Classical Demand Theory

The classical approach begins by specifying preferences over the consumption set $X \subset \mathbb{R}^L_+$.
We assume that $\succsim$ is rational: complete & transitive.

i) <u>completeness</u>: $\forall\, x, y \in X$, $x \succsim y$, $y \succsim x$, or both

ii) <u>transitivity</u>: $\forall\, x, y, z \in X$, if $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$

We make two additional assumptions: <u>desirability</u> & <u>convexity</u>.

iii) desirability: ~~if x ∈ X & y ≥ x~~ larger amounts of commodities are always preferred to smaller ones.

There are two desirability assumptions: <u>monotonicity</u> & <u>local nonsatiation</u>.
For monotonicity, we assume larger amounts of the good are always feasible:

$$\text{if } x \in X \text{ \& } y \geq x \Rightarrow y \in X$$

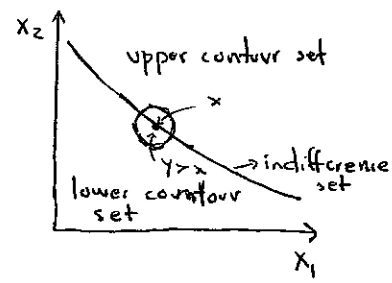
a) Definition. $\succsim$ is <u>monotone</u> if $x \in X$ and $y \gg x$ implies $y \succ x$; it is <u>strongly monotone</u> if $y \geq x$ and $y \neq x \Rightarrow y \succ x$. (<u>More is better.</u>)

(For this assumption, we assume more goods are desirable; for <u>bads</u>, we might define the preferred good as absence of "garbage", minding feaibility.)

For much theory, the weaker assumption, non-satiation, suffices.

b) Definition. $\succsim$ is <u>locally nonsatiated</u> if for every $x \in X$ and every $\epsilon > 0$, there is $y \in X$ such that $\|y - x\| \leq \epsilon$ and $y \succ x$. (<u>Always something better nearby.</u>)

($\|y - x\|$ is <u>Euclidean distance</u>: $\left[\sum_{l=1}^{L} (x_l - y_l)^2\right]^{\frac{1}{2}}$.)



An L-dimensional ball is created around x in which there is some $y \succ x$ (does not require monotonicity; could be in the lower countour set). Prevents

1) thick indifference curves

2) all goods being bads (since the origin becomes a satiation point).

Note the following:

a) if $\succsim$ is strongly monotone, then it is monotone

b) if $\succsim$ is monotone, then it is locally non-satiated

Definition. The indifference set containing x is the set of all consumption bundles that yield indifference w/ x : $\{y \in X : y \sim x\}$.

Definition. The upper contour set is the set of bundles weakly preferred to x: $\{y \in X : y \succsim x\}$.

Definition. The lower contour set is all bundles where x is weakly preferred: $\{y \in X : x \succsim y\}$.

iv) The convexity of $\succsim$ concerns trade-offs the consumer is willing to make among different goods.

Definition. $\succsim$ on X is convex if for every $x \in X$ the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha) z \succsim x$ for any $\alpha \in [0,1]$.

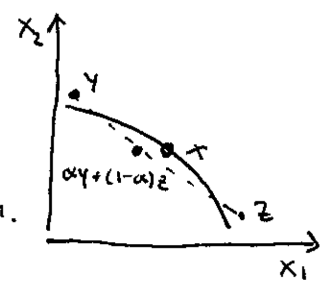
Convexity is a strong but central hypothesis; it derives from

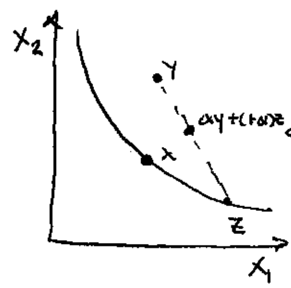
a) diminishing marginal rates of substitution: w/ more, more is required for loss of another

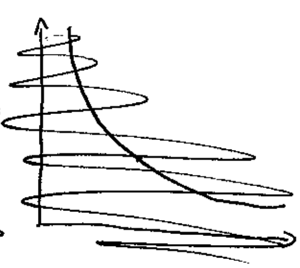
b) a desire for diversification

There are conditions where we logically expect violations of convexity, i.e. toothpaste and orange juice.

We may also desire at times to make a stronger convexity assumption.



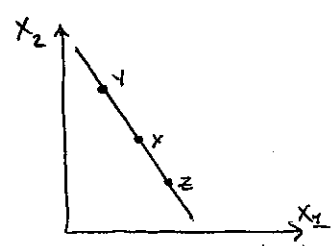




Definition. $\succsim$ on X is strictly convex if for every x, we have $y \succsim x$, $z \succsim x$, and $y \neq z \Rightarrow \alpha y + (1-\alpha) z \succ x \ \forall \alpha \in [0,1]$.

