HW9

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ECON 5700

Due on August 25, 2020.

1 Question 1

Solution:

In this question, it only asks us to prove whether it's orthogonal, not mentioning anything about basis, aks, linearly indepent. Life would be much easier.

1. Let $q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. $q_1 \cdot q_2 = \frac{1}{2} - \frac{1}{2} = 0$. Interesting thing is that we use dot product to justify the orthogonality.

2. let $q_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$, $q_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$, and $q_3 = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$.

   Use dot product between q_1, q_2 and q_3:

$$q_1 \cdot q_2 = \frac{1}{6} - \frac{1}{6} = 0$$

$$q_1 \cdot q_3 = \frac{2}{15} - \frac{2}{15} = 0$$

$$q_2 \cdot q_3 = \frac{1}{10} - \frac{1}{10} = 0$$

   Thus., it's orthogonal.

2 Question 2

Solution:

Let $x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

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Step1: By the G-S process:

$$v_1 = x_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v_2} = \vec{x_2} - (\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}})\vec{v_1} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{v_3} = \vec{x_3} - (\frac{\vec{v_1} \cdot \vec{x_3}}{\vec{v_1} \cdot \vec{v_1}})\vec{v_1} - (\frac{\vec{v_2} \cdot \vec{x_3}}{\vec{v_2} \cdot \vec{v_2}})\vec{v_2} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

Step2: Normalize the vectors:

$$q_1 = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$q_2 = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \begin{bmatrix} \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{6}}{4} \\ \frac{\sqrt{6}}{4} \end{bmatrix}$$

$$q_3 = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \begin{bmatrix} \frac{4\sqrt{3}}{9} \\ -\frac{4\sqrt{3}}{9} \\ \frac{4\sqrt{3}}{9} \end{bmatrix}$$

3 Question 3

Solution:

Since $A = QR \longrightarrow Q^T A = Q^T QR = R$.

$$Q^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 0 \\ 0 & 6 & \frac{2}{3} \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$$

4 Question 4

Solution: The same with 3:

$$R = Q^T A = \begin{bmatrix} \sqrt{6} & 2\sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix}$$

5 Question 5

Solution:

$$\det(A - \lambda I) = (1 + \lambda)^2 - 9 = 0$$

$\lambda_1 = 2, \lambda_2 = -4.$

- When $\lambda = 2:$

  $$|A - \lambda I| = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

  Thus, $x_1 = x_2$, and the eigenvector in this case is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- When $\lambda = -4:$

  $$|A - \lambda I| = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

  In this case the eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

  Normalize the eigenvectors to obtain P:

  $$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

  $$D = P^T A P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

  Leave the final result intentionally.

6 Question 6

Solution:

1. For the first matrix:

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 6 & 3 & 0 \\ 0 & 6 & -6 & 7 \\ -1 & -2 & -9 & 0 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{R_4 - (-1)R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 6 & -6 & 7 \\ 0 & 0 & -6 & -1 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -6 & -1 \end{bmatrix} \xrightarrow{R_4 - (-2)R_3} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix}$$

2. For the second matrix (example):

$$\begin{bmatrix} 2 & 2 & 2 & 1 \\ -2 & 4 & -1 & 2 \\ 4 & 4 & 7 & 3 \\ 6 & 9 & 5 & 8 \end{bmatrix} \xrightarrow[R_2-(-1)R_1\ R_3-2R_1]{R_4-3R_1} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 3 & -1 & 5 \end{bmatrix} \xrightarrow{R_4-\frac{1}{2}R_2} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & \frac{2}{3} & \frac{2}{7} \end{bmatrix} \xrightarrow{R_4-(-\frac{1}{2})R_3} \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

7 Question 7

Solution:

Since the least square method is not required, so I use conventional way to solve this problem. It's not about the approximation, but accurate solution. Once I get the solution, I will check how to make approximation with OLS method.

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & -1 & 2 & 6 \\ 3 & 2 & -1 & 11 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

It's actually not solvable, so we have to use approximation (I KNOW IT, THIS IS THE REASON.).

8 Question 8

Solution:

$$\begin{aligned} A^+ &= (A^T A)^{-1} A^T \\ &= \left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{5}{3} & -\frac{4}{3} \\ -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \end{aligned}$$

9 Question 9

Solution:

- Step 1:

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Step 2- Calculate the eigenvalues:

$$|A^T A - \lambda I| = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$
$$= \lambda^2 - 4\lambda + 3$$
$$= 0$$

Thus, $lambda_1 = 1, \lambda_2 = 3$.

- If $\lambda = 1$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, the eigenvector in this case is: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- If $\lambda = 3$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, the eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus

$$\vec{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Step 3- Calculate u:

$$
\begin{aligned}
u_1 &= 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}
\end{aligned}
$$

- Step 4- A conclusion:

$$
\begin{aligned}
A &= U \Sigma V^T \\
&= \begin{bmatrix} 0 & \frac{\sqrt{6}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}
$$