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5

Multiple Equilibria,
Bubbles, and Stability

We have skipped some difficult issues at various points in the last four chapters. Confronted with saddle point equilibria, we proceeded to focus on the behavior of the economy along the convergent path; in some cases restricting our attention to that path was indeed warranted, but in many others no formal argument was given to rule out other paths. In other places we studied the properties of steady states without checking whether they were stable. We now examine these issues in more detail. The outcome turns out to be more than just a cleaning up of untidy detail. Rather, the chapter opens up broad and fascinating issues, from multiple equilibria to speculative bubbles and chaos.

In section 5.1 we start by analyzing the solution to a simple linear difference equation under rational expectations. This difference equation has various interpretations: it may arise, for example, from an arbitrage relation, from a linearized version of the OLG model with money analyzed in chapter 4, or from the Cagan model also analyzed in chapter 4. The solution to this simple equation is remarkably rich. For some parameter values, the solution may exhibit bubbles, components that explode in expected value over time. For other parameter values, there is an embarrassing wealth of stable solutions, in some of which variables matter just because individuals believe they do. The rest of the chapter is spent analyzing these issues in a general equilibrium context.

In section 5.2 we focus on the question of whether there can be bubbles on real assets in general equilibrium. Bubbles are ruled out when individuals have infinite horizons. However, when individuals have finite horizons, there are circumstances under which bubbles may exist and even be beneficial. To analyze the conditions under which bubbles exist, we use the Diamond overlapping generations model introduced in chapter 3. We conclude the section with a brief discussion of how econometric methods can be used to detect the presence of bubbles in asset markets.

Blanchard, O. and Stanley Fischer, Lectures on Macroeconomics
(MIT Press, 1981), Chapter 5, Multiple equil.,
Bubbles, and Stability pp 213-226
and pp 261-274

From the purely real models of section 5.2 we turn in section 5.3 to the question of whether there can be price level bubbles in monetary models. The main issue is whether general equilibrium considerations allow us to rule out self-generating hyperinflations or deflations. We conclude, using a model in which money provides direct utility services, that there are cases in which self-generating hyperinflations cannot be ruled out.

The most remarkable set of results appears in section 5.4, where we examine general equilibrium models in which there is an infinity of stable equilibria. For convenience, we use the OLG model. We show how and when the equilibrium may have cycles, may exhibit chaos, may exhibit sunspots, and be affected by extrinsic uncertainty. We also discuss whether configurations of parameters that allow for such strange phenomena are likely to occur.

In section 5.5 we study the role and implications of learning. The explicit introduction of learning can help to narrow the range of probable solutions. We conclude in section 5.6 with an assessment of the relevance of the various types of multiplicity of equilibria presented in the chapter.

One word of clarification: this is not the only chapter in the book in which the possibility of multiple equilibria is discussed. We have already in the analysis of seigniorage in chapter 4 examined one case of multiple equilibria. We discuss other examples, consistent with the Keynesian notion that self-justifying "animal spirits" may cause output expansions and contractions, in chapter 8.

5.1 Solutions to a Simple Equation

In this section we characterize the behavior of a variable y that obeys the following expectational difference equation:

$$y_t = aE[y_{t+1}|t] + cx_t, \quad (1)$$

where $E[y_{t+1}|t]$ denotes the expectation of y_{t+1} held at time t so that y depends on the current expectation of its value next period as well as on the variable x .

To solve for the behavior of y , one must specify how individuals form expectations. We will assume in this chapter that individuals have *rational expectations*, that is, expectations equal to the mathematical expectation of y_{t+1} based on information available at time t .¹

We make two further assumptions in defining this rational expectation. The first is that individuals know the model, namely, equation (1) and the parameters a and c . In most real world situations this will obviously not be

the case; individuals will address the model at the same time as the issue of learning later.

The second assumption is that we set at time t so that we can base on “the” information set. For different information sets, we do not have. We will see

We define $E[y_{t+1}|t]$ by

$$E[y_{t+1}|t] = E[y_{t+1}|I_t],$$

where

$$I_t = \{y_{t-i}, x_{t-i}, z_{t-i}, i = 0,$$

$E[y_{t+1}|f]$ is equal to the mean of y given the information set I_t . The information set I_t contains x and y of y and x ; it may also include other information, such as a vector z_t that, though not directly related to the values of x and y . Note that the model does not suffer from a loss of memory, as anything

Before characterizing some of the possible interpretations of this model

Three Examples

Arbitrage

The first interpretation of the relationship between stocks and a riskless asset is that the dividend, and r be the riskless rate, constant over time. Then, if S_t is the price of the stock and the riskless asset, the expected rate of capital gain is equal to the expected rate of capital gain on the riskless rate:

$$\frac{E[p_{t+1}|I_t] - p_t}{p_t} + \frac{d_t}{p_t} = r,$$

or by reorganizing,

$$p_t = aE[p_{t+1}|I_t] + ad_t,$$

we turn in section 5.3 to the bubbles in monetary models. These considerations allow us to draw conclusions. We conclude, using a variety of services, that there are cases that can be ruled out.

In section 5.4, where we show that there is an infinity of stable models. We show how and why models exhibit chaos, may exhibit chaos. We also discuss whether such strange phenomena are

possible. The explicit range of probable solutions. The point of the relevance of the model is stated in the chapter. This chapter in the book in which we find it. We have already in the case of multiple equilibria. The Keynesian notion that self-expansions and contractions,

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(1)

is held at time t so that y is the same next period as well as on

specify how individuals form expectations. If individuals have *rational* mathematical expectation of

given this rational expectation. Namely, equation (1) and the fact that this will obviously not be

the case; individuals will also be learning, and most likely disagreeing, about the model at the same time as they are forming expectations. We return to the issue of learning later in this chapter.

The second assumption is that all individuals have the same information set at time t so that we can indeed talk about "the" mathematical expectation based on "the" information set. However, different individuals often have different information sets, each with a piece of information that the others do not have. We will see examples of such models in the next chapter.²

We define $E[y_{t+1}|t]$ by

$$E[y_{t+1}|t] = E[y_{t+1}|I_t], \quad (2)$$

where

$$I_t = \{y_{t-i}, x_{t-i}, z_{t-i}, i = 0, \dots, \infty\}.$$

$E[y_{t+1}|t]$ is equal to the mathematical expectation of y_{t+1} based on the information set I_t . The information set contains current and lagged values of y and x ; it may also include current and past values of other variables in a vector z_t that, though not present in equation (1), may help predict future values of x and y . Note that this definition of the information set implies no loss of memory, as anything known at time t is still known at time $t+1$.

Before characterizing solutions to (1) and (2), we give three economic interpretations of this model.

Three Examples

Arbitrage

The first interpretation of (1) is as an arbitrage equation, for example, between stocks and a riskless asset. Let p_t be the price of a stock, d_t be the dividend, and r be the rate of return on the riskless asset, assumed constant over time. Then, if risk neutral individuals arbitrage between stocks and the riskless asset, the expected rate of return on the stock, which is equal to the expected rate of capital gain plus the dividend-price ratio, must equal the riskless rate:

$$\frac{E[p_{t+1}|I_t] - p_t}{p_t} + \frac{d_t}{p_t} = r,$$

or by reorganizing,

$$p_t = aE[p_{t+1}|I_t] + ad_t,$$

where

$$a \equiv \frac{1}{1+r} < 1.$$

This is of the same form as (1). The coefficient a in this case is equal to the one-period discount factor and is less than one so long as the interest rate is positive. The price today depends on the expected price tomorrow but by less than one for one.

The Cagan Model

The second interpretation of (1) is as the equilibrium condition in the Cagan model, presented at the end of the previous chapter and analyzed there under the assumption of adaptive expectations. The Cagan money demand function makes the demand for real balances an exponential function of the negative of the expected rate of inflation. In equilibrium money demand must be equal to the real money stock. In discrete time, equation (40) of the previous chapter becomes

$$\frac{M_t}{P_t} = \exp \left[-\alpha \left(\frac{E[P_{t+1}|I_t] - P_t}{P_t} \right) \right],$$

where we have set c in equation (40) of chapter 4 equal to unity and replaced the a in (40) by α .

