"Ans To P. Set 2, Part B"

Econ 6020: Macro Theory I

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Chapter 3

3.1. Re-work the optimal growth solution in terms of the original variables, i.e. without first taking deviations about trend growth.

(a) Derive the Euler equation

(b) Discuss the steady-state optimal growth paths for consumption, capital and output.

Solution

(a) The problem is to maximize

$$\sum_{s=0}^{\infty} \beta^s U(C_{t+s})$$

where $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$, subject to the national income identity, the capital accumulation equation, the production function and the growth of population n:

$$\begin{aligned}
Y_t &= C_t + I_t \\
\Delta K_{t+1} &= I_t - \delta K_t \\
Y_t &= (1+\mu)^t K_t^{\alpha} N_t^{1-\alpha} \\
N_t &= (1+n)^t N_0, \quad N_0 = 1
\end{aligned}$$

The Lagrangian for this problem written in terms of the original variables is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \{\beta^s \left[\frac{C_{t+s}^{1-\sigma}-1}{1-\sigma}\right] + \lambda_{t+s}[\phi^{t+s} K_{t+s}^{\alpha} - C_{t+s} - K_{t+s+1} + (1-\delta)K_{t+s}]\}$$

where $\phi = (1+\mu)(1+n)^{(1-\alpha)} \simeq (1+\eta)^{1-\alpha}$, $\eta = n + \frac{\mu}{1-\alpha}$. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial C_{t+s}} &= \beta^s C_{t+s}^{-\sigma} - \lambda_{t+s} = 0 \qquad\qquad s \geq 0 \\
\frac{\partial \mathcal{L}_t}{\partial K_{t+s}} &= \lambda_{t+s}[\alpha \phi^{t+s} K_{t+s}^{\alpha-1} + 1 - \delta] - \lambda_{t+s-1} = 0 \qquad\qquad s > 0
\end{aligned}$$

Hence the Euler equation is

$$\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}[\alpha \phi^{t+1} K_{t+1}^{\alpha-1} + 1 - \delta] = 1$$

(b) The advantage of transforming the variables as in Chapter 3 is now apparent. It enabled us to derive the steady-state solution in a similar way to static models and hence to use previous results. Now we need a different approach. If we assume that in steady state consumption grows at an arbitrary constant rate γ, then the Euler equation can be rewritten

$$\beta(1+\gamma)^{-\sigma}[\alpha\phi^{t+1}K_{t+1}^{\alpha-1}+1-\delta]=1$$

Hence the steady-state path of capital is

$$\begin{aligned} K_t &= \psi^{-\frac{1}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} \\ &\simeq \psi^{-\frac{1}{1-\alpha}}(1+\eta)^t \end{aligned}$$

where $\psi = \frac{(1+\theta)(1+\gamma)^{\sigma}+\delta-1}{\alpha}$. Hence, in steady state, capital grows approximately at the rate $\eta = n + \frac{\mu}{1-\alpha}$ as before.

The production function in steady state is

$$\begin{aligned} Y_t &= \phi^t K_t^{\alpha} = \psi^{-\frac{\alpha}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} \\ &\simeq \psi^{-\frac{\alpha}{1-\alpha}}(1+\eta)^t \end{aligned}$$

Thus output also grows at the rate η.

The resource constraint for the economy is

$$Y_t = C_t + \Delta K_{t+1} - \delta K_t$$

In steady state this becomes

$$\psi^{-\frac{\alpha}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} = C_t + (\phi^{\frac{1}{1-\alpha}}-1)\psi^{-\frac{1}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} - \delta\psi^{-\frac{1}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t}$$

Hence steady-state consumption is

$$\begin{aligned} C_t &= (2-\phi^{\frac{1}{1-\alpha}}+\delta)\psi^{-\frac{\alpha}{1-\alpha}}\phi^{\frac{1}{1-\alpha}t} \\ &\simeq (2-\phi^{\frac{1}{1-\alpha}}+\delta)\psi^{-\frac{\alpha}{1-\alpha}}(1+\eta)^t \end{aligned}$$

implying that consumption grows at the same constant rate as capital and output, which confirms our original assumption and shows that $\gamma = \eta$. We recall that, as the growth rates of output, capital and consumption are the same, the optimal solution is a balanced growth path.

3.2. Consider the Solow-Swan model of growth for the constant returns to scale production function $Y_t = F[e^{\mu t}K_t, e^{\nu t}N_t]$ where μ and ν are the rates of capital and labor augmenting technical progress.

(a) Show that the model has constant steady-state growth when technical progress is labor augmenting.

