Homework 1

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ECON 7920- Econometrics II

Due on Feb 3, 2022

1 Question 12.1

Solution:

a. From equation (12.2), we know: $y = m(x, \theta_0) + u$, $E(u|x) = 0$. Plug this into (12.4) and take conditional expectation w.r.t x:

$$
\begin{aligned}
E[[y - m(x,\theta)^2]|x] &= E[[y - m(x,\theta_0)]^2|x] + 2E[m(x,\theta_0) - m(x,\theta)|x]E[u|x] + E[[m(x,\theta_0) - m(x,\theta)^2]|x \\
&= E(u^2|x) + 2E[m(x,\theta_0) - m(x,\theta)|x] \cdot 0 + E[[m(x,\theta_0) - m(x,\theta)^2]|x] \\
&= E(u^2|x) + E[[m(x,\theta_0) - m(x,\theta)^2]|x] \\
&=^{LIE} E(u^2|x) + E[m(x,\theta_0) - m(x,\theta)^2]
\end{aligned}
$$

Since the first term is just error term, which is not relevant to the parameter θ, therefore, the only left to consider is the second term. When $\theta = \theta_0$, the second term is 0 obviously. When θ is picked any value in the space excluding θ_0, the second term is alway > 0.

b. Because from the conditional expectation, we can derive and get the unconditional expectation. In other way, if we derive and estimate the parameter value θ given x, we can narrow down the scope of searching, if not, there may be multiple values available and waste of computation power.

2 Question 12.2

Solution:

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a.

$$
\begin{aligned}
E(u^2|x) &= E[(y - E(y|x))^2|x] \\
&= E[y^2 - 2yE(y|x) + (E(y|x))^2|x] \\
&= E(y^2|x) - 2E(y|x)E(y|x) + E(y|x)^2 \\
&= var(y|x) + E(y|x)^2 - E(y|x)^2 \\
&= var(y|x) \\
&= \exp(\alpha_0 + x\gamma_0)
\end{aligned}
$$

b. As $\hat{u}$ is NLS error, our goal is to make the estimation of α_0 and γ_0 consistent. First, Set up the objection function $\min_{\alpha,\gamma} \sum_{i=1}^{N} \{(u_i^2 - \exp(\alpha + x_i\gamma))^2\}$. Substitute $u = y - E(y|x)$ into this objective function, we can obtain:

$$
\min_{\alpha,\gamma} \sum_{i=1}^{N} \{((y_i - m(x,\hat{\theta}))^2 - \exp(\alpha + x_i\gamma))^2\}
$$

From M-estimator method, we can estimate $\hat{\theta}$ to θ_0, and thus, $\hat{u} \to u$, and then the two parameters α_0 and γ_0 can be estimated consistently.

c. Since we know from the last question, we can estimate α and γ, thus, the form is $\min_i^N \frac{\{(y-E(y|x))\}}{\exp(\hat{\alpha}+x_i\hat{\gamma})}$ for WLNS

d. Take square of both sides of v, we can obtain: $v^2 = \exp(-(\alpha_0 + x + \gamma_0))u^2$. Then, take log on both sides: $\log(v^2) = \log(u^2) - (\alpha + x\gamma_0)$. From this question, we know v is independent with x. Thus, Take expectation: $E(\log(u^2)|x) = E(\log(v^2)|x) + \alpha + x\gamma_0$, the first term is constant, thus, γ_0 can be estimated by the regression of $1, x_i$. We can use M-estimator to replace $\hat{u}$ and by doing WNLS to obtain the consistency.

e. I'll make some robust check to test the variance we are suspicious.

