Math Camp
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Homework 2 Solution

Find an equation of the tangent plane to the given surface at the specified point.

$$z = 3y^2 - 2x^2 + x, \quad (2, -1, -3)$$
$$z = \sqrt{xy}, \quad (1, 1, 1)$$
$$z = x\sin(x + y), \quad (-1, 1, 0)$$

Solution

$$z = -7x - 6y + 5$$

$$x + y - 2z = 0$$

$$x + y + z = 0$$

Explain why the function is differentiable at the given point. Then find the linearization of the function at that point.

$$f(x, y) = 1 + x\ln(xy - 5), \quad (2, 3)$$

Solution

$$6x + 4y - 23$$

Find the differential of the function.

$$z = e^{-2x}\cos 2\pi t$$

$$m = p^5 q^3$$

$$R = \alpha\beta^2 \cos\gamma$$

Solution

$$dz = -2e^{-2x}\cos 2\pi t\, dx - 2\pi e^{-2x}\sin 2\pi t\, dt$$

$$dm = 5p^4q^3\, dp + 3p^5q^2\, dq$$

$$dR = \beta^2\cos\gamma\, d\alpha + 2\alpha\beta\cos\gamma\, d\beta - \alpha\beta^2\sin\gamma\, d\gamma$$

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Solution

Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$

For the following two sets of functions with a point in the direction vector **u.**

(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

$$f(x, y) = \sin(2x + 3y), \quad P(-6, 4), \quad \mathbf{u} = \tfrac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right)$$

$$f(x, y, z) = x^2yz - xyz^3, \quad P(2, -1, 1), \quad \mathbf{u} = \left\langle 0, \tfrac{4}{5}, -\tfrac{3}{5}\right\rangle$$

Solution

(a) $\nabla f(x, y) = \langle 2\cos(2x + 3y), 3\cos(2x + 3y)\rangle$

(b) $\langle 2, 3\rangle$ (c) $\sqrt{3} - \frac{3}{2}$

(a) $\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\rangle$

(b) $\langle -3, 2, 2\rangle$ (c) $\frac{2}{5}$

Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

Solution

2/5

Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x, y) = \sin(xy), \quad (1, 0)$$

Solution

$1, \langle 0, 1\rangle$

Find and classify the critical points of the function

$$f(x, y) = 10x^2 y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Solution

The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0$$

$$5x^2 - 4y - 4y^3 = 0$$

we see that either

$$x = 0 \quad \text{or} \quad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

In the second case ($10y - 5 - 2x^2 = 0$), we get

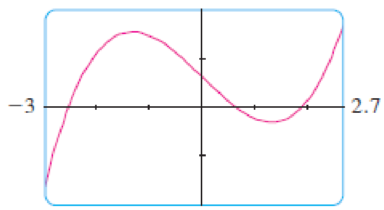
$$x^2 = 5y - 2.5$$

and, putting this in Equation 5, we have $25y - 12.5 - 4y - 4y^3 = 0$. So we have to solve the cubic equation

$$4y^3 - 21y + 12.5 = 0$$

Using a graphing calculator or computer to graph the function

$$g(y) = 4y^3 - 21y + 12.5$$



So we have three roots for y

$$y \approx -2.5452 \qquad y \approx 0.6468 \qquad y \approx 1.8984$$

(Alternatively, we could have used Newton's method or a rootfinder to locate these roots.) From Equation 6, the corresponding x-values are given by

$$x = \pm\sqrt{5y - 2.5}$$

If $y \approx -2.5452$, then x has no corresponding real values. If $y \approx 0.6468$, then $x \approx \pm 0.8567$. If $y \approx 1.8984$, then $x \approx \pm 2.6442$. So we have a total of five critical points, which are analyzed in the following chart. All quantities are rounded to two decimal places.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	0.00	-10.00	80.00	local maximum
$(\pm 2.64, 1.90)$	8.50	-55.93	2488.72	local maximum
$(\pm 0.86, 0.65)$	-1.48	-5.87	-187.64	saddle point

Find the local maximum and minimum values and saddle point(s) of the functions.

$f(x, y) = x^2 + xy + y^2 + y$

$f(x, y) = (x - y)(1 - xy)$

$f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

$f(x, y) = x^3 - 12xy + 8y^3$

Solution

Minimum $f\left(\frac{1}{3}, -\frac{2}{3}\right) = -\frac{1}{3}$

Saddle points at $(1, 1)$, $(-1, -1)$

Maximum $f(0, 0) = 2$, minimum $f(0, 4) = -30$,
saddle points at $(2, 2)$, $(-2, 2)$

Minimum $f(2, 1) = -8$, saddle point at $(0, 0)$

EXERCISE 7.4

1. Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for each of the following functions:

(a) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$

(b) $y = 7x_1 + 6x_1x_2^2 - 9x_2^3$

(c) $y = (2x_1 + 3)(x_2 - 2)$

(d) $y = (5x_1 + 3)/(x_2 - 2)$

Solution

1. (a) $\partial y/\partial x_1 = 6x_1^2 - 22x_1x_2$ $\quad$ $\partial y/\partial x_2 = -11x_1^2 + 6x_2$

(b) $\partial y/\partial x_1 = 7 + 6x_2^2$ $\quad$ $\partial y/\partial x_2 = 12x_1x_2 - 27x_2^2$

(c) $\partial y/\partial x_1 = (2(x_2 - 2)$ $\quad$ $\partial y/\partial x_2 = 2x_1 + 3$

(d) $\partial y/\partial x_1 = 5/(x_2 - 2)$ $\quad$ $\partial y/\partial x_2 = -(5x_1 + 3)/(x_2 - 2)^2$

EXERCISE 7.6

1. Use Jacobian determinants to test the existence of functional dependence between the paired functions.

(a) $y_1 = 3x_1^2 + x_2$

$y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$

(b) $y_1 = 3x_1^2 + 2x_2^2$

$y_2 = 5x_1 + 1$

Solution

1.

(a) $|J| = \begin{vmatrix} 6x_1 & 1 \\ (36x_1^3 + 12x_1x_2 + 48x_1) & (6x_1^2 + 2x_2 + 8) \end{vmatrix} = 0$

The function is dependent.

(b) $|J| = \begin{vmatrix} 6x_1 & 4x_2 \\ 5 & 0 \end{vmatrix} = -20x_2$

Since $|J|$ is not identically zero, the functions are independent