# Game Theory Notes: Maskin

Matthew Basilico

Fall, 2012

**1.1**

**What is the normal form representation of a game? (Pure strategies)**

The *normal form representation* $\Gamma_N$ specifies

1. A set of players $I = \{1, \ldots, I\}$
2. A pure strategy space $S_i$, [a set of strategies $S_i$ (with $s \in S_i$)]
3. Payoff functions $u_i(s_1, \ldots, s_n)$ giving von Neumann-Morgenstern utility levels associated with the (possibly random) outcome arising from strategies $(s_1, \ldots, s_I) \in S_1 \times \ldots \times S_I$.

Formally, we write

$$\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$$

**What is the strategy space $S_i$?**

The strategy spaces $S_i$ represent the possible actions for each player, e.g. $S_i = \{\text{Left, Right}\}$or $S_i = \{\text{Invest, Don't Invest}\}$

**What is an extensive form game?**

Full list of components:

1. A set of players $(I)$
2. An ordering of play $(T)$
   (a) We specify an order of play by defining a game tree $T$.
   (b) We also require that each node has exactly one immediate predecessor, in order for each node to be a complete description of the path preceding it
3. Payoffs as a function of moves $(u_i)$
   (a) Let a *terminal node* be a node that is not a predecessor of any node. Denote the set of such nodes by $Z$.
   (b) By the condition imposed above of unique predecessors, every terminal node corresponds to a *unique complete path history*.

(c) Payoffs can thus be described by $u_i : Z \to \mathbb{R}$

4. Information available at each choice ($h(\cdot)$)

   (a) Define information set $h$ to be a partition of the nodes of the tree.

   i. Philosophically, the information set $h(x)$ represents the nodes at which a player at node $x$ thinks he could be

   ii. e.g. if $h(x) = \{x, x', x''\}$, then a player knows at $x$ he is at either $x$, $x'$ or $x''$, but does not know which node among these

   (b) In order for this interpretation to make sense, we require that

   i. If $x' \in h(x)$, then the same player moves at $x$ and $x'$

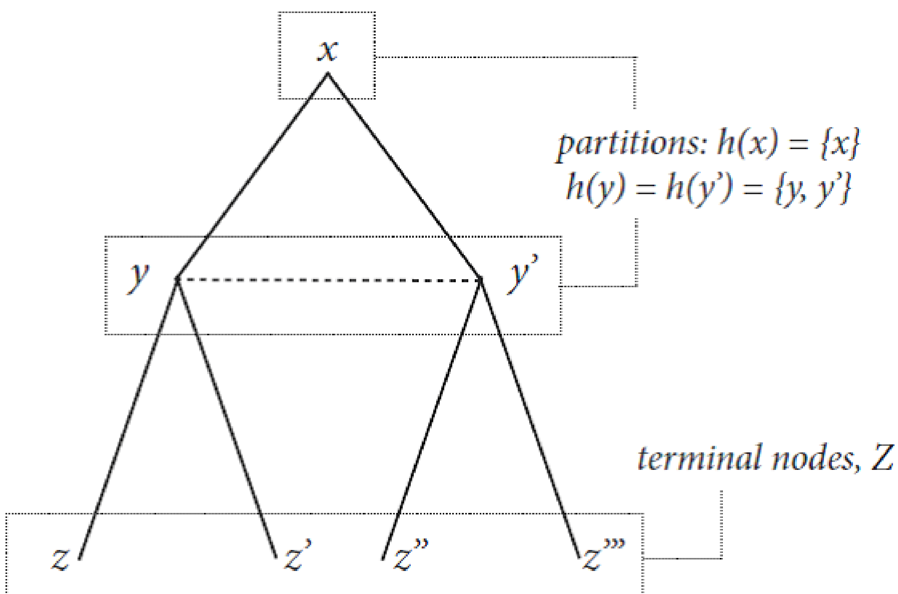
   ii. If $x' \in h(x)$, then the set of feasible **actions** at $x'$ is the same as the set of feasible actions at $x$

5. Sets of feasible actions $A(h_i)$

   (a) Define $A(h_i)$ as the action set for agent $i$ at information set $h_i$.

   (b) A pure strategy for player $i$ is then a map $s_i : \ H_i \to A_i$

6. Probability distributions over exogenous events, i.e. moves of nature




**What is a game of perfect information?**

In these games, all information sets are singletons.
That is, at any given move, the player knows exactly which node he is at.

**What is a game of perfect recall?**

In these games, no information is lost along the road: that is, what a player once knew, he will continue to know.
Formally, we require:

1. If $x' \in h(x)$ then $x \nsucc x'$ and $x' \nsucc x$:
   (a) nodes in the same information set cannot be predecessors of one another
2. If $x'' \in h(x'),\ x \succ x',\ i(x) = i(x')$ then there exists a $\hat{x} \in h(x)$ such that the action from $x$ to $x'$ is the same as the action from $\hat{x}$ to $x''$. [formality]

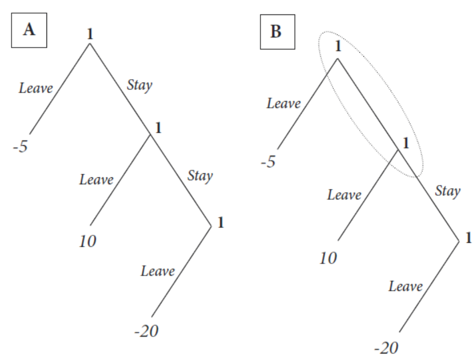



Figure 4: A game of perfect recall (A) and imperfect recall (B).

**What is a multi-stage game with observed actions?**

These games are characterized by stages which fulfill the following requirements:

1. In each stage $k$ every player knows all actions (including those taken by nature) in all previous stages
2. Each player moves at most once within any given stage
3. No information set contained at stage $k$ provides any knowledge of play at that stage

These specifications ensure that there is a well-defined history $h^k$at the start of each stage $k$. Pure strategies therefore specify actions $a_i \in A_i\left(h^k\right)$ for each $h^k$.
[So in multi-stage games, players have history-contingent strategies (contigent on the particular $h^k$ reached at each stage $k$)].

**What are the two definitions for a mixed strategy? How do they differ?**

- *Behavior Strategies*
  - We could define a mixed strategy as an element of $\prod_{h_i \in H_i} \Delta\left(A(h_i)\right)$.
  - Each behavior strategy specifies an independent probability distribution over actions at each information set $h_i$.

- *Mixed Strategies*
  - We could define a mixed strategy as an element of $\Delta(s_i) = \Delta\left(\prod_{h_i \in H_i} A(h_i)\right)$
  - Each mixed strategy specifies a randomization over complete paths of play through the game tree.

Differences

- An agent employing mixed strategies can correlate his decisions at different information sets, but an agent employing behavioral strategies cannot
- Mixed strategies are a more general notion than behavioral strategies
  - There exist mixed strategies for which there is no equivalent behavioral strategy
  - Each mixed strategy corresponds to only one behavioral strategy, but several mixed strategies can generate the same behavioral strategy

**In a game of perfect recall, what is the relationship between mixed and behavioral strategies? Why does this matter for our purposes?**

Theorem 2.1 (Kuhn): In a game of perfect recall, every mixed strategy is equivalent to the unique behavioral strategy it generates, and every behavioral strategy is equivalent to every mixed strategy that generates it.

For our purposes, we will henceforth consider only games of perfect recall, and use the term "mixed strategy" to mean behavioral strategies.

**What is the normal form for mixed strategies?**

$$\Gamma_N = [I, \{\Delta S_i\}, \{u_i(\cdot)\}]$$

**What is a mixed strategy?**

A mixed strategy $\sigma_i$ is a probability distribution over pure strategies. Each player's randomization is statistically independent of those of his opponents.
Definition 7.E.1: Given player $i$'s (finite) pure strategy set $S_i$, a *mixed strategy* for player $i$, $\sigma_i : S_i \to [0, 1]$, assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that will be played, where $\sum_{s \in S_i} \sigma_i(s_i) = 1$.

The *support* of a mixed strategy $\sigma_i$ is the set of pure strategies to which $\sigma_i$ assigns positive probability.

**What is the set of possible mixed strategies?**

MWG:
Player $i$'s set of possible mixed strategies can be associated with the points of the following simplex:

$$\Delta(S_i) = \left\{ \sigma_{1i}, \ldots, \sigma_{Mi} \in \mathbb{R}^M : \sigma_{mi} \geq 0 \text{ for all } m = 1, \ldots, M \text{ and } \sum_{m=1}^{M} \sigma_{mi} = 1 \right\}$$

F&T:
The space of player $i$'s mixed strategies is denoted

$$\sum_i$$

where $\sigma_i(s_i)$ is the probability that $\sigma_i$ assigns to $s_i$.

Note that a pure strategy is a special case of a mixed strategy.

**What is player $i$'s payoff to $\sigma$?**

$\sigma$ is a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ for the $I$ players.
Player $i$'s von Neumann-Morgenstern utility from mixed strategy profile $\sigma$ is:

$$\sum_{s \in S} \left( \prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(s)$$

[Utility from $s$, times the probability of that $s$ occuring, summed over all $s \in S$.]

# 3 ISD & Rationalizability

### 3.1 ISD

**Verbally, what is domination?**

Suppose there is a strategy $s_i'$, which yields a higher payoff for player $i$ than the strategy $s_i$ *regardless of opponents' actions*. Then we should never expect to see him play $s_i$.
Assume: 1) finite strategy spaces, 2) expected utility maximizing agents, 3) common knowledge of the structure of the game and its payoffs

**When is a strategy strictly dominated?**

Pure strategy $s_i$ is *strictly dominated for player $i$* if there exists $\sigma_i \in \Delta(S_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all} \quad s_{-i} \in S_{-i}$$

**Example of a strictly dominated strategy**

| | $C$ | $D$ |
|---|---|---|
| $C$ | $2, 2$ | $-1, 3$ |
| $D$ | $3, -1$ | $0, 0$ |

$C$ is a strictly dominated strategy

**If a strategy is strictly dominated for all pure strategies $s_{-i}$, then is it also strictly dominated for all mixed strategies $\sigma_{-i}$?**

Yes, since mixed strategies are all convex cominbations of pure strategies.

