"BLANCHARD + FISCHER CH 5"

Version of the Baumol-Tobin
. 663–686.

on Mechanism in a General
rnal of Monetary Economics 20,

m Model with Transactions
8.

Loan Model of Interest with
urnal of Political Economy 66,

erns of Growth in a Monetary
534–544.

c Growth." Journal of Political

Prices in a Cash-in-Advance
919–944.

ransactions Demand for Cash."
7.

." Econometrica 32 (Oct.), 671–

d N. Wallace (eds.), Models of
eapolis, 83–90.

for Open Market Operations."

cautionary Demand for Cash."
7.

Vol. 2. Reprinted 1935. London:

quidity and Optimal Inflation."

5 Multiple Equilibria, Bubbles, and Stability

We have skipped some difficult issues at various points in the last four chapters. Confronted with saddle point equilibria, we proceeded to focus on the behavior of the economy along the convergent path; in some cases restricting our attention to that path was indeed warranted, but in many others no formal argument was given to rule out other paths. In other places we studied the properties of steady states without checking whether they were stable. We now examine these issues in more detail. The outcome turns out to be more than just a cleaning up of untidy detail. Rather, the chapter opens up broad and fascinating issues, from multiple equilibria to speculative bubbles and chaos.

In section 5.1 we start by analyzing the solution to a simple linear difference equation under rational expectations. This difference equation has various interpretations: it may arise, for example, from an arbitrage relation, from a linearized version of the OLG model with money analyzed in chapter 4, or from the Cagan model also analyzed in chapter 4. The solution to this simple equation is remarkably rich. For some parameter values, the solution may exhibit bubbles, components that explode in expected value over time. For other parameter values, there is an embarrassing wealth of stable solutions, in some of which variables matter just because individuals believe they do. The rest of the chapter is spent analyzing these issues in a general equilibrium context.

In section 5.2 we focus on the question of whether there can be bubbles on real assets in general equilibrium. Bubbles are ruled out when individuals have infinite horizons. However, when individuals have finite horizons, there are circumstances under which bubbles may exist and even be beneficial. To analyze the conditions under which bubbles exist, we use the Diamond overlapping generations model introduced in chapter 3. We conclude the section with a brief discussion of how econometric methods can be used to detect the presence of bubbles in asset markets.

BLANCHARD, O. and STANLEY FISCHER, LECTURES ON MACROECONOMICS (MIT PRESS, 1981), CHAPTER 5, MULTIPLE EQUIL., BUBBLES, AND STABILITY pp 213-226 and pp 261-274

From the purely real models of section 5.2 we turn in section 5.3 to the question of whether there can be price level bubbles in monetary models. The main issue is whether general equilibrium considerations allow us to rule out self-generating hyperinflations or deflations. We conclude, using a model in which money provides direct utility services, that there are cases in which self-generating hyperinflations cannot be ruled out.

The most remarkable set of results appears in section 5.4, where we examine general equilibrium models in which there is an infinity of stable equilibria. For convenience, we use the OLG model. We show how and when the equilibrium may have cycles, may exhibit chaos, may exhibit sunspots, and be affected by extrinsic uncertainty. We also discuss whether configurations of parameters that allow for such strange phenomena are likely to occur.

In section 5.5 we study the role and implications of learning. The explicit introduction of learning can help to narrow the range of probable solutions. We conclude in section 5.6 with an assessment of the relevance of the various types of multiplicity of equilibria presented in the chapter.

One word of clarification: this is not the only chapter in the book in which the possibility of multiple equilibria is discussed. We have already in the analysis of seigniorage in chapter 4 examined one case of multiple equilibria. We discuss other examples, consistent with the Keynesian notion that self-justifying "animal spirits" may cause output expansions and contractions, in chapter 8.

5.1 Solutions to a Simple Equation

In this section we characterize the behavior of a variable y that obeys the following expectational difference equation:

$$y_t = aE[y_{t+1}|t] + cx_t, \tag{1}$$

where $E[y_{t+1}|t]$ denotes the expectation of y_{t+1} held at time t so that y depends on the current expectation of its value next period as well as on the variable x.

To solve for the behavior of y, one must specify how individuals form expectations. We will assume in this chapter that individuals have *rational expectations*, that is, expectations equal to the mathematical expectation of y_{t+1} based on information available at time t.[1]

We make two further assumptions in defining this rational expectation. The first is that individuals know the model, namely, equation (1) and the parameters a and c. In most real world situations this will obviously not be

turn in section 5.3 to the
bles in monetary models.
onsiderations allow us to
ons. We conclude, using a
rvices, that there are cases
e ruled out.
in section 5.4, where we
ere is an infinity of stable
odel. We show how and
xhibit chaos, may exhibit
. We also discuss whether
h strange phenomena are

s of learning. The explicit
nge of probable solutions.
t of the relevance of the
ed in the chapter.
apter in the book in which
. We have already in the
case of multiple equilibria.
eynesian notion that self-
ansions and contractions,

variable y that obeys the

(1)

held at time t so that y
next period as well as on

ify how individuals form
t individuals have *rational*
thematical expectation of

this rational expectation.
nely, equation (1) and the
this will obviously not be

the case; individuals will also be learning, and most likely disagreeing, about the model at the same time as they are forming expectations. We return to the issue of learning later in this chapter.

The second assumption is that all individuals have the same information set at time t so that we can indeed talk about "the" mathematical expectation based on "the" information set. However, different individuals often have different information sets, each with a piece of information that the others do not have. We will see examples of such models in the next chapter.[2]

We define $E[y_{t+1}|t]$ by

$$E[y_{t+1}|t] = E[y_{t+1}|I_t], \tag{2}$$

where

$$I_t = \{y_{t-i}, x_{t-i}, z_{t-i}, i = 0, \ldots, \infty\}.$$

$E[y_{t+1}|t]$ is equal to the mathematical expectation of y_{t+1} based on the information set I_t. The information set contains current and lagged values of y and x; it may also include current and past values of other variables in a vector z_t that, though not present in equation (1), may help predict future values of x and y. Note that this definition of the information set implies no loss of memory, as anything known at time t is still known at time $t + 1$.

Before characterizing solutions to (1) and (2), we give three economic interpretations of this model.

Three Examples

Arbitrage

The first interpretation of (1) is as an arbitrage equation, for example, between stocks and a riskless asset. Let p_t be the price of a stock, d_t be the dividend, and r be the rate of return on the riskless asset, assumed constant over time. Then, if risk neutral individuals arbitrage between stocks and the riskless asset, the expected rate of return on the stock, which is equal to the expected rate of capital gain plus the dividend–price ratio, must equal the riskless rate:

$$\frac{E[p_{t+1}|I_t] - p_t}{p_t} + \frac{d_t}{p_t} = r,$$

or by reorganizing,

$$p_t = aE[p_{t+1}|I_t] + ad_t,$$

where

$$a \equiv \frac{1}{1+r} < 1.$$

This is of the same form as (1). The coefficient a in this case is equal to the one-period discount factor and is less than one so long as the interest rate is positive. The price today depends on the expected price tomorrow but by less than one for one.

The Cagan Model

The second interpretation of (1) is as the equilibrium condition in the Cagan model, presented at the end of the previous chapter and analyzed there under the assumption of adaptive expectations. The Cagan money demand function makes the demand for real balances an exponential function of the negative of the expected rate of inflation. In equilibrium money demand must be equal to the real money stock. In discrete time, equation (40) of the previous chapter becomes

$$\frac{M_t}{P_t} = \exp\left[-\alpha\left(\frac{E[P_{t+1}|I_t] - P_t}{P_t}\right)\right],$$

where we have set c in equation (40) of chapter 4 equal to unity and replaced the a in (40) by α.

Taking logarithms on both sides, denoting logarithms by lowercase letters, and using the approximation $E[p_{t+1}|I_t] - p_t = (E[P_{t+1}|I_t] - P_t)/P_t$, we get

$$m_t - p_t = -\alpha(E[p_{t+1}|I_t] - p_t).$$

Reorganizing gives

$$p_t = aE[p_{t+1}|I_t] + (1-a)m_t,$$

where

$$a \equiv \frac{\alpha}{1+\alpha}.$$

This is in the same form as (1). The price level depends on the price level expected for next period and on the current nominal money stock. Since in this model the demand for money is necessarily a decreasing function of the expected rate of inflation, α is necessarily positive so that a is between zero and one. The elasticity of the price level today with respect to its expected value tomorrow is less than one.

The OLG Model with Money

The third interpretation is a
with money, also examined
is demanded by the young
when old. Extending equa
uncertainty, we have

$$\frac{M_t}{P_t} = L\left(\frac{E[P_{t+1}|I_t] - P_t}{P_t}\right).$$

The left-hand side is the
the old. The right-hand side
is a function of the expecte
return on money). Taking a
by lowercase letters, using
$(E[P_{t+1}|I_t] - P_t)/P_t$, and igno

$$m_t - p_t = -\alpha(E[p_{t+1}|I_t] -$$

Reorganizing implies that

$$p_t = aE[p_{t+1}|I_t] + (1-a)m$$

where

$$a \equiv \frac{\alpha}{1+\alpha}.$$

This is similar to the eq
important difference: $L(\cdot)$ i
respect to the rate of return
If the substitution effect d
inflation is to decrease savi
and one as in the Cagan mo
can be negative. If α is not
a is greater than one in abso
possibility that, in this mod
to its expected value is gre

In the first two examples
a may be greater than on
solutions are very different
one in absolute value. We
case $|a| > 1$.

nt a in this case is equal to
one so long as the interest
e expected price tomorrow

ium condition in the Cagan
hapter and analyzed there
The Cagan money demand
exponential function of the
quilibrium money demand
te time, equation (40) of the

equal to unity and replaced

logarithms by lowercase
$- p_t = (E[P_{t+1}|I_t] - P_t)/P_t$,

l depends on the price level
minal money stock. Since in
a decreasing function of the
ve so that a is between zero
with respect to its expected

The OLG Model with Money

The third interpretation is as a loglinear approximation to the OLG model with money, also examined in the previous chapter. In that model money is demanded by the young who buy it so as to exchange it against goods when old. Extending equation (3) in the previous chapter to allow for uncertainty, we have

$$\frac{M_t}{P_t} = L\left(\frac{E[P_{t+1}|I_t] - P_t}{P_t}\right).$$

The left-hand side is the real supply of money, supplied inelastically by the old. The right-hand side is the demand for money by the young, which is a function of the expected rate of inflation (the negative of the rate of return on money). Taking a loglinear approximation, denoting logarithms by lowercase letters, using as before the approximation $E[p_{t+1}|I_t] - p_t = (E[P_{t+1}|I_t] - P_t)/P_t$, and ignoring an unimportant constant term, we get

$$m_t - p_t = -\alpha(E[p_{t+1}|I_t] - p_t).$$

Reorganizing implies that

$$p_t = aE[p_{t+1}|I_t] + (1 - a)m_t,$$

where

$$a \equiv \frac{\alpha}{1 + \alpha}.$$

This is similar to the equation derived for the Cagan model, with one important difference: $L(\cdot)$ is now a savings function and its elasticity with respect to the rate of return is, as we saw in chapter 3, ambiguous in sign. If the substitution effect dominates, the effect of an increase in expected inflation is to decrease saving, so that α is positive and a is between zero and one as in the Cagan model. However, if the income effect dominates, α can be negative. If α is not only negative but also less than minus one half, a is greater than one in absolute value. Thus, we cannot exclude a priori the possibility that, in this model, the elasticity of the price level with respect to its expected value is greater than one in absolute value.