Taking logarithms on both sides, denoting logarithms by lowercase letters, and using the approximation $E[p_{t+1}|I_t] - p_t = (E[P_{t+1}|I_t] - P_t)/P_t$, we get

$$m_t - p_t = -\alpha(E[p_{t+1}|I_t] - p_t).$$

Reorganizing gives

$$p_t = aE[p_{t+1}|I_t] + (1-a)m_t,$$

where

$$a \equiv \frac{\alpha}{1+\alpha}.$$

This is in the same form as (1). The price level depends on the price level expected for next period and on the current nominal money stock. Since in this model the demand for money is necessarily a decreasing function of the expected rate of inflation, α is necessarily positive so that a is between zero and one. The elasticity of the price level today with respect to its expected value tomorrow is less than one.

The OLG Model with Money

The third interpretation is as the equilibrium condition in the OLG model with money, also examined in chapter 4. The price level today is demanded by the young when old. Extending equation (1) to include uncertainty, we have

$$\frac{M_t}{P_t} = L \left(\frac{E[P_{t+1}|I_t] - P_t}{P_t} \right).$$

The left-hand side is the real money stock held by the old. The right-hand side is the demand for real balances, which is a function of the expected rate of return on money (the return is $(E[P_{t+1}|I_t] - P_t)/P_t$). Taking a logarithm of both sides, and denoting by lowercase letters, using the approximation $(E[p_{t+1}|I_t] - p_t) = (E[P_{t+1}|I_t] - P_t)/P_t$, and ignoring

$$m_t - p_t = -\alpha(E[p_{t+1}|I_t] - p_t).$$

Reorganizing implies that

$$p_t = aE[p_{t+1}|I_t] + (1-a)m_t,$$

where

$$a \equiv \frac{\alpha}{1+\alpha}.$$

This is similar to the equilibrium condition in the Cagan model. The important difference: $L(\cdot)$ is a decreasing function of the rate of return on money. If the substitution effect dominates the income effect, a decrease in inflation is to decrease saving and one as in the Cagan model. If α is not too large, a can be negative. If α is greater than one in absolute value, a is greater than one in absolute value. It is also possible that, in this model, the price level today is greater than its expected value is greater than one.

In the first two examples a is between zero and one. In the third example a may be greater than one in absolute value. The solutions are very different in the third case. We examine the case $|a| > 1$.

The OLG Model with Money

The third interpretation is as a loglinear approximation to the OLG model with money, also examined in the previous chapter. In that model money is demanded by the young who buy it so as to exchange it against goods when old. Extending equation (3) in the previous chapter to allow for uncertainty, we have

$$\frac{M_t}{P_t} = L\left(\frac{E[P_{t+1}|I_t] - P_t}{P_t}\right).$$

The left-hand side is the real supply of money, supplied inelastically by the old. The right-hand side is the demand for money by the young, which is a function of the expected rate of inflation (the negative of the rate of return on money). Taking a loglinear approximation, denoting logarithms by lowercase letters, using as before the approximation $E[p_{t+1}|I_t] - p_t = (E[P_{t+1}|I_t] - P_t)/P_t$, and ignoring an unimportant constant term, we get

$$m_t - p_t = -\alpha(E[p_{t+1}|I_t] - p_t).$$

Reorganizing implies that

$$p_t = aE[p_{t+1}|I_t] + (1 - a)m_t,$$

where

$$a \equiv \frac{\alpha}{1 + \alpha}.$$

This is similar to the equation derived for the Cagan model, with one important difference: $L(\cdot)$ is now a savings function and its elasticity with respect to the rate of return is, as we saw in chapter 3, ambiguous in sign. If the substitution effect dominates, the effect of an increase in expected inflation is to decrease saving, so that α is positive and a is between zero and one as in the Cagan model. However, if the income effect dominates, α can be negative. If α is not only negative but also less than minus one half, a is greater than one in absolute value. Thus, we cannot exclude a priori the possibility that, in this model, the elasticity of the price level with respect to its expected value is greater than one in absolute value.

In the first two examples a is less than one in absolute value. In the third, a may be greater than one in absolute value. We will see shortly that solutions are very different depending on whether a is greater or less than one in absolute value. We examine first the case $|a| < 1$ and then later the case $|a| > 1$.

bubbles

ations with rational expectations: a convenient method in the revision.

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s and is asked how she would information, the answer must that on average the revision information set I_t , this implies, in

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and take expectations of both

infinity, the expectation of x in of x grows at a rate no sum to converge is that the $(1/a) - 1$. In the case where (1) this requires dividends not to here (1) has the interpretation

of money market equilibrium condition, such as in the last two examples, this requires that the logarithm of money not increase faster than at the rate $(1/a) - 1$. Note that any constant exponential growth rate of the level of money, which implies that the logarithm increases linearly, will satisfy this condition. We shall therefore assume in what follows that the first sum converges. Then, if

$$\lim_{T \rightarrow \infty} a^{T+1} E[y_{t+T+1} | I_t] = 0, \quad (3)$$

the following is a solution:

$$y_t = c \sum_{i=0}^{\infty} a^i E[x_{t+i} | I_t]. \quad (4)$$

Note that equation (4) satisfies condition (3), so it is indeed a solution to equation (1). It gives y as the discounted sum of future expected x 's. In our first example this implies that the price of a stock is the present discounted value of expected future dividends. In the other two examples it implies that the price level depends on the whole sequence of future expected money stocks, with decreasing weights.

If we are willing to specify an expected path for x , we can solve (4) explicitly for y . Equivalently, if we specify a process for x , we can solve for the process for y . We present two examples. The first is that of an increase in x from x_0 to x_T , announced at time t_0 to take place at time $T > t_0$. The path of y is then given by

$$\begin{aligned} y_t &= (1 - a)^{-1} c x_0, & \text{for } t < t_0, \\ &= (1 - a)^{-1} c x_0 + a^{T-t} (1 - a)^{-1} c (x_T - x_0), & \text{for } t_0 \leq t < T, \\ &= (1 - a)^{-1} c x_T, & \text{for } t \geq T. \end{aligned}$$

Consider the interpretation of this equation as deriving from the Cagan model. The path of the nominal money stock, the price level, and real money balances are drawn in figure 5.1.⁵ The equation shows that the announcement of a future increase in the money stock itself increases the price level today. Real money balances decrease, and the price level slowly increases to its new higher level over time. Inflation therefore takes place in advance of the increase in the money stock. This is because individuals look forward. They know that in the period before the money stock is increased, people will anticipate inflation and attempt to reduce their real money balances. In so doing, they will cause the price level to go up before the

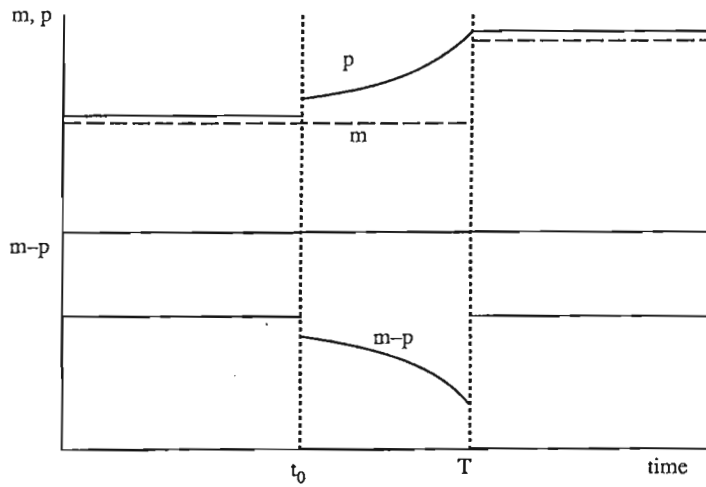


Figure 5.1
Effects of an anticipated increase in nominal money

money stock increases. Working this logic back to the present, current money holders attempt to reduce current real balances, therefore driving up the current price level.

