II. Stationary ARMA Processes.

1.) Suppose that Y_t is covariance stationary. As in Hamilton let γ_j denote the *j*th autocovariance of Y_t and let ρ_j denote its *j*th autocorrelation.

(a) Derive γ_j and ρ_j, $j = 0, 1, 2$, for the case where Y_t is MA(1) and for the case where Y_t is MA(2).

(b) Suppose instead that Y_t is AR(1), that is, $y_t = \phi y_{t-1} + \varepsilon_t$ where ε_t is white noise. Derive γ_j and ρ_j, $j = 0, 1, 2$. What can you conclude (or conjecture) about the relationship between the autoregressive parameter, ϕ, and the autocorrelation, ρ_j.

2.) Suppose that Y_t is a stationary AR(2) process. That is

$$y_t = c + \phi_1 y_{t-1} - \phi_2 y_{t-2} + \varepsilon_t \qquad (1)$$

where c, ϕ_1, and ϕ_2 are constants and ε_t is white noise. Furthermore, suppose that $\phi_1 = (a+b)$ and $\phi_2 = ab$ where $|a|<1$ and $|b|<1$. Derive the $\text{MA}(\infty)$ representation of Y_t and give explicit expressions for the MA coefficients in terms of the constants a and b.

3.) Consider the following stationary ARMA(p,q) process:

$$y_t = c + \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q} \qquad (1)$$

Here, c, ϕ_j, and θ_j are constants and ε_t $i.i.d.(0, \sigma^2)$

(a) Rewrite (1) as the first-order vector difference equation

$$\xi_t = \mathbf{F}\xi_{t-1} + \mathbf{G}\mathbf{v}_t \qquad (2)$$

giving complete definitions of $\xi_t, \mathbf{F}, \mathbf{G}$, and $\mathbf{v}_t$ in terms of the parameters of the original specification in (1).

4.) The present-value model of asset prices gives that the equilibrium price of an asset is the expected present discounted value of its dividend payments. Thus, if P_t denotes the current price of the asset and $r > 0$ denotes the known constant interest rate we have the equilibrium condition

Problem 1

Part (a):

(i) Suppose $Y_t \sim MA(1)$.

Thus $$\boxed{Y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}} \quad (1)$$
where $\varepsilon_t \sim iid(0, \sigma^2)$

Since $E(Y_t) = \mu$
$$\gamma_0 = E[(Y_t - \mu)^2] = E[\varepsilon_t^2 + 2\theta \varepsilon_t \varepsilon_{t-1} + \theta^2 \varepsilon_{t-1}^2]$$
Since ε_t is white noise $E(\varepsilon_t \varepsilon_{t-1}) = 0$ and thus

$$\gamma_0 = E(\varepsilon_t^2) + \theta^2 E(\varepsilon_{t-1}^2) \quad \text{or}$$

$$\gamma_0 = \sigma^2 + \theta^2 \sigma^2 \quad \text{Thus}$$

$$\boxed{\gamma_0 = (1+\theta^2)\sigma^2} \quad (2)$$

$$\gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)] = E[(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-1} + \theta \varepsilon_{t-2})]$$
$$= E[\varepsilon_t \varepsilon_{t-1} + \theta \varepsilon_t \varepsilon_{t-2} + \theta \varepsilon_{t-1}^2 + \theta^2 \varepsilon_{t-1} \varepsilon_{t-2}]$$

As ε_t is white noise $E(\varepsilon_t \varepsilon_{t-1}) = E(\varepsilon_t \varepsilon_{t-2}) = E(\varepsilon_{t-1} \varepsilon_{t-2}) = 0$

And thus

$$\gamma_1 = \cancel{\theta \sigma} \; \theta E(\varepsilon_{t-1}^2) \quad \text{or}$$

$$\boxed{\gamma_1 = \theta \sigma^2} \quad (3)$$

$$\gamma_2 = E[(Y_t - \mu)(Y_{t-2} - \mu)] = E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-2} + \theta\varepsilon_{t-3})]$$

$$= E[\varepsilon_t\varepsilon_{t-2} + \theta\varepsilon_t\varepsilon_{t-3} + \theta\varepsilon_{t-1}\varepsilon_{t-2} + \theta^2\varepsilon_{t-1}\varepsilon_{t-3}]$$

