Hart Notes

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Part I
General Equilibrium

Chapter 15 - General Equilibrium Theory: Examples

- Pure exchange economy with Edgeworth Box
- Production with One-Firm, One-Consumer
- [Small Open Economy]

15B. Pure Exchange: The Edgeworth Box

Definitions and Set Up
Pure Exchange Economy

- An economy in which there are no production opportunities. Agents possess endowments, economic activity consists of trading and consumption

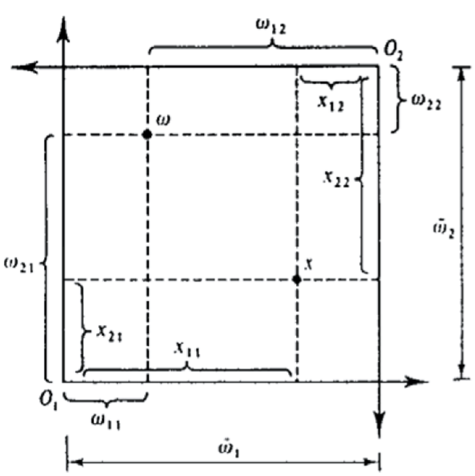
Edgeworth Box Economy

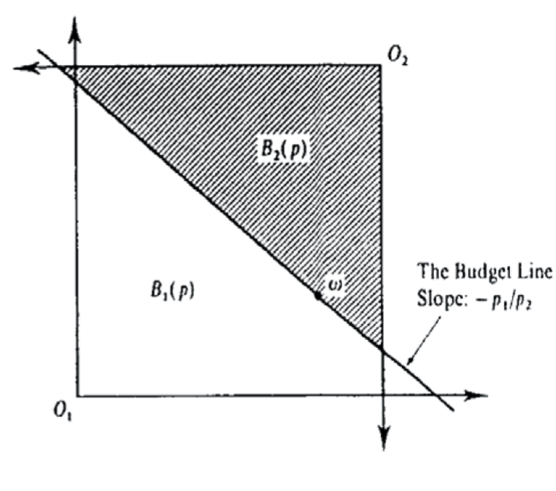
- Preliminaries
  - Assume consumers act as price takers
  - Two consumers $i = 1, 2$; Two commodities $l = 1, 2$
- Consumption and Endowment
  - Consumer i's consumption vector is $x_i = (x_{1i}, x_{2i})$
    * Consumer i's consumption set is $\mathbb{R}^2_+$
    * Consumer i has preference relation $\succsim_i$ over consumption vectors in this set
  - Consumer i's endowment vector is $\omega_i = (\omega_{1i}, \omega_{2i})$
    * Total endowment of good l is $\bar{\omega}_l = \omega_{l1} + \omega_{l2}$
- Allocation

- An *allocation* $x \in \mathbb{R}^4_+$ is an assignment of a nonnegative consumption vector to each consumer
    - $x = (x_1, x_2) = ((x_{11}, x_{21}), (x_{12}, x_{22}))$
- A *feasible allocation* is
    - $x_{l1} + x_{l2} \leq \bar{\omega}_l \quad$ for $l = 1, 2$
- A *nonwasteful allocation* is
    - $x_{l1} + x_{l2} = \bar{\omega}_l \quad$ for $l = 1, 2$
    - These can be depicted in Edgeworth Box

- Wealth and Budget Sets
    - Wealth: Not given exogenously, only endowments are given. Wealth is determined by by prices.
        - $p \cdot \omega_i = p_1\omega_{1i} + p_2\omega_{2i}$
    - Budget Set: Given endowment, budget set is a function of prices
        - $B_i(p) = \left\{x_i \in \mathbb{R}^2_+ : p \cdot x_i \leq p \cdot \omega_i\right\}$
        - Graphically, draw budget line [slope $= -\left(\frac{p_1}{p_2}\right)$]. Consumer 1's budget set consists of all nonnegative vectors below and to the left; consumer 2's is above and to the right

- Graph:
    - Axes: horizontal is good 1, vertial is good 2
    - Origins: consumer 1 in SW corner (as usual), consumer 2 in NE corner (unique to Edgeworth boxes)

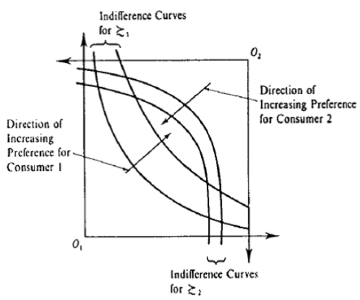


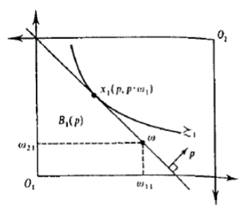


- Offer Curve = Demand (as a function of p)
    - (A depiction of preferences $\succsim_i$ of each consumer)
        - Asume strictly convex, continuous, strongly monotone

– As p varies, budget line pivots around ω. Given this line, consumer demands most preferred point in $B(p)$. This set of demanded consumptions makes up *offer curve*

  * Just like demand function: $x_1(p, p \cdot \omega_1)$
  * Curve passes through the endowment point. Also, since at each point endowment is affordable, every point on offer curve must be at least as good as ω_i.





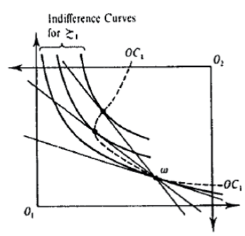


Figure 15.B.3 (top left) Preferences in the Edgeworth box.

Figure 15.B.4 (top right) Optimal consumption for consumer 1 at prices p.

Figure 15.B.5 (bottom) Consumer 1's offer curve.

- Walrasian Equilibrium (a.k.a. Competitive Equilibrium), for an Edgeworth Box:

  – <u>Definition:</u> Price vector p^* and an allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that for $i = 1, 2$

    * $x_i^* \succsim_i x_i$ for all $x_i' \in B_i(p^*)$

  – At equilibrium, the offer curves of the two consumers intersect

    * Any intersection of the offer curves outside of ω corresponds to a WE

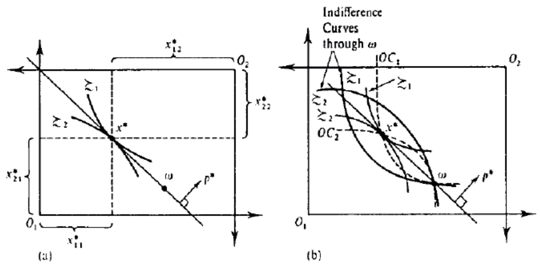
  – Only relative prices $\left(\frac{p_1^*}{p_{2*}}\right)$ are determined in equilibrium, since each consumer's demand is homogenous of degree zero

    * $p^* = (p_1^*, p_2^*)$ is a WE $\implies \alpha p^* = (\alpha p_1^*, \alpha p_2^*)$ is a WE

- Solving for WE strategy:

  1. Optimization: Max utility, using budget constraint to reduce number of variables
  2. Demand: Solve for Demand (OC) as a function of price $x_{li}(p)$
  3. Market Clearing: Set total demand for a good equal to endowment: $x_{l1} + x_{l2} = \omega_l$
     (a) AKA solve for excess demand function $z_l(p) = x_{l1} + x_{l2} - \omega_l$, and then set it equal to 0.
  4. Solve for price ratio $\left(\frac{p_1^*}{p_2^*}\right)$
     (a) To determine p^*, only need to find that for which one market clears. Other will necessarily clear at these prices.

  – *See PS1 Q6*



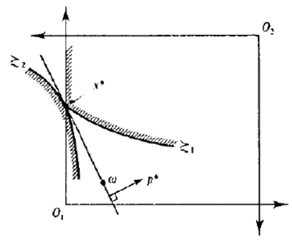


Figure 15.B.7 (top)
(a) A Walrasian equilibrium.
(b) The consumer's offer curves intersect at the Walrasian equilibrium allocation.

Figure 15.B.8 (bottom)
A Walrasian equilibrium allocation on the boundary of the Edgeworth box.

- Multple WE
  - Can certainly have multiple WE, as you will have a WE any time the Offer Curves intersect more than once
    * i.e. quasilinear utility with respect to different numeraires (doctored a bit, MWG 521)
- Non-existance of WE
  - Two classic examples

1. Non-monotonic Preferences
   (a) Endowment on the boundary
   (b) Consumer 2 only desires good 1 and has all good 1.
   (c) Price of good 2 is zero. Consumer 2 strictly prefers receiving more of good 1 (but won't). But also prefers to receive more of good 2 at price 0.
   (d) No p^* at which demands are compatible
      i. [Since Consumer 1's demand for good 2 is infinite at $p_2 = 0$]
      ii. Can also say: there is an $x_1' \succ_1 x_1^*,\ x_1' \in B(p^*)$, hence not a WE
2. Nonconvex Preferences
   (a) In example, consumer 1's offer curve is disconnected, so it does not intersect consumer 2's

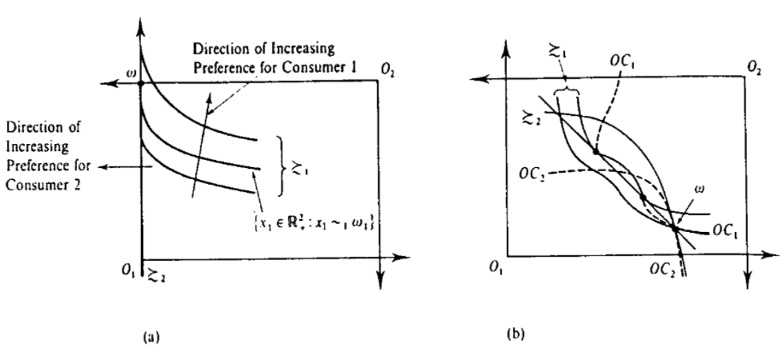
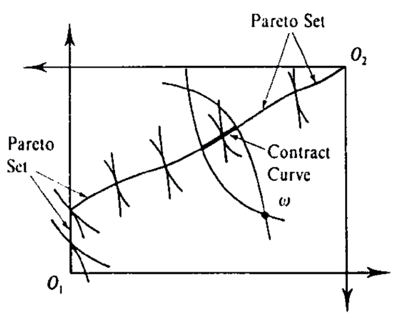


Figure 15.B.10 (a) and (b): Two examples of nonexistence of Walrasian equilibrium.

- Pareto Optimum
  - Definition: An allocation x in the EWB is Pareto Optimal if there is no other allocation x' in the Edgeworth box with $x'_i \succsim_i x_i$ and $i = 1, 2$ and $x'_i \succ_i x_i$ for some i.
    - "An economic outcome is Pareto optimal if there is no alternative feasible outcome at which every individual in the economy is at least as well off, and some individual is strictly better off."
  - Interior solution: consumers' indifference curves through x must be tangent
  - Corner solution: tangency need not hold
- Pareto Set
  - Set of all Pareto Optimal allocations
- Contract Curve
  - The part of the Pareto Set where both consumers do at least as well as at their initial endowments



- 1st Welfare Theorem
  - Any Walrasian equilibirum allocation x^* necessarily belongs to the Pareto Set
- 2nd Welfare Theorem
  - Verbally: Under convexity assumptions (not required for 1st WT), a planner can achieve any desired Pareto optimal allocation by appropriately redistributing wealth in a lump-sum fashion and then "letting the market work"
  - Definition: An allocation x^* in the EWB is supportable as an *equilibrium with transfers* if there is a price system p^* and wealth transfers T_1 and T_2 satisfying $T_1 + T_2 = 0$ such that for each consumer i we have: $x_i^* \succsim x'_i \;\; \forall x'_i \in \mathbb{R}^2_+$ such that $p^* \cdot x'_i \leq p^* \cdot \omega_i + T_i$
    - Note that p^* doesn't change; it is essentially the endowment ω that is moved by the transfers to get to the new x^*
  - Failures
    - Nonconvexities: May fail when preferences aren't convex (given budget line, consumer with nonconvex preference may prefer different point to intended x^*)
    - Monotonicity: (Figure 10(a) above). ω is PO, but cannot be supported as an equilibrium with transfers

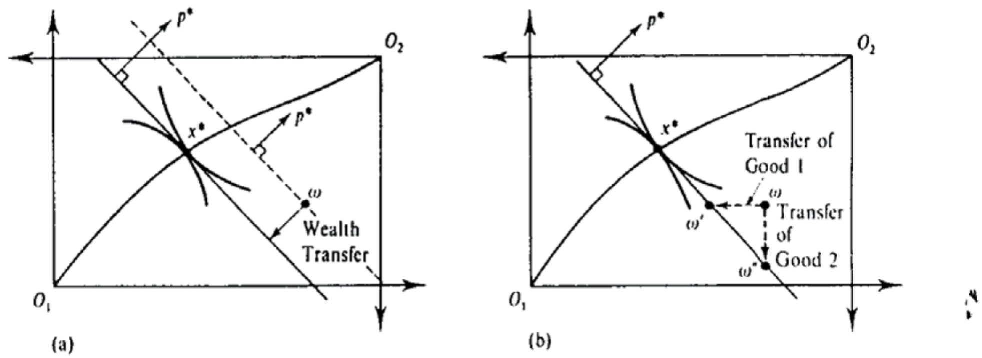


Figure 15.B.13 The second fundamental welfare theorem. (a) Using wealth transfers. (b) Using transfers of endowments.

15C: The One-Consumer, One-Producer Economy

- Set up
  - Single producer, single firm; both are price taking.
  - $x_1 = leisure$ (and $labor = z = \bar{L} - x_1$); $x_2 =$ good produced by firm
  - Endowment: $\bar{L}$ of leisure, 0 of x_2 (although consumer owns firm and gets profits, which are positive)
  - Continuous, convex and strongly monotone $\succsim$ defined over x_1 ($leisure$) and x_2
- Production

$$\max_{z \geq 0} pf(z) - wz$$

  - Given prices (p, w), [p is price of x_2, w is wage] firm has optimal labor demand $z(p, w)$, output $q(p, w)$ and $\pi(p, w)$
- Consumption

$$\max_{(x_1, x_2) \in \mathbb{R}^2_+} u(x_1, x_2)$$

$$\text{s.t. } px_2 \leq w\left(\bar{L} - x_1\right) + \pi(p, w)$$

  - Budget constraint is from wage earning and profit from firm
- Walrasian Equilibrium
  - *Price vector* (p^*, w^*) at which consumption and labor markets clear:

$$x_2(p^*, w^*) = q^*(p^*, w^*)$$

$$\bar{L} - x_1(p^*, w^*) = z(p^*, w^*)$$

- Graphically: 15.C.1(b) is not an equilibrium; 15.C.2 is
- A particular consumption-leisure combination is an WE $\iff$ it maximizes the consumer's utility subject to the economy's technological and endowment constraints

- Graph
  - O_f (producer's origin) in SE corner; takes $x-$axis as "$-z$"
  - O_c (consumer's origin) in SW corner as usual
  - Budget constraint line has slope $-\frac{w}{p}$ and intercept $\frac{\pi(p,w)}{p}$. This is the isoprofit line of the firm's profit maximization problem: $(-z, q): \; pq - wz = \pi(p, w)$

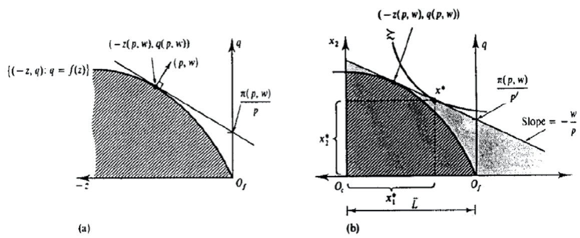


Figure 15.C.1 (a) The firm's problem. (b) The consumer's problem.

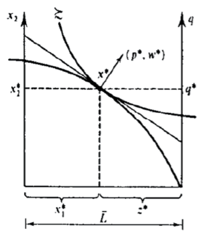


Figure 15.C.2
A Walrasian equilibrium.

16: Equilibrium and Its Basic Welfare Properties

Basic Model Definitions

- I consumers, J consumers, L goods
- Consumer i consumption set $X_i \subset \mathbb{R}^L$
- Preferences $\succsim_i$ defined on X_i, rational (complete & transitive)
- Endowment vector $\bar{\omega} = (\omega_1, \ldots, \omega_L) \in \mathbb{R}^L$
- Firm production set (or "technology") $Y_j \subset \mathbb{R}^L$. Every $Y_j \neq \emptyset$ (nonempty), closed.

Basic data on preferences, technologies and resources for this economy are summarized by $\left(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\omega}\right)$

Feasible Allocation (x, y)

- Definition: A feasible allocation is a vector $(x_1, \ldots, x_I, y_1, \ldots, y_I)$ s.t.