Suppose that x instead follows the first-order stochastic process:

$$x_t - \bar{x} = \rho(x_{t-1} - \bar{x}) + e_t,$$

where e_t belongs to I_t and $E[e_t | I_{t-1}] = 0$. Then by using iterated expectations,

$$E[x_{t+i} | I_t] = \bar{x} + \rho^i(x_t - \bar{x}),$$

so that if ρ is less than $(1/a)$,

$$y_t - \bar{y} = \left(\frac{c}{1 - a\rho} \right) (x_t - \bar{x}),$$

with

$$\bar{y} = \left(\frac{c}{1 - a} \right) \bar{x}.$$

In the arbitrage example this implies that the price of the stock will be a function of current dividends only. It will, however, vary proportionally less than dividends as long as ρ , the degree of persistence, is less than unity. This is because the stock price is the present discounted value of future dividends,

and dividends are expected to grow at rate ρ . In the money examples, if ρ is less than unity, money balances will be high when the money stock is low.

In the case of arbitrage it is as if the present discounted value of future dividends is high. This terminology has now been applied to the case of arbitrage. But, as we now show, the only solution to equation (4) is the fundamental solution.

The Set of Solutions: Bubbles

Although equation (4) is a stochastic difference equation, it was derived by imposing conditions on the solution that are too fast. When we relax this arbitrage condition, we get a set of solutions.

Let y_t^* denote the solution to equation (4) as

$$y_t = y_t^* + b_t.$$

We now examine the restriction that y_t to be also a solution to (1).

If $y_t = y_t^* + b_t$, then $E[y_{t+1} | I_t]$ in (1) implies that

$$y_t^* + b_t = aE[y_{t+1}^* | I_t] + aE[b_{t+1} | I_t].$$

By the definition of y_t^* in (4),

$$b_t = aE[b_{t+1} | I_t],$$

or equivalently,

$$E[b_{t+1} | I_t] = a^{-1}b_t.$$

Thus, for any b_t that satisfies this condition, $y_t = y_t^* + b_t$ is a solution to (1). Note that since a is less than unity,

$$\lim_{i \rightarrow \infty} E[b_{t+i} | I_t] = a^{-i}b_t = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

The following examples of bubbles are based on the popular notion of speculation. The fundamental solution, y_t^* ,

and dividends are expected to return to their normal value at rate $(1 - \rho)$. In the money examples, if ρ is less than one, this solution implies that real money balances will be high when the money stock is high and low when the money stock is low.

In the case of arbitrage it is natural to call the solution that gives the price as the present discounted value of dividends the "fundamental" solution. This terminology has now become standard, even in contexts other than arbitrage. But, as we now show, the fundamental solution is far from being the only solution to equation (1).

The Set of Solutions: Bubbles

Although equation (4) is a solution to (1), it is not the only solution. We derived it by imposing condition (3), that the expectation not explode too fast. When we relax this arbitrary condition, equation (1) admits many other solutions.

Let y_t^* denote the solution given by (4), and let us write any other solution as

$$y_t = y_t^* + b_t.$$

We now examine the restrictions that have to be imposed on b_t in order for y_t to be also a solution to (1).

If $y_t = y_t^* + b_t$, then $E[y_{t+1}|I_t] = E[y_{t+1}^*|I_t] + E[b_{t+1}|I_t]$. Replacing y_t and $E[y_{t+1}|I_t]$ in (1) implies that

$$y_t^* + b_t = aE[y_{t+1}^*|I_t] + aE[b_{t+1}|I_t] + cx_t.$$

By the definition of y_t^* in (4), this reduces to

$$b_t = aE[b_{t+1}|I_t], \quad (5)$$

or equivalently,

$$E[b_{t+1}|I_t] = a^{-1}b_t.$$

Thus, for any b_t that satisfies (5), $y_t = y_t^* + b_t$ is also a solution to (1). Note that since a is less than one, b_t explodes in expected value:

$$\lim_{i \rightarrow \infty} E[b_{t+i}|I_t] = a^{-i}b_t = \begin{cases} +\infty, & \text{if } b_t > 0, \\ -\infty, & \text{if } b_t < 0. \end{cases} \quad (6)$$

The following examples of b_t processes show that b_t embodies quite well the popular notion of speculative bubbles. For that reason, while y_t^* is called the fundamental solution, b_t is called a bubble.

An Ever-Expanding Bubble

In the first example b simply follows a time trend:

$$b_t = b_0 a^{-t}, \quad \text{for arbitrary } b_0.$$

Consider the interpretation of equation (1) as an arbitrage equation and assume for simplicity that dividends, and thus p^* , are constant. If b_t is a time trend, and b_0 is positive, the price of the stock will increase exponentially, though the dividends are constant. What happens is that individuals are ready to pay a higher price for the stock than the price corresponding to the present value of the dividends because they anticipate the price will rise further, resulting in capital gains that precisely offset the low dividend price ratio. This anticipation of ever-increasing prices is self-fulfilling and satisfies the arbitrage condition.

A Bursting Bubble

The ever-expanding bubble has to go on forever, so that it will eventually become very large. In the next example the bubble has a probability of bursting each period. Consider the following process for b_t :

$$\begin{aligned} b_{t+1} &= (aq)^{-1} b_t + e_{t+1}, & \text{with probability } q, \\ &= e_{t+1}, & \text{with probability } 1 - q, \end{aligned}$$

and

$$E[e_{t+1}|I_t] = 0.$$

This process satisfies (5). The bubble bursts with probability $1 - q$ each period and continues with probability q . If it bursts, it returns in expected value to zero. To compensate for the probability of a crash, the expected return, if it does not crash, is higher than in the previous example. The disturbance e allows bubbles to have additional noise and permits new bubbles to form after a bubble has crashed.

Note that e_t can be correlated with unexpected movements in any variable and still satisfy the condition that its conditional expectation be zero. Thus, if the market believes that unexpected sunspots affect the price, they will indeed do so.⁶ The example can be further refined to allow q to be stochastic and for q to be affected by other variables. These modifications provide good accounts of the suggestive informal descriptions of speculative bubbles.⁷

Eliminating Bubbles

An issue that arises is whether, in deriving these solutions, we have not ignored conditions other than (1) that must also be satisfied by a solution.

For instance, in the case of the bubble becomes too large for the economy. There is, as we shall see, a condition that must be satisfied for bubbles to be ruled out. It is possible to rule out some of

Consider, for example, the case of an arbitrage and y gives the price of the asset. If y is positive, then its price cannot be negative. If b is a bubble, then its price cannot be negative. If b is a bubble, then its price cannot be negative. If b is a bubble, then its price cannot be negative.

However, no such simple condition can rule out bubbles. But other conditions can. If y is the price of a physical asset with an elastic supply, possibly at a constant price, then there cannot be positive bubbles. If the price goes to infinity and a perfect substitute is available, which would also go to minus infinity, then there cannot be any bubbles.

If y is the price of a share of a company that issues more shares when there is a bubble, then the bubble does not affect the price, but the price is in the interests of the initial shareholders. However, it is not in the interests of the new shareholders. Thus, if there is an ever-increasing supply of shares, then the likelihood of a bubble is zero.

Thus one would generally expect to find bubbles in assets whose fundamental value is not known, such as in the case of stocks.

If y is subject to a terminal condition, then the bubble must be equal to this value at the end of the period. Working backward in time, one can show that there cannot be bubbles. There are no perpetuities (or "consols").

We have listed here only a few conditions that can be used to eliminate the possibility of bubbles. We have not examined whether and when the elimination of bubbles is possible.

For instance, in the case of the ever-expanding bubble perhaps the value of the bubble becomes too large to be consistent with the finiteness of the economy. There is, as we shall see in this chapter, no general conclusion that bubbles can always be ruled out, but often there are conditions that make it possible to rule out some of the solutions.⁸

Consider, for example, the case where equation (1) is derived from arbitrage and y gives the price of an asset. If the asset can be freely disposed of, then its price cannot be negative. This in turn implies that there cannot be negative bubbles. If b was negative, then, by (6), the expectation of b in the far future would go to minus infinity. Thus the expectation of the price would also go to minus infinity, which is impossible.⁹ Thus b cannot be negative.

