ECON 7020
Practice Problems

Problem 1. Life Cycle Consumption with Quadratic Utility
Suppose that the consumer maximizes the following objective function:

$$\max \sum_{t=0}^{T} \beta^t [u(c_t)] \tag{1}$$

subject to the dynamic budget constraint (Note that this is written slightly different than we did in class; namely income is assumed to be received at the end of period t rather than at the beginning of period $t+1$):

$$A_{t+1} = (1+r)[A_t + y_t - c_t] \tag{2}$$

where

$$u(c_t) = c_t - \frac{b}{2} c_t^2 \tag{3}$$

with $b > 0$. r is a constant net interest rate, A_t is the amount of wealth available at the beginning of period t and y_t is labour income in period t.

1. Do we need a transversality condition? If not, what other constraint do we need? How much wealth should the consumer have when she dies?

   Answer: We don't need an explicit transervsality condition because the time horizon is finite. We do however need something similar: $A_{T+1} = 0$ or $A_T = C_T$ so that all assets are consumed in the last period or all debt paid off, so that the consumer does not die with debt or assets. The consumer will not die with positive assets so long as the marginal utility of an extra unit of consumption is positive. The budget constraint prevents the agent from dying with debt.

2. Assume the agent receives income A_0 in the first period. Show that the dynamic budget constraint together with the constraint you found in (1) implies the following intertemporal budget constraint:

$$\sum_{t=0}^{T} R^{-t} c_t = A_0 + \sum_{t=0}^{T} R^{-t} Y_t \tag{4}$$

where $R = (1 + r)$. Note: Begin with the dynamic budget constraint (dbc) describing A_1. Then substitute this expression for A_1 into the dbc for A_2. Next substitute the expression you got for A_2 into the dbc for period 3, etc. This will allow you to derive the intertemporal budget constraint.

Answer: In order to derive the Intertemporal Budget Constraint, we need to start with the Dynamic Budget Constraint and iterate forward. So, start with the Dynamic Budget Constraint (DBC)

$$A_{t+1} = (1 + r)[A_t + y_t - c_t] \tag{5}$$

Assuming that A_0 is given for the initial asset levels (which could be zero), y_0 is initial income, and c_0 is initial consumption. For simplicity write $1+r = R$ (you can use either style, the answer is the same). The DBC in period 1:

$$A_1 = R[A_0 + y_0 - c_0] \tag{6}$$

now roll forward one period to get the DBC in period 2:

$$A_2 = R[A_1 + y_1 - c_1] \tag{7}$$

Replace A_1 in equation 7 with A_1 from equation Y

$$A_2 = R[R[A_0 + y_0 - c_0] + y_1 - c_1] \tag{8}$$

Now consider A_3

$$A_3 = R[A_2 + y_2 - c_2] \tag{9}$$

replace A_2 with equation 9

$$A_3 = R(R[R[A_0 + y_0 - c_0] + y_1 - c_1] + y_2 - c_2) \tag{10}$$

we can continue doing this, but now we have a pattern that can be generalized. So we can write equation 10 as

$$A_3 = R^3(A_0 + y_0 - c_0) + R^2(y_1 - c_1) + R(y_2 - c_2) \tag{11}$$

We can continue to roll this forward using the same pattern to find:

$$A_{T+1} = R^{T+1}(A_0+y_0-c_0)+R^T(y_1-c_1)+R^{T-1}(y_2-c_2)+...+R(y_T-c_T) \tag{12}$$

Then we notice that $A_{T+1} = 0$ (from above) so:

$$0 = R^{T+1}(A_0+y_0-c_0)+R^T(y_1-c_1)+R^{T-1}(y_2-c_2)+...+R(y_T-c_T) \quad (13)$$

or re-arranging terms,

$$0 = R^{T+1}A_0+R^{T+1}(y_0-c_0)+R^T(y_1-c_1)+R^{T-1}(y_2-c_2)+...+R(y_T-c_T) \quad (14)$$

which we can collect and write as:

$$0 = R^{T+1}A_0 + \sum_{t=0}^{T} R^{T-t+1}(y_t - c_t) \quad (15)$$

Then we can take out a common factor, R^{T+1}:

$$0 = R^{T+1}[A_0 + \sum_{t=0}^{T} R^{-t}(y_t - c_t)] \quad (16)$$

From the fact that the left hand side (LHS) of equation 16 is zero, we can cancel the common factor and re-organize the terms so that:

$$A_0 + \sum_{t=0}^{T} R^{-t}y_t = \sum_{t=0}^{T} R^{-t}c_t \quad (17)$$

which is the intertemporal budget constraint defined above.

