PROBLEM 1:

WRITE eqn (1) as

$$K_{t+2} - \phi K_{t+1} + \frac{1}{\beta} K_t = u_{t+2} \qquad (2)$$

where $\phi \equiv 1 + \frac{1}{\beta} + a_1 b_3 \qquad (3)$

CONSISTENT WITH OUR SOLUTION TO PROBLEM 2 SECTION IC I will use the term "INVERSE ROOTS" TO REFER TO THE Values z_1 and z_2 That Solve

$$1 - \phi z + \frac{1}{\beta} z^2 = 0 \qquad (4)$$

IN PROBLEM 2 SECTION IC we Showed That $\lambda_1 = \frac{1}{z_2}$ and $\lambda_2 = \frac{1}{z_1}$, but I present ~~an alternative~~ a useful ALTERNATIVE proof here.

FACTOR (4) as Follows:

$$\left.\begin{aligned} 1 - \phi z + \frac{1}{\beta} z^2 &= (1 - \lambda_1 z)(1 - \lambda_2 z) \\ &= \lambda_1 \lambda_2 \left(\frac{1}{\lambda_1} - z\right)\left(\frac{1}{\lambda_2} - z\right) = 0 \end{aligned}\right] \qquad (5)$$

Clearly, TWO Values of z Solve (4):

$z_1 = \frac{1}{\lambda_2}$ and $z_2 = \frac{1}{\lambda_1}$ or, generally,

$$\lambda = Z^{-1} \quad (6)$$

Return to (4). Multiply through by Z^{-2} to get

$$Z^{-2} - \phi Z^{-1} + \frac{1}{\beta} = 0$$ or, using (6),

$$\lambda^2 - \phi\lambda + \frac{1}{\beta} = 0 \quad (7)$$

Thus, the roots λ_1 and λ_2 solve (7), or

$$\lambda_{1,2} = \frac{1}{2}\left\{\phi \pm \left[\phi^2 - 4\frac{1}{\beta}\right]^{1/2}\right\} \quad (8)$$

Now, write (2) as

$$\left(1 - \phi L + \frac{1}{\beta} L^2\right) K_{t+2} = u_{t+2} \quad (2')$$

Compare (2′) to the first equality in (5)

$$\left(1 - \phi Z + \frac{1}{\beta} Z^2\right) = (1 - \lambda_1 Z)(1 - \lambda_2 Z) \quad (5')$$

And we can write (2′) as

$$(1 - \lambda_1 L)(1 - \lambda_2 L) K_{t+2} = u_{t+2} \quad (2'')$$

Comparing (2″) to (2′) and using (3) we have

$$\lambda_1+\lambda_2=\phi=1+\frac{1}{\beta}+a_1 b_3 \qquad (9)$$

and
$$\boxed{\lambda_1 \lambda_2=\frac{1}{\beta} \qquad (10)}$$

Choosing subscripts (arbitrarily) so that $|\lambda_1| \leq |\lambda_2|$, we can use (10), (8) and (3) to prove that

$$0<\lambda_1<1<\frac{1}{\beta}<\lambda_2$$

which is the desired result.

(a) First show $\frac{1}{\beta}<\lambda_2$. (This is the hard part.)

We must show that $\frac{1}{2}\left\{\phi+\left[\phi^2-\frac{4}{\beta}\right]^{1/2}\right\}>\frac{1}{\beta}$

or, that
$$\phi+\left[\phi^2-\frac{4}{\beta}\right]^{1/2}>\frac{2}{\beta} \qquad (11)$$

From (3) it follows that

$$\phi+\frac{1}{\beta}-1-a_1 b_3=\frac{2}{\beta} \qquad (12)$$

So to establish (5) it is sufficient to show that

$$\left[\phi^2-\frac{4}{\beta}\right]^{1/2}>\frac{1}{\beta}-1-a_1 b_3 \qquad (13)$$