Since ε_t is white noise

$$E(\varepsilon_t\varepsilon_{t-2}) = E(\varepsilon_t\varepsilon_{t-3}) = E(\varepsilon_{t-1}\varepsilon_{t-2}) = E(\varepsilon_{t-1}\varepsilon_{t-3}) = 0$$

AND Thus

$$\boxed{\gamma_2 = 0 \qquad (4)}$$

~~Applying the same Arg~~

Since $\rho_j \equiv \dfrac{\gamma_j}{\gamma_0}$ we have that

$\rho_0 = 1$ of course

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma} \quad \text{or}$$

$$\boxed{\rho_1 = \left(\frac{\theta}{1+\theta^2}\right) \qquad (5)}$$

and $$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0}{(1+\theta^2)\sigma} \quad \text{or}$$

$$\boxed{\rho_2 = 0 \qquad (6)}$$

(ii) $Y_t \sim MA(2)$

Thus

$$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \qquad (7)$$

where $\varepsilon_t \sim iid(0, \sigma^2)$

$\cancel{\gamma_0 = E(Y_t)}$ Since $E(Y_t) = \mu$

$$\gamma_0 = E\left[(Y_t - \mu)^2\right] = E\left[(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2})^2\right]$$

$$= E\Big[\varepsilon_t^2 + 2\theta_1 \varepsilon_t \varepsilon_{t-1} + 2\theta_2 \varepsilon_t \varepsilon_{t-2} + \theta_1^2 \varepsilon_{t-1}^2 + 2\theta_1\theta_2 \varepsilon_{t-1}\varepsilon_{t-2} + \theta_2^2 \varepsilon_{t-2}^2\Big]$$

Since $E(\varepsilon_t^2) = E(\varepsilon_{t-1}^2) = E(\varepsilon_{t-2}^2) = \sigma^2$

~~and $E(\varepsilon_t \varepsilon_{t-j}) = 0$ for~~

and $E(\varepsilon_t \varepsilon_\tau) = 0$ for $t \neq \tau$

$$\gamma_0 = \sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2 \text{ or}$$

$$\boxed{\gamma_0 = \left[1 + \theta_1^2 + \theta_2^2\right]\sigma^2 \quad (8)}$$

$$\gamma_1 = E\left[(Y_t - \mu)(Y_{t-1} - \mu)\right] = E\left[(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2})(\varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} + \theta_2 \varepsilon_{t-3})\right]$$

Where, ~~Noting that~~ $E(\varepsilon_t \varepsilon_\tau) = 0$ for $t \neq \tau$, so

$$\gamma_1 = E\left[\theta_1 \varepsilon_{t-1}^2 + \theta_2 \theta_1 \varepsilon_{t-2}^2\right] \quad \text{or}$$

$$\gamma_1 = \theta_1 \sigma^2 + \theta_2 \theta_1 \sigma^2 \quad \text{or}$$

$$\boxed{\gamma_1 = [\theta_1 + \theta_1 \theta_2]\sigma^2 \quad (6)}$$

$$\gamma_2 = E\left[(Y_t - \mu)(Y_{t-2} - \mu)\right] =$$

$$E\left[(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2})(\varepsilon_{t-2} + \theta_1 \varepsilon_{t-3} + \theta_2 \varepsilon_{t-4})\right]$$

$$= E\left(\theta_2 \varepsilon_{t-2}^2\right) \quad \text{so}$$

$$\boxed{\gamma_2 = \theta_2 \sigma^2 \quad (7)}$$

Further $\rho_0 = 1$ of course

$$\rho_1 = \left[\frac{\theta_1 + \theta_1 \theta_2}{1 + \theta_1^2 + \theta_2^2}\right] \quad (8)$$

$$\text{and} \quad \rho_2 = \left[\frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}\right] \quad (9)$$