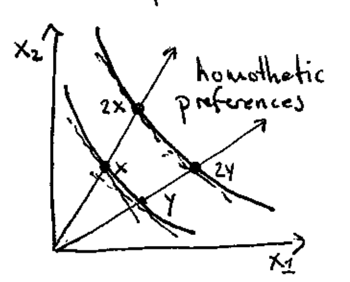
Strict convexity rules out indifference sets w/ flat portions (or straight lines, in $\mathbb{R}^2_+$).

It is common to focus on preferences for which it is possible to deduce the preference relation from a single indifference set. Two examples are the classes of ~~two~~ homothetic & quasi linear preferences.



Convex, but not strictly

Definition. A monotone $\succsim$ on $X = \mathbb{R}^L_+$ is homothetic if all indifference sets are related by proportional expansion rays; that is: if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.



A preference relation is homothetic iff it can be represented by a utility function that is homogeneous of degree one, i.e. $u(\alpha x) = \alpha u(x)$ $\forall x$ and $\alpha > 0$.

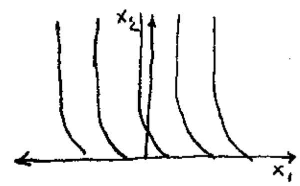
Definition. $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is <u>quasilinear</u> w/ respect to commodity 1 (the numeraire commodity) if

i) all of the indifference sets are paralell displacements of each other along the axis of commodity 1; that is:

if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.

ii) good 1 is desirable; that is:

$x + \alpha e_1 \succ x$ $\forall x$ and $\alpha > 0$.



(Note we assume no lower bound on consumption of the first commodity, i.e. $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$.)

Preference & Utility

Example. Assume $X = \mathbb{R}^2$. Define $x \succsim y$ if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$". This ordering is <u>lexicographic</u>, and cannot be represented by a utility function. (This ordering is the way a dictionary is organized: commodity 1 has <u>first, heirarchical priority</u> in determining the ordering, like the first letter in a word.)

The final assumption needed to ensure the existence of a utility function is that preference relations are <u>continuous</u>.

Definition. $\succsim$ on X is continuous if it is preserved under limits; that is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ w/ $x^n \succsim y^n$ $\forall n$, $x = \lim_{n \to \infty} x^n$, and $y = \lim_{n \to \infty} y^n$, we have $x \succsim y$.

This means preferences cannot exhibit "jumps"; for example, preferring every element in sequence $\{x^n\}$ to $\{y^n\}$ but suddenly reversing this preference at the limiting points of these sequences.

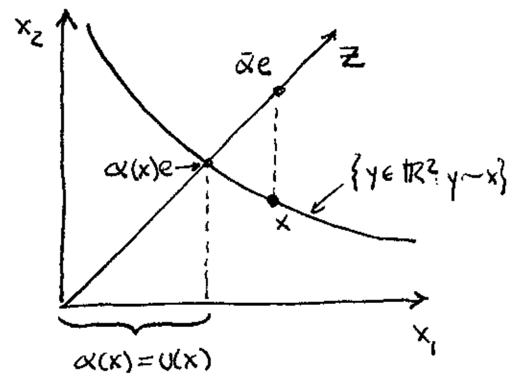
We can now see why lexicographic preferences are <u>not continuous</u>. Let $x^n = (\frac{1}{n}, 0)$ and $y = (0, 1)$ $\forall n$; then $\lim_{n \to \infty} x^n = (0, 0)$ and $\lim y^n = (0, 1)$, but $x^n \succsim y^n$ $\forall n$.

An equivalent way to express continuity is to say that $\forall x$, the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both closed; that is, they include their boundaries. (Continuity is satisfied iff both sets are closed.)

Proposition. Suppose $\succsim$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

Proof. By construction, for $X = \mathbb{R}^L_+$.

Denote a diagonal ray in $\mathbb{R}^L_+$ (the locus of vectors w/ all components equal) by Z. Let e designate the L-vector whose components are all equal to 1; then $\alpha e \in Z$ for all non-negative scalars $\forall \alpha \geq 0$.



For every $x \in \mathbb{R}^L_+$, monotonicity implies that $x \succsim 0$. Also for any $\bar{\alpha}$ such that $\bar{\alpha}e \gg x$, we have $\bar{\alpha}e \succsim x$. Monotonicity & continuity can be shown to imply there is a unique $\alpha(x) \in [0, \bar{\alpha}]$ such that $\alpha(x)e \sim x$.

