Exercises #5

4.B.1 Prove the sufficiency part of Proposition 4.B.1.

> A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the Gorman form with the coefficients w_i the same for every consumer i. That is:
>
> $$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

Show also that if preferences admit the Gorman-form indirect utility functions with the same $b(p)$, then preferences admit expenditure functions of the form $e_i(p, u_i) = c(p, u_i) + d_i(p)$.

4.B.1 By Roy's identity (Proposition 3.G.4) and $v_i(p,w_i) = a_i(p) + b(p)w_i$,

$$x_i(p,w_i) = -\frac{1}{\nabla_{w_i} v_i(p,w_i)} \nabla_p v_i(p,w_i) = -\frac{1}{b(p)} \nabla_p a_i(p) - \frac{w_i}{b(p)} \nabla_p b(p).$$

Thus $\nabla_{w_i} x_i(p,w_i) = -\frac{1}{b(p)} \nabla_p b(p)$ for all i. Since the right-hand side is identical for every i, the set of consumers exhibit parallel, straight expansion paths.

As for the second part, by (3.E.1),

$$e_i(p,u_i) = (u_i - a_i(p))/b(p).$$

Hence, by letting $c(p) = 1/b(p)$ and $d_i(p) = -a_i(p)/b(p)$, we obtain $e_i(p,u_i) = c(p)u_i + d_i(p)$.

Indirect utility function: $v_i(p, w_i) = a_i(p) + b(p) w_i$

By Roy's Identity:

$$x_i(p,w_i) = -\frac{\nabla_p v_i(p,w_i)}{\nabla_w v_i(p,w_i)} = -\frac{\nabla_p a_i(p) + \nabla_p b(p) w_i}{b(p)}$$

$$= -\frac{\nabla_p a_i(p)}{b(p)} - \frac{\nabla_p b(p)}{b(p)} w_i$$

So: $\nabla_{w_i} x_i(p,w_i) = \underbrace{-\frac{\nabla_p b(p)}{b(p)}}_{}$ $\forall i$ → consumers exhibit parallel, straight expansion paths

no i index → same for every consumer

Expenditure function:

$$e_i(p, u_i) = \frac{u_i - a_i(p)}{b(p)} = \underbrace{\frac{1}{b(p)}}_{c(p)} u_i - \underbrace{\frac{a_i(p)}{b(p)}}_{d_i(p)} = c(p)\, u_i - d_i(p)$$

(no i index)

where $c(p) = \frac{1}{b(p)}$ and $d_i(p) = \frac{a_i(p)}{b(p)}$

4.C.3 Give a graphical two-commodity example of a preference relation generating a Walrasian demand that does not satsify the ULD property. Interpret.

4.C.3 A Giffen good will be a most familiar example. In the figure below, good 1 is a Giffen good.

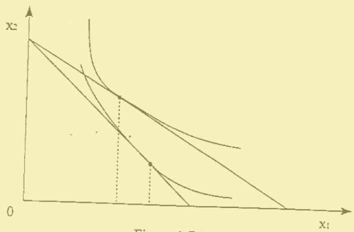
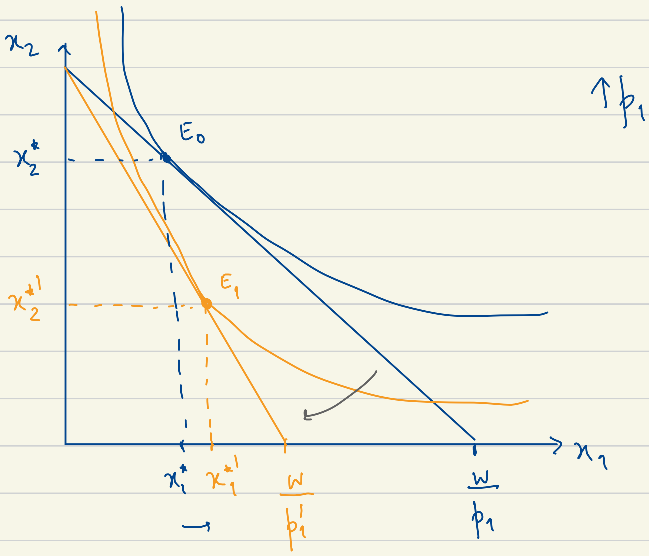


Figure 4.C.3

This example shows that the ULD property is actually not derived from the utility maximization. It is a restriction on preferences.