**Can a pure strategy be strictly dominated by a mixed strategy, even if it is not strictly dominated by any pure strategy?**

Yes (by the above definition).
[Example on p. 6 of S&T]

**Can a mixed strategy be strictly dominated even if none of the pure strategies in its support are dominated?**

Yes.
For example:

| | $L$ | $R$ |
|---|---|---|
| $U$ | $1,3$ | $-2,0$ |
| $M$ | $-2,0$ | $1,3$ |
| $D$ | $0,1$ | $0,1$ |

The mixed strategy $\frac{1}{2}U + \frac{1}{2}D$ is strictly dominated by $D$ even though neither of the pure strategies in its support ($U$ and $M$) are strictly dominated.

**What is ISD? Formally and Verbally**

ISD is exactly what its name suggests: we iteratively eliminate strictly dominated strategies, yielding at each stage a smaller subset of surviving strategies.

Let $S_i$ denote the set of pure strategies available to player $i$
Let $\sum_i$ denote the set of mixed strategies.
Then:
Let $S_i^0 \equiv S_i$ and $\sum_i^0 = \sum_i$ for all $i$. Define recursively:

$$S_i^n = \left\{ s_i \in S_i^{n-1} | \nexists \sigma_i \in \Sigma_i^{n-1} s.t.\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}^{n-1} \right\}$$

$$and\ \Sigma_i^n = \{\sigma_i \in \Sigma_i | \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n\}$$

Then the set of player $i$'s pure strategies which survive ISD is given by $S_i^\infty = \bigcap_{n=0}^\infty S_i^n$

Verbally:
$S_i^n$ is the set of player $i$'s strategies that are not strictly dominated by a strategy in $\Sigma_i^n$, when players $j \neq i$ are constrained to play strategies in $S_j^{n-1}$.
$\Sigma_i^n$ is simply the set of mixed strategies over $S_i^n$.

*So you are iterating on pure strategies; and at any given stage you mix over the remaining pure strategies to know* $\Sigma_i^n$. However, a pure strategy is eliminated if it is dominated by any mixed strategy in $\Sigma_i^{n-1}$

**Does the above inequality hold when opponents use mixed strateges $\sigma_{-i}$ as well?**

Yes. For a given $s_i$, strategy $\sigma_i'$ satisfies the above inequality $\iff$ it satisfies the inequality for all mixed strategies $\sigma_{-1}'$ as well, because player $i$'s payoff when his opponents play mixed strategies is a convex combination of his payoffs when his opponents play pure strategies.

**Can a mixed strategy be strictly dominated, if it assigns positive probabilities only to pure strategies that are not even weakly dominated?**

Yes.
[Example on p. 7 of S&T]

**When is a pure strategy weakly dominated?**

Pure strategy $s_i$ is *weakly dominated for player i* if there exists $\sigma_i \in \Delta(S_i)$ such that

$$u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all} \quad s_{-i} \in S_{-i}$$

and the inequality is strict for at least one $s_{-1}$.

**Does order of elimination matter in ISD?**

Order of elimination does not matter in ISD.
Order of elimination *does* matter when eliminating weakly dominated strategies.

**3.2 Rationalizability**
**Verbally, what is rationalizability?**

Rationalizability is motivated by the question, "what strategies are *reasonable* to play?" (the 'inverse' of the question motivating ISD, 'what strategies are unreasonable for a player to play.')
F&T: A rational player will use *only those strategies that are best responses* to some beliefs he might have about his the strategies of his opponents.

**What is the formal definition of rationalizability?**

Let $\tilde{\Sigma}_i^0 \equiv \Sigma_i$ and define:

$$\tilde{\Sigma}_i^n = \left\{ \sigma_i \in \tilde{\Sigma}_i^{n-1} | \ \exists \sigma_{-i} \in \Pi_{j \neq i} convex\ hull\left(\Sigma_j^{n-1}\right) \ s.t.\ u_i(\sigma_i, \sigma_{-1}) \geq u_i(\sigma_i', \sigma_{-1}) \ \forall \sigma_i' \in \tilde{\Sigma}_i^{n-1} \right\}$$

The set of rationalizable strategies is given by $R_i = \bigcap_{n=0}^{\infty} \tilde{\Sigma}_i^n$

**What is a convex hull?**

The convex hull of a set $X$ is the minimal convex set containing $X$.

#### 3.3 Relationship Between ISD & Rationalizability

**Which is a subset of which?**

$$R_i \subseteq S_i^{\infty}$$

If a strategy $s_i$ is strictly dominated, then it is never a best response.

*Surviving ISD means that a strategy is not strictly dominated, but this doesn't mean it's a best response to an* (uncorralted)

**Under what circumstances do they coincide?**

Rationalizability and ISD coincide in 2-player games.

**When does the equivalence break down?**

The equivalence breaks down in games of three or more players.
The problem is that ISD is conducted over the space which includes *correlated equilibria*, while rationalizability is not.
If one allows for correlated equilibria in rationalizability (called correlated rationalizability), then the two concepts become equivalent.

In ISD, $\sigma_{-i}$ is an arbitrary probability distribution over $S_i^n$, that is, it exists in the space $\Delta\left(S_{-i}^n\right)$ which includes probability distributions that are not acheivable without correlated action by player $i$'s opponents.
In rationalizability, the space of independent mixed strategies for $i$'s opponents is instead $\Pi_{j \neq i} \Delta\left(S_j^n\right)$, which is not convex and is indeed smaller than $\Delta\left(S_i^n\right)$.

# 2 Nash Equilibrium

#### 1.2 Nash Equilibrium

**What type of equilibrium concepts are there? What type is Nash?**

Quoting Annie Liang:
What requirements should a strategy profile meet if it to sustain as an equilibrium? One intuitive answer is that no player should be able to gain from deviation, but it is not obvious what nature of deviations we should consider {single deviations? Pairwise deviations? Coalitional deviations?}. There exist equilibria concepts that correspond to all of these(1), but the concept that has taken clear dominance (perhaps due to tractability) is that of single- player deviations. *Strategy profiles robust to single-player deviations are known as Nash equilibria.*

(1) See coalition-proof and strong-equilibria, for example

As will be discussed in the coming sections, there are many insufiencies to this concept. The most striking, perhaps, is that games often have multiple (unranked) NE. The following definition is thus taken only as a foundational requirement for equilibria, with a great many possible additional restrictions. These restrictions, which will be discussed in subsequent sections, are known collectively as "equilibrium refinements."

**What is the definition of Nash Equilibrium?**

A Nash equilibrium is a profile of strategies such that each player's strategy is an optimal response to the other players' strategies.
Definition 1.2: A mixed-strategy profile $\sigma^*$ is a *Nash equilibrium* if, for all players $i$,

$$u_i\left(\sigma_i^*, \sigma_{-1}^*\right) \geq u_i\left(s_i, \sigma_{-1}^*\right) \text{ for all } s_i \in S_i$$

[A pure strategy Nash equilibrium is a pure strategy profile that satisfies the same conditions]
"If everyone's doin it, then I'll do it"

**What is the relationship between Rationalizability and Nash Equilibrium?**

*Every strategy that is part of a Nash equilibrium profile is rationalizable*, because each player's strategy in a Nash equilibrium can be justified by the Nash equilibrium strategies of other players.

**Why do we only require that the payoff received from playing $\sigma_i^*$be greater than or equal to the payoff received from playing any other *pure* strategy $s_i$, and not any other *mixed* strategy $\sigma_i$?**

Recall that we assumed utility is of expected utility form (and hence linear in probabilities).
The utility received from any mixed strategy is simply a convex combination of the actions in the mixed strategy's support–which cannot exceed the utility of the best action in the support.

**What is one thing that all players in a Nash equilibrium share?**

All players in a Nash equilibrium share a *single common belief* regarding player actions.

**If a player uses a nondegenerate mixed strategy equilibrium, what can be said about the process of randomization that the player uses?**

For mixed strategies, we require additionally that these randomizations are independent of the randomizations of any other player.
(This condition is relaxed when we discuss *correlated equilibria*).

**If a player uses a nondegenerate mixed strategy equilibrium, what can be said about the pure strategies to which she assigns positive probability?**

She must be indifferent between all pure strategies to which she assigns positive probability.

MWG: Proposition 8.D.1: Let $S_i^+ \subset S_i$ denote the set of pure strategies that player $i$ plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$. Strategy profile $\sigma$ is a Nash equilibrium in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}] \iff$ for all $i = 1, \ldots, I$:

1. $$u_i(s_i, \sigma_{-1}) = u_i(s_i', \sigma_{-1}) \quad \forall\ s_i, s_i' \in S_i^+$$

2. $$u_i(s_i, \sigma_{-1}) \geq u_i(s_i'', \sigma_{-1}) \quad \forall\ s_i \in S_i^+ \text{ and } s_i'' \notin S_i^+$$

**When is a Nash equilibrium "strict"? What does this imply about the equilibrium strategy? Is this the same as "unique"?**

A Nash equilibrium is *strict* if each player has a unique best response to his rivals' strategies.
That is, $s^*$ is a strict equilibrium $\iff$ it is a Nash equilibrium and, for all $i$ and all $s_i \neq s_i^*$,

$$u_i\left(s_i^*, s_{-1}^*\right) > u_i\left(s_i, s_{-1}^*\right)$$

By definition, a strict equilibrium is necessarily a pure-strategy equilibrium.
*However, a game can have a* ***unique*** *equilibrium that is in (nondegenerate) mixed strategies, so strict equilibria need not exist.*

**Nash Equilibria and Public Randomization**
*What happens to the acheivable set of Nash equilibria with public randomization?*

With public randomization, the set of achievable equilibria expands to the *convex hull* of Nash equilibria in the uncoordinated game.