(b) What is the effect of the presence of non-labor augmenting technical progress?

Solution

(a) First we recall some key results from Chapter 3. The savings rate for the economy is $s_t = 1 - \frac{C_t}{Y_t} = i_t$, the rate of investment I_t/Y_t. The rate of growth of population is n and of capital is $\frac{\Delta K_{t+1}}{K_t} = s\frac{y_t}{k_t} - \delta$; the growth of capital per capita is $\frac{\Delta k_{t+1}}{k_t} = s\frac{y_t}{k_t} - (\delta + n)$ and the capital accumulation equation is $\Delta k_{t+1} = sy_t - (\delta + n)k_t$ where $y_t = Y_t/N_t$ and $k_t = K_t/N_t$. Hence the sustainable rate of growth of capital per capita is

$$\gamma = \frac{\Delta k_{t+1}}{k_t} = s\frac{y_t}{k_t} - (\delta + n)$$

For the given production function

$$\frac{y_t}{k_t} = e^{\mu t}F[1, e^{(\nu-\mu)t}k_t^{-1}] = e^{\mu t}G[e^{(\nu-\mu)t}k_t^{-1}]$$

and so

$$\gamma = se^{\mu t}G[e^{(\nu-\mu)t}k_t^{-1}] - (\delta + n)$$

For the rate of growth of capital to be constant we therefore require that $\frac{y_t}{k_t}$ is constant. If $\mu = 0$, and hence technical progress is solely labor augmenting, then we simply require that $k_t = e^{\nu t}$. The rate of growth of capital is then $\nu + n$.

Additional Problem 1

1. (a). ~~Product~~ GROWTH RATE of $X_t \cdot Z_t$.

CONTINUOUS TIME: Notation $\frac{dX_t}{dt} \equiv \dot{X}_t$

Let $\frac{\dot{X}_t}{X_t} = x$ and $\frac{\dot{Z}_t}{Z_t} = \eta$

What if $\left[\frac{d(X_t \cdot Z_t)}{dt}\right]\left[\frac{1}{X_t Z_t}\right]$?

$$\frac{d(X_t Z_t)}{d_t} = \frac{dX_t}{d_t} \cdot Z_t + X_t \frac{dZ_t}{dt} \qquad (1)$$

$$\text{So } \frac{\left[\frac{d(X_t \cdot Z_t)}{dt}\right]}{X_t Z_t} = \qquad + \frac{X_t \dot{Z}_t}{X_t Z_t} = \frac{\dot{X}_t}{X_t} + \frac{\dot{Z}_t}{Z_t}$$

Thus

$$\frac{\left[\frac{d(X_t Z_t)}{dt}\right]}{X_t Z_t} = x + \eta$$

Discrete Time: Let $\frac{\Delta X_{t+1}}{X_t} = x$ $\frac{\Delta Z_{t+1}}{Z_t} = \gamma$

Log Approx to percentages

$$\cancel{\Delta(X_t Z_t)}$$

$$\frac{\Delta(X_{t+1} Z_{t+1})}{X_t Z_t} \cong \ln(X_{t+1} Z_{t+1}) - \ln(X_t Z_t)$$

$$= \ln(X_{t+1}) + \ln(Z_{t+1}) - (\ln X_t + \ln Z_t)$$

$$= (\ln X_{t+1} - \ln X_t) + (\ln Z_{t+1} - \ln Z_t)$$

$$\cong \frac{\Delta X_{t+1}}{X_t} + \frac{\Delta Z_{t+1}}{Z_t}$$

Thus

$$\frac{\Delta(X_{t+1} \cdot Z_{t+1})}{X_t Z_t} \cong x + \gamma.$$

Taylor Series Expansion.

$$\frac{\Delta(X_{t+1} Z_{t+1})}{X_t Z_t} = \frac{(X_{t+1} Z_{t+1}) - (X_t Z_t)}{X_t Z_t}$$ or

$$\frac{\Delta X_{t+1} Z_{t+1}}{X_t Z_t} = \frac{X_{t+1} Z_{t+1}}{X_t Z_t} - 1 \qquad (2)$$

