Practice Exam for Math Camp Final

Part I. Calculus (55 points)

1. (6 points) Given the two functions

$$y_1 = f_1(x_1, x_2) = (x_1^2 - 3x_2)(x_1 - 2)$$
$$y_2 = f_2(x_1, x_2) = 3x_1 ln x_2 + e^{x_1 x_2}$$

   (a) (4 points) Compute the gradient of f_1, f_2, respectively.

   (b) (2 points) Form the Jacobian matrix and find the determinant of it. Are the two functions dependent?

2. (4 points) Determine the total derivative $\frac{dz}{dt}$ for the following function

$$z = x^2 - 8xy - y^3$$

where $x = 2t, y = 1 - 2t$.

3. (6 points) Solve the following constrained optimization. Find the extrema for

$$U(c_1, c_2) = (5c_1 - 2)^2 c_2^4$$
$$s.t. c_1 + c_2 = 30\ (unit - 1{,}000\ USD)$$

   Where $U(c_1, c_2)$ is the utility function of consumptions at time $t = 1$ and $t = 2$.

4. (4 points) Use Taylor's expansion to express a second order approximation around $x_0 = 1$ for the following function:

$$f(x) = xe^x$$

5. (15 points) Compute the integral in each case

   (a) (4 points) $\int_1^6 \frac{dx}{x-2}$; (b) (5 points) $\int \quad 4xe^{x^2+3}dx$; (c) (6 points) $\int_3^{x^2} \frac{dt}{t}$

6. (5 points) Derive the relative extrema of the following by the second derivative test and note whether you have found a relative minimum or maximum (and briefly note why it is a min or max):

$$y = 2x^3 - x^2 + 3$$

7. (5 points) Find the implicit differentiation $\frac{dy}{dx}$

$$F(x, y) = 5x^3 + x^2y + 5y^2$$

8. (5 points) Determine if $\sum_{n=1}^{\infty} \frac{5^{n+1}}{n^2}$ is a convergent series.

9. (5 points) What is the degree of homogeneity of the following function:

$$f(x_1, x_2) = \frac{1}{2} x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

Part II. Real Analysis (10 points)

10. (10 points) Examine the following claims.

(a) (5 points) Show that $(\mathbb{R}^n, d_\infty)$, $where\ d_1(x, y) = |x - y|$ is a metric space.

(b) (5 points) True or False.

(i) $\emptyset$ is both open and closed.

(ii) The union of finite collection of open subsets of $\mathbb{R}^n$ is open

(iii) The intersection of any collection of open subsets of $\mathbb{R}^n$ is open.

(iv) The closure and interior of $\mathbb{R}^n$ is $\mathbb{R}^n$.

(v) A Cauchy sequence is convergent.

Part III. Linear Algebra (55 points)

11. (23 points) Use matrices A through D to answer the following questions:

$$A = \begin{pmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 3 \\ 3 & 2 \\ 1 & 5 \end{pmatrix}$$

(a) (2 points) Compute BC^T.

(b) (3 points) Compute $det\ A$ using Laplace Expansion Theorem.

(c) (5 points) Compute A^{-1}. What is $trace(A), rank(A)$?

(d) (4 points) Give $adj\ B$.

(e) (4 points) Give two equivalent statements to the claim that an $n \times n$ square matrix is invertible.

(f) (5 points) Use Cramer's rule to solve the system again. Compare with part (e) to verify your answer.

$$\begin{gathered} 5x_1 - 3x_2 + 4x_3 = 4 \\ x_1 + 2x_2 = 7 \\ -x_2 + 3x_3 = 3 \end{gathered}$$

12. (32 points) Suppose matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, compute

(a) (3 points) Find $null(A)$.

(b) (3 points) Determine if A is positive definite.

(c) (3 points) If a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be defined as $T(\vec{x}) = A\vec{x}, \forall\ \vec{x} \in \mathbb{R}^2$. Find the standard matrix of its inverse T^{-1}.

(d) (3 points) Find $ker(A)$

(e) (4 points) Find the eigenvalues and eigenvectors of A.

(f) (3 points) Diagonalize A.

(g) (3 points) Find A^8

(h) (4 points) By Grant-Schmidt process, find the orthonormal set of column vectors of A.

(i) (3points) QR factorize A.

(j) (3 points) Compute A^{+}.

Bonus (15 points)

For problem 12.

1. (3 points) LU factorize A.

2. (5 points) Cholesky decompose A.

3. (7 points) Find the singular value decomposition of A.