Part(b) Suppose Y_t is AR(1), That is,

$$\boxed{Y_t = \phi Y_{t-1} + \varepsilon_t \qquad (9)}$$

$\varepsilon_t \sim iid(0, \sigma^2)$

TAKING EXPECTATIONS

$$E(Y_t) = \phi E(Y_{t-1}) + \underbrace{E(\varepsilon_t)}_{\nearrow 0} \quad \text{~~or~~}$$

SINCE Y_t IS STATIONARY $E(Y_t) = E(Y_{t-1}) = \mu$ SO

$\mu = \phi \mu$ or

$(1-\phi)\mu = 0$. STATIONARITY REQUIRES $|\phi| < 1$

SO $(1-\phi) \neq 0$ THEREFORE $\boxed{\mu = 0}$ (10)

~~If~~ TO DERIVE AUTOCOVARIANCES IT IS USEFUL TO WORK WITH THE MA(∞) REPRESENTATION. WRITE (9) AS

$(1-\phi L) Y_t = \varepsilon_t$ or

$Y_t = \left(\dfrac{1}{1-\phi L}\right) \varepsilon_t$ or

$$Y_t = \varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots \qquad (11)$$

Since $\mu = 0$ we have

$$\gamma_0 = E(Y_t^2) = E\left[(\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \cdots)^2\right]$$

Since $E(\varepsilon_t \varepsilon_\tau) = 0$ for $t \neq \tau$ this gives

$$\gamma_0 = E\left[(\varepsilon_t^2 + \phi^2\varepsilon_{t-1}^2 + (\phi^2)^2\varepsilon_{t-2}^2 + \cdots)\right]$$

$$= \sigma^2 + \phi^2\sigma^2 + \phi^4\sigma^2 + \cdots \text{ or}$$

$$\gamma_0 = (1 + \phi^2 + \phi^4 + \phi^6 + \cdots)\sigma^2$$

or since $|\phi^2| < 1$

$$\boxed{\gamma_0 = \left[\frac{1}{1-\phi^2}\right]\sigma^2} \quad (12)$$

Next, consider

$$\gamma_1 = E(Y_t Y_{t-1}) = E\left[(\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \cdots)(\varepsilon_{t-1} + \phi\varepsilon_{t-2} + \cdots)\right]$$

Since $E(\varepsilon_t \varepsilon_{t-j}) = 0$ for $j = 1, 2, 3, \ldots$ This gives

$$\gamma_1 = E\left[\phi(\varepsilon_{t-1} + \phi\varepsilon_{t-2} + \phi^2\varepsilon_{t-3} + \cdots)(\varepsilon_{t-1} + \phi\varepsilon_{t-2} + \cdots)\right]$$

or

$$\gamma_1 = \phi E\left[\varepsilon_{t-1}^2 + \phi^2 \varepsilon_{t-2} + \phi^4 \varepsilon_{t-3} + \cdots\right]$$

$$\gamma_1 = \phi\left[1 + \phi^2 + \phi^4 + \cdots\right]\sigma^2$$ and Thus

$$\boxed{\gamma_1 = \left[\frac{\phi}{1-\phi^2}\right]\sigma^2 \quad (13)}$$

Next, consider

$$\gamma_2 = E(Y_t Y_{t-2}) = E\left[(\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \cdots)(\varepsilon_{t-2} + \phi\varepsilon_{t-3} + \cdots)\right]$$

Since $E(\varepsilon_t \varepsilon_{t-j}) = 0$ for $j = 1, 2, 3, \ldots$

and $E(\varepsilon_{t-1}\varepsilon_{t-j}) = 0$ for $j = 2, 3, 4, \ldots$

$$\gamma_2 = \phi^2 E\left[(\varepsilon_{t-2} + \phi\varepsilon_{t-3} + \phi^2\varepsilon_{t-4} + \cdots)^2\right]$$ or

$$\gamma_2 = \phi^2\left[1 + \phi^2 + \phi^4 + \cdots\right]\sigma^2$$ or

$$\boxed{\gamma_2 = \left[\frac{\phi^2}{1-\phi^2}\right]\sigma \quad (14)}$$

Since $\rho_j = \dfrac{\gamma_j}{\gamma_0}$ from (12) (13) and (14)