In the first two examples a is less then one in absolute value. In the third, a may be greater than one in absolute value. We will see shortly that solutions are very different depending on whether a is greater or less than one in absolute value. We examine first the case $|a| < 1$ and then later the case $|a| > 1$.

Solutions When $|a| < 1$: Fundamentals and Bubbles

The "Fundamental" Solution

Various methods available to solve linear equations with rational expectations are described in appendix A. The most convenient method in the simplest cases, such as (1), is repeated substitution.

All the methods of solution rely on the following statistical fact, known as *the law of iterated expectations*:[3] let Ω be an information set and ω be a subset of this information set. Then for any variable x,

$$E[E[x|\Omega]|\omega] = E[x|\omega].$$

Or, heuristically, if one has rational expectations and is asked how she would revise her expectation were she given more information, the answer must be that she is as likely to revise it up or down so that on average the revision will be equal to zero. Applied to the information set I_t, this implies, in particular, that[4]

$$E[E[x|I_{t+1}]|I_t] = E[x|I_t].$$

Today's expectation of next period's expectation of the variable x is the same as today's expectation of x.

We now write equation (1) at time $t+1$ and take expectations of both sides conditional on information at time t:

$$E[y_{t+1}|I_t] = aE[E[y_{t+2}|I_{t+1}]|I_t] + cE[x_{t+1}|I_t].$$

Using the law of iterated expectations,

$$E[y_{t+1}|I_t] = aE[y_{t+2}|I_t] + cE[x_{t+1}|I_t].$$

Replacing in (1) gives

$$y_t = a^2E[y_{t+2}|I_t] + acE[x_{t+1}|I_t] + cx_t.$$

Solving recursively up to time T,

$$y_t = c\sum_{i=0}^{T} a^iE[x_{t+i}|I_t] + a^{T+1}E[y_{t+T+1}|I_t].$$

For the first term to converge as T tends to infinity, the expectation of x must not increase too fast. If the expectation of x grows at a rate no faster than exponential, the condition for this sum to converge is that the expectation of x grow at rate no larger than $(1/a) - 1$. In the case where (1) has the interpretation of an arbitrage relation, this requires dividends not to grow faster than the interest rate. In the case where (1) has the interpretation

of money market equilibriu
this requires that the logaritl
$(1/a) - 1$. Note that any cc
money, which implies that t
condition. We shall therefc
converges. Then, if

$$\lim_{T\to\infty} a^{T+1}E[y_{t+T+1}|I_t] = 0,$$

the following is a solution:

$$y_t = c\sum_{i=0}^{\infty} a^iE[x_{t+i}|I_t].$$

Note that equation (4) sa
equation (1). It gives y as th
first example this implies th
value of expected future div
the price level depends on
stocks, with decreasing wei

If we are willing to spe
explicitly for y. Equivalentl
the process for y. We prese
in x from x_0 to x_T, announce
of y is then given by

$$y_t = (1-a)^{-1}cx_0,$$

$$= (1-a)^{-1}cx_0 + a^{T-t}(1$$

$$= (1-a)^{-1}cx_T,$$

Consider the interpretati
model. The path of the n
money balances are drawr
announcement of a future
price level today. Real mon
increases to its new higher
advance of the increase in tl
forward. They know that ir
people will anticipate infla
balances. In so doing, they

bles

tions with rational expecta-
convenient method in the
on.
wing statistical fact, known
nformation set and ω be a
iable x,

and is asked how she would
ormation, the answer must
that on average the revision
ation set I_t, this implies, in

ion of the variable x is the

d take expectations of both

nfinity, the expectation of x
n of x grows at a rate no
sum to converge is that the
a) $-$ 1. In the case where (1)
his requires dividends not to
ere (1) has the interpretation

of money market equilibrium condition, such as in the last two examples, this requires that the logarithm of money not increase faster than at the rate $(1/a) - 1$. Note that any constant exponential growth rate of the level of money, which implies that the logarithm increases linearly, will satisfy this condition. We shall therefore assume in what follows that the first sum converges. Then, if

$$\lim_{T\to\infty} a^{T+1}E[y_{t+T+1}|I_t] = 0, \tag{3}$$

the following is a solution:

$$y_t = c\sum_{i=0}^{\infty} a^i E[x_{t+i}|I_t]. \tag{4}$$

Note that equation (4) satisfies condition (3), so it is indeed a solution to equation (1). It gives y as the discounted sum of future expected x's. In our first example this implies that the price of a stock is the present discounted value of expected future dividends. In the other two examples it implies that the price level depends on the whole sequence of future expected money stocks, with decreasing weights.

If we are willing to specify an expected path for x, we can solve (4) explicitly for y. Equivalently, if we specify a process for x, we can solve for the process for y. We present two examples. The first is that of an increase in x from x_0 to x_T, announced at time t_0 to take place at time $T > t_0$. The path of y is then given by

$$\begin{aligned} y_t &= (1-a)^{-1}cx_0, && \text{for } t < t_0,\\ &= (1-a)^{-1}cx_0 + a^{T-t}(1-a)^{-1}c(x_T - x_0), && \text{for } t_0 \leqslant t < T,\\ &= (1-a)^{-1}cx_T, && \text{for } t \geqslant T. \end{aligned}$$

Consider the interpretation of this equation as deriving from the Cagan model. The path of the nominal money stock, the price level, and real money balances are drawn in figure 5.1.[5] The equation shows that the announcement of a future increase in the money stock itself increases the price level today. Real money balances decrease, and the price level slowly increases to its new higher level over time. Inflation therefore takes place in advance of the increase in the money stock. This is because individuals look forward. They know that in the period before the money stock is increased, people will anticipate inflation and attempt to reduce their real money balances. In so doing, they will cause the price level to go up before the

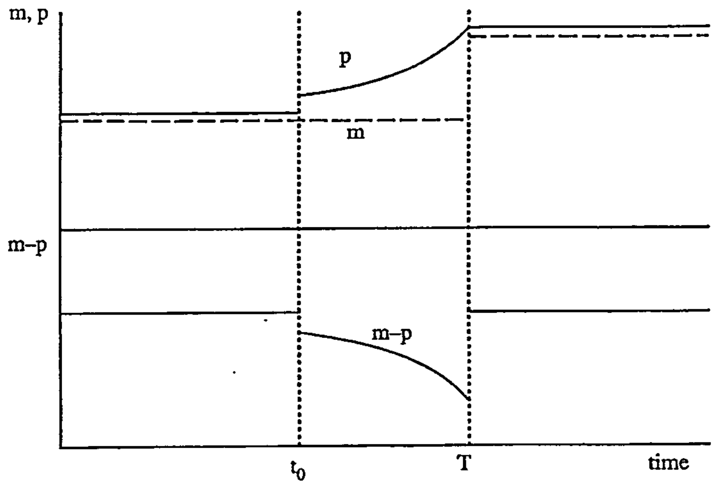

Figure 5.1
Effects of an anticipated increase in nominal money

money stock increases. Working this logic back to the present, current money holders attempt to reduce current real balances, therefore driving up the current price level.

Suppose that x instead follows the first-order stochastic process:

$$x_t - \bar{x} = \rho(x_{t-1} - \bar{x}) + e_t,$$

where e_t belongs to I_t and $E[e_t|I_{t-1}] = 0$. Then by using iterated expectations,

$$E[x_{t+i}|I_t] = \bar{x} + \rho^i(x_t - \bar{x}),$$

so that if ρ is less than $(1/a)$,

$$y_t - \bar{y} = \left(\frac{c}{1 - a\rho}\right)(x_t - \bar{x}),$$

with

$$\bar{y} = \left(\frac{c}{1 - a}\right)\bar{x}.$$

In the arbitrage example this implies that the price of the stock will be a function of current dividends only. It will, however, vary proportionally less than dividends as long as ρ, the degree of persistence, is less than unity. This is because the stock price is the present discounted value of future dividends,

and dividends are expected t
the money examples, if ρ i
money balances will be hig
the money stock is low.

In the case of arbitrage it
as the present discounted v
This terminology has now
arbitrage. But, as we now sh
the only solution to equatic

The Set of Solutions: Bubbles
Although equation (4) is a
derived it by imposing con
fast. When we relax this arb
solutions.

Let y_t^* denote the solution
as

$$y_t = y_t^* + b_t.$$

We now examine the restric
y_t to be also a solution to (

If $y_t = y_t^* + b_t$, then $E[y$
$E[y_{t+1}|I_t]$ in (1) implies that

$$y_t^* + b_t = aE[y_{t+1}^*|I_t] + aE[$$

By the definition of y_t^* in (4

$$b_t = aE[b_{t+1}|I_t],$$

or equivalently,

$$E[b_{t+1}|I_t] = a^{-1}b_t.$$

Thus, for any b_t that sat
Note that since a is less tha

$$\lim_{i\to\infty} E[b_{t+i}|I_t] = a^{-i}b_t = \{$$

The following examples
the popular notion of specu
the fundamental solution, b

time

ack to the present, current
alances, therefore driving up

r stochastic process:

using iterated expectations,

e price of the stock will be a
ver, vary proportionally less
tence, is less than unity. This
ed value of future dividends,

and dividends are expected to return to their normal value at rate $(1 - \rho)$. In the money examples, if ρ is less than one, this solution implies that real money balances will be high when the money stock is high and low when the money stock is low.