However, no such simple argument allows us to eliminate positive bubbles. But other conditions may be present that rule out positive bubbles. If y is the price of a physical asset and if a substitute is available in infinitely elastic supply, possibly at a very high price (think of oil and solar energy), then there cannot be positive bubbles. If b is positive, then the expected price goes to infinity and consequently exceeds the price at which the substitute is available, which is impossible. Thus with free disposal and the existence of a perfect substitute in infinitely elastic supply at some price, there cannot be any bubbles at all.¹⁰

If y is the price of a share, the question arises of whether firms will issue more shares when there is a bubble on share prices. If issuing more shares does not affect the bubble, for example, does not make the bubble crash, it is in the interests of the initial shareholders to issue more shares and invest the proceeds. However, it seems unlikely that the markets would absorb an ever-increasing supply of an asset at an unchanging price. This decreases the likelihood of a bubble on an easily reproducible asset.¹¹

Thus one would generally expect bubbles when fundamentals are difficult to ascertain, such as in the gold, art, or foreign exchange markets, rather than on assets whose fundamentals are clearly defined, such as blue chip stocks.

If y is subject to a terminal condition at some future time, then since y must be equal to this value at the terminal time, b must then be equal to zero. Working backward in time, b must be equal to zero always, and there cannot be bubbles. There therefore cannot be bubbles on bonds, except on perpetuities (or "consols").

We have listed here only partial equilibrium arguments that can be used to eliminate the possibility of bubbles. In sections 5.2 and 5.3 we will examine whether and when general equilibrium considerations also lead to the elimination of bubbles.

Solutions When $|a| > 1$: Indeterminacies

We have until now considered the case where $|a|$ was less than one and concluded that there was an infinity of solutions. If, however, we were ready to impose a nonexplosion condition, we would be left with only one solution—the fundamental solution. We turn now to the case $|a| > 1$, which, as we showed earlier, could be consistent with equation (1) interpreted as the equilibrium condition of an OLG model with money when the income effect is sufficiently strong.

This radically changes the nature of the results. Now the fundamental solution is no longer well defined. More precisely, the sum in (4) is unlikely to converge in general. And there is an infinity of bubbles, which are now stable rather than exploding. For example, suppose x is identically equal to one for all t . Then the set of solutions is given by

$$y_t = (1 - a)^{-1}c + b_t,$$

where

$$b_t = a^{-1}b_{t-1} + e_t, \quad E[e_t|I_{t-1}] = 0. \quad (7)$$

This implies, in particular, that if we make e identically equal to zero, then y will converge to $(1 - a)^{-1}c$ for any initial value of y_0 . Without a more detailed specification, it is difficult to think of reasons why the economy will (or economists analyzing the model should) choose one solution over another. Various criteria have been offered to choose among solutions, but none of them is very convincing.¹² The multiplicity of solutions is definitely more perplexing in this case. We will return to this, as well as to related issues, in section 5.4.

Extensions

Higher-Dimensional Systems

The behavior of y in (1) depends on whether $|a|$ is less or greater than one. The more likely case is that in which $|a|$ is less than one. We now examine how this condition extends to higher-dimensional systems and whether it is likely to be satisfied.

Returning to equation (1), ignoring uncertainty and expectations, the condition $|a| < 1$ can be stated as the condition that the difference equation that gives y_{t+1} as a function of y_t should be *unstable* or have a root $1/a$ that is strictly greater than one in absolute value. In this case, for a given sequence of x , there is a unique value of y^* for which y does not explode.

This condition generalizes the equation system.¹³ Suppose at time t , and m variables (sometimes with optimal control problem) are considered. Then the system must have one root between -1 and 1 in order to have a unique solution. Equivalently, it must have the

All the dynamic systems exhibit a point property. When we assume a path, we were in effect choosing the value of the variable fundamental solution and a positive root of the system.

The fact that all the system point stable and thus satisfies an indication that this condition holds to this point in the conclusion.

Nonlinear Dynamics

We end this section with a discussion of nonlinear systems. Indeed, most at least under uncertainty. It shows that even if a nonlinear system around a steady state, it may

The expectations difference

$$y_t^2 = \left(\frac{1}{2}\right) E[y_{t+1}|y_t], \quad y_t.$$

The value $y = 1/2$ is a solution around $y = 1/2$, then dy_t/dy linear, $y = 1/2$ would be the case that the following is also a solution

$$y_{t+1} = y_t^2, \quad \text{with probability } \frac{1}{2},$$

$$= \frac{1}{2}, \quad \text{with probability } \frac{1}{2}.$$

This condition generalizes if the economy is characterized by a difference equation system.¹³ Suppose there are n predetermined, state, variables at time t , and m variables (sometimes called "jumping" variables, or by analogy with optimal control problems, "costate" variables) that are not predetermined. Then the system must have exactly m roots outside the unit circle in order to have a unique nonexploding solution. For example, if we consider a model of money and capital in which there is one predetermined variable, capital, and one jumping variable, the price level, the system must have one root between -1 and $+1$ and one root outside that range. Equivalently, it must have the saddle point property.¹⁴

All the dynamic systems examined in the previous chapters had the saddle point property. When we assumed that the economy chose the saddle point path, we were in effect choosing the fundamental solution. On any other path the value of the variables could have been expressed as the sum of the fundamental solution and a bubble growing at the rate determined by the positive root of the system.

The fact that all the systems we have looked at in this chapter were saddle point stable and thus satisfied the extension of the condition $|a| < 1$ is an indication that this condition is often likely to be satisfied. We will return to this point in the conclusion of this chapter.

Nonlinear Dynamics

We end this section with a caveat. Thus far we have limited ourselves to linear systems. Indeed, most of what we know is limited to such systems, at least under uncertainty. The following example, from Azariadis (1981), shows that even if a nonlinear system satisfies locally the condition $|a| < 1$ around a steady state, it may have more than one nonexploding solution.

The expectations difference equation is

$$y_t^2 = \left(\frac{1}{2}\right) E[y_{t+1}|y_t], \quad y_t \in [0, 1].$$

The value $y = 1/2$ is a solution to the equation. If we linearize the system around $y = 1/2$, then $dy_t/dE[y_{t+1}|y_t] = 1/2$ so that, if the system were linear, $y = 1/2$ would be the only nonexploding solution. Note, however, that the following is also a solution:

$$y_{t+1} = y_t^2, \quad \text{with probability } q_t = \frac{1 - 4y_t^2}{1 - 2y_t^2},$$

$$= \frac{1}{2}, \quad \text{with probability } 1 - q_t.$$

By construction, q_t is always between zero and one. So the system has at least two nonexploding solutions. The first is $y = 1/2$. In the second, y follows stochastic cycles, although there is no intrinsic uncertainty. We will return to issues related to nonlinearity in section 5.4.

5.2 Bubbles on Assets in General Equilibrium

Whether there can be bubbles on real assets in general equilibrium depends on whether individuals have finite or infinite horizons and, if they have finite horizons, on whether the economy is dynamically efficient. After showing the conditions under which bubbles on real assets can exist, we draw parallels between results derived here and various results obtained in chapters 2 and 3. We end the section by discussing the empirical evidence on the presence or absence of bubbles.

The Case of Infinite Horizons

Bubbles are not unlike Ponzi games; assets are bought only on the anticipation that they can be resold at a higher price to somebody else who will buy them for the same reason. It is therefore not surprising that bubbles cannot arise when there is a finite number of individuals who have infinite horizons.

The proof of this very general proposition was given by Tirole (1982). The logic is as follows. Suppose that there is a finite number of infinitely lived individuals. The asset yields dividends or services every period. If it yields services, they can be rented out to the person who values them most that period. This implies that the fundamental value, p_t^* , is the same for all individuals at all times. Finally, the services or dividends do not depend on the price. This excludes money, whose services depend on the price level.

Suppose that, under these assumptions, there is a negative bubble, with $p_t < p_t^*$. Then all individuals will want to buy and keep the asset forever. Purchasing the asset costs p_t , holding the asset forever, and renting it out every period yields p_t^* in present value. There would therefore be excess demand for the asset, and $p_t < p_t^*$ cannot be an equilibrium.

Suppose, alternatively, that there is a positive bubble so that $p_t > p_t^*$. If short selling is allowed, an argument symmetrical to the preceding one implies excess supply and rules out positive bubbles. But it is possible to exclude positive bubbles even without short selling. If p exceeds p^* , an individual who buys the asset must do so with the anticipation of eventually realizing his capital gain by selling the asset in the future. Let t_i be the date

by which individual i intends to sell. By T all individuals plan to sell, and nobody plans to be holding the asset in equilibrium.

