Microeconomic Theory II

Preference, Utility, & Choice

Let X be the choice set, or list of potential alternatives.
Then we define the preference relation $x \succsim y \Leftrightarrow x$ at least as good as y.
The strict preference relation $x \succ y \Leftrightarrow x \succsim y$ but not $y \succsim x$.
And the indifference relation $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$.

We say that a preference relation ($\succsim$) is rational if it is:

i) complete: $\forall x, y \in X$ we have $x \succsim y$, $y \succsim x$, or both.

ii) transitive: $\forall x, y \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Properties: If $\succsim$ is rational, then

i) $\succ$ is irreflexive $\Leftrightarrow$ ($x \succ x$ never holds,)
and transitive $\Leftrightarrow$ ($x \succ y$ and $y \succ z \Rightarrow x \succ z$)

ii) $\sim$ is reflexive $\Leftrightarrow$ ($x \sim x \ \forall x$)
is transitive $\Leftrightarrow$ ($x \sim y$ and $y \sim z \Rightarrow x \sim z$)
is symmetric $\Leftrightarrow$ ($x \sim y \Leftrightarrow y \sim x$)

iii) if $x \succ y$ and $y \succsim z$, then $x \succ z$

A utility function assigns a numerical value to each $x \in X$ ranking these elements of X consistent w/ an individuals preferences.

Definition: A function $u: X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if $\forall x, y \in X$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$.

Proposition: A preference relation can be represented by a utility function only if it is rational.

Proof: To proove, show that if $\exists$ a utility function that represents $\succsim$, then $\succsim$ must be (a) complete and (b) transitive.

a) Because $u(\cdot)$ is real-valued on X it must be that $\forall x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. By definition above, this implies either $x \succsim y$ or $y \succsim x$ (completeness).

b) WLOG, suppose $x \succsim y$ and $y \succsim z$. Because $u(\cdot)$ represents $\succsim$ we must have $u(x) \geq u(y)$ and $u(y) \geq u(z) \Longrightarrow u(x) \geq u(z)$. Because $u(\cdot)$ represents $\succsim$, this implies $x \succsim z$ (transitivity).

You can think of preferences and utility as underlying primatives. These are both things that we do not observe. What we do observe is choice.

Choice Rules

A choice structure $(\mathcal{B}, C(\cdot))$ consists of:

i) $\mathcal{B}$ is a set of non-empty subsets of X.
Every element of $\mathcal{B}$ is a set $B \subset X$. As an example, a specific $B \in \mathcal{B}$ could be a budget set, which may not contain all subsets of X, i.e. the consumer cannot afford everything.

ii) $C(\cdot)$ is a choice correspondence that assigns a non-empty set of choseen elements $C(B) \subset B$ for every set $B \in \mathcal{B}$. When $C(B)$ contains a single element, that is the individuals' choice from B. But, $C(B)$ may not contain a unique element; typically $C(\cdot) \neq \emptyset$.

Review: X is the set of elements
$\mathcal{B}$ is the set of subsets of X.
B is a specific subset of $\mathcal{B}$.
$C(B)$ yields a choice made from B.

Example 1: Suppose $X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$
Then $C_1(\{x, y\}) = x$ and $C_1(\{x, y, z\}) = x$, so x is chosen no matter what budget the decision-maker faces.

Example 2: Suppose $X = \{x, y, z\}$, $\mathcal{B} = \{\{x,y\}, \{x,y,z\}\}$ as before.

Then $C_2(\{x,y\}) = x$ and $C_2(\{x,y,z\}) = \{x, y\}$, so that x is chosen between x and y, but when z is available, either x or y is chosen.

While we allow these choices, example 2 seems "odd" to us because x and y are available under both budgets, and yet the availability of z seems to affect the ~~preference as reflected in the~~ choice.

Thus, we want to add some structure to $C(\cdot)$ to impose some "reasonable" restrictions on behavior.

Definition: $\mathcal{B}, C(\cdot)$ satisfy the weak axiom of revealed preference if the following holds:

If, for some $B \in \mathcal{B}$ w/ $x, y \in B$, we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ w/ $x, y \in B'$ and $y \in C(B')$, we must have $x \in C(B')$.

In words:

If x is chosen when y is available in one budget set, then there cannot be a budget set containing x and y where y is chosen and x is not.

This rules out $C(\{x,y\}) = x$ and $C(\{x,y,z\}) = y$, and is related to independence of irrelevant alternatives.

Definition: Given $B, C(\cdot)$, the revealed preference relation $\succsim^*$ is defined by

$$x \succsim^* y \iff \exists\, B \in \mathcal{B} \text{ such that } x, y \in B \text{ and } x \in C(B).$$

We read this as "x is revealed weakly preferred to y", or "x is revealed to be at least as good a y".

We can restate WARP as follows: "if x is revealed at least as good as y, then y cannot be revealed as preferred to x".

What can we say about the examples we considered earlier?

In the first example, $x \succsim^* y$ and $x \succsim^* z$; this choice structure $(\mathcal{B}, C_1)$ satisfies the weak axiom because y and z are never chosen. (Revealed preference relations need not be complete or transitive.)

In the second example, because $C_2(\{x,y,z\}) = \{x,y\}$, we have $y \succ^* x$ (as well as $x \succsim^* y$ ~~and y~~, $x \succsim^* z$, and $y \succsim^* z$). But, because $C_2(\{x,y\}) = \{x\}$, x is revealed preferred to y, and WARP is violated. (WARP would be satisfied if $C_2(\{x,y\}) = \{x,y\}$; the absence of y in the choice ~~rule~~ where it was previously available is what violates WARP.)