we have

$\rho_0 = 1$ of course,

$\rho_1 = \phi$

$\rho_2 = \phi^2$

we can infer (or conjecture) that

For the AR(1) Process

$$\boxed{\rho_j = \phi^j}$$

Problem 2. Consider

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \qquad (1)$$

Write as

$$y_t - \phi_1 y_{t-1} + \phi_2 y_{t-2} = c + \varepsilon_t \qquad (2)$$

Since $\phi_1 = (a+b)$ and $\phi_2 = ab$ we have

$$y_t - (a+b) y_{t-1} + ab\, y_{t-2} = c + \varepsilon_t \quad \text{or}$$

$$\left(1 - (a+b)L + ab L^2\right) y_t = c + \varepsilon_t \quad \text{or}$$

$$(1 - aL)(1 - bL) y_t = c + \varepsilon_t \quad \text{or}$$

$$y_t = \frac{c}{(1-a)(1-b)} + \frac{1}{(1-aL)(1-bL)} \varepsilon_t \qquad (3)$$

Note that, since $|b| < 1$ we can write

$$\frac{1}{(1-aL)(1-bL)} \varepsilon_t = \left(\frac{1}{1-aL}\right)\left[1 + bL + b^2L^2 + \cdots\right] \varepsilon_t \quad \text{or}$$

$$\frac{1}{(1-aL)(1-bL)}\varepsilon_t = \left(\frac{1}{1-aL}\right)\left[\varepsilon_t + b\varepsilon_{t-1} + b^2\varepsilon_{t-2} + \cdots\right]$$

And, since $|a|<1$ we have

$$\frac{1}{(1-aL)(1-bL)}\varepsilon_t = \left[1 + aL + a^2L^2 + a^3L^3 + \cdots\right]\left[\varepsilon_t + b\varepsilon_{t-1} + b^2\varepsilon_{t-2} + \cdots\right]$$

$$\begin{aligned} = {} & \varepsilon_t + b\varepsilon_{t-1} + b^2\varepsilon_{t-2} + b^3\varepsilon_{t-3} + \cdots \\ & + a\varepsilon_{t-1} + ab\,\varepsilon_{t-2} + ab^2\varepsilon_{t-3} + \cdots \\ & + a^2\varepsilon_{t-2} + a^2b\,\varepsilon_{t-3} + \cdots \\ & + a^3\varepsilon_{t-3} + \cdots \end{aligned}$$

$$\begin{aligned} = {} & \varepsilon_t + (a+b)\varepsilon_{t-1} + (b^2+ab+a^2)\varepsilon_{t-2} + \\ & + (b^3+ab^2+a^2b+a^3)\varepsilon_{t-3} \\ & + (b^4+ab^3+a^2b^2+a^3b+a^4)\varepsilon_{t-4} \\ & + (b^5+ab^4+a^2b^3+a^3b^2+a^4b+a^5)\varepsilon_{t-5} \\ & + \cdots \end{aligned}$$

Collecting we have

$$y_t = \frac{c}{(1-a)(1-b)} + \sum_{j=0}^{\infty} \psi_j \, \varepsilon_{t-j} \qquad (4)$$

where

$$\psi_0 = 1$$

$$\psi_1 = (a+b)$$

$$\psi_2 = (b^2 + ab + a^2)$$

$$\psi_3 = (b^3 + ab^2 + a^2 b + a^3)$$

$$\vdots$$

$$\psi_j = (b^j + ab^{j-1} + a^2 b^{j-2} + \dots + a^{j-1} b + a^j)$$

PROBLEM 3 Y_t follows A STATIONARY ARMA(p, q) PROCESS:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \quad (1)$$

FIRST, TAKE EXPECTATIONS AND NOTE THAT

$$E(y_t) = E(y_{t-1}) = \cdots = E(y_{t-p}) = \mu$$

SO THAT

$$\mu = c + \phi_1 \mu + \cdots + \phi_p \mu \quad \text{or}$$

$$c = (1 - \phi_1 - \phi_2 - \cdots - \phi_p)\mu \quad (2)$$

This in (1) gives

$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \quad (3)$$

where $Z_{t-j} = (Y_{t-j} - \mu)$ for $j = 0, 1, 2, \ldots, p$.

