"SARGENT, MACROTHEORY CH IX"

CHAPTER IX

DIFFERENCE EQUATIONS AND LAG OPERATORS[1]

Linear difference equations underlie much work in macroeconomics and applied economic dynamics. Lag operators provide a powerful tool for solving systems of linear difference equations, and for gaining insights about their structure. Lag operators constitute a language which much of macroeconomics and dynamic econometrics takes for granted. This chapter provides an introduction to the analysis of nonstochastic difference equations via lag operators. While a variety of economic examples occur in this chapter, additional examples continue to occur throughout the remainder of this book.

1. LAG OPERATORS

The backward shift or lag operator is defined by

$$\begin{aligned} LX_t &= X_{t-1} \\ L^n X_t &= X_{t-n} \qquad \text{for} \quad n = \cdots, -2, -1, 0, 1, 2, \ldots. \end{aligned} \tag{1}$$

Multiplying a variable X_t by L^n thus gives the value of X shifted back n periods. Notice that if $n < 0$ in (1), the effect of multiplying X_t by L^n (more precisely, the effect of "operating on X_t with L^n") is to shift X *forward* in time by $-n$ periods.[2] This language is loose. Actually, we are starting out with a *sequence* $\{X_t\}_{t=-\infty}^{\infty}$ which associates a real number X_t with each integer t. We are operating on the

[1] It would be useful for the reader to be familiar with the material on difference equations in Allen (1960) and Baumol (1959).

[2] This chapter aims to teach the reader to manipulate lag operators, while devoting little or no attention to describing their mathematical foundations. The key Riesz-Fischer theorem which justifies these methods is discussed briefly in Chapter XI, pp. 249–253. The reader interested in increasing his proficiency with these techniques is urged to consult Gabel and Roberts (1973, Chapter 4).

Sargent, Thomas J. Macroeconomic Theory, 2nd ed.
Academic Press, 1987

sequence $\{X_t\}$ with the operator L^n to obtain the new sequence $\{y_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}$. Formally, the operator L^n maps one sequence into another sequence.

We shall consider polynomials in the lag operator

$$A(L) = a_0 + a_1 L + a_2 L^2 + \cdots = \sum_{j=0}^{\infty} a_j L^j$$

where the a_j's are constants and $L^0 \equiv 1$. Operating on X_t with $A(L)$ yields a moving sum of X's:

$$\begin{aligned} A(L)X_t &= (a_0 + a_1 L + a_2 L^2 + \cdots)X_t \\ &= a_0 X_t + a_1 X_{t-1} + a_2 X_{t-2} + \cdots = \sum_{j=0}^{\infty} a_j X_{t-j}. \end{aligned}$$

It is generally convenient to work with polynomials $A(L)$ that are "rational," meaning that they can be expressed as the ratio of two (finite order) polynomials in L:

$$A(L) = B(L)/C(L)$$

where

$$B(L) = \sum_{j=0}^{m} b_j L^j, \qquad C(L) = \sum_{j=0}^{n} c_j L^j$$

where the b_j and c_j are constants. Assuming that $A(L)$ is rational amounts to imposing a more economical and restrictive parametrization on the a_j.

To take the simplest example of a rational polynomial in L, consider[3]

$$A(L) = \frac{1}{1 - \lambda L}. \tag{2}$$

For the scalar $|C| < 1$, we know that

$$\frac{1}{1 - C} = 1 + C + C^2 + \cdots. \tag{3}$$

Thus suggests treating λL of (2) exactly like the C of (3) to get

$$\frac{1}{1 - \lambda L} = 1 + \lambda L + \lambda^2 L^2 + \cdots, \tag{4}$$

[3] Actually, we should write $A(L) = I/(I - \lambda L)$ where I is the identity lag operator defined by $I \equiv 1 + 0L + 0L^2 + \cdots$. So I satisfies $Ix_t = x_t$, and thus acts like unity.

an expansion which is sometimes only "useful" so long as $|\lambda| < 1$. To motivate the equality (4), assume that $|\lambda| < 1$ and operate on both sides of (4) with $1 - \lambda L$ to obtain

$$\frac{1-\lambda L}{1-\lambda L} = 1 = (1 + \lambda L + \lambda^2 L^2 + \cdots) - \lambda L(1 + \lambda L + \lambda^2 L^2 + \cdots) = 1.$$

The reason that we say that (4) is sometimes "useful" only if $|\lambda| < 1$ derives from the following argument. We intend often to multiply $1/(1 - \lambda L)$ by X_t to obtain the infinite moving sum

$$\frac{1}{1-\lambda L} X_t = (1 + \lambda L + \lambda^2 L^2 + \cdots)X_t = \sum_{i=0}^{\infty} \lambda^i X_{t-i}. \tag{5}$$

Consider this sum for a path of X that is constant over time, so that $X_{t-i} = \bar{X}$ for all i and all t. Then the sum (5) becomes

$$\frac{1}{1-\lambda L} X_t = \bar{X} \sum_{i=0}^{\infty} \lambda^i.$$

The sum $\sum_{i=0}^{\infty} \lambda^i$ equals $1/(1 - \lambda)$ if $|\lambda| < 1$. But if $|\lambda| \geq 1$ that sum is unbounded, being $+\infty$ if $\lambda \geq 1$. We shall sometimes (though not always) be applying the polynomial in the lag operator (4) in situations in which it is appropriate to go infinitely far back in time; and we sometimes find it necessary to insist that in such cases the infinite sum in (5) exist where X has been constant through time. This is what leads to the requirement sometimes imposed that $|\lambda| < 1$ in (4). As we shall see, however, in standard analyses of difference equations, which take the starting point of all processes as some point only finitely far back into the past, the requirement that $|\lambda| < 1$ need not be imposed in (4).

It is useful to note that there is an alternative expansion for the "geometric" polynomial $1/(1 - \lambda L)$. For notice that formally

$$\frac{1}{1-\lambda L} = \frac{-(\lambda L)^{-1}}{1-(\lambda L)^{-1}} = \frac{-1}{\lambda L}\left(1 + \frac{1}{\lambda} L^{-1} + \left(\frac{1}{\lambda}\right)^2 L^{-2} + \cdots\right)$$
$$= \frac{-1}{\lambda} L^{-1} - \left(\frac{1}{\lambda}\right)^2 L^{-2} - \left(\frac{1}{\lambda}\right)^3 L^{-3} - \cdots, \tag{6}$$

an expansion which is especially "useful" where $|\lambda| > 1$, i.e., where $|1/\lambda| < 1$. So (6) implies that

$$\frac{1}{1-\lambda L} X_t = -\frac{1}{\lambda} X_{t+1} - \left(\frac{1}{\lambda}\right)^2 X_{t+2} - \cdots = -\sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i X_{t+i},$$

which shows $(1/(1 - \lambda L))X_t$ to be a geometrically declining weighted sum of *future* values of X. Notice that for this infinite sum to be finite for a constant time path $X_{t+i} = \bar{X}$ for all i and t, the series $-\sum_{i=1}^{\infty} (1/\lambda)^i$ must be convergent, which requires that $|1/\lambda| < 1$.

More generally, let $\{x_t\}_{t=-\infty}^{\infty}$ be any bounded sequence of real numbers, i.e. for some $M > 0$, $|x_t| < M$ for all t. Then applying the operator $1/(1 - \lambda L)$ to the sequence $\{x_t\}_{t=-\infty}^{\infty}$ can be taken to give either the sequence $\{y_t\}_{t=-\infty}^{\infty}$ where

$$y_t = \frac{1}{1-\lambda L} x_t = \sum_{j=0}^{\infty} \lambda^j x_{t-j}$$

or the sequence $\{z_t\}_{t=-\infty}^{\infty}$ where

$$z_t = \frac{-(\lambda L)^{-1}}{1-(\lambda L)^{-1}} x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j x_{t+j}.$$

In general, $\{y_t\}$ is a bounded sequence if $|\lambda| < 1$, while $\{z_t\}$ is a bounded sequence if $|\lambda| > 1$. In many (though not all) contexts, we want application of $(1 - \lambda L)^{-1}$ to map all bounded sequences into bounded sequences, so that we choose the "backward" expansion if $|\lambda| < 1$ and the forward expansion if $|\lambda| > 1$.

To illustrate how polynomials in the lag operator can be manipulated consider the difference equation

$$Y_t = \lambda Y_{t-1} + bX_t + a, \qquad t = -\infty, \ldots, 0, 1, 2, \ldots, \tag{7}$$

where X_t is an exogenous variable and Y_t is an endogenous variable and $\lambda \neq 1$. Here, X_t is a sequence of real numbers $t = \cdots, -1, 0, 1, 2, \ldots$. Write the above equation as

$$(1 - \lambda L)Y_t = a + bX_t.$$

Operating on both sides of this equation by $(1 - \lambda L)^{-1}$ gives

$$Y_t = \frac{a}{1-\lambda L} + \frac{b}{1-\lambda L} X_t + c\lambda^t$$
$$= \frac{a}{1-\lambda} + b \sum_{i=0}^{\infty} \lambda^i X_{t-i} + c\lambda^t, \tag{8}$$

since $a/(1 - \lambda L) = a \sum_{i=0}^{\infty} \lambda^i = a/(1 - \lambda)$. Here c is any constant. The reason that we must include the term $c\lambda^t$ in (8) is that for any constant c, $(1 - \lambda L)c\lambda^t = c\lambda^t - c\lambda\lambda^{t-1} = 0$.[4] Therefore, it follows that application of $1 - \lambda L$ to both sides of (8) gives Equation (7) once again. Consequently, (8) is the general "solution" of the difference equation (7) and describes the entire time path of Y associated with a given time path of X. In order to get a "particular solution" we must be able to tie down the constant c. This requires an additional bit of information in the form of a specified value of Y_t at some particular time or some conditions on

[4] Technically, we are free to add to the solution any function of time $f(t)$ for which $(1 - \lambda L)f(t) = 0$. It can be proved that $f(t) = c\lambda^t$ is the only such function.

the path of $\{Y_t\}$ such as boundedness. Notice that for the Y_t defined by (8) to be finite, $\lambda^i X_{t-i}$ must be "small" for large i. More precisely, we require

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \lambda^i X_{t-i} = 0 \qquad \text{for all } t. \tag{9}$$

For the case of X constant for all time, $X_{t-i} = \bar{X}$ all i and t, this condition requires $|\lambda| < 1$. Notice also that the infinite sum $a \sum_{i=0}^{\infty} \lambda^i$ in (8) is finite only if $|\lambda| < 1$, in which case it equals $a/(1-\lambda)$, or if $a = 0$, in which case it equals zero regardless of the value of λ. We tentatively assume that $|\lambda| < 1$.

For analyzing difference equations with arbitrary initial conditions given, it is convenient to rewrite (8) for $t > 0$ as

$$Y_t = a\sum_{i=0}^{t-1} \lambda^i + a\sum_{i=t}^{\infty} \lambda^i + b\sum_{i=0}^{t-1} \lambda^i X_{t-i} + b\sum_{i=t}^{\infty} \lambda^i X_{t-i} + c\lambda^t$$

$$= \frac{a(1-\lambda^t)}{1-\lambda} + \frac{a\lambda^t}{1-\lambda} + b\sum_{i=0}^{t-1} \lambda^i X_{t-i} + b\lambda^t \sum_{i=0}^{\infty} \lambda^i X_{0-i} + c\lambda^t,$$

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b\sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t \left\{ \frac{a}{1-\lambda} + b\sum_{i=0}^{\infty} \lambda^i X_{0-i} + c\lambda^0 \right\}, \qquad t \geq 1. \tag{10}$$

The term in braces equals Y_0, as reference to expression (8) will confirm. So (10) becomes

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b\sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t Y_0$$

or

$$Y_t = \frac{a}{1-\lambda} + \lambda^t \left(Y_0 - \frac{a}{1-\lambda} \right) + b\sum_{i=0}^{t-1} \lambda^i X_{t-i}, \qquad t \geq 1. \tag{11}$$

Now textbooks on difference equations often analyze the special case in which $X_t = 0$ for all $t \geq 0$. Under this special circumstance (11) becomes

$$Y_t = \frac{a}{1-\lambda} + \lambda^t \left(Y_0 - \frac{a}{1-\lambda} \right), \tag{12}$$

which is the solution of the first-order difference equation $Y_t = a + \lambda Y_{t-1}$ subject to the initial condition that Y equals the arbitrary given value Y_0 at time 0. Notice that if $Y_0 = a/(1-\lambda)$, then (12) implies $Y_t = Y_0$ for all $t \geq 0$, which shows $a/(1-\lambda)$ to be a "stationary point" or long-run equilibrium value of Y. Notice also that if, as we are assuming, $|\lambda| < 1$, then (12) implies that

$$\lim_{t\to\infty} Y_t = \frac{a}{1-\lambda},$$

which shows that the system is "stable," tending to approach the stationary point as time passes.