Treat $\frac{X_{t+1} Z_{t+1}}{X_t Z_t}$ as a function of X_{t+1} and Z_{t+1}, given X_t and Z_t. Expand this function around (X_t, Z_t)

$$\left[\frac{X_{t+1} Z_{t+1}}{X_t Z_t}\right] \stackrel{\sim}{=} \frac{X_t Z_t}{X_t Z_t} + \left(\frac{Z_t}{X_t Z_t}\right)\left(X_{t+1} - X_t\right) + \left(\frac{X_t}{X_t Z_t}\right)\left(Z_{t+1} - Z_t\right)$$

$$= 1 + \frac{X_{t+1} - X_t}{X_t} + \frac{Z_{t+1} - Z_t}{Z_t} \quad \text{so}$$

$$\frac{X_{t+1} Z_{t+1}}{X_t Z_t} \stackrel{\sim}{=} 1 + x + \eta \qquad (3)$$

use (3) in (2) to get

(b) Growth Rate of $\left(\frac{X_t}{Z_t}\right)$. Continuous Time

$$\frac{d\left(\frac{X_t}{Z_t}\right)}{dt} = \frac{\left(\frac{dX}{dt}\right) Z - \left(\frac{dZ}{dt}\right) X}{(Z)^2} = \frac{\dot{X} Z - \dot{Z} X}{(Z)^2} \qquad (4)$$

$$\frac{\left[d\left(\frac{X_t}{Z_t}\right)/dt\right]}{(X_t/Z_t)} = \frac{\dot{X} Z}{(Z)^2} \cdot \left(\frac{Z}{X}\right) - \frac{\dot{Z} X}{(Z)^2}\left(\frac{Z}{X}\right)$$

$$= \left(\frac{\dot{X}}{X}\right) - \left(\frac{\dot{Z}}{Z}\right) = x - \eta.$$

Discrete time

Log Approx To percentages

$$\frac{\Delta\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(X_t/Z_t\right)} \cong \ln\left(\frac{X_{t+1}}{Z_{t+1}}\right) - \ln\left(\frac{X_t}{Z_t}\right)$$

$$= \left(\ln X_{t+1} - \ln Z_{t+1}\right) - \left(\ln X_t - \ln Z_t\right)$$

$$= \left(\ln X_{t+1} - \ln X_t\right) - \left(\ln Z_{t+1} - \ln Z_t\right)$$

$$\cong \frac{\Delta X_{t+1}}{X_t} - \frac{\Delta Z_{t+1}}{Z_t}$$

Thus

$$\frac{\Delta\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(X_t/Z_t\right)} \cong x - \eta$$

Taylor Series Expansion

$$\frac{\Delta\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(X_t/Z_t\right)} = \frac{\left(\frac{X_{t+1}}{Z_{t+1}}\right) - \left(\frac{X_t}{Z_t}\right)}{\left(\frac{X_t}{Z_t}\right)} = \frac{\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(\frac{X_t}{Z_t}\right)} - 1$$

So

$$\frac{\Delta\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(\frac{X_t}{Z_t}\right)} = \frac{\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(\frac{X_t}{Z_t}\right)} - 1 \qquad (5)$$

Treat $\left(\frac{X_{t+1}}{Z_{t+1}}\right)$ as a function of X_{t+1} and Z_{t+1} and expand around (X_t, Z_t).

$$\left(\frac{X_{t+1}}{Z_{t+1}}\right) \cong \left(\frac{X_t}{Z_t}\right) + \left(\frac{1}{Z_t}\right)\left(X_{t+1} - X_t\right) + \left(\frac{-X_t}{Z_t^2}\right)\left(Z_{t+1} - Z_t\right)$$

$$= \left(\frac{X_t}{Z_t}\right) + \left(\frac{X_t}{Z_t}\right)\left(\frac{X_{t+1} - X_t}{X_t}\right) - \left(\frac{X_t}{Z_t}\right)\left(\frac{Z_{t+1} - Z_t}{Z_t}\right)$$

so

$$\left(\frac{X_{t+1}}{Z_{t+1}}\right) \cong \left(\frac{X_t}{Z_t}\right)\left[1 + x - \eta\right] \qquad (6)$$

(6) in (5) gives

$$\frac{\Delta\left(\frac{X_{t+1}}{Z_{t+1}}\right)}{\left(\frac{X_t}{Z_t}\right)} \cong x - \eta$$

c) $Z_t = X_t^{\alpha}$

Continuous Time

$$\frac{\dot{Z}_t}{Z_t} = \frac{\left(d Z_t/dt\right)}{Z_t} = \frac{d\left(X_t^{\alpha}\right)/dt}{Z_t} = \left[\frac{d\exp(\alpha \ln X_t)}{dt}\right] / Z_t$$

$$= \frac{\left[\alpha \exp(\alpha \ln X_t)\right] \frac{d \ln X_t}{dt}}{Z_t} = \left[\alpha \left(X_t^{\alpha}\right) \frac{dX_t/dt}{X_t}\right] / Z_t$$