**Is there a relationship between a single strategy profile surviving iterated deletion of strictly dominated strategies, and Nash equilibrium?**

Yes:
If a single strategy profile survives iterated deletion of strictly dominated strategies, then it is a Nash equilibrium of the game.
Conversely, any Nash-equilibrium profile must put weight only on strategies that are not strictly dominated (or more generally, do not survive iterated deletion of strictly dominated strategies), because a player could increase his payoff by replacing a dominated strategy with one that dominates it.
[However, Nash equilibria may assign positive probability to weakly dominated strategies]

### 1.3 Existance and Properties of Nash Equilibria

**What is the statement of the existance theorem?**

Every finite strategic-form game has a mixed strategy equilibrium.

**What is Kakutani's theorem?**

Kakutani's fixed point theorem:

AL Theorem 3.2:
The following conditions are sufficient for $r : \sum \rightrightarrows \sum$ to have a fixed point:

1. $\sum$ is a compact, convex and nonempty subset of a finite-dimensional Euclidean space.
2. $r(\sigma)$ is nonempty for all $\sigma$.
3. $r(\sigma)$ is convex for all $\sigma$.
4. $r(\sigma)$ has a closed graph: if $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in r(\sigma^n)$ then $\hat{\sigma} \in r(\sigma)$. [a.k.a. $r(\sigma)$ is upper-hemicontinuous]

Also think about it as :

1. $\sum$ is a <u>convex, nonempty and compact</u> subset of a finite-dimensional Euclidean space.
2. $r(\sigma)$ is <u>convex, nonempty and upperhemicontinuous</u> for all $\sigma$.

**How do we prove the existance of a mixed-stategy equilibrium?**

We define two things:

1. Define player $i$'s *reaction correspondance* $r_i$ as the mapping of each strategy profile $\sigma$ to the set of strategies that maximizes player $i$'s payoff for the given $\sigma_{-1}$.
   (a) Think of this as the ***best response correspondance***
2. Define $r : \sum \rightrightarrows \sum$ as the Cartesian product of $r_i$ for all $i$.

Then, a fixed point of $r$ is a $\sigma$ such that $\sigma \in r(\sigma)$ (and hence, for each player, $\sigma_i \in r(\sigma_i)$).
*This fixed point is a Nash Equilibrium.*
[In order to prove the existance of a NE, we only need to prove the existance of such a fixed point]
We show that $r$ fulfuills each of Kakutani's four conditions [below]

**How do we demonstrate that $r$ fulfills each of Kakutani's four conditions?**

1. $\sum$ is a compact, convex and nonempty subset of a finite-dimensional Euclidean space
   (a) $\sum$ is a simplex of dimension $\#S_i - 1$ for all $i$.
2. $r(\sigma)$ is nonempty for all $\sigma$
   (a) Each player's payoff function is linear and therefore continuous in his own mixed strategy.
   (b) *Continuous functions obtain maxima on compact sets*; therefore there exists some strategy $\sigma_i'$ for every profile $\sigma_{-1}$ that maximizes $i$'s payoff. [It follows that $r(\sigma)$ is nonempty]
3. $r(\sigma)$ is convex for all $\sigma$
   (a) Suppose $r(\sigma)$ is not convex. Then, there exists $\sigma', \sigma'' \in r(\sigma)$ and $\lambda[0,1]$ such that $\lambda\sigma' + (1-\lambda)\sigma'' \notin r(\sigma)$.

(b) Utility is assumed to take EU (expected utility) form, so for each $i$:

$$u_i(\lambda\sigma' + (1-\lambda)\sigma'') = \lambda u_i(\sigma, \sigma_{-1}) + (1-\lambda)u_i(\sigma'', \sigma_{-1})$$

(c) But since $\sigma'$ and $\sigma''$ are both best responses to $\sigma_{-i}$, their weighted average must also be a best response.

(d) It follows that $\lambda\sigma' + (1-\lambda)\sigma'' \in r(\sigma)$, and we have the desired contradiction.

4. $r(\sigma)$ has a closed graph: if $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in r(\sigma^n)$ then $\hat{\sigma} \in r(\sigma)$. [aka $r(\sigma)$ is upper hemicontinuous]

   - Think: $(\underbrace{\sigma^n}_{x}, \underbrace{\hat{\sigma}^n}_{y}) \to (\underbrace{\sigma}_{x}, \underbrace{\hat{\sigma}}_{y})$, and then $\underbrace{\hat{\sigma}}_{y} \in r(\underbrace{\sigma}_{x})$

   (a) Suppose $r(\sigma)$ is not closed.

   (b) Then there exists some sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ such that $\hat{\sigma}^n \in r(\sigma^n)$ and $\hat{\sigma} \notin r(\sigma)$.

   (c) This implies the existance of some player $i$ for which $\hat{\sigma}_i \notin r_i(\sigma)$. Equivalently, there exists an $\epsilon > 0$ and $\sigma_i'$ such that $u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon$. Then, for $n$ sufficiently large we have:

      i. $u_i\left(\sigma_i', \sigma_{-i}^n\right) > u_i(\sigma_i', \sigma_{-i}) - \epsilon$
      ii. $> u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon - \epsilon$
      iii. $= u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon$
      iv. $> u_i\left(\hat{\sigma}_i^n, \sigma_{-i}^n\right) + \epsilon$

   (d) where i. and iv. follow from the continuity of $u$, and ii. follows from the result above.

   (e) $\sigma_i'$ therefore does strictly better than $\hat{\sigma}_i^n$ against $\sigma_i^n$, contradicting $\hat{\sigma}_i^n \in r_i(\sigma^n)$

   $r: \sum \rightrightarrows \sum$ thus fulfills all four conditions of Kakutani's Theorem and has a fixed point. Equivalently, the corresponding game has a Nash equilibrium.

# 4 Bayesian Games

**What is happening in a Bayesian Game?**

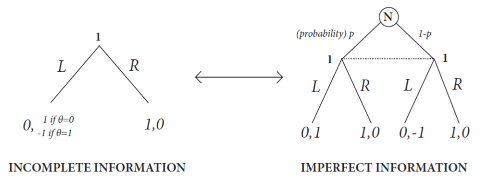
- Players maximize expected utility through *type contingent strategies:* $EU(s_i|\theta_i)$
- Payoff grid
  - Like Nash Equilibria, except:
    - Each row/column specifies outcomes from set of "type-contingent strategies," i.e. $UD$ or $DD$, etc if there are 2 types
    - Outcomes in each box are determined by expected probabilities
      - [In 2 type, 2 strategy case, there will be four terms to get the utility for each player]
- Select Nash Equilibria from payoff grid as in typical NE

**What is motivating the Bayesian game?**

We've assumed complete information about the structure of the game as well as the payoff functions of the players.
Bayesian games relax this latter assumption: want to see what is happening when agents have *incomplete information about others' payoff functions.*

**What can be termed the "approach" of a Bayesian game?**

Turn a game of *incomplete information* to a game of *imperfect information.*
This is accomplished through an external randomizing agent–nature.
Nature selects the *type* of each agent, which corresponds to his *payoff function.*
Ex:




**What are the formal components of a Bayesian game?**

- A set of agents $I = \{1, \ldots, n\}$
- Agent types $\theta_i$ independently drawn from (possibly agent-specific) type spaces $\Theta_i$
- A (common) prior probability distribution over types $p(\theta_1, \ldots, \theta_I)$
- Pure strategy spaces $S_i$
- Payoff functions $u_i(s_1, \ldots, s_n; \theta_1, \ldots, \theta_n)$

Assume common knowledge regarding all aspects of the game except for private realizations of $\theta_i$ (e.g. strategy spaces, possible types, and so forth)

**Does Bayesian equilibrium allow for possibility of erroneous or differing beliefs? Where is the uncertainty in Bayesian equilibrium?**

Unlike ISD and rationalizability, there is not room for erroneous beliefs.
In equilibrium, **every player knows the strategy $s_i(\theta_i)$ of every other player–the uncertainty is over the realizations of $\theta_i$, *not* over the mapping from $\Theta_i$ to $S_i$ itself.**

**What are players in a Bayesian game trying to maximize in equilibrium?**

In Bayesian equilibrium, players will maximize their *expected payoffs.*
That is, in equilibrium, each agent will play the action that yields the (weakly) highest action in expectation to the equilibrium strategies $s_{-i}: \Theta_{-i} \to S_{-i}$ of the other agents.

**How is expected utility from a Bayesian game expressed formally?**

Recall that we assumed common priors over $\Theta_i$. For any player, there are two ways of calculating expected utility:

1. **Ex-ante expected utility**

   Before realization of his own type $\theta_i$, the expected utility to $i$ of playing $s_i(\theta_i) \in S_i$ is given by:

   $$EU(s_i) = \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) u_i(s_i(\theta_i),\ s_{-i}(\theta_{-i});\ \theta_i, \theta_{-i})$$

   (a) Note to self: I am almost sure that this is a "strategy set $s_i$" in the sense of a set of contingent strategies, dependent on each $\theta_i$. Otherwise we wouldn't get equivalence between the two definitions, especially in regards to the NE

2. **Interim expected utility**

   Following the realization of his type $\theta_i$, the expected utility to $i$ of playing $s_i \in S_i$ is given by:

   $$EU(s_i|\theta_i) = \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i',\ s_{-i}(\theta_{-i});\ \theta_i, \theta_{-i})$$

Also, note that (1) is the expectation of (2).