This argument allows us to conclude that in a general equilibrium, it must be the case that the new generation is born every

Finite Horizons

The following argument is used to show that bubbles in general equilibrium cannot exist if the economy is dynamically efficient. This suggests that at some point relative to the economy, an argument is not quite right (if the interest rate is less than the marginal product of capital, the economy will be inefficient), the economy will not be dynamically efficient. That the above argument does not rule out bubbles and extend this intuitive conclusion to the case of finite horizons. Tirole (1985) and Weil (1985) presented in chapter 3.

Recall that in the Diamond model, individuals are consuming and saving in each period. The return on saving is the marginal product of capital. In the steady state, the accumulation are given by

$$k_{t+1} = (1+n)^{-1} \{s[w(k_t), r], r\}$$

Or by expressing savings in terms of capital, we have

$$k_{t+1} = (1+n)^{-1} s(k_t, k_{t+1}),$$

The capital stock at time t , which depends on the savings r . Since w_t depends directly as a function of k_t , this guarantees that the equilibrium is nonoscillating, namely, that

y of stable solutions, but
 e to any of these solutions.
 s of yet. But the preceding
 the literature. Where the
 adamental solution appears
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ne has been to explore the
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for each to arise are not identical. It has been argued that chaos offers an alternative to the now prevalent formalization of business cycles as resulting from the dynamic effects of stochastic shocks through propagation mechanisms (a view we will develop at length in the rest of the book). It has been argued that chaos, which can be generated by simple deterministic systems, offers a less ad hoc explanation of fluctuations than one based on unexplained shocks. Some work has examined empirically whether the behavior of economic variables is better explained by chaotic or linear stochastic processes (see Brock 1986), but without clear conclusions as of yet.

Another reaction, associated with Grandmont, has been to concentrate on cycles and reduce the dimension of indeterminacy. As we noted earlier, Grandmont has shown that under some additional assumptions there may exist a unique unstable cycle that is stable under simple learning rules. Grandmont argues that such deterministic cycles provide an alternative to the linear stochastic process view of cycles. If he is correct, policy can have very drastic effects on the dynamics of the economy by changing the specific form of the nonlinearity.

We are not at this stage convinced by either of these last two approaches. Although the nonlinearity needed to obtain multiple stable equilibria, sunspots, cycles, or chaos is consistent with optimizing behavior, the conditions for such equilibria still appear unlikely. In the models considered in this chapter, for example, they require implausibly large income effects.³⁶ Thus, for the time being, though we find the phenomena analyzed in this chapter both interesting and disturbing, we are willing to proceed on the working assumption that the conditions needed to generate stable multiplicities of equilibria are not met in practice.

Appendix: A Tool Kit of Solutions to Linear Expectational Difference Equations

In section 5.1 we solved a difference equation with rational expectations by using the method of repeated substitution. That method is convenient in simple cases but rapidly becomes unwieldy. In this appendix we present the two methods that are most often used to solve such difference equations analytically. We make no attempt at generality or rigor. Surveys by Taylor (1985) on methods of solution in small models and by Blanchard (1985) on analytical and numerical methods of solution in large models give both a more exhaustive presentation and further references.

We will solve the following equation:

$$p_t = a_0 E[p_{t+1}|t] + a_1 p_{t-1} + a_2 E[p_t|t-1] + a_3 m_t + e_t. \quad (A1)$$

We use the notation $E[p_{t+i}|t-j]$ to denote the rational expectation of p_{t+i} based on information available at time $t-j$. The information set is assumed to include at least current and lagged values of m , e , and p . The variable p is endogenous, the variable m is exogenous, and e is a stochastic disturbance. For the moment we do not need to specify the processes followed by either m or e .

Such an equation, in which a variable depends both on itself lagged and on past expectations of current values and current expectations of future values of itself, is fairly typical. One interpretation is that p is the logarithm of the price level and m the logarithm of the nominal money stock. In this case one may want to impose the additional homogeneity restriction $a_0 + a_1 + a_2 + a_3 = 1$.

It is convenient to define

$$x_t \equiv a_3 m_t + e_t \quad (\text{A2})$$

so that

$$p_t = a_0 E[p_{t+1}|t] + a_1 p_{t-1} + a_2 E[p_t|t-1] + x_t. \quad (\text{A1}')$$

The Method of Undetermined Coefficients

The method of undetermined coefficients consists of guessing the form of the solution and then solving for the coefficients. The guess may come from experience or from attempts at repeated substitution. An educated guess here is that p will depend on itself lagged once, and on current and once-lagged expectations of once-lagged current and future values of x :

$$p_t = \lambda p_{t-1} + \sum_{i=0}^{\infty} c_i E[x_{t+i}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1]. \quad (\text{A3})$$

The method is to find values of λ , c_i , and d_i such that (A3) is a solution to (A1'). The first step is to derive $E[p_t|t-1]$ and $E[p_{t+1}|t]$ implied by (A3). By taking expectations of both sides of (A3), both at time t and $t+1$, and using the law of iterated expectations, we get

$$E[p_t|t-1] = \lambda p_{t-1} + \sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1], \quad (\text{A4})$$

$$E[p_{t+1}|t] = \lambda p_t + \sum_{i=0}^{\infty} c_i E[x_{t+i+1}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t]. \quad (\text{A5})$$

Now, by substituting (A4) and (A5) into (A1'), we get

$$\begin{aligned} p_t = & a_0 \left(\lambda p_t + \sum_{i=0}^{\infty} c_i E[x_{t+i+1}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t] \right) + a_1 p_{t-1} \\ & + a_2 \left(\lambda p_{t-1} + \sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] + \sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1] \right) + x_t, \end{aligned} \quad (\text{A6})$$

or

$$\begin{aligned} p_t = & (1 - a_0 \lambda)^{-1} \left\{ a_0 \left(\sum_{i=0}^{\infty} c_i E[x_{t+i+1}|t] \right) \right. \\ & \left. + a_2 \left(\sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] \right) \right\} \\ & + (1 - a_0 \lambda)^{-1} \left\{ a_1 p_{t-1} + a_2 \left(\sum_{i=0}^{\infty} d_i E[x_{t+i-1}|t-1] \right) \right\} \end{aligned}$$

For (A3) to be a solution we equate the coefficients for

$$\lambda = (1 - a_0 \lambda)^{-1} (a_1 + a_2 \lambda)$$

or, equivalently,

$$a_0 \lambda^2 + (a_2 - 1) \lambda + a_1 = 0.$$

There will generally be two solutions. The one with the smaller absolute value will satisfy the extension of the saddle point property.³⁷ Based on p_{t-1} , we are in effect choosing

Suppose for example that $a_0, a_1, a_2, a_0 + a_1 + a_2 < 1$. Then from

$$\Psi(\lambda) = a_0 \lambda^2 + (a_2 - 1) \lambda + a_1$$

it follows that $\Psi(0) > 0$, $\Psi(1) < 0$, and one root is smaller than one and the other is larger than one.

We will assume that the root smaller than one is satisfied and proceed. Let λ_1 be the root smaller than one, and let λ_2 be the other root. We note that $\lambda_1 + \lambda_2 = (1 - a_2)/a_0$. We now derive the equations for these coefficients. We have

$$\begin{aligned} x_t: & c_0 = (1 - a_0 \lambda_1) \\ E[x_{t+1}|t]: & c_1 = (1 - a_0 \lambda_1) \\ E[x_{t+i}|t]: & c_i = (1 - a_0 \lambda_1) \\ x_{t-1}: & d_0 = (1 - a_0 \lambda_1) \\ E[x_t|t-1]: & d_1 = (1 - a_0 \lambda_1) \\ E[x_{t+i}|t-1]: & d_{i+1} = (1 - a_0 \lambda_1) \end{aligned}$$

Noting that $d_0 = 0$, and with

$$c_0 = (1 - a_0 \lambda_1)^{-1},$$

$$c_i = \left(\frac{\lambda_1 a_0}{a_1} \right) c_{i-1} = \lambda_1^{-1} c_{i-1},$$

nal expectation of p_{t+i} based on information set is assumed to include at time t . For the moment we do not consider e_t .

on itself lagged and on past values of future values of itself, is a function of the price level and m_t . One may want to impose $a_3 = 1$.

