ECON 5700

Lecture 1

Theorem: $f^{-1}\ \exists$ iff f is one-to-one.

Injective: $\forall x_1, x_2 \in X,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Surjective (onto): $\forall y \in Y$, $\exists$ at least 1 $x \in X$, st $f(x) = y$.

§ 2. Limit.

1) Limit of sequence $\{x_n\}_{n=1}^{\infty}$

· literal: $\lim\limits_{n\to\infty} x_n = a \qquad x_n \to a\ (n\to\infty)$

· Analytical: (ε-δ) $\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall n > N_\varepsilon$

$|x_n - a| < \varepsilon$.

2) Theorems:

Suppose $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$ $\quad \lim\limits_{n\to\infty} z_n = c$

① a is unique.

② a is bounded. $\Rightarrow \exists M,\ |a| < M$.

③ $\lim\limits_{n\to\infty} x_n \pm y_n = a \pm b \qquad \lim\limits_{n\to\infty} x_n \cdot y_n = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$

$\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ where $\lim\limits_{n\to\infty} y_n \neq 0$

④ Monotone theorem.

$y_n \leq x_n \quad (\forall n \in \mathbb{N})$

$\Rightarrow b \leq a$.

⑤ Squeese theorem:

$$\begin{array}{ccccc} y_n & \leq & x_n & \leq & z_n \\ \downarrow & & & & \downarrow \\ a & & & & a. \end{array} \Rightarrow \lim_{n\to\infty} x_n = a.$$

if.

[Sandwich] !!!

⑥ if $x_n \uparrow$ bounded up, then $\{x_n\}_n$ is convergent if $x_n \downarrow$ and Bounded Below, then x_n converges

3) Limit of function:

- Literal: $\lim_{x \to x_0} f(x) = A$
- Analytical: $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$, $\forall x$, s.t. $|x - x_0| < \delta_\varepsilon$

then we have $|f(x) - A| < \varepsilon$ like select from $B(x_0, \delta_\varepsilon)$
center radius.

4) Theorem:

(1) uniqueness (2) Boundness (3) $\lim_{x \to x_0} f(x) \pm g(x)$ $f(x)/g(x)$

(4) monotonicity $f(x) \le g(x)$ $\lim_{x \to x_0} f(x) \le \lim_{x \to x_0} g(x)$.

(5) Squeeze Theorem.

5) Important Limits:

① $\lim_{n \to \infty} (1 + \frac{1}{n})^n = \lim_{n \to 0} (1 + n)^{\frac{1}{n}} = e = 2.718\ldots$

②. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

③ $\lim_{x \to \infty} (1 + \frac{a}{x})^x = e^a$

6) Continuity

Def: $\lim_{x \to x_0} f(x) = f(x_0)$ we say $f(x)$ is continuous at x_0

If $f(x)$ is continuous at any $x \in [a, b]$, then $f(x)$ is continuous on $[a, b]$

Remark: If $f(x)$ is continuous at x_0,

i) ① $f(x_0 + 0) = \lim_{x \to x_0^+} f(x)$, exists
just notation.

② $f(x_0 - 0) = \lim_{x \to x_0^-} f(x)$ exists

ii) $f(x_0 - 0) = f(x_0 + 0) = f(x_0)$

(i) ad (ii) $\Leftrightarrow$ continuous

what if (iii) doesn't hold. $\Rightarrow$ there is a jump!!!

or just jump at x_0

Ex 1: $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 2$.

Ex 2: $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t} = \lim_{t\to 0} \frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{t^2}{t(\sqrt{t^2+9}+3)}$

$= \lim_{t\to 0} \frac{t}{\sqrt{t^2+9}+3}$

$= \frac{\lim_{t\to 0} t}{\lim_{t\to 0}\sqrt{t^2+9}+3} = 0$

Ex 3: $\lim_{x\to 0} \frac{|x|}{x}$ exists?

$= \lim_{x\to 0} \frac{|x|\cdot x}{x\,x} = \lim_{x\to 0} = \frac{|x|\cdot x}{x^2} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$

it's not continuous!! D:

So it doesn't exist.

§3. Differentiation.

