ECON 7020
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Problem Set 3
Due date: March 4, 2022

Quadratic: $u(c) = c - \frac{b}{2} c^2$

Problem 1. Assume a quadratic utility, rational expectations framework and assume that the rate of time preference, ρ equals the interest rate, r. $\rho = r$
Assume that labour income follows the following stochastic process:

$$y_{t+1} = \lambda y_t + (1-\lambda)\bar{y} + \epsilon_{t+1} \tag{1}$$

$c_t + a_{t+1} = (1+r) a_t + y_t$

where $E_t \epsilon_{t+1} = 0$ and ϵ_{t+1} is an income innovation, $0 \geq \lambda \leq 1$and $\bar{y}$ is the unconditional mean of labour income.

1. Prove that the consumption function in this case has the following form:

$$c_t = rA_t + \frac{r}{1+r-\lambda} y_t + \frac{1-\lambda}{1+r-\lambda} \bar{y}. \tag{2}$$

2. What happens if $\lambda = 1$? Explain.

3. What happens if $\lambda = 0$? Explain.

1. In quadratic utility function:

$u(c_t) = c_t - \frac{b}{2} c_t^2$

From (1.5) in lecture notes and in 2/17/2022 class contents

$u'(c_t) = \left(\frac{1+r}{1+\rho}\right) u'(c_{t+1})$

when $r = \rho$

$\Rightarrow u'(c_t) = u'(c_{t+1})$

under rational expectation and PIH

$c_t = rA_t + \frac{r}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i E_t y_{t+i}$

$\because y_{t+1} = \lambda y_t + (1-\lambda)\bar{y} + \varepsilon_{t+1}$

$E_t y_{t+1} = \lambda y_t + (1-\lambda)\bar{y}$

$y_{t+2} = \lambda y_{t+1} + (1-\lambda)\bar{y} + \varepsilon_{t+2}$

$E_t y_{t+2} = \lambda E_t y_{t+1} + (1-\lambda)\bar{y} = \lambda[\lambda y_t + (1-\lambda)\bar{y}] + (1-\lambda)\bar{y}$

$= \lambda^2 y_t + (\lambda+1)(1-\lambda)\bar{y}$

$y_{t+3} = \lambda y_{t+2} + (1-\lambda)\bar{y} + \varepsilon_{t+3}$

$E_t y_{t+3} = \lambda E_t y_{t+2} + (1-\lambda)\bar{y}$

$= \lambda[\lambda^2 y_t + (1+\lambda)(1-\lambda)\bar{y}] + (1-\lambda)\bar{y}$

$= \lambda^3 y_t + (1-\lambda)[\lambda(\lambda+1)+1]\bar{y}$

$y_{t+4} = \lambda y_{t+3} + (1-\lambda)\bar{y} + \varepsilon_{t+4}$

$E_t y_{t+4} = \lambda E_t y_{t+3} + (1-\lambda)\bar{y}$

$= \lambda[\lambda^3 y_t + (1-\lambda)[\lambda(\lambda+1)+1]\bar{y}] + (1-\lambda)\bar{y}$

$= \lambda^4 y_t + (1-\lambda)[\lambda(\lambda(\lambda+1)+1)+1]\bar{y}$

$= \lambda^4 y_t + (1-\lambda)[1+\lambda+\lambda^2+\lambda^3]\bar{y}$

$\therefore E_t y_{t+i} = \lambda^i y_t + (1-\lambda) \cdot \frac{1-\lambda^i}{1-\lambda} \bar{y}$

$= \lambda^i y_t + (1-\lambda^i)\bar{y}$

put $E_t y_{t+i}$ into c_t.

$$c_t = rA_t + \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i\left[\lambda^i y_t + (1-\lambda^i)\bar{y}\right]$$

$$= rA_t + \frac{r}{1+r}\sum_{i=0}^{\infty}\left[\left(\frac{1}{1+r}\right)^i \lambda^i y_t + \left(\frac{1}{1+r}\right)^i \bar{y} - \left(\frac{1}{1+r}\right)^i \lambda^i \bar{y}\right]$$

$$= rA_t + \frac{r}{1+r}\left[\frac{1}{1-\frac{\lambda}{1+r}} y_t + \frac{1}{1-\frac{1}{1+r}}\bar{y} - \frac{1}{1-\frac{\lambda}{1+r}}\bar{y}\right]$$

$$= rA_t + \frac{r}{1+r}\left[\frac{1}{\frac{1+r-\lambda}{1+r}} y_t + \frac{1}{\frac{1+r-1}{1+r}}\bar{y} - \frac{1}{\frac{1+r-\lambda}{1+r}}\bar{y}\right]$$

$$= rA_t + \frac{r}{1+r}\left[\frac{1+r}{1+r-\lambda} y_t + \frac{1+r}{r}\bar{y} - \frac{1+r}{1+r-\lambda}\bar{y}\right]$$