Example: Walrasian demand that does not satisfy Uncompensated Law of Demand.

Giffen good : as the price increases, the individual consumes more of it.



$\uparrow p_1 \Rightarrow x_1 \uparrow$ and $x_2 \downarrow$

4.C.11 Suppose that there are two consumers, 1 and 2, with utility functions over two goods, 1 and 2, of $u_1(x_{11}, x_{21}) = x_{11} + 4\sqrt{x_{21}}$ and $u_2(x_{12}, x_{22}) = 4\sqrt{x_{12} + x_{22}}$. The two consumers have identical wealth levels, $w_1 = w_2 = w/2$.

(a) Calculate the individual demand functions and the aggregate demand functions.

(b) Compute the individual Slutsky matrices $S_i(p, w/2)$ (for $i = 1, 2$) and the aggregate Slutsky matrix $S(p, w)$. [*Hint*: Note that for this two-good example, only one element of each matrix must be computed to determine the entire matrix.] Show that $dp \cdot S(p, w)dp < 0$ for all $dp \neq 0$ not proportional to p. Conclude that aggregate demand satisfies WARP.

4.C.11 (a) When deriving individual demands from the first-order conditions of utility maximization, we will neglect the nonnegativity constraints (which is investigated in Exercise 3.D.4(c)). In fact, we will later see that, for prices and wealths under consideration, the demands are always in the interior of the nonnegative orthant.

It follows directly from the first-order conditions that

$$x_1(p, w/2) = (x_{11}(p, w/2), x_{21}(p, w/2)) = (w/2p_1 - 4p_1/p_2,\ 4p_1^2/p_2^2),$$

$$x_2(p, w/2) = (x_{12}(p, w/2), x_{22}(p, w/2)) = (4p_2^2/p_1^2,\ w/2p_2 - 4p_2/p_1),$$

Hence

$$x(p, w) = x_1(p, w/2) + x_2(p, w/2)$$
$$= (w/2p_1 - 4p_1/p_2 + 4p_2^2/p_1^2,\ w/2p_2 - 4p_2/p_1 + 4p_1^2/p_2^2).$$

(b) Denote the (ℓ,k) entry of the Slutsky matrix $S_i(p,w)$ of consumer i by $s_{\ell ki}(p,w)$. Since $\partial x_{21}(p,w/2)/\partial w_1 = 0$, $s_{221}(p,w/2) = \partial x_{21}(p,w/2)/\partial p_2 = -8p_1^2/p_2^3$. Hence by Proposition 2.F.3, $s_{211}(p,w/2) = s_{121}(p,w/2) = 8p_1/p_2^2$, and hence $s_{111}(p,w/2) = -8/p_2$. Thus

$$S_1(p,w/2) = \begin{bmatrix} -8/p_2 & 8p_1/p_2^2 \\ 8p_1/p_2^2 & -8p_1^2/p_2^3 \end{bmatrix}.$$

Similarly, we can show that

$$S_2(p,w/2) = \begin{bmatrix} -8p_1^2/p_2^3 & 8p_2/p_1^2 \\ 8p_2/p_1^2 & -8/p_1 \end{bmatrix}.$$

We can also apply Proposition 2.F.3 to derive the Slutsky matrix $S(p,w)$ of the aggregate demand function:

$$S(p,w) = \begin{bmatrix} -w/4p_1^2 - 6/p_2 - 6p_2^2/p_1^3 & w/4p_1p_2 + 6p_1/p_2^2 + 6p_2/p_1^2 \\ w/4p_1p_2 + 6p_1/p_2^2 + 6p_2/p_1^2 & -w/4p_2^2 - 6/p_1 - 6p_1^2/p_2^3 \end{bmatrix}.$$