1) Def: If $y=f(x)$. $\Delta y = f(x_0+\Delta x) - f(x_0)$.

$$f'(x_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$ Differential Quotient

$$= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$f(x_0+\Delta x) = f(x_0) + \Delta y$

$= f(x_0) + f'(x_0)\Delta x \quad \Delta x \to 0$

if no $\Delta x \to 0$ $\quad f(x_0+\Delta x) \doteq f(x_0) + f'(x_0)\cdot\Delta x \quad x_0+\Delta x = y$

$f(y) \doteq f(x_0) + f'(x_0)(y - x_0)$

1st order of Taylor Exp

linearization of $f(x)$ at $x = x_0$

2) $(\sin x)' = \cos x \qquad (x^a)' = a x^{a-1}$

$(\cos x)' = -\sin x \qquad (\tan x)' = \dfrac{1}{\cos^2 x}$

$(e^x)' = e^x \qquad (a^x)' = a^x \ln a$

$(\ln x)' = \dfrac{1}{x}$

(3) (i) Rule of Product: $u(x), v(x)$

$u(x), v(x)$ differentiable at $x = x_0$

$(uv)' = u'v + uv'$

(ii) Rule of division

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

(4) chain Rule!

$y = f(u(x)) \Rightarrow y_x' = y_u' u_x'$

Ex: Find derivative of $\sqrt{x^2+1}$

$(x^2+1)^{\frac{1}{2}} \Rightarrow \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x$

Ex2: Find derivative $f(x) = \ln(\sin x)$

$$f'(x) = \frac{1}{\sin x} \cos x =$$

(5) Higher order derivative.

$$f^{(n)}(x) = [f^{(n-1)}(x)]' = \frac{d^n y}{dx^n}$$

(6) Differentiable.

$f(x)$ has derivative at x_0

$\Updownarrow$

$f(x)$ is differentiable at x_0

$$\frac{dy}{dx} = f'(x_0)$$

(7) Implicit differentiation.

$y = y(x)$

$x^2 + y^2 = 25$

step 1: Take derivative to x on both sides

$$\frac{d(x^2+y^2)}{dx} = \frac{d\,25}{dx} = 0$$

$$\frac{dx^2}{dx} \qquad \frac{dy^2}{dx}$$

$$\frac{dx^2}{dx} = 2x$$

$$\frac{dy^2}{dx} = \frac{d\,(y(x))^2}{dx} = 2y(x)\frac{dy}{dx}$$

step 2: Derive $\frac{dy}{dx}$

rough: $2x + 2y \cdot y' = 0$

definite: $2x + 2y(x) \cdot y_x = 0$

(8) Derivative of function inverse

Def: suppose $f(x) \in C[a,b]$

$f(x)$ is invertible if $\exists f^{-1}$ s.t. $y = f^{-1}(x)\ \forall y \in Y$

we have $x \in \bar{X}$ s.t. $f(x) = y$

$$\frac{d f^{-1}(x)}{dx} = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

R.V. Y, the pdf of Y is $f_Y(y)$ at $Y=y$

R.V. $\bar{X}$, x $f_{\bar{X}}(x)$ $\bar{X}=x$

if $y = g(x)$

$$f_Y(y) = f_{\bar{X}}(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

Transformation of R.V

Ex: $y = f(x) = \frac{x+2}{x}$ $\Leftrightarrow$ $y = 1 + \frac{2}{x}$

$\Rightarrow f^{-1}(y) = \frac{2}{y-1}$

$f'(x) = -\frac{2}{x^2}$

$$\frac{d f^{-1}(x)}{dx} = \frac{1}{-2x^{-2}}$$

$$= \frac{1}{-2\left(\frac{2}{x-1}\right)^{-2}}$$

(9) L' Hospital's Rule

suppose: f, g are differentiable $g'(x) \neq 0$ on $[a, b]$

For some point $x_0 \in [a, b]$

(i) $\frac{0}{0}$ suppose $f(x) \to 0$ $(x \to x_0)$

$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x)$

(ii) $\frac{\pm\infty}{\pm\infty}$ suppose $\lim_{x \to x_0} f(x) = \pm\infty$, $\lim_{x \to x_0} g(x) = \pm\infty$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Ex: $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} = \lim_{x\to 1}\frac{1}{x} = 1$

Ex: $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to 0}\frac{e^x}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \infty$

§4 Partial Differentiation.

1) Def of limit of multivariate function

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0$ s.t $\forall (x,y)\in D$ and

$0 < \sqrt{(x-a)^2+(y-b)^2} < \delta_\varepsilon$

and we have $|f(x,y) - L| < \varepsilon$

we say $\lim f(x,y) = L$ where $(x,y)\to(a,b)$