**What is the formal definition of a Bayesian Nash Equilibrium?**

A Bayesian Nash Equilibrium is any strategy profile in which every player $i$ maximizes either (1) or (2) from above. That is

1. Using Ex-ante definition

   $$s_i \in \arg\max_{s_i' \in S_i} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) u_i(s_i(\theta_i),\ s_{-i}(\theta_{-i});\ \theta_i, \theta_{-i})$$

2. Using interim definition

   $$s_i(\theta_i) \in \arg\max_{s_i' \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(s_i',\ s_{-i}(\theta_{-i});\ \theta_i, \theta_{-i})$$

Since all types have positive probability, the two maximizations are equivalent.

**Where do we see a difference between the interim and ex-ante approaches? What is this difference?**

In ISD.
*ISD is at least as strong in the ex-ante interpretation as in the interim interpretation.*

In the ex-ante approach, the pooling of consequences allows the large potential benefits to outweigh the small potential costs.

## 5 Subgame Perfection

**Subgame Definition**
*What is the definition of a subgame?*

A proper subgame $G$ of an extensive-form game $T$ consists of *a single node and all its successors in $T$*, with the property that if $x' \in G$ and $x'' \in h(x')$ then $x'' \in G$.

**Backward Induction**
*What is the key solution strategy in SPE? When does it apply?*

Backward induction can be used in finite games that are subgame perfect. Proceed from the terminal nodes to determine winners, and proceed "backwards," replacing the nodes with the corresponding terminal payoffs at each step.

**Subgame Perfection**
*When is a strategy profile SPE?*

Definition: A strategy profile of an extensive-form game is a subgame-perfect equilibrium if the restriction of $\sigma$ to $G$ is a Nash equilibrium of $G$ for every proper subgame $G$.

Another way of saying: strategies must define a Nash equilibria in the smaller continuation game beginning at any singleton information set.

**OSDP**
*What is the OSDP?*

Definition: Strategy profile $s$ satisfies the one-stage deviation condition if there is no player $i$ and no strategy $\hat{s}_i$ such taht:

- $\hat{s}_i$ agrees with $s_i$ except at a single $h^t$
- $\hat{s}_i$ is a better response to $s_{-i}$ than $s_i$ conditional on that history $h^t$ being reached

Another way of saying: $s$ satisfies the one-stage deviation principle if there is no agent who could gain by deviating from $s$ at one stage, and one stage only, and reverting subsequently back to $s$.

**OSDP and Finite Multistage Games**
*What is the relationship between SPNE and OSDP in finite multi-stage games?*

In a finite multi-stage game with observed actions, strategy profile $s$ is sub-game perfect $\iff$ it satisfies the one-stage deviation condition.

**OSDP and Infinite Multistage Games**
*What is the relationship between SPNE and OSDP in infinite multi-stage games?*

1. Definition: Continuous at Infinity

   A game is continuous at infinity if for each player $i$ the utility function $u_i$ satisfies

$$\sup_{h,\tilde{h}\ s.t.\ h^t=\tilde{h}^t} |u_i(h) - u_i(\tilde{h})| \to 0 \quad as \quad t \to \infty$$

   [Note that in the above, $u_i(h) - u_i(\tilde{h})$ given the constraint means that the the two strategies agree until time $t$, and then can be whatever]

2. ODSP and Infinite Multistage Games

   Definition: In an infinite-horizon multi-stage game with observed actions *that is continuous at infinity*, profile $s$ is subgame perfect $\iff$ it satisfies the one-stage deviation condition.

# 6 Repeated Games

## Preliminary Definitions

**Repeated Games Definition**

*What are repeated games?*

The repeated game with observed actions is a special class of multistage games with observable actions in which a *stage game* is repeated in every period.
It has the following characteristics:

- Each stage game is a finite $I$-player simaltaneous-move game with finite action spaces $A_i$ and payoff functions $g_i:\ A \to \mathbb{R}$, where $A = \prod_{i\in\mathcal{I}} A_i$.
- *All actions are observed at the end of each stage game, so that players can condition strategies on past histories* $h^t = (a^0, a^1, \ldots, a^{t-1})$
- Pure strategies $s_i$ are sequences of mappings $s_i^t$ from possible period-t histories $h^t \in H^t$ to actions $a_i \in A_i$, and mixed strategies are sequences of mappings $\sigma_i^t$ from $H^t$ to mixed actions $\alpha_i \in A_i$.
  - [Note that players condition on action histories (observed), not strategies directly. If all agents are playing pure strategies, then these are the same thing, but not if agents are playing mixed strategies]

**Set of Equilibrium Outcomes in Repeated Games**

*What is the intuition behind the set of equilibrium outcomes in repeated games?*

*Because players can condition on histories, the set of equilibrium outcomes for the repeated stage game can exceed* that of the equilibrium outcomes in the one-shot stage game.
The purpose of the Folk Theorems is to fully characterize the set of payoffs that can be achieved in the equilibrium of a repeated game. [There are many, here we see a few foundational ones]

**Reservation Utility**
*What is a player's reservation utility in a game?*

A player's *reservation utility*, also known as the *minmax*, is given by:

$$\underline{v}_i = \min_{\alpha_{-i}} \left[ \max_{\alpha_i} g_i(\alpha_i, \alpha_{-i}) \right]$$

This represents the greatest payoff player $i$ can guarantee through his own action. Irregardless of his opponents' play, so long as $i$ chooses action $\arg\min_{\alpha_{-i}} [\max_{\alpha_i} g_i(\alpha_i, \alpha_{-i})]$, he will receive payoff of at least $\underline{v}_i$.

**Set of Feasible Payoffs [in repeated game]**
*What is the set of feasible payoffs acheivable through some combination of actions in $A$?*

The set of *feasible payoffs* is given by

$$V = \text{convex hull } \{v | \exists a \in A \text{ with } g(a) = v\}$$

That is, $V$ is the convex hull of the set of payoffs that are achievable through some combination of actions in $A$.
*Note: in calling this a "feasible" set, we are assuming the possibility of a public randomizing device. The problem is that some convex combinations of pure strategy payoffs correspond to *correlated strategies* and cannot be achieved by independent randomizations.

## Infinitely Repeated Games

**Aggregate Payoff**

We normalize total payoffs so that one util earned every period results in an aggregate payoff of 1:

$$u_i = E_\sigma (1 - \delta) \sum_{t=0}^{\infty} \delta^t g_i \left(\sigma^t \left(h^t\right)\right)$$

where $(1 - \delta)$ is the normalization factor, and $\delta^t g_i (\sigma^t (h^t))$ is the discounted payoff.

**Folk Theorem 1**

Theorem: For every feasible payoff vector $v$ for $v_i > \underline{v}_i$ for all players $i$, there exists a $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a Nash equilibrium of $G(\delta)$ with payoffs $v$.

Verbally:

- So long as players are sufficiently patient ($\delta$ sufficiently high), then any feasible individually rational payoff (by individually rational, we mean the payoff exceeds the reservation utility) can be sustained in a Nash Equilibrium.

- The strategy profile that corresponds to this payoff is the one in which players play the strategy that yields $v$, *and are minmaxed if they ever deviate.* (This is also known as a *grim-trigger strategy*).
- Note that this Folk Theorem ensures the existance of an NE, but *not an SPNE.* The intuition here is that players may want to deviate as punishers, because min-maxing their opponents may be a costly act.
- "NE for $v > minmax$"

**Friedman Folk Theorem (1.2)**

Theorem: Let $\alpha^*$ be a static equilibrium (an equilibrium of the stage game) with payoffs $e$. Then, for any $v \in V$ with $v_i > e_i$, for all players $i$ there is a $\underline{\delta}$ such that for all $\delta > \underline{\delta}$ there is a *subgame perfect equilibrium* of $G(\delta)$ with payoffs $v$.

Verbally

- The first folk theorem is adapted to make an SPNE as follows:
  - Instead of minmaxing players that deviate, it is agreed that all players will revert to the NE that yields payoffs $e$ if ever a deviation occurs
  - (Sometimes known as "Nash Threats")
- "SPNE for $v > Nash\ Threat$"

**Fudenberg and Maskin Theorem (1.3)**

Theorem: Assume that the dimension of the set $V$ of feasible payoffs equals the number of players. Then, for any $v \in V$ with $v_i > \underline{v}_i$, for all $i$, there is a discount factor $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a *SPNE* of $G(\delta)$ with payoffs $v$.

Verbally:

- In these cases, usually there will be a punishment for a finite number of periods before going back to playing $v$
  - (Can also be reward schemes, but we didn't see these in class)
- Hence with this slightly tigher set of restrictions, can get an *SPNE over the feasible set of payoffs from Folk Theorem 1*
  - According to Annie, these conditions are always satisfied in 2 player games
- "SPNE for $v > minmax$"

### Finite Horizon Repeated Games

#### Set Up

- Generally assume no discounting
- Punishments and rewards can unravel backwards. Therefore, the set of payoffs for finite and infinite games can be very different.

#### Set of Equilibrium Payoffs

Like the infinite case, there are circumstances under which the set of equilibrium payoffs can be *expanded*

- Play in the last period must constitute a Nash Equilibrium
  - The power to achieve other payoffs in rounds comes from the flexilibilty in specifying *which* NE is played
  - (Strategies depend on histories of observed play in previous stages)
- Hence, *it is imperative that there exist multiple equilibria*
  - If the stage game has a unique Nash Equilibrium, this NE will necessarily be played at every period

# 7 Equilibrium Refinements

## Perfect Bayesian Equilibrium

Motviation: introduce the requirement that players play optimally at each information set. That is, by limiting our analysis to proper subgames, we consider continuation games beginning at any information set. In order to constitute a PBE, strategies must yield a Bayesian equilibrium in each of these continuation games.