By Exercise 2.F.9(b) (and $S(p,w)p = 0$), if $dp \in \mathbb{R}^2$, $dp \neq 0$, and dp is not proportional to p, then $dp \cdot S(P,w)dp < 0$. Thus, according to the small-type discussion after Proposition 2.F.3, the aggregate demand function $x(p,w)$ satisfies the WA.

a) $u_1(x_{11}, x_{21}) = x_{11} + 4\sqrt{x_{21}}$

$u_2(x_{12}, x_{22}) = 4\sqrt{x_{12}} + x_{22}$

$w_1 = w_2 = \frac{w}{2}$

Individual Demand:

Consumer 1:

$$\max_{\{x_{11}, x_{21}\}} u_1(x_{11}, x_{21}) = x_{11} + 4\sqrt{x_{21}}$$

s.t.: $p_1 x_{11} + p_2 x_{21} = w_1$

$$\mathcal{L} = x_{11} + 4\sqrt{x_{21}} + \lambda\left[w_1 - p_1 x_{11} - p_2 x_{21}\right]$$

$$(x_{11}):\quad 1 - \lambda p_1 = 0 \iff \lambda = \frac{1}{p_1}$$

$$(x_{21}):\quad 2\, x_{21}^{-0.5} - \lambda p_2 = 0 \iff \lambda = \frac{2x_{21}^{-0.5}}{p_2}$$

$$(\lambda):\quad w_1 - p_1 x_{11} - p_2 x_{21} = 0$$

$$(x_{11}) + (x_{21}):\quad \frac{1}{p_1} = \frac{2x_{21}^{-0.5}}{p_2} \iff x_{21} = \left[\frac{1}{2}\left(\frac{p_2}{p_1}\right)\right]^{-2} \iff x_{21} = 4\frac{p_1^2}{p_2^2}$$

$$\text{into } (\lambda):\quad p_1 x_{11} + \cancel{p_2}\, \frac{4p_1^2}{p_2^{\cancel{2}}} = w_1 \iff x_{11} = \frac{1}{p_1}\left(w_1 - \frac{4p_1^2}{p_2}\right)$$

$$\iff x_{11} = \frac{w_1}{p_1} - \frac{4p_1}{p_2}$$

$$x_1(p_1, w_1) = \begin{bmatrix} x_{11}(p_1, w_1) \\ x_{21}(p_1, w_1) \end{bmatrix} = \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix}$$

Consumer 2:

$$\max_{\{x_{12}, x_{22}\}} u(x_{12}, x_{22}) = 4\sqrt{x_{12}} + x_{22}$$

$$\text{s.t.: } p_1 x_{12} + p_2 x_{22} = w_2$$

$$\mathcal{L} = 4\sqrt{x_{12}} + x_{22} + \lambda\left[w_2 - p_1 x_{12} - p_2 x_{22}\right]$$

$$(x_{12}): \quad 2x_{12}^{-0.5} - \lambda p_1 = 0 \quad \Leftrightarrow \quad \lambda = \frac{2x_{12}^{-0.5}}{p_1}$$

$$(x_{22}): \quad 1 - \lambda p_2 = 0 \quad \Leftrightarrow \quad \lambda = \frac{1}{p_2}$$

$$(\lambda): \quad w_2 - p_1 x_{12} - p_2 x_{22} = 0$$

$$(x_{12}) + (x_{22}): \quad \frac{2x_{12}^{-0.5}}{p_1} = \frac{1}{p_2} \quad \Leftrightarrow \quad x_{12} = \left[\frac{1}{2}\left(\frac{p_1}{p_2}\right)\right]^{-2} \quad \Leftrightarrow \quad x_{12} = 4\frac{p_2^2}{p_1^2}$$

