Final Exam

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ECON 5700

Due on August 28, 2021.

1 Question 1

Solution:

1. Rearrange the functions of the question to become:

$$f_2(x_1, x_2) = 2x_1^2 + 5x_1x_2 + 2x_2^2$$

$$\triangledown f_1 = (f_{1,x_1}, f_{2,x_2}) = (-2x_1 - \frac{5}{2}x_2, -\frac{5}{2}x_1 - 2x_2)$$

$$\triangledown f_2 = (f_{2,x_1}, f_{2,x_2}) = (4x_1 + 5x_2, 5x_1 + 4x_2)$$

2.

$$\begin{aligned}
\mathcal{J} &= \begin{bmatrix} -x_1 - \frac{5}{2}x_2 & -\frac{5}{2}x_1 - 2x_2 \\ 4x_1 + 5x_2 & 5x_1 + 4x_2 \end{bmatrix} \\
&= -(5x_1 + 4x_2)(2x_1 + \frac{5}{2}x_2) + (4x_1 + 5x_2)(\frac{5}{2}x_1 + 2x_2) \\
&= -(10x_1^2 + \frac{25}{2}x_1x_2 + 8x_1x_2 + 10x_2^2) + (10x_1^2 + \frac{25}{2}x_1x_2 + 8x_1x_2 + 10x_2^2) \\
&= 0
\end{aligned}$$

Thus, these two functions are dependent.

2 Question 2

Solution:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

*I worked on my final exam on my own. There may be many typos, and all the errors are my own. Email: wye22@fordham.edu

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Thus, we can derive:

$$\begin{aligned}
\frac{dz}{dt} &= (3x^2 + 8xy - y^2)(-3) + (4x^2 - 2xy + 1)\cdot\frac{1}{2} \\
&= -9x^2 - 24xy + 3y^2 + 2x^2 - xy + \frac{1}{2} \\
&= -7x^2 - 25xy + 3y^2 + \frac{1}{2} \\
&= -7(-3t)^2 - 25(-3t)(1 + \frac{1}{2}t) + 3(1 + \frac{1}{2}t)^2 + \frac{1}{2} \\
&= -\frac{9}{4}t^2 + 78t + \frac{7}{2}
\end{aligned}$$

3 Quesiton 3

Solution:

We set up a lagrangian equation from this equation:

$$\mathcal{L} = -c_1^2 + c_1c_2 - 2c_2^2 + \lambda(16 - c_1 - c_2)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = -2c_1 + c_2 - \lambda = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = c_1 - 4c_2 - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 16 - c_1 - c_2 = 0 \tag{3}$$

Combine the above 3 equations, we can deduce the optimal point of c_1 and c_2. $c_1^* = 10$, $c_2^* = 6$.

$$U_{c_1,c_1} = -2 < 0$$

$$U_{c_2,c_2} = -4$$

$$U_{c_1,c_2} = 1$$

$$D = -2\cdot(-4) - 1 = 7 > 0$$

Thus, the points we got are local maximum point. Also, it's easy to justify, this point pair is also global maxmimum point, because put these two point pair to the utility function, and combine with the budget contraint. We can only get the max utility at this point.

4 Question 4

Solution:

Form Taylor's expansion at $x_0 = 0$:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + o(x - x_0)^n \\ &= 3 + 6x + 9x^2 + 12x^3 + o(x)^n \end{aligned}$$

5 Question 5

Solution:

1.
$$\int_0^{10} (e^{3x} + 4x)dx = \frac{1}{3}e^{3x}\Big|_0^{10} + 2x^2\Big|_0^{10} = \frac{1}{3}e^{30} + \frac{149}{3}$$

2.
$$\int_0^1 6x^2 e^{x^3} dx = 2\int_0^1 e^{x^3} dx^3 = 2e^{x^3}\Big|_0^1 = 2e - 2$$

3.
$$\int_1^{3x^3} e^t dt = e^{3x^3} - e + c = e^{3x^3} + c^1$$

6 Question 6

Solution:

$$\begin{aligned} \lim_{x\to 0} &= \lim_{x\to 0} \frac{e^x \sin x + (e^x - 1)\cos x}{3x^2 + 6x} \\ &= \lim_{x\to 0} \frac{e^x \cos x + e^x \sin x + e^x \cos x - e^x \sin x + \sin x}{6x + 6} \\ &= \lim_{x\to 0} \frac{2e^x \cos x + \sin x}{6x + 6} \\ &= \frac{1}{3} \end{aligned}$$

[1] I do think this method is the same but much easier than the complicated and tedious standard solution.