  1. $x_i \in X_i \;\; \forall i$

2. $y_j \in Y_j \quad \forall j$
3. $\sum_i x_i = \bar{\omega} + \sum_j y_j$

The set of feasible allocations is denoted by $A = \left\{(x, y) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_I : \sum_i x_i = \bar{\omega} + \sum_j y_j\right\} \subset \mathbb{R}^{L(I+J)}$

Pareto Optimum

- Definition: A feasible allocation $(x, y) \in A$ is Pareto Optimal if $\nexists (x', y') \in A$ s.t. $x'_i \succsim_i x_i \quad \forall i$ and $x'_i \succ x_i$ for some i.

Private Ownership Economies

- Endowment: Each consumer i has initial endowment $\omega_i \in \mathbb{R}^L$
- Shareholding: Each consumer i has shareholding θ_{ij} in firm j
  - $\theta_{ij} \in [0, 1] \quad \forall i, j$
  - $\sum_i \theta_{ij} = 1 \quad \forall j$

Data on preferences, technology, resources and ownership are summarized by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{\omega_i, \theta_{i1}, \ldots \theta_{iJ}\}_{i=1}^I\right)$

Walrasian Equilibrium
Given a private ownership economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \left\{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})_{i=1}^I\right\}\right)$,

An allocation (x^*, y^*) and a price vector $p = (p_1, \ldots p_L)$

constitute a Walrasian (or competitive) Equilibrium if:

1. $p \cdot y_j \leq p \cdot y_j^* \quad \forall y_j \in Y_j$
   (a) [π max: That is, for every j, y_j^* maximizes profits in y_j]
2. For every i, x_i^* is maximal for $\succsim_i$ in the budget set $\left\{x_i \in X_i : \ p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\right\}$
   (a) [Hence $x_i \succ_i x_i^* \implies p^* x_i > p^* \omega_i + \sum_j \theta_{ij} p \cdot y_j^*$]
   (b) ["Consumers maximize utility subject to budet constraint, and anything strictly better is unattainable"]
3. $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$
   (a) [Demand = supply (L equations, one for each good)]

Note that if $p^* = 0$, then all are satisfied except (2), since can be another bundle with more of a good that is affordable.

Local Nonsatiation

- Definition: The preference relation $\succsim_i$ on the consumption set X_i is LNS if for every $x_i \in x_i$ and every $\varepsilon > 0$, $\exists \ x'_i \in X_i$ s.t. $\|x'_i - x_i\| \leq \varepsilon$ and $x'_i \succ_i x_i$

Note that this assumes divisibility of goods, so it can't happen in a real economy.

1st Theorem of Welfare Economics

Statement:

- Suppose $\succsim_i$ is LNS $\forall i$
- Let (x^*, y^*, p^*) be a WE
- Then (x^*, y^*) is PO

Proof:

- Suppose that (x^*, y^*, p) is a price equilibrium with transfers and that the associated wealth levels are $(w_1, \ldots, w_I)$. Recall that $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$.

  1. We have that if $x_i \succ_i x_i^* \implies p \cdot x_i > w_i$
  2. LNS implies (which can be verified) $x_i \succsim_i x_i^* \implies p \cdot x_i \geq w_i$

- Consider an allocation $\left(x^\dagger, y^\dagger\right)$ that Pareto dominates (x^*, y^*). That is, $x_i^\dagger \succsim_i x_i^*$ for all i and $x_i^\dagger \succ_i x_i^*$ for some i.

- By (2) we must have $p \cdot x_i^\dagger \geq w_i \quad \forall i$, and by (1) we must have $p \cdot x_i^\dagger > w_i$ for some i.

  – Hence, $\sum_i p x_i^\dagger > \sum_i w_i$
  – (and since $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$) we have

$$\sum_i p x_i^\dagger > \sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$$

- Moreover, because y_j^* is profit maximizing for firm j at price vector p,we have $p \cdot \bar{\omega} + \sum_j p y_j^* \geq p \cdot \bar{\omega} + \sum_j p y_j$

$$\text{Thus } \sum_i p \cdot x_i^\dagger > p \cdot \bar{\omega} + \sum_j p \cdot y_j$$

$$\implies \textit{contradiction} \text{ of (3) in WE definition; } \left(x^\dagger, y^\dagger\right) \text{ cannot be } \textbf{feasible}$$

  – Since (3), $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$, implies $\sum_i p \cdot x_i^* = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$
  – [Right Argument:

  By feasibility, "$x - \bar{\omega} - y = 0$" $\left[\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*\right]$

  "$\therefore p^* (x - \omega - y) = 0$" $\left[\sum_i p \cdot x_i^* = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*\right]$

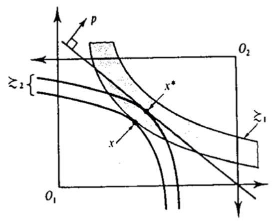
  Contradiction

  – [Wrong Argument: $p^* (x - \omega - y) > 0 \therefore x - \omega - y > 0$, *contradicts* $x - \omega - y = 0$]

Importance of LNS Assumption

- Assumption of Local Nonsatiation rules out situation with "thick indifference curves," as can be seen in EWB example.

– See graph below: consumer 1 is indifferent about a move to allocation x, and consumer 2, having strongly monotone preferences, is strictly better off.



MRS at Prices
[Calculus Proof - downside is corner solutions, 2nd order differentiability, doesn't make as clear what you need. But it does have advantages, including helping us think in terms of marginal benefits and costs]
If consumers maximize utility subject to budget constaint, then MRS between any two goods will equal the price ratio between those goods.
Since we all face same prices, all of our MRS are the same.
Also, if firms are maximizing profit, the MRT (marginal rate of transformation) will equal relative prices.
So MRS equalized across consumers, MRT equalized across producers, and MRTs = MRSs = price ratios.
This is FOC for Pareto Optimum.
Best way to do this is to give people utility functions. Maximize one person's utility s.t. all other utilities being at least at some level. ($\max u_1$ s.t. $u_i \geq \bar{u}_i$, technological constraints). The FOC will say MRT = MRS.
Conclusion: WE satisfies FOCs for a PO.

If then assume everything is convex, then FOCs will also be sufficient for a global optimum. We still need to worry about corner solutions (for MRT=MRS)

2nd Welfare Theorem

Price Equilibrium with Transfers:

- Definition: $(x^*, y^*, p^*)$is a *price equilibrium with transfers* if $\exists$ an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = p^*\bar{\omega} + \sum_j p^* \cdot y_j^*$ such that

1. $\forall j, \quad y_j^* \in Y_j$ and $p^* \cdot y_j^* \geq p^* \cdot y_j \quad \forall y_j \in Y_j$
   (a) "y_j^* maximizes profits"
2. $\forall i, \quad x_i^* \in \mathbb{R}_+^L$, $p^* \cdot x_i^* \leq w_i$ and $x_i \succ_i x_i^* \implies p^* x_i > w_i$
   (a) MWG style: For every i, x_i^* is maximal for $\succsim_i$ in the budget set $\{x_i \in X_i : \ p \cdot x_i \leq w_i\}$
   (b) "Anything strictly better off is unattainable"
3. $\sum_i x_i^* - \sum_j y_j^* - \bar{\omega} = 0$

Price Quasiequilibrium with Transfers:
Only part that changes is from $>$ to $\geq$ in (2)

- Definition: $(x^*, y^*, p^*)$is a *price equilibrium with transfers* if $\exists$ an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = p^*\bar{\omega} + \sum_j p^* \cdot y_j^*$ such that

1. $\forall j, \quad y_j^* \in Y_j$ and $p^* \cdot y_j^* \geq p^* \cdot y_j \quad \forall y_j \in Y_j$
   (a) "y_j^* maximizes profits"
2. $\forall i, \quad x_i^* \in \mathbb{R}_+^L$, $p^* \cdot x_i^* \leq w_i$ and $\underline{x_i \succ_i x_i^* \implies p^* x_i \geq w_i}$
3. $\sum_i x_i^* - \sum_j y_j^* - \bar{\omega} = 0$

Statement 1 of 2nd WT:

- Suppose $\begin{array}{rl} \text{i)} & Y_j \text{is convex } \forall j \\ \text{ii)} & \succsim_i \text{is convex } \forall i \\ \text{iii)} & \succsim_i \text{is LNS } \forall i \end{array}$
- Let (x^*, y^*) be Pareto Optimal.
- Then there exists a price vector $p^* \neq 0$ s.t. (x^*, y^*, p^*) is a *quasiequilibrium* with transfers.

["If you have a quasiequilibrium, then competitive market equilibrium if distribution of wealth is appropriate" (15:40)]

[Proof preliminaries:]
The mathematical tool is something called separating hyperplane theorem. Took years to realize this was the right tool; they used to use calculus (i.e. Samulson). Would say Pareto optimum, use FOC. From this MRS must be equal between consumers for any good; this must be equal to MRT across firms. This is assuming no corner solutions.
So if all marginal rates equal, then can set prices equal to all marginal rates. These prices satisfy all FOCs. If we assume everything so that FOCs are necc and suff, then consumers will be maximizing according to their budget constraint. So we can adjust budgets to get consumers to consume at desired level.
*In both approaches, assume **convexity.***

The advantage is that this proof shows what is really driving things. Arrow-Debreu realized in early 1950s that you could do it this way.

[Basic idea of proof:] (21:00)

Suppose have two goods (Figure 16.D.1)
Hyperplane is something which has one dimension less than full space $(n-1)$ *dimensional*
Dissecting Hyperplane Theorem:
If you have two convex sets which don't intersect, then you can find a dissecting hyperplane.
Careful! If two convex sets intersect in one point, then can't necessarily separate them (look at two lines that intersect)

What we will do is see that feasible production set does not intersect *__what lies above__* the set that is preferred to the (consumption set?)*]* (26:00)

Proof:
Define

$$V_i = \left\{ \sum_i x_i \in \mathbb{R}^L_+ | x_i \succ_i x_i^* \right\}$$

$$\text{Let } V = \sum V_i$$

$$\text{Let } Y = \sum Y_i$$

- [Notes]
  - For every i, the set V_i of consumptions preferred to x_i^*
  - V is the aggregate consumption bundles that could be split into I individual consumptions, each preferred by its corresponding consumer to x_i^*
  - Y is simply the aggregate production set
  - Note that $Y + \{\bar{\omega}\}$–geometrically the aggregate production set with its origin shifted to $\bar{\omega}$–is the set of aggregate bundles producible with given technology and endowments (and therefore usable for consumption).
  - Set addition: When you add sets, $A + B = \sum x | x = a + b$ for some $a \in A, b \in B$, i.e. $(1, 0, -1/2)+(0, 1, -1/4)=(1, 1, -3/4)$]

__Step 1:__
V_i is convex $\forall i$

- *Proof:*
  - Suppose that $x_i \succ_i x_i^*$, $x_i' \succ_i x_i^*$. Take $0 \leq \alpha \leq 1$
  - WOLG assume $x_i \succsim_i x_i'$ (by completeness)
  - Since $\succsim_i$ is convex, $\alpha x_i + (1-\alpha)x_i' \succsim x_i' \succ_i x_i^*$
  - By transitivity: $\alpha x_i + (1-\alpha)x_i' \succ_i x_i^*$

  [$\therefore \alpha x_i + (1-\alpha)x_i' \in V_i$]

__Step 2:__
V is convex, $Y + \{\omega\}$ is convex

- *Proof:*
  - (The sum of convex sets is convex)

__Step 3:__
$V \cap (Y + \{\omega\}) = \emptyset$

- *Proof:*
  - This is a consequence of the Pareto Optimality of (x^*, y^*).

[If not there exists $x \in V,\ x \in Y + \{\omega\}$
Can write $x = \sum_i x_i,\ x_i \in V_i \forall i,\ x = \sum_j y_j + \omega,\ y_j \in Y_j \forall j$
This would mean that with the given endowments and technologies it would be possible to produce an aggregate vector that could be uesd to give every consumer i a consumption bundle that is preferred to x_i^*.]

<u>*Step 4:*</u> (41:49)
$\exists\ p^* = (p_1, \ldots, p_L) \neq 0$ and a number r s.t.

$$p^* \cdot z \geq r\ \forall z \in V$$

$$p^* \cdot z \leq r\ \forall z \in Y + \{\omega\}$$

- *Proof:*
  - This follows directly from *SHT* (Separating Hyperplane Theorem).

[Hyperplane is $\left\{z \in \mathbb{R}^L |\ p^* z = r\right\}$]

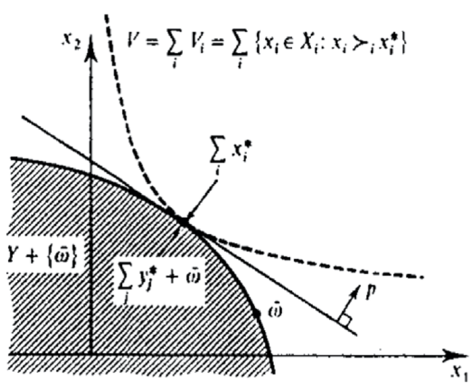


Figure 16.D.1
The separation argument in the proof of the second welfare theorem.

<u>*Step 5:*</u>
If $x_i \succsim_i x_i^*\ \forall i$, then $p^* \left(\sum_i x_i\right) \geq r$

[This is saying that if I put $\succ_i$ then we'd already know if from above, since it would be in V. Only additional element here is going from $\succ$ to $\succsim$]

- *Proof:*
  - By local nonsatiation, we can find $(x_i')_{i=1,\ldots,I}$ arbitrarily close to x_i s.t. $x_i' \succ_i x_i$
    - $\implies x_i' \succ_i x_i^*$
    - $\implies x_i' \in V_i \implies \sum_i x_i' \in V$

- $\therefore p^* \cdot (\sum_i x_i') \geq r$
- Take limits as $x_i' \to x_i \quad \forall i$
- By continuity $p^* \cdot (\sum_i x_i) \geq r$

<u>*Step 6:*</u>
$p^* \cdot (\sum_i x_i^*) = r = p^* \cdot \left(\sum_j y_j^* + \bar{\omega}\right)$

- *Proof:*
  - By Step 5, $p^* \cdot (\sum_i x_i^*) \geq r$
  - But we also know:$\sum_i x_i^* = \sum_j y_j^* + \bar{\omega}$ (feasible allocation)
  - $p^* \cdot (\sum_i x_i^*) = p^* \cdot \left(\sum_j y_j^* + \bar{\omega}\right) \leq r$

$\therefore\ p^* \cdot (\sum_i x_i^*) = r = p^* \cdot \left(\sum_j y_j^* + \bar{\omega}\right)$

<u>*Step 7:*</u>
$\forall j,\ p^* \cdot y_j^* \geq p^* \cdot y_j \quad \forall y_j \in Y_j$

- *Proof:*
  - Consider a particular j. Let $y_j \in Y_j$
  - Then $y_j + \sum_{k \neq j} y_k^* \in Y$
  - Therefore, $p^* \cdot \left(y_j + \bar{\omega} + \sum_{k \neq j} y_k^*\right) \leq r = p^* \cdot \left(y_j^* + \bar{\omega} + \sum_{k \neq j} y_k^*\right)$
  - Hence, $p^* \cdot y_j \leq p^* \cdot y_j^*$

[If it's profit maximizing in the aggregate, then it must be maximizing at the individual level. If this weren't the case, then let some firm maximize profit, keep everyone else the same, then profit increases in the aggregate]

<u>*Step 8:*</u>
Let $w_i = p^* \cdot x_i^*$
$\sum_i w_i = p^* \cdot (\sum_i x_i^*) = r = p^* \cdot (\sum_i y_i^* + \bar{\omega})$

- *Proof:*
  - [Last two equalities from previous steps]

<u>*Step 9:*</u>
$x_i \succ_i x_i^* \implies p^* \cdot x_i \geq w_i$

- *Proof:*
  - Suppose $x_i \succ_i x_i^*$
  - Then certainly $x_i \succsim_i x_i^*$

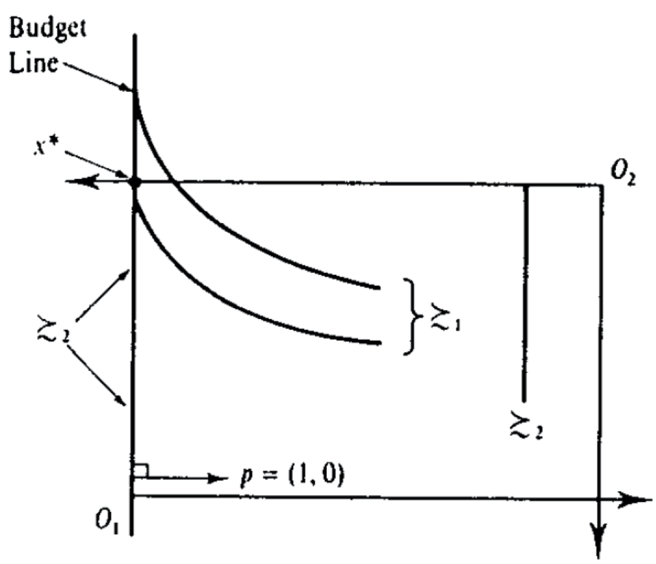
- $\therefore p^* \cdot \left(x_i + \sum_{k \neq i} x_i^*\right) \geq p^* \cdot \left(x_i^* + \sum_{k \neq i} x_i^*\right)$ (by step 5)
- $\implies p^* \cdot x_i \geq p^* \cdot x_i^* = w_i$

Conclusion

- (1) follows from Step 7 $\left[\forall j, \quad y_j^* \in Y_j \text{ and } p^* \cdot y_j^* \geq p^* \cdot y_j \quad \forall y_j \in Y_j\right]$
- (2) follows from Step 9 $\left[\forall i, \quad x_i^* \in \mathbb{R}_+^L, \ x_i \succ_i x_i^* \implies p^* x_i \geq w_i\right]$
- (3) follows from feasibility of Pareto Optimal allocation (x^*, y^*) $\left[\sum_i x_i^* - \sum_j y_j^* - \bar{\omega} = 0\right]$

Price Quasiequilibrium that is not a Price Equilibrium
Can we conclude that (x^*, y^*, p^*) is a WE with transfers?