$$\text{into } (\lambda): \quad \cancel{p_1}\, 4\frac{p_2^2}{p_1^{\cancel{2}}} + p_2 x_{22} = w_2 \quad \Leftrightarrow \quad x_{22} = \frac{1}{p_2}\left(w_2 - 4\frac{p_2^2}{p_1}\right)$$

$$\Leftrightarrow \quad x_{22} = \frac{w_2}{p_2} - 4\frac{p_2}{p_1}$$

$$x_2(p, w_2) = \begin{bmatrix} x_{12}(p, w_2) \\ x_{22}(p, w_2) \end{bmatrix} = \begin{bmatrix} 4\dfrac{p_2^2}{p_1^2} \\ \dfrac{w_2}{p_2} - \dfrac{4p_2}{p_1} \end{bmatrix}$$

<u>Aggregate Demand</u>:

$$x(p, w) = x_1(p, w_1) + x_2(p, w_2) = \begin{bmatrix} x_{11}(p, w_1) + x_{12}(p, w_2) \\ x_{21}(p, w_1) + x_{22}(p, w_2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} + \frac{4p_2^2}{p_1^2} \\ \frac{4p_1^2}{p_2^2} + \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix} = \begin{bmatrix} \frac{w}{2p_1} - \frac{4p_1}{p_2} + \frac{4p_2^2}{p_1^2} \\ \frac{4p_1^2}{p_2^2} + \frac{w}{2p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

$w_1 = \frac{1}{2}w$, $w_2 = \frac{1}{2}w$

b) Slutsky Substitution matrix:

$$S(p,w) = D_p h(p,u) = D_p x(p,w) + D_w x(p,w)\left[x(p,w)\right]^T$$

$$s_{\ell k} = \frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)$$

Individual Slutsky Matrix:

Consumer 1:

$$S_1(p,w_1) = \begin{bmatrix} s_{111}(p,w_1) & s_{121}(p,w_1) \\ s_{211}(p,w_1) & s_{221}(p,w_1) \end{bmatrix} \qquad x_1(p,w_1) = \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix}$$

$$= D_p \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix} + D_{w_1} \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix} \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} & \frac{4p_1^2}{p_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial p_1}\left[\frac{w_1}{p_1} - \frac{4p_1}{p_2}\right] & \frac{\partial}{\partial p_2}\left[\frac{w_1}{p_1} - \frac{4p_1}{p_2}\right] \\ \frac{\partial}{\partial p_1}\left[\frac{4p_1^2}{p_2^2}\right] & \frac{\partial}{\partial p_2}\left[\frac{4p_1^2}{p_2^2}\right] \end{bmatrix} +$$

$$+ \begin{bmatrix} \frac{\partial}{\partial w_1}\left[\frac{w_1}{p_1} - \frac{4p_1}{p_2}\right] \\ \frac{\partial}{\partial w_1}\left[\frac{4p_1^2}{p_2^2}\right] \end{bmatrix} \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} & \frac{4p_1^2}{p_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{w_1}{p_1^2} - \frac{4}{p_2} & \frac{4p_1}{p_2^2} \\ \frac{8p_1}{p_2^2} & -\frac{8p_1^2}{p_2^3} \end{bmatrix} + \begin{bmatrix} \frac{1}{p_1} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} & \frac{4p_1^2}{p_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{w_1}{p_1^2} - \frac{4}{p_2} & \frac{4p_1}{p_2^2} \\ \frac{8p_1}{p_2^2} & -\frac{8p_1^2}{p_2^3} \end{bmatrix} + \begin{bmatrix} \frac{w_1}{p_1^2} - \frac{4}{p_2} & \frac{4p_1}{p_2^2} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{p_2} & \frac{8p_1}{p_2^2} \\ \frac{8p_1}{p_2^2} & -\frac{8p_1^2}{p_2^3} \end{bmatrix}$$