In the case of arbitrage it is natural to call the solution that gives the price as the present discounted value of dividends the "fundamental" solution. This terminology has now become standard, even in contexts other than arbitrage. But, as we now show, the fundamental solution is far from being the only solution to equation (1).

The Set of Solutions: Bubbles

Although equation (4) is a solution to (1), it is not the only solution. We derived it by imposing condition (3), that the expectation not explode too fast. When we relax this arbitrary condition, equation (1) admits many other solutions. ?

Let y_t^* denote the solution given by (4), and let us write any other solution as

$$y_t = y_t^* + b_t.$$

We now examine the restrictions that have to be imposed on b_t in order for y_t to be also a solution to (1).

If $y_t = y_t^* + b_t$, then $E[y_{t+1}|I_t] = E[y_{t+1}^*|t] + E[b_{t+1}|I_t]$. Replacing y_t and $E[y_{t+1}|I_t]$ in (1) implies that

$$y_t^* + b_t = aE[y_{t+1}^*|I_t] + aE[b_{t+1}|I_t] + cx_t.$$

By the definition of y_t^* in (4), this reduces to

$$b_t = aE[b_{t+1}|I_t], \tag{5}$$

or equivalently,

$$E[b_{t+1}|I_t] = a^{-1}b_t.$$

Thus, for any b_t that satisfies (5), $y_t = y_t^* + b_t$ is also a solution to (1). Note that since a is less than one, b_t explodes in expected value:

$$\lim_{i\to\infty} E[b_{t+i}|I_t] = a^{-i}b_t = \begin{cases} +\infty, & \text{if } b_t > 0, \\ -\infty, & \text{if } b_t < 0. \end{cases} \tag{6}$$

The following examples of b_t processes show that b_t embodies quite well the popular notion of speculative bubbles. For that reason, while y_t^* is called the fundamental solution, b_t is called a bubble.

An Ever-Expanding Bubble

In the first example b simply follows a time trend:

$$b_t = b_0 a^{-t}, \qquad \text{for arbitrary } b_0.$$

Consider the interpretation of equation (1) as an arbitrage equation and assume for simplicity that dividends, and thus p^*, are constant. If b_t is a time trend, and b_0 is positive, the price of the stock will increase exponentially, though the dividends are constant. What happens is that individuals are ready to pay a higher price for the stock than the price corresponding to the present value of the dividends because they anticipate the price will rise further, resulting in capital gains that precisely offset the low dividend price ratio. This anticipation of ever-increasing prices is self-fulfilling and satisfies the arbitrage condition.

A Bursting Bubble

The ever-expanding bubble has to go on forever, so that it will eventually become very large. In the next example the bubble has a probability of bursting each period. Consider the following process for b_t:

$$
\begin{aligned}
b_{t+1} &= (aq)^{-1} b_t + e_{t+1}, && \text{with probability } q,\\
&= e_{t+1}, && \text{with probability } 1 - q,
\end{aligned}
$$

and

$$E[e_{t+1} \mid I_t] = 0.$$

This process satisfies (5). The bubble bursts with probability $1 - q$ each period and continues with probability q. If it bursts, it returns in expected value to zero. To compensate for the probability of a crash, the expected return, if it does not crash, is higher than in the previous example. The disturbance e allows bubbles to have additional noise and permits new bubbles to form after a bubble has crashed.

Note that e_t can be correlated with unexpected movements in any variable and still satisfy the condition that its conditional expectation be zero. Thus, if the market believes that unexpected sunspots affect the price, they will indeed do so.[6] The example can be further refined to allow q to be stochastic and for q to be affected by other variables. These modifications provide good accounts of the suggestive informal descriptions of speculative bubbles.[7]

Eliminating Bubbles

An issue that arises is whether, in deriving these solutions, we have not ignored conditions other than (1) that must also be satisfied by a solution.

For instance, in the case of
the bubble becomes too la
economy. There is, as we sh
bubbles can always be rule
possible to rule out some o
Consider, for example,
arbitrage and y gives the pr
of, then its price cannot be
be negative bubbles. If b w
the far future would go to
would also go to minus
be negative.
However, no such sim
bubbles. But other conditio
If y is the price of a physica
elastic supply, possibly at
then there cannot be posit
price goes to infinity and
substitute is available, whic
existence of a perfect subs
there cannot be any bubble
If y is the price of a shar
more shares when there is
does not affect the bubble,
is in the interests of the ini
the proceeds. However, it
ever-increasing supply of
the likelihood of a bubble
Thus one would generall
to ascertain, such as in the
than on assets whose func
stocks.
If y is subject to a term
must be equal to this valu
zero. Working backward in
cannot be bubbles. There t
perpetuities (or "consols").
We have listed here onl
to eliminate the possibilit
examine whether and whe
the elimination of bubbles.

d:

an arbitrage equation and
, are constant. If b_t is a time
will increase exponentially,
ens is that individuals are
the price corresponding to
anticipate the price will rise
ffset the low dividend price
is self-fulfilling and satisfies

r, so that it will eventually
ubble has a probability of
ocess for b_t:

$- q,$

with probability $1 - q$ each
ursts, it returns in expected
ity of a crash, the expected
the previous example. The
al noise and permits new

l movements in any variable
expectation be zero. Thus,
s affect the price, they will
d to allow q to be stochastic
modifications provide good
s of speculative bubbles.[7]

ese solutions, we have not
o be satisfied by a solution.

For instance, in the case of the ever-expanding bubble perhaps the value of the bubble becomes too large to be consistent with the finiteness of the economy. There is, as we shall see in this chapter, no general conclusion that bubbles can always be ruled out, but often there are conditions that make it possible to rule out some of the solutions.[8]

Consider, for example, the case where equation (1) is derived from arbitrage and y gives the price of an asset. If the asset can be freely disposed of, then its price cannot be negative. This in turn implies that there cannot be negative bubbles. If b was negative, then, by (6), the expectation of b in the far future would go to minus infinity. Thus the expectation of the price would also go to minus infinity, which is impossible.[9] Thus b cannot be negative.

However, no such simple argument allows us to eliminate positive bubbles. But other conditions may be present that rule out positive bubbles. If y is the price of a physical asset and if a substitute is available in infinitely elastic supply, possibly at a very high price (think of oil and solar energy), then there cannot be positive bubbles. If b is positive, then the expected price goes to infinity and consequently exceeds the price at which the substitute is available, which is impossible. Thus with free disposal and the existence of a perfect substitute in infinitely elastic supply at some price, there cannot be any bubbles at all.[10]

If y is the price of a share, the question arises of whether firms will issue more shares when there is a bubble on share prices. If issuing more shares does not affect the bubble, for example, does not make the bubble crash, it is in the interests of the initial shareholders to issue more shares and invest the proceeds. However, it seems unlikely that the markets would absorb an ever-increasing supply of an asset at an unchanging price. This decreases the likelihood of a bubble on an easily reproducible asset.[11]

Thus one would generally expect bubbles when fundamentals are difficult to ascertain, such as in the gold, art, or foreign exchange markets, rather than on assets whose fundamentals are clearly defined, such as blue chip stocks.

If y is subject to a terminal condition at some future time, then since y must be equal to this value at the terminal time, b must then be equal to zero. Working backward in time, b must be equal to zero always, and there cannot be bubbles. There therefore cannot be bubbles on bonds, except on perpetuities (or "consols").

We have listed here only partial equilibrium arguments that can be used to eliminate the possibility of bubbles. In sections 5.2 and 5.3 we will examine whether and when general equilibrium considerations also lead to the elimination of bubbles.

Solutions When $|a| > 1$: Indeterminacies

We have until now considered the case where $|a|$ was less than one and concluded that there was an infinity of solutions. If, however, we were ready to impose a nonexplosion condition, we would be left with only one solution—the fundamental solution. We turn now to the case $|a| > 1$, which, as we showed earlier, could be consistent with equation (1) interpreted as the equilibrium condition of an OLG model with money when the income effect is sufficiently strong.

This radically changes the nature of the results. Now the fundamental solution is no longer well defined. More precisely, the sum in (4) is unlikely to converge in general. And there is an infinity of bubbles, which are now stable rather than exploding. For example, suppose x is identically equal to one for all t. Then the set of solutions is given by

$$y_t = (1 - a)^{-1}c + b_t,$$

where

$$b_t = a^{-1}b_{t-1} + e_t, \qquad E[e_t | I_{t-1}] = 0. \tag{7}$$

This implies, in particular, that if we make e identically equal to zero, then y will converge to $(1 - a)^{-1}c$ for any initial value of y_0. Without a more detailed specification, it is difficult to think of reasons why the economy will (or economists analyzing the model should) choose one solution over another. Various criteria have been offered to choose among solutions, but none of them is very convincing.[12] The multiplicity of solutions is definitely more perplexing in this case. We will return to this, as well as to related issues, in section 5.4.

Extensions

Higher-Dimensional Systems

The behavior of y in (1) depends on whether $|a|$ is less or greater than one. The more likely case is that in which $|a|$ is less than one. We now examine how this condition extends to higher-dimensional systems and whether it is likely to be satisfied.

Returning to equation (1), ignoring uncertainty and expectations, the condition $|a| < 1$ can be stated as the condition that the difference equation that gives y_{t+1} as a function of y_t should be *unstable* or have a root $1/a$ that is strictly greater than one in absolute value. In this case, for a given sequence of x, there is a unique value of y^* for which y does not explode.

: $|a|$ was less than one and
. If, however, we were ready
uld be left with only one
now to the case $|a| > 1$,
ent with equation (1) inter-
nodel with money when the

ults. Now the fundamental
ly, the sum in (4) is unlikely
of bubbles, which are now
pose x is identically equal to
by

(7)

lentically equal to zero, then
alue of y_0. Without a more
asons why the economy will
choose one solution over
hoose among solutions, but
city of solutions is definitely
o this, as well as to related

| is less or greater than one.
than one. We now examine
onal systems and whether it

ainty and expectations, the
that the difference equation
ble or have a root $1/a$ that is
is case, for a given sequence
oes not explode.