- The answer is not always. Famous counterexample that we saw in Ch 15.
  - The price vector that solves the quasiequilibrium is $p = (p_1, 0)$.
  - Consumer 1's consumption bundle $x_1^* = (0, \omega_2)$ satisfies (2) of the definition of price quasiequilibrium.
    - * "Any better bundle is at least as expensive"
  - However, it is not consumer 1's preference-maximizing bundle in her budget set. (She would have infinite demand for good 2)
    - * "Not WE because better bundle doesn't cost more"
  - An important reason for failure is that *consumer 1's wealth level at the quasiequilibrium is zero.*



Final note: Problem with not assuming covexity

From Quasiequilibrium to Equilibrium in 2nd Welfare Theorem

1. Assume $p^* \cdot x^* > 0$

Lemma: ***If not at zero wealth, then a strictly preferred bundle is more expensive*** (2:00)
Suppose $x^* \in X_i = \mathbb{R}^L_+$ and $p,\ w_i$ s.t.

$$x_i \succ_i x_i^* \implies px_i \geq w_i$$

Then if there exists $x_i' \in X_i$ s.t. $px_i' < w_i$, it follows that

$$x_i \succ_i x_i^* \implies px_i > w_i$$

[Anything strictly prefered to x_i^* is outside the budget constraint]

- *Proof:* (by contradiction)
  - Suppose $x_i \succ_i x_i^*$ and $p \cdot x_i = w_i$ (We know can't be $<$)
  - Consider $y = \alpha x_i + (1-\alpha)\, x_i' \in X_i$, $0 < \alpha < 1$
  - $p \cdot y < w_i$
  - By continuity of $\succsim_i \implies y \succ_i x_i^*$ for α close to 1.
  - This contradicts $x_i \succ_i x_i^* \implies px_i \geq w_i$ (above).

From last time had a price quasiequilibrium with transfers.
Let's just add: if $p^* x_i^* > 0\ \ \forall i$, then (x^*, y^*, p^*) is a price equilibrium with transfers. (9:50)
Problem with this is that it is endogenous.

2. Make Assumptions that will give Equilibrium

3 Assumptions for Equilibrium (One approach):

1. Assume $\succsim_i$ are strictly monotonic (which means if I give you more than any good, you are strictly better off)
2. Also assume $\bar{\omega} \gg 0$
3. And $0 \in Y_j \forall j$ (any firm could shut down)

   (Quite strong assumptions, but then we are ready to make some progress)

- Then $p^* \gg 0$ in any price quasiequilibrium

Proof:
We have in price quasiequilirbrium

$$(*)\ \ x_i \succ_i x_i^* \implies p^* x_i \geq p^* x_i^*$$

1. $p^* \geq 0$
   (a) Since if have negative prices for any good, just consume more of that good which violates the above (*)

2. $\sum_i w_i = \sum_i \left(p^*\omega_i + \sum_j \theta_{ij} p^* y_j^*\right)$ (this is just total wealth in the economy $\sum_i w_i$)

   (a) $= p^*\bar{\omega} + \sum_j p^* y_j^*$

   (b) $\geq p^*\bar{\omega} > 0 \implies w_i > 0$ for at least one i, say k

      i. For $k: \; x_k \succ x_k^* \implies p^* x_k > p^* x_k^*$

      ii. Consumer k would buy an infinite amount of good l if it was free

   (c) $\implies p^* \gg 0$

      i. For consumer $i \neq k$: 2 cases

         A. $x_i^* \neq 0$, then $p^* \cdot x_i^* > 0$ (cost of your bundle has to be strictly positive

         B. By Lemma: $x_i \succ_i x_i^* \implies p^* x_i > p^* x_i^*$

         C. $x_i^* = 0$

         D. They are starving. But they are still maximizing utility subject to their budget constraint. They have no wealth, $p^* \gg 0$, hence the best they can do is to consume nothing.

3. Everyone is maximizing utility subject to their budget constraint, so we have established that this is a full equilibrium

   (a) This is the kind of thing you can do if you don't want to just assume $p^* x_i^* > 0$, which is endogenous.

   (b) Here we've shown everything works if you have strongly monotone preferences, $\bar{\omega} \gg 0$.

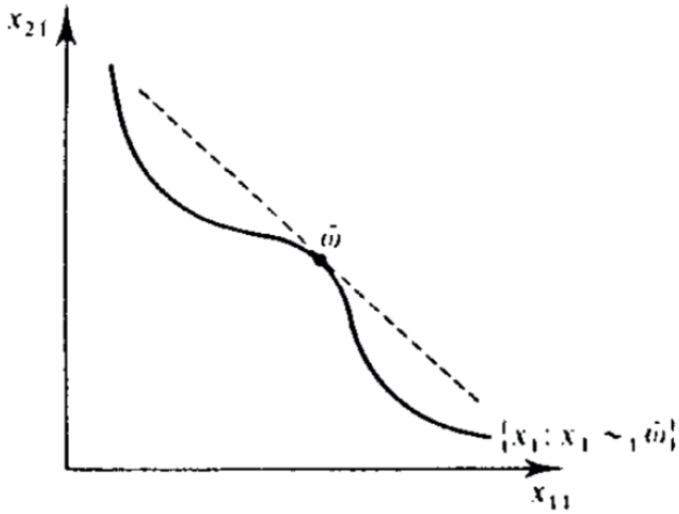
      i. These are pretty strong assumptions. You can weaken them and there was a lot of work on this in 50s and 60s.

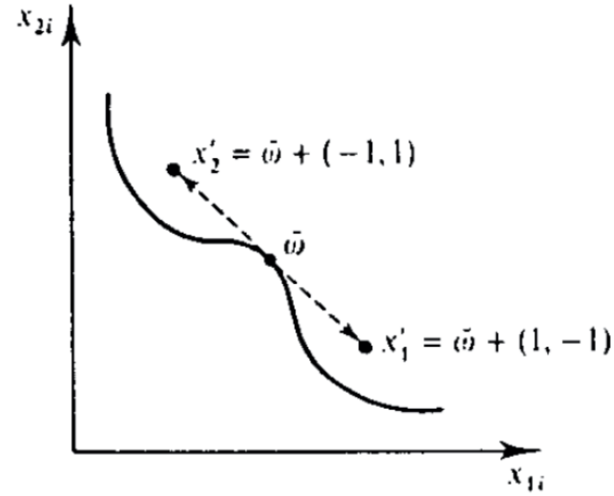
Importance of Convexity Assumptions

How important are convexity assumptions? We've seen they are important
However, if we do look at a large economy, these convexity assumptions becomes much less serious

- [Don't have time to go into the details, but sketch. There's much more about this in the literature]
- Take upper contour sets, and take convex hull of those sets
- Show modified economy; want to show that feasible allocation is feasible for the original economy (only if there are a large number of people)
  - Take pareto optimum; then go back and say how can you sustain? It's by this trick of having some above and some below.
- The way this is done is to not have anyone actually consuming that bundle, but to have half below and half above (x_1' and x_2') – Figure 16.D.5
- You can also do this with production (34:00)

  - However can only do this if convexities are bounded. If there are increasing returns to scale forever, then firms will want to be huge





[How interesting is the 2nd welfare theorem?]
At first, seen as more important because the math is more subtle (separating hyperplane theorem)
Hard to say what it means: interested in a benevolent planner, who has found out somehow what a pareto optimum is, who reads this theorem, and says could achieve pareto optimum. But they could also mandate the equilibrium without using prices.
But planners, even if benevolent, don't have perfect information. Then you're in 2nd best world, then can't achieve pareto optimum, and then 2nd welfare theorem doesn't apply.
For example; Income tax: people know own ability, planner doesn't. Planner wants to redistribute from high to low ability. But you can't have a lump sum transfer, b/c you don't know who's who. If you ask, high types have incentive to lie. Given this, you end up with an income tax. Won't just have T=100, but rather $T = t(y_i)$, earned income. But with these taxes you have an incentive effect, don't have incentive to work as hard in the model.
Idea is that once you do all this, *you are quite far away from 2nd welfare theorem.*
So moral of the story is that is it not really clear to know where the 2nd welfare theorem is applicable.
Hart: thinks that for economics, the first is more important. "Formalization of the invisible hand"

Ch. 17: Existance of Walrasian Equilibrium

Expressing Equilibria as the solution to a system of equations

- Will give us a little sense of the literature, but there are tons of things that won't be covered.
- For purposes of formal analysis, extremely helpful to be able to express equilibria as *a system of equations.*
  - Much of what follows is studying how this can be done

Excess Demand Function

Pure Exchange Economy Set Up

- Focus on a *pure exchange economy.*
  - $X_i = \mathbb{R}_+^L \forall i$
  - $\succsim_i$ rational, continuous $\forall i$
  - $\omega_i \in R_+^L \forall i$
  - $\bar{\omega} = \sum_i \omega_i \gg 0$

Walrasian Equilibrium

- Walrasian Equilibrium

$$(x_1^*, \ldots, x_I^*, p^*) \text{ s.t.}$$

1. $\forall i,\ p^* x^* \le p^* w_i$ and $x_i \succ_i x_i^* \implies px_i > p^* w_i$
2. $\sum_i x_i^* = \bar{\omega}$

Further Assumptions: Monotonicity, Convexity

- We also assume
  - $\succsim_i$ is strictly monotonic $\forall i$
  - $\succsim_i$ is strictly convex $\forall i$
  - these imply $p^* \gg 0$

So we can talk about demand functions.

Excess Demand $z(p)$

- Demand functions for each i
- **Excess demand:** $z(p)$

$$z(p) \equiv \sum_i x_i (p, p\omega_i) - \bar{\omega}$$

$$x_i (p, w) \in R_+^L \text{ solves } : \max \succsim_i \text{ s.t. } px \le w,$$

  - with $p \gg 0,\ w \ge 0$

Walrasian Equilibrium, Restated:

- Walrasian Equilibrium

$$p^* \text{ s.t. } z(p^*) = 0$$

(can also be stated as $z_l(p^*) = 0$ for every $l = 1, \ldots, L$)

- If some component is not 0, this means that market is not clearing.

Properties of Excess Demand Function
Proposition: Excess demand function z satisfies:

1. z is continuous
   (a) Excess demand functions are continuous since *it is a sum of demand functions that are continuous*
2. z is homogenous of degree zero $z(\lambda p) = z(p) \quad \forall \lambda > 0$
3. $p \cdot z(p) = 0$
   (a) Walras' law
4. $\exists s > 0$ s.t. $z_l(p) > -s \forall l, p$
   (a) Whole z function is bounded below by $-s$
   (b) Since $x_i \geq 0$, which means $x_i - \omega_i \geq -\omega_i$, which means that $\sum_i (x_i - \omega_i) \geq -\bar{\omega}$, and we see that $\sum_i (x_i - \omega_i) = z$.
5. If $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then

$$\max(z_1(p^n), \ldots, z_L(p^n)) \to \infty$$

   (a) We're interested in convergence where some (but not all) prices are going to zero. This tells us that some excess demand will go to infininty.
   (b) Suppose the prices good one are going to 0, and other prices are positive, and they like good one, then consumers would load up on it (buying more and more of it). So then since demand for one good is going to ∞, the max is going to ∞
   (c) More subtle case: suppose 2 goods $[p_1, \ p_2]$ are becoming very cheap, and others are not. p_2 might be going to zero much faster than p_1 (so $z_1(p^n)$ not necessarily going to ∞) what you can prove is that *you'll load up on something, but may not loading up on everything.*
      i. This is why we need the "max" in the above expression

Existance Question:

- ***Is there such a p^*? (or are there more than one?)***

$$
\begin{gathered}
z_1\left(p_1^*, \ldots, p_L^*\right)=0 \\
z_2\left(p_1^*, \ldots, p_L^*\right)=0 \\
\vdots \\
z_L\left(p_1^*, \ldots, p_L^*\right)=0
\end{gathered}
$$

Equation Counting:
L equations, L unknowns. But more complicated than this:
Result: If $(x_1^*, \ldots, x_I^*, p^*)$ is a WE, then so is $(x_1^*, \ldots, x_I^*, \lambda p^*)\, \forall \lambda > 0$
This is just saying that if I double prices, the budget constraint gets doubled, so whatever you were doing before is still optimal. There is no money in this economy.
For the equation counting approach, this poses some difficulties. We can just normalize the price of one good to 1.
So we'll start out saying $p_L^* = 1$. But now have decreased number of unknowns. But now system is overdetermined (more equations than unknowns).
However, turns out we don't need to worry about this because of *Walras' law,* which will get rid of 1 equation. This will leave us with $(N-1)$ equations, $(N-1)$ unknowns.

Proof of Walras' Law:

- Back to consumers' problem: (1:04:00)

$$\max \succsim_i \text{ s.t. } px_i \leq w_i$$

$$\text{Solution: } x_i\left(p, w_i\right)$$

$$px_i\left(p, w_i\right) = w_i = p\omega_i$$

- Walras' Law (Proof):

$$pz(p) = p\left(\sum_i x_i\left(p, p\omega_i\right) - \sum_i \omega_i\right) = \sum_i \underbrace{p\left(x_i\left(p, p\omega_i\right) - \omega_i\right)}_{=0} = 0$$

It is telling us in and out of equilibrium: whatever prices are out there, if people report excess demands, if you add them up, the value of excess demands is 0. Everybody is spending everything they have, which means that net value of demands is 0. If you're buying something, you must be selling something to satisfy your budget constraint. Can't have a net deficit or a net surplus.

Formally:
Because of Walras' law, to verify that a price vector $p \gg 0$ clears all markets, it suffices to check that it clears all markets but one.

- If $p \gg 0$ and $z_1(p) = \cdots = z_{L-1}(p) = 0$, then because $p \cdot z(p) = \sum_l p_l z_l(p) = 0$ and $p_L > 0$, we must also have $z_L(p) = 0$.

This gets rid of an equation, so we we're back to $L-1$ equations, $L-1$ unknowns.
[Though remember that equation counting is just a first step, need to go further]

A simple proof of existance: 2 consumers, 2 goods

We can do everything in the edgeworth box.
Draw indifference curves from endowment point. Price line will cut in between them towards the center., with price line
Draw

- Endowment point
- Indifference curves (through initial endowment point)
- Price line (such that it needs to go through the initial endowment

How do we know there is such an outcome?

- What you do is look at the "contract curve"
    - The Pareto optima when everyone's preferences are strictly convex.
        * The set of Pareto optima in between the two indifference curves are called the ***contract curve***
        * That is, the set of Pareto allocations that are individually rational
            · So each consumer is at least as well off as if they walked away with their initial endowment
- Move along the contract curve. At each point can draw the common tangent. As long as everything is *continuous*, it common tangent must go through contract curve somewhere.
    - Where it does, we have a WE, by definition

Informal Proof, but quite instructive.
It doesn't rule out multiple equilibria.