3. Set up the consumer's maximization problem. What is Bellman's Equation?

   Answer:

$$V_t(A_t, y_t) = \max_{c_t}[u(C_t) + \beta V_{t+1}(A_{t+1}, y_{t+1})] \quad (18)$$

   (This is Bellman's Equation.)
   subject to:

$$A_{t+1} = (1 + r)[A_t + y_t - c_t] \quad (19)$$

4. Find the first order conditions (use the Envelope theorem).

   Answer:

$$u'(C_t) + \beta V^A_{t+1}(A_{t+1}, y_{t+1})\frac{\partial A_{t+1}}{\partial C_t} = 0 \quad (20)$$

since $\frac{\partial A_{t+1}}{\partial C_t} = -R$ we have:

$$u'(C_t) = \beta R V_{t+1}^A(A_{t+1}, y_{t+1}) \tag{21}$$

Then by the envelope theorem we have the second (simplified) condition:

$$V_t^A(A_t, y_t) = \beta R V_{t+1}^A(A_{t+1}, y_{t+1}) \tag{22}$$

This is sometimes called the Envelope Condition.

5. Derive the Euler Equation.

   Answer: To derive the Euler Equation we substitute $V_t^A(A_t, y_t)$ for $\beta R V_{t+1}^A(A_{t+1}, y_{t+1})$ in the first order condition (equation 21) or

$$u'(C_t) = V_t^A(A_t, y_t) \tag{23}$$

   Roll this expression forward one period

$$u'(C_{t+1}) = V_{t+1}^A(A_{t+1}, y_{t+1}) \tag{24}$$

   Then use this fact in the original first order condition

$$u'(C_t) = \beta R u'(C_{t+1}) \tag{25}$$

   This is the Euler Equation, it describes the relationship between optimal consumption in two adjacent periods.

6. Derive a closed form solution for c_0.

   Answer: Now we can use the quadratic utility function where the marginal utility is $u'(C_t) = 1 - bC_t$. Replace this is in the Euler Equation:

$$1 - bC_t = \beta R(1 - bC_{t+1}) \tag{26}$$

   In order to generate a closed form solution, we need $\beta^{-1} = R$, then we have:

$$C_t = C_{t+1} \tag{27}$$

   Thus consumption is constant over time. In order to get an explicit consumption function, we need to use the IBC

$$A_0 + \sum_{t=0}^{T} R^{-t} y_t = \sum_{t=0}^{T} R^{-t} C_t \tag{28}$$

where $C_t = C_{t+1} = ... = \bar{c}$ We can substitute this into the IBC to get:

$$\bar{c}\sum_{t=0}^{T} R^{-t} = A_0 + \sum_{t=0}^{T} R^{-t} y_t \tag{29}$$

One way of simplifying this further, is to take the limit as $t \to \infty$ so that we have:

$$\bar{c}\sum_{t=0}^{\infty} R^{-t} = A_0 + \sum_{t=0}^{\infty} R^{-t} y_t \tag{30}$$

Then, the sum of the infinite series $\sum_{t=0}^{\infty} R^{-t} = \frac{1+r}{r}$

$$\frac{1+r}{r}\bar{c} = A_0 + \sum_{t=0}^{\infty} R^{-t} y_t \tag{31}$$

Which simplifies to

$$\bar{c} = \frac{r}{1+r}[A_0 + \sum_{t=0}^{\infty} R^{-t} y_t] \tag{32}$$

There are other ways to go about finding a closed form solution. For those of you wondering what a "closed form solution" is, here is a fairly general definition: In mathematics, an equation or system of equations is said to have a closed-form solution if, and only if, at least one solution can be expressed analytically in terms of a bounded number of certain "well-known" functions. Typically, these well-known functions are defined to be elementary functions; so infinite series, limits, and continued fractions are not permitted. In our solution for consumption, we do have an infinite series. But since the evolution of income is unknown (I didn't give you a process for income, though it would have been appropriate for you to assume one), you cannot simplify this solution further. The fundamental point of this problem was to recognize that an Euler equation provides you with an algorithm or rule that guides the choice of consumption, i.e., an optimal consumption path must satisfy the Euler equation in each time period. But that an Euler equation isn't a consumption function or a solution, in the sense that it does not tell you what consumption will be in any given period. In this problem with an analytically tractable utility function, we can find this type of solution, provided we are willing to make a few assumptions.

In most problems where there is explicit uncertainty regarding income and a more realistic utility function (such as a CRRA function) you will not be able to determine the solution to the consumption function analytically. The only way to determine such solutions is to numerically solve for them with a program like Matlab or Mathematica.

Problem 2. Life Cycle Utility over Consumption and Leisure Suppose that the consumer maximizes the following objective function:

$$\max \sum_{t=0}^{T} \beta^t [u(c_t) + d(l_t)] \tag{33}$$

where c_t is consumption and l is leisure. Assume that both felicity functions, u and d are increasing and concave.