This condition generalizes if the economy is characterized by a difference equation system.[13] Suppose there are n predetermined, state, variables at time t, and m variables (sometimes called "jumping" variables, or by analogy with optimal control problems, "costate" variables) that are not predetermined. Then the system must have exactly m roots outside the unit circle in order to have a unique nonexploding solution. For example, if we consider a model of money and capital in which there is one predetermined variable, capital, and one jumping variable, the price level, the system must have one root between -1 and $+1$ and one root outside that range. Equivalently, it must have the saddle point property.[14]

All the dynamic systems examined in the previous chapters had the saddle point property. When we assumed that the economy chose the saddle point path, we were in effect choosing the fundamental solution. On any other path the value of the variables could have been expressed as the sum of the fundamental solution and a bubble growing at the rate determined by the positive root of the system.

The fact that all the systems we have looked at in this chapter were saddle point stable and thus satisfied the extension of the condition $|a| < 1$ is an indication that this condition is often likely to be satisfied. We will return to this point in the conclusion of this chapter.

Nonlinear Dynamics

We end this section with a caveat. Thus far we have limited ourselves to linear systems. Indeed, most of what we know is limited to such systems, at least under uncertainty. The following example, from Azariadis (1981), shows that even if a nonlinear system satisfies locally the condition $|a| < 1$ around a steady state, it may have more than one nonexploding solution.

The expectations difference equation is

$$y_t^2 = \left(\frac{1}{2}\right) E[y_{t+1} \mid y_t], \qquad y_t \in [0, 1].$$

The value $y = 1/2$ is a solution to the equation. If we linearize the system around $y = 1/2$, then $dy_t/dE[y_{t+1} \mid y_t] = 1/2$ so that, if the system were linear, $y = 1/2$ would be the only nonexploding solution. Note, however, that the following is also a solution:

$$\begin{aligned} y_{t+1} &= y_t^2, && \text{with probability } q_t = \frac{1 - 4y_t^2}{1 - 2y_t^2}, \\ &= \frac{1}{2}, && \text{with probability } 1 - q_t. \end{aligned}$$

By construction, q_t is always between zero and one. So the system has at least two nonexploding solutions. The first is $y = 1/2$. In the second, y follows stochastic cycles, although there is no intrinsic uncertainty. We will return to issues related to nonlinearity in section 5.4.

5.2 Bubbles on Assets in General Equilibrium

Whether there can be bubbles on real assets in general equilibrium depends on whether individuals have finite or infinite horizons and, if they have finite horizons, on whether the economy is dynamically efficient. After showing the conditions under which bubbles on real assets can exist, we draw parallels between results derived here and various results obtained in chapters 2 and 3. We end the section by discussing the empirical evidence on the presence or absence of bubbles.

The Case of Infinite Horizons

Bubbles are not unlike Ponzi games; assets are bought only on the anticipation that they can be resold at a higher price to somebody else who will buy them for the same reason. It is therefore not surprising that bubbles cannot arise when there is a finite number of individuals who have infinite horizons.

The proof of this very general proposition was given by Tirole (1982). The logic is as follows. Suppose that there is a finite number of infinitely lived individuals. The asset yields dividends or services every period. If it yields services, they can be rented out to the person who values them most that period. This implies that the fundamental value, p_t^*, is the same for all individuals at all times. Finally, the services or dividends do not depend on the price. This excludes money, whose services depend on the price level.

Suppose that, under these assumptions, there is a negative bubble, with $p_t < p_t^*$. Then all individuals will want to buy and keep the asset forever. Purchasing the asset costs p_t, holding the asset forever, and renting it out every period yields p_t^* in present value. There would therefore be excess demand for the asset, and $p_t < p_t^*$ cannot be an equilibrium.

Suppose, alternatively, that there is a positive bubble so that $p_t > p_t^*$. If short selling is allowed, an argument symmetrical to the preceding one implies excess supply and rules out positive bubbles. But it is possible to exclude positive bubbles even without short selling. If p exceeds p^*, an individual who buys the asset must do so with the anticipation of eventually realizing his capital gain by selling the asset in the future. Let t_i be the date

by which individual i inten
i. By T all individuals plar
nobody plans to be hold
equilibrium.

This argument allows us
in which individuals have ir
in a general equilibrium, it r
over time. We therefore tu
new generation is born eve

Finite Horizons

The following argument is
bubbles in general equilibri
This suggests that at some
relative to the economy,
argument is not quite righ
interest rate is less than tl
inefficient), the economy w
that the above argument
and extend this intuitive c
Tirole (1985) and Weil (1
presented in chapter 3.

Recall that in the Diamc
consuming, and saving in
suming their savings in the
its marginal product. In tl
accumulation are given by

$k_{t+1} = (1 + n)^{-1}\{s[w(k_t), r$

Or by expressing savings

$k_{t+1} = (1 + n)^{-1}s(k_t, k_{t+1})$

The capital stock at tim
time t, which depend on tl
savings r. Since w_t depends
directly as a function of k
guarantees that the equili
nonoscillating, namely, tha

of stable solutions, but
to any of these solutions.
of yet. But the preceding
the literature. Where the
lamental solution appears
ive learning schemes. We
ls. One result, obtained by
ed in the previous section,
a simple learning process.
are sunspot equilibria, see

:ity of solutions studied in
very different types of
depending on which one

5.2 and 5.3; it arises when
ble in systems of higher
solution, the fundamental
ions, the bubbles. These
ut by partial or general
often rely on a degree of
sent in practice. Our brief
ests that the fundamental
bbles.
ot the following research
bubbles, assume that the
the fundamental solution.
d will do in the rest of the
es, aimed both at finding
their implications.
licity of stable equilibria
led to different reactions.
iply little that can be said
uch conditions.
ne has been to explore the
appear together with the
have seen, the conditions

for each to arise are not identical. It has been argued that chaos offers an alternative to the now prevalent formalization of business cycles as resulting from the dynamic effects of stochastic shocks through propagation mechanisms (a view we will develop at length in the rest of the book). It has been argued that chaos, which can be generated by simple deterministic systems, offers a less ad hoc explanation of fluctuations than one based on unexplained shocks. Some work has examined empirically whether the behavior of economic variables is better explained by chaotic or linear stochastic processes (see Brock 1986), but without clear conclusions as of yet.

Another reaction, associated with Grandmont, has been to concentrate on cycles and reduce the dimension of indeterminacy. As we noted earlier, Grandmont has shown that under some additional assumptions there may exist a unique unstable cycle that is stable under simple learning rules. Grandmont argues that such deterministic cycles provide an alternative to the linear stochastic process view of cycles. If he is correct, policy can have very drastic effects on the dynamics of the economy by changing the specific form of the nonlinearity.

We are not at this stage convinced by either of these last two approaches. Although the nonlinearity needed to obtain multiple stable equilibria, sunspots, cycles, or chaos is consistent with optimizing behavior, the conditions for such equilibria still appear unlikely. In the models considered in this chapter, for example, they require implausibly large income effects.[36] Thus, for the time being, though we find the phenomena analyzed in this chapter both interesting and disturbing, we are willing to proceed on the working assumption that the conditions needed to generate stable multiplicities of equilibria are not met in practice.

Appendix: A Tool Kit of Solutions to Linear Expectational Difference Equations

In section 5.1 we solved a difference equation with rational expectations by using the method of repeated substitution. That method is convenient in simple cases but rapidly becomes unwieldy. In this appendix we present the two methods that are most often used to solve such difference equations analytically. We make no attempt at generality or rigor. Surveys by Taylor (1985) on methods of solution in small models and by Blanchard (1985) on analytical and numerical methods of solution in large models give both a more exhaustive presentation and further references.

We will solve the following equation:

$$p_t = a_0 E[p_{t+1}|t] + a_1 p_{t-1} + a_2 E[p_t|t-1] + a_3 m_t + e_t. \qquad (A1)$$

We use the notation $E[p_{t+i}|t-j]$ to denote the rational expectation of p_{t+i} based on information available at time $t-j$. The information set is assumed to include at least current and lagged values of m, e, and p. The variable p is endogenous, the variable m is exogenous, and e is a stochastic disturbance. For the moment we do not need to specify the processes followed by either m or e.

Such an equation, in which a variable depends both on itself lagged and on past expectations of current values and current expectations of future values of itself, is fairly typical. One interpretation is that p is the logarithm of the price level and m the logarithm of the nominal money stock. In this case one may want to impose the additional homogeneity restriction $a_0 + a_1 + a_2 + a_3 = 1$.

It is convenient to define

$$x_t \equiv a_3 m_t + e_t \tag{A2}$$

so that

$$p_t = a_0 E[p_{t+1}|t] + a_1 p_{t-1} + a_2 E[p_t|t-1] + x_t. \tag{A1'}$$

The Method of Undetermined Coefficients

The method of undetermined coefficients consists of guessing the form of the solution and then solving for the coefficients. The guess may come from experience or from attempts at repeated substitution. An educated guess here is that p will depend on itself lagged once, and on current and once-lagged expectations of once-lagged current and future values of x:

$$p_t = \lambda p_{t-1} + \sum_{i=0}^{\infty} c_i E[x_{t+i}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1]. \tag{A3}$$

The method is to find values of λ, c_i, and d_i such that (A3) is a solution to (A1'). The first step is to derive $E[p_t|t-1]$ and $E[p_{t+1}|t]$ implied by (A3). By taking expectations of both sides of (A3), both at time t and $t+1$, and using the law of iterated expectations, we get

$$E[p_t|t-1] = \lambda p_{t-1} + \sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1], \tag{A4}$$

$$E[p_{t+1}|t] = \lambda p_t + \sum_{i=0}^{\infty} c_i E[x_{t+i+1}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t]. \tag{A5}$$