Now consider the first-order system (7) under the assumption that $a = 0$, so that $a\sum_{i=0}^{\infty} \lambda^i$ equals zero regardless of the value of λ. Then the appropriate counterpart to (10) is

$$Y_t = b\sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t \left\{ b\sum_{i=0}^{\infty} \lambda^i X_{0-i} + c\lambda^0 \right\}.$$

Assuming that condition (9) is met even where $|\lambda| > 1$ (so that the second term in the equation is finite), the above equation becomes

$$Y_t = b\sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t Y_0, \qquad t \geq 1.$$

As before we analyze the special case where $X_t = 0$ for all $t > 0$. Then the above equation becomes

$$Y_t = \lambda^t Y_0, \qquad t \geq 1.$$

The stationary point of this solution is zero since if $Y_0 = 0$, Y will remain equal to zero forever, regardless of the value of λ. However, if $|\lambda| > 1$, the system will diverge farther and farther from this stationary point if either $Y_0 > 0$, or $Y_0 < 0$. If $\lambda > 1$, Y_t will tend toward $+\infty$ as $t \to \infty$ provided $Y_0 > 0$; Y_t will tend toward $-\infty$ as $t \to \infty$ if $Y_0 < 0$. If $\lambda < -1$, Y_t will display explosive oscillations of periodicity two time periods.

We can also solve the difference equation (7) by applying the "forward inverse" of $1 - \lambda L$ to get the general solution

$$Y_t = \frac{-\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}\, a + b\left(\frac{-\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}} \right) X_t + d\lambda^t,$$

$$Y_t = \frac{a}{1-\lambda} - b\sum_{i=0}^{\infty} \left(\frac{1}{\lambda} \right)^{i+1} X_{t+i+1} + d\lambda^t, \tag{8'}$$

where d is a constant to be determined from some side condition on the path of Y_t, such as an initial condition or terminal condition. If $a = 0$, then for *any* value of $\lambda \neq 1$, in general (8) and (8′) *both* represent solutions of the difference equation (7). They are simply alternative representations of the solution in the sense that for any given initial condition or other side condition, one can generally find values of d and c which guarantee that both (8) and (8′) satisfy (7). The equivalence of the solutions (8) and (8′) will hold whenever $(b/(1-\lambda L))X_t$ and $(b\lambda^{-1}L^{-1}/(1-\lambda^{-1}L^{-1}))X_t$ are both finite for all t.

It often happens, however, that either $(b/(1-\lambda L))X_t$ or

$$\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}} X_t$$

fails to be finite, i.e., the infinite sum fails to converge. In this case, one or the other of the representations (8) or (8′) breaks down, i.e., fails to give a Y_t sequence that is finite for all finite t. For example, if the sequence $\{X_t\}$ is bounded, this is sufficient to imply that $\{(b/(1-\lambda L))X_t\}$ is a bounded sequence if $|\lambda| < 1$, but not sufficient to imply that

$$\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t$$

is a convergent sum for all t. Similarly, if $|\lambda| > 1$, boundedness of the sequence $\{X_t\}$ is sufficient to imply that

$$\left\{\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t\right\}$$

is a bounded sequence, but fails to guarantee finiteness of $(b/(1-\lambda L))X_t$. In instances where one of $(b/(1-\lambda L))X_t$ or

$$\frac{b\lambda^{-1}L^{-1}}{1-\lambda^{-1}L^{-1}}X_t$$

is always finite and the other is not we shall take as our solution to (7) either (8) where the backward sum in X_t is finite, or (8′) where the forward sum in X_t is finite. This procedure assures us that we shall find the unique solution of (7) that is finite for all finite t, provided that such a solution exists. Such a solution is guaranteed to exist where $\{X_t\}$ is a bounded sequence.

Now if we desired to impose that the $\{Y_t\}$ sequence given by (8) or (8′) is bounded—as we are free to do if no other side condition has been imposed—then it is evident that we must set $c = 0$ in (8) or $d = 0$ in (8′). This is necessary since if $\lambda > 1$ and $c > 0$,

$$\lim_{t\to\infty} c\lambda^t = \infty;$$

while if $\lambda < 1$ and $c > 0$,

$$\lim_{t\to-\infty} c\lambda^t = \infty.$$

It follows that Y_t will be bounded for all t only if c or d is zero. For the solution in the forward direction to be finite for all finite t, we clearly require a condition analogous to (9)

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\left(\frac{1}{\lambda}\right)^i X_{t+i} = 0. \tag{9′}$$

The principle of solving "stable roots" ($|\lambda| < 1$) backward and "unstable roots" ($|\lambda| > 1$) forward was encountered in Chapter I. It is a device designed to ensure that the solution of the differential (or difference) equation maps

bounded functions (or sequences) as driving processes into bounded functions (or sequences). Below, we shall see that a formal justification for this procedure is sometimes available in the context of difference equations that emerge from optimum problems.

2. SECOND-ORDER DIFFERENCE EQUATIONS

Consider the second-order difference equation

$$Y_t = t_1 Y_{t-1} + t_2 Y_{t-2} + a + bX_t, \tag{13}$$

where $\{X_t\}$ is again an exogenous sequence of real numbers for $t = \cdots, -1, 0, 1, \ldots$. Using lag operators, (13) can be written as

$$(1 - t_1 L - t_2 L^2)Y_t = a + bX_t.$$

A solution to this difference equation is given by

$$Y_t = \frac{a}{1 - t_1 L - t_2 L^2} + \frac{b}{1 - t_1 L - t_2 L^2}X_t \tag{14}$$

where we have temporarily ignored the terms analogous to $c\lambda^t$ which appeared in the general solution of the first-order equation. By long division it is easy to verify that

$$\frac{b}{1 - t_1 L - t_2 L^2} = \sum_{i=0}^{\infty} w_i L^i \tag{15}$$

where $w_0 = b$, $w_1 = bt_1$, and

$$w_j = t_1 w_{j-1} + t_2 w_{j-2} \qquad \text{for} \quad j \geq 2.$$

That is,

$$\begin{array}{r|l}
 & 1 + t_1 L + (t_2 + t_1^2)L^2 + (t_1(t_2 + t_1^2) + t_1 t_2)L^3 + \cdots \\ \hline
1 - t_1 L - t_2 L^2 & 1 \\
 & \underline{1 - t_1 L - t_2 L^2} \\
 & \quad t_1 L + t_2 L^2 \\
 & \quad \underline{t_1 L - t_1^2 L^2 \qquad - t_1 t_2 L^3} \\
 & \qquad (t_2 + t_1^2)L^2 + t_1 t_2 L^3 \\
 & \qquad \underline{(t_2 + t_1^2)L^2 - t_1(t_2 + t_1^2)L^3 - t_2(t_2 + t_1^2)L^4} \\
 & \qquad \cdots
\end{array}$$

Notice that the weights in (15) follow a geometric pattern if $t_2 = 0$, as we would expect, since then (13) collapses to a first-order equation.

It is convenient to write the polynomial $1 - t_1 L - t_2 L^2$ in an alternative way, given by the *factorization*

$$\begin{aligned} 1 - t_1 L - t_2 L^2 &= (1 - \lambda_1 L)(1 - \lambda_2 L) \\ &= 1 - (\lambda_1 + \lambda_2)L + \lambda_1 \lambda_2 L^2, \end{aligned} \tag{16}$$

so that $\lambda_1 + \lambda_2 = t_1$ and $-\lambda_1\lambda_2 = t_2$. To see how λ_1 and λ_2 are related to the *roots* or *zeros* of $1 - t_1 z - t_2 z^2$, notice that

$$(1 - \lambda_1 z)(1 - \lambda_2 z) = \lambda_1\lambda_2\left(\frac{1}{\lambda_1} - z\right)\left(\frac{1}{\lambda_2} - z\right).$$

Therefore the equation

$$0 = (1 - \lambda_1 z)(1 - \lambda_2 z) = \lambda_1\lambda_2\left(\frac{1}{\lambda_1} - z\right)\left(\frac{1}{\lambda_2} - z\right)$$

is satisfied at the two roots $z = 1/\lambda_1$ and $z = 1/\lambda_2$. Given the polynomial $1 - t_1 z - t_2 z^2$, the roots $1/\lambda_1$ and $1/\lambda_2$ are found from solving the *characteristic equation*

$$1 - t_1 z - t_2 z^2 = 0 \qquad \text{or} \qquad t_2 z^2 + t_1 z - 1 = 0$$

for two values of z. The roots are given by the quadratic formula

$$z = \frac{-t_1 \pm \sqrt{t_1^2 + 4t_2}}{2t_2}. \tag{17}$$

Formula (17) enables us to obtain the reciprocals of λ_1 and λ_2 for given values of t_1 and t_2.

We assume that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 1$. Then without loss of generality we can write the second-order difference equation as

$$(1 - \lambda_1 L)(1 - \lambda_2 L)Y_t = a + bX_t.$$

The general solution to this difference equation is

$$Y_t = \frac{a}{(1 - \lambda_1 L)(1 - \lambda_2 L)} + \frac{b}{(1 - \lambda_1 L)(1 - \lambda_2 L)} X_t + c_1\lambda_1^{\,t} + c_2\lambda_2^{\,t} \tag{18}$$

where c_1 and c_2 are any constants. To see that (18) solves the difference equation for any values of c_1 and c_2, operate on both sides of (18) with

$$(1 - \lambda_1 L)(1 - \lambda_2 L)$$

and notice that

$$(1 - \lambda_1 L)(1 - \lambda_2 L)c_1\lambda_1^{\,t} = (1 - \lambda_1 L)(1 - \lambda_2 L)c_2\lambda_2^{\,t} = 0.$$

In order to determine a particular solution to the difference equation, we now need two side conditions on the path of Y_t. For example, two initial conditions in the forms of given values for Y at time 0 and 1 are sufficient to determine c_1 and c_2.

Notice that since $\lambda_1 \neq \lambda_2$,

$$\frac{1}{(1 - \lambda_1 L)(1 - \lambda_2 L)} = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L}\right),$$

which can be verified directly. Thus (18) can be written

$$\begin{aligned} Y_t &= \frac{a}{(1 - \lambda_1)(1 - \lambda_2)} + \frac{\lambda_1 b}{\lambda_1 - \lambda_2}\frac{1}{1 - \lambda_1 L}X_t \\ &\quad - \frac{\lambda_2 b}{\lambda_1 - \lambda_2}\frac{1}{1 - \lambda_2 L}X_t + c_1\lambda_1^{\,t} + c_2\lambda_2^{\,t} \\ &= a\sum_{i=0}^{\infty}\lambda_1^{\,i}\sum_{j=0}^{\infty}\lambda_2^{\,j} + \frac{\lambda_1 b}{\lambda_1 - \lambda_2}\sum_{i=0}^{\infty}\lambda_1^{\,i}X_{t-i} \\ &\quad - \frac{\lambda_2 b}{\lambda_1 - \lambda_2}\sum_{i=0}^{\infty}\lambda_2^{\,i}X_{t-i} + c_1\lambda_1^{\,t} + c_2\lambda_2^{\,t} \end{aligned} \tag{19}$$

where we are making use of the fact that for a constant a

$$H(L)a = \sum_{i=0}^{\infty} h_i L^i a = a\sum_{i=0}^{\infty} h_i = aH(1).$$

Notice that

$$\frac{1}{1 - \lambda_1 L}\frac{1}{1 - \lambda_2 L} = \sum_{i=0}^{\infty}\lambda_1^{\,i}L^i\sum_{j=0}^{\infty}\lambda_2^{\,j}L^j,$$

so that the sum of the distributed lag weights $\sum_{i=0}^{\infty}\lambda_1^{\,i}\sum_{i=0}^{\infty}\lambda_2^{\,i}$ is finite and equals $1/((1 - \lambda_1)(1 - \lambda_2))$ provided that both $|\lambda_1| < 1$, $|\lambda_2| < 1$. So in writing (19) we require either that both $|\lambda_1|$ and $|\lambda_2|$ be less than unity or that $a = 0$, so that $a\sum_{i=0}^{\infty}\lambda_1^{\,i}\sum_{j=0}^{\infty}\lambda_2^{\,j}$ is defined. Furthermore, we require that

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}\lambda_j^{\,i}X_{t-i} = 0, \qquad \text{all } t,$$

hold for $j = 1, 2$, so that the geometric sums in (19) are both finite.

Suppose that $a = 0$. On this assumption write (19) for $t \geq 1$ as

$$\begin{aligned} Y_t &= \frac{\lambda_1 b}{\lambda_1 - \lambda_2}\sum_{i=0}^{t-1}\lambda_1^{\,i}X_{t-i} - \frac{\lambda_2 b}{\lambda_1 - \lambda_2}\sum_{i=0}^{t-1}\lambda_2^{\,i}X_{t-i} \\ &\quad + \frac{\lambda_1 b}{\lambda_1 - \lambda_2}\sum_{i=t}^{\infty}\lambda_1^{\,i}X_{t-i} - \frac{\lambda_2 b}{\lambda_1 - \lambda_2}\sum_{i=t}^{\infty}\lambda_2^{\,i}X_{t-i} + c_1\lambda_1^{\,t} + c_2\lambda_2^{\,t} \end{aligned}$$

or

$$Y_t = \frac{\lambda_1 b}{\lambda_1 - \lambda_2}\sum_{i=0}^{t-1}\lambda_1^{\,i}X_{t-i} - \frac{\lambda_2 b}{\lambda_1 - \lambda_2}\sum_{i=0}^{t-1}\lambda_2^{\,i}X_{t-i} + \lambda_1^{\,t}\theta_0 + \lambda_2^{\,t}\eta_0, \; t \geq 1, \tag{20}$$

where

$$\theta_0 = \left\{c_1 + \frac{b\lambda_1}{\lambda_1 - \lambda_2}\sum_{i=0}^{\infty}\lambda_1^{\,i}X_{0-i}\right\} \quad \text{and} \quad \eta_0 = \left\{c_2 - \frac{b\lambda_2}{\lambda_1 - \lambda_2}\sum_{i=0}^{\infty}\lambda_2^{\,i}X_{0-i}\right\}.$$

The case in which $X_t = 0$ for $t \geq 1$ is often analyzed, as for the first-order case. On this assumption (20) becomes

$$Y_t = \lambda_1{}^t\theta_0 + \lambda_2{}^t\eta_0, \qquad t \geq 1. \tag{21}$$

If $\theta_0 = \eta_0 = 0$, $Y_t = 0$ for all $t \geq 1$, regardless of the values of λ_1 and λ_2. So $Y = 0$ is the stationary point or long-run equilibrium value of (21).

