Microeconomic Theory II

Classical Demand Theory, Part 1 of 2

1. axiomatic approach
2. does there always exist $u(\ )$ represents $\succsim$? properties?
3. utility maximization problem (UMP) $\Rightarrow$ implications for $x(p,w)$ & $v(p,w)$

Preference Assumptions

- consumption set $X \subset \mathbb{R}^L_+$

$\succsim$ is rational if it is complete & transitive

i. complete: $\forall x, y \in X$, $x \succsim y$, $y \succsim x$, or both

ii. transitive: $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$

- three classes of additional assumptions:
  - a. desirability
  - b. convexity
  - c. special cases (aggregation)

Desirability: monotonicity & local nonsatiation (LNS)
(stronger) (weaker)

a. monotonicity

- we need to assume that larger consumption is always feasible

if $x \in X$ & $y \geq x \Rightarrow y \in X$

Def. $\succsim$ is <u>monotone</u> if $x \in X$ and $y \gg x$ implies $y \succ x$; it is <u>strongly monotone</u> if $y \geq x$ and $y \neq x \Rightarrow y \succ x$.

$$\begin{array}{ccc} x & y & z \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array}$$

monotonicity $\Rightarrow y \succ x$
$\not\Rightarrow z \succ x$

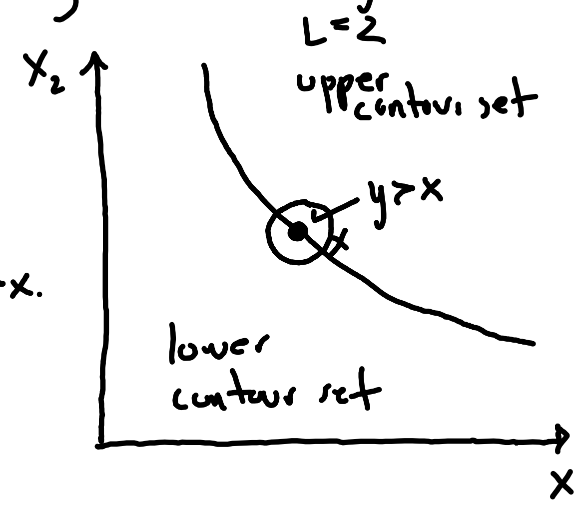
strong monotonicity $\Rightarrow y \succ x$
$\Rightarrow z \succ x$

We only use monotonicity when <u>goods</u> are desirable, or "absense of" a <u>bad</u>.

b Def. $\succsim$ is LNS if for every $x \in X$ and every $\epsilon > 0$, there is $y \in X$ $\ni$ $\underbrace{\|y-x\|}$ $\leq \epsilon$ and $y \succ x$. (Always something better nearby.)

Euclidean Distance: $\left[\sum_i (x_i - y_i)^2\right]^{\frac{1}{2}}$

An L-dimensional ball around x contains $y \succ x$.



- Prevents:
  1. thick indifference curves
  2. <u>all</u> goods being bad (but some can be)

Def. <u>indifference</u> set to x: $\{y \in X : y \sim x\}$.

Def. upper <u>contour</u> set to x: $\{y \in X : y \succsim x\}$.

Def. lower <u>contour</u> set to x: $\{y \in X : y \precsim x\}$.

Note:

a) strongly monotone $\succsim$ $\implies$ monotone $\succsim$

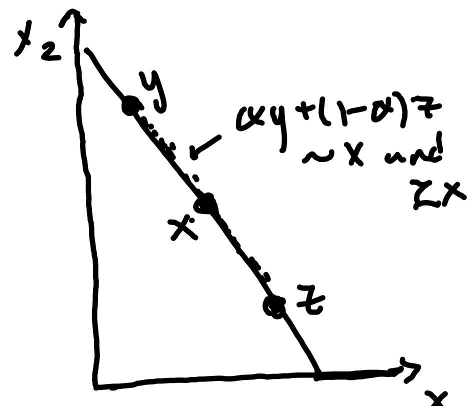
b) monotone $\succsim$ $\implies$ LNS $\succsim$

(not necessarily in reverse)

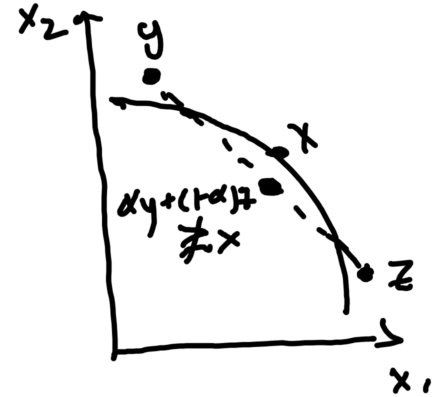
Covexity (tradeoff): convexity (weaker) & strict convexity (stronger)

Def. $\succsim$ is convex if for every $x \in X$ the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha) z \succsim x$ for $\alpha \in [0,1]$.