#### Strategy for Finding PBE [in 2x2 Signaling Game]
*What is the strategy for finding PBEs?*

**Note: Handout from MIT on this is *excellent*. Has examples; also implemented in my solutions to Problem 4 of pset 4.
0. Decide whether you're looking for a separating, pooling, or semi-separating equilibrium

- Pooling Equilibrium: both types of Player 1 play the same strategy [don't forget about off-equilibrium beliefs!]

- Separating Equilibrium: both types of Player 1 play a different strategy
- Semi-Separating Equilibrium: one type of Player 1 plays a pure strategy, while the other type of Player 1 plays a mixed strategy

1. Assign a strategy to Player 1
2. Derive Beliefs for Player 2
   (a) According to Bayes' rule at each information set reached with positive probability along the information set
   (b) Arbitrary beliefs (i.e. by using "$\lambda \in [0,1]$") at information sets off the equilibrium path
3. Determine Player 2's Best Response
4. Equilibrium Check
   (a) $\implies$ Check Player 1's strategy, in light of Player 2's response, to see if there incentive to deviate

[Rinse and repeat for each different type of PBE]

## Sequential Equilibria

Note: $(\sigma, \mu)$ is a profile of strategies $(\sigma)$ and beliefs $(\mu)$

Sequential equilibria *consider the limit of equilibria in a sequence of perturbed games.* (Hence it is similar to THPE in spirit, which we didn't discuss; THPE and SE coincide for almost all games).

There are two key conditions:

1. (S) An assessment $(\sigma, \mu)$ is <u>sequentially rational</u> if, for any information set $h$ and alternative strategy $\sigma'_{i(h)}$,

$$u_{i(h)}\left(\sigma | h, \mu\left(h\right)\right) \geq u_{i(h)}\left(\sigma'_{i(h)}, \sigma_{-\sigma(h)} | h, \mu\left(h\right)\right)$$

   (a) Sequentially rational: you're playing what makes sense at every information set.

2. (C) An assessment $(\sigma, \mu)$ is consistent if $(\sigma, \mu) = \lim_{n\to\infty}\left(\sigma^n, \mu^n\right)$ for some sequence $(\sigma^n, \mu^n)$ in $\Phi^0$. At each totally mixed step, Bayes Rule completely pins down beliefs.

*Definition:* A sequential equilibrium is an assessment $(\sigma, \mu)$ that satisfies conditions $S$ and $C$.

[Let $\Sigma^0$ denote the set of completely mixed strategies and let $\Phi^0$ denote the set of all assessments $(\sigma, \mu)$ such that $\sigma \in \Sigma^0$ and $\mu$ is uniquely defined from $\sigma$ by Bayes' rule.]

- **Properties of Sequential Equilibria**

  1. For any finite extensive-form game there exists at least one sequential equilibrium

2. As with NE, the SE correspondance is upper hemi-continuous with respect to payoffs. That is, for any sequence of utility functions $u^n \to u$, if $(\sigma^n, \mu^n)$ is an SE of the game with payoffs $u^n$, and $(\sigma^n, \mu^n) \to (\sigma, \mu)$, then $(\sigma, \mu)$ is a SE of the game with payoffs $u$.
3. *The set of SE can change with addition of "irrelevant moves."*
   (a) See figure below.
   (b) Roughly, this is because figure $B$ creates a new information set at which play must be optimal.
   (c) *Another way of seeing this:* We have not placed restrictions on the relative weights an agent can place on off-equilibrium moves. Therefore we can place positive weight on the strictly dominated strategy $L_1$ in $\boxed{\text{A}}$.

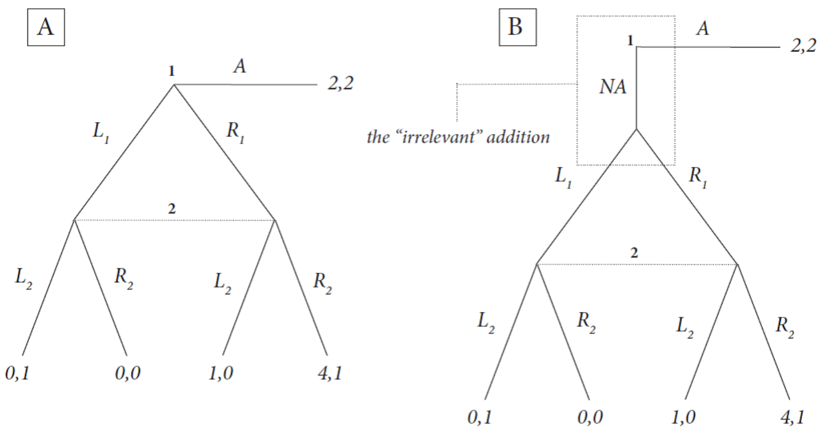


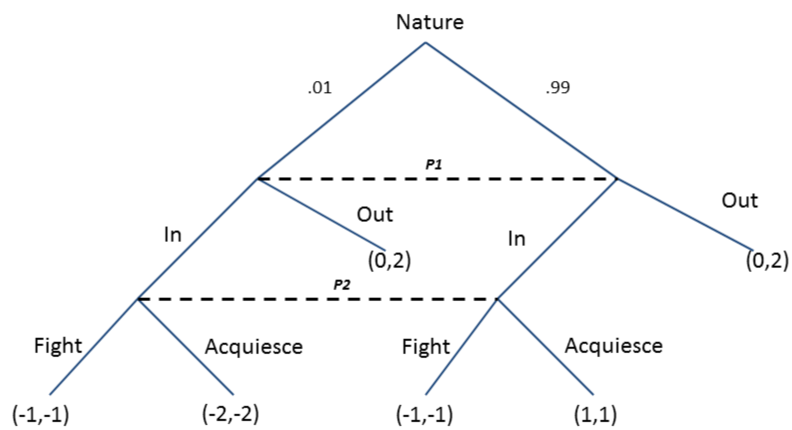

Figure 4: $A$ is a SE outcome in [A], but not in [B].

## Example: Firm Entry

*Example from Lecture*

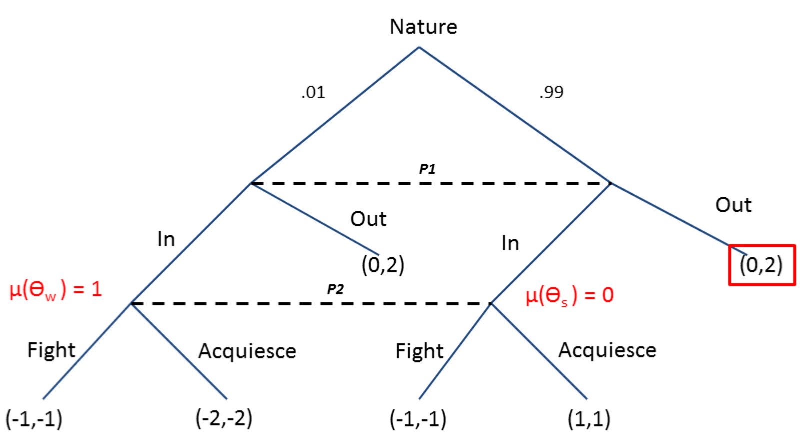
Game Set Up

- P1 is firm deciding to “enter market”
- P2 is firm in market deciding to fight or acquiesce if P1 enters




PBE, but not Sequential Equilibrium:

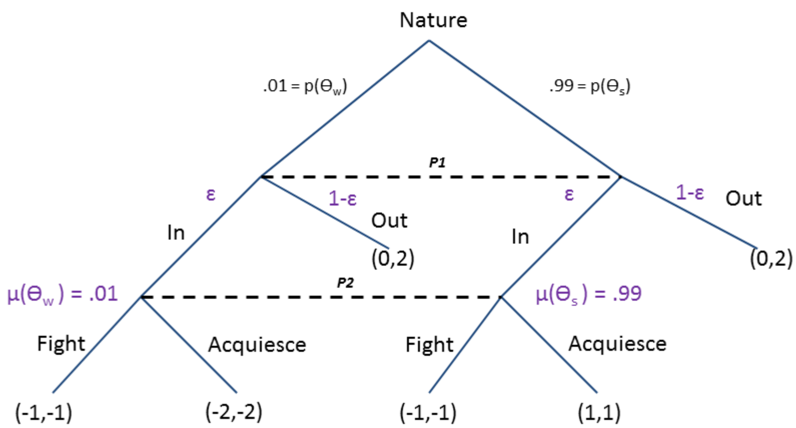
*P1: Out, P2: Fight*



(Out, Fight) not Sequential Equilibrium:

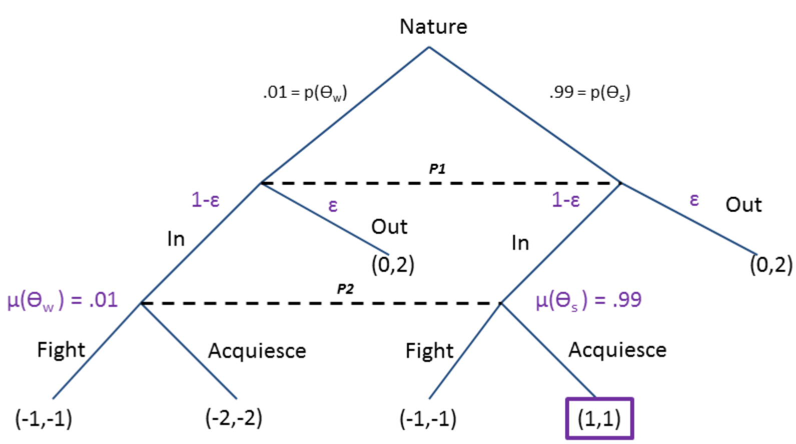
- Player 1's strategy is totally mixed, ε weight on *In*
- Player 2 Beliefs now consistent with Bayes Rule
  - *BR2*(In) = *Acquiesce*
- Player 1 now has profitable deviation to *In*

Hence (Out, Fight) is not an SE




Sequential Equilibrium:
(In, Acquiesce)

- Player 1 has totally mixed strategy
- Player 2's beliefs consistent with Bayes' rule
- Each player sequentially rational
- Taking lim ε → 0, get ∂ⁿ→ "*In*"
- Sequence of beliefs has μ(Θs ) = .99




*Additional Notes:*
Note that on sequential equilibria, one way to do this in practice is the following:

- Say there is an equilibrium action ($a_1$) and two off equilibrium actions ($a_2$ and $a_3$).