If λ_1 and λ_2 are real, then $\lim_{t\to\infty} Y_t$ will equal zero if and only if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$, regardless of the values of the parameters θ_0 and η_0, so long as they are finite.

If $\lambda_1 > 1$, $|\lambda_2| < |\lambda_1|$, and $\theta_0 > 0$, then $\lim_{t\to\infty} Y_t = +\infty$. If $\lambda_1 > 1$, $|\lambda_2| < |\lambda_1|$, and $\theta_0 < 0$, then $\lim_{t\to\infty} Y_t = -\infty$. Thus, Y will tend toward the stationary point zero as time passes provided that both $|\lambda_1| < 1$ and $|\lambda_2| < 1$. If one or both of the λ's exceed one in absolute value, the behavior of Y will eventually be "dominated" by the term in (21) associated with the λ that is larger in absolute value; i.e., eventually Y will grow approximately as $\lambda_m{}^t$, where λ_m is the λ_j with the larger absolute value.

Now suppose that the roots are complex. If the roots are complex, they will occur as a complex conjugate pair, as the quadratic formula (17) verifies. So assume that the roots are complex, and write them as

$$\lambda_1 = re^{iw} = r(\cos w + i \sin w), \qquad \lambda_2 = re^{-iw} = r(\cos w - i \sin w)$$

where the real part is $r \cos w$ and the imaginary part is $\pm r \sin w$. Notice that

$$\lambda_1 - \lambda_2 = r(e^{iw} - e^{-iw}) = 2ri \sin w. \tag{22}$$

Equation (21) becomes

$$\begin{aligned} Y_t &= \theta_0(re^{iw})^t + \eta_0(re^{-iw})^t \\ &= \theta_0(r^t e^{iwt}) + \eta_0(r^t e^{-iwt}) \\ &= \theta_0 r^t[\cos wt + i \sin wt] + \eta_0 r^t[\cos wt - i \sin wt] \\ &= (\theta_0 + \eta_0)r^t \cos wt + i(\theta_0 - \eta_0)r^t \sin wt. \end{aligned} \tag{23}$$

Since Y_t must be a real number for all t, it follows that $\theta_0 + \eta_0$ must be real and $\theta_0 - \eta_0$ must be imaginary. Therefore, θ_0 and η_0 must be complex conjugates, say $\theta_0 = pe^{i\theta}$, $\eta_0 = pe^{-i\theta}$. Therefore we can write

$$\begin{aligned} Y_t &= pe^{i\theta}r^t e^{iwt} + pe^{-i\theta}r^t e^{-iwt} = pr^t[e^{i(wt+\theta)} + e^{-i(wt+\theta)}] \\ &= 2pr^t \cos(wt + \theta). \end{aligned} \tag{24}$$

This is the solution of the "unforced" (i.e., $X_t = 0$ for all t) second-order difference equation with complex roots. The parameters p and θ are chosen to satisfy two side conditions on the path of Y_t, say two initial conditions. The path of Y_t oscillates with a frequency determined by w. The "damping factor" r^t is determined by the amplitude r of the complex roots. The value $Y_t = 0$ is the

stationary point of the difference equation, which will be approached at $t \to \infty$ for arbitrary initial conditions if $r < 1$. If $r > 1$, the path of Y_t displays explosive oscillations, unless the initial conditions are, say, $Y_0 = 0$, $Y_1 = 0$, so that Y starts out at the stationary point for two successive values. If $r < 1$, the system displays damped oscillations provided $w \neq 0$, which is so when the roots are complex. If $r = 1$, Y_t displays repeated oscillations of unchanging amplitude, and the solution is "periodic."

If λ_1 and λ_2 are complex, the distributed lag weights of (19) are easily shown to oscillate. For we have

$$\begin{aligned} \frac{b}{\lambda_1 - \lambda_2}\sum_{j=0}^{\infty}(\lambda_1^{j+1} - \lambda_2^{j+1})X_{t-j} &= \frac{b}{re^{iw} - re^{-iw}}\sum_{j=0}^{\infty}[(re^{iw})^{j+1} - (re^{-iw})^{j+1}]X_{t-j} \\ &= \frac{b}{2ri \sin w}\sum_{j=0}^{\infty}2r^{j+1}i\{\sin w(j+1)\}X_{t-j} = b\sum_{j=0}^{\infty} r^j \frac{\sin w(j+1)}{\sin w}X_{t-j}. \end{aligned}$$

Since the roots are complex, $\sin w \neq 0$. Notice that the damping factor weighting the sine curve is r^j, so that the amplitude of the lag weights decreases as j increases, provided that $r < 1$.

As noted above, the roots λ_1 and λ_2 are the reciprocals of the roots of the polynomial

$$1 - t_1 z - t_2 z^2 = 0. \tag{25}$$

For we know that $1 - t_1 z - t_2 z^2 = (1 - \lambda_1 z)(1 - \lambda_2 z)$, with roots $1/\lambda_1$ and $1/\lambda_2$. Alternatively, multiply the above equation by z^{-2} to obtain

$$z^{-2} - z^{-1}t_1 - t_2 = 0 = (z^{-1} - \lambda_1)(z^{-1} - \lambda_2)$$

or

$$s^2 - t_1 s - t_2 = 0 \tag{26}$$

where $s = z^{-1}$. Notice that the roots of (26) are the reciprocals of the roots of (25). Thus, λ_1 and λ_2 are the roots of (26).

It is interesting to know what values of t_1 and t_2 yield complex roots. Using the quadratic formula we have that the roots of (26) are

$$\lambda_i = s = \frac{t_1 \pm \sqrt{t_1{}^2 + 4t_2}}{2}.$$

For the roots to be complex, the term whose square root is taken must be negative, i.e.,

$$t_1{}^2 + 4t_2 < 0, \tag{27}$$

which implies that $t_2 < 0$. In case (27) is satisfied, the roots are

$$\lambda_1 = \frac{t_1}{2} + \frac{i\sqrt{-(t_1{}^2 + 4t_2)}}{2} = a + bi, \qquad \lambda_2 = \frac{t_1}{2} - \frac{i\sqrt{-(t_1{}^2 + 4t_2)}}{2} = a - bi.$$

To write $a + bi$ in polar form we recall that

$$a + bi = r \cos w + ri \sin w = re^{iw}$$

where $r = \sqrt{a^2 + b^2}$ and where $\cos w = a/r$. Thus we have that

$$r = \sqrt{\left(\frac{t_1}{2}\right)^2 - \frac{(t_1{}^2 + 4t_2)}{4}} = \sqrt{-t_2}.$$

We also have that

$$\cos w = \frac{t_1}{2\sqrt{-t_2}} \qquad \text{or} \qquad w = \cos^{-1}\left(\frac{t_1}{2\sqrt{-t_2}}\right).$$

For the oscillations to be damped we require that $r = \sqrt{-t_2} < 1$, which requires that $-t_2 < 1$.

The periodicity of the oscillations is $2\pi/\cos^{-1}(t_1/2\sqrt{-t_2})$; i.e., this is the number of periods from peak to peak in the oscillations.

If the roots are real, movements will be damped if both roots are less than one in absolute value. That requires

$$-1 < \frac{t_1 + \sqrt{t_1{}^2 + 4t_2}}{2} < 1 \qquad \text{and} \qquad -1 < \frac{t_1 - \sqrt{t_1{}^2 + 4t_2}}{2} < 1.$$

The condition $\frac{1}{2}(t_1 + \sqrt{t_1{}^2 + 4t_2}) < 1$ implies

$$\sqrt{t_1{}^2 + 4t_2} < 2 - t_1, \qquad t_1{}^2 + 4t_2 < 4 + t_1{}^2 - 4t_1,$$

$$t_1 + t_2 < 1. \tag{28}$$

The condition $\frac{1}{2}(t_1 - \sqrt{t_1{}^2 + 4t_2}) > -1$ implies that

$$-\sqrt{t_1{}^2 + 4t_2} > -2 - t_1, \qquad \sqrt{t_1{}^2 + 4t_2} < 2 + t_1,$$

$$t_1{}^2 + 4t_2 < t_1{}^2 + 4 + 4t_1,$$

$$t_2 < 1 + t_1. \tag{29}$$

Conditions (28) and (29) must be satisfied for the roots, if real, to be less than unity in absolute value.

Notice that both roots are negative and real if

$$t_1{}^2 + 4t_2 > 0 \qquad \text{and} \qquad \frac{t_1 + \sqrt{t_1{}^2 + 4t_2}}{2} < 0,$$

which implies

$$t_1 < -\sqrt{t_1{}^2 + 4t_2}, \qquad t_1{}^2 > t_1{}^2 + 4t_2, \qquad 0 > t_2.$$

Figure 1 depicts regions of the t_1, t_2 plane for which conditions (27), (28), or (29) are or are not satisfied. The graph shows combinations of t_1 and t_2 that give rise to damped oscillations, explosive oscillations, etc.

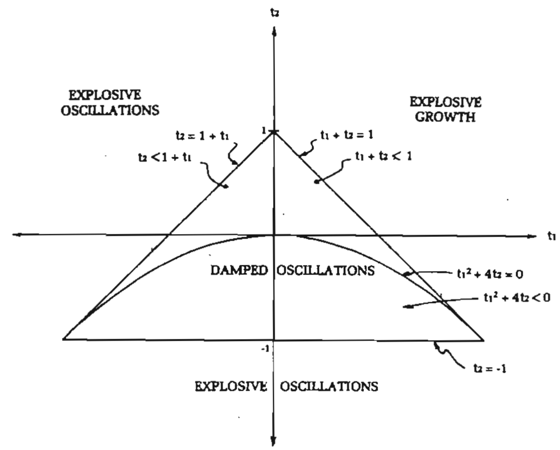

FIGURE 1 (Source: W. Baumol, *Economic Dynamics*, 2nd ed., p. 221, New York: Macmillan Copyright 1959 by Macmillan Publishing Co., Inc.)

A. An Example

Maybe the most famous second-order difference equation in economics is the one associated with Samuelson's multiplier accelerator model (see Samuelson, 1944). Samuelson posited the model

$$C_t = cY_{t-1} + \alpha, \qquad 1 > c > 0 \quad \text{(consumption function)}$$

$$I_t = \gamma(Y_{t-1} - Y_{t-2}), \qquad \gamma > 0 \quad \text{(accelerator)}$$