Existance of GE

Conditions:

- $z(p)$ is a function defined for all strictly positive price vectors, satisfying conditions 1-5 above
- $\underline{\succsim_i \text{ continuous, strictly convex, strongly monotone } \forall i}$
- $\sum_i \omega_i \gg 0$

Proposition:

$$\exists p \text{ s.t. } z(p) = 0$$

Proof:
Preliminary:

$$\text{Let } \Delta = \left[p \in \mathbb{R}_+^L | \sum_i p_i = 1 \right]$$

[Just normalizing prices in a conventient way.]

$$\text{Construct } f: \ \Delta \to \Delta$$

***Step 1*:**
Construction of the fixed-point correspondence for $p \in interior \Delta$

$$\text{For } p \in interior\Delta$$

$$\text{Let } f(p) = \{q \in \Delta | q \cdot z(p) \geq q' \cdot z(p) \quad \forall q' \in \Delta\}$$

In words: q will maximize (among the permissible price vectors–those in Δ) the value of the excess demand vector. Intuitively, this will assign the highest prices to the commodities that are most in excess demand.
[$z_1(p) \gg 0$, i.e. $z(p) = (1, 2, -5, -6)$. The value of q will be $(0, 1, 0, 0)$]

In particular, we have $f(p) = \{q \in \Delta : \ q_l = 0 \text{ if } z_l(p) < \max\{z_1(p), \ldots, z_L(p)\}\}$

- If $z(p) \neq 0$ for $p \gg 0$
  - then because of Walras' law $[p \cdot z(p) = 0]$ we have $z_l(p) < 0$ for some l and $z_{l'}(p) > 0$ for some $l' \neq l$.
  - Thus, for such a p, any $q \in f(p)$ has $q_l = 0$ for some l.

$$z(p) \neq 0 \quad \Longrightarrow \quad f(p) \subset boundary\Delta$$

- In contrast

$$z(p) = 0 \quad \Longrightarrow \quad f(p) = \Delta$$

Step 2:
Construction of the fixed-point correspondance for $p \in boundary\Delta$

$$\text{For } p \in boundary\Delta$$

$$f(p) = \{q \in \Delta |\ q \cdot p = 0\}$$

- Because $p_l = 0$ for some l, we have $f(p) \neq \emptyset$
- Note that no price on the boundary can be a fixed point
  - That is, $p \in boundary\Delta$ and $p \in f(p)$ cannot occur because $p \cdot p > 0$ while $p \cdot q = 0$ for all $q \in f(p)$

[Ex: $p = \left(0, \frac{1}{4}, \frac{3}{4}\right)$; $q = (1, 0, 0)$]

Step 3:
A fixed point of $f(\cdot)$ is an equilibrium

- Suppose that $p^* \in f(p^*)$
  - Since by step 2 we saw that we cannot have $p^* \in boundary\Delta$ if $p^* \in f(p^*)$
    - We must have $p^* \gg 0$
    - We cannot have $z(p^*) \neq 0$
      - Since we saw in step 1 that $f(p^*) \in boundary\Delta$, which is incompatible with $p^* \in f(p)^*$ and $p^* \gg 0$.
- Hence if $p^* \in f(p^*) \quad \Longrightarrow \; z(p^*) = 0$

Is $f(p)$ convex?
Yes. (Kakutani's FPT)

Step 4:
The fixed-point correspondence is convex-valued and upper hemicontinuous

[See MWG]

Step 5:
A fixed point exists.

- Kakutani's fixed point theorem:
  - A *convex-valued, upper hemicontinuous correspondence* from *a non-empty, compact, convex set into itself* has a fixed point.
- Since $f(\cdot)$ is a convex-valued, upperhemicontinuous correspondance from Δ to Δ , and Δ is a nonempty, convex and compact set, we conclude that there is a $p^* \in \Delta$ with $p^* \in f(p^*)$.

Examples of Non-Existance

- Two classic examples (same as Ch 15)

   1. Non-monotonic Preferences

      (a) Endowment on the boundary

      (b) Consumer 2 only desires good 1 and has all good 1.

      (c) Price of good 2 is zero. Consumer 2 strictly prefers receiving more of good 1 (but won't). But also prefers to receive more of good 2 at price 0.

      (d) No p^* at which demands are compatible

         i. [Since Consumer 1's demand for good 2 is infinite at $p_2 = 0$]

         ii. Can also say: there is an $x_1' \succ_1 x_1^*,\ x_1' \in B(p^*)$, hence not a WE

   2. Nonconvex Preferences

      (a) In example, consumer 1's offer curve is disconnected, so it does not intersect consumer 2's

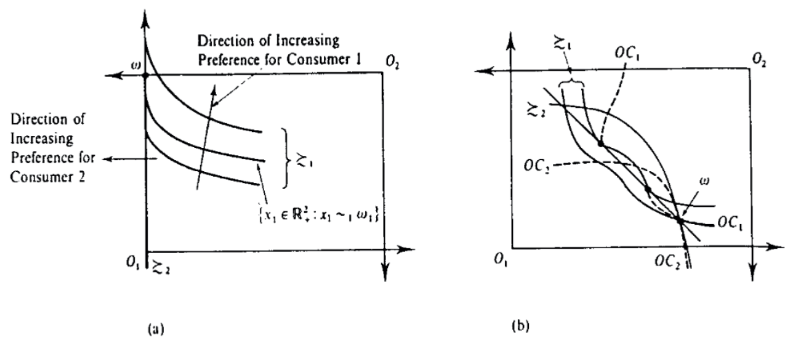


Figure 15.B.10 (a) and (b): Two examples of nonexistence of Walrasian equilibrium.

Extra Issues

Uniqueness?
Would be very nice but it's hard to get. (see graph 3 - "nothing pathological about this")

2 cases where you get it

1. Assume aggregration of consumers

   (a) If we assume a normative representative consumer

   (b) $\rightarrow$unique pareto optimum (PO)

   (c) $\therefore$ unique WE allocation

      i. [sort of approach that macroeconomists rely on quite a bit]

2. Gross substitutes

   (a) If z satisfies GS if: if p' and p are s.t. for some l, $p'_l > p_l$ and $p'_k = p_k \forall k \neq l$ then $z_k(p') > z_k(p) \forall k \neq l$

   (b) This is a strong assumption, but it does get you what you want.

   Proof:

   Suppose $z(p) = z(p') = 0$, $p' \nsim p$. By homogeneity of degree zero, we can assume that $p' \geq p$ and $p_l = p'_l$ for some l.

   Now, consider altering the price vector p' to obtain the price vector p in $L-1$ steps, lowering (or keeping unaltered) the price of every commodity $k \neq l$ one at a time.

   By gross substitution, the excess demand of good l cannot decrease in any step, and because $p \neq p'$, it will actually increase in at least one step.

   Hence $z_l(p) > z_l(p')$

Locally unique if

- Finite number
- Odd number
- Eq set is continuous in underlying geometry

Sonnershien-Debreu-Mantel Theorem: "Anything Goes"
(See book)
Roughly: Given a function $z: R^L_{++} \to R^L$
that satisfies

- z homogeneous of deree 0 in p
- $pz(p) = 0 \ \ \forall p \in R^L_{++}$

we can find an economy that has z as its excess demand function
(Can also go much further and see that this economy is well behaved...)

Ch. 18: Core

Alternative notion of equilibrium based on the idea of cooperative behavior
Discovered by Edgeworth (1881), but then lay fallow until 1950s by cooperative game theorists.

Set up:

- Exchange Economy (for simplicity)
  - I consumers $\succsim_i$
  - L goods

- ω_i

- A coalition is a subset S of $\{1, \ldots, I\} = I$, $S \subset I$.

Blocking Definition:

- Consider a feasible allocation $(x_1, \ldots, x_I)$

  1. $x_i \in R_+^L \forall i$
  2. $\sum_i x_i = \sum_i \omega_i$

- We say that a 'coalition' S blocks $(x_1, \ldots, x_I)$ if $\exists\, (x_i')_{i \in S}$ s.t.

  1. $x_i' \succsim_i x_i \forall i \in S$, $x_i' \succ_i x_i$ for at least one $i \in S$
  2. $\sum_{i \in S} x_i' = \sum_{i \in S} \omega_i$

Must be feasible allocation just to the subset to whom there is one person who can benefit (and no one is worse off)

Core Definition

- Definition

  - An allocation is in the *core* if it can't be blocked by *any* coalition

Is core nonempty?

- Indeed, in many games it can be empty.
- But in economic environments we're alright: under a few assumptions it isn't empty.

Core in Edgeworth box
Contract curve lying between Pareto optima

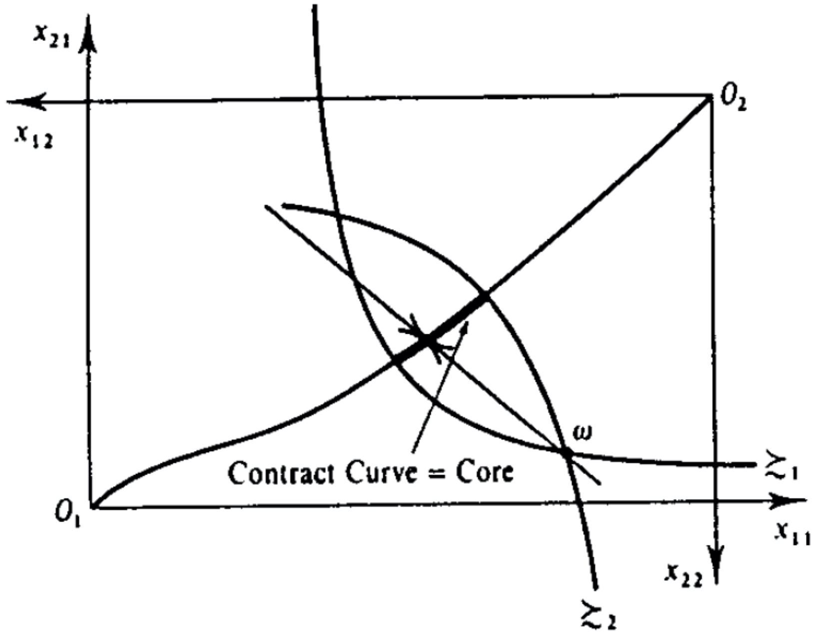


Figure 18.B.1
The core equals the contract curve in the two-consumer case.

Core when $i = 2$:
The core for for $i = 2$ is just the set of Pareto optimal allocations, which can be found by equating the marginal utilities of the two consumers.

Important Core Propositions

Any Core Allocation is Pareto Optimal

- Since have the set $S =$ everybody

Any Walrasian Equilibrium is in the Core

Proposition:

- Any Walrasian Equilibrium is in the core (Assuming all consumers are LNS)

Proof:

- Let (x^*, p^*) be a WE.
- Suppose not in the core. Then we can find $S \subset I$ and $(x_i')_{i \in S}$ s.t. [from above]
  1. $x_i' \succsim_i x_i \forall i \in S,\ x_i' \succ_i x_i$ for at least one $i \in S$
  2. $\sum_{i \in S} x_i' = \sum_{i \in S} \omega_i$
- We also established earlier

$$x_i' \succ_i x_i^* \implies p^* x_i' > p\omega_i$$

$$x_i' \succsim_i x_i^* \implies p^* x_i' \geq p\omega_i$$

- If we just add over people in the coalition

$$p^* \left(\sum_{i \in S} x_i' - \sum_{i \in S} \omega_i \right) > 0 \implies p^* \sum_{i \in S} x_i' > p^* \sum_{i \in S} \omega_i$$

  - We see this violates (2)

- Intuition: The value of the alternative allocation must be greater than the value of our endowment. Which means it's not feasible for us.
- Converse: Converse (that every core allocation is WE) of course is <u>not true</u>. It will later be argued that the converse does approximately hold if consumers are numerous.

Core Convergence

In a very large economy, the core and the set of WE are the same. We'll look at economies with N of each type.
(figure 3)

Start with EW Box economy. Now think of economy with 2 2's and 2 1's.
Look at a coalition with 2 2's and 1 1. A new equilibria will be proposed.
...
So we see that core shrinks as you add consumers.

N-Replica Economy

Debreu-Scarf 1963

- Set up
  - H <u>types</u> of consumers $h = 1, \ldots, H$
  - N <u>number of consumers of each type</u>
  - Consumer h has consumption set R^L_+, $\succsim_h$ strictly convex$\forall h = 1, ..., h$, $\omega_h \in R^L_+$
  - Replica economy E_N:
    - $I = NH$, N consumers of type $h = 1, ..., H$

WE of E_N = WE of E_1

- <u>Observation 1</u>:

- WE of $E_N \equiv$ WE of E_1

  Let's start with something that is a WE in E_1. There are prices, people are utility maximizing, demand = supply.

  Now larger economy. Each consumer will do what their type did in smaller economy. Markets will clear again – old WE will still be a WE.

  Converse is also true. Because of strictly convex preferences, will be a unique solution is all maximizing utility subject to budget constraints.

  So if demand = supply, it must be that demand=supply for all cross-sections of the economy.

Equal Treatment Property

- Proposition:
  - Any allocation $(x_{hn})\begin{matrix} h=1,\ldots,H \\ n=1,\ldots,N \end{matrix}$ in the core of E_N has the *equal treatment property*
    - The equal treatment property: $x_{hj} = x_{hk} \quad \forall h,j,k$
    - Different members of the same type will get the same bundle
- Proof:
  - Suppose not.
  - Form coalition of losers:
    - Choose *from every type* the person who is treated worst in that type.
      - (Can compare within each type, since have same preferences).
  - Let $x'_h = \frac{1}{N}\sum_{j=1}^{N} x_{hj} \quad h=1,\ldots,H$
    - $\sum_h x'_h = \frac{1}{N}\sum_h\sum_j x_{hj} = \frac{1}{N}(N\sum_h \omega_h) = \sum_h \omega_h$
- Formally
  - WLOG suppose that for every type h, consumer $h1$ is the worse-off individual
    - $[x_{hn} \succsim_h x_{h1} \quad \forall h,n]$
  - Define the average consumption for each type: $\hat{x}_h = \frac{1}{N}\sum_n x_{hn}$
  - Form the coalition $S = \{11,...,h1,...H1\}$ (the worst off members) attaining the average bundle $(\hat{x}_1,...,\hat{x}_h)$
    - See that $\hat{x}_h \succsim_h x_{h1}$ for all h and for the group not treated identically (WLOG $h=1$) $\hat{x}_1 \succ_1 x_{11}$

      * So will block as long as feasible

  - Check feasibility:

      * Note that since x is feasible, there is $y \in Y$ such that $\sum_h \sum_n x_{hn} = y + N\left(\sum_h \omega_h\right)$
          · Dividing by N: $\sum_h \hat{x}_h = \frac{1}{N}\sum_h \sum_n x_{hn} = \frac{1}{N}y + \left(\sum_h \omega_h\right)$
      * By constant returns assumption on Y: $\frac{1}{N}y \in Y$
          · Hence we conclude $(\hat{x}_1, ..., \hat{x}_H) \in \mathbb{R}_+^{LH}$ is feasible for coalition S.

Core Convergence Theorem

Can represent the core of E_N, $\quad C(E_N) = \left\{(x_h)_{h=1,\dots,H}\right\}$

Proposition:

- If the feasible allocation $x^* = (x_1^*, \dots, x_H^*) \in \mathbb{R}_4^{LH}$ has the core property for all $N = 1, 2, \dots$, that is $x^* \in C_N$ for all N,
- then x^* is a WE allocation

[Telling us is that $C(E_N)$ shrinks to set of WE]

Proof:

- Assume that people (consumer h) have continuously differentiable utility functions U_h.
- Proof by contradiction
  - Suppose that $x^* \in C(E_N)\,\forall N$
  - By the 2nd Welfare Theorem, x^* must be a Pareto Optimum for E_1.
- Assume (p^*, x^*) is not a WE.
- Form coalition kicking out a "most favored"
  - [Form a blocking coalition by getting rid of one of the most favored and have everybody else in the coalition]
  - $\implies p^* x_h^* > p^* \omega_h$ for some h, WLOG $h = 1$ "the most favored"
  - Let S = everybody except one type h.
- Each coalition consumer gets new bundle x_h' by dividing "most favored"'s surplus over the rest of the coalition
  - $x_h' = x_h^* + \frac{1}{N-1+N(H-1)}\left(x_1^* - \omega_1\right)$.
  - (Dividing what we were handing over to 1, and giving it to everyone else. This will ensure that demand = supply for people in the coalition)

- <u>Argue that for N large enough, this is an unambiguous gain.</u>

  1. First we see that $p \cdot (x_1 - \omega_1) > 0$ implies $\nabla u_h(x_h) \cdot (x_1 - \omega_1) > 0$

     (a) Since $p \propto \nabla u_h(x_h)$

     (b) [The first equation means the "most favored" has wealth above endowment]

  2. From Figure 18.B.3 (or Taylor expansion), we see that there is an $\bar{\alpha} > 0$ such that for every h, $u_h(x_h + \alpha(x_1 - \omega_1)) > u_h(x_h)$ when $0 < \alpha < \bar{\alpha}$.