By continuity, the upper & lower contour sets of x are closed; then $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ are nonempty & closed. By completeness of $\succsim$, we know $\mathbb{R}_+ \subset (A^+ \cup A^-)$; the nonemptyness & closedness of A^+ & A^-, along w/ the fact that $\mathbb{R}_+$ is connected, imply $A^+ \cap A^- \neq \emptyset$. Thus $\exists$ a scalar α such that $\alpha e \sim x$. Further, by monotonicity, $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Hence, there can be at most one scalar satisfying $\alpha e \sim x$; this scalar is $\alpha(x)$.

We now assign a utility value $u(x) = \alpha(x)$ to every x. We need to verify that this function represents a preference & that it is continuous.

i) $\alpha(x)$ represents a preference by construction. Suppose $\alpha(x) \geq \alpha(y)$. By monotonicity, this implies $\alpha(x)e \succsim \alpha(y)e$, and since $\alpha(x)e \sim x$ and $\alpha(y)e \sim y$, $x \succsim y$.

If we instead suppose $x \succsim y$, then $(\alpha(x)e \sim x) \succsim (\alpha(y)e \sim y)$, so by monotonicity, we have $\alpha(x) \geq \alpha(y)$. Hence, $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$.

ii) That $\alpha(x)$ is continuous is demonstrated in the text, pg. 48–49.

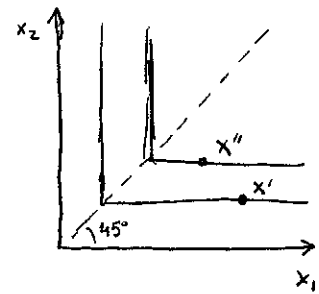
From now on, we will assume that the preference relation is continuous, & hence representable by a continuous utility function.

It is convenient at this point to make some notes.

i) $u(\cdot)$ is <u>not unique</u> for a given $\succsim$; any strictly increasing transformation of $u(\cdot)$, say $v(x) = f(u(x))$, where $f(\cdot)$ is strictly increasing, also represents $\succsim$.

ii) If $\succsim$ is continuous, $\exists$ some continuous $u(\cdot)$ representing $\succsim$, but not all utility functions representing ~~the~~ $\succsim$ are continuous. In fact, any strictly increasing, discontinuous transformation of a continuous $u(\cdot)$ also represents $\succsim$.

iii) $u(\cdot)$ need not be differentiable. For example, Leontief preferences have $x'' \succsim x'$ iff $\min\{x_1'', x_2''\} \geq \min\{x_1', x_2'\}$, causing a kink in indifference curves where $x_1 = x_2$.



Generally, we will assume that utility functions are twice differentiable.

iv) Restrictions on preferences translate into restrictions on the form of utility functions.

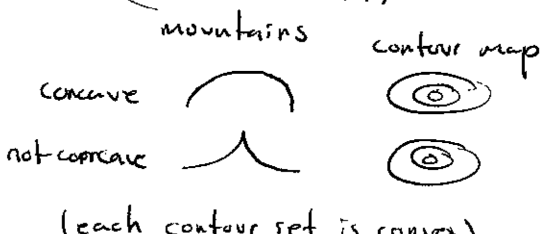
a) monotonicity implies $u(x) > u(y)$ if $x \gg y$; that is, $u(\cdot)$ is increasing

b) convexity of preferences implies $u(\cdot)$ is quasiconcave (strict convexity $\Rightarrow$ ~~concavity~~ strictly quasiconcave)

$u(\cdot)$ is quasiconcave if $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex for all x, or equivalently $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$ for any x, y, and all $\alpha \in [0,1]$.

(If the inequality is strict for all $x \neq y$ & $\alpha \in (0,1) \Rightarrow$ strict quasiconcavity.)

Further notes (refresher)



For a function f: $\forall \alpha$ the set of points x_1, x_2 such that $f(x_1, x_2) \geq \alpha$ is convex is said to be quasiconcave.

a) quasiconcave $\Rightarrow$ convex contour set

b) strictly quasiconcave $\Rightarrow$ strictly convex contour set

Utility Maximization Problem (UMP)

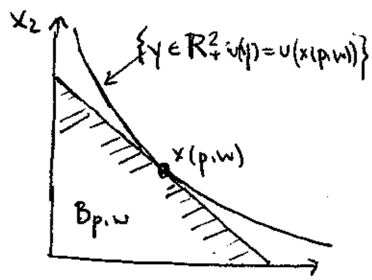
We assume a rational, continuous, locally non-satiated $\succsim$ represented by $u(x)$. In the UMP, the consumer chooses from $X = \mathbb{R}^L_+$ her most preferred bundle in the Walrasian budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ where $p \gg 0$ and $w > 0$, to maximize her utility.

$$\max_{x \geq 0} \; u(x)$$
$$\text{s.t. } p \cdot x \leq w$$

Proposition. If $p \gg 0$ and $u(\cdot)$ is continuous, then $\exists$ a solution to the UMP.

