

Next, let  $V_1 \equiv \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$  denote the eigenvector of  $P$  associated with  $\lambda_2 = 1$ . Thus

$$\begin{bmatrix} (p_{11}-1) & p_{21} \\ p_{12} & (p_{22}-1) \end{bmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, using  $p_{21} = 1 - p_{22}$  and initial normalization  $v_{11} = 1$  we have

$$(p_{11}-1) + (1-p_{22})v_{12} = 0 \quad \text{or}$$

$$v_{12} = \frac{1-p_{11}}{1-p_{22}}. \quad \text{Thus}$$

$$V_1 = \begin{bmatrix} 1 \\ \left( \frac{1-p_{11}}{1-p_{22}} \right) \end{bmatrix}. \quad \text{Renormalizing so that the elements of the eigenvector sum to 1}$$

$$W_1 = \left[ \frac{1}{1 + \left( \frac{1-p_{11}}{1-p_{22}} \right)} \right] V_1 = \left[ \frac{1}{\left( \frac{1-p_{22}+1-p_{11}}{1-p_{22}} \right)} \right] V_1$$

$$= \left( \frac{1-p_{22}}{2-p_{11}-p_{22}} \right) V_1$$



$$\text{So } W_1 = \begin{bmatrix} \frac{1-p_{22}}{2-p_{11}-p_{22}} \\ \left( \frac{1-p_{22}}{2-p_{11}-p_{22}} \right) \cdot \left( \frac{1-p_{11}}{1-p_{22}} \right) \end{bmatrix} \quad \text{or}$$

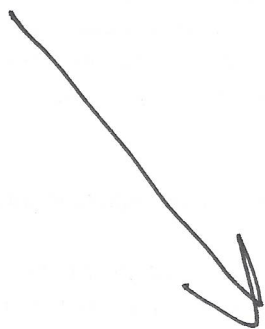
$$W_1 = \begin{bmatrix} \left( \frac{1-p_{22}}{2-p_{11}-p_{22}} \right) \\ \left( \frac{1-p_{11}}{2-p_{11}-p_{22}} \right) \end{bmatrix}$$

which verifies the  
Second column of  
HAMILTON'S MATRIX  $T$ .

Note that  $W_1 = \Pi$ , The vector of ergodic probabilities.

To Verify  $T^{-1}$  we need simply show that

$$T \cdot T^{-1} = I_2$$



$$T \cdot T^{-1} = \begin{bmatrix} \frac{1-p_{22}}{2-p_{11}-p_{22}} & -1 \\ \frac{1-p_{11}}{2-p_{11}-p_{22}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{-(1-p_{11})}{2-p_{11}-p_{22}} & \frac{(1-p_{22})}{2-p_{11}-p_{22}} \end{bmatrix}$$

$$(1,1) \text{ element} = \frac{1-p_{22}}{2-p_{11}-p_{22}} + \frac{1-p_{11}}{2-p_{11}-p_{22}} = 1 \quad \checkmark$$

$$(1,2) \text{ element} = \frac{1-p_{22}}{2-p_{11}-p_{22}} - \frac{1-p_{22}}{2-p_{11}-p_{22}} = 0 \quad \checkmark$$

$$(2,1) \text{ element} = \frac{1-p_{11}}{2-p_{11}-p_{22}} + \frac{-(1-p_{11})}{2-p_{11}-p_{22}} = 0 \quad \checkmark$$

$$(2,2) \text{ element} = \frac{1-p_{11}}{2-p_{11}-p_{22}} + \frac{1-p_{22}}{2-p_{11}-p_{22}} = 1 \quad \checkmark$$