  3. Hence for any N large enough that $\frac{1}{N-1+N(H-1)} < \bar{\alpha}$, the coalition will actually be blocking

- Intuitively, if N is large, each consumer in the coalition is taking only a very small fraction of the "most favored's" surplus. Hence the individual gains will be "at the margin" and thus individually favorable.

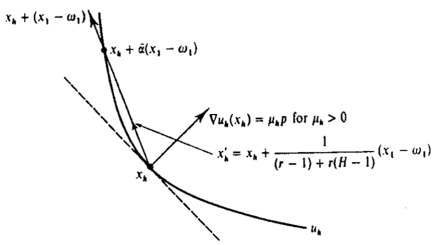


Figure 18.B.3
The consumption change of a consumer of type h in the blocking coalition.

One application of the core result

- Argument for free trade

  – Can a country be worse off after trade with ROW than before?

    * [Look at 2 extremes – from no trade, to completely free trade. Could that be bad?]

  – Core: tells us it is impossible for everyone to be worse off.

    * *Because, if that was true, then country could "block", and go back to their old way*

    * So Core says somebody must be better off. And actually says there's a net gain.

  – Then we can have an equilibrium with transfers that makes everyone better off

Ch 19: General Equilibrium Under Uncertainty

Introduce Uncertainty (& time)

- Suppose that there is a finite number of states of the world. $s = 1, \ldots, S$
- A state of the world is a complete (exogenous) description of a possible outcome of uncertainty.
- States of the world are mutually exclusive
  - Easiest examples of states of the world involve things like weather. Because it is a complete description.
- Can represent these in a tree diagram (points a one node a $t = 0$, S nodes at $t = 1$)
  - Double tree diagram – intermediate time nodes are not states of the world because *they are not complete descriptions*

We can extend the previous analysis if we are prepared to (make a big assumption) assume a complete set of contingent commodity markets. [Arrow (1953), Debreu (1959)]

Example:

- Say price of apple at date 0 is \$1, and price of apple at date 1 state 3 is \$.50. Why?
  1. People like apples now as opposed to tomorrow? (discounting)
  2. People don't believe state 3 is likely? (probability)
- What if they don't deliver?
  - Arrow-Debreu assume that there is very tough legal framework.

Arrow-Debreu

We have **contingent commodity markets.** Contingent commodities are those whose delivery is conditional on a certain state of the world being realized. All contingent commodity markets operate at date 0. People's initial endowments, producers' production plans, and peoples' shares in firms are also state-contingent. The **First and Second Welfare Theorems still hold** in terms of ex ante utility/welfare. **Existence is also a-ok** as long as we have the convexity assumptions above.
Remember that MRS is the ratio of marginal utilities of goods. This comes in handy in certain problems. If consumers are risk-neutral, their MRS will equal the probabilities of realizing each state (i.e. $MRS = \frac{\pi_1}{\pi_2}$). If there is aggregate risk, the MRS will not be the same as the probability ratio. [Thanks Heather S]

Arrow-Debreu Equilibrium Looking at contingent commodity markets where consumers buy and sell commitments to receive or deliver certain amounts of contingent commodities if a certain state occurs. The Walrasian equilibrium in these markets is called an Arrow-Debreu equilibrium.
Formally, an allocation $(x_i^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ is an Arrow-Debreu eqm if:

1. producers are maximizing: $py_j^* \geq py_j \;\; \forall y_j \in Y_J$
2. consumers are maximizing given their budget sets: $x_i^* \in X_i : px_i \leq p\omega_i + \sum \theta_{ij} py_j$ and $x_i' \succeq x_i^* \Rightarrow px_i' > w_i$
3. markets clear: $\sum x_i^* = \sum y_j^* + \bar{\omega}_i$

Properties of Arrow-Debreu:

- Walrasian Equilibrium
  - WE: $(p^*, x_1^*, \ldots, x_I^*)$ s.t.
    1. $x_i^* \in \mathbb{R}_+^{L(S+1)}, \;\; p^* x_i^* \leq p^* \omega_i, \, x_i \succ_i x_i^* \implies p^* x_i > p^* \omega_i$
    2. $\sum_i x_i^* = \sum_i \omega_i$
- 1st and 2nd Welfare Theorems Hold
  - This is for *ex-ante* beliefs
  - ex-ante Pareto optimality implies ex-post Pareto optimality, and hence no ex-post trade

[Risk aversion – makes things nice. Recently have been efforts to incorporate. We're not going to worry about this here]

Example:

- $T = 2, \; X_i = \mathbb{R}_+^{L(S+1)}, \; x_i = \left(x_i^0, \left(x_{si}^1\right)_{s=1,\ldots,S}\right), \; \omega_i = \left(\omega_i^0, \left(\omega_{si}^1\right)_{s=1,\ldots,S}\right)$, i has preferences $\succsim_i$ over X_i
  - Leading expression: $\sum_s \pi_{si} u_{si}\left(x_i^0, x_{si}^1\right)$
    - Consumer has state independent preferences: $\sum_s \pi_{si}\left(V_i\left(x_i^0\right) + \delta_i V_i\left(x_{si}^1\right)\right)$

Transition to Arrow: That we have all these contingent commodity markets seems crazy.

Sequential Trade - Arrow Simplification

Intuition: Arrow - we can interpret the contingent commodity market equilibrium as:
Instead of contingent commodity markets, assume that at date 0, markets open for L date 0 goods, and markets for securities.

- Date 0: $L + S$ markets
- Date 1: L markets

Set up

- Suppose we have two dates only (simple tree)
  - Exchange economy
    - L goods in each state at date 1
    - $X_i = \mathbb{R}_+^{LS} \quad \forall i$
    - Agent $i's$ preferences: $\sum \pi_{si} u_{si} (x_{si})$
    - Endowments: $\omega_i = (\omega_{1i}, \ldots, \omega_{Si})$
- **System of Prices** $(q, p) \in \mathbb{R}^S \times \mathbb{R}^{LS}$
  - q_s =price of Arrow security
  - p_s = goods price vector at date 1 in state s
- Contingent Commodity trading plan
  - $z_i = (z_{1i}, \ldots, z_{Si})$
- Set of Spot commodity trading plans:
  - $x_i = (x_{1i}, \ldots, x_{Si})$, $x_{1i}, x_{si} \in \mathbb{R}_+^L$
- At date 0, markets for S "Arrow" securities (z_{si}) pay (with price q_s)
  - $\begin{array}{cc} \$1 & \text{in state } s \\ \$0 & \text{in all other states} \end{array}$
- At date 1, markets for L goods opens

Consumer's Problem

- At date 0, Consumer i solves: (z_i^*, x_i^*) must solve

$$\max \sum_i \pi_{si} U_{si} (x_{si}) \quad \text{s.t.}$$

  - $S+1$ budget constraints for each consumer i, (1) for security market, and (2) for each spot market

1.
$$\sum_s q_s z_{si} \leq 0$$

2.
$$p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + z_{si}$$

(a) consumer believes that at state s this will be the price vector

(b) must obey budget constraint for each state of the world. as long as you're correct about predicting these prices you'll be able to honor the promises that you've made.

A REE (Rational Expectation Equilibrium)

- $q^* \in \mathbb{R}^S$
- $p^* = (p_1^*, \ldots, p_s^*) \in R^{LS}$
- $(z_i^*)_{i=1,\ldots,I}$, $z_i^* \in R^S \forall i$ collection of portfolios for consumers
- $(x_i^*)_{i=1,\ldots,I}$, where $x_i^* \in R_+^{LS} \forall i$ list of consumption plans for the consumers
- s.t.

1. consumer optimization problem above holds $\forall i$
2. $\sum_i z_i^* = 0$, $\sum_i x_{si}^* = \sum_i \omega_{si} \quad \forall s$

Transition:
1:1 relationship between this equilibrium an the complete market equilibrium (what we got assuming contingent commodity markets) (30:00)

Arrow-Debreu complete contingent commodity market equilibrium $\equiv$ Arrow sequential trade equilibium (with <u>RE</u>)

[OH wants just sketch of proof]

Arrow $\rightarrow$ Arrow-Debreu

1. Arrow Optimization Problem

$$\max \sum_i \pi_{si} u_{si}\left(x_{si}\right) \quad \text{s.t.}$$

(a)
$$\sum_s q_s z_s = 0$$

(b)
$$p_s x_{si} = p_s \omega_{si} + z_{si}$$

2. Substitute for $z_{si} = p_s x_{si} - p_s \omega_{si}$, so both budget constraints become:

$$\sum_s q_s \left(p_s x_{si} - p_s \omega_{si}\right) = 0$$

3. Re-write as

$$\sum_s \left(\left(q_s p_s\right) x_{si} - \left(q_s p_s\right) \omega_{si}\right) = 0$$

4. In other words it's the same as, Let $\hat{p}_s = q_s p_s$

$$\sum_s \hat{p}_s x_{si} - \sum_s \hat{p}_s \omega_{si} = 0$$

Which is also

$$\hat{p} x_i = \hat{p} \omega_i \quad \text{where } \hat{p} = \left(\hat{p}_1, \ldots, \hat{p}_S\right)$$

This is the identical construction as before.

Arrow-Debreu $\rightarrow$ Arrow

1. Arrow-Debreu Optimization Problem

   (a) Have some prices $\hat{p}$

   (b) Agents solving

$$\max_{x_i} \sum_s \pi_{si} u_{si}\left(x_{si}\right)$$

$$\text{s.t.} \quad \sum_s \hat{p}_s x_{si} \leq \sum_s \hat{p}_s \omega_{si}$$

2. Define p_s, z_{si}

   (a) Arrow: Let $\varepsilon_s = 1 \ \ \forall s$, $p_s = \hat{p}_s$.

   (b) Claim: let $z_{si} = p_s x_{si} - p_s \omega_{si}$
   We know that in this economy, consumer i will choose the same x_i^*'s as before...$z_{si} = p_s x_{si}^* - p_s \omega_{si}$

3. Re-write maximization problem with p_s, z_{si} in Arrow style:

$$\max \sum_i \pi_{si} u_{si}\left(x_{si}\right) \quad \text{s.t.}$$

   (a)

$$\sum_s q_s z_{si} \leq 0$$

(b)

$$p_s x_{si} \leq p_s \omega_{si} + z_{si}$$

4. Substitute $z_{si} = p_s x_{si}^* - p_s \omega_{si}$

$$\sum_s q_s z_{si} = \sum_s (q_s p_s) x_{si}^* - \sum (q_s p_s) \omega_{si}$$

$$= \sum_s \hat{p}_s x_{si}^* - \sum_s \hat{p}_s \omega_{si} = \hat{p} x_i^* - \hat{p} \omega_i = 0$$

All that we have to know now is that markets clear.

[We can define the z's to satisfy the budget state-by-state]

$$\sum_i p_s \left(\sum_i x_i^* - \sum_i \omega_{si} \right) = 0$$

Incomplete Markets (Not edited\cleaned)

An obvious reaction to what we've done so far is that yes, Arrow has brought us closer to reality (although RE assumption); but clearly don't have all these futures markets.
So look at incomplete markets.

Assume some securities exist and others don't.
Early contributions: Diamond (1967), Radner (1972). Some of this is called Radner equilibrium.

One of the things you can do is to assume...

- Same set up as in Arrow
- But at date 0 you have K securities are traded
- Security k pays $(r_{1k}, \ldots, r_{sk})$ where $r_{sk} \in R_+^L$
  - Best way of thinking of this is buying a share in firm. What are you getting? If the firm is producing a vector of goods. Can think of it as a shareholder you're getting some portion of that share of goods.
  - This is a vector of goods in every state
- Goods markets reopen at date 1
- RE (rational expectation)
- Radner equilibrium
  - List of security prices $q \in R^K$
  - $(p_s)_{s=1,\ldots,S}$

- Optimal portfolios $z_i^* \in R^K \quad \forall i$
- $x_i^* = (x_{i1}^*, \ldots, x_{iS}^*) \in R_+^{LS}$

1. (z_i^*, x_i^*) solves:

$$\max_x \sum_s \pi_{si} u_{si}\left(x_{si}\right)$$

$$\text{s.t. } qz \leq 0$$

Now:

$$p_s x_{si} \leq p_s \omega_{si} + \sum_{k=1}^{K} z_{k_i} p_s r_{sk}$$

2. Markets clear

$$\sum_i z_i^* = 0$$

$$\sum_i x_{si}^* = \sum_i \omega_{si} \quad \forall i$$

All we're trying to capture here is that the number of trading opportunities around may be limited
If only one security, no one would trade (there's nothing to trade)

Questions:

- Does a Radner eq exist?
- Is it Pareto Optimal?

Clearly not generally first-best PO

- First-best - means planner can do better than market, since planner can do things that are not available in the market
  - e.g. $L = 1,\ S = 2,\ K = 0, 1$
- Andy: If $K < S$ then you can't fully insure (1:14:00)

But what about constrained Pareto Optimality?

- Can a planner, s.t. same constraints as market do better?
  - First person to consider was Peter Diamond 1967 paper
  - Looked at case where $L = 1$. Can set $p_s = 1$
  - Get: $x_{si} = \omega_{si} + \sum_{k=1}^{K} z_{ki} r_{sk}$ for b.c.

$$\max \sum_s \pi_{si} u_{si} \left(\omega_{si} + \sum_{k=1}^{K} z_{ki} r_{sk} \right)$$

$$s.t.\ qz \leq 0$$

$$2. \quad \sum_i z_i^* = 0$$

What's happening is that people are buying goods

- i.e. $S = 4$ set $(1, 1, 3, 7)$ and $K = 2$ sec $\left(2, 8, 9, \frac{1}{2}\right)$
  * "redefining this economy in kind of a characteristic space"
- Planner can't do any better than market does. Planner won't be able to make improvements.
- Next time: if have more than 1 good, result goes out the window.

From last time:
K new goods
Let

$$V_i\left(z_{1i}, \ldots, z_{Ki}\right) = \sum_s \pi_{si} u_{si} \left(\omega_{si} + \sum_k z_{ki} r_{sk} \right)$$

Treat economy as a standard one with K goods:

- Prices $q_1, \ldots, q_L$
- Consumer i:
  -
$$\max V_i\left(z_i\right)$$

$$\text{s.t. } q_{zi} \leq 0$$

CE in PO
Meaning?
A planner cannot find $\left(z_i'\right)_{i=1,\ldots,I}$ s.t. $\sum_i z_i'$=0 and $V_i\left(z_i'\right) \geq V_i\left(z_i\right) \quad \forall i$
Eq q $\sum_i z_i = 0$

True if $L = 1$. What happens if $L > 1$?

See **Hart** '75 JET

- Suppose $K = 0$ (or 1)
- $S = 2,\ I = 2,\ L = 2$
  - $\pi_1 u_1\left(x_{11}\right) + \pi_2 u_2\left(x_{21}\right)$

- $\pi_1 u_2(x_{12}) + \pi_2 u_2(x_{22})$

- If you could find 2 market equilibria that are pareto ranked, then you can argue it
- Assume 3 equilibria in each state (see graph), A, B (medium), C in state 1, A', B', C' in state 2
  - Assume (A, C') is an equilibrium in economy
  - Claim: this is likely to be pareto dominated by another equilibrium (B, B')

Can this be generalized?

- See Geankopolos-Polemaschakis '86

Convey the fact that when you have incomplete markets you're going to have constrained markets, that this will fall apart when you have more than two goods (and/or more than two dates).

- Furthermore, can have multiple equilibria
- Assume two securities $K = 2$
  - $(1, 2+\epsilon)$, $(1, 2)$

Can have two euqilibrium for whole economy, one at these prices, but if you choose your parameters,

- Can also have **no equilibrium** (not robust – knife-edge case)
- Prices may not be linearly independent

you could have no trading in securities at all, since the prices will be linearly dependent
Start with incomplete markets, add a market, then everyone is worse off

Another thing that is bad about these settings:

- Firm behavior is hard to predict
  - No clear maximization statement for firm behavior
- With incomplete markets
  - Profits will be a random variable
- What do shareholders want?
  - May disagree

A big open issue here: why are markets incomplete?