$$C_t + I_t = Y_t,$$

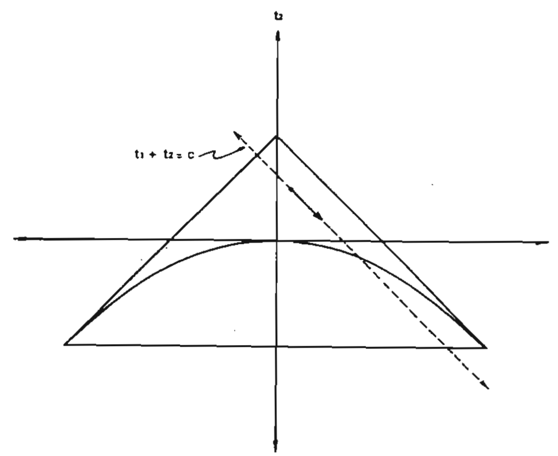

FIGURE 2

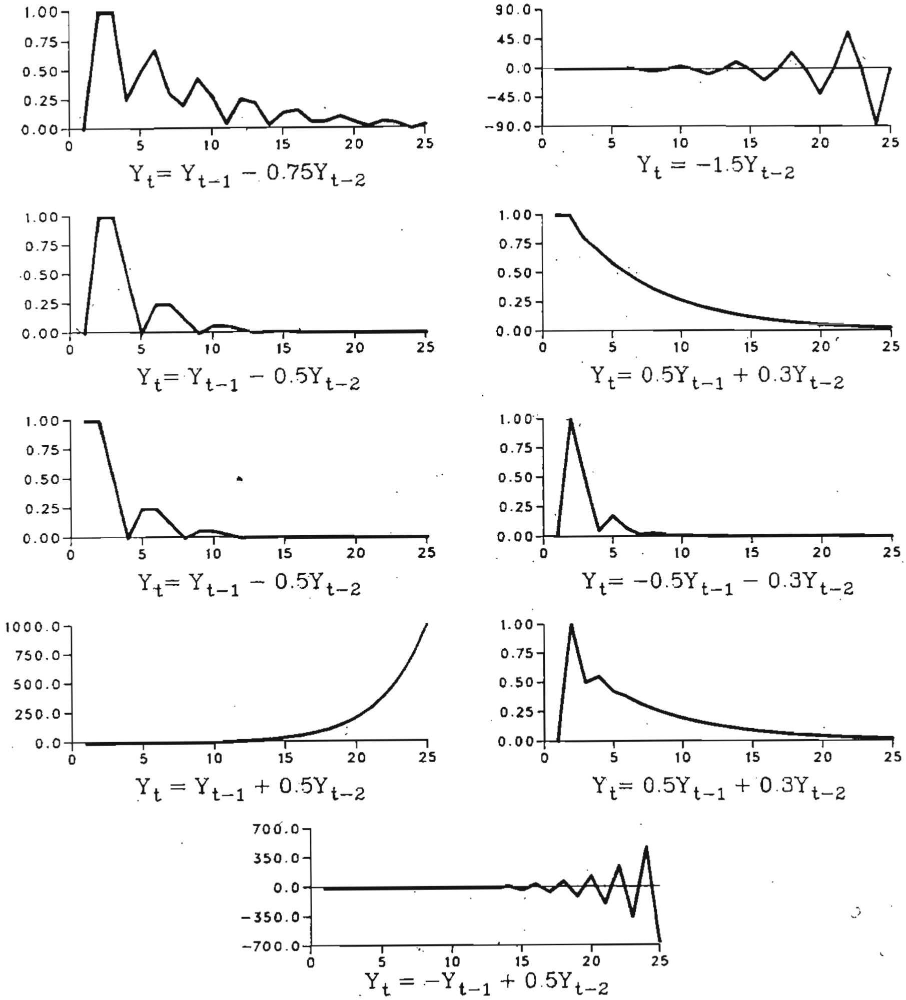

FIGURE 3 Second-order difference equations.

where C_t is consumption and I_t is investment. Substituting the first two equations into the third gives

$$Y_t = (c + \gamma)Y_{t-1} - \gamma Y_{t-2} + \alpha$$

or

$$Y_t = t_1 Y_{t-1} + t_2 Y_{t-2} + \alpha$$

where $t_1 = c + \gamma$, $t_2 = -\gamma$. Notice that $t_1 + t_2 = c$. So increases in the parameter γ move the parameters t_1 and t_2 downward and to the right along the line

$t_1 + t_2 = c$ in Figure 2. Using Figure 2, the values of c and γ compatible wit damped oscillations, explosive oscillations, and so on, can easily be determined

Figure 3 shows some realizations of second-order difference equations fo various values of t_1 and t_2. In each case the values Y_1 and Y_2 were the two initia conditions which we specified arbitrarily. The reader can check that th behavior of these realizations matches that predicted by Figure 1.

3. SECOND-ORDER DIFFERENCE EQUATIONS (EQUAL ROOTS)

The preceding treatment assumed that $\lambda_1 \neq \lambda_2$. (Notice that we divided by $\lambda_1 - \lambda_2$ to obtain (19).) If $\lambda_1 = \lambda_2$, then the polynomial we must study is

$$\begin{aligned}\frac{1}{(1-\lambda L)(1-\lambda L)} &= \frac{1}{1-\lambda L}(1 + \lambda L + \lambda^2 L^2 + \cdots)\\ &= (1 + \lambda L + \lambda^2 L^2 + \cdots) + \lambda L(1 + \lambda L + \lambda^2 L^2 + \cdots)\\ &\quad + \lambda^2 L^2(1 + \lambda L + \lambda^2 L^2 + \cdots) + \cdots\\ &= 1 + 2\lambda L + 3\lambda^2 L^2 + \cdots,\\ \frac{1}{(1-\lambda L)^2} &= \sum_{i=0}^{\infty} (i+1)\lambda^i L^i. \end{aligned} \tag{30}$$

The lag distribution generated by the polynomial in (30) is called a second-order Pascal lag distribution. It is the lag operator product or "convolution" of two geometric lag distributions with the same decay parameter[5] λ.

With the aid of (30) we can study the solution to difference equations of the form

$$(1 - \lambda L)^2 Y_t = a + bX_t.$$

The general solution is

$$Y_t = a\sum_{i=0}^{\infty}(i+1)\lambda^i + b\sum_{i=0}^{\infty}(i+1)\lambda^i X_{t-i} + c_1\lambda^t + c_2 t\lambda^t \tag{31}$$

where c_1 and c_2 are any real constants. To verify that (31) is the solution, operate on both sides by $(1-\lambda L)^2$ and verify that $(1-\lambda L)^2\{c_1\lambda^t + c_2 t\lambda^t\} = 0$ for any choices of c_1 and c_2. To get a particular solution, we require two side conditions on the path of Y_t to determine the two constants c_1 and c_2. For $a\sum_{i=0}^{\infty}(1+i)\lambda^i$ to be finite either $|\lambda| < 1$ or $a = 0$ must be satisfied.

Assume that $a = 0$ and notice that (31) can be rewritten for $t \geq 1$ as

$$Y_t = b\sum_{i=0}^{t-1}(1+i)\lambda^i X_{t-i} + b\sum_{i=t}^{\infty}(1+i)\lambda^i X_{t-i} + c_1\lambda^t + c_2 t\lambda^t. \tag{32}$$

[5] Let $w(L) = \sum_{j=-\infty}^{\infty} w_j L^j$, $v(L) = \sum_{j=-\infty}^{\infty} v_j L^j$. The *convolution* of the two sequences $\{w_j\}_{j=-\infty}^{\infty}$, $\{v_j\}_{j=-\infty}^{\infty}$ is the sequence whose jth element is $h_j = \sum_{k=-\infty}^{\infty} w_k v_{j-k}$. The reader should verify that $\sum_{j=-\infty}^{\infty} h_j L^j = h(L) = w(L)v(L) = v(L)w(L)$.

The second sum can be written as

$$b\sum_{j=0}^{\infty}(j+1+t)\lambda^{t+j}X_{0-j} = b\lambda^t\sum_{j=0}^{\infty}(j+1)\lambda^j X_{0-j} + bt\lambda^t\sum_{j=0}^{\infty}\lambda^j X_{0-j}.$$

Therefore (32) becomes

$$Y_t = b\sum_{i=0}^{t-1}(1+i)\lambda^i X_{t-i} + \lambda^t\theta_0 + t\lambda^t\eta_0 \tag{33}$$

where

$$\theta_0 = \left\{c_1 + b\sum_{j=0}^{\infty}(j+1)\lambda^j X_{0-j}\right\} \quad \text{and} \quad \eta_0 = \left\{c_2 + b\sum_{j=0}^{\infty}\lambda^j X_{0-j}\right\}.$$

As for our earlier cases, (33) displays the solution for $t \geq 1$ as the sum of the distributed lag in $X_1, X_2, \ldots, X_t$ and two initial conditions. If $X_t = 0$ for $t \geq 1$, the sequence Y_t will approach its stationary value of zero if $|\lambda| < 1$. If $|\lambda| > 1$, the sequence Y_t will diverge from the stationary point of zero unless $\theta_0 = \eta_0 = 0$.

4. Nth-ORDER DIFFERENCE EQUATIONS (DISTINCT ROOTS)

Consider a rational polynomial with nth-order denominator:

$$A(L) = \frac{F(L)}{G(L)} = \frac{F(L)}{(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_n L)}.$$

We assume that the degree of the numerator polynomial is less than the degree of the denominator polynomial. The zeros of $G(z) = 0$ are $z_1 = 1/\lambda_1$, $z_2 = 1/\lambda_2, \ldots, \ldots, z_n = 1/\lambda_n$ since each of these values for z satisfies the equation

$$G(z) = (1-\lambda_1 z)(1-\lambda_2 z)\cdots(1-\lambda_n z) = 0.$$

Suppose the n roots are distinct. Now the method of partial fractions enables us to express $A(L)$ as

$$\frac{F(L)}{(1-\lambda_1 L)\cdots(1-\lambda_n L)} = \frac{A_1}{1-\lambda_1 L} + \frac{A_2}{1-\lambda_2 L} + \cdots + \frac{A_n}{1-\lambda_n L} \tag{34}$$

where $A_1, A_2, \ldots, A_n$ are constants to be determined. To determine them, multiply both sides of the above equation by $(1-\lambda_1 L)\cdots(1-\lambda_n L)$ to get

$$F(L) = A_1(1-\lambda_2 L)\cdots(1-\lambda_n L) + A_2(1-\lambda_1 L)(1-\lambda_3 L)\cdots(1-\lambda_n L)$$
$$+ \cdots + A_n(1-\lambda_1 L)\cdots(1-\lambda_{n-1}L).$$

Evaluating the above equation at $L = 1/\lambda_1$, the first root of $G(L) = 0$, gives

$$A_1 = \frac{F(1/\lambda_1)}{(1-(\lambda_2/\lambda_1))\cdots(1-(\lambda_n/\lambda_1))}.$$

For the general term A_i, we obtain

$$A_i = \frac{F(1/\lambda_i)}{(1-(\lambda_1/\lambda_i))\cdots(1-(\lambda_{i-1}/\lambda_i))(1-(\lambda_{i+1}/\lambda_i))\cdots(1-(\lambda_n/\lambda_i))}. \tag{35}$$

As an example, consider applying (34) to the second-order denominator polynomial

$$A(L) = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{A_1}{1-\lambda_1 L} + \frac{A_2}{1-\lambda_2 L}.$$

We have

$$A_1 = \frac{1}{1-(\lambda_2/\lambda_1)} = \frac{\lambda_1}{\lambda_1-\lambda_2} \quad \text{and} \quad A_2 = \frac{1}{1-(\lambda_1/\lambda_2)} = \frac{\lambda_2}{\lambda_2-\lambda_1}.$$

Thus we obtain

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{1-\lambda_1 L} - \frac{\lambda_2}{1-\lambda_2 L}\right),$$

which we earlier used on the way to deriving (19).

Suppose we have an nth-order difference equation

$$(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_n L)Y_t = bX_t. \tag{36}$$

The general solution to (36) is obtained by "dividing"[6] by $(1-\lambda_1 L)\cdots(1-\lambda_n L)$ to obtain

$$Y_t = \frac{b}{(1-\lambda_1 L)\cdots(1-\lambda_n L)}X_t + c_1\lambda_1{}^t + \cdots + c_n\lambda_n{}^t,$$

where $c_1, \ldots, c_n$ are any constants. That this is the solution can be verified by operating on both sides of the above equation with $(1-\lambda_1 L)\cdots(1-\lambda_n L)$. We now require n side conditions on the path of Y_t to determine the n constants $c_1, \ldots, c_n$. We suppose that the λ_j are all distinct. Then application of (34) to the above equation gives

$$Y_t = b\sum_{r=1}^{n}\left(\frac{A_r}{1-\lambda_r L}\right)X_t + \sum_{j=1}^{n}c_j\lambda_j{}^t, \tag{37}$$

which shows that Y_t can be expressed as the weighted sum of n geometric distributed lags with decay coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$.

[6] That is, "operating on both sides with the operator inverse of

$$(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_n L)."$$

Given n initial values of Y, and assuming $X_t = 0$ always, it is possible to start up difference equation (36) finitely far back in the past and to obtain a solution of the form

$$Y_t = \lambda_1{}^t\eta_1 + \lambda_2{}^t\eta_2 + \cdots + \lambda_n{}^t\eta_n$$

where $\eta_1, \ldots, \eta_n$ are constants chosen to satisfy the n initial values. The above equation can be derived from (37) by applying calculations analogous to those applied above in the first- and second-order cases.

5. Nth-ORDER DIFFERENCE EQUATIONS (N EQUAL ROOTS)

Consider the nth-order difference equation

$$(1 - \lambda L)^n Y_t = bX_t, \tag{38}$$

which has the solution

$$Y_t = \frac{b}{(1 - \lambda L)^n} X_t + c_1\lambda^t + c_2 t\lambda^t + \cdots + c_n t^{n-1}\lambda^t.$$

The polynomial $1/(1 - \lambda L)^n$ is the one associated with an nth-order Pascal lag distribution, which is formed by multiplying (convolving) n geometric lag distributions with the same decay parameter λ. We have already studied the second-order Pascal distribution. The binomial expansion with negative exponent is

$$\begin{aligned}(1 - x)^{-n} &= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \\ &+ \frac{n(n+1)\cdots(n+i-1)}{i!}x^i + \cdots, \qquad |x| < 1.\end{aligned}$$

Notice that $n(n + 1)\cdots(n + i - 1) = (n + i - 1)!/(n - 1)!$ The coefficient on x in the above expansion is thus

$$\binom{n+i-1}{i} \equiv \frac{(n+i-1)!}{i!(n-1)!}.$$

Therefore, the expansion can be written

$$(1 - x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i.$$

Consequently, for a lag-generating function[7] with n equal roots, we have

$$\frac{1}{(1 - \lambda L)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} \lambda^i L^i, \tag{39}$$

which agrees with our earlier formulas for the special cases $n = 1$ and $n = 2$.