Additional Problem 3

(a)  $A$  is invertible. Thus

$$(A^{-1})A = I.$$

TRANSPOSING gives  $A'(A^{-1})' = I' = I$

Thus  $(A^{-1})' = (A')^{-1} \quad (1) \quad \text{QED}$

(b) Begin From  $P = T \Lambda T^{-1}$

TRANSPOSE TO get  $P' = (T^{-1})' \Lambda T'$

Post Multiply BY  $(T')^{-1}$  TO get

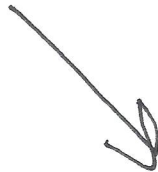
$$P'(T')^{-1} = (T^{-1})' \Lambda \quad \text{or, using (1),}$$

$$P'(T^{-1})' = (T^{-1})' \Lambda. \quad (2)$$

Denote the columns of  $(T^{-1})'$  as follows:

$$(T^{-1})' = [y_{(N \times 1)}, y_{2(N \times 1)}, \dots, y_{N(N \times 1)}]$$

Then eqn (2) gives



$$P' [y_1, y_2, \dots, y_N] = [y_1, y_2, \dots, y_N] \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & & \lambda_N \end{bmatrix}$$

which, multiplying out, gives

$$P' y_1 = y_1$$

$$P' y_2 = y_2 \lambda_2$$

$$\vdots$$

$$P' y_N = y_N \lambda_N$$

The first of these equations,  $P' y_1 = y_1$ , confirms that the first column of  $(T^{-1})'$  is the eigenvector of  $P'$  associated with its eigenvalue of 1. The first row of  $T^{-1}$  is, of course,  $y_1' (1 \times N)$ .

Additional Problem 4

(a) For this Markov chain we define  $\xi_{t+j}$  as follows:  
(3x1)

$$\xi_t = \begin{cases} (1, 0, 0)' & \text{if } S_t = S_1 \\ (0, 1, 0)' & \text{if } S_t = S_2 \\ (0, 0, 1)' & \text{if } S_t = S_3 \end{cases} \quad (3)$$

(b) Given (3) we can write the stochastic process for dividends as

$$d_{t+j} = \mu \xi_{t+j} + \varepsilon_{t+j} \quad (4)$$

where  $\mu_{(1 \times 3)} = [\mu_1, \mu_2, \mu_3]$ .

Note that

$$E(\varepsilon_{t+j} | \xi_t) = 0 \quad \text{for } j=1, 2, 3, \dots \quad (5)$$

Note Also that

$$E(\xi_{t+j} | \xi_t) = P^j \xi_t \quad \text{for } j=1, 2, 3, \dots \quad (6)$$



Thus

$$E(d_{t+1} | \mathcal{F}_t) = E[\mu \mathcal{F}_{t+1} + \varepsilon_{t+1} | \mathcal{F}_t] = \mu P \mathcal{F}_t \quad (7a)$$

$$E(d_{t+2} | \mathcal{F}_t) = E[\mu \mathcal{F}_{t+2} + \varepsilon_{t+2} | \mathcal{F}_t] = \mu P^2 \mathcal{F}_t \quad (7b)$$

and, in general,

$$E(d_{t+j} | \mathcal{F}_t) = E[\mu \mathcal{F}_{t+j} + \varepsilon_{t+j} | \mathcal{F}_t] = \mu P^j \mathcal{F}_t \quad (7c)$$

for  $j = 1, 2, 3, \dots$

C.) Based on eqn 4, two time series variables determine  $d_{t+j}$ ,  $\mathcal{F}_{t+j}$  and  $\varepsilon_{t+j}$ . Since  $\varepsilon_{t+j}$  is white noise nothing in the current information set is useful in forecasting  $\varepsilon_{t+j}$  for  $j = 1, 2, 3, \dots$ . Since  $\mathcal{F}_{t+j}$  is a Markov chain ~~the only variable in the current information set that is useful in forecasting~~

$$E(\mathcal{F}_{t+j} | \mathcal{F}_t, \mathcal{F}_{t-1}, \mathcal{F}_{t-2}, \dots) = E(\mathcal{F}_{t+j} | \mathcal{F}_t).$$