- One answer: states not verifyable
  - To have these Arrow securities, we better be able to verify that the state occurred. It is plausible that some of these states are not so easy to verify

      - Not a very good story for the following reason:
        - $S = 3$, $L = 1$, $K = 1$.
        - $(1, 2, 3)$ single firm producing this bundle
      - Argue: with a security lke this, even if states are not verifyable, ways of creating new securities
        - $\implies$ trade **derivatives.** I.e. I'll pay out 1 in state 1 if my share pays out 1. If it pays 2 or 3 then I know I don't need to pay.
        - Or: option to buy 1 unit of sec 1 at date 1 at price 2.5
        - "can go all the way as long as you can find one firm in the economy that will payout differently in a different state of the economy"

Part II

2nd Half of Course: Contract Theory

- Moral Hazard
- (Assymetric Information)
- Incomplete contracts

1 Principal-Agent Problem (Moral Hazard version)

Problem Solving Formula*

- 1st best
  - Max P's utility s.t. IR holds
- 2nd best
  - Max p's utility s.t. IR, IC
    - Substute FOC of IC

1.1 CARA Utility Model

Lecture, BD 4.2
Not going to try to be very general. Look at a simple situation where you have a principle and an agent.

- P wants A to do something.
  - P doesn't observe A's action a
  - P puts A on an incentive scheme
- Assume P risk neutral (owner), A risk averse
  - i.e. P could be insurer, A could be homeowner
- Assume $q = a + \varepsilon$
  - Variables
    * q is revenue
    * a is effort (one dimensional)
    * ε is noise
- Assume ε is normally distributed with mean 0
  $\varepsilon \sim \mathcal{N}\left(0, \sigma^2\right)$
- P has all the bargaining power (WLOG)
  - (you can trace the pareto frontier by varying $\bar{\omega}$)
- Won't go much beyond 2 people since "we have our hands full"

***Utility Functions*:**

- Agent:

$$U_A = -e^{-r\left(\omega - \frac{1}{2}ca^2\right)}$$

  - CARA, ω minus some cost of effort, quadratic
  - Assume agent not wealth constrained: utility function defined on $(-\infty, \infty)$
- Principal

$$U_P = q - \omega$$

At $t = 0$, M1 and M2 meet; Can trade assets:

P&A sign contract	A chooses a	ε realized	$q$realized
0	1	2	3

First Best

a is verifyable

- P thinks as follows: suppose I want agent to choose $\hat{a}$. What do I have to pay him?

$$-e^{-r\left(\hat{\omega}-\frac{1}{2}c\hat{a}^2\right)} \geq -e^{-r\bar{\omega}}$$

$$\hat{\omega} - \frac{1}{2}c\hat{a}^2 = \bar{\omega} \quad \text{(IR)}$$

  - (IR) is the Individual Rationality constraint
  - This gives us:

$$\hat{\omega} = \bar{\omega} + \frac{1}{2}c\hat{a}^2$$

- P will pay using a wage. There is no moral hazard since a is verifyable here. Wage will be best for both because agent is risk averse, P is risk neutral.
- Hence Principal has problem
  - Principal's utility:

$$U_P = q - \hat{\omega} = \hat{a} - \frac{1}{2}c\hat{a}^2 - \bar{\omega}$$

  - Maximization Problem:

$$\max U_P = \max_{\hat{a}} \hat{a} - \frac{1}{2}c\hat{a}^2$$

$$FOC: \quad 1 = c\hat{a} \implies \hat{a} = \frac{1}{c}$$

Second Best

P doesn't observe a or ε

- Concise Version:
  - Prinicipal Maximization:

$$\max U_P = \max_{t,s} \mathbb{E}\left(q - t - sq\right)$$

$$\text{s.t.} \;\; a \in \arg\max \left[t + sa - \frac{1}{2}ca - \frac{1}{2}rs^2\sigma^2\right] \Rightarrow a = \frac{s}{c} \quad \text{(IC)}$$

$$t + sa - \frac{1}{2}ca^2 - \frac{1}{2}rs^2\sigma^2 = \bar{\omega} \text{ (IR)}$$

- Substitute *both* constraints:

$$\Rightarrow \max_s \frac{s}{c} - \frac{1}{2}\frac{s^2}{c} - \frac{1}{2}rs^2\sigma^2 - \bar{\omega}$$

- FOC

$$s = \frac{1}{1 + rc\sigma^2}$$

- Background:
  - This is moral hazard classical problem
  - Principal uses an incentive scheme $\omega = t + sq$
    - t is constant, s is slope on the incentive scheme
    - Remember:$U_P = q - \omega$ and $q = a + \varepsilon$
- Getting IC: (Agent's Problem:)

$$\max_a \mathbb{E}\left(-e^{-r\left(t+sa+s\varepsilon-\frac{1}{2}ca^2\right)}\right)$$

[Since Agent's utility is: $U_A = -e^{-r\left(\omega-\frac{1}{2}ca^2\right)}$, where $\omega = t + sq = t + sa + s\varepsilon$]

- Certainty Equivalance
  - Agent's problem looks complicated but turns out it isn't, since we can use certainty equivalent:

  where $X \sim (m, v)$

$$\mathbb{E}\left(-e^{-rx}\right) = -e^{-r\left(m-\frac{1}{2}rv\right)}$$

$$\text{C.E. of } X = m - \frac{1}{2}rV$$

- The maximization problem is now

$$\max_a t + sa - \frac{1}{2}ca^2 - \frac{1}{2}rs^2\sigma^2$$

  - Agent just ends up maximizing (since last term follows out, t follows out):

$$\max_a sa - \frac{1}{2}ca^2$$

$$a = \frac{s}{c}$$

- *Agent works harder the steeper the slope of the incentive scheme*

- *Intuition:*
  - $t + sa - \frac{1}{2}ca^2 - \frac{1}{2}rs^2\sigma^2$ is the certainty equivalent compensation, which is thus equal to her expected compensation $(t + sa)$ net of her effort $\left(\frac{1}{2}ca^2\right)$ and a risk premium $\left(\frac{1}{2}rs^2\sigma^2\right)$
    - The risk premium $\left(\frac{1}{2}rs^2\sigma^2\right)$, for a given s, is increasing in the coefficient or relative risk aversion (r) and in the variance of output (σ). It is also increasing in s, since the higher the s, the more the agent bears the risk associated to q.

- Principal solves
  - $\max U_P = \max q - \omega$ which is:

$$\max_{s,t} \mathbb{E}\left(q - t - sq\right)$$

$$\text{s.t. } a \in \arg\max \left[t + sa - \frac{1}{2}ca - \frac{1}{2}rs^2\sigma^2\right] \Rightarrow a = \frac{s}{c} \quad \text{(IC)}$$

$$t + sa - \frac{1}{2}ca^2 - \frac{1}{2}rs^2\sigma^2 = \bar{\omega} \text{ (IR)}$$

  - Substitute in from (IR) to remove t:
    - (IR): $t = -sa + \frac{1}{2}ca^2 + \frac{1}{2}rs^2\sigma^2 + \bar{\omega}$

$$\max_{s,t} \mathbb{E}\left(q - t - sq\right)$$

$$= \max_{s,t} a - \mathbb{E}\left(t + sa\right)$$

$$= \max_{s} a - \mathbb{E}\left(-sa + \frac{1}{2}ca^2 + \frac{1}{2}rs^2\sigma^2 + \bar{\omega} + sa\right)$$

$$= \max_{s} a - \frac{1}{2}ca^2 - \frac{1}{2}rs^2\sigma^2 - \bar{\omega}$$

$$\text{s.t. } a = \frac{s}{c} \quad \text{(IC)}$$

  - Substite in for (IC) so that a is chosen optimally:

$$= \max_{s} \frac{s}{c} - \frac{1}{2}\frac{s^2}{c} - \frac{1}{2}rs^2\sigma^2 - \bar{\omega}$$

- FOC$_s$:

$$\frac{1}{c} - \frac{s}{c} - rs\sigma^2 = 0$$

$$1 - s - rs\sigma^2 = 0 \Rightarrow 1 = s + crs\sigma^2 = s\left(1 + rc\sigma^2\right)$$

$$s = \frac{1}{1 + rc\sigma^2}$$

- Intuition:
  - We see s (the effort and variable compensation component) thus go down when c (cost of effort), r (degree of risk aversion) and σ^2 (randomness of perfomance) go up, a result that is intuitive.

If $s = 1$, you are essentially selling the firm to the agent. Will only be $s = 1$ if $r = 0$ or $\sigma^2 = 0$)

Firm is worth much more to principal, since agent doesn't like bearing so much risk.

1.2 Two Performance Outcomes Model

Section, BC

- Output q can only take 2 values, $q \in \{0, 1\}$
  - When $q = 1$, "success", when $q = 0$ "failure
- Probabilities
  - Probability of success: $\Pr(q = 1|a) = p(a)$
    * p strictly increasing and concave in a, $p(0) = 0,\ p(\infty) = 1,\ p'(0) > 1$
- Utility functions
  - Principal

$$U_P = V(q - w)$$

    * where $V'(\cdot) > 0,\ V''(\cdot) \leq 0$
  - Agent

$$U_A = u(w) - \psi(a)$$

    * where $u'(\cdot) > 0,\ u''(\cdot) \leq 0$ and $\psi'(\cdot) > 0,\ \psi''(\cdot) \geq 0$

- simplify by assuming $\psi(a) = a$

First Best
Agent's choice of action is observable **(a is observable)**

- Strategy
  - Max Principal's utility (choosing a, w_i), s.t. IR
- Principal's maximization problem

$$\max_{a,w_i} p(a) V(1 - w_1) + (1 - p(a)) V(0 - w_0)$$

$$\text{s.t. } p(a) u(w_1) + [1 - p(a)] u(w_0) - a \geq \bar{u} = 0 \quad \text{(IR)}$$

  - Write as Lagrangian:

$$\max_{a,w_i} p(a) V(1 - w_1) + (1 - p(a)) V(0 - w_0) + \lambda (p(a) u(w_1) + [1 - p(a)] u(w_0) - a)$$

- 3 FOCs:
  - $\text{FOC}_{w_1}: \; p(a) V'(1 - w_1) = \lambda p(a) u'(w_1)$
  - $\text{FOC}_{w_0}: \; [1 - p(a)] V'(-w_0) = \lambda [1 - p(a)] u'(w_0)$
  - $\text{FOC}_a: \; p'(a)(V(1 - w_1) - V(-w_0)) + p'(a) \lambda (u(w_1) - u(w_0)) - \lambda = 0$
- Determinants of Optimal Action a^*
  - ***Borch Rule*:**

$$\frac{V'(1 - w_1)}{u'(w_1)} = \lambda = \frac{V'(-w_0)}{u'(w_0)}$$

    - From FOC_{w_1} and FOC_{w_0}
  - $\text{FOC}_a: \; p'(a)(V(1 - w_1) - V(-w_0)) + p'(a) \lambda (u(w_1) - u(w_0)) - \lambda = 0$
  - Together, FOC_a and the Borch Rule pin down a^*

Alternative Set Up:

- *Surplus to the Agent, Outside option for Principal*
- We can formulate the same problem through the Agent's maximization problem.
- Strategy:
  - Max Agent's utility (choosing a, w_i), s.t. IR for Principal

- Agent's maximization problem

$$\max \{p(a)\, u(w_1) + [1 - p(a)]\, u(w_0) - a\}$$

$$\text{s.t. } p(a)\, V(1 - w_1) + [1 - p(a)]\, V(-w_0) \geq \bar{V}$$

First-Best: Examples

1. Risk-Neutral Principal

    - Have $V(x) = x$
    - Strategy: Max Principal's utility (choosing a, w_i), s.t. IR
    - [Can also skip to FOCs/Borch rule and substitute there]
    - Principal's maximization problem

$$\max_{a, w_i} p(a)(1 - w_1) + (1 - p(a))(0 - w_0)$$

$$\text{s.t. } p(a)\, u(w_1) + [1 - p(a)]\, u(w_0) - a \geq \bar{u} = 0 \quad \text{(IR)}$$

    - Write as Lagrangian:

$$\max_{a, w_i} p(a)(1 - w_1) + (1 - p(a))(0 - w_0) + \lambda (p(a)\, u(w_1) + [1 - p(a)]\, u(w_0) - a)$$

    - 3 FOCs:
        - $\text{FOC}_{w_1}: \ p(a) = \lambda p(a)\, u'(w_1)$
        - $\text{FOC}_{w_0}: \ [1 - p(a)] = \lambda [1 - p(a)]\, u'(w_0)$
        - $\text{FOC}_{a}: \ p'(a)((1 - w_1) + (w_0)) + p'(a)\lambda (u(w_1) - u(w_0)) - \lambda = 0$
    - Determinants of Optimal Action a^*
        - ***Borch Rule*:**

$$\frac{1}{u'(w_1)} = \lambda = \frac{1}{u'(w_0)}$$

This implies $w_1 = w_0 = w^*$

        - From FOC_{w_1} and FOC_{w_0}
        - $\text{FOC}_{a}: \ p'(a)((1 - w_1) + (w_0)) + p'(a)\lambda (u(w_1) + u(w_0)) - \lambda = 0$
            * $\text{FOC}_{a}: \ p'(a) - \lambda = 0$

- Together, FOC_a and the Borch Rule pin down a^*
    - This gives us:

$$\lambda = p'(a^*) = \frac{1}{u'(w^*)}$$

    - Re-writing the Agent's IR with $w^* = w_1 = w_2$ we also see that $p(a)\,u(w^*) + [1-p(a)]\,u(w^*) - a = 0$ gives us

$$a^* = u(w^*)$$

- Interpretation
  - Optimum entails full insurance of agent [$u(w^*)$ in one state $= u(w^*)$ in the other]
  - Constant wage w^*, constant effort level a^*

2. Risk Neutral Agent

- Have $u(x) = x$
- Strategy: Max Principal's utility (choosing a, w_i), s.t. IR
- [Can also skip to FOCs/Borch rule and substitute there]
- Principal's maximization problem

$$\max_{a,w_i} p(a)\,V(1-w_1) + (1-p(a))\,V(0-w_0)$$

$$\text{s.t. } p(a)\,w_1 + [1-p(a)]\,w_0 - a \geq \bar{u} = 0 \quad \text{(IR)}$$

  - Write as Lagrangian:

$$\max_{a,w_i} p(a)\,V(1-w_1) + (1-p(a))\,V(0-w_0) + \lambda\left(p(a)\,w_1 + [1-p(a)]\,w_0 - a\right)$$

- 3 FOCs:
  - $\text{FOC}_{w_1}: \; p(a)\,V'(1-w_1) = \lambda p(a)$
  - $\text{FOC}_{w_0}: \; [1-p(a)]\,V'(-w_0) = \lambda\,[1-p(a)]$
  - $\text{FOC}_a: \; p'(a)\,(V(1-w_1) - V(-w_0)) + p'(a)\,\lambda\,(w_1 - w_0) - \lambda = 0$
- Determinants of Optimal Action a^*
  - ***Borch Rule*:**

$$V'(1-w_1) = \lambda = V'(-w_0)$$

$$\text{This implies: } 1 - w_1 = -w_0 \implies w_1^* - w_0^* = 1$$

- $\text{FOC}_a: \; p'(a)\left(V(1-w_1) - V(-w_0)\right) + p'(a)\lambda(w_1 - w_0) - \lambda = 0$
  * $\text{FOC}_a: \; p'(a)\lambda = \lambda \implies$

$$p'(a^*) = 1$$

- Together, FOC_a and the Borch Rule pin down a^*

• Interpretation

  - Once again, marginal productivity of effort is equated with its marginal cost for the principal

Second-Best
Agent's choice of action is un̲observable **(a is unobservable)**

• Strategy

  - Max Principal's utility (choosing a, w_i), s.t. IR, IC of Agent
  - Solve IC of Agent via FOC
  - Plug result into Principal's problem to solve
    * Note that a is a choice variable for principal, since IC contraint says that a will be chosen maximally as the agent would choose

• Principal's maximization problem

$$\max_{a, w_i} p(a) V(1 - w_1) + (1 - p(a)) V(0 - w_0)$$

$$\text{s.t. } p(a) u(w_1) + [1 - p(a)] u(w_0) - a \geq \bar{u} = 0 \quad \text{(IR)}$$

$$\text{and } a \in \arg\max_{\hat{a}} p(\hat{a}) u(w_1) + [1 - p(\hat{a})] u(w_0) - \hat{a} \quad \text{(IC)}$$

• Solve IC of Agent via FOC

$$\max_{\hat{a}} p(\hat{a}) u(w_1) + [1 - p(\hat{a})] u(w_0) - \hat{a}$$

$$p'(a)[u(w_1) - u(w_0)] = 1$$

• Plug result into Principal's Problem

$$\max_{a,w_i} p(a)\,V(1-w_1) + (1-p(a))\,V(0-w_0)$$

$$\text{s.t. } p(a)\,u(w_1) + [1-p(a)]\,u(w_0) - a \geq \bar{u} = 0 \quad \text{(IR)}$$

$$\text{and } p'(a)\,[u(w_1) - u(w_0)] = 1 \quad \text{(IC)}$$

- Write as Lagrangian:

$$\max_{a,w_i} p(a)\,V(1-w_1)+(1-p(a))\,V(0-w_0)+\lambda\,(p(a)\,u(w_1) + [1-p(a)]\,u(w_0) - a)+\mu\,(p'(a)\,[u(w_1) - u(w_0)])$$

  - 3 FOCs:
    - $\text{FOC}_{w_1}:\ p(a)\,V'(1-w_1) = \lambda p(a)\,u'(w_1) + \mu\,(p'(a)\,u'(w_1))$

$$\frac{V'(1-w_1)}{u'(w_1)} = \lambda + \mu\frac{p'(a)}{p(a)}$$

    - $\text{FOC}_{w_0}:\ [1-p(a)]\,V'(-w_0) = \lambda\,[1-p(a)]\,u'(w_0) - \mu\,(p'(a)\,u'(w_0))$

$$\frac{V'(-w_0)}{u'(w_0)} = \lambda - \mu\frac{p'(a)}{1-p(a)}$$

    - $\text{FOC}_{a}:\ p'(a)\,(V(1-w_1) - V(-w_0))+p'(a)\,\lambda\,(u(w_1) - u(w_0))-\lambda+\mu\,(p''(a)\,[u(w_1) - u(w_0)]) = 0$
  - Interpretation
    - We can see that if $\mu = 0$, then the Borch rule is recovered. However, the optimum is $\mu > 0$ under quite general conditions.
    - With $\mu > 0$, optimal insurance is distorted.
      - Agent gets a larger share of the surplus in the case of high performance
      - Agent gets a smaller share of the surplus in the case of low performance

This is a model without assymetric information.