Now (4) can be written

$$\underset{(p\times 1)}{\xi_t} = \underset{(p\times p)}{F}\ \underset{(p\times 1)}{\xi_{t-1}} + G_{(p\times q)}\, V_{t\,(q\times 1)} \qquad (5)$$

where

$$\xi_t = \begin{bmatrix} y_t-\mu \\ y_{t-1}-\mu \\ \vdots \\ y_{t-(p-1)}-\mu \end{bmatrix}_{(p\times 1)} \qquad F = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 \end{bmatrix}_{(p\times p)}$$

$$G = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_q \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{[p\times(q+1)]}$$

$$V_t = \begin{bmatrix} \varepsilon_t & \varepsilon_{t-1} & \varepsilon_{t-2} & \cdots & \varepsilon_{t-q} \end{bmatrix}^T_{[(q+1)\times 1]}$$

Define $\xi_t = \begin{bmatrix} Z_t \\ Z_{t-1} \\ \vdots \\ Z_{t-(p-1)} \end{bmatrix}_p$ so that $\xi_{t-1} = \begin{bmatrix} Z_{t-1} \\ Z_{t-2} \\ \vdots \\ Z_{t-p} \end{bmatrix}_{(p\times 1)}$

and (3) can be written as

$$\xi_t = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{(p\times p)} \xi_{t-1} + \left(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}\right) \qquad (4)$$

let $V_t \underset{(q\times 1)}{} = \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix}_{[(q+1)\times 1]}$

and $G = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_q \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{[p\times(q+1)]}$

Problem 4

Let $R \equiv 1+r$ hence $\left(\frac{1}{1+r}\right) = R^{-1}$ and write (1) as

$$\boxed{P_t = E_t \sum_{j=1}^{\infty} R^{-j} d_{t+j} \qquad (2)}$$

or

$$\boxed{P_t = E_t \left[R^{-1} d_{t+1} + R^{-2} d_{t+2} + R^{-3} d_{t+3} + \cdots\right] \qquad (2')}$$

(a) $d_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

Here $E_t d_{t+1} = \mu + \theta \varepsilon_t$

and $E_t d_{t+j} = \mu$ for $j = 2, 3, 4, \ldots$

Thus (2') Gives

$$P_t = R^{-1}(\mu + \theta \varepsilon_t) + R^{-2}\mu + R^{-3}\mu + \cdots$$

$$P_t = R^{-1}\theta \varepsilon_t + R^{-1}\left[\mu + R^{-1}\mu + R^{-2}\mu + \cdots\right] \qquad (3)$$

Since $|R^{-1}| < 1$

$$\left[\mu + R^{-1}\mu + R^{-2}\mu + \cdots\right] = \left[\frac{1}{1-R^{-1}}\right]\mu = \left(\frac{1+r}{r}\right)\mu \qquad (4)$$

SUBST from (4) INTO (3) TO get

$$P_t = R^{-1}\Theta \varepsilon_t + \frac{1}{1+r}\left[\frac{1+r}{r}\right]\mu \quad \text{or}$$

$$\boxed{P_t = \frac{\mu}{r} + \Theta R^{-1}\varepsilon_t \qquad (5)}$$

(b) $d_t = \mu + \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \Theta_2 \varepsilon_{t-2}$

Here $E_t d_{t+1} = \mu + \Theta_1 \varepsilon_t + \Theta_2 \varepsilon_{t-1}$

$E_t d_{t+2} = \mu + \Theta_2 \varepsilon_t$

AND $E_t d_{t+j} = \mu$ for $j = 3, 4, 5, \ldots$

SUBSTITUTING INTO (2') gives

$$P_t = R^{-1}[\mu + \Theta_1 \varepsilon_t + \Theta_2 \varepsilon_{t-1}] + R^{-2}[\mu + \Theta_2 \varepsilon_t]$$
$$+ R^{-3}\mu + R^{-4}\mu + R^{-5}\mu + \cdots \quad \text{or}$$

$$P_t = R^{-1}[\Theta_1 \varepsilon_t + \Theta_2 \varepsilon_{t-1}] + R^{-2}\Theta_2 \varepsilon_t +$$
$$R^{-1}[\mu + R^{-1}\mu + R^{-2}\mu + \cdots] \qquad (6)$$