Now, by substituting (A4) and (A5) into (A1'), we get

$$p_t = a_0\left(\lambda p_t + \sum_{i=0}^{\infty} c_i E[x_{t+i+1}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t]\right) + a_1 p_{t-1}$$
$$+ a_2\left(\lambda p_{t-1} + \sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1]\right) + x_t, \tag{A6}$$

or

$$p_t = (1 - a_0\lambda)^{-1}\Big\{a_0\Big(\sum_{i=0}^{\infty} c_i E[$$

$$+ a_2\Big(\sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1]$$

For (A3) to be a solutior
we equate the coefficients for

$$\lambda = (1 - a_0\lambda)^{-1}(a_1 + a_2\lambda)$$

or, equivalently,

$$a_0\lambda^2 + (a_2 - 1)\lambda + a_1 = 0.$$

There will generally be tw
with satisfies the extension of
will be smaller than one in abs
the saddle point property.[37] E
on p_{t-1}, we are in effect choo

Suppose for example that
money and that a_0, a_1, a_2,
$a_0 + a_1 + a_2 < 1$. Then from

$$\Psi(\lambda) = a_0\lambda^2 + (a_2 - 1)\lambda +$$

it follows that $\Psi(0) > 0$, $\Psi(1)$
and one and the other is large

We will assume that the co
is satisfied and proceed. Let
value, and let λ_2 be the oth
$\lambda_1 + \lambda_2 = (1 - a_2)/a_0$. We n
the equations for these coeffic
We have

x_t: $c_0 = (1 - a_0\lambda$

$E[x_{t+1}|t]$: $c_1 = (1 - a_0\lambda$

$E[x_{t+i}|t]$: $c_i = (1 - a_0\lambda_$

x_{t-1}: $d_0 = (1 - a_0\lambda$

$E[x_t|t-1]$: $d_1 = (1 - a_0\lambda$

$E[x_{t+i}|t-1]$: $d_{i+1} = (1 - a$

Noting that $d_0 = 0$, and wi

$$c_0 = (1 - a_0\lambda_1)^{-1},$$

$$c_i = \left(\frac{\lambda_1 a_0}{a_1}\right)c_{i-1} = \lambda_2^{-1} c_{i-1},$$

$$p_t = (1 - a_0\lambda)^{-1}\left\{a_0\left(\sum_{i=0}^{\infty} c_i E[x_{t+i+1}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t]\right) + (a_1 + a_2\lambda)p_{t-1}\right.$$

$$\left. + a_2\left(\sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1]\right) + x_t\right\}. \tag{A6'}$$

For (A3) to be a solution to (A1), (A6′) and (A3) must be identical. Thus we equate the coefficients for each variable. Starting with the coefficient on p_{t-1},

$$\lambda = (1 - a_0\lambda)^{-1}(a_1 + a_2\lambda)$$

or, equivalently,

$$a_0\lambda^2 + (a_2 - 1)\lambda + a_1 = 0. \tag{A7}$$

There will generally be two solutions for λ in (A7). If the model we are dealing with satisfies the extension of the condition $|a| < 1$ in section 5.1, one of the roots will be smaller than one in absolute value and the other larger than one. It will have the saddle point property.[37] By choosing the smaller of the roots as the coefficient on p_{t-1}, we are in effect choosing the stable, nonexploding solution.

Suppose for example that the equation gives the price level as a function of money and that a_0, a_1, a_2, and a_3 are all positive and sum to one, so that $a_0 + a_1 + a_2 < 1$. Then from the definition

$$\Psi(\lambda) = a_0\lambda^2 + (a_2 - 1)\lambda + a_1,$$

it follows that $\Psi(0) > 0$, $\Psi(1) < 0$, and $\Psi(\infty) > 0$ so that one root is between zero and one and the other is larger than one.

We will assume that the condition for the existence of a unique stable solution is satisfied and proceed. Let λ_1 be the root that is less than one in absolute value, and let λ_2 be the other. Note, for later use, that $\lambda_1\lambda_2 = a_1/a_0$ and that $\lambda_1 + \lambda_2 = (1 - a_2)/a_0$. We now solve for c_i and d_i, using the assumption that λ in the equations for these coefficients is λ_1, the root that implies the stable solution. We have

$$\begin{aligned}
&x_t: && c_0 = (1 - a_0\lambda_1)^{-1}[1 + a_0 d_0],\\
&E[x_{t+1}|t]: && c_1 = (1 - a_0\lambda_1)^{-1}[a_0(c_0 + d_1)],\\
&E[x_{t+i}|t]: && c_i = (1 - a_0\lambda_1)^{-1}[a_0(c_{i-1} + d_i)],\\
&x_{t-1}: && d_0 = (1 - a_0\lambda_1)^{-1}[a_2 d_0],\\
&E[x_t|t-1]: && d_1 = (1 - a_0\lambda_1)^{-1}[a_2(c_0 + d_1)],\\
&E[x_{t+i}|t-1]: && d_{i+1} = (1 - a_0\lambda_1)^{-1}[a_2(c_i + d_{i+1})].
\end{aligned}$$

Noting that $d_0 = 0$, and with some manipulation, we get

$$c_0 = (1 - a_0\lambda_1)^{-1},$$

$$c_i = \left(\frac{\lambda_1 a_0}{a_1}\right)c_{i-1} = \lambda_2^{-1}c_{i-1}, \qquad \text{for } i = 1, \ldots,$$

$$d_i = \left(\frac{a_2}{a_0}\right) c_i, \qquad \text{for } i = 1, \ldots .$$

Thus, if λ_2 is larger than one, the sequences c_i and d_i converge to zero as i gets large.

We have solved for p_t as a function of lagged p_t and past and current expectations of current and future x. Sometimes the process for x is specified. Then we would want to solve directly for p as a function of observable variables. There are two procedures. One is to derive the solution for p as a function of expectations of x as we have just done, and then to solve for expectations of x as a function of observable variables in (A3). The other is to use the method of undetermined coefficients to solve directly for p_t as a function of observable variables.

Suppose, for example, that e is identically equal to zero and that m (and therefore x) follows

$$m_t = \rho m_{t-1} + v_t,$$

where v_t is white noise. We would then guess that the solution is of the form

$$p_t = \lambda p_{t-1} + c m_t + d m_{t-1} \tag{A3'}$$

and solve for λ, c, and d as above.

Despite its widespread use the method of undetermined coefficients suffers from a few handicaps. First, the initial guess may fail to include a solution or may inadvertently discard other solutions. Second, the method reveals only indirectly whether the model has the desirable saddle point property. Third, like repeated substitution, it can become somewhat unwieldy.

Factorization

The method of factorization was introduced to economics by Sargent (see Sargent 1979 for a detailed presentation). It is best seen as a convenient shortcut to the method of z-transforms (see Whiteman 1983).

The method proceeds in three steps.

The *first* [which is needed only if the equation includes both current and lagged expectations, if a_2 is different from zero in equation (A1′)] is to take expectations on both sides of (A1′) conditional on the farthest lagged information set in (A1). In (A1′) we take expectations based on information at time $t-1$. This implies

$$E[p_t|t-1] = a_0 E[p_{t+1}|t-1] + a_1 p_{t-1} + a_2 E[p_t|t-1] + E[x_t|t-1], \tag{A1''}$$

or

$$(1-a_2)E[p_t|t-1] = a_0 E[p_{t+1}|t-1] + a_1 p_{t-1} + E[x_t|t-1].$$

In the *second* step we factor equation (A1″) to express $E[p_t|t-1]$ as a lagged function of itself, and of expectations of current and future values of x, $E[x_{t+i}|t-1]$, $i \geqslant 0$. To do so, we introduce the lag operator, L, which operates on the time subscript of a variable (not on the time at which the expectation of that variable is held):

$LE[p_{t+i}|t-1] = E[p_{t+i-1}|t -$

so that, in particular,

$LE[p_{t+1}|t] = E[p_t|t] = p_t.$

For convenience, we also int

$FE[p_{t+i}|t-1] = E[p_{t+i+1}|t].$

Using the definitions of F

$[-a_0F + (1-a_2) - a_1 L]E[$

The next step is to factor (A8) as

$\left[F^2 - \left(\frac{1-a_2}{a_0}\right)F + \left(\frac{a_1}{a_0}\right)\right]$

We can factor the polyn $(F-\lambda_2)$, where

$\lambda_1 + \lambda_2 = \frac{1-a_2}{a_0}$ and

Note that λ_1 and λ_2 are undetermined coefficients. T λ_1 is less than one in absolut

We can rewrite (A9) as

$(F-\lambda_1)(F-\lambda_2)LE[p_t|t-1$

or

$(1-\lambda_1 L)E[p_t|t-1] = \Big(\frac{1}{a_0}$

Since $|\lambda_2^{-1}| < 1$, we can ex

$E[p_t|t-1] = \lambda_1 p_{t-1} + \Big(\frac{1}{a_0}$

Equation (A12) gives the ex itself. [Note again that if a_2 the first step, so (A12) wo expectations of current and be no need for the third ste

The *third* step is to deriv an expression for $E[p_{t+1}|t]$ This gives

'erge to zero as i gets large.
st and current expectations
specified. Then we would
e variables. There are two
tion of expectations of x as
as a function of observable
ndetermined coefficients to
s.
to zero and that m (and

solution is of the form

(A3')

ed coefficients suffers from
nclude a solution or may
nod reveals only indirectly
perty. Third, like repeated

ics by Sargent (see Sargent
convenient shortcut to the

es both current and lagged
1')] is to take expectations
l information set in (A1). In
e $t-1$. This implies

$] + E[x_t|t-1],$ (A1")

$|t-1].$

ess $E[p_t|t-1]$ as a lagged
re values of x, $E[x_{t+i}|t-1]$,
hich operates on the time
xpectation of that variable

$$LE[p_{t+i}|t-1] = E[p_{t+i-1}|t-1],$$

so that, in particular,

$$LE[p_{t+1}|t] = E[p_t|t] = p_t.$$

For convenience, we also introduce the forward operator, $F = L^{-1}$. Thus

$$FE[p_{t+i}|t-1] = E[p_{t+i+1}|t].$$

Using the definitions of F and L, we can rewrite (A1") as

$$[-a_0F + (1-a_2) - a_1L]E[p_t|t-1] = E[x_t|t-1]. \tag{A8}$$

The next step is to factor the polynomial in parentheses. To do so, we rewrite (A8) as

$$\left[F^2 - \left(\frac{1-a_2}{a_0}\right)F + \left(\frac{a_1}{a_0}\right)\right]LE[p_t|t-1] = \left(\frac{-1}{a_0}\right)E[x_t|t-1]. \tag{A9}$$

We can factor the polynomial $\{F^2 - [(1-a_2)/a_0]F + (a_1/a_0)\}$ as $(F-\lambda_1) \times (F-\lambda_2)$, where

$$\lambda_1 + \lambda_2 = \frac{1-a_2}{a_0} \qquad \text{and} \qquad \lambda_1\lambda_2 = \frac{a_1}{a_0}. \tag{A10}$$

Note that λ_1 and λ_2 are the same as λ_1 and λ_2 derived in the method of undetermined coefficients. Thus the same discussion applies, and we assume that λ_1 is less than one in absolute value and that λ_2 is larger than one in absolute value.

