} - \beta(\kappa-1) q_1 = 0 \Longleftrightarrow \widehat{h}_0 = \frac{2}{\beta(\kappa-1) q_1}
$$

(i.e. one equation in one unknown). In that case,

$$
\begin{aligned}
\widehat{c}_0 &= \left[\frac{\kappa q_1}{1+r} - q_0\right] \frac{2}{\beta(\kappa-1) q_1}, \\
\widehat{c}_1 &= y - \frac{2}{\beta}.
\end{aligned}
$$

- Comparing the allocation with (left) commitment to without (right) commitment (i.e. the unconstrained problem to the constrained problem) we see

$$\begin{aligned}
\frac{1}{\beta(1+r)} &= c_0^* > \widehat{c}_0 = \left[\frac{\kappa q_1}{1+r} - q_0\right]\frac{2}{\beta(\kappa-1)q_1} \iff \\
1 &= c_0^* > \widehat{c}_0 = \left[\kappa q_1 - q_0(1+r)\right]\frac{2}{(\kappa-1)q_1} \qquad \text{(CC0)} \\
\frac{1}{\beta\left[q_0(1+r)-q_1\right]} &= h_0^* < \widehat{h}_0 = \frac{2}{\beta(\kappa-1)q_1} \iff \\
& \quad (\kappa-1)q_1 < 2\left[q_0(1+r)-q_1\right] \qquad \text{(CH0)} \\
y - \frac{2}{\beta} &= c_1^* = \widehat{c}_1 = y - \frac{2}{\beta} \qquad \text{(CC1)}
\end{aligned}$$

  Intuitively, in the case without commitment $h_0^* < \widehat{h}_0$ since raising housing loosens the constraint. Note that the parameterization such that the collateral constraint is binding (BC) implies $2q_0(1+r) - q_1 > \kappa q_1$ and re-writing (CH0) gives $\kappa q_1 < 2q_0(1+r) - q_1$. Hence we know $h_0^* < \widehat{h}_0$ and $c_0^* > \widehat{c}_0$.

- Since $h_0^* < \widehat{h}_0$, one would expect that housing prices (q_0 and q_1) would change. For example, one might expect that since demand rose, $\widehat{q}_0$ would rise.

- Note that during the 2008 financial crisis, which was characterized by a precipitous drop in housing prices q_1, people found it hard to borrow. That is consistent with the collateral constraint (CC); as q_1 falls, borrowing becomes more constrained.

3.2 Exclusion

- In the previous static setting, without the ability to punish we saw there was no lending. In any finite problem, there is the same unraveling; in the last period T the HH will not pay back, so the lender will not lend. But the same occurs at $T-1$, knowing that there is no lending in T.

- Suppose now that there are an infinite number of time periods with the two subperiods described above (i.e. repetitions of the static model above where for simplicity housing completely depreciates across time periods). Let $\delta < 1$ be the HH's discount factor across periods (not subperiods).

- Here we ask if it is possible to sustain the commitment solution with trigger strategies that punish a defaulter with permanent exclusion without having to impose the collateral constraint?[2]

[2] With finite y_1 and $r > 0$, feasibility requires that a HH would always face a finite limit on how much they could borrow in any given period.

- To implement the trigger strategy, we would again (as in the government commitment problem notes) create a private history H for each agent where $H = 0$ means they do not have a default in their records and $H = 1$ means they do have a default. Such histories happen in the real world. Credit records keep track of household delinquency and default and credit scores use private credit histories to infer the likelihood a person will default. When people default, their score falls, and they find it harder (or more expensive) to borrow.

- Let
$$V^U = \frac{u(c_0^*, h_0^*, c_1^*)}{1-\delta}$$
denote the value of the commitment allocation along the equilibrium path where the household does not default. Let
$$V^D = u(c_0^*, h_0^*, y_1 + q_1 h_0^*) + \frac{\delta u(0, 0, y_1)}{1-\delta}$$
denote the utility the HH receives by defaulting and being punished with permanent autarky. While $u(c_0^*, h_0^*, y_1 + q_1 h_0^*) > u(c_0^*, h_0^*, c_1^*)$ (the incentive to default), given log utility $u(0, 0, y_1) = -\infty$, so a deviation to default is suboptimal for any $\delta > 0$.

Econ 712 Macroeconomics I
Fall 2020, University of Wisconsin-Madison
Instructor: Dean Corbae

Summary of Topics/Methods Covered in First Half of Macro 1

Topics

- Incomplete markets due to matching, commitment, or information frictions.
- The role of government to implement pareto superior outcomes in competitive economies with frictions.
  - One example is the introduction of fiat money in overlapping generations models. Fiat money is an example of a rational bubble (its equilibrium value can differ from its fundamental value (which is zero)).
- Ricardian equivalence (conditions for the irrelevance of the timing of lump sum taxes).
- The effect of distortionary taxes on equilibrium output.
- Time inconsistent government policy.

Methods

- Environment: A statement of population, preferences, technologies (which includes not only endowment/production, but also commitment, matching, and information)
- Social Planner's Problem: A planner solves a constrained optimization problem (objective is weighted population welfare subject to constraints (e.g. economywide resources) to obtain a pareto optimal allocation.
- Competitive Equilibrium: Allocation and price sequence which satisfies atomless agent optimization (both households and firms) and market clearing. In the presence of government (which is not atomless), we need to include its budget constraint.
  - Atomless agent actions do not affect aggregate outcomes. With a unit measure of identical agents, the solution method is to take all other's actions as given (say K), allow an atomless agent to choose their own best action (say k), and then check equilibrium consistency (i.e. $k = K$). In macro, this is called a Big K, little k problem.

- In macro, often the planner's problem is simpler to solve than the competitive equilibrium (since we do not have to compute prices). The second welfare theorem, when applicable, provides an implementation result.
- Labor/leisure choice problems.
- Indirect utility functions, the envelope condition, and finite dynamic programming.
- Existence and uniqueness of competitive equilibrium. Pareto ranked equilibria and equilibrium selection in the presence of multiple equilibria.
- Stationary and nonstationary equilibria. Nonlinear difference equations.
- Irrelevance results for government policy with perfect financial markets:
  - Conditions for a sequence of budget constraints to be equivalent to a consolidated budget constraint.
  - Conditions for equivalence between overlapping generations economies with bequests and the infinitely lived agent problem.
- Government policy with commitment (Ramsey equilibrium) versus sequentially rational behavior (sustainable equilibrium) without the ability to commit.
  - Dynamic punishment can help sustain good outcomes which are not sustainable in static environments. We saw how one shot deviations which are punished by reversion to bad equilibrium outcomes "off-the-equilibrium path" may sustain good behavior "on-the-equilibrium path".
- Incentive compatibility constraints (illustrated in the Diamond-Dybvig model of banking):
  - Example where the first best allocation actually satisfies incentive compatibility.
  - Example of multiple pareto ranked equilibria and how government policy (deposit insurance) can implement the superior equilibrium.