PROB SET III
MARKOV CHAINS

S+B 23.31. We seek the long-run values of y for the systems given in eqns (5) and (6) on p114. The most efficient way to do this is to solve for y in the general system and then just "plug in" the values. The general system, as given in eqn (4) is

$$x_{t+1} = q x_t + p y_t$$
$$y_{t+1} = (1-q) x_t + (1-p) y_t \qquad (4)$$

~~Since~~ The stationary values of x and y solve

$$x = q x + p y \qquad (7a)$$
$$y = (1-q) x + (1-p) y \qquad (7b)$$

Further, since $[x, y]'$ is a probability vector we have

$$x + y = 1 \qquad (7c)$$

Since (7a) and (7b) are linearly dependent, we can use (7a) and (7c) as 2 eqns in x and y to solve for y.

(7c) gives $x = 1-y$. This in (7a) gives

$(1-y) = q(1-y) + py$ or

$1 - y = q - qy + py$ or

$1 - q = (1 + p - q)y$ or

$$y = \left[\frac{1-q}{1+p-q}\right] \qquad (\#)$$

In eqn (4) $q = .998$ and $p = .136$

So $y = \dfrac{.002}{.138} = 0.0145$

In eqn (5) $q = .996$ and $p = .102$

So $y = \dfrac{.004}{.106} = 0.0377$

Cf p115

StB 23.32 Let $\underset{(K\times 1)}{C} \equiv [1, 1, \ldots, 1]'$.

Since X is a prob vector it's elements are non negative and they sum to 1. That is, using $\underset{(K\times 1)}{X} = [x_1, x_2, \cdots x_K]'$, $C = [1, 1, \ldots 1]'$,

$X'C = 1$ and $x_i \geq 0$ for $i = 1, 2, \ldots k$.

Since M is a Markov matrix it's elements are non negative and each of its columns sum to 1. That is

using $M_{(K\times K)} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1K} \\ m_{21} & m_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ m_{K1} & \cdots & & m_{KK} \end{bmatrix}$

$M'C = C$ and $m_{ij} \geq 0$ for $i = 1, 2, \ldots k$; $j = 1, 2, \ldots k$.

We must prove that $\underset{(K\times 1)}{MX}$ is a probability vector.

FIRST, NOTE That its elements Sum TO 1

$$(Mx)'c = 1 \quad \text{because}$$

$$(Mx)'c = x'M'c = x'c = 1$$

Second, Note That each element of Mx can be written as

$$[m_{i1}, m_{i2}, \dots m_{ik}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{bmatrix}$$

$$= \sum_{j=1}^{k} m_{ij} x_j$$

Since $m_{ij} \geq 0$ and $x_i \geq 0 \quad \forall i, j$

it follows That $\sum_{j=1}^{k} m_{ij} x_j \geq 0, \forall i$

QED

SEB 23.33 Let S_1 = SUNNY, S_2 = RAINY, S_3 = CLOUDY

PART (a)

IF SUNNY, NEXT DAY SUNNY OR CLOUDY W/ EQUAL PROB BUT WON'T RAIN.

$$m_{11} = \text{PROB}(S_{N+1} = S_1 \mid S_N = S_1) = 0.5$$
$$m_{21} = \text{PROB}(S_{N+1} = S_2 \mid S_N = S_1) = 0$$
$$m_{31} = \text{PROB}(S_{N+1} = S_3 \mid S_N = S_1) = 0.5$$

IF CLOUDY, NEXT DAY 50% CHANCE RAIN, 25% CHANCE SUNNY, 25% CHANCE CLOUDY

$$m_{13} = \text{PROB}(S_{N+1} = S_1 \mid S_N = S_3) = 0.25$$
$$m_{23} = \text{PROB}(S_{N+1} = S_2 \mid S_N = S_3) = 0.50$$
$$m_{33} = \text{PROB}(S_{N+1} = S_3 \mid S_N = 3) = 0.25$$

IF RAINS, NEXT DAY WON'T BE SUNNY, 50-50 CHANCE RAINS (SO 50% CHANCE CLOUDY).

$$m_{12} = \text{PROB}(S_{N+1} = S_1 \mid S_N = S_2) = 0$$
$$m_{22} = \text{PROB}(S_{N+1} = S_2 \mid S_N = S_2) = 0.5$$
$$m_{32} = \text{PROB}(S_{N+1} = S_3 \mid S_N = S_2) = 0.5$$

$$\text{So } M = \begin{bmatrix} 0.5 & 0 & 0.25 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.25 \end{bmatrix}$$

Part(b)

$$M^2 = \begin{pmatrix} 1/2 & 0 & 1/4 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/4 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/4 \end{pmatrix} = \begin{bmatrix} \frac{1}{4}+\frac{1}{8} & \frac{1}{8} & \frac{1}{8}+\frac{1}{16} \\ \frac{1}{4} & \frac{1}{4}+\frac{1}{4} & \frac{1}{4}+\frac{1}{8} \\ \frac{1}{4}+\frac{1}{8} & \frac{1}{4}+\frac{1}{8} & \frac{1}{8}+\frac{1}{4}+\frac{1}{16} \end{bmatrix}$$

So $$M^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} & \frac{3}{16} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{7}{16} \end{bmatrix}$$

Note that M^2 has only positive entries. Thus M is Regular

Part(c) The long Run probabilities,

$X = [x^1, x^2, x^3]'$ Must Satisfy

$X = MX$. Sinc these three equations are linearly dependent, we can use any two together with the ~~Restriction~~ Restriction that $x^1 + x^2 + x^3 = 1$ to solve for x^1, x^2, and x^3.