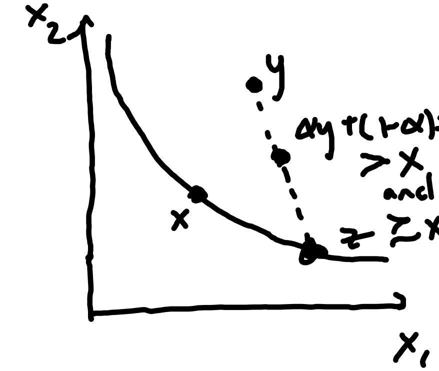
$L = 2$



$\succsim$ is convex but not strictly convex



X is non-convex



$\succsim$ is convex & strictly convex

- convexity derives from
  a. diminishing marginal rates of substitution
  b. a desire for diversification

} may not hold! mints & orange juice

Def. $\succsim$ on X is strictly convex if for every $x \in X$, we have $y \succsim x$, $z \succsim x$, and $y \neq z \Rightarrow \alpha y + (1-\alpha) z \succ x$ $\forall \alpha \in [0,1]$.

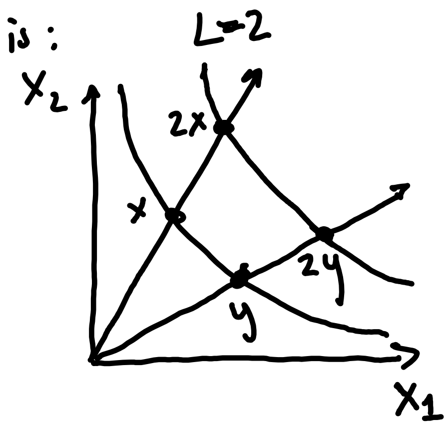
Note: strict convexity $\Rightarrow$ convexity

Special Cases: homothetic & quasilinear preferences

We want to be able to represent entire $\succsim$ from a single indifference set, b/c this is useful for aggregation, i.e., across people.

Def. A <u>monotone</u> $\succsim$ on $X = \mathbb{R}^L_+$ is <u>homothetic</u> if all indifference sets are rel. by proportional expansion rays: that is:

if $x \sim y$ then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$



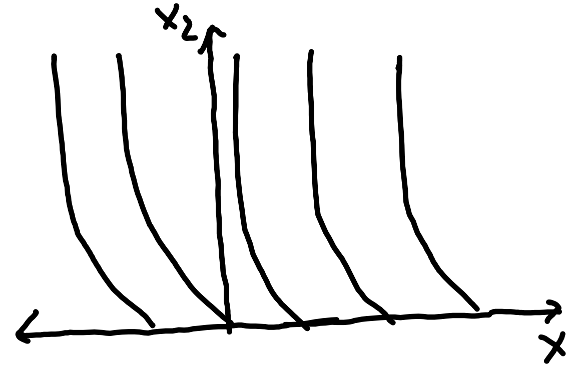
A $\succsim$ is homothetic iff it can be represented by a $u(\cdot)$ fn. that is homogeneous of degree one, i.e.

$u(\alpha x) = \alpha u(x)$ $\forall x$ and $\alpha > 0$.

Def. $\succsim$ on $X = \underbrace{(-\infty, \infty)}_{\text{commodity } 1} \times \underbrace{\mathbb{R}^{L-1}_+}_{\substack{L-1 \text{ other} \\ \text{goods}}}$ is quasi<u>lin</u>ear w/ respect to commodity 1 (the numeraire commodity) if

i.) all indif. sets are parallel displacements along the com. 1 axis, that is

if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.



ii) good 1 is desirable, that is

$x + \alpha e_1 \succ x \; \forall x$ and $\alpha > 0$

Preference & Utility

We have already shown that it is necessary for $\succsim$ to be rational for $\succsim$ to be represented by a $u(\cdot)$ fn. This is not sufficient.

Ex. Assume $X = \mathbb{R}^2$. Define $x \succsim y$ if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$". This is lexicographic — these are rational but cannot be represented by a $u(\cdot)$ fn.