$$= \left[\alpha Z_t x\right] / Z_t = \alpha x .$$

$$\frac{d X_t^\alpha}{dt} = \frac{d \exp(\alpha \ln X_t)}{dt} = \alpha \exp(\alpha \ln X_t) \frac{d \ln X_t}{dt}$$

$$= \alpha X_t^\alpha \cancel{x} \frac{dX/dt}{X_t}$$

$$= \alpha X_t^\alpha x = \alpha (Z_t) x$$

$$\frac{\Delta Z_{t+1}}{Z_t} \cong \ln Z_{t+1} - \ln Z_t$$

$$= \alpha \ln X_{t+1} - \alpha \ln X_t$$

$$\cong \alpha \frac{\Delta X_{t+1}}{X_t} = \alpha x$$

$$\frac{\Delta Z_{t+1}}{Z_t} = \frac{Z_{t+1}}{Z_t} - 1$$

$$Z_{t+1} \cong X_t^\alpha + \alpha (X_t^{\alpha-1})(X_{t+1} - X_t)$$

$$= Z_t + \alpha X_t^\alpha \cdot x$$

Discrete Time

Log Approx to percentages

$$\frac{\Delta Z_{t+1}}{Z_t} \cong \ln Z_{t+1} - \ln Z_t = \alpha \ln X_{t+1} - \alpha \ln X_t$$

$$= \alpha \left(\ln X_{t+1} - \ln X_t \right) \cong \alpha \frac{\Delta X_{t+1}}{X_t} = \alpha x$$

Taylor Series Expansion

$$\frac{\Delta Z_{t+1}}{Z_t} = \frac{Z_{t+1}}{Z_t} - 1 \qquad (7)$$

$$Z_{t+1} = X_{t+1}^{\alpha} \qquad \text{Expand around } X_{t+1} = X_t$$

$$Z_{t+1} \cong X_t^{\alpha} + \alpha X_t^{\alpha-1} \left(X_{t+1} - X_t \right)$$

$$= Z_t + \alpha Z_t \left(\frac{X_{t+1} - X_t}{X_t} \right) \qquad \text{so}$$

$$Z_{t+1} \cong Z_t + \alpha Z_t x \qquad (8)$$

use (8) in (7) to get

$$\frac{\Delta Z_{t+1}}{Z_t} \cong 1 + \alpha x - 1 \qquad \text{or}$$

$$\frac{\Delta Z_{t+1}}{Z_t} \cong \alpha x$$

Additional problem 2 In class we derived the modified Capital accummulation equation

$$(1+\eta) K_{t+1} = (1-\delta) K_t + K_t^{\alpha} - C_{et} \qquad (1)$$

and the modified Euler eqn (Intertemporal optimality condition)

$$(1+\eta) = \tilde{\beta} \left[\frac{C_{et+1}}{C_{et}} \right]^{-\sigma} \left[1-\delta + \alpha K_{t+1}^{\alpha-1} \right] \qquad (2)$$

Recall that $(1+\eta) = (1+\omega)(1+n)$ (3)

$\Delta K_{t+1} = 0$ locus

Re-write (1) as

$$K_{t+1} = \left(\frac{1-\delta}{1+\eta}\right) K_t + \left(\frac{1}{1+\eta}\right) K_t^{\alpha} - \left(\frac{1}{1+\eta}\right) C_{et}$$

Use $\left(\frac{1-\delta}{1+\eta}\right) \cong (1-\delta-\eta)$ ~~to get~~ and $\left(\frac{1}{1+\eta}\right) \cong (1-\eta)$

