HW7

Wei Ye[*]
ECON 5700

Due on August 22, 2020.

1 Question 1

Solution:

1. let $\vec{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, and $\vec{m} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$.

$$\begin{aligned}
\mathcal{T}(\vec{u} + \vec{v} + \vec{m}) &= \mathcal{T}(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}) \\
&= \mathcal{T}(\begin{pmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \\ z_1 + z_2 + z_3 \end{pmatrix}) \\
&= \begin{pmatrix} x_1 - y_1 + z_1 \\ 2x_1 + y_1 - 3z_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 - y_2 + z_2 \\ 2x_2 + y_2 - 3z_2 \\ z_2 \end{pmatrix} + \begin{pmatrix} x_3 - y_3 + z_3 \\ 2x_3 + y_3 - 3z_3 \\ z_3 \end{pmatrix} \\
&= \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v}) + \mathcal{T}(\vec{m})
\end{aligned}$$

Now, we prove under scalar mulplication, TL still exists. Sill let $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{aligned}
\mathcal{T}(c\vec{v}) &= \mathcal{T}(\begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}) \\
&= \begin{pmatrix} c(x - y + z) \\ c(2x + y - 3z) \\ cz \end{pmatrix} \\
&= c\mathcal{T}(\vec{v})
\end{aligned}$$

[*]I worked on my assignment sololy. Email: wye22@fordham.edu

Thus, we can prove this is linear transformation.

2. Let $\vec{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, and $\vec{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

$$\begin{aligned} \mathcal{T}(\vec{u}+\vec{v}) &= \mathcal{T}(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}) \\ &= \begin{pmatrix} -y_1-y_2 \\ x_1+x_2+2y_1+2y_2 \\ 3x_1+3x_2-4y_1-4y_2 \end{pmatrix} \\ &= \begin{pmatrix} -y_1 \\ x_1+2y_1 \\ 3x_1-4y_1 \end{pmatrix} + \begin{pmatrix} -y_2 \\ x_2+2y_2 \\ 3x_2-4y_2 \end{pmatrix} \\ &= \mathcal{T}(\vec{u}) + \mathcal{T}(\vec{v}) \end{aligned}$$

New, we prove even in scalar multiplication, LT still exists.

$$\begin{aligned} \mathcal{T}(c\begin{pmatrix} x \\ y \end{pmatrix}) &= \left(c\begin{pmatrix} -y \\ x+2y \\ 3x-4y \end{pmatrix}\right) \\ &= c\begin{pmatrix} -y \\ x+2y \\ 3x-4y \end{pmatrix} \\ &= c\mathcal{T}(\begin{pmatrix} x \\ y \end{pmatrix}) \end{aligned}$$

All in one, we can prove LT exist in our case.☺

2 Question 2

Solution: Since T_A is matrix transformation.

$$T_A(u) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$
$$T_A(v) = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix}$$

.

3 Question 3

Solution:
The first and the second matrices are both not linear transformation.

Because the first matrix does not exist under scalar mulplication.

$$\mathcal{T}(c\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} cy \\ c^2x^2 \end{bmatrix}$$
$$\neq c\mathcal{T}(\begin{bmatrix} x \\ y \end{bmatrix})$$

For the second matrix, it doesn't exist due to the same reason.

$$\mathcal{T}(c\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} cxcy \\ c(x+y) \end{bmatrix}$$
$$\neq c\mathcal{T}(\begin{bmatrix} x \\ y \end{bmatrix})$$

Thus, they are not linear transformation.

4 Question 4

Solution:

1. By direct subsititution:

$$\begin{aligned} S^{\circ}T(a) &= S\begin{bmatrix} x_1 + 2x_2 \\ 2x_2 - x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + 2x_2 - (2x_2 - x_3) \\ x_1 + 2x_2 + 2x_2 - x_3 \\ -x_1 - 2x_2 + 2x_2 - x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_3 \\ x_1 + 4x_2 - x_3 \\ -x_1 - x_3 \end{bmatrix} \end{aligned}$$

2. By matrix multiplication:

$$S = y_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y_2\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$[S] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} x_1 + 2x_2 \\ 2x_2 - x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

Thus:

$$\begin{aligned} [S][T] &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 4 & -1 \\ -1 & 0 & -1 \end{bmatrix} \end{aligned}$$

5 Question 5

Solution:

$$\begin{aligned} |A - \lambda I| &= \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \\ &= -\lambda^3 + 3\lambda + 2 = 0 \end{aligned}$$

Thus, $\lambda_1 = -1$, and $\lambda_2 = 2$.

- When $\lambda = -1$:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] &= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &= x_1 + x_2 + x_3 = 0 \end{aligned}$$

Let $x_1 = t$, $x_2 = m$, and $x_3 = -t - m$.

$$\begin{bmatrix} t \\ m \\ -t-m \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + m \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Thus, the eigenvectco associate with the eigenvalue of -1 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$.

- When $\lambda = 2$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Thus, in the case the eigenvector is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$