To arrive at (39) in an alternative way, write

$$f(\lambda L) = \frac{1}{(1 - \lambda L)} = \sum_{i=0}^{\infty} (\lambda L)^i.$$

Differentiating with respect to λL gives

$$f'(\lambda L) = \frac{1}{(1 - \lambda L)^2} = \sum_{i=0}^{\infty} i(\lambda L)^{i-1} = \sum_{i=0}^{\infty} (i + 1)(\lambda L)^i,$$

which is (39) with $n = 2$. Differentiating again with respect to λL gives

$$f''(\lambda L) = \frac{2}{(1 - \lambda L)^3} = \sum_{i=0}^{\infty} i(i - 1)(\lambda L)^{i-2}$$

or

$$\frac{1}{(1 - \lambda L)^3} = \sum_{i=0}^{\infty} \frac{(i+1)(i+2)}{2}(\lambda L)^i = \sum_{i=0}^{\infty} \binom{i+2}{i}(\lambda L)^i,$$

which is (39) with $n = 3$.

With the aid of (39), the solution to (38) can be written

$$Y_t = b\sum_{i=0}^{\infty} \binom{n+i-1}{i} \lambda^i X_{t-i} + c_1\lambda^t + c_2 t\lambda^t + \cdots + c_n t^{n-1}\lambda^t.$$

6. AN EXAMPLE OF A FIRST-ORDER SYSTEM

Consider the following model studied by Cagan (1956). Let m_t be the log of the money supply, p_t the log of the price level and p_{t+1}^e the log of the price expected to prevail at time $t + 1$ given information available at time t. The model is

$$m_t - p_t = \alpha(p_{t+1}^e - p_t), \qquad \alpha < 0, \tag{40}$$

which is a portfolio equilibrium condition. The demand for real balances varies inversely with expected inflation $p_{t+1}^e - p_t$. The variable m_t is exogenous.

Suppose first that

$$p_{t+1}^e - p_t = \gamma(p_t - p_{t-1}), \tag{41}$$

[7] A term synonymous with "polynomial in the lag operator."

so that the public expects inflation next period to be the current rate of inflation, $p_t - p_{t-1}$, multiplied by the constant γ. Then (40) becomes

$$m_t - p_t = \alpha\gamma p_t - \alpha\gamma p_{t-1}.$$

Using lag operators, this can be written as

$$[(\alpha\gamma + 1) - \alpha\gamma L]p_t = m_t$$

or

$$\left[1 - \frac{\alpha\gamma}{1+\alpha\gamma}L\right]p_t = \frac{1}{1+\alpha\gamma}m_t.$$

The solution can be written

$$p_t = \frac{1}{1+\alpha\gamma}\sum_{i=0}^{\infty}\left(\frac{\alpha\gamma}{1+\alpha\gamma}\right)^i m_{t-i} + \left(\frac{\alpha\gamma}{1+\alpha\gamma}\right)^t c$$

where c is a constant to be determined by, say, a given value of p at some time. The solution for p_t will be finite for the time path $m_t = \bar{m}$ for all t, provided that

$$\left|\frac{\alpha\gamma}{1+\alpha\gamma}\right| < 1.$$

The above inequality is in the spirit of the "stability condition" developed by Cagan in his paper. It is a condition that delivers a bounded p_t for all $t \geq 0$ for $\{m_t\}$ sequences that are bounded. Notice that

$$\frac{1}{1+\alpha\gamma}\sum_{i=0}^{\infty}\left(\frac{\alpha\gamma}{1+\alpha\gamma}\right)^i = \frac{(1+\alpha\gamma)^{-1}}{1-\alpha\gamma(1+\alpha\gamma)^{-1}} = 1.$$

Thus, the long-run effect of a once-and-for-all jump in m is to drive p up by an equal amount (provided the above "stability condition" is met).

Returning to (40), let us abandon (41) and now assume perfect foresight:

$$p^e_{t+1} = p_{t+1}. \tag{42}$$

Substituting (42) into (40) gives

$$m_t - p_t = \alpha p_{t+1} - \alpha p_t \qquad \text{or} \qquad \alpha p_{t+1} + (1-\alpha)p_t = m_t.$$

Write this as

$$\left(L^{-1} + \frac{1-\alpha}{\alpha}\right)p_t = \frac{1}{\alpha}m_t$$

or

$$\left(1 - \frac{\alpha-1}{\alpha}L\right)p_t = \frac{1}{\alpha}m_{t-1}. \tag{43}$$

Notice that since $\alpha < 0$, it follows that $(\alpha - 1)/\alpha > 1$. This fact is an invitation to solve (43) in the "forward" direction, i.e., to use (6). Dividing both sides of (43) by $(1 - ((\alpha - 1)/\alpha)L)$ gives

$$p_t = \left(\frac{\alpha^{-1}}{1-(\alpha-1)\alpha^{-1}L}\right)m_{t-1} + c\left(\frac{\alpha-1}{\alpha}\right)^t$$

where c is any constant. Using (6), this becomes

$$p_t = \left(\frac{\alpha^{-1}(-\alpha(\alpha-1)^{-1}L^{-1})}{1-\alpha(\alpha-1)^{-1}L^{-1}}\right)m_{t-1} + c\left(\frac{\alpha-1}{\alpha}\right)^t$$

$$= -\frac{1}{\alpha-1}\left(\sum_{i=0}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^i L^{-i}\right)m_t + c\left(\frac{\alpha-1}{\alpha}\right)^t,$$

$$p_t = \frac{1}{1-\alpha}\sum_{i=0}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^i m_{t+i} + c\left(\frac{\alpha-1}{\alpha}\right)^t. \tag{44}$$

Notice that since $\alpha < 0$, $0 < \alpha/(\alpha - 1) < 1$, so that the sum of the lag weights is finite. Equation (44) expresses the log of the current price as a moving sum of current and *future* values of the log of the money supply. Notice that

$$\frac{1}{1-\alpha}\sum_{i=0}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^i = \frac{(1-\alpha)^{-1}}{1-\alpha(\alpha-1)^{-1}} = 1,$$

so that p is a weighted *average* of current and future values of m. The solution for p_t given by (44) will be finite for all finite t if there is a constant $K > 0$ and an x, $1 \leq x < (\alpha - 1)/\alpha$ such that $|m_t| < Kx^t$ for all t. This is a condition that the money supply not grow too fast. For the solution given by (44) to be bounded for money supply paths that are bounded, we would require that $c = 0$. This amounts to an arbitrary terminal condition that rules out the occurrence of runaway inflations in the absence of runaway growth in the money supply.

7. AN EXAMPLE OF A SECOND-ORDER SYSTEM

Consider the following model studied by Muth (1961). Let p_t be the price of a commodity at t, C_t the demand for current consumption, I_t the stock of inventories of the commodity, Y_t the output of the commodity, and p_t^e the price previously expected to prevail at time t; $\{X_t\}$ is a bounded sequence of real numbers that represents the effects of the weather on supply. The model is

$$\begin{array}{lll} C_t = -\beta p_t, & \beta > 0 & \text{(demand curve)} \\ Y_t = \gamma p_t^e + X_t, & \gamma > 0 & \text{(supply curve)} \\ I_t = \alpha(p^e_{t+1} - p_t), & \alpha > 0 & \text{(inventory demand)} \\ Y_t = C_t + (I_t - I_{t-1}) & & \text{(market clearing).} \end{array}$$

Let us suppose that there is perfect foresight so that $p_t^e = p_t$ for all t. Making this assumption and substituting the first three equations into the fourth gives

$$\gamma p_t + X_t = \alpha(p_{t+1} - p_t) - \alpha(p_t - p_{t-1}) - \beta p_t$$

or

$$\alpha p_{t+1} - (2\alpha + \beta + \gamma)p_t + \alpha p_{t-1} = X_t.$$

Dividing by α gives

$$p_{t+1} - \frac{2\alpha + \beta + \gamma}{\alpha} p_t + p_{t-1} = \alpha^{-1}X_t$$

or

$$(L^{-1} - \phi + L)p_t = \alpha^{-1}X_t$$

where $\phi = ((\beta + \gamma)/\alpha) + 2 > 0$. Multiplying by L gives

$$(1 - \phi L + L^2)p_t = \alpha^{-1}X_{t-1}. \tag{45}$$

We need to factor the polynomial $1 - \phi L + L^2$ as

$$1 - \phi L + L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L) = 1 - (\lambda_1 + \lambda_2)L + \lambda_1\lambda_2 L^2$$

so that we require

$$\lambda_1 + \lambda_2 = \phi, \qquad \lambda_1\lambda_2 = 1.$$

The second equality establishes that $\lambda_1 = 1/\lambda_2$, so that the two roots appear as a reciprocal pair. So we can write

$$1 - \phi L + L^2 = (1 - \lambda L)(1 - \lambda^{-1}L)$$

where λ is chosen to satisfy $\lambda + \lambda^{-1} = \phi$. So (45) can be written

$$(1 - \lambda L)(1 - \lambda^{-1}L)p_t = \alpha^{-1}X_{t-1}. \tag{46}$$

Since $(\beta + \gamma)/\alpha > 0$, it follows that $\phi = ((\beta + \gamma)/\alpha) + 2 > 2$. That implies that λ does not equal 1 since $\lambda + \lambda^{-1} = \phi$. Notice that if $\lambda > 1$, $\lambda^{-1} < 1$. So one of our roots necessarily exceeds 1, the other necessarily is less than 1.

We divide both sides of (46) by $(1 - \lambda L)(1 - \lambda^{-1}L)$ to obtain

$$p_t = \frac{1}{\alpha}\frac{1}{(1 - \lambda L)(1 - \lambda^{-1}L)}X_{t-1} + c_1\lambda^t + c_2\left(\frac{1}{\lambda}\right)^t \tag{47}$$

where c_1 and c_2 are any constants. Without loss of generality, suppose $\lambda < 1$ and let $\lambda_2 = 1/\lambda$. Use (6) and (19) to write the solution as

$$p_t = \frac{\alpha^{-1}\lambda}{\lambda - \lambda_2}\left(\frac{1}{1 - \lambda L}\right)X_{t-1} - \frac{\alpha^{-1}\lambda_2}{(\lambda - \lambda_2)}\left(\frac{-(\lambda_2 L)^{-1}}{1 - (\lambda_2)^{-1}L^{-1}}\right)X_{t-1} + c_1\lambda^t + c_2\left(\frac{1}{\lambda}\right)^t$$

$$= \frac{\alpha^{-1}\lambda}{\lambda - \lambda^{-1}}\left(\frac{1}{1 - \lambda L}\right)X_{t-1} + \frac{\alpha^{-1}\lambda^{-1}}{\lambda - \lambda^{-1}}\left(\frac{\lambda L^{-1}}{1 - \lambda L^{-1}}\right)X_{t-1} + c_1\lambda^t + c_2\lambda^{-t}$$

$$= \frac{\alpha^{-1}\lambda}{\lambda - \lambda^{-1}}\left(\frac{1}{1 - \lambda L}\right)X_{t-1} + \frac{\alpha^{-1}}{\lambda - \lambda^{-1}}\left(\frac{1}{1 - \lambda L^{-1}}\right)X_t + c_1\lambda^t + c_2\lambda^{-t},$$

$$p_t = \frac{\alpha^{-1}}{\lambda - \lambda^{-1}}\sum_{i=1}^{\infty}\lambda^i X_{t-i} + \frac{\alpha^{-1}}{\lambda - \lambda^{-1}}\sum_{i=0}^{\infty}\lambda^i X_{t+i} + c_1\lambda^t + c_2\lambda^{-t},$$

$$p_t = \frac{\alpha^{-1}}{\lambda - \lambda^{-1}}\sum_{i=-\infty}^{\infty}\lambda^{|i|}X_{t-i} + c_1\lambda^t + c_2\lambda^{-t}. \tag{48}$$

The solution (48) expresses p_t as a "two-sided" distributed lag of X, i.e., as a weighted sum of past, present, and future values of X. In this model the current price depends on the entire path of exogenous shock X over the entire past *and* the entire future. As usual, the constants c_1 and c_2 are determined from two side conditions on the path of p_t. For example, if we were to impose the side conditions $\lim_{t\to-\infty}|p_t| < \infty$, $\lim_{t\to+\infty}|p_t| < \infty$, i.e., if we imposed boundedness on the p_t path for all bounded $\{X_t\}$ sequences, then we would require $c_1 = c_2 = 0$.

8. AN OPTIMIZATION EXAMPLE: SOLVING A SYSTEM OF EULER EQUATIONS

Thus far, our advice to solve stable roots backward and unstable roots forward has been to a certain extent arbitrary, being partly based on the desire to have solutions that are bounded. As it happens, when a linear difference equation emerges from a dynamic quadratic optimization problem, there are present additional necessary conditions for optimality which have precisely the effect of causing us to solve the stable root(s) backward and the unstable root(s) forward. We shall illustrate this for the case of a firm employing a single factor of production, labor, and facing a quadratic technology and quadratic costs of adjusting its labor force.