USING (4) IN (6) gives

$$P_t = R^{-1}\left[\Theta_1 + R^{-1}\Theta_2\right]\varepsilon_t + R^{-1}\Theta_2\varepsilon_{t-1}$$

$$+ \frac{1}{1+r}\left[\frac{1+r}{r}\right]\mu \quad \text{or}$$

$$\boxed{P_t = \frac{\mu}{r} + R^{-1}\left[\Theta_1 + R^{-1}\Theta_2 \ \varepsilon_t + R^{-1}\Theta_2\varepsilon_{t-1}\right] \quad (7)}$$

IN (5) and (7) ~~Note~~ NOTE That if DIVIDENDS ARE MA(1) The Asset Price is MA(1) and if Dividends are MA(2) The Asset Price is MA(2).

(C) $d_t = c + \phi d_{t-1} + \varepsilon_t$ (8)

write $X_t \equiv \begin{bmatrix} c \\ d_t \end{bmatrix}$ and (8) Becomes

$$X_t = F X_{t-1} + V_t \quad (9)$$

where $F \equiv \begin{bmatrix} 1 & 0 \\ 1 & \phi \end{bmatrix}$ and $V_t = \begin{bmatrix} 0 \\ \varepsilon_t \end{bmatrix}$

ASO

c) $d_t = c + \phi d_{t-1} + \varepsilon_t$ (8)

"Brute-Force" Method.

Work from (2')

$$P_t = E_t\left[R^{-1}d_{t+1} + R^{-2}d_{t+2} + R^{-3}d_{t+3} + \cdots\right] \quad (2')$$

From (8) it follows that

$$E_t d_{t+1} = c + \phi d_t \quad (9.1)$$

$$E_t d_{t+2} = c + \phi E_t d_{t+1} \quad \text{so}$$

$$E_t d_{t+2} = c + \phi c + \phi^2 d_t \quad (9.2)$$

$$E_t d_{t+3} = c + \phi E_t d_{t+2} \quad \text{so}$$

$$E_t d_{t+3} = c + \phi c + \phi^2 c + \phi^3 d_t \quad (9.3)$$

$$E_t d_{t+4} = c + \phi c + \phi^2 c + \phi^3 c + \phi^4 d_t \quad (9.4)$$

etc.

Substitute from (9.1)-(9.4) into (2') to get

$$P_t = R^{-1}(c + \phi d_t) + R^{-2}(c + \phi c + \phi^2 d_t) + R^{-3}(c + \phi c + \phi^2 c + \phi^3 d_t) + R^{-4}(c + \phi c + \phi^2 c + \phi^3 c + \phi^4 d_t) + \cdots$$

or

$$P_t = R^{-1}c + R^{-2}(1+\phi)c + R^{-3}(1+\phi+\phi^2)c + R^{-4}(1+\phi+\phi^2+\phi^3)c + \cdots$$
$$+ R^{-1}\phi[1 + R^{-1}\phi + R^{-2}\phi^2 + \cdots]d_t \qquad (10)$$

Now, note that

$$R^{-1}\phi[1 + R^{-1}\phi + R^{-2}\phi + \cdots] = \frac{\phi}{1+r}\left[\frac{1}{1-R^{-1}\phi}\right] = \frac{\phi}{1+r}\left[\frac{1}{1-\left(\frac{\phi}{1+r}\right)}\right]$$

$$= \frac{\phi}{1+r}\left[\frac{1}{\left(\frac{1+r-\phi}{1+r}\right)}\right] = \frac{\phi}{1+r}\ \frac{1+r}{1+r-\phi} = \frac{\phi}{1+r-\phi}$$

This in (10) gives

$$\boxed{P_t = R^{-1}c\left[1 + R^{-1}(1+\phi) + R^{-2}(1+\phi+\phi^2) + R^{-3}(1+\phi+\phi^2+\phi^3) + \cdots\right] + \left(\frac{\phi}{1+r-\phi}\right)d_t \qquad (11)}$$