Briefly: Mirlees Contract
Don't need to memorize this part:
Is restriction to linear schemes justified?
Unfortunately, no.
This was first observed by Jim Murlees in '74.
Realized that in the innocuous looking model, you can approximate the first-best with a non-linear scheme
OH not going to go through details, but give basic idea:
Pay the agent a flat wage, but with a penalty if q falls below some critical level.
So the scheme is captured by 3 numbers

$$(\bar{\omega}, \underline{\omega}, \underline{q})$$

Solve for the first best:

$$a_{FB} = \frac{1}{c}$$

If agent works this hard, then $q = \frac{1}{c} + \varepsilon$
We can choose $\underline{\text{q}}$ s.t. $Prob\left[\frac{1}{c} + \varepsilon < \underline{q}\right]$ is very small.
What is less obvious if that if we look at agent's marginal incentives.
Suppose he slacks a little bit, then probability of disaster goes up.
What you can show looking at normal density $\frac{\frac{d}{da}Prob\left[a+c\leq \underline{q}\right]}{Prob\left[a+\varepsilon\leq \underline{q}\right]}$ is large.
If he deviates a little bit, the risk is still small, but it goes up by quite a bit.
(Intuition 22:30)
Balton-Duetrepon Book Ch 10

Suppose $\varepsilon \sim [-K, K]$
What we can do is to say that we want the guy to choose $\frac{1}{c}$.

$$q \in \left[\frac{1}{c} - K, \frac{1}{c} + K\right]$$

So what we can say is that we'll pay you a flat wage unless q falls below $\frac{1}{c} - K$

$$w = \bar{w} \text{ unless } \varepsilon < \frac{1}{c} - K$$

$$Prob\left[\frac{1}{c} + \varepsilon < \underline{q}\right]$$

So this makes us think that restricting ourselves to linear schemes is restrictive.
However, looking at these nonlinear schemes has some problems. (Milgrom 1987)
Dynamic Version
P sees only final ε
Holmstrom-Milstrom 1987 Econometrica
If things go badly, then will get much lower effort. Have non-stationary properties.
There are a lot of properties in this that will help us get the results that we want.
CARA utility function
Also consumption takes place at the end.

Over time, became more comfortable to assume linearity.

Extension: Multi-tasking (Not edited/cleaned)

What is connection with GE? Looking at firms more closely, who may have different incentives than their shareholders. Is πmax the behavior? Motivating the agent is hard. So they'll end up with something less than maximum value in the first best.

So far, trade-off between effort and risk-sharing. People haven't found this to be such an important trade-off. How do incentive schemes vary with the environment that they are in? Prendergast found that $s \uparrow$ in riskier environments. [Perhaps in risky environment it is harder to motivate the agent]

Different model, where effort isn't the issue, but where choice of task is the right issue.

$t+sq$; $q-t-sq$
s will be the same as the situation in which we had.

Typically, agent has a choice of tasks: how to allocate effort
[Holmstrom-Milgrom JLEO '91]
Agent chooses a_1, a_2.
P's benefit$=B_1a_1+B_2a_2+m$
P risk neutral
Expected benefit$=B_1a_1+B_2a_2$

B is not verifiable.
What is verifyable?
$z_1=a_1+\varepsilon_1$
$z_2=a_2+\varepsilon_2$

- Assume that V of $\varepsilon_2=\infty$

Hence z_2 gives us no information.
So really the only thing we have is
$z_1=a_1+\varepsilon_1$
Now assume $\varepsilon\sim N\left(0,\sigma^2\right)$

Agent's const on $=C\left(a_1,a_2\right)$ (quadratic & strictly convex)
C may not be decreasing in c_1, c_2 over whole range.
You'll see very shortly why this allows us to solve everything out.

Incentive Scheme:
Assume $B_1>0$, $B_2>0$

$$w=t+sz$$

$E\left[w\right]=t+sa$

$$U_P=B_1a_1+B_2a_2-t-sa_1$$

What's this worth to agent? Need to do certainty equivalent:

$$U_A=t+sa_1-\frac{1}{2}rs^2\sigma^2-C\left(a_1,a_2\right)$$

This is strictly concave.
I can max some of these, subject to the the incentive constraint:

$$\max B_1a_1+B_2a_2-\frac{1}{2}rs^2\sigma^2-C\left(a_1,a_2\right)$$

$$\text{s.t. } s=C_1;\quad 0=C_2\quad\text{(IC)}$$

These constraints came from $\max sa_1-C\left(a_1,a_2\right)$
Solve by writing FOCs:
But before getting fo FOCs:
$s=C_1;\quad 0=C_2\quad$ (IC) These are two equations, two unknowns
Solution of IC can be written as $a_1\left(s\right), a_2\left(s\right)$

$$\max B_1 a_1 + B_2 a_2 - \frac{1}{2} r s^2 \sigma^2 - C\left(a_1\left(s\right), a_2\left(s\right)\right)$$

FOC

$$B_1 \frac{da_1}{ds} + B_2 \frac{da_2}{ds} - r s \sigma^2 - s \frac{da_1}{ds} = 0$$

Can solve below and plug into the above:

$$1 = C_{11} \frac{da_1}{ds} + C_{12} \frac{da_2}{ds}$$

$$0 = C_{12} \frac{da_1}{ds} + C_{22} \frac{da_2}{ds}$$

When you plug in, get a linear equation in s, and so get a formula for s:

$$s = \frac{B_1 - B_2 \frac{C_{12}}{C_{22}}}{1 + r\sigma^2 \left(C_{11} - \frac{C_{12}^2}{C_{22}}\right)}$$

It is a closed form solution, since everything on RHS is a constant.
C is strictly convex:
$C_{11} > 0$, $C_{22} > 0$, $C_{11}C_{22} - C_{12}^2 > 0$ [so $C_{12}$is >0 or <0]
$C_{11} - \frac{C_{12}^2}{C_{22}} > 0$.

Special cases:

1. $C_{12} = 0$

$$s = \frac{B_1}{1 + r\sigma^2 C_{11}}$$

   (a) This is telling us that he fact that we have a different task doesn't matter.

   (b) *benchmark.

2. $C_{12} < 0$

   (a) I have rasied the numerator, reduced the demoninator relative to benchmark

   (b) $\implies s \uparrow$ relative to the benchmark

      i. C_{12} in economic terms. $C_{12} < 0$ completemetns in the sense of Edgeworth.

3. $C_{12} > 0$

   (a) substitutes

   (b) What you can show is that $\frac{\partial s}{\partial C_{12}} < 0$ at $C_{12} = 0$

      i. means that $s \downarrow$ (but not much)

   (c) **this is economically interesting, because it gives justification for low-powered incentive schemes. What one might be concerned about is that we're incentivizing only one thing, we're going to have the agent move away from things that we might care about.

i. If you incentivize doctors according to costs, may compromise on quality.

ii. If you incentivize teachers on test scores, could see cheating.

iii. Executives with high-powered incentives may also cheat

Thurs: brief discussion of theory of the firm.
At end: something about different of looking at the closed form solution for s above.
And then something about measurability.

2 Incomplete Contracting

Theory of the Firm and Incomplete Contracts

Strategies

- First best
  - All choice variables at disposal of social planner (allows for complete contracting)
- Second Best
  1. Gains from Trade (Ex-Post Surplus)
  2. Bargaining Outcome for each agent
  3. Ex-Ante Individual Maximization
  4. [Ex-Ante Social Surplus for Analysis]

Nash Bargaining

- Let
  - $\bar{u}_i$ be i's outside option
  - S be the size of the total pie
- Nash Bargaining:

$$U_i^* = \frac{(S - \sum_i \bar{u}_i)}{n} + \bar{u}_i$$

Intuition: Important Pieces in Property Rights Model

- Indescribability of the widget is critical, as well as non-verifiability of investments
  - (this is what "no long-term contract means")
  - Otherwise: Suppose widget *is* describable ex-ante

        * If they agree that B will purchase S for price p
            · Trade is always ex-post efficient
            · No renegotiation $\implies$ no spillover
    - Fixed price contract doesn't work if:
        * value and cost of widget are uncertain.
            · Trade may not be always efficient
            · Renegotiation in some states of the world
- First Best can often be achieved by both
    - Designing renegotiation game (end of 2008 example)
    - Specifying default terms of trade

Example: GHM Model
Spring 2010 General

- Setup
    - Two workers invest in a venture
    - Cost
        * Each $i \in \{A, B\}$ can acquire skill v_i at cost $\frac{v_i^2}{2\theta_i}$
    - Payoff
        * Yes trade
            · If both parties participate, the ex-post value of the venture is $v_A + v_B$
        * No trade
            · If partnership breaks down
            · The owner of the venture gets v_i
            · The other worker gets 0
    - Ex-Post Bargaining
        * Ex-post Nash bargaining, assuming 50/50 split
            · "50/50 split from the net benefit of entering an agreement"
- First Best
    - First best allows for complete contracting beforehand

– Each agent's problem

$$\max_{v_A, v_B} \sum_i \left(v_i - \frac{v_i^2}{2\theta_i} \right)$$

* FOC

$$1 = \frac{v_i}{\theta_i} \implies v_i = \theta_i$$

– Surplus from trade

$$\sum_i \left(\theta_i - \frac{\theta_i^2}{2\theta_i} \right) = \frac{1}{2}\theta_A + \frac{1}{2}\theta_B$$

- Second Best
  - Set up
    * Let i be the owner and j be the other worker
    * Renegotiation *threat points:*
      · v_i and 0, respectively
    * Total gross surplus:
      · $v_i + v_j$

1. Gains from Trade: Ex-Post Total Trade Surplus from Nash Bargaining
   (a) Ex-Post Trade Surplus = Gross surplus - Outisde option
   (b) $((v_i + v_j) - (v_i + 0)) = v_j$
2. Nash Bargaining Outcome: Ex-Post Individual Trade Surplus from Nash Bargaining (to i and j)
   – Nash Bargaining Formula:
     * Individual ExPNS = *threat point* + $\frac{1}{2}$ [Total Ex-Post Trade Surplus]
     * [See above for Nash Bargaining: $U_i^* = \frac{(S - \sum_i \bar{u}_i)}{n} + \bar{u}_i$]

   (a) $i: \; v_i + \frac{1}{2}\left[(v_i + v_j) - (v_i + 0)\right] = v_i + \frac{v_j}{2}$
   (b) $j: \; 0 + \frac{1}{2}\left[(v_i + v_j) - (v_i + 0)\right] = \frac{v_j}{2}$
3. Ex-Ante Individual Maximization Problem:
   [Using Payoffs from (1)], $\max \textit{individual trade surplus} - \textit{cost}$

(a)

$$\max_{v_i} \left[v_i + \frac{v_j}{2}\right] - \frac{v_i^2}{2\theta_i}$$

(b)

$$\max_{v_j} \left[\frac{v_j}{2}\right] - \frac{v_j^2}{2\theta_i}$$

i. FOCs

(c)

$$1 = \frac{v_i}{\theta_i} \implies v_i = \theta_i$$

(d)

$$\frac{1}{2} = \frac{v_j}{\theta_j} \implies v_j = \frac{1}{2}\theta_j$$

4. Ex-Ante <u>Total</u> <u>Net</u> Surplus

   (a) Total net surplus is <u>*ex-ante:*</u> NB surplus - cost

      i. Add surplus for each agent, plugging in FOC values from (2)

$$\left[\theta_i + \frac{\theta_j}{4} - \frac{\theta_i^2}{2\theta_i}\right] + \left[\frac{\theta_j}{4} - \frac{\left(\frac{\theta_j}{2}\right)^2}{2\theta_j}\right] = \left[\theta_i - \frac{\theta_i}{2}\right] + \left[\frac{\theta_j}{4} - \frac{\theta_j}{8}\right] = \frac{1}{2}\theta_i + \frac{3}{8}\theta_j$$

5. Interpretation

   (a) Who should own the firm?

      i. The worker with higher θ should own the venture, as the net total surplus $\left[\frac{1}{2}\theta_i + \frac{1}{8}\theta_j\right]$ depends more on θ_i

   (b) Would you describe the worker's investment as asset specific? Person specific?

      i. Asset specific: from owner's point of view, his investment is independent of whether he will work with person j

Example: Property Rights Model
Spring 2008 General

- Setup
  - A Buyer and Seller trade a widget
    - B =buyer, S =seller

- Cost
    - Cost to seller = 0
    - Cost to buyer = $\frac{i^2}{2}$
- Payoff
    - Yes trade
        - If both parties participate, Ex-post value the widget is i
        - "Value to the buyer is i"
    - No trade
        - If partnership breaks down
        - The Buyer gets 0
        - The Seller gets 0
- Ex-Post Bargaining
    - Ex-post Nash bargaining, assuming 50/50 split
- "i is observable but not contractable"

- First Best
    - First best allows for complete contracting beforehand
        - In this case, with contracting, the Buyer will be able to take all the surplus
    - Buyer's

$$\max_i i - \frac{i^2}{2}$$

        - FOC

$$i = 1$$

- Second Best: Scenario 1 (No Outside Options)
    - Set up
        - Renegotiation *threat points:*
            - 0 and 0, respectively

* Total gross surplus:
    - i

1. Gains from Trade: Ex-Post Total Trade Surplus from Nash Bargaining
    (a) $(i+0)-(0+0)=i$
2. Nash Bargaining Outcome: Ex-Post Individual Trade Surplus from Nash Bargaining (to i and j)
    - Individual ExPNS = *threat point* + $\frac{1}{2}$ [Total Ex-Post Trade Surplus]

    (a) $B: \; 0+\frac{1}{2}[i]=\frac{1}{2}i$
    (b) $S: \; 0+\frac{1}{2}[i]=\frac{1}{2}i$
3. Ex-Ante Individual Maximization Problem:
    [Using Payoffs from (1)], $\max$ *individual trade surplus* − *cost*
    (a) Here, only maximizing for Buyer:
    (b)
    $$B: \; \max_i \frac{1}{2}i-\frac{i^2}{2}$$
    (c)
    $$S: \; \max_i \frac{1}{2}i-0$$
    i. FOCs
    (d)
    $$i=\frac{1}{2}$$
4. Ex-Ante Total Net Surplus
    (a) Total net surplus is *ex-ante:* NB surplus - cost

$$\left[\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{4}\cdot\frac{1}{2}\right]+\left[\frac{1}{2}\cdot\frac{1}{2}-0\right]=\frac{3}{8}$$

* Second Best: Scenario 2 (Outside Option)
    - *Suppose instead B can can λi with $\lambda \in (0,1)$, by getting the widget from an alternative seller*
    - Set up
        * Renegotiation *threat points:*

· λi for B, 0 for S

∗ Total gross surplus:

· i

1. Gains from Trade: Ex-Post Total Trade Surplus from Nash Bargaining

   (a) $(i+0)-(\lambda i+0)=(1-\lambda)\,i$

2. Nash Bargaining Outcome: Ex-Post Individual Trade Surplus from Nash Bargaining (to i and j)

   – Individual ExPNS = *threat point* + $\frac{1}{2}$ [Total Ex-Post Trade Surplus]

   (a) $B:\ \lambda i+\frac{1}{2}\left[(1-\lambda)\,i\right]=\left(\frac{1+\lambda}{2}\right)i$

   (b) $S:\ 0+\frac{1}{2}\left[(1-\lambda)\right]=\left(\frac{1-\lambda}{2}\right)i$

3. Ex-Ante Individual Maximization Problem:
   [Using Payoffs from (1)], max *individual trade surplus* − *cost*

   (a) Here, only maximizing for Buyer:

   (b)
   $$B:\ \max_i\left(\frac{1+\lambda}{2}\right)i-\frac{i^2}{2}$$

   (c)
   $$S:\ \max_i\left(\frac{1-\lambda}{2}\right)i-0$$

   i. FOCs

   (d)
   $$i=\frac{1+\lambda}{2}$$

4. Ex-Ante Total Net Surplus

   (a) Total net surplus is *ex-ante:* NB surplus - cost

$$\left[\frac{1}{2}\cdot\frac{1+\lambda}{2}-\left(\frac{1+\lambda}{2}\right)^2\cdot\frac{1}{2}\right]+\left[\frac{1}{2}\cdot\frac{1-\lambda}{2}-0\right]=\frac{1}{2}-\frac{(1+\lambda)^2}{4}$$

- Suppose there is no alternative seller but you can design the bargaining game at date 1. What game would achieve the first-best?