Consumption is typically measured using expenditure x_t on some group of goods and services. In reality, consumption requires expenditure plus time spent on 'home production'and households can substitute between the two. For example, it is more expensive to eat out or buy pre-prepared food, but it is less costly in terms of time. It is cheaper to buy raw foods and prepare them yourself but it is more costly in terms of time.

Assume that consumption is related to expenditure and time inputs:

$$c_t = f(x_t, s_t) \tag{34}$$

where s_t represents time spent in home production. Assume that f is increasing and concave in both of its inputs. Assume that the household can borrow or lend at the risk free rate, $R = 1 + r$. The household's dynamic budget constraint is

$$A_{t+1} = R[A_t + w_t h_t - x_t] \tag{35}$$

where h_t is hours spend working outside the home, w_t is the wage rate, which is exogenous, A_t is financial assets at the beginning of period t, and x_t is expenditure. There is no uncertainty, at time 0 households know the entire sequence of wages with certainty from 0 to T.

Assume that households have an endowment of one unit of time each period, which they spend on leisure, working for wages, and in home production. Thus leisure time satisfies:

$$l_t = 1 - h_t - s_t \tag{36}$$

1. What are the state variables? What are the control variables?

   Answer: The state variable is A_t, assets, and the control variables are x_t, s_t, h_t which are, respectively, expenditure, time spent in home production, and hours spent working for wages.

2. Write down the Bellman equation for this problem.

   Answer: The Bellman equation is:

   $$V_t(A_t) = \max_{x_t,s_t,h_t} [u(f(x_t, s_t)) + d(1 - s_t - h_t) + \beta V_{t+1}(A_{t+1})] \tag{37}$$

   Note that since $l_t = 1 - h_t - s_t$ we can replace l_t in $d(l_t)$ so that we can optimize by directly choosing s_t, h_t.

3. Derive the first order conditions (hint: there will be 3 in addition to the envelope condition)

   Answer: First order conditions:

   $$u'(f(x_t, s_t))f_{x_t} = R\beta V^A_{t+1}(A_{t+1}) \tag{38}$$
   $$u'(f(x_t, s_t))f_{s_t} = d'(1 - s_t - h_t) \tag{39}$$
   $$d'(1 - s_t - h_t) = Rw_t\beta V^A_{t+1}(A_{t+1}) \tag{40}$$

   And the Envelope condition (using the Envelope Theorem)

   $$V^A_t(A_t) = R\beta V^A_{t+1}(A_{t+1}) \tag{41}$$

4. Find the Euler Equation.

   Answer: Start with equation 38, notice that the RHS of equation 38 is the same as the RHS of equation 41, use this fact to replace $R\beta V^A_{t+1}(A_{t+1})$ in the first FOC:

   $$u'(f(x_t, s_t))f_{x_t} = V^A_t(A_t) \tag{42}$$

   Roll forward one period (lead one period)

   $$u'(f(x_{t+1}, s_{t+1}))f_{x_{t+1}} = V^A_{t+1}(A_{t+1}) \tag{43}$$

   Replace in the first FOC, equation 38:

   $$u'(f(x_t, s_t))f_{x_t} = R\beta u'(f(x_{t+1}, s_{t+1}))f_{x_{t+1}} \tag{44}$$

   This is the Euler Equation.

5. Show that $\frac{x_t}{s_t}$ is an increasing function of wages, w_t. That is, as wages rise people substitute expenditures for time in the production of consumption. Provide intuition for this result.

   Answer: Divide the first FOC, equation 38, by the second FOC, equation 39, to give

$$\frac{f_{x_t}}{f_{s_t}} = \frac{R\beta V_{t+1}^A(A_{t+1})}{d'(1 - s_t - h_t)} \tag{45}$$

   From the third FOC, equation 40, note that $V_{t+1}^A(A_{t+1}) = \frac{d'(1-s_t-h_t)}{Rw_t\beta}$ Use this to substitute above for $d'(1 - s_t - h_t)$ and simplify to give

$$\frac{f_{x_t}}{f_{s_t}} = \frac{1}{w_t} \tag{46}$$

   Thus if wages for work outside the home rise, $\frac{f_{x_t}}{f_{s_t}}$ must fall. Since $f(x_t, s_t)$ is concave, this implies $\frac{x_t}{s_t}$ must rise. This means that when w_t rises, the opportunity cost of time spent in home production rises (your time is now more valuable). Therefore it is optimal for the consumer to substitute expenditures on goods produced outside the home for the time used in the production of home consumption, which is a pretty intuitive idea, eh?