3 Question 12.3

Solution:

a. We assume the elasticity of $\hat{E(\cdot)}$ to z_1 is ϵ_1

$$
\begin{aligned}
\epsilon_1 &= \frac{\partial \hat{E}(y|z)}{\partial z_1} \cdot \frac{z_1}{E(\hat{y}|z)} \\
&= \exp(\cdot)\frac{\hat{\theta_2}}{z_1}\frac{z_1}{\exp(\cdot)} \\
&= \hat{\theta_2}
\end{aligned}
$$

b. Since we want to know what percentage of $E(y|z)$ would be. We take log on both sides first.

$$\log(\hat{E}(y|z)) = \hat{\theta_1} + \hat{\theta_2}\log(z_1) + \hat{\theta_3}z_2 \tag{12.3.1}$$

From (12.3.1), we take the partial differential equation w.r.t. z_2

$$\frac{\partial \hat{E}(\cdot)}{\partial z_2} = \hat{\theta_3}$$

Thus, when $\Delta z_2 = 1$, it will cause $\hat{E}(y|z)$ to change $\theta_3\%$

c. Take the first order differentiation w.r.t. z_2:

$$\exp(\cdot)(\hat{\theta_3} + 2\hat{\theta_4}z_2)$$

We can estimate $z_2 = \frac{-\hat{\theta_3}}{2\hat{\theta_4}}$, by the consistency of the parameters estimation, we can conclude: $z_2 = -\frac{\theta_3}{2\theta_4}$.

d. From the question, we know $\exp(x\theta) = \exp(x_1\theta_1 + x_2\theta_2)$, so we set $m(x,\theta) = \exp(x\theta)$. Thus, $\Delta_{\theta_i} m_i(x,\theta) = \exp(x_i\theta_i)x_i$. First, we make regression with $m(x,\theta)$ with its estimation to get the value of residuals $\hat{u}$, then make regressions on $\hat{u}$ with the gradidents of m function to gain LM value. Finally, we compare this value with our threshold to test.

4 Question 12.17

Solution:

a. For this we can get a similar equation below the equation (12.15) in textbook.

$$N^{-\frac{1}{2}}\sum_{i=1}^{N} g(w_i,\theta) = N^{-\frac{1}{2}}\sum_{i=1}^{N} g(w_i,\theta_0) + N^{-1}\sum_{i=1}^{N} G_i\sqrt{N}(\hat{\theta} - \theta_0) \qquad 12.17.1$$

Where G is Jacobian w.r.t θ. Then follow the steps in textbook form (12.15) to (12.17), we can obtain $\sqrt{N}(\hat{\theta} - \theta_0) = o_p(1)$. By LLN, we assume $N^{-1}\sum_{i=1}^{N} G_i = G_0$. Thus:

$$\sqrt{N}\hat{\delta} = N^{-\frac{1}{2}}\sum_{i=1}^{N}(g(w_i,\theta_0) - G_0A_0^{-1}s_i(\theta_0)) + o_p(1) \qquad 12.17.2$$

Rewrite 12.17.2,

$$\sqrt{(\delta \hat{-} \delta_0)} = N^{-\frac{1}{2}}\sum_{i=1}^{N}(g(w_i,\theta_0) - \delta_0 - G_0A_o^{-1}s_i(\theta_0)) + o_p(1) \qquad 12.17.3$$

Where the expectation of $g(w_i,\theta_0) - \delta_0 - G_0A_o^{-1}s_i(\theta_0)$ is equal to 0. Thus, we can obtain:

$$\sqrt{N}(\hat{\delta} - \delta_0) \ {}^a\mathcal{N}(0, (g_i - \delta_0 - G_0A_0^{-1}s_i)'(g_i - \delta_0 - G_0A_0^{-1}s_i)) \qquad 12.17.4$$

b. If we want to estimate the parameters consistently, we need to keep $\hat{A} \to A_0$ and $G_i \to G_0$. Then we can replace G_0 and A_0 with $\hat{G}$ and $\hat{A}$ in 12.17.4.

c. As given in the question $E(s(w, \theta_0)|x) = 0$, and we assume $g(w, \theta)$, $w = (x, y)$, so $cov(g, s) = 0$. Thus, $cov(g_i - s_0, G_0 A_0^{-1} s_i) = 0$, Therefore: $var(g_i - \delta_0 - G_0 A_0^{-1} S_I) = var(g_i) + G_0(Avar\sqrt{N}(\hat{\theta} - \theta_0))G_0'$