Consumer 2:

$$S_2(p, w_2) = \begin{bmatrix} s_{112}(p, w_2) & s_{122}(p, w_2) \\ s_{212}(p, w_2) & s_{222}(p, w_2) \end{bmatrix} \qquad x_2(p, w_2) = \begin{bmatrix} \frac{4p_2^2}{p_1^2} \\ \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

$$= D_p \begin{bmatrix} \frac{4p_2^2}{p_1^2} \\ \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix} + D_{w_2} \begin{bmatrix} \frac{4p_2^2}{p_1^2} \\ \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix} \begin{bmatrix} \frac{4p_2^2}{p_1^2} & \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial p_1}\left[\frac{4p_2^2}{p_1^2}\right] & \frac{\partial}{\partial p_2}\left[\frac{4p_2^2}{p_1^2}\right] \\ \frac{\partial}{\partial p_1}\left[\frac{w_2}{p_2} - \frac{4p_2}{p_1}\right] & \frac{\partial}{\partial p_2}\left[\frac{w_2}{p_2} - \frac{4p_2}{p_1}\right] \end{bmatrix} +$$

$$+ \begin{bmatrix} \frac{\partial}{\partial w_2}\left[\frac{4p_2^2}{p_1^2}\right] \\ \frac{\partial}{\partial w_2}\left[\frac{w_2}{p_2} - \frac{4p_2}{p_1}\right] \end{bmatrix} \begin{bmatrix} \frac{4p_2^2}{p_1^2} & \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8p_2^2}{p_1^3} & \frac{8p_2}{p_1^2} \\ \frac{4p_2}{p_1^2} & -\frac{w_2}{p_2^2} - \frac{4}{p_1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{p_2} \end{bmatrix} \begin{bmatrix} \frac{4p_2^2}{p_1^2} & \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8p_2}{p_1^3} & \frac{8p_2}{p_1^2} \\ \frac{4p_2}{p_1^2} & -\frac{w_2}{p_2^2} - \frac{4}{p_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{4p_2}{p_1^2} & \frac{w_2}{p_2^2} - \frac{4}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8p_2}{p_1^3} & \frac{8p_2}{p_1^2} \\ \frac{8p_2}{p_1^2} & -\frac{8}{p_1} \end{bmatrix}$$

Aggregate Slutsky Matrix:

$$x(p, w) = \begin{bmatrix} \frac{w}{2p_1} - \frac{4p_1}{p_2} + \frac{4p_2^2}{p_1^2} \\ \frac{4p_1^2}{p_2^2} + \frac{w}{2p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

where $w = w_1 + w_2 \rightarrow$ aggregate wealth

$$x_1(p, w_1) = \begin{bmatrix} \frac{w_1}{p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix} = \begin{bmatrix} \frac{w}{2p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix} = x_1(p, w)$$

$$x_2(p, w_2) = \begin{bmatrix} \frac{4p_2^2}{p_1^2} \\ \frac{w_2}{p_2} - \frac{4p_2}{p_1} \end{bmatrix} = \begin{bmatrix} \frac{4p_2^2}{p_1^2} \\ \frac{w}{2p_2} - \frac{4p_2}{p_1} \end{bmatrix} = x_2(p, w)$$

$$S_A(p,w) = \overbrace{\begin{bmatrix} \frac{\partial}{\partial p_1} x(p,w) & \frac{\partial}{\partial p_2} x(p,w) \end{bmatrix}}^{D_p x(p,w)} + \overbrace{\frac{\partial}{\partial w} x(p,w)}^{D_w x(p,w)} \begin{bmatrix} x(p,w) \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{\partial}{\partial p_1} x_1(p,w) + \frac{\partial}{\partial p_1} x_2(p,w) & \frac{\partial}{\partial p_2} x_1(p,w) + \frac{\partial}{\partial p_2} x_2(p,w) \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial w} x_1(p,w) \\ \frac{\partial}{\partial w} x_2(p,w) \end{bmatrix} \begin{bmatrix} x(p,w) \end{bmatrix}^T$$