Proof of (13):

$$\phi^2 - \frac{4}{\beta} = \left(1 + a_1 b_3 + \frac{1}{\beta}\right)^2 - \frac{4}{\beta}$$

$$= \left(1 + \frac{1}{\beta}\right)^2 + (a_1 b_3)^2 + 2\left(1 + \frac{1}{\beta}\right) a_1 b_3 - \frac{4}{\beta}$$

$$= \left(\frac{1}{\beta}\right)^2 + \frac{2}{\beta} + 1 - \frac{4}{\beta} + (a_1 b_3)^2 + 2\left(1 + \frac{1}{\beta}\right) a_1 b_3$$

$$= \left(\frac{1}{\beta}\right)^2 - \frac{2}{\beta} + 1 + (a_1 b_3)^2 + 2\left(1 + \frac{1}{\beta}\right) a_1 b_3$$

$$= \left(\frac{1}{\beta} - 1\right)^2 + (a_1 b_3)^2 + 2\left(1 + \frac{1}{\beta}\right) a_1 b_3$$

$$= \left(\frac{1}{\beta} - 1\right)^2 + (a_1 b_3)^2 + 2\left(\frac{1}{\beta} - 1\right) a_1 b_3 + 4 a_1 b_3$$

$$= \left(\frac{1}{\beta} - 1 + a_1 b_3\right)^2 + 4 a_1 b_3$$

Collecting, we have

$$\phi^2 - \frac{4}{\beta} = \left(\frac{1}{\beta} - 1 + a_1 b_3\right)^2 + 4 a_1 b_3 \qquad (14)$$

Since $a_1 b_3 > 0$ it follows from (14) that

$$\left[\phi^2 - \frac{4}{\beta}\right]^{1/2} > \left(\frac{1}{\beta} - 1 + a_1 b_3\right) > \frac{1}{\beta} - 1 - a_1 b_3$$

QED(13)

Thus we have $\lambda_2 > \frac{1}{\beta}$ and, since $0 < \beta < 1$,

$$\boxed{1 < \frac{1}{\beta} < \lambda_2 \qquad (15)}$$

b From (10), since $\frac{1}{\beta} > 0$ we have $\boxed{\lambda_1 \cdot \lambda_2 > 0}$

and, since (15) gives $\lambda_2 > 0$, we have

That

$$\boxed{\lambda_1 > 0 \qquad (16)}$$

c Finally, (10) gives $\lambda_1 \lambda_2 = \frac{1}{\beta}$ or

$$\lambda_1 = \frac{1}{\beta \lambda_2}$$

which since (15) gives $\lambda_2 > \frac{1}{\beta}$ and

Thus $1 > \frac{1}{\beta \lambda_2}$ we have

$$\boxed{\lambda_1 = \frac{1}{\beta \lambda_2} < 1 \qquad (17)}$$

(d) Collecting, (15), (16), and (17) Establish

$$0 < \lambda_1 < 1 < \frac{1}{\beta} < \lambda_2$$

which is the Result we seek

Comparing eqns (7) and (3) here to, say, Eqn (12) from Prob 3, Sect I C we see that, if a_1 and b_3 are the same across the 2 problems and if b_1 in Prob 3 Sect I C is $\frac{1}{\beta}$, if $b_1 = \frac{1}{\beta}$, then the roots of Eqn (2) here are the same as the roots of the system formed by (1a, b) in Prob 3 Sect I C.

Note then that the 2nd order DE here, ~~and the~~ eqn (1), and the system of 2 first order DE's in Prob 3 Sect I C both have one stable root $|\lambda_1| < 1$, and one unstable root $|\lambda_2| > 1$. This, as we will see, is the basis for the $\boxed{\text{SADDLE PATH EQUILIBRIUM}}$ that (famously) characterizes the Ramsey-Cass-Koopmans Model of Growth.

PROBlem 2 (HAMILTON p 29)

Consider $$y_t = a_0 \phi^t + \sum_{i=0}^{\infty} \phi^i w_{t-i} \qquad (1)$$

as a Soln TO $$y_t = \phi y_{t-1} + w_t \qquad (2)$$

From (1) $$y_{t-1} = a_0 \phi^{t-1} + \sum_{i=0}^{\infty} \phi^i w_{t-1-i} \qquad (1')$$

Subst From y_{t-1} into (2) and The Result is

$$y_t = \phi\left[a_0 \phi^{t-1} + \sum_{i=0}^{\infty} \phi^i w_{t-1-i}\right] + w_t \quad \text{or}$$

$$y_t = a_0 \phi^t + \sum_{i=0}^{\infty} \phi^{i+1} w_{t-1-i} + w_t \quad \text{or}$$

$$\underbrace{y_t = a_0 \phi^t + \sum_{i=0}^{\infty} \phi^i w_{t-i}}_{y_t}$$ subscript change

which using (1) gives

$$y_t = y_t$$

which is Clearly True. So (1) Solves (2).