7 Question 7

Solution:

From the implicit theorem, we can get:

$$F_x(x,y)dx + F_y(x,y)dy = 0$$

From this, we can deduce the function:

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{6x^2 - 2xy}{-x^2 + \frac{1}{y}} \\ &= \frac{2xy - 6x^2}{-x^2 + \frac{1}{y}}\end{aligned}$$

8 Question 8

Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{(\frac{1}{2})^{n+2}(n+1)^2}{(\frac{1}{2})^{n+1}n^2} &= \lim_{x\to\infty} \frac{1}{2}(1+\frac{1}{n})^2 \\ &= \frac{1}{2} < 1\end{aligned}$$

Thus, it's a convergent sereis.

9 Question 9

Solution:

$$f'(x) = 9x^2 - 4x + 1$$

$$f''(x) = 18x - 4$$

If $f''(x) = 0 \longrightarrow x = \frac{2}{9}$, then it would be an inflection point. If $f''(x) < 0 \longrightarrow \frac{2}{9}$, then it's concavity. If $f''(x) > 0 \longrightarrow x > \frac{2}{9}$, then it's convexity.

10 Question 10

Solution:

1.
   - Because $|x-y| \geq 0$, and $1 > 0$, thus, $\min\{1, |x-y|\} \geq 0$ always for any $x, yin\mathcal{R}$.
   - Becaues $d_1(y,x) = \min\{1, |y-x|\} = \min\{1, |x-y|\} = d_1(x,y)$ for any $x, y \in \mathcal{R}$(or say in real field.),
   - $d_1(x,z) = \min\{1, |x-z|\} \leq \min\{1, |x-y|\} + \min\{1, |y-z|\}$. For this case, it's easy to prove,but I don't know is it required to say specificly. But the main idea is that using y as bridge variable to prove the triangle inequality holds for any $x, y, z \in \mathcal{R}$.
2. (a) **T**
   (b) **T**
   (c) **T**
   (d) **T**
   (e) **F**

11 Question 11

Solution:

1.

$$\begin{aligned} BC^T &= \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \end{aligned}$$

2.

$$\begin{aligned} \det A &= 1 \cdot (2-2) + (-1)^{3+1}(-1-2) \\ &= -3 \end{aligned}$$

3.

$$
\begin{aligned}
\det A &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{pmatrix} \\
&= -3 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \\
&= -3
\end{aligned}
$$

4.

$$
\begin{aligned}
\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] &= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -3 & -3 & -1 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right]
\end{aligned}
$$

Thus, $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ -2 & 0 & 1 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

Trace $(A) = 1 + 1 + 2 = 4$, From question 3: row operations, we can easlity get rank $(A) = 3$, it's full rank.

5.
- The rank of the matrix is n.
- The determinant of this matrix is non-zero, which means the matrix needs to be invertible.

6.

$$
D = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} = -5
$$

$$
D_{x_1} = \begin{bmatrix} 2 & -3 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} = -10
$$

$$D_{x_2} = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix} = -5$$

$$D_{x_3} = \begin{bmatrix} 2 & -3 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 5$$

Thus, $x_1 = \frac{D_{x_1}}{D} = 2$, $x_2 = \frac{D_{x_2}}{D} = 1$, and $x_3 = \frac{D_{x_3}}{D} = -1$.

12 Question 12

Solution:

1. We do row echelon of matrix B, thus, we get:

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

From the definition of null of matrix, we can get x_3 could be any number, $x_2 = x_3$, and $x_1 = -x_3$. One possible basis is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

2. $3 > 0$, and $3 \cdot 3 - 1 = 8 > 0$, thus, A is positive definite.

3. Since the transformation is $T(\vec{x}) = A\vec{x}$, thus, the standard matrix of its inverse T^{-1} can be written as A^{-1}.

$$A^{-1} = \frac{1}{8} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

4. By reduced row echelon, we can derive B as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By the denination of kernal, we need to find $Ax = 0$, $Ax_1 = 0 \to x_1 = 0$, $Ax_2 = 0 \to x_2 = 0$, and $A_2x_3 = 0 \to x_3 = 0$. Thus, the $\ker(B) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

5.

$$\begin{aligned} |A - \lambda I| &= \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda)^2 - 1 \\ &= 0 \end{aligned}$$

Thus, $\lambda_1 = 2, \lambda_2 = 4$.