Def. $\succsim$ on X is continuous if it is preserved under limits; that is for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ w/ $x^n \succsim y^n \ \forall n$, $x = \lim_{n \to \infty} x^n$ and $y = \lim_{n \to \infty} y^n$, we have $x \succsim y$.

This rules out discontinuous "jumps" in preferences, or sudden reversals.

Lexicographic preferences are not continuous. Let $x^n = (\frac{1}{n}, 0)$ and $y=(0,1) \ \forall n$; then $\lim_{n \to \infty} x^n = (0,0)$ and $\lim y^n (0,1)$, but $x^n \succsim y^n \ \forall n$.

Equivalently, continuity holds when $\forall x$, the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : y \precsim x\}$ are both closed — that is, they include their boundaries.

(*) Proposition. Suppose $\succsim$ on X is continuous. Then there $\exists$ a continuous utility fn that represents $\succsim$.

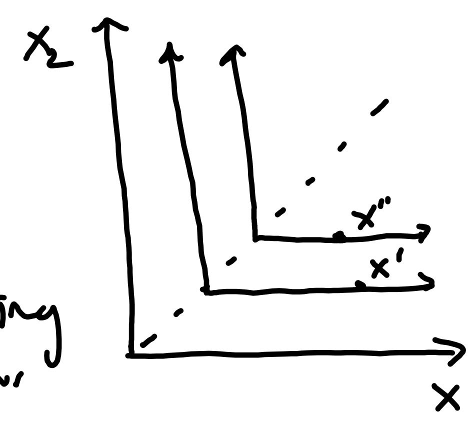
Proof. In text. Keep in mind: MWG assume monotonicity, but it is not necessary — it's only a shortcut to make the proof "digestible".

Important Implications about the relationship b/w $\succsim$ and utility

1. $u(\cdot)$ is <u>not a unique</u> representation of $\succsim$; any strictly increasing transform of $u(\cdot)$, say $v(x) = f(u(x))$ where $f(\cdot)$ is strictly inc. also represents $\succsim$. ($\ln(\cdot)$ is strictly increasing — this is often applied to $u(\cdot)$)

2. If $\succsim$ is continuous, $\exists$ a continuous $u(\cdot)$ representing $\succsim$, but not all $u(\cdot)$ representing continuous $\succsim$ are continous.

3. $u(\cdot)$ need not be differentiable, e.g. Leontief preferences have $x'' \succsim x'$ iff $\min\{x_1'', x_2''\} \geq \min\{x_1', x_2'\}$.



4. Restrictions on $\succsim$ "carry through" to $u(\cdot)$ fn.

a.) Monotonicity $\Rightarrow u(x) > u(y)$ if $x \gg y \Rightarrow u(\cdot)$ is increasing

b.) convexity of $\succsim \;\Rightarrow u(\cdot)$ is quasiconcave (convex contour set)

$u(\cdot)$ is <u>quasiconcave</u> if $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex for all x, or equivalently $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$.

strict convexity $\Rightarrow u(\cdot)$ is strictly quasiconcave: $>$ rather than $\geq$

Utility Maximization Problem (UMP)

- when does it have a solution?
- what are the characteristics of the solution?

We assume a rational, continuous, LNS, $\succsim$ represented by $u(x)$. The consumer choses from $X = \mathbb{R}^L_+$ her most preferred bundle in the Walrasian budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ where $p \gg 0$ and $w > 0$ to maximize her utility.

$$\max_{x \geq 0} u(x) \quad \text{s.t.} \quad px \leq w$$

Proposition. If $p \gg 0$ and $u(\cdot)$ is continuous, then $\exists$ a solution to the UMP.

Proof. If $p \gg 0$, $B_{p,w}$ is compact b/c it is both bounded and closed.

for any $l = 1, \ldots, L$ we have $x_l \leq w/p_l \;\forall\; x\, B_{p,w}$

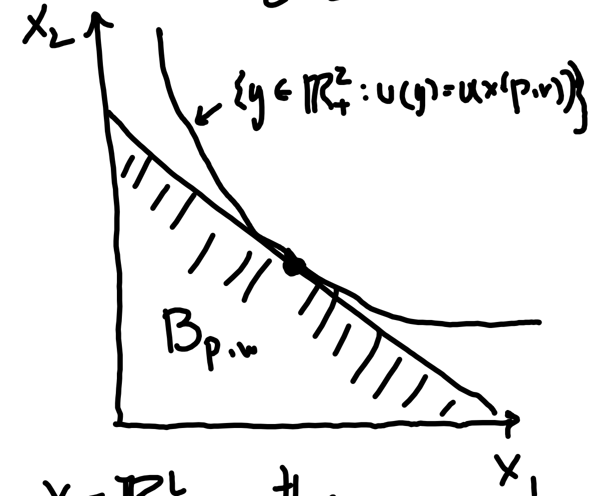
A cont. fn always has a max on a compact set. (MWG Math Appendix M.F.)

