Homework 1

Wei Ye[*]
ECON 7020- Macroeconomics II

Due on Feb 3, 2022

1 Problems 1

Solution:

a. Since the transition function is $x' = g(x, u, z, \phi)$, it means next period of state variable is predetermined by this period of control variable u, state variable x, one time shocks z and ϕ (Impulse Response).

b. Set up Bellman Equation (Since there is only on time shock, we don't take them into value function):

$$\begin{aligned} V(x) &= \max_u \{r(x,u) + \beta E_t[V(x')]\} \\ &= \max_u \{r(x,u) + \beta E_t[V(g(x,u,z,\phi))]\} \end{aligned}$$

c.
- FOC with respect to u:

$$\frac{\partial r(x,u)}{\partial u} + \beta E_t\{V'(x')\frac{\partial g(x,u,z,\phi)}{\partial u}\} = 0$$

- Benveniste-Sheinkman Condition:

$$v'(x) = \frac{\partial r(x,u)}{\partial x} + \beta E_t\{V'(x')\frac{\partial g(x,u,z,\phi)}{\partial x}\}$$

d. In order to satisfy Banach Fixed Point Theorem, aka, Contraction Mapping Theorem, it should have monotonicity and discounting properties. At the same time, the process should be bounded on some value, and is σ-measurable.

[*]1st year PhD student in Economics Department at Fordham University. Email: wye22@fordham.edu

2 Problem 2

Solution:

a. Set up a bellman equation:

$$\begin{aligned} V(k,z) &= \max_c \{\ln(c) + \beta E_t[V(k', z')|z]\} \\ &= \max_c \{\ln(c) + \beta E_t[V(zk^\alpha - c, z')|z]\} \end{aligned}$$

b. The explicit form of Bellman Equation:

$$V(k,z) = \max_c \{\ln(c) + \beta[V((p^H z^H + (1-p^H)z^L)k^\alpha - c)]\}$$

c. WLOG, we guess and assume $V(k_{t+1}) = \ln(k_{t+1})$. For iterations in Bellman Equation, We assume $V^0(k') = 0$ like what we did in the lecture. In the meanwhile, WLOG, we denote $\bar{z} = p^H z^H + (1-p^H)z^L$.

The **first** iteration:

$$V^1(k) = \max_{k'} \{\ln(\bar{z}k^\alpha - k')\}$$

In this case, it's obvious that when $k' = 0$, we can get the maximum value $V^1(k) = \ln(\bar{z}k^\alpha)$. Plug this value into our second iteration:

The **second** iteration:

$$V^2(k) = \max_{k'} \{\ln(\bar{z}k^\alpha - k') + \beta(\ln(\bar{z}) + \alpha \ln(k'))\}$$

Taking the first order derivative w.r.t k', we can obtain $\frac{-1}{\bar{z}k^\alpha - k'} + \frac{\beta\alpha}{k'} = 0$, thus, $k' = \frac{\alpha\beta\bar{z}k^\alpha}{1+\alpha\beta}$. And optiam value function is $V^2(k) = \ln(\bar{z}k^\alpha - \frac{\alpha\beta\bar{z}k^\alpha}{1+\alpha\beta}) + \beta\ln(\bar{z}(\frac{\alpha\beta\bar{z}k^\alpha}{1+\alpha\beta})^\alpha) = \alpha(1+\alpha\beta)\ln(k) + \phi_1$, where $\phi_1 = \ln(\bar{z}) - \ln(1+\alpha\beta) + \beta\ln(\bar{z}) + \beta\alpha\ln(\alpha) + \alpha\beta\ln(\beta) - \alpha\ln(1+\alpha\beta)$

The **Third** iteration: Take $V^2(k)$ into bellman equation:

$$V^3(k) = \max_{k'} \{\ln(\bar{z}k^\alpha - k') + \beta\alpha(1+\alpha\beta)\ln(k') + \beta\phi_1\}$$

Take the first order derivative w.r.t k', we can get the optimal $K' = \frac{\alpha\beta(1+\alpha\beta)\bar{z}k^\alpha}{1+\alpha\beta(1+\alpha\beta)}$, and plug k' into $V^3(\cdot)$ to gain the optimal $V^3(x) = \alpha(1+\alpha\beta(1+\alpha\beta)\ln(k)) + \phi_2$, where $\phi_2 = \alpha\ln(\bar{z}) - \ln(1+\alpha\beta(1+\alpha\beta)) + \alpha\beta(1+\alpha\beta)(\ln(\alpha\beta(1+\alpha\beta)\bar{z}) - \ln(1+\alpha\beta(1+\alpha\beta)))$.
Take the **Fourth** iteration:

$$V^4(k) = \max_{k'} \{\ln(\bar{z}k^\alpha - k') + \beta\alpha(1+\alpha\beta(1+\alpha\beta)\ln(k') + \beta\phi_2)\}$$

Take the first order derivative w.r.t. K', And the optimal $k' = \frac{\alpha\beta(1+\alpha\beta(1+\alpha\beta))\bar{z}k^\alpha}{1+\alpha\beta(1+\alpha\beta(1+\alpha\beta))}$, and the optimal value function of $V^4(k) = \alpha(1+\alpha\beta(1+\alpha\beta(1+\alpha\beta)))\ln(k) + \phi_3$, ϕ_3 is the term that is not revelant to k. If the initial guess is different, we may get different functions[1].

[1] See `https://lhendricks.org/econ720/ih1/Dp_ln.pdf`

3 Problem 3

Solution:

a. Before we set up a bellman equation, we need to rearrange the sequence problem to:

$$W = \sum_{t=0}^{\infty} \beta^t u(f(k_t, n_t) - k_{t+1}, 1 - n_t)$$

We then set up a bellman equation:

$$V(k) = \max_{n,k'} \{u(f(k,n) - k', 1 - n) + \beta V(k')\}$$

b. We hope to gain the policy function,i.e., a control variable can be expressed by the state variable k. Once We know n and k, we know c automatically.

c. First, we take the first order condition w.r.t n:

$$u_c(c,n)(\frac{\partial f(k,n)}{\partial n} - \frac{\partial f(k,n)}{\partial n}) + u_n(c,n)(-1) + \beta V'(k')\frac{\partial f(k,n)}{\partial k} = 0$$

Second, we use Envelop Theorem:

$$V'(k) = u_c(c,n)\frac{\partial f(k,n)}{\partial k}$$

d. Back to our sequence problem, and set up a lagrangian equation:

$$\mathcal{L} := \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) + \lambda_t (f(k_t, n_t) - c_t - k_{t+1})$$

Take first order derivative w.r.t c and n respectively.

$$\beta^t u_c(c_t, 1 - n_t) = -\lambda_t$$

$$\beta^t u_n(c_t, 1 - n_t)(-1) = \lambda_t f_n(k_t, n_t)$$

Thus, we can get:

$$\frac{MU_c}{MU_n} = \frac{1}{f_n(k_t, n_t)}$$

e. Replace value function with the equation we derived in envelop theorem:

$$u_n(c,n) = \beta \frac{\partial f(k,n)}{\partial k} u_c(c', n') \frac{\partial f(k', n')}{\partial k}$$