Now, notice that

$$1+\phi = (1+\phi+\phi^2+\phi^3+\cdots)$$
$$-\ (\phi^2+\phi^3+\phi^4+\cdots)$$
$$= \left(\frac{1}{1-\phi}\right) - \left(\frac{\phi^2}{1-\phi}\right) \quad \text{so}$$

$$(1+\phi) = \left(\frac{1-\phi^2}{1-\phi}\right) \qquad (12.1)$$

A similar argument gives

$$1+\phi+\phi^2 = \frac{1-\phi^3}{1-\phi} \qquad (12.2)$$

$$1+\phi+\phi^2+\phi^3 = \frac{1-\phi^4}{1-\phi} \qquad (12.3)$$

etc.

Using $1 = \frac{1-\phi}{1-\phi}$ and (12.1) - (12.3) we have that

$$R^{-1}c\left[1+R^{-1}(1+\phi)+R^{-2}(1+\phi+\phi^2)+\cdots\right] =$$

$$R^{-1}c\left[\frac{1-\phi}{1-\phi}+R^{-1}\left(\frac{1-\phi^2}{1-\phi}\right)+R^{-2}\left(\frac{1-\phi^3}{1-\phi}\right)+\cdots\right] =$$

$$cR^{-1}\left[\frac{1}{1-\phi}+R^{-1}\left(\frac{1}{1-\phi}\right)+R^{-2}\left(\frac{1}{1-\phi}\right)+\cdots\right]$$

$$-cR^{-1}\left[\frac{\phi}{1-\phi}+\frac{R^{-1}\phi^2}{1-\phi}+\frac{R^{-2}\phi^3}{1-\phi}+\cdots\right]$$

$$= cR^{-1}\left(\frac{1}{1-\phi}\right)\left(\frac{1}{1-R^{-1}}\right) - cR^{-1}\left(\frac{\phi}{1-\phi}\right)\left(\frac{1}{1-R^{-1}\phi}\right)$$

$$= \left(\frac{c}{1+r}\right)\left(\frac{1}{1-\phi}\right)\left(\frac{1+r}{r}\right) - \left(\frac{c}{1+r}\right)\left(\frac{\phi}{1-\phi}\right)\left(\frac{1+r}{1+r-\phi}\right)$$

$$= \left(\frac{c}{1-\phi}\right)\left[\frac{1}{r} - \frac{\phi}{1+r-\phi}\right]$$

$$= \left(\frac{c}{1-\phi}\right)\left[\frac{1+r-\phi-\phi r}{r(1+r-\phi)}\right] = \frac{c}{1-\phi}\left[\frac{(1+r)(1-\phi)}{r(1+r-\phi)}\right]$$

$$= \frac{1+r}{(1+r-\phi)r}\, c$$

~~Using this~~ Thus

$$R^{-1}c\left[1+R^{-1}(1+\phi)+R^{-2}(1+\phi+\phi^2)+\cdots\right]$$

$$= \frac{1+r}{(1+r-\phi)r}\, c \qquad (13)$$

use (13) in (11) to get

$$\boxed{P_t = \frac{1+r}{(1+r-\phi)r}\, c + \left(\frac{\phi}{1+r-\phi}\right) d_t \qquad (14)}$$

Method using Selector MATRIX

Define

$$X_t = \underset{(2\times1)}{\begin{bmatrix} c \\ d_t \end{bmatrix}} \qquad F = \underset{(2\times2)}{\begin{bmatrix} 1 & 0 \\ 1 & \phi \end{bmatrix}} \qquad V_t = \underset{(2\times1)}{\begin{bmatrix} 0 \\ \varepsilon_t \end{bmatrix}}$$

and $S = \underset{(1\times2)}{\begin{bmatrix} 0 & 1 \end{bmatrix}}$ so that (8)

can be written as

$$X_t = F X_{t-1} + V_t \qquad (15)$$

and

$$d_t = S \cdot X_t \qquad (16)$$

Begin from (2)

$$P_t = \underset{t}{E} \sum_{j=1}^{\infty} R^{-j} \underset{t}{E}\, d_{t+j} \qquad (2)$$