Relationship Between Preference Relations & Choice Rules

Let's pose two questions:

i) If a decision-maker has a rational ordering, ~~de~~ $\succsim$, do her choices from $\mathcal{B}$ satisfy WARP? (Yes.)

ii) If $\mathcal{B}, C(\cdot)$ satisfy WARP, is there a rational preference relation consistent w/ this choice structure? (Maybe.)

First, a note on notation: suppose that an individual has a rational preference relation $\succsim$ on X. If facing $B \subset X \Rightarrow C^*(B, \succsim) = \{x \in B : x \succsim y \ \forall y \in B\}$, then we say that the rational preference relation $\succsim$ generates the choice correspondence $(\mathcal{B}, C^*(\cdot, \succsim))$. This simply says that preference-maximizing behavior is to choose the most preferred alternatives.

Proposition: Suppose that $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathcal{B}, C^*(\cdot, \succsim))$, satisfies WARP.

Proof: Suppose that for some $B \in \mathcal{B}$ we have $x, y \in B$ and $x \in C^*(B, \succsim)$. By definition of C^* this implies $x \succsim y$. Now suppose that for some $B' \in \mathcal{B}$ w/ $x, y \in B'$ we have $y \in C^*(B, \succsim)$. This implies $y \succsim z \ \forall z \in B'$. But we know $x \succsim y \Rightarrow x \succsim z \ \forall z \in B'$ $\Rightarrow x \in C^*(B', \succsim)$ as well, which is precisely what is demanded of WARP.

Now, on to the second question. We will need a new piece of terminology to start.

<u>Definition</u>: Given a choice structure $(\mathcal{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ <u>rationalizes</u> $C(\cdot)$ relative to $\mathcal{B}$ if $C(B) = C^*(B, \succsim)\ \forall\ B \in \mathcal{B}$; that is, if $\succsim$ generates the choice structure $(\mathcal{B}, C(\cdot))$.

In other words, the rational preference relation $\succsim$ rationalizes $C(\cdot)$ on $\mathcal{B}$ if the optimal choices generated by $\succsim$ (captured by $C^*(\cdot, \succsim)$) coincide with $C(\cdot)$ for all budget sets in $\mathcal{B}$.

In a sense, we can interpret a decision maker's choices as if she were a preference maximizer. However, in general, there may be more than one rationalizing preference relation for a given choice structure.

<u>Example 3</u>: Suppose that $X = \{x, y, z\}$, $\mathcal{B} = \{\{x,y\}, \{y,z\}, \{x,z\}\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, and $C(\{x,z\}) = \{z\}$.

This choice structure satisfies WARP. (WARP does not require transitivity.)

To have rational $\succsim$, we would need $x \succ y$, ~~$y \succ z$~~ and $y \succ z \Rightarrow$ $x \succ z$ by transitivity, but this contradicts $C(\{x,z\}) = z$.

The more budget sets, the more opportunities for contradictory behavior. Note also the exclusion of $\{x, y, z\}$ from $\mathcal{B}$.

Consider now the proposition proven by Arrow (1959):

Proposition: If $(\mathcal{B}, C(\cdot))$ is a choice structure such that

i) the weak axiom is satisfied, and
ii) $\mathcal{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathcal{B}$; that is, $C(B) = C^*(B, \succsim)\ \forall\ B \in \mathcal{B}$. Further, this rational preference relation is unique.

Proof: We will use the revealed preference relation $\succsim^*$ for the first part. We need to show that $\succsim^*$ (i) is rational and (ii) it rationalizes $C(\cdot)$ relative to $\mathcal{B}$. Then, we can (iii) show uniqueness.

i) We first check that $\succsim^*$ is rational, i.e. that it satisfies completeness & transitivity.

Completeness: By assumption $\{x,y\} \in \mathcal{B}$. Because either x or y must be in $C(\{x,y\})$, we must have $x \succsim^* y$, $y \succsim^* x$, or both.

Transitivity: WOLOG, let $x \succsim^* y$ and $y \succsim^* z$. Consider the set of $\{x,y,z\} \in \mathcal{B}$ ~~Because either x or y must be in $C\{x,y\}$ we must have $x \succsim^* y$~~ It suffices to show that $x \in C(\{x,y,z\})$ b.c. this implies $x \succsim^* z$. Suppose that $y \in C(\{x,y,z\})$. Because $x \succsim^* y$, WARP $\Rightarrow x \in C(\{x,y,z\})$. Instead suppose that $z \in C(\{x,y,z\})$; b.c. $y \succsim^* z$, WARP $\Rightarrow y \in C(\{x,y,z\})$, and then $x \in C(\{x,y,z\})$ as above.

ii) We need to show that $C(B) = C^*(B, \succsim^*)\ \forall B \in \mathcal{B}$. First, we will show $C(B) \subset C^*(B, \succsim^*)$, then $C^*(B, \succsim^*) \subset C(B) \Rightarrow C(B) = C^*(B, \succsim^*)$.

1) Suppose that $x \in C(B)$. Then $x \succsim^* y\ \forall y \in B$. This implies $x \in C^*(B, \succsim^*)$, and means that $C(B) \subset C^*(B, \succsim^*)$.

2) Suppose that $x \in C^*(B, \succsim^*) \Rightarrow x \succsim^* y\ \forall y \in B$. Because $C(B) \neq \emptyset$, WARP $\Rightarrow x \in C(B) \Rightarrow C^*(B, \succsim^*) \subset C(B)$.

iii) Because $\mathcal{B}$ includes all two-element subsets of X the choice behavior in $C(\cdot)$ lists all pairwise rankings over X. Therefore, there cannot be another choice function distinct from $C^*(B, \succsim^*)$, and so the rational preference relation is unique.