We consider a firm that at time t chooses employment to maximize its present value

$$v_t = \sum_{j=0}^{\infty} b^j\left\{(f_0 + a_{t+j})n_{t+j} - \frac{f_1}{2}n_{t+j}^2 - \frac{d}{2}(n_{t+j} - n_{t+j-1})^2 - w_{t+j}n_{t+j}\right\}, \tag{49}$$

n_{t-1} given. Here $f_0, f_1, d > 0$ and the discount factor b obeys $0 < b < 1$. Here w_{t+j} is the real wage faced by the firm at time $t+j$, while n_{t+j} is the firm's employment at $t+j$. The firm faces a known sequence of $\{a_{t+j}\}_{j=0}^{\infty}$, a_{t+j} being a shock to the marginal product of labor at time $t+j$. We assume for all t that for some $K > 0$, $|w_t| < K(x)^t$, $|a_t| < K(x)^t$, where $1 \leq x < 1/\sqrt{b}$; sequences that satisfy these inequalities for some $K > 0$ and $1 \leq x < 1\sqrt{b}$ will be termed of exponential order less than $1/\sqrt{b}$. The firm faces costs of adjusting its labor force rapidly, which we model by deducting the term $\frac{1}{2}d(n_{t+j} - n_{t+j-1})^2$ from real revenue each period. The firm faces known sequences $\{w_{t+j}\}_{j=0}^{\infty}$ and $\{a_{t+j}\}_{j=0}^{\infty}$ and is assumed to choose a sequence $\{n_{t+j}\}_{j=0}^{\infty}$ to maximize v_t.

In order to discover the appropriate solution to this problem it is convenient to consider the finite horizon problem: maximize

$$v_t^T = \sum_{j=0}^{T} b^j\left\{(f_0 + a_{t+j})n_{t+j} - \frac{f_1}{2}n_{t+j}^2 - \frac{d}{2}(n_{t+j} - n_{t+j-1})^2 - w_{t+j}n_{t+j}\right\} \tag{50}$$

subject to n_{t-1} being given. This problem matches our infinite horizon problem when we drive T to infinity. If a sequence $\{n_{t+j}\}_{j=0}^{T}$ is to maximize (50), it must satisfy the following system of first-order necessary conditions, which we derive by differentiating (50) with respect to $n_t, n_{t+1}, \ldots, n_{t+T}$:

$$\begin{aligned} f_0 + a_{t+j} - w_{t+j} - f_1 n_{t+j} - d(n_{t+j} - n_{t+j-1}) \\ + db(n_{t+j+1} - n_{t+j}) = 0, \qquad j = 0, 1, \ldots, T-1, \end{aligned} \tag{51}$$

$$b^T[f_0 + a_{t+T} - w_{t+T} - f_1 n_{t+T} - d(n_{t+T} - n_{t+T-1})] = 0. \tag{52}$$

Equation (51) is a system of second-order linear difference equations known as the "Euler equations," which we write as

$$dbn_{t+j+1} - (f_1 + d(1+b))n_{t+j} + dn_{t+j-1} = w_{t+j} - a_{t+j} - f_0$$

or

$$bn_{t+j+1} + \phi n_{t+j} + n_{t+j-1} = \frac{1}{d}(w_{t+j} - a_{t+j} - f_0), \qquad j = 0, 1, \ldots, T-1,$$

$$\phi = -\left(\frac{f_1}{d} + (1+b)\right). \tag{53}$$

To solve this second-order difference equation, we need two boundary conditions. One boundary condition is supplied by the historically given initial level of n_{t-1}, and the other by the terminal condition (52). That terminal condition is known as the transversality condition and is a necessary condition for optimality. To get the terminal condition in the infinite horizon problem, it is appropriate to take

$$\lim_{T\to\infty} b^T[f_0 + a_{t+T} - w_{t+T} - f_1 n_{t+T} - d(n_{t+T} - n_{t+T-1})]n_{t+T} = 0 \tag{54}$$

as the transversality condition. We obtain this condition by multiplying (52) by the terminal stock of employment n_{t+T}, and taking the limit of the resulting equation as $T \to \infty$. Sufficient conditions for the transversality condition (54) to hold are, first, that $\{a_{t+j}\}$ and $\{w_{t+j}\}$ be of exponential order less than $1/\sqrt{b}$, and, second, that the solution for $\{n_{t+j}\}$ be of exponential order less than $1/\sqrt{b}$. Thus, suppose that $\{w_t\}$, $\{a_t\}$, and $\{n_t\}$ are each of exponential order less than $1/\sqrt{b}$. Then notice that

$$\begin{aligned} &|b^T[f_0 n_{t+T} + a_{t+T}n_{t+T} - w_{t+T}n_{t+T} - (f_1 + d)n_{t+T}^2 + d(n_{t+T}n_{t+T-1})]| \\ &\quad \leq b^T f_0 |n_{t+T}| + b^T|a_{t+T}n_{t+T}| + b^T|w_{t+T}n_{t+T}| \\ &\quad\quad + b^T(f_1 + d)|n_{t+T}^2| + b^T d|n_{t+T}n_{t+T-1}| \leq b^T f_0 K x^{t+T} \\ &\quad\quad + b^T K x^{2(t+T)} + b^T K x^{2(t+T)} + b^T(f_1 + d)K x^{2(t+T)} \\ &\quad\quad + b^T d K x^{2(t+T)-1} \\ &\quad = f_0 K x^t b^T x^T + K(2 + f_1 + d)b^T x^{2T} x^{2t} + dKb^T x^{2T} x^{2t-1}. \end{aligned}$$

From the definition of a sequence of exponential order less than $1/\sqrt{b}$, we have that $0 < x\sqrt{b} < 1$. This condition implies that $xb < \sqrt{b} < 1$. Therefore the limit as $T \to \infty$ of the expression above equals zero. This proves that the conditions that $\{a_t\}$, $\{w_t\}$ and $\{n_t\}$ are of exponential order less than $1/\sqrt{b}$ are sufficient to imply that the transversality condition (54) is satisfied. The necessary conditions for optimality for the infinite horizon problem are then satisfied if we can find a solution to the difference equation

$$bn_{t+j+1} + \phi n_{t+j} + n_{t+j-1} = d^{-1}(w_{t+j} - a_{t+j} - f_0)$$

subject to the transversality condition (54) and the known initial value of n_{t-1}.

To solve the difference equation, write it as

$$b\left(1 + \frac{\phi}{b}L + \frac{1}{b}L^2\right)n_{t+j+1} = \frac{1}{d}(w_{t+j} - a_{t+j} - f_0).$$

We seek a factorization

$$\left(1 + \frac{\phi}{b}L + \frac{1}{b}L^2\right) = (1 - \lambda_1 L)(1 - \lambda_2 L) = 1 - (\lambda_1 + \lambda_2)L + \lambda_1\lambda_2 L^2.$$

Equating powers of L gives

$$-\phi/b = (\lambda_1 + \lambda_2), \qquad 1/b = \lambda_1\lambda_2, \qquad \text{or} \qquad 1/\lambda_1 b = \lambda_2.$$

Thus we have that λ_1 must satisfy

$$-\frac{\phi}{b} = \left(\lambda_1 + \frac{1}{\lambda_1 b}\right) \qquad \text{or} \qquad \frac{f_1}{d} + (1+b) = -\phi = \lambda_1 b + \frac{1}{\lambda_1}.$$

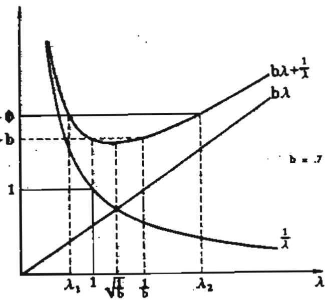

FIGURE 4

Figure 4 depicts the determination of λ_1 and λ_2. The function $\lambda b + \lambda^{-1}$ attains a minimum at $\lambda = \sqrt{1/b}$, attaining a value of $2\sqrt{b}$ there. For $0 < b < 1$, we have that $1 + b \geq 2\sqrt{b}$ with equality at $b = 1$, so that even with $f_1 = 0$, in which case $\phi = -(1 + b)$, we have $-\phi \geq 2\sqrt{b}$. This assures that with $f_1 > 0$ the solutions for λ_1 and λ_2 as depicted in Figure 4 are real and distinct. Without loss of generality, let λ_1 be the smaller root. Then notice that the preceding implies that $\lambda_1 < 1/\sqrt{b} < \lambda_2$. By using Figure 4 we can establish that $\lambda_1 < 1 < 1/b < \lambda_2$. From $-\phi = ((f_1/d) + 1 + b)$, we have that $-\phi > 1 + b$, since $f_1/d > 0$. Now if $-\phi$ were to equal $1 + b$, we would have λ_1 and λ_2 being determined from

$$1 + b = \lambda b + \lambda^{-1}$$

or

$$b\lambda^2 - (1 + b)\lambda + 1 = 0, \qquad (\lambda b - 1)(\lambda - 1) = 0,$$

so that $\lambda_1 = 1$, $\lambda_2 = 1/b$. Since $-\phi > 1 + b$, inspection of Figure 4 shows that unity must be an upper bound for λ_1 and $1/b$ a lower bound for λ_2.[8]

Having achieved this factorization, we write the difference equation as

$$b(1 - \lambda_1 L)(1 - \lambda_2 L)n_{t+j+1} = d^{-1}(w_{t+j} - a_{t+j} - f_0).$$

[8] A version of Figure 4 with $b = 1$ determines the intriguing golden ratio or golden section. The golden ratio is the unique positive number λ whose reciprocal equals itself plus unity: $\lambda^{-1} = 1 + \lambda$. This equation can be rearranged to read $\lambda^{-1} + \lambda = 1 + 2\lambda$. From the quadratic formula, the golden ratio equals $(\sqrt{5} - 1)/2 = 0.618034$ and is found as the intersection in the positive quadrant of the line $1 + 2\lambda$ with the curve $\lambda + \lambda^{-1}$. The golden ratio, which appears repeatedly in nature and mathematics, fascinated ancient people, and is reflected in the design of the Parthenon. One place that the number occurs in mathematics is as the limit as t goes to infinity of the ratio of successive terms in a Fibonacci sequence. A Fibonacci sequence is generated by the difference equation $x_{t+1} = x_t + x_{t-1}$ with initial conditions $x_0 = 1$, $x_{-1} = 0$. The characteristic polynomial $(1 - L - L^2)$ associated with this equation can be factored as $(\lambda - L)(\lambda^{-1} + L)$ where λ is the golden ratio. For more on the golden ratio, see "Math and Music: The Deeper Links," *The New York Times*, Sunday, August 29, 1982.

To satisfy the transversality condition, operate on both sides of this equation with the "forward" inverse of $1 - \lambda_2 L$ to get

$$(1 - \lambda_1 L)n_{t+j+1} = \frac{-(b\,d\lambda_2)^{-1}L^{-1}}{1 - \lambda_2^{-1}L^{-1}}(w_{t+j} - a_{t+j} - f_0) + c\lambda_2^{t}$$

where c is a constant. However, since $\lambda_2 > 1/b > 1/\sqrt{b}$, we must set $c = 0$ in order to satisfy the transversality condition, for otherwise $\{n_{t+j}\}$ would fail to be of exponential order less than $1/\sqrt{b}$. Setting $c = 0$ and observing that $1/\lambda_2 = b\lambda_1$, we have

$$n_{t+j+1} = \lambda_1 n_{t+j} - \frac{\lambda_1}{d}\sum_{i=0}^{\infty}\left(\frac{1}{\lambda_2}\right)^{i}(w_{t+j+1+i} - a_{t+j+1+i} - f_0), \qquad j = 0, 1, \ldots. \tag{55}$$

This is a demand schedule for employment that expresses current employment as a function of once-lagged employment and current and all future values of the real wage and the shock to productivity. Since $\lambda_2 > 1/b$, we have that $1/\lambda_2 < b$, so that the infinite weighted sum on the right-hand side of (55) converges for $\{a_{t+j}\}$ and $\{w_{t+j}\}$ sequences that are of exponential order of less than $1/b$. We have assumed the stronger condition that $\{a_t\}$, $\{w_t\}$ are sequences of exponential order less than $1/\sqrt{b}$, so that the infinite weighted sum on the right side of (55) converges.

It remains to show that (55) holds for $j = -1$ as well. To show this, take the solution (55) for $j = 0$ and use it to eliminate n_{t+1} from the Euler equation (53) for $j = 0$, thereby obtaining

$$\{b\lambda_1 + \phi\}n_t + n_{t-1} = z_t + \frac{b\lambda_1^{*}}{1 - \lambda_2^{-1}L^{-1}} z_{t+1}$$

where $z_t \equiv d^{-1}(w_t - a_t - f_0)$. Since $\lambda_1 + \lambda_2 = -\phi/b$ and $b\lambda_1 = \lambda_2^{-1}$, we can rearrange the above equation to read

$$n_t = \lambda_1 n_{t-1} - \frac{\lambda_1}{1 - \lambda_2^{-1}L^{-1}} z_t,$$

which asserts that (55) holds for $j = -1$. Thus, (55) gives the firm's optimal plan for setting n_{t+j+1} for $j = -1, 0, 1, \ldots$.