Thus

$$E(d_{t+j} | \mathcal{D}_t) = E(d_{t+j} | \mathcal{F}_t).$$

d.) Now

$$\begin{aligned}
 A_t &= E_t \left[ \left( \frac{1}{1+r} \right) d_{t+1} + \left( \frac{1}{1+r} \right)^2 d_{t+2} + \dots \right] \\
 &= E_t \left[ \left( \frac{1}{1+r} \right) d_{t+1} + \left( \frac{1}{1+r} \right)^2 d_{t+2} + \dots \mid \mathcal{F}_t \right] \\
 &= \left( \frac{1}{1+r} \right) \left[ E(d_{t+1} \mid \mathcal{F}_t) + \left( \frac{1}{1+r} \right) E(d_{t+2} \mid \mathcal{F}_t) + \left( \frac{1}{1+r} \right)^2 E(d_{t+3} \mid \mathcal{F}_t) \right. \\
 &\quad \left. + \dots \right]
 \end{aligned}$$

$$= \left( \frac{1}{1+r} \right) \left[ \mu P \mathcal{F}_t + \left( \frac{1}{1+r} \right) \mu P^2 \mathcal{F}_t + \left( \frac{1}{1+r} \right)^2 \mu P^3 \mathcal{F}_t + \dots \right]$$

$$A_t = \left( \frac{1}{1+r} \right) \mu \left[ I + \left( \frac{1}{1+r} \right) P + \left( \frac{1}{1+r} \right)^2 P^2 + \dots \right] P \mathcal{F}_t \quad (8)$$

Note that, since the Markov chain is Ergodic, its largest eigenvalue in absolute value is 1. Thus all of the eigenvalues of  $\left( \frac{1}{1+r} \right) P$  lie inside the unit circle. Thus

$$\left[ I + \left( \frac{1}{1+r} \right) P + \left( \frac{1}{1+r} \right)^2 P^2 + \dots \right] = \left[ I - \left( \frac{1}{1+r} \right) P \right]^{-1}$$

and (8) becomes <sup>(cf. Hamilton p 732)</sup>

$$A_t = \left( \frac{\mu}{1+r} \right) \left[ I - \left( \frac{1}{1+r} \right) P \right]^{-1} P \mathcal{F}_t \quad (9)$$

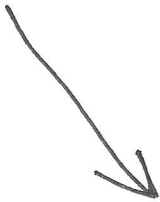


# Additional Problem 5

(a)

	From $\rightarrow$	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	
TO $\downarrow$							
$S_1$		1	$\frac{1}{2}$	0	0	0	] = P
$S_2$		0	0	$\frac{1}{2}$	0	0	
$S_3$		0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	
$S_4$		0	0	$\frac{1}{2}$	0	0	
$S_5$		0	0	0	$\frac{1}{2}$	1	

The ~~is~~ MARKOV chain is NON ERGODIC. It is Not Possible to go from every STATE to every STATE: STATES  $S_1$  and  $S_5$  are absorbing states. Since All Regular MARKOV CHAINS Are Ergodic and Since this MARKOV chain is NonErgodic it is NOT Regular.



(b)

$$\begin{array}{c}
 \text{From} \longrightarrow \\
 \text{TO} \searrow
 \end{array}
 \begin{array}{c}
 S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \\
 \begin{array}{c}
 S_1 \\
 S_2 \\
 S_3 \\
 S_4 \\
 S_5
 \end{array}
 \begin{bmatrix}
 1 & 1/3 & 0 & 0 & 0 \\
 0 & 0 & 1/3 & 0 & 0 \\
 0 & 2/3 & 0 & 1/3 & 0 \\
 0 & 0 & 2/3 & 0 & 1 \\
 0 & 0 & 0 & 2/3 & 0
 \end{bmatrix}
 \end{array}
 = P$$

Once Again The Markov Chain is Non Ergodic. It is Not possible to go from State  $S_1$  to every other (to any other) state as state  $S_1$  is an Absorbing state. The Markov chain is Not Regular because it is Non Ergodic and All Regular Chains are Ergodic.