$$= rA_t + \frac{r}{1+r-\lambda} y_t + \bar{y} - \frac{r}{1+r-\lambda}\bar{y}$$

$$= rA_t + \frac{r}{1+r-\lambda} y_t + \frac{1+r-\lambda-r}{1+r-\lambda}\bar{y}$$

$$= rA_t + \frac{r}{1+r-\lambda} y_t + \frac{1-\lambda}{1+r-\lambda}\bar{y}$$

2. if $\lambda = 1$, which means $\bar{y}$ can not affect C_t

In other words, only current income and capital returns determine current consumption. Average income (Expected Average future income) is not a factor in current consumption's determination.

3. if $\lambda = 0$

$$C_t = rA_t + \frac{r}{1+r} y_t + \frac{1}{1+r} \bar{y}$$

which means, C_t is determined by three factors, Assets, current income and average income.

However, for income, there is weight for y_t and $\bar{y}$. for y_t, the weight is $\frac{r}{1+r}$, for $\bar{y}$, it's $\frac{1}{1+r}$, By the way these all three factors are affected by r.

Problem 2. Suppose a consumer maximizes the following objective function:

$$max E_t \sum_{i=0}^{\infty} \beta^i u(c_{t+i}) \tag{3}$$

subject to the dynamic budget constraint:

$$A_{t+i+1} = (1+r)[A_{t+i} + y_{t+i} - c_{t+i}] \tag{4}$$

where

$$y_{t+1} = y_t + \epsilon_{t+1} \tag{5}$$

and $\epsilon_{t+1} \sim N(0, \sigma^2)$.

1. Under what circumstances do we get a "certainty equivalent result"?

2. Now assume that the utility function is of the exponential form, e.g., $u(c_t) = -(\frac{1}{\alpha})e^{-\alpha c_t}$ where $\alpha > 0$. Calculate the measure of relative risk aversion.

3. For a general utility function $u(c_t)$, *derive* the coefficient of absolute prudence. What is the coefficient of absolute prudence for the utility function mentioned above?

4. How does the existence of prudent behavior alter the optimal consumption path found under the certainty equivalent result?

1. From Lungqvist & Sargent's RMT, 2024 page 115: "the property of CE is "future disturbance have zero mean conditional on the current state"". Thus, back to our question, the only future shock is from y_t, which means if at time t, the only info we can infer is y_{t+1}'s shock is zero in the future.

2. $u'(c_t) = (-\frac{1}{\alpha}) e^{-\alpha c_t} \cdot (-\alpha) = e^{-\alpha c_t}$.

$u''(c_t) = -\alpha e^{-\alpha c_t}$

Thus, $-\frac{u''(c_t)}{u'(c_t)} = -\frac{-\alpha e^{-\alpha c_t}}{e^{-\alpha c_t}} = \alpha$.

3. $E_t u'(C_{t+1}) = E_t u'(C_t) + \underbrace{E_t u''(C_t)(C_{t+1}-C_t) + E_t \frac{u'''(C_t)}{2}(C_{t+1}-C_t)^2}_{}$

when these terms $=0$

$$E_t(C_{t+1}-C_t) = -\frac{u'''(C_t)}{2\,u''(C_t)} E_t(C_{t+1}-C_t)^2.$$

Thus, we extract $\frac{-u'''(C_t)}{u''(C_t)}$ from the above eqn,

and assue it be coefficient of absolute prudence.

in the utility function in (2)

coefficient of absolute prudence is

$$= \frac{-u'''(C_t)}{u''(C_t)} = -\frac{\alpha^2 e^{-\alpha C_t}}{-\alpha e^{-\alpha C_t}} = \alpha$$

4. This part, my reference is mainly from Princeton University Fin501, slide 04-68 Theorem 4.8. And UC-Berkeley Notes for ECON 202A: consumption by Pierre-Olivier Gourinchas.

$P(c)\cdot c = -\frac{u'''(c)}{u''(c)}\cdot c$

From Theorem 4.8 of Princeton's slides

if $cP(c) \le 2$ $S_A > S_B$ where S_A is the saving of $\tilde{R}_A$ (return distribution)

S_B - - - $\tilde{R}_B$

if $cP(c) > 2$ $S_A < S_B$

And $\tilde{R}_A$ SSD $\tilde{R}_B$

$\therefore$ $cP(c) \Rightarrow$ Saving $\Rightarrow$ consumption path

By the way, in this question, back to consumption path

The path effect may possibly be like if a representative consumer more prudent, they're more risk averse, under CE, future shock is 0 under current period, so he will save less than the scenario under non-CE. I possibly think there will be less deviation from non-prudent consumption under CE, but deviation can exist.