To get

$$K_{t+1} = (1-\delta-\eta) K_t + (1-\eta) K_t^{\alpha} - (1-\eta) C_{et}$$

So that

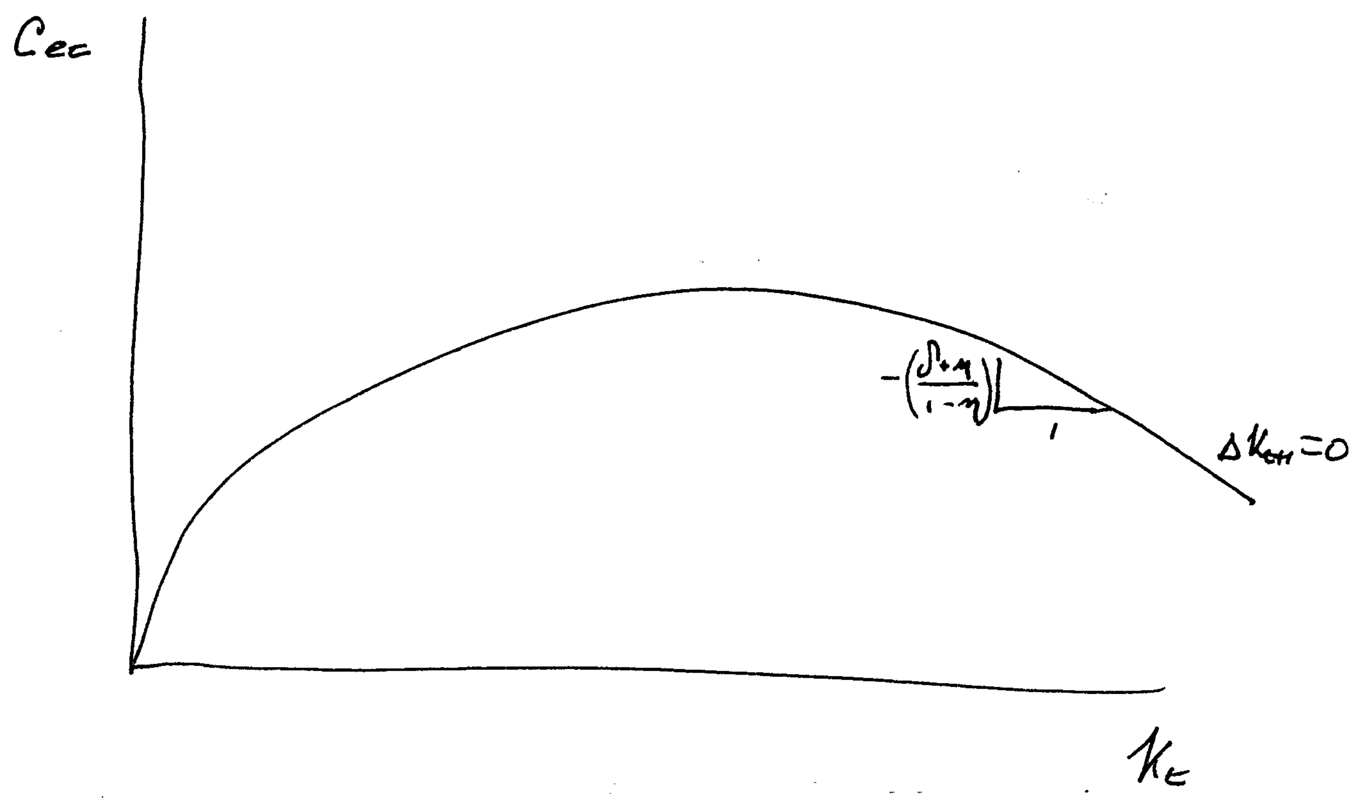
$$\Delta K_{t+1} = -(\delta + \eta) K_t + (1-\eta) K_t^{\alpha} - (1-\eta) C_{et} \qquad (4)$$

Thus $\underline{\Delta K_{t+1} = 0}$ where

$$(1-\eta) C_{et} = (1-\eta) K_t^{\alpha} - (\delta + \eta) K_t \quad \text{or}$$

$$C_{et} = K_t^{\alpha} - \left[\frac{\delta + \eta}{1-\eta}\right] K_t \qquad (5)$$

