# Final Exam for Math Camp

Total: 140 points + Bonus 20 points

## Part I. Calculus (58 points)

1. (8 points) Given the two functions

$$y_1 = f_1(x_1, x_2) = -x_1^2 - \frac{5}{2}x_1x_2 - x_2^2$$
$$y_2 = f_2(x_1, x_2) = (2x_1 + x_2)(x_1 + 2x_2)$$

(a) (4 points) Compute the gradient of $f_1, f_2$, respectively.

(b) (4 points) Form the Jacobian matrix and find the determinant of it. Are the two functions dependent?

2. (5 points) Determine the total derivative $\frac{dz}{dt}$ for the following function

$$z = x^3 + 4x^2y - xy^2 + y$$

where $x = -3t, y = 1 + \frac{1}{2}t$.

3. (12 points) Solve the following constrained optimization.

$$\max U(c_1, c_2) = -c_1^2 + c_1c_2 - 2c_2^2$$
$$s.t. c_1 + c_2 = 16 \ (unit - 1{,}000\ USD)$$

where $U(c_1, c_2)$ is the utility function of consumptions at time $t = 1$ and $t = 2$. Use the second derivative test to determine if you have maximum or minimum, as below

| $\boldsymbol{D(x,y) = f_{xx}f_{yy} - f_{xy}^2}$ | |
|---|---|
| $\boldsymbol{D > 0, f_{xx} > 0}$ | Local minimum |
| $\boldsymbol{D > 0, f_{xx} < 0}$ | Local maximum |
| $\boldsymbol{D < 0}$ | Saddle point |
| $\boldsymbol{D = 0}$ | No information |

4. (5 points) Use Taylor's expansion to express a third order approximation around $x_0 = 0$ for the following function:

$$f(x) = \frac{3}{(1-x)^2}$$

5. (10 points) Compute the integral in each case
   (a) (3 points) $\int_0^{10}(e^{3x} + 4x)dx$ ; (b) (3 points) $\int_0^1 6x^2 e^{x^3} dx$; (c) (4 points) $\int_1^{3x^3} e^t dt$

6. (5 points) Compute $\lim_{x\to 0}\left(\frac{e^x - 1}{x^3+3x^2}\right)\sin(x)$
   Hint: use L'Hopital's rule and the special limit of $\frac{\sin(x)}{x}$ when $x \to 0$.

7. (5 points) Find the implicit differentiation $\frac{dy}{dx}$ :

$$F(x,y) = 2x^3 - x^2y + lny$$

   where $y = y(x)$.

8. (5 points) Determine if $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n+1} n^2$ is a convergent series.

| For infinite series $\Sigma_{n=1}^{\infty} a_n$ | |
|---|---|
| $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$ | Convergence. |
| $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} > 1$ | Divergence. |
| $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$ | No information. |

9. (3 pts) For function $f(x) = 3x^3 - 2x^2 + x - 1$, determine the convexity and concavity of the function for different parts and give the inflection point

## Part II. Real Analysis (10 points)

10. (15 points) Examine the following claims.

   (a) (5 points) Show that $(\mathbb{R}^n, d_1)$, $d_1(x,y) = \min\{1, |x-y|\}$ is a metric space.

(b) (5 points) True or False.

Let $(X, d)$ be metric space.

(i) $X$ is both open and closed.

(ii) The union of finite collection of closed subsets of $\mathbb{R}^n$ is closed

(iii) The intersection of any collection of closed subsets of $\mathbb{R}^n$ is closed.

(iv) $\overline{(0,1)} \cap (0,1)^{\circ} = (0,1)$

(v) If there is a Cauchy sequence in $X$ is convergent, then $(X, d)$ is complete.

## Part III. Linear Algebra (72 points)

11. (23 points) Use matrices A through D to answer the following questions:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) (2 points) Compute $BC^T$.

(b) (3 points) Compute $det\ A$ using Laplace Expansion Theorem.

(c) (4 points) Compute $det\ A$ using row operations

(d) (5 points) Compute $A^{-1}$. What is $trace(A), rank(A)$?

(e) (4 points) Give two equivalent statements to the claim that an $n \times n$ square matrix is of full rank.

(f) (5 points) Use Cramer's rule to solve the system.

$$2x_1 - 3x_2 - x_3 = 2$$
$$x_1 + 2x_3 = 0$$
$$x_1 + x_2 + 2x_3 = 1$$

12. (32 points) Suppose matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$, compute:

(a) (3 points) Find $null(B)$.

(b) (3 points) Determine if $A$ is positive definite.

(c) (3 points) If a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be defined as $T(\vec{x}) = A\vec{x}, \forall \vec{x} \in \mathbb{R}^2$. Find the standard matrix of its inverse $T^{-1}$.

(d) (3 points) Find $ker(B)$

(e) (4 points) Find the eigenvalues and eigenvectors of $A$.

(f) (3 points) Diagonalize $A$.

(g) (3 points) Find $A^4$

(h) (4 points) By Grant-Schmidt process, find the orthonormal column vectors set of $A$.

(i) (3points) When a square matrix $A$ has linearly independent columns, then $A$ can be factored as $A = QR$, where $Q$ is a matrix with orthonormal columns, and $R$ is an invertible upper triangular matrix. Evaluate the $QR$ factorization of $A$.

Hint: directly use the conclusion from (h). Note $Q^T Q = I$, this will help you find $R$.

(j) (3 points) Compute the generalized inverse of matrix $A^+$.

13. True of False (7 pts)

(i) When Q is an orthogonal matrix, $|Q| = 1 \; or -1$

(ii) If $Q$ is an orthogonal matrix, then $Q^{-1}$ is orthogonal

(iii) Multiplication of two elementary matrices is elementary matrix

(iv) If $A_{n\times n}$ has n linearly independent variables, $A$ is diagonalizable

(v) $A_{n\times n}$ has inverse iff. $\lambda_i \neq 0, \forall i$

(vi) If A is positive definite, it has $LU$ factorization, $LDL^T$ decomposition, and Cholesky decomposition

(vii) Composite of linear transformation is not necessarily a linear transformation

14. Factorization and Decomposition (10 pts)
    For matrix

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix}$$

    1. (4 points) $LU$ factorize A.
    2. (3 points) Find the $LDL^T$ decomposition of A
    3. (3 points) Cholesky decompose A.

## Bonus (20 points)

For problem 11

1. (4 points) Give $adj\ B$.

For problem 12.

2. (4 points) Spectral decompose A

3. (12 points) Find the singular value decomposition of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$.

    (i) Compute $A^T A$, and the singular values, and $\Sigma$ (3 points)
    (ii) Compute $V$(3 points)
    (iii) Compute $U$(3 points)