We can rewrite (A9) as

$$(F-\lambda_1)(F-\lambda_2)LE[p_t|t-1] = \left(\frac{-1}{a_0}\right)E[x_t|t-1],$$

or

$$(1-\lambda_1L)E[p_t|t-1] = \left(\frac{1}{a_0\lambda_2}\right)(1-\lambda_2^{-1}F)^{-1}E[x_t|t-1]. \tag{A11}$$

Since $|\lambda_2^{-1}| < 1$, we can expand $(1-\lambda_2^{-1}F)^{-1}$ as $\sum_{i=0}^{\infty}\lambda_2^{-i}F^i$ to get

$$E[p_t|t-1] = \lambda_1 p_{t-1} + \left(\frac{1}{a_0\lambda_2}\right)\sum_{i=0}^{\infty}\lambda_2^{-i}E[x_{t+i}|t-1]. \tag{A12}$$

Equation (A12) gives the expectation of p_t as of $t-1$. The last step is to derive p_t itself. [Note again that if a_2 were equal to zero, we would not have gone through the first step, so (A12) would give p_t as a function of p_{t-1} as well as current expectations of current and future x. This would be the solution, and there would be no need for the third step.]

The *third* step is to derive the solution for p_t. To do so, we use (A12) to get an expression for $E[p_{t+1}|t]$ and replace both $E[p_{t+1}|t]$ and $E[p_t|t-1]$ in (A1'). This gives

$$p_t = a_0\lambda_1 p_t + \left(\frac{1}{\lambda_2}\right)\sum_{i=0}^{\infty} \lambda_2^{-i}E[x_{t+i+1}|t] + (a_1 + a_2\lambda_1)p_{t-1}$$
$$+ \left(\frac{a_2}{a_0\lambda_2}\right)\sum_{i=0}^{\infty} \lambda_2^{-i}E[x_{t+i}|t-1] + x_t.$$

Reorganizing, and using the fact that, from the definition of λ_1, $(a_1 + a_2\lambda_1)/(1 - a_0\lambda_1) = \lambda_1$, gives

$$p_t = \lambda_1 p_{t-1} + \left(\frac{1}{1 - a_0\lambda_1}\right)\sum_{i=0}^{\infty} \lambda_2^{-i}E[x_{t+i}|t]$$
$$+ \left(\frac{1}{1 - a_0\lambda_1}\right)\left(\frac{a_2}{a_0}\right)\sum_{i=0}^{\infty} \lambda_2^{-i-1}E[x_{t+i}|t-1]. \tag{A13}$$

This solution is the same as that obtained by the method of undetermined coefficients.

Problems

1. Assume that the simple linear difference equation of section 5.1 is derived from an arbitrage equation between stocks and bonds and that the real interest rate is constant. Assume that dividends follow the stochastic process

$$d_t = (1 - \rho)d_0 + \rho d_{t-1} + v_t, \qquad d_0 > 0,\ 0 < \rho < 1,$$

where v_t is white noise. The variance of v_t is σ^2.

(a) Solve for the current price of the stock as a function of current and past dividends. Explain.
(b) Calculate the unconditional variance of the stock price as a function of σ^2 and other relevant parameters.
(c) How does the variance of the stock price change as ρ increases?

2. In section 5.2 we showed that if the money stock follows a first-order autoregressive process with $c < 1$, then in the Cagan model real balances will be high when the money stock is high and low when the money stock is low.

(a) Give the economic intuition behind this result.
(b) Suppose that the *growth rate* of money follows a stable first-order autoregressive process. Solve for the process for the price level.
(c) Does the same characterization hold with the addition of the words "relative to trend" following high and low?

3. *Land and bubbles.*

In an overlapping generations model in which people live for two periods, with the population growing at rate n and no production (all goods come from the exogenous endowment of the young), there is a given amount of land. The land has no productive use.

(a) Can there be a bubble on l
your answer and the efficien
that land is valued in the real
(b) Suppose now that the econ
only endowment is one unit c
function is Cobb-Douglas, wi
capital. Can there be a stead
(c) Can there be a bubble on la

4. (a) Return to problem 1. C
of the "ex post" price of the
the actual price.

(b) Suppose now that $\rho = 1$
happens to the unconditiona

5. Suppose that, in the mo
instantaneous utility functior

$U(c, m) = a\ln(c) + b\ln(m).$

(a) Could a self-generating hy
families?

(b) In evaluating this possibili
should be attached to the fact
growth did not become extr

6. A learning problem in wh
economy is solved in sectior
information on current value
Answer, in particular, the q
fundamentals equilibrium.

Notes

1. Muth (1961) was the first t
tion." Until his article, and al
arbitrary expectation forma
adaptive expectations (of wh
important paper (1960) Muth
tions about a variable y wou

2. In the classic rational ex
individuals do not all have t
why policymakers may not b
prices convey information to
chapter.

-1

nition of λ_1, $(a_1 + a_2\lambda_1)/$

(A13)

method of undetermined

of section 5.1 is derived
d that the real interest rate
c process

current and past dividends.

ice as a function of σ^2 and

p increases?

follows a first-order auto-
l real balances will be high
y stock is low.

le first-order autoregressive

n of the words "relative to

live for two periods, with
(all goods come from the
. amount of land. The land

(a) Can there be a bubble on land in this economy? Discuss the relationship between your answer and the efficiency of equilibrium in the real world in view of the fact that land is valued in the real world.

(b) Suppose now that the economy becomes a production economy, that individuals' only endowment is one unit of labor in the first period of life and that the production function is Cobb-Douglas, with constant returns to scale in terms of land, labor, and capital. Can there be a steady state?

(c) Can there be a bubble on land in this economy in which land is a productive asset?

4. (a) Return to problem 1. On the assumption that $\rho < 1$, show that the variance of the "ex post" price of the stock (defined in section 5.2) exceeds the variance of the actual price.

(b) Suppose now that $\rho = 1$, with the dividend following a random walk. What happens to the unconditional variance of the price of the stock?

5. Suppose that, in the model of section 5.3, the representative family has an instantaneous utility function

$U(c, m) = a \ln(c) + b \ln(m)$.

(a) Could a self-generating hyperinflation develop in an economy populated by such families?

(b) In evaluating this possibility, and the problem of multiple equilibria, what weight should be attached to the fact that there is no known hyperinflation in which money growth did not become extremely high?

6. A learning problem in which individuals do not know the current state of the economy is solved in section 5.5. Using that model, assume that individuals have information on current values of Y and v, and solve for the dynamics of the model. Answer, in particular, the question of whether the economy converges to the fundamentals equilibrium.

Notes

1. Muth (1961) was the first to use this assumption and the term "rational expectation." Until his article, and also for a long time after, researchers used plausible but arbitrary expectation formation mechanisms, the most popular being that of adaptive expectations (of which we saw an example in the last chapter). In another important paper (1960) Muth found the conditions under which adaptive expectations about a variable y would indeed be rational.

2. In the classic rational expectations macroeconomic article by Lucas (1973), individuals do not all have the same information set. Lucas showed in this article why policymakers may not be able to use the Phillips curve trade-off and also how prices convey information to market participants. We present this model in the next chapter.

3. The importance of this law for economics and finance was demonstrated by Samuelson (1965), who used it to show that future prices would follow a martingale.

4. Note that we are using here the assumption of no memory loss. The result does not go through without it.

5. The solution to the Cagan model under rational expectations was first obtained by Sargent and Wallace (1973).

6. Sunspots have become the generic example of a variable that affects the equilibrium only because individuals believe it does. We will see later other examples of extrinsic uncertainty potentially affecting the equilibrium. However, Jevons (1884) who introduced sunspots into economics believed that they mattered because they affected agricultural output.

7. Among the most famous historical episodes are the Dutch tulip mania (1634–1636) and the South Sea Bubble (1720). See Charles MacKay (1841) and Charles Kindleberger (1978) for accounts of these and other fascinating episodes.

8. These issues are discussed at greater length in Blanchard and Watson (1982) and in Fischer and Merton (1984).

9. Note that if the bubble is stochastic, the probability that the price will become negative may be very small. There is some evidence that individuals systematically ignore very small probabilities. This weakens the argument made here for eliminating bubbles as well as some of the arguments made later in the chapter.

10. The remark of the previous note applies here as well.

11. This is similar to the conclusion for a physical asset available in infinitely elastic supply.

12. Taylor (1977) proposed one such criterion.

13. Precise statements are given in Blanchard and Kahn (1981) and in Whiteman (1983).

14. There are corresponding conditions in differential equation systems. The general condition becomes that the system must have exactly m roots with positive real parts (see Buiter 1984). In the example given here the differential system in prices and capital should have one positive and one negative root.

15. In figure 5.2, all we needed to do was to plot the combinations of constant (b, k) which satisfied (10) and (11). Here, because we want to characterize the dynamics, we must first derive the loci of (k_t, b_t) along which $b_{t+1} = b_t$ and $k_{t+1} = k_t$ respectively. Hence the derivation of (13) and (14).