So the $\Delta K_{t+1} = 0$ locus looks like this

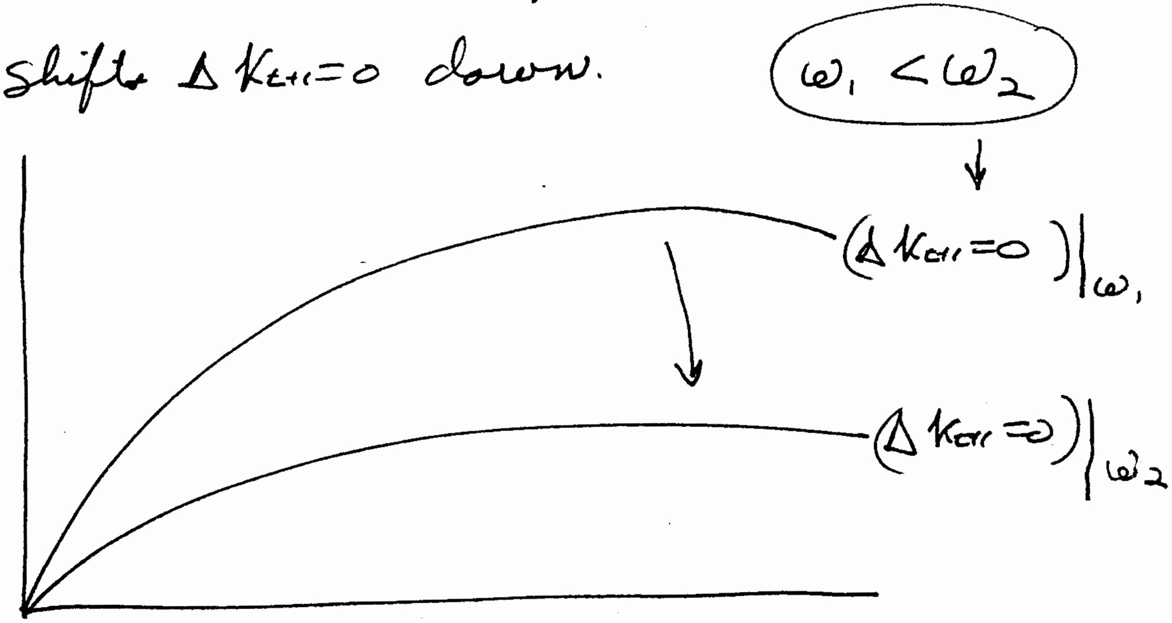


(I have assumed the shape of the curves here - You should be able to derive it.)

Note that an increase in ω causes $\uparrow \eta$, via eqn 3.

Since $\uparrow \eta$ causes $\uparrow \left[\frac{\delta+\eta}{1-\eta}\right]$ we see from (5)

that $\uparrow \omega$ shifts $\Delta k_{t+1}=0$ down.



~~Δc_{t+1}~~ $\Delta c_{t+1}=0$ locus

From (2), at the steady state where $c_{t+1}=c_t=c_s$ so that $\Delta c_{t+1}=0$, we have

$$(1+\eta) = \tilde{\beta}\left[\frac{c_s}{c_s}\right]^{-\sigma}\left[1-\delta+\alpha k_s^{\alpha-1}\right] \quad \text{or}$$

$$(1+\eta) = \tilde{\beta}\left[1-\delta+\alpha k_s^{\alpha-1}\right] \qquad (6)$$

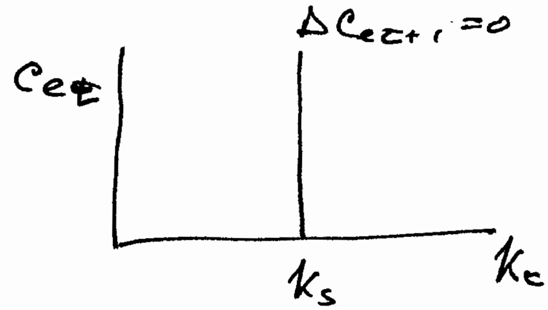
Thus $\Delta C_{et+1}=0$ at the value of K_s that satisfies (6)
(You should be able to derive the ~~A~~ equation for ΔC_{et+1} from (6)).

In (6) Note that $\tilde{\beta} = (1+\gamma)^{1-\sigma}$. Thus

$$(1+\eta) = \beta(1+\gamma)^{1-\sigma}\left[1-\delta+\alpha K_s^{\alpha-1}\right] \quad \text{or}$$

$$\left[(1+\theta)(1+\gamma)^{\sigma} - 1+\delta\right] = \alpha K_s^{\alpha-1} \qquad (7)$$

For now it is sufficient to note that (7) defines a unique $K_s \ni \Delta C_{et+1}=0$



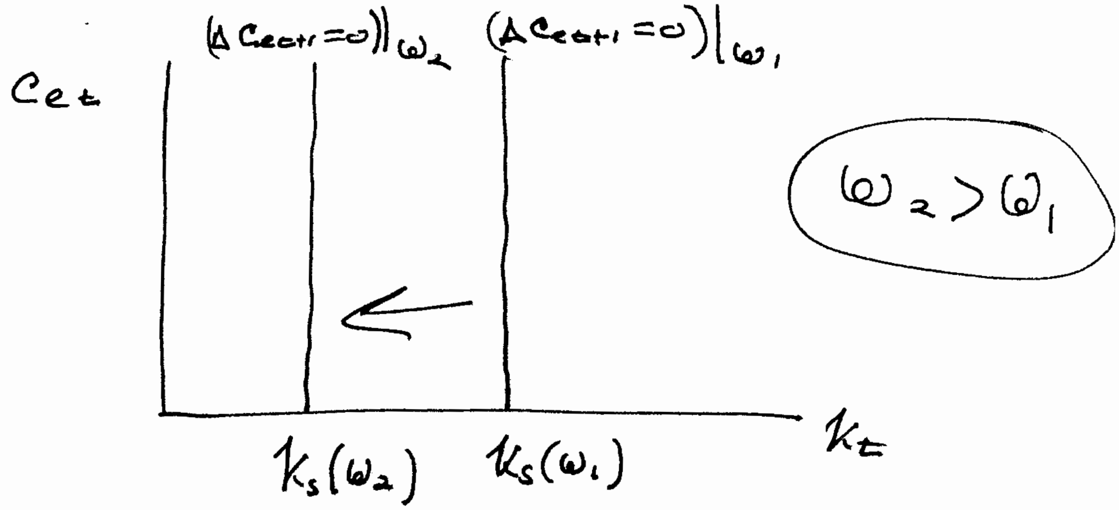
Note that $\frac{\partial RHS(}{\partial} < 0$