$$x_1(p,w) = \begin{bmatrix} \frac{w}{2p_1} - \frac{4p_1}{p_2} \\ \frac{4p_1^2}{p_2^2} \end{bmatrix} \qquad \frac{\partial x_1}{\partial p_1} = \begin{bmatrix} -\frac{w}{2p_1^2} - \frac{4}{p_2} \\ \frac{8p_1}{p_2^2} \end{bmatrix} \qquad \frac{\partial x_1}{\partial p_2} = \begin{bmatrix} \frac{4p_1}{p_2^2} \\ -\frac{8p_1^2}{p_2^3} \end{bmatrix}$$

$$x_2(p,w) = \begin{bmatrix} \frac{4p_2^2}{p_1^2} \\ \frac{w}{2p_2} - \frac{4p_2}{p_1} \end{bmatrix} \qquad \frac{\partial x_2}{\partial p_1} = \begin{bmatrix} -\frac{8p_2^2}{p_1^3} \\ \frac{4p_2}{p_1^2} \end{bmatrix} \qquad \frac{\partial x_2}{\partial p_2} = \begin{bmatrix} \frac{8p_2}{p_1^2} \\ -\frac{w}{2p_2^2} - \frac{4}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{w}{2p_1^2} - \frac{4}{p_2} - \frac{8p_2^2}{p_1^3} & \frac{4p_1}{p_2^2} + \frac{8p_2}{p_1^2} \\ \frac{8p_1}{p_2^2} + \frac{4p_2}{p_1^2} & -\frac{8p_1^2}{p_2^3} - \frac{w}{2p_2^2} - \frac{4}{p_1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2p_1} \\ \frac{1}{2p_2} \end{bmatrix} \begin{bmatrix} \frac{w}{2p_1} - \frac{4p_1}{p_2} + \frac{4p_2^2}{p_1^2} & \frac{4p_1^2}{p_2^2} + \frac{w}{2p_2} - \frac{4p_2}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{w}{2p_1^2} - \frac{4}{p_2} - \frac{8p_2^2}{p_1^3} & \frac{4p_1}{p_2^2} + \frac{8p_2}{p_1^2} \\ \frac{8p_1}{p_2^2} + \frac{4p_2}{p_1^2} & -\frac{8p_1^2}{p_2^3} - \frac{w}{2p_2^2} - \frac{4}{p_1} \end{bmatrix} + \begin{bmatrix} \frac{w}{4p_1^2} - \frac{2}{p_2} + \frac{2p_2^2}{p_1^3} & \frac{2p_1}{p_2^2} + \frac{w}{4p_1p_2} - \frac{2p_2}{p_1^2} \\ \frac{w}{4p_1p_2} - \frac{2p_1}{p_2^2} + \frac{2p_2}{p_1^2} & \frac{2p_1^2}{p_2^3} + \frac{w}{4p_2^2} - \frac{2}{p_1} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{w}{4p_1^2} - \frac{6}{p_2} - \frac{6p_2^2}{p_1^3} & \frac{6p_1}{p_2^2} + \frac{6p_2}{p_1^2} + \frac{w}{4p_1p_2} \\ \frac{6p_1}{p_2^2} + \frac{6p_2}{p_1^2} + \frac{w}{4p_1p_2} & -\frac{6p_1^2}{p_2^3} - \frac{w}{4p_2^2} - \frac{6}{p_1} \end{bmatrix}$$

Show: $dp\, S(p,w)\, dp < 0$ for all $dp \neq 0$

wealth compensation for price change: $dw = x(p,w)\, dp$

Chain rule: $dx = D_p\, x(p,w)\, dp + D_w\, x(p,w)\, dw$

$$= D_p\, x(p,w)\, dp + D_w\, x(p,w)\, [x(p,w)]^T\, dp$$

Aggregate Demand

For $x(p,w)$ to satisfy the Uncompensated Law of Demand we need:

$$dp \cdot dx \leq 0$$

$$\Leftrightarrow \quad dp\, \left[D_p\, x(p,w)\, dp + D_w\, x(p,w)\, [x(p,w)]^T\, dp \right] \leq 0$$

$$\Leftrightarrow \quad dp\, \left[\underbrace{D_p\, x(p,w) + D_w\, x(p,w)\, [x(p,w)]^T}_{S(p,w)} \right] dp \leq 0$$

$$\Leftrightarrow \quad dp\, S(p,w)\, dp \leq 0 \quad \left(\text{and } dp\, S(p,w)\, dp < 0 \text{ when } dp \neq 0 \right)$$