16. Care must be taken in using a phase diagram to analyze the dynamics of a *difference* equation system. The economy will not, as in the case of a differential equation system, move continuously along one of the trajectories, but rather it will jump from point to point on that trajectory. An equilibrium that appears stable on the phase diagram may be in fact unstable. The economy, though staying on the

path that converges to the e
of increasing size. Thus we r
saddle point stable around E
sytem linearized around each

17. What is important in th
depends on the price but that
of prices (here, a random wal

18. Shiller (1984) and Summ
long overvaluations or und
rational bubbles.

19. See Merton (1987) for a r
two-step volatility test that
inequality. He concludes that

20. The analysis in this sectio
Obstfeld (1984), and Obstfel

21. Despite the fact that m
pathological properties of th
enters the utility function an
as a vehicle for saving.

22. The case where σ is neg
transversality condition is dis

23. Obstfeld and Rogoff (19
is of the form

$$u(m) = \frac{m^{1-\gamma}}{1-\gamma};$$

this condition is satisfied if γ

24. Note that we saw in the t
the model can have a multip
look at the dynamics of the n

25. The model is a simplifie
Grandmont allows for endog
given. We also draw in what

26. In chapter 4 we considere

27. If there were population
occur where the offer curve
where prices are falling at rat

28. This construction is due t

29. This is an implication of S

ance was demonstrated by
would follow a martingale.

emory loss. The result does

ectations was first obtained

able that affects the equilib-
see later other examples of
m. However, Jevons (1884)
they mattered because they

Dutch tulip mania (1634–
MacKay (1841) and Charles
cinating episodes.

ard and Watson (1982) and

that the price will become
t individuals systematically
at made here for eliminating
the chapter.

ll.

available in infinitely elastic

n (1981) and in Whiteman

uation systems. The general
m roots with positive real
lifferential system in prices
root.

binations of constant (b, k)
characterize the dynamics,
h $b_{t+1} = b_t$ and $k_{t+1} = k_t$

analyze the dynamics of a
n the case of a differential
rajectories, but rather it will
rium that appears stable on
ny, though staying on the

path that converges to the equilibrium, may oscillate back and forth in oscillations of increasing size. Thus we must check in this case whether the system is indeed saddle point stable around E and stable around A by computing the roots of the sytem linearized around each of the two equilibria. This check is left to the reader.

17. What is important in the Marsh-Merton example is not that the dividend depends on the price but that the particular dividend policy implies nonstationarity of prices (here, a random walk).

18. Shiller (1984) and Summers (1986) have pointed out that fads, if they lead to long overvaluations or undervaluations of the stock, may look very much like rational bubbles.

19. See Merton (1987) for a review of the evidence. West (1987) has constructed a two-step volatility test that first tests the arbitrage relation and then the variance inequality. He concludes that the rejection does not come from a failure of arbitrage.

20. The analysis in this section is based on Brock (1975), Calvo (1978), Gray (1982), Obstfeld (1984), and Obstfeld and Rogoff (1983).

21. Despite the fact that money is the only asset, the model has none of the pathological properties of the OLG model with money. This is because money enters the utility function and is used both (implicitly) for transaction services and as a vehicle for saving.

22. The case where σ is negative and hyperdeflation cannot be ruled out by the transversality condition is discussed by Brock (1975).

23. Obstfeld and Rogoff (1983) discuss this condition at greater length. If $u(m)$ is of the form

$$u(m) = \frac{m^{1-\gamma}}{1-\gamma};$$

this condition is satisfied if $\gamma > 1$.

24. Note that we saw in the third example of section 5.1 that a loglinear version of the model can have a multiplicity of convergent paths, when $\alpha < -1$. We now look at the dynamics of the model without linearization.

25. The model is a simplified version of the model used by Grandmont (1985). Grandmont allows for endogenous labor supply, but we take the endowments as given. We also draw in what follows on the survey by Woodford (1984).

26. In chapter 4 we considered the case where e_2 was equal to zero.

27. If there were population growth at rate n, the steady state equilibrium would occur where the offer curve intersects the line $m_{t+1} = (1 + n)m_t$, that is, the line where prices are falling at rate $1 + n$.

28. This construction is due to Azariadis and Guesnerie (1984).

29. This is an implication of Sarkovskii's theorem, presented by Grandmont (1983).

30. For a relatively simple presentation of the theory of periodic and aperiodic behavior of one-dimensional dynamic systems, see Grandmont (1983). The possibility of chaos in deterministic systems has been explored by various authors; see, for example, Day (1982, 1983).

31. See Guesnerie (1986).

32. The example comes from Azariadis (1981). The proof follows Woodford (1987).

33. The model is a simplified and slightly modified version of Evans (1985).

34. This assumption, which differs from that of section 5.1, makes the problem more interesting. The case where the information set includes current values of y and v is easier and is left to the reader.

35. Marcet and Sargent (1987) show, in the context of the Cagan hyperinflation model with two equilibria (studied at the end of chapter 4), that with least squares learning the economy converges to the low-inflation equilibrium.

36. Multiple stable equilibria may also emerge in a different class of models, models that allow for increasing returns and/or externalities in labor and goods markets. Multiplicity in those models does not rely on the presence of income effects. We present and discuss such models, and the likelihood of multiple equilibria, in chapter 8.

37. The intuition for this is as follows: Suppose that individuals have perfect foresight, so that equation (A1) is simply a difference equation in p_{t-1}, p_t, p_{t+1}, and x_t. Given p_{t-1} at time t, for p_t to be uniquely determined by the condition that the equation does not explode, the equation must have one root smaller and one root greater than 1 in absolute value. If both roots were, for example, smaller than one in absolute value, the difference equation would converge for any value of p_t. The system would have the type of multiplicity studied in section 5.4. [The specific condition for saddle point stability in systems such as (A1) is given by Blanchard 1985.] The roots of equation (A7) turn out to be the inverses of the roots of the difference equation obtained by assuming perfect foresight in (A1). Thus, for saddle point stability, they must also be such that one is smaller and one larger than 1 in absolute value.

References

Azariadis, Costas (1981). "Self-Fulfilling Prophecies." *Journal of Economic Theory* 25, 3 (Dec.), 380–396.

Azariadis, Costas, and Roger Guesnerie (1984). "Sunspots and Cycles." CARESS Working Paper 83-22R. University of Pennsylvania. March.

Black, Fischer (1974). "Uniqueness of the Price Level in Monetary Growth Models with Rational Expectations." *Journal of Economic Theory* 7, 1 (Jan.), 53–65.

Blanchard, Olivier, and Mark
Financial Markets." In P. Wac
Lexington, MA.: Lexington B

Blanchard, Olivier (1985). "M
Models: A Survey." *Mathema*

Blanchard, Olivier, and Charl
Models under Rational Expec

Bray, Margaret (1982). "Learn
tions Models." *Journal of Econ*

Bray, Margaret, and N. Savin (
Model Specification." *Econome*

Brock, William (1975). "A Sin
Monetary Economics 1, 2 (Apri

Brock, William (1986). "Disti
Expanded Version." Mimeo. U

Buiter, Willem (1984). "Saddl
pectations Models: A Gener
Econometrica 52, 3 (May), 665-

Burmeister, Edwin, Christophe
"The 'Saddlepoint Property'" a
Good Models." *Econometrica* 4

Burmeister, Edwin, Robert Flo
Solutions in Rational Expectati

Calvo, Guillermo (1978). "On
Perfect Foresight." *Journal of E*

Campbell, John, and Robert Sh
Value Relation." *Journal of Poli*

Cass, David, M. Okuno, and I
the Pareto Optimality of Co
Models." *Journal of Economic T*

Day, Richard (1982). "Irregul
406–414.

Day, Richard (1983). "The Eme
Quarterly Journal of Economics 9

DeCanio, Stephen (1979). "Rat
Quarterly Journal of Economics 9

y of periodic and aperiodic
andmont (1983). The possi-
red by various authors; see,

e proof follows Woodford

rsion of Evans (1985).

.1, makes the problem more
es current values of y and v

of the Cagan hyperinflation
er 4), that with least squares
quilibrium.

rent class of models, models
in labor and goods markets.
sence of income effects. We
multiple equilibria, in chapter

at individuals have perfect
quation in p_{t-1}, p_t, p_{t+1}, and
ed by the condition that the
e root smaller and one root
r example, smaller than one
rge for any value of p_t. The
in section 5.4. [The specific
(A1) is given by Blanchard
inverses of the roots of the
ght in (A1). Thus, for saddle
ler and one larger than 1 in

urnal of Economic Theory 25,

pots and Cycles." CARESS
arch.

Monetary Growth Models
7, 1 (Jan.), 53–65.

Blanchard, Olivier, and Mark Watson (1982). "Bubbles, Rational Expectations and Financial Markets." In P. Wachtel (ed.), *Crises in the Economic and Financial Structure.* Lexington, MA.: Lexington Books.

Blanchard, Olivier (1985). "Methods of Solution for Dynamic Rational Expectations Models: A Survey." *Mathematical Programming Study* 23, 210–225.

Blanchard, Olivier, and Charles Kahn (1981). "The Solution of Linear Difference Models under Rational Expectations." *Econometrica* 48 (July), 1305–1311.

Bray, Margaret (1982). "Learning, Estimation and the Stability of Rational Expectations Models." *Journal of Economic Theory* 26, 318–339.

Bray, Margaret, and N. Savin (1986). "Rational Expectations Equilibria, Learning and Model Specification." *Econometrica* 54, 5 (Sept.), 1129–1160.

Brock, William (1975). "A Simple Perfect Foresight Monetary Model." *Journal of Monetary Economics* 1, 2 (April), 133–150.

Brock, William (1986). "Distinguishing Random and Deterministic Systems: An Expanded Version." Mimeo. University of Wisconsin.

Buiter, Willem (1984). "Saddlepoint Problems in Continuous Time Rational Expectations Models: A General Method and Some Macroeconomic Examples." *Econometrica* 52, 3 (May), 665–680.

Burmeister, Edwin, Christopher Caton, A. Rodney Dobell, and Stephen Ross (1973). "The 'Saddlepoint Property'" and the Structure of Dynamic Heterogeneous Capital Good Models." *Econometrica* 41, 1 (Jan.), 79–95.

Burmeister, Edwin, Robert Flood, and Peter Garber (1982). "On the Equivalence of Solutions in Rational Expectations Models." *Economic Letters.*

Calvo, Guillermo (1978). "On the Indeterminacy of Interest Rates and Wages with Perfect Foresight." *Journal of Economic Theory* 19 (Dec.), 321–337.

Campbell, John, and Robert Shiller (1987). "Cointegration and Tests of the Present Value Relation." *Journal of Political Economy* 95, 1062–1088.

Cass, David, M. Okuno, and I. Zilcha (1979). "The Role of Money in Supporting the Pareto Optimality of Competitive Equilibrium in Consumption-Loan Type Models." *Journal of Economic Theory* 20, 41–80.