We have in effect imposed the transversality condition first by using the "unstable" root λ_2 to solve in the forward direction and second by setting $c\lambda_2^{t} = 0$ in (55). To see this, write (55) as

$$(1 - \lambda_1 L)n_{t+j+1} = \varepsilon_{t+j+1}, \qquad j = -1, 0, 1, 2, \ldots, \tag{56}$$

where

$$\varepsilon_{t+j+1} \equiv -\frac{\lambda_1}{d}\frac{1}{1 - \lambda_2^{-1}L^{-1}}(w_{t+j+1} - a_{t+j+1} - f_0).$$

Since $\{w_{t+j}\}$ and $\{a_{t+j}\}$ are of exponential order less than $1/\sqrt{b}$, it follows[9] that the sequence $\{\varepsilon_{t+j+1}\}$ is of exponential order less than $1/\sqrt{b}$ and is finite for finite $t+j$. The solution of the difference equation (55) with n_{t-1} given is

$$n_{t+j+1} = \lambda_1^{j+2} n_{t-1} + \sum_{i=0}^{j+1} \lambda_1{}^i \varepsilon_{t+j+1-i}, \qquad j = -2, -1, 0, 1, \ldots. \tag{57}$$

This is the unique solution to the Euler equation that satisfies the initial condition and the transversality condition (54). It is straightforward to verify that both the initial condition and the transversality condition are satisfied by this solution, the latter condition being satisfied by virtue of the exponential order being less than $1/\sqrt{b}$ for the $\{\varepsilon_{t+j}\}$ sequence and the fact that $|\lambda_1| < 1$. For notice that

$$\left|\sum_{i=0}^{j+1} \lambda_1{}^i \varepsilon_{t+j+1-i}\right| < \sum_{i=0}^{j+1} \lambda_1{}^i K(x)^{t+j+1-i} \le K(x)^{t+j+1}\left(\frac{1}{1-(\lambda_1/x)}\right),$$

which proves that $\sum_{i=0}^{j+1} \lambda_1{}^i \varepsilon_{t+j+1-i}$ is of exponential order less than $1/\sqrt{b}$. It follows that (57) implies that n_{t+j} is of exponential order less than $1/\sqrt{b}$, and therefore that the transversality condition (54) is satisfied.

It should be emphasized that the transversality condition compels us to solve the unstable root forward. If we had solved the unstable root λ_2 "backward," then eventually the solution for n_{t+j+1} would come to be dominated by terms of exponential order λ_2. Since $\lambda_2 > 1/\sqrt{b}$, that solution would violate the transversality condition (54).

In conclusion, the solution of our infinite horizon maximum problem is for the firm to set its employment according to the demand schedule or decision rule

$$n_{t+j+1} = \lambda_1 n_{t+j} - \frac{\lambda_1}{d}\frac{1}{1-\lambda_2^{-1}L^{-1}}(w_{t+j+1} - a_{t+j+1} - f_0)$$

for $j = -1, 0, 1, 2, \ldots$.

9. A MORE GENERAL UNIVARIATE OPTIMIZATION PROBLEM

The problem of the preceding section is a special case of a more general quadratic optimization problem, one version of which follows. We define the polynomial in the lag operator,

$$d(L) = d_0 + d_1 L + \cdots + d_m L^m, \tag{58}$$

[9] For example, since $|w_{t+j+1}| < K(x)^{t+j+1}$ for some $K > 0$ and for $1 \le x < 1/\sqrt{b}$, we have

$$F_{t+j+1} \equiv \left|\sum_{i=0}^{\infty}\left(\frac{1}{\lambda_2}\right)^i w_{t+j+i+1}\right| < K \sum_{i=0}^{\infty}\left(\frac{1}{\lambda_2}\right)^i (x)^{t+j+i+1} = K\left(\frac{1}{1-\lambda_2^{-1}x}\right)x^{t+j+1}$$

where $1/\lambda_2 < \sqrt{b}$ and $x < 1/\sqrt{b}$. Therefore, F is of exponential order less than $1/\sqrt{b}$.

where $d_0 \neq 0$, $d_m \neq 0$. We assume that g_t is a sequence of exponent
than $1/\sqrt{b}$, where $0 < b < 1$. Then the problem is to choose
$\{y_t, t \ge 0\}$ to maximize

$$\sum_{t=0}^{\infty} b^t\{g_t y_t - \tfrac{1}{2}h y_t{}^2 - \tfrac{1}{2}[d(L)y_t]^2\}$$

where $h > 0$, subject to $y_{-1}, y_{-2}, \ldots, y_{-m}$ given. The problem of S
version of problem (59) with $g_t = (f_0 + a_t - w_t)$, $m = 1$, $y_t = n_t$,
$\sqrt{d}(1-L)$, $h = f_1$.

The Euler equation for this problem is

$$[d(bL^{-1})d(L) + h]y_t = g_t.$$

We invite the reader to verify that this is the Euler equation by di
(59) with respect to y_t and rearranging. Note that (53) is a special
with $d(L) = (1-L)\sqrt{d}$, $h = f_1$, $g_t = (f_0 + a_t - w_t)$.

In addition to the Euler equation (60), the necessary and suffi
tions for maximization of (59) are completed by the condition

$$\frac{h}{2}\sum_{t=0}^{\infty} b^t y_t{}^2 < +\infty.$$

Inspection of (59) reveals that any $\{y_t\}$ path that violates (61), even
the Euler equation, gives a very bad outcome for the criterio
Imposing conditions (61) is equivalent to imposing transversality
Transversality conditions could be derived as in Section 8, by diffe
finite T version of (59) with respect to $y_T, \ldots, y_{T-m+1}$, multiplying t
m expressions by $y_T, \ldots, y_{T-m+1}$, respectively, and setting the limits a
zero. As do the transversality conditions, (61) forces the solution {
exponential order less than $1/\sqrt{b}$.

We briefly describe how (60) is solved subject to (61). We first n
but important feature of the characteristic polynomial $h + d(bL^{-}$
appears in (60). This polynomial evaluated at any value z_0 equals
mial evaluated at bz_0^{-1}. In particular, if z_0 is a zero of this polynomia
bz_0^{-1}. To prove this, suppose that z_0 is a zero, which means setting I
$h + d(bz_0^{-1})d(z_0) = 0$. The claim is that this implies that settin
will also set the polynomial to zero. Making this substitu
$h + d(z_0)d(bz_0^{-1})$, which equals the characteristic polynomial evalu
which equals zero by assumption. Thus, the zeros of the characte
nomial of the Euler equation come in pairs of the form z_k, bz_k^{-1}, k

Let us assume that the zeros of the characteristic polynomial a
(The reader can apply the methods of Section 5 to analyze the case
zeros, if this becomes necessary.) There are $2m$ zeros of the
$h + d(bz^{-1})d(z)$. Denote these zeros $z_1, z_2, \ldots, z_{2m}$. Without loss of

13. NONANTICIPATIVE REPRESENTATIONS AND LUCAS'S CRITIQUE

We return to a version of the difference equation associated with Cagan's model of hyperinflation, which we studied in Section 6 above. In particular, we have

$$y_t = \lambda y_{t+1} + x_t, \qquad |\lambda| < 1,$$

where $\{x_t\}$ is a sequence of exponential order less than $1/\lambda$. The unique bounded solution of this equation is given by

$$y_t = \sum_{j=0}^{\infty} \lambda^j x_{t+j}. \tag{110}$$

This equation represents the solution for y_t in terms of an infinite number of future values of the "forcing variable" x_t. In practice, one often prefers a "nonanticipative" representation of a solution, in which y_t is expressed as a solution only of x's dated t and earlier. We now briefly show how representation (110) can be converted to an alternative representation expressing y_t as such a function of current and past x's only. Given (110), such a representation in terms of current and past x's can be attained by positing a specific difference equation for $\{x_t\}$ itself, using this difference equation to express x_{t+j} for $j \geq 1$ as functions of $\{x_t, x_{t-1}, \ldots\}$, and then using these functions to eliminate x_{t+j} for $j \geq 1$ from (110). The parameters of the representation for y_t as a function of current and past x's will depend on the parameters of the difference equation for x_t. This is so because the representation expressing y_t as a function of $\{x_t, x_{t-1}, \ldots\}$ incorporates the solution of the difference equation expressing x_{t+j}, $j \geq 1$, in terms of $\{x_t, x_{t-1}, x_{t-2}, \ldots\}$. For example, suppose that x_t is governed by the difference equation

$$x_t = \rho x_{t-1}, \qquad |\rho\lambda| < 1.$$

Then we have that

$$x_{t+j} = \rho^j x_t, \qquad j \geq 1.$$

Substituting this into (110) gives

$$y_t = \left(\frac{1}{1-\lambda\rho}\right) x_t, \tag{111}$$

which is an expression for y_t in terms of x_t in which the parameter ρ of the difference equation for x_t appears.

More generally, assume that x_t is governed by the difference equation

$$a(L)x_t = 0 \tag{112}$$

where $a(L) = 1 - a_1 L - a_2 L^2 - \cdots - a_r L^r$ and where the zeros of $a(z)$ exceed $1/\lambda$ in absolute value. Then Hansen and Sargent (1980) show that the solution of (110) can be represented as

$$y_t = a(\lambda)^{-1}\left[1 + \sum_{j=1}^{r-1}\left(\sum_{k=j+1}^{r} \lambda^{k-j} a_k\right)L^j\right]x_t. \tag{113}$$

(We shall show how to derive this formula in Chapter XI.) Equation (113) is evidently a generalization of (111). That is, (111) is (113) for the case $r = 1$, $a_0 = 1$, $a_1 = \rho$.

Equations (112) and (113) display a hallmark of economic models incorporating foresight as in (110), namely the presence of cross-equation restrictions between, on the one hand, the parameters of the process $\{x_t\}$ of the forcing variables, and, on the other hand, the solution for y_t in terms of $\{x_t, x_{t-1}, \ldots\}$. This characteristic of models with foresight means that it will not be possible to find a representation expressing y_t as a function of $\{x_t, x_{t-1}, \ldots\}$ that is independent of the law of motion (difference equation) governing the forcing process $\{x_t\}$. Alterations in the law of motion for x_t will alter the *function* describing the dependence of y_t on current and past x's.

The presence of such cross-equation restrictions in models with foresight is the source of Robert E. Lucas's (1976) criticism of some econometric policy evaluation procedures of the past. Lucas criticized methods that took representations of a variable like y_t as a function of $\{x_t, x_{t-1}, \ldots\}$ and treated them as being invariant with respect to alterations in the difference equation generating the $\{x_t\}$ sequence.

EXERCISES

1. Verify that the presence of both *lagged* employment and *future* values of w_t and a_t on the right-hand side of the employment decision rule (55) (demand schedule) depends on having the adjustment cost parameter d strictly positive. (Set $d = 0$ and rework the firm's optimum problem.)

2. Determine the effect of an increase in d on the speed of adjustment parameters λ_1 and λ_2. Does a firm facing a small d adjust its labor force more or less quickly in response to current conditions than a firm facing a larger value of d? (*Hint*: use Figure 4.)

3. (*A Keynesian investment schedule*) A firm chooses a sequence of capital $\{k_{t+j}\}_{j=0}^{\infty}$ to maximize

$$v_t = \sum_{j=0}^{\infty} b^j\left\{a_0 k_{t+j} - \frac{a_1}{2}k_{t+j}^2 - J_{t+j}(k_{t+j} - k_{t+j-1}) - \frac{d}{2}(k_{t+j} - k_{t+j-1})^2\right\},$$

given $k_{t-1} > 0$, where $a_0, a_1, d > 0$ and where $\{J_{t+j}\}_{j=0}^{\infty}$ is a known sequence of the price of capital relative to the price of the firm's output. The sequence $\{J_{t+j}\}$ is of exponential order less than $1/b$, where $0 < b < 1$ is the discount factor (the reciprocal of one plus the real rate of interest).

A. Derive the Euler equations and the transversality condition.

B. Show that the optimum decision rule of the firm is of the form

$$k_{t+j+1} = \lambda_1 k_{t+j} + c_0 + \frac{c_1}{1 - \lambda_2^{-1}L^{-1}}(bJ_{t+j+2} - J_{t+j+1}),$$

where $0 < \lambda_1 < 1 < \lambda_2$, and c_0 and c_1 are constants. Find c_0 and c_1; show how to find λ_1 and λ_2; and prove that the λ's obey the inequalities just stated.

4. (*Keynesian stabilization policy*) The reduced form for GNP (Y) is

$$Y_t = \alpha + BS_t + cg_t, \qquad B > 0, \quad c > 0,$$

where g is government purchases and S_t is exports, an exogenous variable outside the government's control. The sequence of exports $\{S_t\}_{t=0}^{\infty}$ is of exponential order less than $1/b$ where $0 < b < 1$. Suppose that the government sets g_t to minimize the loss function

$$T = \sum_{t=0}^{\infty} b^t\{(Y_t - Y_t^*)^2 + d(g_t - g_{t-1})^2\}, \qquad d \geq 0,$$

where $\{Y_t^*\}_{t=0}^{\infty}$ is a target sequence of GNPs that is of exponential order less than $1/b$, and d is non-negative and measures the cost of changing the setting of g from its previous value.