So use $x^3 = 1 - (x^1 + x^2)$ to eliminate x^3 from the first two eqns of $X = MX$ and write

$$x^1 = \frac{3}{8}x^1 + \frac{1}{8}x^2 + \frac{3}{16}\left[1 - x^1 - x^2\right] \qquad (1)$$

$$x^2 = \frac{1}{4}x^1 + \frac{1}{2}x^2 + \frac{3}{8}\left[1 - x^1 - x^2\right] \qquad (2)$$

From (1)

$$\frac{5}{8}x^1 = \frac{3}{16} - \frac{1}{16}x^2 - \frac{3}{16}x^1$$

$$\frac{13}{16}x^1 = \frac{13}{16} - \frac{1}{16}x^2$$

Additional Problem 1

$$P = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \qquad \text{Note} \qquad \begin{matrix} p_{11} + p_{12} = 1 \\ p_{21} + p_{22} = 1 \end{matrix} \qquad (1)$$

Eigenvalues, λ_1 and λ_2, solve $\mathrm{Det}(P - \lambda I_2) = 0$.

$$\mathrm{Det}\begin{bmatrix} (p_{11}-\lambda) & p_{21} \\ p_{12} & (p_{22}-\lambda) \end{bmatrix} = (p_{11}-\lambda)(p_{22}-\lambda) - p_{12}\,p_{22}$$

$$= \lambda^2 - (p_{11}+p_{22})\lambda + p_{11}p_{22} - p_{12}p_{21}.$$ Thus, eigenvalues solve

$$\lambda_{1,2} = \frac{(p_{11}+p_{22}) \pm \left[(p_{11}+p_{22})^2 - 4(p_{11}p_{22} - p_{12}p_{21})\right]^{1/2}}{2} \qquad (2)$$

From (1), $p_{12} = 1 - p_{11}$ and $p_{21} = 1 - p_{22}$ and so

$$p_{12}\,p_{21} = (1-p_{11})(1-p_{22}) = 1 + p_{11}p_{22} - p_{11} - p_{22}$$

Thus

$$\left[(p_{11}+p_{22})^2 - 4(p_{11}p_{22} - p_{12}p_{21})\right] = \left[(p_{11}+p_{22})^2 - 4(p_{11}p_{22} - 1 - p_{11}p_{22} - p_{11} - p_{22})\right]$$

$$= (p_{11}+p_{22})^2 + 4 - 4(p_{11}+p_{22}) = \left[2 - (p_{11}+p_{22})\right]$$

and eqn (2) gives

$$\lambda_{1,2} = \frac{(p_{11}+p_{22}) \pm \left\{\left[2 - (p_{11}+p_{22})\right]^2\right\}^{1/2}}{2}$$

OR

$$\lambda_{1,2} = \frac{(p_{11}+p_{22}) \pm [2-(p_{11}+p_{22})]}{2}$$

So $\lambda_1 = \frac{2(p_{11}+p_{22})-2}{2}$ or $\boxed{\lambda_1 = (p_{11}+p_{22})-1 \quad (4)}$

Note here that, unless $(p_{11}+p_{22})=2$ and, therefore, ~~p_{11} and p_{22}~~ unless both $p_{11}=1$ and $p_{22}=1$, then $|\lambda_1|<1$.

Hamilton Ch 22 Exercise 22.1

Note that the columns of T are the (appropriately normalized) eigenvectors of P. Note further that the eigenvector of P associated with $\lambda_2=1$ is π, the vector of ergodic probabilities. Since the second column of T has elements that sum to zero, Hamilton has organized T so that the first column is the eigenvector associated with $\lambda_2=1$ and the second column is the eigenvector associated with $\lambda_1=(p_{11}+p_{22})-1$.

Let $V_2 \equiv \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$ denote the eigenvector of P that is associated with $\lambda_1 = (p_{11}+p_{22})-1$. Thus

$$\left(\begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\right)\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ So

$(p_{11}-\lambda_1)v_{21} + p_{21}v_{22} = 0$. Using initial normalization $v_{21} = 1$ we have

$(p_{11}-\lambda_1) + p_{21}v_{22} = 0$ or

$$v_{22} = \left[\frac{p_{11}-\lambda_1}{-p_{21}}\right] = \frac{\lambda_1 - p_{11}}{p_{21}} = \frac{p_{11}+p_{22}-1-p_{11}}{1-p_{22}}.$$

Thus $v_{22} = -1$ and

$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Renormalize as $W_2 = (-1)V_2$ and we have $W_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ which verifies the second column of Hamilton's matrix T.