  - [Say $a_1$ goes to node $x_1$, $a_2$ goes to node $x_2$, and $a_3$ goes to node $x_3$]

- Place the following weights on each strategy: $1-\epsilon-\epsilon'$ on $a_1$, $\epsilon$ on $a_2$, and $\epsilon'$ on $a_3$.
- By Bayes' rule, this will give us
  $\frac{\epsilon}{\epsilon+\epsilon'}$ at node $x_2$, and $\frac{\epsilon'}{\epsilon+\epsilon'}$ at node $x_3$.
- *Note this means we can set $\epsilon$ and $\epsilon'$ to whatever they need to be to fulfill beliefs*
  - [No restriction on strategy weighting off the equilibrium path]

# 8 Forward Induction

*Forward induction refinements instead suppose that when an off-equilibrium information set is reached, players use this information to instruct their beliefs.*

This is in contrast to all the equilibrium refinements we've seen so far, which have supposed that players assume opponents will continue to play according to equilibrium strategies even after a deviation is observed.

*Notation:*
Consider a simple class of ***2-player signaling game*** in which:

- Player 1 has private information regarding his type $\theta \in \Theta$ and chooses action $a_1 \in A_1$
- Player 2 observes $a_1$ and chooses $a_2 \in A_2$
- for a nonempty subset $T \in \Theta$, $BR(T, a_1)$ is the set pure strategy best responses for player 2 to action $a_1$ for beliefs $\mu(\cdot|a_1)$ such that $\mu(T|a_1) = 1$; that is:

$$BR(T, a_1) = \cup_{\mu:\mu(T|a_1)=1} BR(\mu, a_1)$$

  - where $BR(\mu, a_1) = \arg\max_{a_2} \sum_{\theta\in\Theta} \mu(\theta|a_1)\, u_2(a_1, a_2, \theta)$
  - "Everything that is outside of $T$ has probability 0."

***Intuitive Criterion*:**

- Fix a vector of equilibrium payoffs $u_1^*$ for the sender.

1. For <u>each strategy $a_1$, let $J(a_1)$</u> be the set of all $\theta$ such that

$$u_1^*(\theta) > \max_{a_2 \in BR(\Theta, a_1)} u_1(a_1, a_2, \theta)$$

   - "I would never do this $a_1$ if I was in $\theta \in J(a_1)$"

2. If for some $a_1$ there exists a $\theta' \in \Theta$ such that

$$u_1^*(\theta') < \min_{a_2 \in BR(\Theta \setminus J(a_1), a_1)} u_1(a_1, a_2, \theta')$$

   then the equilibrium <u>fails the intuitive criterion</u>.

- "If I am $\theta'$, and you knew I wasn't in $J(a_1)$ *and* I played $a_1$, then your best response would automatically give me a better payoff than I am getting in equilibrium."
  - Then can use this to say:

- ***Generalized Speech:***
  - "I am not in $J(a_1)$, since then I would have never played $a_1$. Furthermore, now that you trust that I'm outside $J(a_1)$, you should change your strategy $a_2$ as we will both do better than if you stuck to your equilibrium strategy $a_2^*$."

*In words*:

- $J(a_1)$ is the set of all types that could not gain from deviating from equilibrium to $a_1$, *no matter* player 2's beliefs regarding his type
- An equilibrium will then fail the Intuitive Criterion if the only way to sustain it (i.e. the only way to keep player 1 from having a profitable deviation) is if player 2 holds the belief that player 1 is of some type $\theta \in J(a_1)$

## Example: Beer-Quiche Game

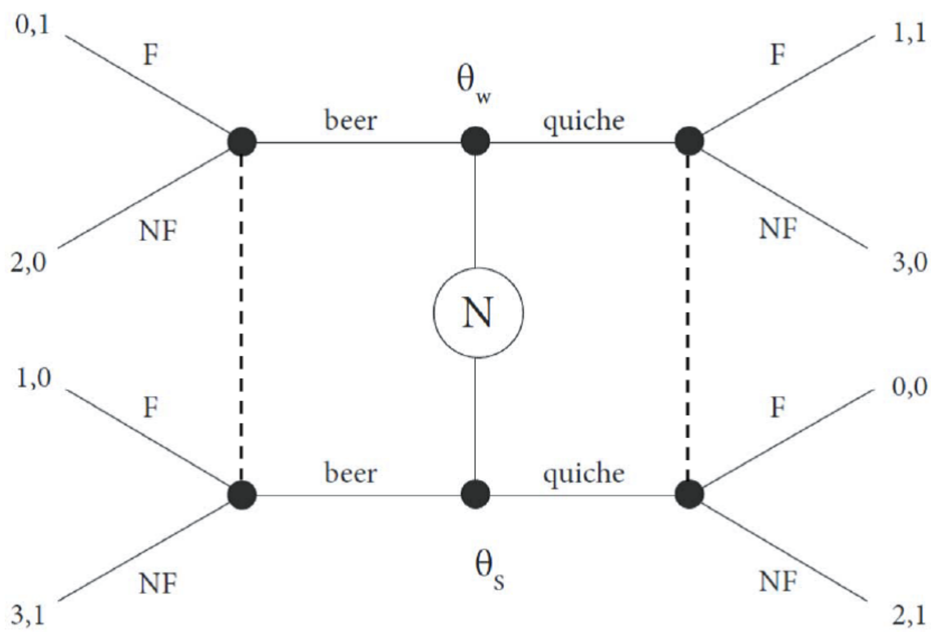



Figure 1: The Beer-Quiche Game.

For example: $\theta_w = .1, \quad \theta_s = .9$

- Beer-Quiche Game has *2 pooling PBE*
  1. Both Choose Beer
  2. Both Choose Quiche
- *Off Equilibrium Beliefs:*
  - In both PBE, off-equilibrium beliefs are that if unexpected breakfast is observed, there is probability of at least $\frac{1}{2}$ that player 1 was weak.
    - Hence, in off-equilibrium, player 2 chooses to fight.
- Forward Induction: ***Equilibrium in which both types choose Quiche not reasonsable***
  - This equilibrium fails the Intuitive Criterion
    - $\theta_w \in J(a_1 = beer)$
      - Weak type$\theta_w$ does better in equilibrium $[u_1(quiche, NF, \theta_w) = 3]$ than could do for any outcome playing beer $\left[\max_{a_2 \in BR(\Theta, beer)} u_1(beer, a_2, \theta_w) = 2\right]$
      - "Given that player 2 will not think player 1 to be the weak type $\theta_w$ if he observes a deviation from equilibrium, player 1 of type $\theta_s$ may indeed want to deviate."
    - $2 = u_1^*(\theta_s) < \min_{a_2 \in BR(\theta_s, beer)} u_1(beer, a_2, \theta_s) = u_1(beer, NF, \theta_s) = 3$
      - $\Theta \backslash J(a_1) = \theta_s$
      - Strong type does worse in equilibrium (2) than any outcome where he deviates to $a_1 = beer$, and player 2 plays a best response knowing that player 1 must not be $\theta_w$ by the above.
  - Speech:
    - "I am having beer, and you should infer from this that I am strong for so long as it is common knowledge that you will not fight if I eat quiche, I would have no incentive to drink beer and make this speech if I were $\theta_w$."

**Strategy:**

1. Set an equilibrium (i.e. Pooling Quiche)
2. See if there is a $J(a_1)$ for any $a_1$
   (a) Especially off-equilibrium $a_1$
3. If there is a $J(a_1)$ situation:
   (a) Look at what happens in that $a_1$, now that player 2 knows to play $BR(\Theta \backslash J(a_1), a_1)$
4. If this refined $BR$ gives some $\theta$ of player 1 ($\theta \in \Theta \backslash J(a_1)$) a better payoff than in equilibrium $\Longrightarrow$ this equilibrium fails the intuitive criterion.

# 9 Auctions

## Auctions as a Bayesian Game

- Set of players $N = \{1, 2, \ldots, n\}$
- Type set $\Theta_i = [\underline{v}, \bar{v}]$, $\underline{v} \geq 0$
- Action set, $A_i = \mathbb{R}^+$
- Beliefs
  - Opponents' valuations are independent draws from a distribution function $F$
  - $F$ is strictly increasing and continuous
- Payoff Function

$$u_i(a, v) = \left\{ \begin{array}{cc} \frac{v_i - P(a)}{m} & \text{if } a_j \leq a_i \forall j \neq i, \text{and } |\{j : a_j = a_i\}| = m \\ 0 & \text{if } a_j > a_i \text{for some } j \neq i \end{array} \right\}$$

  - $P(a)$ is the price paid by the winner if the bid profile is $a$

## Second Price (Vickrey) Auctions

*What is a Second Price auction? What is the optimal strategy? How is this proved? What is the seller's expected revenue? What type of open bid auction is similar?*

*Overview*:
In a Vickrey, or Second Price, auction, bidders are asked to submit sealed bids $b_1, \ldots, b_n$. The bidder who has the highest bid is awarded the object, and pays the amount of the second highest bid.