(A2)

(A1')

of guessing the form of the solution may come from experience. A good guess here is that p will be a function of once-lagged expectations of

(A3)

at (A3) is a solution to (A1'). By taking the limit as $t \rightarrow \infty$, and using the law of

(A4)

(A5)

get

 p_{t-1}

(A6)

$$p_t = (1 - a_0\lambda)^{-1} \left\{ a_0 \left(\sum_{i=0}^{\infty} c_i E[x_{t+i}|t] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t] \right) + (a_1 + a_2\lambda)p_{t-1} + a_2 \left(\sum_{i=0}^{\infty} c_i E[x_{t+i}|t-1] + \sum_{i=0}^{\infty} d_i E[x_{t+i}|t-1] \right) + x_t \right\}. \quad (A6')$$

For (A3) to be a solution to (A1), (A6') and (A3) must be identical. Thus we equate the coefficients for each variable. Starting with the coefficient on p_{t-1} ,

$$\lambda = (1 - a_0\lambda)^{-1}(a_1 + a_2\lambda)$$

or, equivalently,

$$a_0\lambda^2 + (a_2 - 1)\lambda + a_1 = 0. \quad (A7)$$

There will generally be two solutions for λ in (A7). If the model we are dealing with satisfies the extension of the condition $|a| < 1$ in section 5.1, one of the roots will be smaller than one in absolute value and the other larger than one. It will have the saddle point property.³⁷ By choosing the smaller of the roots as the coefficient on p_{t-1} , we are in effect choosing the stable, nonexploding solution.

Suppose for example that the equation gives the price level as a function of money and that a_0, a_1, a_2 , and a_3 are all positive and sum to one, so that $a_0 + a_1 + a_2 < 1$. Then from the definition

$$\Psi(\lambda) = a_0\lambda^2 + (a_2 - 1)\lambda + a_1,$$

it follows that $\Psi(0) > 0$, $\Psi(1) < 0$, and $\Psi(\infty) > 0$ so that one root is between zero and one and the other is larger than one.

We will assume that the condition for the existence of a unique stable solution is satisfied and proceed. Let λ_1 be the root that is less than one in absolute value, and let λ_2 be the other. Note, for later use, that $\lambda_1\lambda_2 = a_1/a_0$ and that $\lambda_1 + \lambda_2 = (1 - a_2)/a_0$. We now solve for c_i and d_i , using the assumption that λ in the equations for these coefficients is λ_1 , the root that implies the stable solution. We have

$$x_t: \quad c_0 = (1 - a_0\lambda_1)^{-1}[1 + a_0d_0],$$

$$E[x_{t+1}|t]: \quad c_1 = (1 - a_0\lambda_1)^{-1}[a_0(c_0 + d_1)],$$

$$E[x_{t+i}|t]: \quad c_i = (1 - a_0\lambda_1)^{-1}[a_0(c_{i-1} + d_i)],$$

$$x_{t-1}: \quad d_0 = (1 - a_0\lambda_1)^{-1}[a_2d_0],$$

$$E[x_t|t-1]: \quad d_1 = (1 - a_0\lambda_1)^{-1}[a_2(c_0 + d_1)],$$

$$E[x_{t+i}|t-1]: \quad d_{i+1} = (1 - a_0\lambda_1)^{-1}[a_2(c_i + d_{i+1})].$$

Noting that $d_0 = 0$, and with some manipulation, we get

$$c_0 = (1 - a_0\lambda_1)^{-1},$$

$$c_i = \left(\frac{\lambda_1 a_0}{a_1} \right) c_{i-1} = \lambda_2^{-1} c_{i-1}, \quad \text{for } i = 1, \dots,$$

$$d_i = \left(\frac{a_2}{a_0}\right) c_i, \quad \text{for } i = 1, \dots$$

Thus, if λ_2 is larger than one, the sequences c_i and d_i converge to zero as i gets large.

We have solved for p_t as a function of lagged p_t and past and current expectations of current and future x . Sometimes the process for x is specified. Then we would want to solve directly for p as a function of observable variables. There are two procedures. One is to derive the solution for p as a function of expectations of x as we have just done, and then to solve for expectations of x as a function of observable variables in (A3). The other is to use the method of undetermined coefficients to solve directly for p_t as a function of observable variables.

Suppose, for example, that e is identically equal to zero and that m (and therefore x) follows

$$m_t = \rho m_{t-1} + v_t,$$

where v_t is white noise. We would then guess that the solution is of the form

$$p_t = \lambda p_{t-1} + c m_t + d m_{t-1} \quad (\text{A3}')$$

and solve for λ , c , and d as above.

Despite its widespread use the method of undetermined coefficients suffers from a few handicaps. First, the initial guess may fail to include a solution or may inadvertently discard other solutions. Second, the method reveals only indirectly whether the model has the desirable saddle point property. Third, like repeated substitution, it can become somewhat unwieldy.

Factorization

The method of factorization was introduced to economics by Sargent (see Sargent 1979 for a detailed presentation). It is best seen as a convenient shortcut to the method of z -transforms (see Whiteman 1983).

The method proceeds in three steps.

The *first* [which is needed only if the equation includes both current and lagged expectations, if a_2 is different from zero in equation (A1')] is to take expectations on both sides of (A1') conditional on the farthest lagged information set in (A1). In (A1') we take expectations based on information at time $t-1$. This implies

$$E[p_t|t-1] = a_0 E[p_{t+1}|t-1] + a_1 p_{t-1} + a_2 E[p_t|t-1] + E[x_t|t-1], \quad (\text{A1}'')$$

or

$$(1 - a_2)E[p_t|t-1] = a_0 E[p_{t+1}|t-1] + a_1 p_{t-1} + E[x_t|t-1].$$

In the *second* step we factor equation (A1'') to express $E[p_t|t-1]$ as a lagged function of itself, and of expectations of current and future values of x , $E[x_{t+i}|t-1]$, $i \geq 0$. To do so, we introduce the lag operator, L , which operates on the time subscript of a variable (not on the time at which the expectation of that variable is held):

$$LE[p_{t+i}|t-1] = E[p_{t+i-1}|t-1]$$

so that, in particular,

$$LE[p_{t+1}|t] = E[p_t|t] = p_t.$$

For convenience, we also intro

$$FE[p_{t+i}|t-1] = E[p_{t+i+1}|t].$$

Using the definitions of F a

$$[-a_0 F + (1 - a_2) - a_1 L]E[p_t|t-1]$$

The next step is to factor t (A8) as

$$\left[F^2 - \left(\frac{1 - a_2}{a_0} \right) F + \left(\frac{a_1}{a_0} \right) \right] L$$

We can factor the polyno (F - λ_2), where

$$\lambda_1 + \lambda_2 = \frac{1 - a_2}{a_0} \quad \text{and}$$

Note that λ_1 and λ_2 are undetermined coefficients. Th λ_1 is less than one in absolute

We can rewrite (A9) as

$$(F - \lambda_1)(F - \lambda_2)LE[p_t|t-1]$$

or

$$(1 - \lambda_1 L)E[p_t|t-1] = \left(\frac{1}{a_0 \lambda_2} \right)$$

Since $|\lambda_2^{-1}| < 1$, we can expa

$$E[p_t|t-1] = \lambda_1 p_{t-1} + \left(\frac{1}{a_0 \lambda_2} \right)$$

Equation (A12) gives the exp itself. [Note again that if a_2 w the first step, so (A12) wou expectations of current and fu be no need for the third step.