which we can now write as

$$P_t = \underset{t}{E} \sum_{j=1}^{\infty} R^{-j} S X_{t+1}$$ or, since R^{-1} is a scalar,

$$P_t = S \sum_{j=1}^{\infty} R^{-j} \underset{t}{E} X_{t+1} \qquad (17)$$

Since $\underset{t}{E} V_{t+1} = \underset{t}{E} V_{t+2} = \underset{t}{E} V_{t+3} = \cdots = 0$

we can use (15) in (17) to get

$$P_t = S \sum_{j=1}^{\infty} R^{-j} F^{j} X_t$$ or

$$P_t = S R^{-1} F \left[\sum_{j=0}^{\infty} R^{-j} F^{j} \right] X_t \qquad (18)$$

The Eigenvalues of F are 1 and ϕ both of which are less than $(R^{-1})^{-1} = R = 1 + r$ in Absolute Value. Therefore we can invoke Hamilton's Proposition 1.3 (page 20) and write (18) as

$$P_t = S R^{-1} F \left[I - R^{-1} F \right]^{-1} X_t \qquad (19)$$

Since $\left[I - R^{-1}F\right] = \begin{bmatrix} (1-R^{-1}) & 0 \\ -R^{-1} & (1-R^{-1}\phi) \end{bmatrix}$ it follows that

$$\left[I - R^{-1}F\right]^{-1} = \frac{1}{(1-R^{-1})(1-R^{-1}\phi)} \begin{bmatrix} (1-R^{-1}\phi) & 0 \\ +R^{-1} & (1-R^{-1}) \end{bmatrix} \quad (20)$$

Use (20) in (19) to write

$$P_t = \begin{bmatrix} 0 & 1 \end{bmatrix} R^{-1} \begin{bmatrix} 1 & 0 \\ 1 & \phi \end{bmatrix} \frac{1}{(1-R^{-1})(1-R^{-1}\phi)} \begin{bmatrix} 1-R^{-1}\phi & 0 \\ +R^{-1} & (1-R^{-1}) \end{bmatrix} \begin{bmatrix} c \\ d_t \end{bmatrix}$$

or

$$P_t = \underset{(1\times2)}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \underset{(1\times1)}{\frac{R^{-1}}{(1-R^{-1})(1-R^{-1}\phi)}} \underset{(2\times2)}{\begin{bmatrix} (1-R^{-1}\phi) & 0 \\ (1-R^{-1}\phi+\phi R^{-1}) & \phi(1-R^{-1}) \end{bmatrix}} \underset{(2\times1)}{\begin{bmatrix} c \\ d_t \end{bmatrix}}$$

or

$$P_t = \frac{R^{-1}}{(1-R^{-1})(1-R^{-1}\phi)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} (1-R^{-1}\phi)c \\ c + \phi(1-R^{-1}\phi)d_t \end{bmatrix}$$

or

$$P_t = \frac{R^{-1}}{(1-R^{-1})(1-R^{-1}\phi)} \left[c + \phi(1-R^{-1})d_t \right] \quad (21)$$

OR

Note that $\left[\frac{R^{-1}\phi}{1-R^{-1}\phi}\right] = \frac{R}{R}\left[\frac{R^{-1}\phi}{1-R^{-1}\phi}\right] = \frac{\phi}{R-\phi} = \frac{\phi}{1+r-\phi}$

This in (21) gives

$$P_t = \frac{R^{-1}}{(1-R^{-1})(1-R^{-1}\phi)} c + \left(\frac{\phi}{1+r-\phi}\right) d_t \quad (22)$$

$$\frac{R^{-1}}{(1-R^{-1})(1-R^{-1}\phi)} = \frac{\left(\frac{1}{1+r}\right)}{\left(\frac{r}{1+r}\right)\left(\frac{1+r-\phi}{1+r}\right)} = \frac{(1+r)}{(1+r-\phi)r}$$

This in (22) gives

$$\boxed{P_t = \frac{(1+r)}{r(1+r-\phi)} c + \left(\frac{\phi}{1+r-\phi}\right) d_t \quad (23)}$$

cf (14)