***Equilibrium*:**

- In a second price auction, it is a weakly dominant strategy to bid one's value, $b_i(v_i) = v_i$

*Proof*:
Proof. Suppose $i$'s value is $s_i$, and she considers bidding $b_i > s_i$. Let $\hat{b}$ denote the highest bid of the other bidders $j \neq i$ (from $i$'s perspective this is a random variable). There are three possible outcomes from $i$'s perspective: (i) $\hat{b} > b_i, s_i$; (ii) $b_i > \hat{b} > s_i$; or (iii) $b_i,\ s_i > \hat{b}$. In the event of the first or third outcome, $i$ would have done equally well to bid $s_i$ rather than $b_i > s_i$. In (i) she won't win regardless, and in (iii) she will win, and will pay $\hat{b}$ regardless. However, in case (ii), $i$ will win and pay more than her value if she bids $\hat{b}$, something that won't happen if she bids $s_i$. Thus, $i$ does better to bid $s_i$ than $b_i > s_i$. A similar argument shows that $i$ also does better to bid $s_i$ than to bid $b_i < s_i$. Q.E.D.

*Uniqueness of Equilibrium*

The above equilibrium is the unique *symmetric* Bayesian Nash equilibrium of the second price auction. There are also assymetric equilibria that involve players using weakly dominated strategies. One such equilibrium is for some player $i$ to bid $b_i(v_i) = \bar{v}$ and all the other players to bid $b_j(v_j) = 0$.

*Expected Value*

Since each bidder will bid their value, the seller's revenue (the amount paid in equilibrium) will be equal to the second highest value. Let $S^{i:n}$ denote the $i$th highest of $n$ draws from distribution $F$ (so $S^{i:n}$ is a random variable with typical realization $s^{i:n}$). Then the seller's expected revenue is

$$\mathbb{E}\left[S^{2:n}\right]$$

*English Auctions:*

While Vickrey auctions are not used very often in practice, open ascending (or English) auctions are used frequently. One way to model such auctions is to assume that the price rises continuously from zero and players each can push a button to "drop out" of the bidding. In an independent private values setting, the Nash equilibria of the English auction are the same as the Nash equilibria of the Vickrey auction.

## First Price Auctions

*Overview*
In a sealed bid, or first price, auction, bidders submit sealed bids $b_1, \ldots, b_n$. The bidders who submits the highest bid is awarded the object, and pays his bid. Under these rules, it should be clear that bidders will not want to bid their true values. By doing so, they would ensure a zero profit. By bidding somewhat below their values, they can potentially make a profit some of the time.

- Bidding less than your value is known as "bid shading"

*Bayesian Equilibrium*

1. $n$ bidders
2. You are player 1 and your value is $v > 0$
3. You believe the other bidders' values are independently and uniformly distributed over $[0, 1]$
4. You believe the other bidders use strategy $\beta(v_j) = av_j$
5. Expected Payoff if you bid $b$

$$(v - b) \cdot prob\,(\text{you win})$$

$$(v - b) \cdot prob\,(b > av_2 \text{ and } b > av_3 \text{ and ... } b > av_n)$$

$$(v - b) \cdot prob\,(b > av_2) \cdot prob\,(b > av_3) \cdots prob\,(b > av_n)$$

$$= (v - b) \left(\frac{b}{a}\right)^{n-1}$$

- ***Intuition:*** You can think of this as:

$$= \underbrace{(v-b)}_{\text{gain from winning}} \quad \underbrace{\left(\frac{b}{a}\right)^{n-1}}_{\text{probability of winning}}$$

  - *Trade-off in b:* notice that the gain from winning is decreasing in $b$, while the probability of winning is increasing in $b$. So want to find the optimal bid that will balance these two.

6. Maximize Expected Payoff:

$$\max_b (v-b)\left(\frac{b}{a}\right)^{n-1}$$

$$\text{FOC}_b: \quad -\left(\frac{b}{a}\right)^{n-1} + (n-1)\frac{v-b}{a}\left(\frac{b}{a}\right)^{n-2} = 0$$

7. Solving for $b$:

$$b = \frac{n-1}{n}v$$

*Dutch Auction*
Open auction; price descends until someone bids. Seen as equivalent to first price auction.

## Revenue Equivalence

*Overview*:
Any auction in which

- the number of bidders is the same
- the bidders are risk-neutral
- the object always goes to the bidder with the highest value
- the bidder with the lowest value expects zero surplus

yields the same expected revenue.

*Allows us to draw connections about the four auctions we've seen: all have same revenue in expectation*

*Fails Under Risk Aversion*:

- Second price auction: unaffected by risk aversion
  - You can demonstrate that strategy of bidding your valuation is still optimal (proof holds since utility function remains monotonic)
- First price auction: Risk averse bid *always higher than risk neutral bid*

  - Risk is the possibility of losing

- $\implies$ Hence revenue equivalence fails in the risk averse case

- Demonstration of bid change in First Price Auction in risk-averse case:

  - $\theta_i$ is true type, $\hat{\theta}_i$ is the type $i$ pretends to be for the bid, $s$ is the bid strategy, $F(\hat{\theta})$ is the probability of $\hat{\theta}$ winning the auction [Note that Maskin also refers to this as $G(\hat{\theta})$, and will then sometimes use $F(\cdot)$ as the CDF]

$$\max_{\hat{\theta}} u_i\left(\theta_i - s\left(\hat{\theta}_i\right)\right) F\left(\hat{\theta}\right)$$

$$FOC_{\hat{\theta}}: \quad -u_i'\left(\theta_i - s\left(\hat{\theta}_i\right)\right) s'\left(\hat{\theta}_i\right) F\left(\hat{\theta}\right) + u_i\left(\theta_i - s\left(\hat{\theta}_i\right)\right) F'\left(\hat{\theta}\right) = 0$$

$$u_i'\left(\theta_i - s\left(\hat{\theta}_i\right)\right) s'\left(\hat{\theta}_i\right) F\left(\hat{\theta}\right) = u_i\left(\theta_i - s\left(\hat{\theta}_i\right)\right) F'\left(\hat{\theta}\right) \implies s'\left(\hat{\theta}_i\right) F\left(\hat{\theta}\right) = \frac{u_i\left(\theta_i - s\left(\hat{\theta}_i\right)\right)}{u_i'\left(\theta_i - s\left(\hat{\theta}_i\right)\right)} F'\left(\hat{\theta}\right) \implies$$

$$s'\left(\hat{\theta}_i\right) = \frac{F'\left(\hat{\theta}\right)}{F\left(\hat{\theta}\right)} \left\{ \underbrace{\frac{u_i\left(\theta_i - s\left(\hat{\theta}_i\right)\right)}{u_i'\left(\theta_i - s\left(\hat{\theta}_i\right)\right)}}_{(*)\text{ Term}} \right\}$$

- In the risk neutral case, the (*) term is 1. However, in the risk averse case, the (*) term $< 1$, so $s'\left(\hat{\theta}_i\right)$ increases with risk aversion.

  - Hence, given that the bid strategy increases with risk aversion for the first price auction, but stays the same with risk aversion for the second price auction, the revenue equivalence theorem fails with risk aversion

**Revelation Principle**

Look at a BE of a game
Then there is an equivalent Bayesian game in which agents truthfully reveal their types, so that the social planner can restrict attention to mechanisms with this property.

## Probability in Auctions - 2 player

*From Annie's Office Hours 12/13/12*

Say that we are player 1 with $v_1$, and are in equilibrium. The following will help us define the expected revenue if:

$$v_1 \text{ given}$$

$$\text{In first price auction: Expected Revenue } \int_0^{v_1} b(v_1)f(v_2)dv_2 = b(v_1)\int_0^{v_1} f(v_2)dv_2 = b(v_1)F(v_1)$$

We can decompose this into the value of the bid (which is just a function of $v_1$) in the first price auction, times the probability that $v_1$ is the highest valuation drawn, which is defined as $\int_0^{v_1} f(v_2)dv_2 = F(v_1)$.

$$\text{In second price auction: Expected Revenue } \int_0^{v_1} v_2 f(v_2)dv_2$$

Here, things are a bit different since the price paid when player two draws some $v_2$ also determines what player 1 pays. Hence the two terms in the integral

$$\int_0^{v_1} \underbrace{v_2}_{payment} \underbrace{f(v_2)}_{prob\ of\ this\ payment} dv_2$$

Definitions

- $F(x)$ is the cdf (the cumulative distribution function, so $F(y)$ is the probability that a given draw $x_1 \leq y$ )
- $f(x) = F(x)$ is the pdf (the probability distribution function, so $f(y)$ is the probability that a given draw $x_1 = y$)

We have the important identity:

$$F'(x) = f(x)$$

Example
We could have the cdf $F(x) = x^2$, so that $F(\frac{1}{2}) = \frac{1}{4}$. We would then have $f(x) = 2x$.

Note that all of the above has been defined given that we know $v_1$ (i.e. $v_1 = \frac{3}{4}$). To get total expected revenue, we'd need to integrate over all of the possible $v_1$ realizations (though she didn't show how to do this).