Proof. If $p \gg 0$, $B_{p,w}$ is ~~bounded~~ compact b/c it is both bounded (for any $l = 1, \ldots, L$ we have $x_l \leq w/p_l$ $\forall\, x \in B_{p,w}$) and closed. A continuous function always has a maximum value on any compact set. (M.F of Math Appendix.)

We now focus our attention on the solution set to the UMP (the set of optimal consumption bundles) and the maximal utility value (the value function).



The Walrasian demand correspondence is the rule that assigns the set of optimal consumption vectors in the UMP to each $(p,w) \gg 0$, denoted $x(p,w) \in \mathbb{R}^L_+$. When single-valued for all (p,w), we call it the demand function.

Proposition. If $u(\cdot)$ is continuous representing a locally-non-satiated $\succsim$ on $X = \mathbb{R}^L_+$, then the Walrasian demand correspondence $x(p,w)$ satisfies:

i) homogeneity of degree zero in (p,w): $x(\alpha p, \alpha w) = x(p,w)$ for any $p, w,$ & $\alpha > 0$.

ii) Walras' law: $p \cdot x = w$ $\forall\, x \in x(p,w)$

iii) if $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set; if $\succsim$ is strictly convex, so $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single element

Proof. See formal proof on pp. 52–55.

Example. Suppose $L=2$ & the consumer possesses Cobb-Douglas utility $u(x_1,x_2) = k x_1^{\alpha} x_2^{1-\alpha}$ for some $\alpha \in (0,1)$ & $k>0$.

It is increasing at all $(x_1, x_2) \gg 0$ & is homogenous of degree one. We use the log transformation to $\alpha \ln x_1 + (1-\alpha)\ln x_2$.

The UMP is thus

$$\max_{x_1,x_2} \quad \alpha \ln x_1 + (1-\alpha)\ln x_2$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = w$$

Since $u(\cdot)$ is increasing, the budget constraint will hold w/ strict equality at any solution. We will impose this below.

The Lagrangian is

$$\mathcal{L} = \alpha \ln x_1 + (1-\alpha)\ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - w)$$

and the first-order conditions (FOCs) are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0 \implies \lambda = \frac{\alpha}{x_1 p_1} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1-\alpha}{x_2} - \lambda p_2 = 0 \implies \lambda = \frac{1-\alpha}{x_2 p_2} \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 = w \quad (3)$$

So, clearly from (1) and (2), $\frac{\alpha}{x_1 p_1} = \frac{1-\alpha}{x_2 p_2} \implies (1-\alpha) x_1 p_1 = \alpha x_2 p_2$.

Then, by $x_2 p_2 = w - x_1 p_1$ from (3), we have

$$(1-\alpha) x_1 p_1 = \alpha (w - x_1 p_1)$$

$$x_1 p_1 = \alpha w$$

$$x_1 = \frac{\alpha w}{p_1}$$

And using the budget constraint again, $x_2 = \frac{(1-\alpha)w}{p_2}$.

For homework, verify homogeneity, homotheticity, homogeneity, Walras' law, & the uniqueness of the solution for Cobb-Douglas utility.

The Indirect Utility Function

For each $(p, w) \gg 0$, the utility value of the UMP is denoted $v(p, w) \in \mathbb{R}$. It is equal to $u(x^*)$ for any $x^* \in x(p, w)$, and is called the <u>indirect utility function</u>.

Proposition. The indirect utility function ~~is~~, for $u(\cdot)$ continuous representing locally non-satiated $\succsim$ on $X \in \mathbb{R}^L_+$, is $v(p, w)$ satisfying

i) homogenous of degree zero

ii) strictly increasing in w & non-increasing in p_l for any l

iii) quasiconvex; that is, the set $\{(p, w) : v(p, w) < \bar{v}\}$ is convex for any $\bar{v}$.

iv) continuous in p & w

Proof. In text, pp. 56–57.

Example. Indirect Utility Function for Cobb–Douglas

We have $v(p, w) = u(x(p, w))$. By substitution & simplication,

$$\begin{aligned} v(p, w) &= \alpha \ln (x_1^*) + (1-\alpha) \ln (x_2^*) \\ &= \alpha \ln \left(\frac{\alpha w}{p_1}\right) + (1-\alpha) \ln \left(\frac{(1-\alpha) w}{p_2}\right) \\ &= \alpha \ln \alpha + \ln w - \alpha \ln p_1 + (1-\alpha) \ln (1-\alpha) - (1-\alpha) \ln p_2 \end{aligned}$$