Note also that $\frac{\partial LHS(7)}{\partial} = \sigma(1+\theta)(1+\gamma)^{\sigma-1}\frac{\partial\gamma}{\partial\omega}$

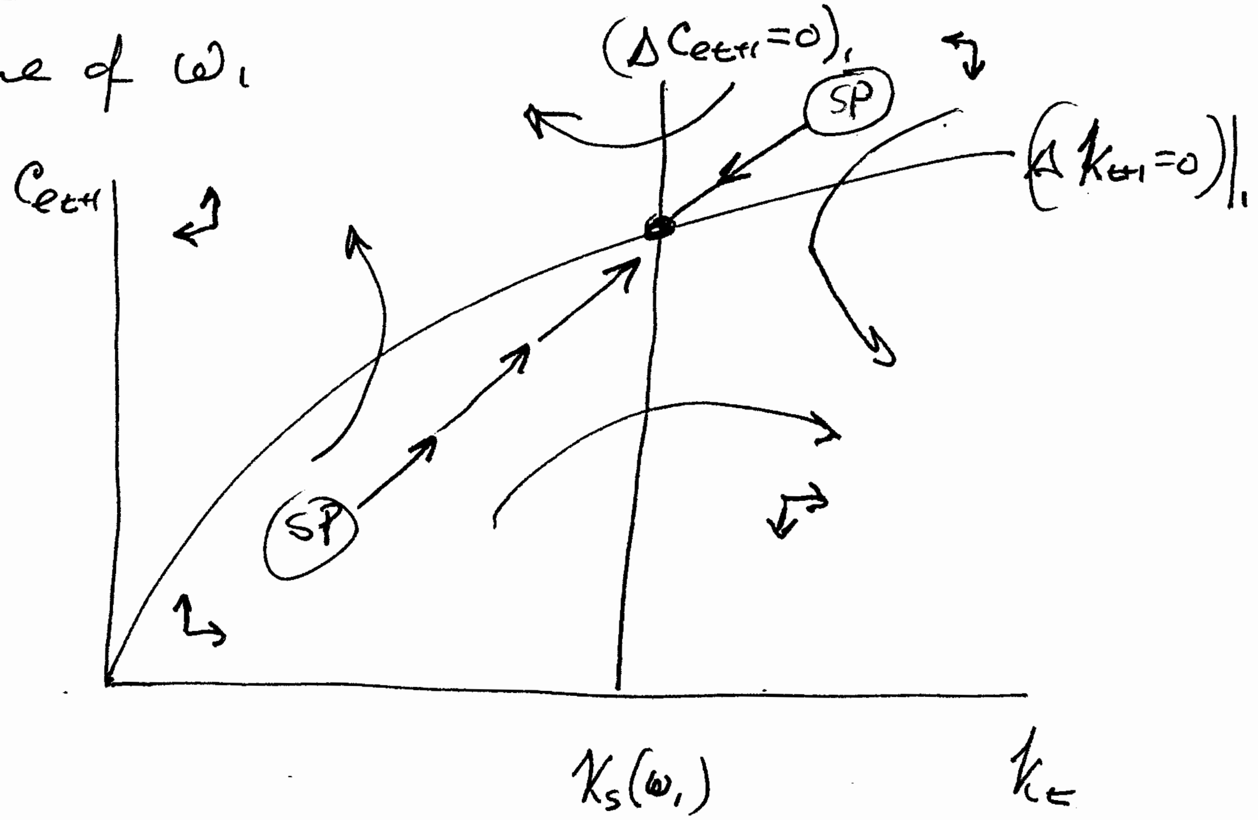
and that $\frac{\partial\gamma}{\partial\omega} = (1+n) > 0$

Thus $\frac{\partial LHS(7)}{\partial \omega} > 0$. It follows that, for (7) to hold an increase in ω requires a decline in k_s.