*Calculating total expected revenue*

- One good way to do this for the first price auction is to simply use the result for the second price auction and substitute it in, using the revenue equivalence theorem

*Calculating Expectation*
Expectation of $x$ when $x < b_1$ and when it is drawn from the pdf $f(x)$

$$\mathbb{E}(x) = \int_0^{b_1} xf(x)dx$$

# 10 Bargaining

## Infinite Horizon Bilateral Bargaining

- Set Up:
  - 2 Player bargaining game
    - 2 Players divide up $1 by taking turns proposing offers to each other.
  - Each offer consists of $(x, 1-x)$, where $x \in [0,1]$
  - $\delta$ is discount factor.
    - At each period, $(1-\delta)$ of the remaining pie is lost
- Example:
  1. P1 offers $(x, 1-x)$ to P2
     (a) If P2 accepts, then the allocation is $(x, 1-x)$
     (b) If P2 declines, then this round ends
  2. P2 offers $(y, 1-y)$ to P1
     (a) If P1 accepts, then the allocation is $(\delta y, \delta(1-y))$
     (b) If P1 declines, then this round ends
  3. ...
- Claim:
  - There is a unique Subgame Perfect Nash Equilibrium (SPNE) of the alternating offer bargaining game:

$$\text{Player } i \text{Proposes: } \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

$$\text{Player } j \text{Accepts iff offered at least } \frac{\delta}{1+\delta}$$

**Proof:**

- To show uniqueness, we suppose there exist multiple SPNE and show that they are the same:

1. Define:
   (a) $x^H$ = the most P1 can get in an SPNE
   (b) $x^L$ = the least P1 can get in SPNE
2. Consider a SPNE in which $(x^H, x^*)$
   P1 gets $x^H$, P2 gets $x^*$
   (a) *Feasibility Condition 1*:

$$x^* \geq \delta x^L \qquad (1)$$

   i. P2 must recieve more than her lowest possible payment, discounted by $\delta$

(b) *Feasibility Condition 2:*

$$1 - x^H \geq x^* \qquad (2)$$

i. Portions must add up to 1 or less

(c) Combining these conditions (1) and (2)

$$1 - x^H \geq \delta x^L \qquad (3)$$

3. We now look at the SPNE in which $\left(x^L, 1 - x^L\right)$
   P1 gets $x^L$, P2 gets $1 - x^L$.

   (a) We Prove the Following Condition:

   $$1 - x^L \leq \delta x^H \qquad (4)$$

   (b) Proof by contradictionn

   i. Suppose $1 - x^L > \delta x^*$

   ii. Then $1 - x^L - \epsilon > \delta x^*$

   iii. This implies $\left(x^L + \epsilon, 1 - x^L - \epsilon\right)$ is accepted.
   $\Longrightarrow \rightarrow\leftarrow$ . This is a contradiction because P1 can improve his equilibrium payoff by deviating unilaterally. Hence $x^L$ is not an equilibrium payoff. [We had assumed $x^L$ is the lowest possible equilibrium payoff in an SPNE]

4. Hence we have the two inequalities:

$$\left\{ \begin{array}{ll} 1 - x^H \geq \delta x^L & (3) \\ 1 - x^L \leq \delta x^H & (4) \end{array} \right\}$$

5. Algebra: To $x^H \leq \frac{1}{1+\delta}$
   $1 - x^L \leq \delta x^H$ [4]
   $1 - \delta x^H \leq x^L$ [Rearrange]
   $\delta\left(1 - \delta x^H\right) \leq \delta x^L \leq 1 - x^H$ [Multiply by $\delta$ to use 3]
   $\delta + \delta^2 x^H \leq 1 - x^H$ [Distribute]
   $x^H + \delta^2 x^H \leq 1 - \delta$ [Rearrange]
   $\left(1 - \delta^2\right) x^H \leq 1 - \delta$ [Factor]
   $(1 - \delta)(1 + \delta) x^H \leq 1 - \delta$ [Factor]

$$x^H \leq \frac{1}{1+\delta} \qquad (5)$$

6. Algebra: To $x^L \geq \frac{1}{1+\delta}$
   $1 - x^L \leq \delta x^H$ [4]
   $1 - x^L \leq \frac{\delta}{1+\delta}$ [5]
   $x^L \geq 1 - \frac{\delta}{1+\delta}$ [Rearrange]
   $x^L \geq \frac{1+\delta}{1+\delta} - \frac{\delta}{1+\delta}$

$$x^L \geq \frac{1}{1+\delta} \qquad (6)$$

7. Combining (5) and (6)

$$\frac{1}{1+\delta} \leq x^L \leq x^H \leq \frac{1}{1+\delta}$$

Hence, $x^L = x^H$. The SPNE is unique.

- Remark:
  - There exist infinite Nash Equilibria in this game. Both P1 and P2 can choose anything to threaten to offer. Credibility is immaterial for NE as long as the threats are "believed."
- Extensions
  - What if P1 and P2 have different discount rates?
    * Then player with lower discount rate holds the advantage.
  - Finite game
  - What if rather than alternating, the proposer was chosen randomly at each period?

## Finite Horizon Bilateral Bargaining

- Set Up:
- 2 Player finite horizon bargaining game
  - 2 Players divide up $v$ by taking turns proposing offers to each other.
  - Each offer consists of $(x, 1-x)$, where $x \in [0,1]$
  - $\delta$ is discount factor.
  - At each period, $(1-\delta)$ of the remaining pie is lost
    * Hence, a dollar received in period $t$ is worth $\underline{\delta^{t-1} \text{ in period 1 dollars}}$
  - *Finite Horizon:* If after $T$ periods an agreement has not been reached, barganing is terminated and players each receive nothing
- Claim:
  - There is a unique Subgame Perfect Nash Equilibrium (SPNE) of the finite horizon alternating offer bargaining game

- Solution: ***Backward Induction***
- Suppose first $T$ is **odd**.
- Period $T$
  - Player 2 will accept any offer $\geq 0$

- Hence Player 1 offers to take all remaining value for himself, and Player 2 accepts 0

$$\left(\delta^{T-1}v, 0\right)$$

- Period $T-1$
  - Player 1 will only accept offer $\geq \delta^{T-1}v$ [which is what he can get by rejecting and going in to next period]
  - Hence, player 2 offers $\delta^{T-1}v$ to player 1, keeping the rest for herself

$$\left(\delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v\right)$$

- Period $T-2$
  - Player 2 will only accept offer $\geq \delta^{T-2}v - \delta^{T-1}v$
  - Hence, player 1 offers $\delta^{T-3}v - \left(\delta^{T-2}v - \delta^{T-1}v\right) = \delta^{T-3}v - \delta^{T-2}v + \delta^{T-1}v$, keeping the rest for himself

$$\left(\delta^{T-3}v - \delta^{T-2}v + \delta^{T-1}v, \delta^{T-2}v - \delta^{T-1}v\right)$$

- SPNE
  - Continuing the game in this fashion, we can determine the unique SPNE when $T$ is odd:
    * Player 1 Payoff:

$$v_1^*(T) = v\left[1 - \delta + \delta^2 - \ldots + \delta^{T-1}\right]$$

$$= v\left[(1-\delta)\left(\frac{1-\delta^{T-1}}{1-\delta^2}\right) + \delta^{T-1}\right]$$

      · [Using series formula: see appendix]
    * Player 2 Payoff:

$$v_2^*(T) = 1 - v_1^*(T)$$

- Suppose $T$ is **even**
  - Look at Odd Game beginning at Period 2, of length $T-1$:
    * Player 2 will get $\delta v_1^*(T-1)$
  - Hence, Player 1 must offer: $v - \delta v_1^*(T-1)$ to keep, $\delta v_1^*(T-1)$ for player 2:

$$\left(v - \delta v_1^*(T-1), \delta v_1^*(T-1)\right)$$

- Convergence and Relationship to $\infty-$Horizon Game

- Note that as $T \to \infty$,

$$v_1^*(T) \to v\left[\frac{1-\delta}{(1-\delta)(1+\delta)}\right] = \frac{v}{(1-\delta)}$$

$$v_2^*(T) \to \delta\left[v_1^*(T-1)\right] = \frac{\delta v}{1-\delta}$$

  * [For $v_1$ used odd case, and $v_2$ used even case ???]
- We see that this limits to the $\infty-$horizon case

# Appendix: Math Reminders

**Bayes's Rule:**

$$p(A|B) = \frac{p(B|A)\,p(A)}{p(B)} = \frac{p(B|A)\,p(A)}{p(B|A)+p(B|A')}$$

**Infinite Geometric Series:**
For $\delta < 1$:

$$\sum_{t=1}^{\infty} \delta^t = \frac{1}{1-\delta}$$

Hence:
$\frac{1}{1-\delta} = 1+\delta+\delta^2+\delta^3+...$
$\frac{\delta}{1-\delta} = \delta+\delta^2+\delta^3+...$

**Finite Geometric Series:**

$$\sum_{t=0}^{T} z^t = \frac{1-z^{t+1}}{1-z}$$

Also can think of it as:
$a+ar+ar^2+...+ar^{n-1} = \sum_{i=1}^{n} ar^{n-1}$

$$\sum_{i=1}^{n} ar^{n-1} = a\frac{1-r^n}{1-r}$$

*Finite Bargaining Example*:
For: $1-\delta+\delta^2-\ldots+\delta^{T-1}$
We have:

1. $1+\delta^2+...+\delta^{T-3}$

$$\sum_{i=1}^{\frac{1}{2}(T-1)} \left(\delta^2\right)^{i-1} = \frac{1-\left(\delta^2\right)^{\frac{1}{2}(T-1)}}{1-\delta^2}$$

2. $-\delta - \delta^3 - ... - \delta^{T-2} = (-1)\left(\delta + \delta^3 + ... + \delta^{T-2}\right)$

$$(-1)\sum_{i=1}^{\frac{1}{2}(T-1)} \delta\left(\delta^2\right)^{i-1} = -\delta\frac{1-\left(\delta^2\right)^{\frac{1}{2}(T-1)}}{1-\delta^2}$$

3. $+\delta^{T-1}$

Adding together:

$$-\delta\frac{1-\left(\delta^2\right)^{\frac{1}{2}(T-1)}}{1-\delta^2} + \frac{1-\left(\delta^2\right)^{\frac{1}{2}(T-1)}}{1-\delta^2} + \delta^{T-1}$$

$$= \frac{-\delta + \delta^{(T-1)+1} + 1 - \delta^{(T-1)}}{1-\delta^2} + \delta^{T-1}$$

$$= \frac{-\delta + \delta^{T} + 1 - \delta^{T-1}}{1-\delta^2} + \delta^{T-1}$$

$$= \frac{(1-\delta)\left(1-\delta^{T-1}\right)}{1-\delta^2}$$

*As desired.*