To verify the WA, take any (p, w), (p', w') with $x(p, w) \neq x(p', w')$ and $p \cdot x(p', w') \leq w$.[9] Define $p'' = (w/w')p'$. By homogeneity of degree zero, we have $x(p'', w) = x(p', w')$. From $(p'' - p) \cdot [x(p'', w) - x(p, w)] < 0$, $p \cdot x(p'', w) \leq w$, and Walras' law, it follows that $p'' \cdot x(p, w) > w$. That is, $p' \cdot x(p, w) > w'$. ■

<u>Show</u> : if aggregate demand $x(p,w)$ satisfies the Uncompensated Law of Demand, then it satisfies WARP.

Definition: WARP : if $p \cdot x(p',w') \leq w$ and $x(p,w) \neq x(p',w')$ then $p' \cdot x(p,w) > w'$.

ULD : $x(p,w)$ satisfies the uncompensated law of demand if $(p'-p)\left[x(p',w') - x(p,w)\right] \leq 0$.

- Consider (p,w), (p',w') with $x(p,w) \neq x(p',w')$ and

$x(p',w')$ affordable at (p,w) $\longleftarrow$ $p \cdot x(p',w') \leq w$.

$x(p,w)$ not affordable at (p',w')

We need to show that $p' \cdot x(p,w) > w'$ for WARP to hold.

- Let $p'' = \frac{w}{w'} p'$.

- By homogeneity of degree zero, we have

$$x(p',w') = x(\alpha p', \alpha w') \quad \text{for } \alpha > 0, \text{ so let } \alpha = \frac{w}{w'}$$

$$= x\left(\frac{w}{w'} p', \frac{w}{w'} w'\right) = x(p'', w)$$

and so: $x(p'',w) = x(p',w')$.

- Assume Walras' Law :

$p \cdot x(p,w) = w$ and $p'' \cdot x(p'',w) = w$

$x(p', w')$ affordable at (p, w)

- because $x(p', w') = x(p'', w)$ and we have assumed $p \cdot x(p', w') \leq w$:

$$p \cdot x(p'', w) \leq w. \longrightarrow x(p'', w) \text{ affordable at } (p, w).$$

- Uncompensated Law of Demand:

$$(p'' - p) \cdot \left[x(p'', w) - x(p, w) \right] < 0$$

which expanded becomes:

$$\underbrace{\left[\underbrace{p'' \cdot x(p'', w)}_{\substack{= w \\ \text{by (iii) WL}}} - \underbrace{p'' \cdot x(p, w)}_{\substack{> w \\ \text{(required)}}} \right]}_{< 0} + \underbrace{\left[\underbrace{p \cdot x(p, w)}_{\substack{= w \\ \text{by (iii) WL}}} - \underbrace{p \cdot x(p'', w)}_{\substack{\leq w \\ \text{by (ii)}}} \right]}_{\geq 0} < 0$$

$x(p, w)$ not affordable at (p'', w)

So: $p'' \cdot x(p, w) > w$ is needed for ULD to hold.

- And because $p'' = \frac{w}{w'} p'$:

$x(p, w)$ not affordable at (p', w').

$$p'' \cdot x(p, w) > w \iff \frac{\cancel{w}}{w'} p' \cdot x(p, w) > \cancel{w} \iff p' \cdot x(p, w) > w'$$

- We have shown that $p' \cdot x(p, w) > w'$.

So $x(p, w)$ satisfies WARP.