The *third* step is to derive an expression for $E[p_{t+1}|t]$ a This gives

$$LE[p_{t+i}|t-1] = E[p_{t+i-1}|t-1],$$

so that, in particular,

$$LE[p_{t+1}|t] = E[p_t|t] = p_t.$$

For convenience, we also introduce the forward operator, $F = L^{-1}$. Thus

$$FE[p_{t+i}|t-1] = E[p_{t+i+1}|t].$$

Using the definitions of F and L , we can rewrite (A1'') as

$$[-a_0 F + (1 - a_2) - a_1 L]E[p_t|t-1] = E[x_t|t-1]. \quad (A8)$$

The next step is to factor the polynomial in parentheses. To do so, we rewrite (A8) as

$$\left[F^2 - \left(\frac{1-a_2}{a_0} \right) F + \left(\frac{a_1}{a_0} \right) \right] LE[p_t|t-1] = \left(\frac{-1}{a_0} \right) E[x_t|t-1]. \quad (A9)$$

We can factor the polynomial $\{F^2 - [(1-a_2)/a_0]F + (a_1/a_0)\}$ as $(F - \lambda_1) \times (F - \lambda_2)$, where

$$\lambda_1 + \lambda_2 = \frac{1-a_2}{a_0} \quad \text{and} \quad \lambda_1 \lambda_2 = \frac{a_1}{a_0}. \quad (A10)$$

Note that λ_1 and λ_2 are the same as λ_1 and λ_2 derived in the method of undetermined coefficients. Thus the same discussion applies, and we assume that λ_1 is less than one in absolute value and that λ_2 is larger than one in absolute value.

We can rewrite (A9) as

$$(F - \lambda_1)(F - \lambda_2)LE[p_t|t-1] = \left(\frac{-1}{a_0} \right) E[x_t|t-1],$$

or

$$(1 - \lambda_1 L)E[p_t|t-1] = \left(\frac{1}{a_0 \lambda_2} \right) (1 - \lambda_2^{-1} F)^{-1} E[x_t|t-1]. \quad (A11)$$

Since $|\lambda_2^{-1}| < 1$, we can expand $(1 - \lambda_2^{-1} F)^{-1}$ as $\sum_{i=0}^{\infty} \lambda_2^{-i} F^i$ to get

$$E[p_t|t-1] = \lambda_1 p_{t-1} + \left(\frac{1}{a_0 \lambda_2} \right) \sum_{i=0}^{\infty} \lambda_2^{-i} E[x_{t+i}|t-1]. \quad (A12)$$

Equation (A12) gives the expectation of p_t as of $t-1$. The last step is to derive p_t itself. [Note again that if a_2 were equal to zero, we would not have gone through the first step, so (A12) would give p_t as a function of p_{t-1} as well as current expectations of current and future x . This would be the solution, and there would be no need for the third step.]

The *third* step is to derive the solution for p_t . To do so, we use (A12) to get an expression for $E[p_{t+1}|t]$ and replace both $E[p_{t+1}|t]$ and $E[p_t|t-1]$ in (A1'). This gives

$$p_t = a_0 \lambda_1 p_t + \left(\frac{1}{\lambda_2} \right) \sum_{i=0}^{\infty} \lambda_2^{-i} E[x_{t+i+1}|t] + (a_1 + a_2 \lambda_1) p_{t-1} \\ + \left(\frac{a_2}{a_0 \lambda_2} \right) \sum_{i=0}^{\infty} \lambda_2^{-i} E[x_{t+i}|t-1] + x_t.$$

Reorganizing, and using the fact that, from the definition of λ_1 , $(a_1 + a_2 \lambda_1)/(1 - a_0 \lambda_1) = \lambda_1$, gives

$$p_t = \lambda_1 p_{t-1} + \left(\frac{1}{1 - a_0 \lambda_1} \right) \sum_{i=0}^{\infty} \lambda_2^{-i} E[x_{t+i}|t] \\ + \left(\frac{1}{1 - a_0 \lambda_1} \right) \left(\frac{a_2}{a_0} \right) \sum_{i=0}^{\infty} \lambda_2^{-i-1} E[x_{t+i}|t-1]. \quad (\text{A13})$$

This solution is the same as that obtained by the method of undetermined coefficients.

Problems

1. Assume that the simple linear difference equation of section 5.1 is derived from an arbitrage equation between stocks and bonds and that the real interest rate is constant. Assume that dividends follow the stochastic process

$$d_t = (1 - \rho)d_0 + \rho d_{t-1} + v_t, \quad d_0 > 0, 0 < \rho < 1,$$

where v_t is white noise. The variance of v_t is σ^2 .

(a) Solve for the current price of the stock as a function of current and past dividends. Explain.

(b) Calculate the unconditional variance of the stock price as a function of σ^2 and other relevant parameters.

(c) How does the variance of the stock price change as ρ increases?

2. In section 5.2 we showed that if the money stock follows a first-order autoregressive process with $c < 1$, then in the Cagan model real balances will be high when the money stock is high and low when the money stock is low.

(a) Give the economic intuition behind this result.

(b) Suppose that the *growth rate* of money follows a stable first-order autoregressive process. Solve for the process for the price level.

(c) Does the same characterization hold with the addition of the words "relative to trend" following high and low?

3. Land and bubbles.

In an overlapping generations model in which people live for two periods, with the population growing at rate n and no production (all goods come from the exogenous endowment of the young), there is a given amount of land. The land has no productive use.

- (a) Can there be a bubble on land? Give your answer and the efficiency condition that land is valued in the real world.
 (b) Suppose now that the economy has only one unit of land. The production function is Cobb-Douglas, with capital. Can there be a steady state?
 (c) Can there be a bubble on land?

4. (a) Return to problem 1. Or, of the "ex post" price of the stock, the actual price.

(b) Suppose now that $\rho = 1$. What happens to the unconditional variance?

5. Suppose that, in the model, the instantaneous utility function

$$U(c, m) = a \ln(c) + b \ln(m).$$

(a) Could a self-generating hypothesis exist for families?

(b) In evaluating this possibility should be attached to the fact that growth did not become extremely high?

6. A learning problem in which the economy is solved in section 5.2. Information on current values of c and m . Answer, in particular, the question of fundamentals equilibrium.

Notes

1. Muth (1961) was the first to introduce "rational expectations." Until his article, and also arbitrary expectation formation, adaptive expectations (of which Muth's important paper (1960) Muth's rational expectations about a variable y would

2. In the classic rational expectations model, individuals do not all have the same information. Why policymakers may not be able to convey information to the public is the subject of chapter 10.

(a) Can there be a bubble on land in this economy? Discuss the relationship between your answer and the efficiency of equilibrium in the real world in view of the fact that land is valued in the real world.

(b) Suppose now that the economy becomes a production economy, that individuals' only endowment is one unit of labor in the first period of life and that the production function is Cobb-Douglas, with constant returns to scale in terms of land, labor, and capital. Can there be a steady state?

(c) Can there be a bubble on land in this economy in which land is a productive asset?

4. (a) Return to problem 1. On the assumption that $\rho < 1$, show that the variance of the "ex post" price of the stock (defined in section 5.2) exceeds the variance of the actual price.

(b) Suppose now that $\rho = 1$, with the dividend following a random walk. What happens to the unconditional variance of the price of the stock?

5. Suppose that, in the model of section 5.3, the representative family has an instantaneous utility function

$$U(c, m) = a \ln(c) + b \ln(m).$$

(a) Could a self-generating hyperinflation develop in an economy populated by such families?

(b) In evaluating this possibility, and the problem of multiple equilibria, what weight should be attached to the fact that there is no known hyperinflation in which money growth did not become extremely high?

6. A learning problem in which individuals do not know the current state of the economy is solved in section 5.5. Using that model, assume that individuals have information on current values of Y and v , and solve for the dynamics of the model. Answer, in particular, the question of whether the economy converges to the fundamentals equilibrium.

Notes

1. Muth (1961) was the first to use this assumption and the term "rational expectation." Until his article, and also for a long time after, researchers used plausible but arbitrary expectation formation mechanisms, the most popular being that of adaptive expectations (of which we saw an example in the last chapter). In another important paper (1960) Muth found the conditions under which adaptive expectations about a variable y would indeed be rational.

2. In the classic rational expectations macroeconomic article by Lucas (1973), individuals do not all have the same information set. Lucas showed in this article why policymakers may not be able to use the Phillips curve trade-off and also how prices convey information to market participants. We present this model in the next chapter.

3. The importance of this law for economics and finance was demonstrated by Samuelson (1965), who used it to show that future prices would follow a martingale.