Day, Richard (1982). "Irregular Growth Cycles." *American Economic Review* 72, 406–414.

Day, Richard (1983). "The Emergence of Chaos from Classical Economic Growth." *Quarterly Journal of Economics* 98, 201–213.

DeCanio, Stephen (1979). "Rational Expectations and Learning from Experience." *Quarterly Journal of Economics* 93, 1 (Feb.), 47–58.

Evans, George (1985). "Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models." *Quarterly Journal of Economics* 100, 4 (Dec.), 1217–1233.

Fischer, Stanley, and Robert Merton (1984). "Macroeconomics and Finance: The Role of the Stock Market." *Carnegie Rochester Conference Series. Vol. 21. Essays on Macroeconomic Implications of Financial and Labor Markets and Political Processes.* Autumn, 57–108.

Flavin, Marjorie (1983). "Excess Volatility in the Financial Markets: A Reassessment of the Empirical Evidence." *Journal of Political Economy* 91, 6 (Dec.), 929–956.

Flood, Robert, and Peter Garber (1980). "Market Fundamentals versus Price Level Bubbles: The First Tests." *Journal of Political Economy* 88, 747–770.

Friedman, Benjamin (1979). "Optimal Expectations and the Extreme Information Assumptions of 'Rational Expectations' Macromodels." *Journal of Monetary Economics* (Jan.), 23–41.

Geanakopolos, John, and Herakles Polemarchakis (1983). "Walrasian Indeterminacy and Dynamic Macroeconomic Equilibrium: The Case of Certainty." Mimeo. August.

Grandmont, Jean Michel (1983). "Periodic and Aperiodic Behavior in Discrete Dynamical Systems." CEPREMAP Working Paper 8317. September.

Grandmont, Jean Michel (1985). "On Endogenous Business Cycles." *Econometrica* 53, 5 (Sept.), 995–1045.

Gray, JoAnna (1982). "Dynamic Instability in Rational Expectations Models: An Attempt to Clarify." International Finance Discussion Paper 197. Federal Reserve Board. January.

Guesnerie, Roger (1986). "Stationary Sunspot Equilibria in an *n* Commodity World." Mimeo. Presented at the Conference on Non-Linear Economic Dynamics. Paris. June 1985.

Hahn, Frank (1968). "On Warranted Growth Paths." *Review of Economic Studies* 35, 175–184.

Jevons, W. Stanley (1884). *Investigations in Currency and Finance.* London: Macmillan.

Kehoe, Timothy, and David Levine (1983). "Indeterminacy of Relative Prices in Overlapping Generations Models." MIT Working Paper 313. April.

Kehoe, Timothy, and David Levine (1985). "Comparative Statics and Perfect Foresight in Infinite Horizon Economies." *Econometrica* 53, 433–453.

Kindleberger, Charles (1978). *Manias, Panics and Crashes.* New York: Basic Books.

LeRoy, Stephen, and Richard Porter (1981). "The Present Value Relation: Tests Based on Variance Bounds." *Econometrica* 49, 3 (May), 555–574.

Lucas, Robert (1973). "Som
offs." *American Economic Rev*

MacKay, Charles (1841). *Ext*
Reprinted by Farrar, Strauss

Mankiw, N. Gregory, David
Reexamination of Stock Mar

Marcet, Albert, and Thoma
Learning Mechanisms in Self

Marcet, Albert, and Thoma
Dynamics of Hyperinflation.

Marsh, Terry, and Robert
Bounds Tests for the Rationa
76, 3 (June), 483–498.

Merton, Robert (1987). "Or
Hypothesis." In Rudiger Dor
in Honor of Franco Modigliani

Muth, John (1960). "Optim
Journal of the American Statist

Muth, John (1961). "Rationa
Econometrica 39, 315–334.

Obstfeld, Maurice (1984). "
Foresight Model." *Econometr*

Obstfeld, Maurice, and Ken
Macroeconomic Models: Car
4 (Aug.), 675–687.

Samuelson, Paul (1965). "Pr
domly." *Industrial Manageme*

Sargent, Thomas, and Neil W
Growth." *Econometrica* 41, (N

Sargent Thomas (1979). *Mac*

Shiller, Robert (1981). "Do
Subsequent Changes in Divi
436.

Shiller, Robert (1984). "Stoc
Economic Activity 2, 457–510

Summers, Lawrence (1986).
Values?" *Journal of Finance* 41

Multiple Equilibria Problem
al of Economics 100, 4 (Dec.),

onomics and Finance: The
ce Series. Vol. 21. Essays on
rkets and Political Processes.

l Markets: A Reassessment
91, 6 (Dec.), 929–956.

mentals versus Price Level
, 747–770.

the Extreme Information
urnal of Monetary Economics

. "Walrasian Indeterminacy
Certainty." Mimeo. August.

odic Behavior in Discrete
. September.

iness Cycles." Econometrica

Expectations Models: An
Paper 197. Federal Reserve

n an n Commodity World."
omic Dynamics. Paris. June

view of Economic Studies 35,

inance. London: Macmillan.

nacy of Relative Prices in
r 313. April.

rative Statics and Perfect
3, 433–453.

. New York: Basic Books.

ent Value Relation: Tests
55–574.

Lucas, Robert (1973). "Some International Evidence on Output-Inflation Trade-offs." *American Economic Review* 63, 3 (June), 326–334.

MacKay, Charles (1841). *Extraordinary Popular Delusions and the Madness of Crowds.* Reprinted by Farrar, Strauss and Giroux, 1932.

Mankiw, N. Gregory, David Romer, and Matthew Shapiro (1985). "An Unbiased Reexamination of Stock Market Volatility." *Journal of Finance* 40 (July), 677–687.

Marcet, Albert, and Thomas J. Sargent (1986). "Convergence of Least Square Learning Mechanisms in Self Referential Linear Stochastic Models." Mimeo. October.

Marcet, Albert, and Thomas J. Sargent (1987). "Least Squares Learning and the Dynamics of Hyperinflation." Mimeo. June.

Marsh, Terry, and Robert Merton (1986). "Dividend Variability and Variance Bounds Tests for the Rationality of Stock Market Prices." *American Economic Review* 76, 3 (June), 483–498.

Merton, Robert (1987). "On the Current State of the Stock Market Rationality Hypothesis." In Rudiger Dornbusch et al. (eds.), *Macroeconomics and Finance: Essays in Honor of Franco Modigliani.* Cambridge, MA: MIT Press.

Muth, John (1960). "Optimal Properties of Exponentially Weighted Forecasts." *Journal of the American Statistical Association* 55, 290–306.

Muth, John (1961). "Rational Expectations and the Theory of Price Movements." *Econometrica* 39, 315–334.

Obstfeld, Maurice (1984). "Multiple Stable Equilibria in an Optimizing Perfect Foresight Model." *Econometrica* 52, 1 (Jan.), 223–228.

Obstfeld, Maurice, and Kenneth Rogoff (1983). "Speculative Hyperinflations in Macroeconomic Models: Can We Rule Them Out?" *Journal of Political Economy* 91, 4 (Aug.), 675–687.

Samuelson, Paul (1965). "Proof that Properly Anticipated Prices Fluctuate Randomly." *Industrial Management Review* 6 (Spring), 41–49.

Sargent, Thomas, and Neil Wallace (1973). "The Stability of Models of Money and Growth." *Econometrica* 41, (Nov.), 1043–1048.

Sargent Thomas (1979). *Macroeconomic Theory.* New York: Academic Press.

Shiller, Robert (1981). "Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends?" *American Economic Review* 71, 3 (June), 421–436.

Shiller, Robert (1984). "Stock Prices and Social Dynamics." *Brookings Papers on Economic Activity* 2, 457–510.

Summers, Lawrence (1986). "Do Market Prices Accurately Reflect Fundamental Values?" *Journal of Finance* 41, 3 (July), 591–601.

Taylor, John (1975). "Monetary Policy During a Transition to Rational Expectations." *Journal of Political Economy* 83, 5, 1009–1021.

Taylor, John B. (1977). "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations." *Econometrica* 33, (Oct.), 671–684.

Taylor, John B. (1985). "New Econometric Techniques for Macroeconomic Policy Evaluation." In Zvi Griliches and Michael D. Intriligator (eds.), *Handbook of Econometrics*. Vol. 3. Amsterdam: North-Holland.

Tirole, Jean (1982). "On the Possibility of Speculation under Rational Expectations." *Econometrica* 50 (Sept.), 1163–1181.

Tirole, Jean (1985). "Asset Bubbles and Overlapping Generations." *Econometrica* 53, 6 (Dec.), 1499–1528.

Townsend, Robert (1978). "Market Anticipations, Rational Expectations and Bayesian Analysis." *International Economic Review* 19 (June), 481–494.

Weil, Philippe (1987). "Confidence and the Real Value of Money in Overlapping Generation Models." *Quarterly Journal of Economics* 102, 1 (Feb.), 1–22.

West, Kenneth (1988). "Dividend Innovations and Stock Price Volatility." *Econometrica* 56, 1 (Jan.), 37–61.

West, Kenneth (1987). "A Specification Test for Speculative Bubbles." *Quarterly Journal of Economics* 102, 553–580.

Whiteman, Charles (1983). *Linear Rational Expectations Models: A User's Guide*. Minneapolis: University of Minnesota Press.

Woodford, Michael (1984). "Indeterminacy of Equilibrium in the Overlapping Generations Model: A Survey." Mimeo. Columbia. May.

Woodford, Michael (1987), "Learning to Believe in Sunspots." Mimeo. Graduate School of Business, Chicago.

6

The next four chapters are
fluctuations in output, em
chapter 1. This chapter sets
optimal decisions by firms
to the case of uncertainty.
of aggregate fluctuations.
we start with a brief overv
of the current state of econ

The wealth of business
in the pre-Keynesian perio
Depression,[2] first published i
in the economics literature
table of contents are The P
Theories, Changes in Cost,
ness, Under-consumption
Theories.

Although abounding in
purely theoretical, for early
long been noted, for exam
more cyclical than that of
invoked the accelerator me
output of investment good
cycle facts was assembled
Research sponsored project
book on cycles was publis
1946 Arthur Burns and Mit
Business Cycles. Using a refe
documented the existence
of a large number of price
cycle.