An $\uparrow \omega$ causes the $\Delta c_{et+1} = 0$ locus to shift left.

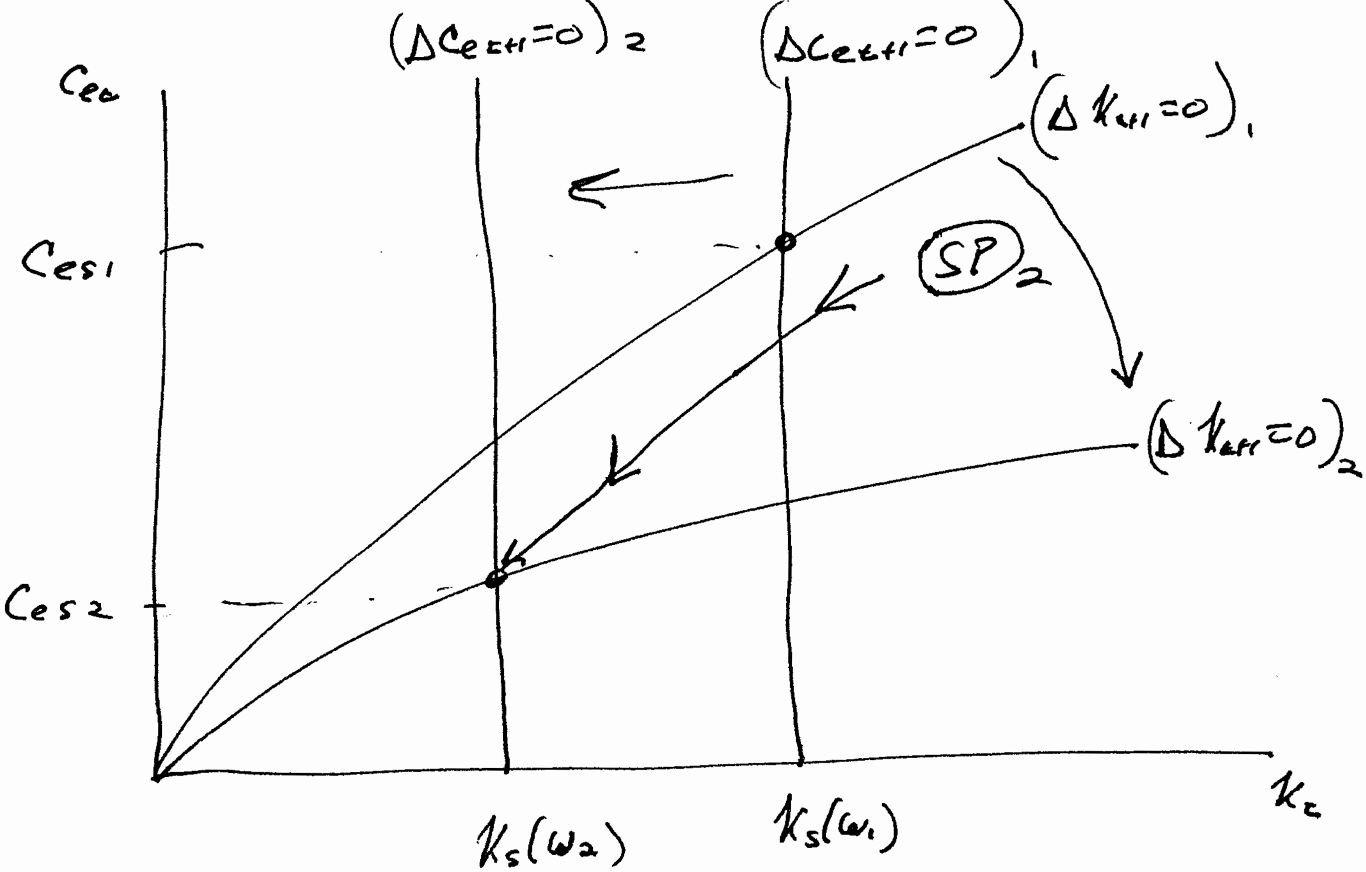


<u>Putting it Together.</u> For the initial value of ω_1



~~Since the~~ (Given the strong similarity to the model where $n = \omega = 0$ I assume the same dynamics but, again, you should be able to derive the dynamics)

Given that an $\uparrow \omega$ shifts the $\Delta C_{t+1} = 0$ locus left and the $\Delta k_{t+1} = 0$ locus down the effect of the $\uparrow \omega$ will look like the Figure below



$\omega_2 > \omega_1$