4. Note that we are using here the assumption of no memory loss. The result does not go through without it.

5. The solution to the Cagan model under rational expectations was first obtained by Sargent and Wallace (1973).

6. Sunspots have become the generic example of a variable that affects the equilibrium only because individuals believe it does. We will see later other examples of extrinsic uncertainty potentially affecting the equilibrium. However, Jevons (1884) who introduced sunspots into economics believed that they mattered because they affected agricultural output.

7. Among the most famous historical episodes are the Dutch tulip mania (1634–1636) and the South Sea Bubble (1720). See Charles MacKay (1841) and Charles Kindleberger (1978) for accounts of these and other fascinating episodes.

8. These issues are discussed at greater length in Blanchard and Watson (1982) and in Fischer and Merton (1984).

9. Note that if the bubble is stochastic, the probability that the price will become negative may be very small. There is some evidence that individuals systematically ignore very small probabilities. This weakens the argument made here for eliminating bubbles as well as some of the arguments made later in the chapter.

10. The remark of the previous note applies here as well.

11. This is similar to the conclusion for a physical asset available in infinitely elastic supply.

12. Taylor (1977) proposed one such criterion.

13. Precise statements are given in Blanchard and Kahn (1981) and in Whiteman (1983).

14. There are corresponding conditions in differential equation systems. The general condition becomes that the system must have exactly m roots with positive real parts (see Buiter 1984). In the example given here the differential system in prices and capital should have one positive and one negative root.

15. In figure 5.2, all we needed to do was to plot the combinations of constant (b, k) which satisfied (10) and (11). Here, because we want to characterize the dynamics, we must first derive the loci of (k_t, b_t) along which $b_{t+1} = b_t$ and $k_{t+1} = k_t$ respectively. Hence the derivation of (13) and (14).

16. Care must be taken in using a phase diagram to analyze the dynamics of a difference equation system. The economy will not, as in the case of a differential equation system, move continuously along one of the trajectories, but rather it will jump from point to point on that trajectory. An equilibrium that appears stable on the phase diagram may be in fact unstable. The economy, though staying on the

path that converges to the equilibrium of increasing size. Thus we must be saddle point stable around E in the system linearized around each equilibrium.

17. What is important in the model depends on the price but that the path of prices (here, a random walk) is a martingale.

18. Shiller (1984) and Summers (1982) discuss long overvaluations or undervaluations of rational bubbles.

19. See Merton (1987) for a recent two-step volatility test that finds no inequality. He concludes that the market is efficient.

20. The analysis in this section is due to Obstfeld (1984), and Obstfeld and Rogoff (1983).

21. Despite the fact that most of the pathological properties of the model enter the utility function and the budget constraint as a vehicle for saving.

22. The case where σ is negative is ruled out by the transversality condition is discussed in Obstfeld and Rogoff (1983).

23. Obstfeld and Rogoff (1983) show that the form of the utility function is of the form

$$u(m) = \frac{m^{1-\gamma}}{1-\gamma};$$

this condition is satisfied if $\gamma > 0$.

24. Note that we saw in the text that the model can have a multiplicity of equilibria. Look at the dynamics of the model.

25. The model is a simplified version of Grandmont allows for endogenous money. We also draw in what follows.

26. In chapter 4 we considered the case of a bubble.

27. If there were population growth, bubbles would occur where the offer curve is tangent to the budget line where prices are falling at rate σ .

28. This construction is due to Obstfeld and Rogoff (1983).

29. This is an implication of Samuelson's law of iterated expectations.

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path that converges to the equilibrium, may oscillate back and forth in oscillations of increasing size. Thus we must check in this case whether the system is indeed saddle point stable around E and stable around A by computing the roots of the system linearized around each of the two equilibria. This check is left to the reader.

17. What is important in the Marsh-Merton example is not that the dividend depends on the price but that the particular dividend policy implies nonstationarity of prices (here, a random walk).

18. Shiller (1984) and Summers (1986) have pointed out that fads, if they lead to long overvaluations or undervaluations of the stock, may look very much like rational bubbles.

19. See Merton (1987) for a review of the evidence. West (1987) has constructed a two-step volatility test that first tests the arbitrage relation and then the variance inequality. He concludes that the rejection does not come from a failure of arbitrage.

20. The analysis in this section is based on Brock (1975), Calvo (1978), Gray (1982), Obstfeld (1984), and Obstfeld and Rogoff (1983).

21. Despite the fact that money is the only asset, the model has none of the pathological properties of the OLG model with money. This is because money enters the utility function and is used both (implicitly) for transaction services and as a vehicle for saving.

22. The case where σ is negative and hyperdeflation cannot be ruled out by the transversality condition is discussed by Brock (1975).

23. Obstfeld and Rogoff (1983) discuss this condition at greater length. If $u(m)$ is of the form

$$u(m) = \frac{m^{1-\gamma}}{1-\gamma};$$

this condition is satisfied if $\gamma > 1$.

24. Note that we saw in the third example of section 5.1 that a loglinear version of the model can have a multiplicity of convergent paths, when $\alpha < -1$. We now look at the dynamics of the model without linearization.

25. The model is a simplified version of the model used by Grandmont (1985). Grandmont allows for endogenous labor supply, but we take the endowments as given. We also draw in what follows on the survey by Woodford (1984).

26. In chapter 4 we considered the case where e_2 was equal to zero.

27. If there were population growth at rate n , the steady state equilibrium would occur where the offer curve intersects the line $m_{t+1} = (1+n)m_t$, that is, the line where prices are falling at rate $1+n$.

28. This construction is due to Azariadis and Guesnerie (1984).

29. This is an implication of Sarkovskii's theorem, presented by Grandmont (1983).

30. For a relatively simple presentation of the theory of periodic and aperiodic behavior of one-dimensional dynamic systems, see Grandmont (1983). The possibility of chaos in deterministic systems has been explored by various authors; see, for example, Day (1982, 1983).

31. See Guesnerie (1986).

32. The example comes from Azariadis (1981). The proof follows Woodford (1987).

33. The model is a simplified and slightly modified version of Evans (1985).

34. This assumption, which differs from that of section 5.1, makes the problem more interesting. The case where the information set includes current values of y and v is easier and is left to the reader.

35. Marcat and Sargent (1987) show, in the context of the Cagan hyperinflation model with two equilibria (studied at the end of chapter 4), that with least squares learning the economy converges to the low-inflation equilibrium.

36. Multiple stable equilibria may also emerge in a different class of models, models that allow for increasing returns and/or externalities in labor and goods markets. Multiplicity in those models does not rely on the presence of income effects. We present and discuss such models, and the likelihood of multiple equilibria, in chapter 8.

37. The intuition for this is as follows: Suppose that individuals have perfect foresight, so that equation (A1) is simply a difference equation in p_{t-1} , p_t , p_{t+1} , and x_t . Given p_{t-1} at time t , for p_t to be uniquely determined by the condition that the equation does not explode, the equation must have one root smaller and one root greater than 1 in absolute value. If both roots were, for example, smaller than one in absolute value, the difference equation would converge for any value of p_t . The system would have the type of multiplicity studied in section 5.4. [The specific condition for saddle point stability in systems such as (A1) is given by Blanchard 1985.] The roots of equation (A7) turn out to be the inverses of the roots of the difference equation obtained by assuming perfect foresight in (A1). Thus, for saddle point stability, they must also be such that one is smaller and one larger than 1 in absolute value.

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6

The next four chapters are about fluctuations in output, employment, and interest rates. Chapter 1. This chapter sets out the optimal decisions by firms and households to the case of uncertainty. Theories of aggregate fluctuations. The next two chapters start with a brief overview of the current state of economic theory.

The wealth of business cycles in the pre-Keynesian period. The *Great Depression*,² first published in 1933, is a landmark in the economics literature. The table of contents are The Phillips Curve, Theories, Changes in Cost, Interest Rates, Under-consumption Theories, and Theories.

Although abounding in purely theoretical, for early on it has long been noted, for example, that the business cycle is more cyclical than that of the economy. The accelerator mechanism. The output of investment goods. The business cycle facts was assembled by the Research sponsored projects. The book on cycles was published in 1946 by Arthur Burns and Mitchell. *Business Cycles*. Using a reference to the documented existence of a large number of price and output cycles.