To evaluate the boundedness of $\{y_t\}_{t=-\infty}^{\infty}$ begin with the 2^{nd} term on the RHS of (1). Since $\{w_t\}_{t=-\infty}^{\infty}$ is bounded it is true by definition that $\exists$ a ~~$\bar{w}>0$~~ positive constant, $\bar{w}>0$, $\ni -\bar{w} \le w_t \le \bar{w}$ $\forall t$. Since $|\phi|<1$ it follows that

$$\frac{-1}{1-\phi}\bar{w} \le \sum_{i=0}^{\infty} \phi^i w_{t-i} \le \frac{1}{1-\phi}\bar{w}.$$

Thus $\sum_{i=0}^{\infty} \phi^i w_{t-i}$ is bounded.

Next consider the term $a_0 \phi^t$. Since $|\phi|<1$ then $|\phi^{-1}|>1$. Since the range of y_t includes $\lim_{t\to-\infty} y_t$ we must consider $\lim_{t\to\infty} |a_0 \phi^t|$.

With $|\phi^{-1}|>1$, if $a_0 \neq 0$ then $\lim_{t\to-\infty} |a_0 \phi^t| = +\infty$.

So, unless $a_0 = 0$ the sequence of solutions, $\{y_t\}_{t=-\infty}^{\infty}$ is not bounded.

Problem 3: (a) If $|\phi| < 1$ ~~So~~ then we can write

$$\left(\frac{1}{1-\phi L}\right)c = \left[1 + \phi L + (\phi L)^2 + (\phi L)^3 + \cdots\right] c \text{ or}$$

$$\left(\frac{1}{1-\phi L}\right)c = c + \phi L c + \phi^2 L^2 c + \phi^3 L^3 c + \cdots \quad (1)$$

As c is a constant $Lc = L^2 c = L^3 c = \cdots = c$

Thus (1) gives

$$\left(\frac{1}{1-\phi L}\right)c = \left[c + \phi c + \phi^2 c + \cdots\right] = \left[1 + \phi + \phi^2 + \phi^3 + \cdots\right] c$$

which, since $|\phi| < 1$ gives

$$\boxed{\left(\frac{1}{1-\phi L}\right)c = \left[\frac{1}{1-\phi}\right] c}$$

(b) If $|\phi| > 1$ then $|\phi^{-1}| < 1$ and we can write

$$\left(\frac{-\phi^{-1}L^{-1}}{1-\phi^{-1}L^{-1}}\right)c = -\phi^{-1}\left[1 + (\phi^{-1}L^{-1}) + (\phi^{-1}L^{-1})^2 + \cdots\right] L^{-1} c$$

Note that $L^{-1}c = c$, because c is constant, and we have that

$$\left(\frac{-\phi^{-1}L^{-1}}{1-\phi^{-1}L^{-1}}\right)c = \frac{-1}{\phi}\left[c+\phi^{-1}L^{-1}c+\phi^{-2}L^{-2}c+\cdots\right]$$

$$= \frac{-1}{\phi}\left[c+\phi^{-1}c+\phi^{-2}c+\phi^{-3}c+\cdots\right]$$

Since $|\phi^{-1}|<1$ we then have

$$\left(\frac{-\phi^{-1}L^{-1}}{1-\phi^{-1}L^{-1}}\right)c = \frac{-1}{\phi}\left[\frac{1}{1-\phi^{-1}}\right]c$$

$$= \frac{-1}{\phi}\left[\frac{1}{1-\frac{1}{\phi}}\right]c = \frac{-1}{\phi}\left[\frac{1}{\frac{\phi-1}{\phi}}\right]c$$

$$= \frac{-1}{\phi}\left[\frac{\phi}{\phi-1}\right]c$$ and thus

$$\boxed{\left(\frac{-\phi^{-1}L^{-1}}{1-\phi^{-1}L^{-1}}\right)c = \left(\frac{1}{1-\phi}\right)c}$$