- When $\lambda = 2 : x_1 = -x_2$, and the eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- When $\lambda = 4 : x_1 = x_2$, thus, the eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6. $\operatorname{diag}(A) = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\operatorname{diag}(A) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

7.

$$\begin{aligned} A^4 &= PD^4P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

8. We let $\vec{x_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{x_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   By Grant-Schmidt Process:

$$\vec{v_1} = \vec{x_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{v_2} = \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right) \cdot \vec{v_1} = \begin{bmatrix} -\frac{4}{5} \\ \frac{12}{5} \end{bmatrix}$$

   Normalize the basis:

$$\vec{q}_1 = \frac{\vec{v}_1}{\|v_1\|} = \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|v_2\|} = \begin{bmatrix} -\frac{\sqrt{10}}{10} \\ \frac{3\sqrt{10}}{10} \end{bmatrix}$$

9. From (h)/(8), we can get the Q:

$$Q = \begin{bmatrix} \frac{3\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} \\ \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \end{bmatrix}$$

Since $A = QR \longrightarrow Q^T A = Q^T QR \to R = Q^T A$.

$$R = Q^T A = \begin{pmatrix} \frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \\ -\frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{10}}{5} & \frac{3\sqrt{10}}{5} \\ 0 & \frac{4\sqrt{10}}{5} \end{pmatrix}$$

10.

$$\begin{aligned} A^+ &= (A^T A)^{-1} A^T \\ &= (\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix})^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \frac{1}{64} \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \end{aligned}$$

13 Question 13

1. **F**
2. **T**
3. **F**
4. **T**
5. **T**
6. **T**
7. **F**

14 Question 14

Solution:

1.

$$\begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix} \xrightarrow{R_3 - \frac{1}{3}R_1} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & \frac{10}{3} & 1 \end{pmatrix} \xrightarrow{R_3 - \frac{10}{3}R_2} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{10}{3} & 1 \end{pmatrix}$$

2. First, we need to find the eigenvalues of matrix A:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 3-\lambda & 2 & 0 \\ 0 & 1-\lambda & 3 \\ 1 & 4 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)((1-\lambda)(3-\lambda)) - 3(4(3-\lambda) - 2) \\ &= -\lambda^3 + 5\lambda^2 + 5\lambda - 27 \\ &= 0 \end{aligned}$$

Thus, from above we can get [2] $\lambda_1 = 4.89449$, $\lambda_2 = -2.29654$, and $\lambda_3 = 2.40205$.

Thus:

$$D = \begin{bmatrix} 4.89449 & 0 & 0 \\ 0 & -2.29654 & 0 \\ 0 & 0 & 2.40205 \end{bmatrix}$$

$$LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{10}{3} & 1 \end{bmatrix} \begin{bmatrix} 4.89449 & 0 & 0 \\ 0 & -2.29654 & 0 \\ 0 & 0 & 2.40205 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{10}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

3. From 2, we can get there is no cholesky decomposition with respect to A. Because

$$D = dd$$

However, we can't get square root of negative eigenvalue, so the cholesky decomposition doesn't exist in this case.

[2] I use online calculator to compute the eigenvalues, because it's impossible to compute these eigenvaluse manually.

15 Bonus Questions

Solution:

1.
$$\operatorname{adj} B = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$
.

2. Use the results of question 12 directly, the spectrum decompose of A:

$$\begin{aligned} \text{Decompose } A &= QDQ^T \\ &= \begin{bmatrix} \frac{3\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} \\ \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \\ -\frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \end{bmatrix} \end{aligned}$$

3. (a)
$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} |A^T A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)[(1-\lambda)^2 - 1] \\ &= -\lambda^3 + 6\lambda^2 - 8\lambda \\ &= 0 \end{aligned}$$

Thus, $\lambda_1 = 4$, $\lambda_2 = 0$, and $\lambda_3 = 2$. $\sigma_1 = 2, \sigma_2 = 0, \sigma_3 = \sqrt{2}$.

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

(b)
- When $\lambda = 4$, the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- When $\lambda = 2$, the eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- When $\lambda = 0$, the eigenvector is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

  By G-S decomposition:

$$V = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$Q = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

(c)

$$\vec{U}_1 = \frac{1}{2} A q_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{U}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$