A. Derive an optimal rule for setting g_t under the assumption that $d = 0$.

B. Derive the optimal rule for setting g_t under the assumption that $d > 0$.

5. (*Cass-Koopmans optimum growth problem*) A planner wants to choose (consumption, capital) sequences that maximize

$$\sum_{t=0}^{\infty} \beta^t\left(u_0 c_t - \frac{u_1}{2} c_t^2\right), \qquad u_0, u_1 > 0$$

subject to $c_t + k_{t+1} \leq f_0 k_t$, where k_0 is given and satisfies $0 < k_0 < (u_0/u_1)(1/(f_0 - 1))$. Here c_t is per capita consumption and k_t is per capita capital. Assume that $f_0 > 1/\beta > 1$, where β is the discount factor. The planner imposes the side condition that the $\{k_t\}$ sequence be bounded. (Actually, all we need is that k_t be required to be nonnegative, but boundedness does the job and is easier to handle technically.)

A. Find the Euler equation and the transversality condition. Interpret the transversality condition. (*Hint*: use the constraint to eliminate c_t from the objective function and differentiate with respect to successive k's.)

B. Solve the Euler equation for the steady state value of k, say $\bar{k}$. Find the steady-state value of $\bar{c}$. (*Hint*: you should get

$$\bar{k} = \left(\frac{1}{f_0 - 1}\right)\frac{u_0}{u_1}, \qquad \bar{c} = \frac{u_0}{u_1}.)$$

Interpret the steady-state value of $\bar{c}$.

C. Show that the optimal feedback rule for setting k is

$$k_{t+1} = \frac{1}{f_0\beta} k_t - \left(\frac{1}{f_0 - 1}\right)\left(\frac{u_0}{u_1}\left(\frac{1}{\beta f_0} - 1\right)\right). \tag{*}$$

(*Hint*: Solve the Euler equation.) Does this solution satisfy the transversality condition? If you had solved the "other" root backward, rather than forward, would the transversality condition be satisfied?

D. Prove that as $t \to \infty$ the solution for k_{t+1} in (*) converges to $\bar{k}$.

E. Prove that consumption increases as capital increases toward its steady-state value.

6. A consumer is assumed to face the sequence of budget constraints

$$A_{t+1} = (1 + r)A_t + (1 + r)(y_t - c_t), \qquad t = 0, 1, 2, \ldots, \tag{†}$$

where A_t is his asset holdings at the beginning of period t, y_t is exogenous income, and c_t is consumption at t. Here $r > 0$ is the real rate of return on assets, assumed constant over time. The consumer earns (pays) each period $1 + r$ times his initial assets plus $1 + r$ times the addition

(subtraction) to his assets made by consuming less (more) than his income. Assume that both c_t and y_t are of exponential order less than $1 + r$.

Suppose we impose upon the consumer the boundary condition

$$\lim_{t\to\infty} (1 + r)^{-t} A_t = 0.$$

Note TYPO "$t=0$" SHOULD Be "$i=0$"

Show that then the sequence of budget constraints (†) implies

$$A_t + \sum_{t=0}^{\infty} \frac{y_{t+i}}{(1 + r)^i} = \underbrace{\sum_{t=0}^{\infty}}_{i=0} \frac{c_{t+i}}{(1 + r)^i}, \qquad t = 0, 1, 2, \ldots.$$

Interpret this result.

Assume that the consumer starts out with initial assets of A_0 at time 0. Show that the sequence of budget constraints (†) implies

$$A_t = \sum_{i=0}^{t-1} (1 + r)^{i+1}(y_{t-1-i} - c_{t-i-1}) + (1 + r)^t A_0.$$

Interpret this result.

7. Suppose that portfolio equilibrium is described by Cagan's equation:

$$m_t - p_t = -1(p_{t+1}^e - p_t), \qquad t = 0, 1, 2, \ldots,$$

where m_t is the log of the money supply, p_t the log of the price level at t, and p_{t+1}^e the public's expectation of p_{t+1}, formed at time t. Suppose that expectations are "rational," so that

$$p_{t+1}^e = p_{t+1}.$$

A. Suppose that $\{m_t\}$, $t = 0, 1, \ldots$, is given by

$$m_t = 10\lambda^t, \qquad t = 0, 1, 2, \ldots.$$

Compute the equilibrium value of p_t for $t = 0, 1, 2$, and $t = 5, 6$ for the following values of λ:

(i) $\lambda = 1$;
(ii) $\lambda = 1.5$;
(iii) $\lambda = 2.0$.

B. Suppose that m_t follows the path

$$m_t = 10\lambda^t, \qquad t = 0, 1, 2, 3, 4$$
$$m_t = 11\lambda^t, \qquad t = 5, 6, 7, 8, \ldots.$$

Compute the equilibrium values of p_t for $t = 0, 1, 2, 3, 4, 5, 6$ assuming that $\lambda = 1.5$. How does this time path for $\{p_t\}$ compare with that computed in A(ii)? Graph the two paths.

8. (Advertising)

A monopolist faces the following demand curve for his product,

$$p_t = A_0 - A_1 Q_t + g(L)a_t + u_t, \qquad A_0, A_1 > 0,$$

where p_t is price, Q_t is output, a_t is advertising, u_t is a sequence of shocks to demand, and $g(L) = g_0 + g_1 L + \cdots + g_m L^m$, where $g_j > 0$ for $j = 0, \ldots, m$. The firm maximizes

$$\sum_{t=0}^{\infty} \beta^t\left\{p_t Q_t - Q_t s_t - (1/2)[d(L)Q_t]^2 - \frac{\gamma}{2}a_t^2 - a_t w_t\right\}, \qquad 0 < \beta < 1, \tag{1}$$

where $d(L) = \sum_{j=0}^{n} d_j L^j$. In (1), s_t is a shock to costs, $\frac{1}{2}[d(L)Q_t]^2$ represents costs of rapid adjustment, and the marginal costs of advertising at t are $(w_t + \gamma a_t)$, where w_t is a known sequence. We assume that (u_t, s_t, w_t) are known sequences of exponential order less than $1/\sqrt{\beta}$. The criterion (1) is to be maximized over sequences for $\{Q_s, a_s, s \geq 0\}$ taking as given $\{Q_s, a_s, s < 0\}$.

A. Find the Euler equations for this problem.

B. Argue that the solution will be linear laws of motion for (Q_t, a_t) in which each of (Q_t, a_t) depends on lagged values of both Q and a, and current and future values of all of (u, s, w).

9. (Time to build with two processes)

Consider a monopolist whose output satisfies

$$Q_t = f(L)n_{1t} + g(L)n_{2t} \tag{1}$$

where

$$f(L) = \sum_{j=0}^{m} f_j L^j, \; g(L) = \sum_{j=0}^{r} g_j L^j, \quad f_j \geq 0, \quad g_j \geq 0,$$

for all j. In (1), n_{1t} is the amount of labor at t that is assigned to process 1, while n_{2t} is the amount that is assigned to process 2. The idea is that output can be produced via two processes, with different timing characteristics, e.g., to represent the notion that the first process is fast but wasteful, while the other is efficient but time consuming, we might set $f(L) = L$, $g(L) = (\frac{1}{4})[L + L^2 + L^3 + L^4]$. The firm faces the demand curve

$$p_t = A_0 - A_1 Q_t + u_t, \qquad A_0, A_1 > 0, \tag{2}$$

where u_t is a known sequence of exponential order less than $1/\sqrt{\beta}$. The firm hires labor at the wage rate w_t, where w_t is a known sequence of exponential order less than $1/\sqrt{\beta}$. The firm's problem is to maximize

$$\sum_{t=0}^{\infty} \beta^t \{p_t Q_t - w_t(n_{1t} + n_{2t})\}, \qquad 0 < \beta < 1, \tag{3}$$

subject to (1) and (2), with $\{n_{1s}, s = -1, \ldots, -m\}$, $\{n_{2s}, s = -1, \ldots, -r\}$ given.

A. Find the Euler equations for this problem.

B. Indicate the form of the optimum decision rules for (n_{1t}, n_{2t}).

10. At time t, a farmer plants A_t units of land from which he produces $y_{t+1} = f_1 A_t$ units of output available to be sold at time $(t + 1)$ at price p_{t+1} where $f_1 > 0$. If A_t units of land are planted, the farmer's costs at time t are $c_0 A_t + (c_1/2)A_t^2 + c_2 A_t A_{t-1}$, where $c_0, c_1, c_2 > 0$. The term $c_2 A_t A_{t-1}$ reflects the wearing out of the land from two successive heavy plantings. The price of output p_t obeys the first-order difference equation

$$p_{t+1} = \alpha p_t$$

with p_0 given and $|\alpha| < 1/\sqrt{\beta}$. The farmer faces the p_t sequence as a price-taker. The farmer chooses $A_0, A_1, \ldots$ to maximize

$$\sum_{t=0}^{\infty} \beta^{t+1} p_{t+1}(f_1 A_t) - \sum_{t=0}^{\infty} \beta^t \left[c_0 A_t + \frac{c_1}{2} A_t^2 + c_2 A_t A_{t-1} \right], \qquad 0 < \beta < 1,$$

subject to A_{-1} given.

A. Obtain the first-order necessary conditions for optimization.

B. Assume that $c_1 = 5/2$, $c_2 = 1$, $\beta = 1$, $f_1 = 10$, and $c_0 = 1$. Find an optimal acreage plan for the farmer of the form

$$A_t = L(A_{t-1}, p_t, \text{constant})$$

where L is a linear function. Compute actual numerical values for the linear coefficients under the alternative assumptions $\alpha = 0.5$, $\alpha = 0$, and $\alpha = -0.5$.

C. Using your answer to B, describe how you would expect the farmer's supply of output next period, $y_{t+1} = f_1 A_t$, to vary with this period's price, p_t. When is supply y_{t+1} to vary inversely with p_t?

D. Use this example to illustrate the general observation that decision rules change whenev the economic environment changes.

11. A social planner wants to maximize the criterion

$$\sum_{t=0}^{\infty} b^t \{u_1(c_t - a) - (\tfrac{1}{2})u_2(c_t - a)^2 - u_3 n_t - (\tfrac{1}{2})u_4 n_t^2\}, \qquad ($$

$$0 < b < 1, \qquad u_1, u_2, u_3, u_4 > 0, \quad a > 0,$$

subject to

$$\begin{aligned} &\text{(a)}\; c_t + n_{t+1} + g = f n_t, \qquad g > 0, \quad f > 1, \quad fb > 1, \\ &\text{(b)}\; n_0 > 0, \quad \text{given.} \end{aligned} \qquad (2$$

Here c_t is consumption, n_t is labor supplied, g is government purchases, which are fixed and outsid the control of the planner, and a is a "bliss level" of consumption. (In the background in (1)–(2), it i understood that output at t equals f min(k_t, n_t) where k_t is capital at t. This fixed proportion technology makes it optimal to set $n_t = k_t$. We have substituted this into an original set c constraints to get (2).)

A. Show that the optimal path for $\{n_t\}$ converges as $t \to \infty$. Call the limit point the "stationar optimal value of n."

B. Give an explicit formula for the stationary optimal value of n, call it $\bar{n}$.

C. Stationary values of n_t must satisfy (2a) with $n_{t+1} = n_t = n$:

$$c + n + g = fn. \qquad (3$$

Consider the following "Golden rule" problem: to maximize

$$u_1(c - a) - (\tfrac{1}{2})u_2(c - a)^2 - u_3 n - (\tfrac{1}{2})u_4 n^2$$

subject to (3), by choice of c and n. (In words, find the utility maximizing sustainable levels o consumption and labor.) Solve this problem, finding an explicit formula for the "Golden rule" employment level, n_g.

D. Does $n_g = \bar{n}$, in general? If not, describe special settings of the parameter values for which $n_g = \bar{n}$. Interpret these special settings.

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CHAPTER X

LINEAR LEAST SQUARES PROJ (REGRESSIONS)

The concept of linear regression has many importan nomics, several of which we shall illustrate in subsequent important application will be its use in modeling the problem faced by agents in an environment in which information about a variable affecting their welfare. By usin agents can estimate that unobserved variable in a manner certain sense. Two leading applications of the signal macroeconomics are Robert Lucas's model of the Phillip Friedman's theory of the consumption function. Another linear regression is to characterize and study the optimal co monetary and fiscal authorities.

1. LINEAR LEAST SQUARES REGRESSION: THE OR CONDITION

We consider a set of random variables $y, x_1, x_2, \ldots,$ means of this list of random variables are denoted $Ey, Ex_1,$ that these means are finite, as are the population sec $Ex_1^2, \ldots, Ex_n^2$. By the Cauchy-Schwarz inequality the foll moments exist and are finite:

$$\begin{matrix} Eyx_1 & Ex_1^2 & Ex_1x_2 & \cdots & Ex_1x_n \\ Eyx_2 & Ex_2x_1 & Ex_2^2 & \cdots & Ex_2x_n \\ \vdots & \vdots & & & \\ Eyx_n & Ex_nx_1 & Ex_nx_2 & \cdots & Ex_n^2. \end{matrix}$$

Consider estimating the random variable y on the basi only for the random variables $x_1, \ldots, x_n$ as well as knowing