The solution set to the UMP is the optimal consumption bundle $x^*(p,w)$ and $v(p,w)$.

$x^*(p,w)$: Walrasian demand correspondence

$v(p,w)$: value of the utility fn at optimum $x^*(p,w)$

$L=2$



Proposition. If $u(\cdot)$ is continuous representing a LNS $\succsim$ on $X = \mathbb{R}^L_+$, then the Walrasian demand correspondence $x(p,w)$ satisfies:

i.) homogeneity of degree zero (HDZ) in (p,w): $x(\alpha p, \alpha w) = x(p,w)$ for any p, w, & $\alpha > 0$.

ii) Walras' law: $p \cdot x = w \;\; \forall x \in x(p,w)$

iii) if $\succsim$ is convex so $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set, and

if $\succsim$ is strictly convex, so $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ has a single element, that is, $x(p,w)$ is a <u>demand function</u>, not just a correspondence

Proof. In MWG, pp. 52-55.

The Indirect Utility (Value) Fn

For each $(p, w) \gg 0$, the utility value of the UMP is denoted $v(p, w) \in \mathbb{R}$.
It is equal to $u(x^*)$ for any $x^* \in x(p, w)$.

Proposition. The value fn for $u(\cdot)$ continuous representing LNS $\succsim$ on $X \in \mathbb{R}^L_+$ is $v(p, w)$ satisfying

1. homogenous of degree zero in (p, w)
2. strictly increasing in w & non-decreasing in p_ℓ for any ℓ
3. quasiconvex ; that is, the set $\{(p, w) : v(p, w) < \bar{v}\}$ is convex for any $\bar{v}$
4. continuous in p & w

Proof. In MWG, pp. 56-57.

Ex 1. Suppose $L=2$ and Cobb-Douglas utility $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ for some $\alpha \in (0,1)$ & $k > 0$.

$u(\cdot)$ is increasing at all $(x_1, x_2) \gg 0$ & homogeneous of degree one. WLOG, we use the log transformation.

The UMP is

$$\max_{x_1, x_2} \alpha \ln x_1 + (1-\alpha) \ln x_2, \text{ s.t. } p_1 x_1 + p_2 x_2 \leq w.$$

The Lagrangian is

$$\mathcal{L} = \alpha \ln x_1 + (1-\alpha) \ln x_2 - \lambda (p_1 x_1 + p_2 x_2 - w).$$

Since $u(\cdot)$ is increasing, the budget constraint will hold w/ equality. Imposing this,

$$\left.\begin{aligned} \partial \mathcal{L} / \partial x_1 &= \frac{\alpha}{x_1} - \lambda p_1 = 0 \\ \partial \mathcal{L} / \partial x_2 &= \frac{1-\alpha}{x_2} - \lambda p_2 = 0 \end{aligned}\right\} \overset{(1.)}{\Longrightarrow} \frac{\alpha}{x_1 p_1} = \frac{1-\alpha}{x_2 p_2} \overset{(2.)}{\Longrightarrow} (1-\alpha) x_1 p_1 = \alpha x_2 p_2$$

$$\partial \mathcal{L} / \partial \lambda = p_1 x_1 + p_2 x_2 = w \overset{(3.)}{\Longrightarrow} x_2 p_2 = w - x_1 p_1 \overset{w/(2.)}{\Longrightarrow} (1-\alpha) x_1 p_1 = \alpha (w - x_1 p_1)$$

$$\vdots$$

$$x_1^*(p, w) = \alpha \frac{w}{p_1}, \quad x_2^*(p, w) = (1-\alpha) \frac{w}{p_2}$$

Ex 3. From Ex. 1, find $v(p,w)$.

$$v(p,w) = u(x^*(p,w))$$

$$= \left[\frac{\alpha w}{p_1}\right]^{\alpha} \cdot \left[\frac{(1-\alpha)w}{p_2}\right]^{1-\alpha}$$