  – **Buyer will make the following take-it-or-leave-it offer:**

        * **He will get the widget for free.**

    - [The buyer maximizes $i - \frac{i^2}{2}$]
    - *Note: this does not work if both B and S invest*

[Class Notes on Theory of the Firm]

How do we regulate these relationships?
Contract, ideally long-term

- Complete, contingent contract
- *Very hard to write such a contract*
    - Future is very uncertain (i.e. power-coal contracts can be 30-50 years)

Things that are hard to contract:

1. thinking costs
2. negotiation costs
3. enforcement costs

- Owner of a non-human asset has residual control rights
    - I can't force you to go from dept X to dept Y, but can exclude you from dept X
- In model, ex-post renegotiation takes place under symmetry and is costless
    - So why are there problems?
    - Ex-ante noncontractable investments
        * –> hold-up problem
        * Suppose the electricity company can install a boiler which is more or less specific to the coal or coal mine. Suppose that we can't contract on this in advance. Because we write an incomplete contract; you can make me pay a lot for this coal because with this investment, I have very poor outside options, and can therefore jack up price of coal.
            · One I invest in you, my willingness to pay for your coal will be very high
            · $p = v$, $v - p - i = i < 0$
            · But if I buy the coal mine you can't do this
            · Coal mine will not have that hold-up power
        * Regardless of the asset ownership structure

Formal Model

- Taken from 1995 book:
- Referred to as the "Property Rights Theory of the Firm"

At $t = 0$, M1 and M2 meet; Can trade assets:

|————————————|—————————————|
0 $\frac{1}{2}$invest 1: (S will supply B with one widget)

$$\begin{array}{cccccc} & & M1 & & M2 & \text{2 managers} \\ \text{consumers} & \leftarrow & a1 & ^{widget}\leftarrow & a2 & \text{2 assets} \end{array}$$

Interpretation

- $M1$ working with a1
- $M2$ working with a2

- No long-term contract
- No discounting
- Risk neutral, wealth parties
- Symmetric info

M1: Invests i $\rightarrow$

- $R(i) - p(-i)$ if trade occurs
- r$(i; A) - p(-i)$ if no trade occurs
- $R'(i) > r'(i, A) \quad \forall A$ The more A you have, the better off you'll do when contract breaks
  - p is price of widget (determined at date 1/2, so when get to date 1 it's a sunk cost)
  - $R' > 0,\ R'' < 0$
  - r is a different revenue function
  - A is set of assts that M1 owns

M2: Invests $e \rightarrow$

Driving force in this model is firing grocer when don't own the store (they walk away with groceries) and when you do own the store
3.5
M1: $i \rightarrow payoff = \left\{ \begin{array}{cc} R(i) - p\ \ (-i) & \text{if trade with M2} \\ r\left(i; \underbrace{A}\right) - \bar{p}\ \ (-i) & \text{if trade elsewhere} \end{array} \right\}$

A =assets $M1$ owns

M2: $e \rightarrow payoff = \left\{ \begin{array}{lll} p - C\left(e\right) & \left(-e\right) & \text{if trade with M1} \\ \bar{p} - c(e;B) & \left(-e\right) & \text{if trade elsewhere} \end{array} \right\}$
B=Assets M2 owns

- $R' > 0,\ R'' < 0$
- $r' > 0,\ r'' < 0$
- $R'\left(i\right) > r'\left(i;A\right) \quad \forall i, A$

Leading cases

- NI: $A = \left\{a1\right\},\ B = \left\{a2\right\}$
- VI(1): $A = \left\{a1, a2,\right\},\ B = \emptyset$
- VI(2): $A = \emptyset,\ B = \left\{a1, a2\right\}$

Net surplus if trade $= R\left(i\right) - C\left(e\right) - i - e$

First-Best

$$\max_{i,e} R\left(i\right) - C\left(e\right) - i - e$$

FOCs

$$R'\left(i\right) = 1$$

$$-C'\left(e\right) = 1$$

Second-Best

Can't contract on widget price. Bargain about this at date 1 in advance.
Can't contract on i,e.

Go to date 1. At this point, i, e are sunk.
"Ex post"
Suppress them for now. If no trade surplus $= r - c$
If trade... $= R - C$

Gains from trade
$G = (R - C) - (r - c)$. We're going to split them 50/50

1. M1's payoff: $= r - \bar{p} + \frac{1}{2}\left(R - C - r + c\right) - i$
2. M2's payoff: $= \bar{p} - c + \frac{1}{2}\left(R - C - r + c\right) - e$

The assumption is at date 1/2, M1 and M2 choose i, e noncooperatively.

- This suggests we should be looking at nash equilibrium
- But you also have nash bargaining once you reach date = 1

Nash Equilibria:
M1's:

$$\max_i r - \bar{p} + \frac{1}{2}(R - C - r + c) - i$$

$$FOC: \ \frac{1}{2}R'(i) + \frac{1}{2}r'(i; A) = 1$$

M2:
By parellel argument:

$$FOC: \ -\frac{1}{2}C'(e) - \frac{1}{2}c'(e; A) = 1$$

$$i_{SB} < i_{FB}$$

$$e_{SB} < e_{FB}$$

Assume $r'(i; A)$ increasing in A.
As we go from NI to VI(1), because A gets bigger, i goes up. $i_{SB}\uparrow$. But we will always have underinvestment.
(We're getting closer to the first best, but not all the way there)
However, $e_{SB}\downarrow$.

What drives this model, is that once I've fixed α, outside options play a role.

How do we evaluate?
Surplus

- $S = R(i_{SB}) - C(e_{SB}) - i_{SB} - e_{SB}$

Main result:

- If $i$is "important", then VI(1) is optimal
- If e is "important," then VI(2) is optimal
- If $i, \ e$ both somewhat important, NI is optimal

Other results

- If $a_1, \ a_2$ highly complementary, VI(1) or VI(2) optimal
- If $a_1, \ a_2$ are independent, NI is optimal

Good things about theory

- It's a model, It emphasizes costs as well as benefits, Workhorse-model (Antras, Helpman)

Bad things about theory

- Hard to test, Ex-post efficiency seems to miss a lot of things, Foundational issues
  - This story of hard to specify widget in advance–can we do something about that?

"Not the be-all and end-all, but still the one that is used the most"

3 Financial Contracting

In Arrow-Debreu, no one ever breaches a contract. But we see in the world obviously that they do.

We'll be talking about branch of literature that instead starts from first principals.

3.1 Costly State Verification ($K > 0$) Approach

3.1.1 Background and Set-Up

- Background
  - Borrower E needs $I - A$ of investment
  - A is E's wealth
    - I is cost of investment
  - Return on investment is R, stochastic with distribution $p(R)$
  - Lender L can lend $I - A$ and expects to get:
    - Payment D if $R > D$
    - Audit $R - K$ if $R < D$
      - Since K is cost of Audit
  - Important tool: *Revelation Prinicipal*
    - Restrict analysis to when E tells truth. Will report truth when $R > D$, will be audited when $R < D$
  - Assume 1 E, many L, so L gets 0 profit in (PC, L)
- Procedure
  1. E maximizes over D subject to:
     (a) PC, L
     (b) PC, E
  2. Constraints can be combined into a single constraint to show less than FB

3.1.2 Maximization Problem

1. E Maximization:

$$\max_D \int_{R\geq D} (R-D)\,p(R)\,dR$$

$$\text{s.t.} \quad \int_{R\geq D} Dp(R)\,dR + \int_{R<D} (R-K)\,p(R)\,dR = I - A \text{ (PC, } L\text{)}$$

$$\int_{R\geq D} (R-D)\,p(R)\,dR \geq A \quad \text{(PC,}E\text{)}$$

2. Summing over constraints:

$$\int_R Rp(R)\,dR - \int_{R<D} Kp(R)\,dR \geq I$$

   (a) Compare to First-Best funding level:

$$\int_R Rp(R)\,dR \geq I$$

   (b) We see that I will be lower for $K > 0$ than perfect information case

3.1.3 [Class Notes]

Literature starts with Townsend '79, then Gale Hellwig '85

- Basic idea
  - Entrepreneur's <u>income is private info and can be hidden</u>
  - But income can be learned and seized for $K > 0$

- Set up (Tirole Model)
  - Borrower E (entrepreneur)
    * Has wealth $A \geq 0$
    * Investment oppotunity that costs $I > A$
  - Lender L can supply funds
  - Return on investment opportunity R, stochastic with pdf $p(R)$
  - Both parties are risk-neutral
  - <u>Cost of audit: K</u>

- *Use Revelation principal*
  - Can restrict attention to situation where people divulging private information tell the truth
  - Applying revelation principle:
    - WLOG E tells the truth
    - Focus on deterministic mechanisms

Invest — Income $R$is realized, only E observes, density fn is $p(r)$ — E reports on R and maybe makes payment — Audit? and gets R, which becomes verifyable

|——————————-|———————————-|———————————-|

0 1 2 3

- E decides to Report or Not
  - E Reports: No Audit and Payment
    - *payment must be constant* over this region, otherwise E will report R that minimizes payment
  - E Doesn't Report: Audit
    - Can show L will <u>confiscate all of R</u>
- Implication:
  - If I announce something that gets me audited, I will get nothing
  - No audit $\iff R > D$

Armed with this we assume:

- One E, many potential L's
- E solves:

$$\max_D \int_{R \geq D} (R - D)\, p\,(R)\, dR$$

$$\text{s.t.} \int_{R \geq D} Dp\,(R)\, dR + \int_{R<D} (R - K)\, p\,(R)\, dR = I - A \text{ (IR, } L)$$

  - Top Equation: Entrepreneur is maximizing his expected revenue $(R - D)$, integrating over the possible values of R. He gets no revenue for $R < D$ since he will be audited. D will be chosen to fulfill the lender L's participation constraint:
  - (IR, L): Lender needs amount loaned (RHS) to equal the amount expected in return (LHS). The LHS is broken into two terms: that expected if $R \geq D$ and that expected if $R < D$.

- *(IR, L) not monotonic in D*
- Maximizes social surplus satisfying (IR, L)

- Applying E's (IR, E)
  - For a project go go ahead, we need to be able to find a D s.t. (IR, L) is satisfied and

$$\int_{R\geq D}(R-D)\,p(R)dR \geq A \quad (\text{IR},E)$$

  - Summing over (IR,L) and (IR,E)

$$\int_{R\geq D} Rp\,(R)\,dR + \int_{R<D}(R-K)\,p\,(R)\,dR \geq I$$

  - We see that this will be less than the risk-averse funding of an investment (First Best):

$$\int_{R} Rp\,(R)\,dR \geq I$$

    * Unless K never impacts

A few comments about this

- First serious model of financial contracting
- But has weaknesses
  - Stochasatic schemes do better
    * I make a report, then we have some randomization device to determine whether I get inspected or not
      · Randomization smooths things, because there is this fixed cost K
    * But doesn't always look so much like a debt contract
  - No outside equity
    * The insider is the equity holder, and the outsider has debt
      · In the world, we see that lender has a mix of debt, equity, other things
  - Reliance on K not entirely satisfactory
    * Not much economics in K. Just a fixed cost.

3.2 Moral Hazard Approach ($\bar{A}$ model)

Holmstrom-Tirole

3.2.1 Concise

1. Principal's Problem:

$$\max_{w_H, w_L} p_2 w_H + (1 - p_2) w_L$$

$$\boxed{\text{s.t. } p_2 w_H + (1 - p_2) w_L - e \geq p_1 w_H + (1 - p_1) w_L \quad (\text{IC}, E)}$$

$$\boxed{p_2 (R_H - w_H) + (1 - p_2)(R_L - w_L) = I - A \quad (\text{PC}, L)}$$

   (a) PC binds with equality, IC does not.

2. Assume $w_L = 0$. Problem becomes:

$$\max p_2 w_H$$

$$\text{s.t. } p_2 w_H - e \geq p_1 w_H \quad (\text{IC})$$

$$p_2 (R_H - w_H) + (1 - p_2)(R_L) = I - A \quad (\text{PC})$$

3. Use (PC), (IC) to solve for $\bar{A}$:
   $w_H = \frac{p_2 R_H + (1-p_2) R_L - (I-A)}{p_2} \geq \frac{e}{p_2 - p_1}$

$$A \geq \bar{A} = \frac{e p_2}{p_2 - p_1} + [I - p_2 R_H - (1 - p_2) R_L]$$

4. Interpret:

   (a) Can be above FB funding level

$$p_2 R_H + (1 - p_2) R_L - e - I \geq 0$$

   yet still have positive $\bar{A}$.

3.2.2 More Detail

Holmstrom-Tirole (originally comes from Innes, 1990)
In Tirole's book

In debt contract–at least in good states, I get every extra dollar that I make.

- E has project that costs I
- E's wealth is $= A < I$
  - E works:
    - $R = R_H$ with prob p_2
    - $R = R_L$ with prob 1-p_2
  - E shirks:
    - $R = R_H$ with prob p_1
    - $R = R_L$ with prob $1 - p_1$
- Cost of effort $= e$
- Only E observes effort decision
  - (hidden action)
- First-Best
  - $p_2 R_H + (1 - p_2) R_L - e > I$
    - $p_1 R_H + (1 - p_1) R_L < I$
- Incentive scheme (w_L, w_H)
- Optimal Contract
  - [Principal solves]

$$\max p_2 w_H + (1 - p_2) w_L$$

$$\text{s.t. } p_2 w_H + (1 - p_2) w_L - e \geq p_1 w_H + (1 - p_1) w_L \quad \text{(IC)}$$

$$p_2 (R_H - w_H) + (1 - p_2)(R_L - w_L) \geq I - A \quad \text{(PC)}$$

- **(PC) must hold with equality.**
- w_L and w_H must both be ≥ 0

(some intuition 32:00)

Simplifying Assumption: $w_L = 0$
WLOG $w_2 = 0$ [This makes life easier, if we can not worry about w_L]

Solving for $\bar{A}$:
From substituting constraints (IC) and (IR)

1. Agent Solves:

$$\max p_2 w_H$$

$$\text{s.t. } p_2 w_H - e \geq p_1 w_H \quad \text{(IC)}$$

$$p_2 (R_H - w_H) + (1 - p_2)(R_L) = I - A \quad \text{(PC)}$$

   (a) Important to remember (PC) holds with equality, keep inequality on (IC) for below

2. Use participation constraint to solve for w_H:
   From (PC), (IC)

$$w_H = \frac{p_2 R_H + (1 - p_2) R_L - (I - A)}{p_2} \geq \frac{e}{p_2 - p_1}$$

   [Algebra: re-write (IC), (PC) in terms of p_2, substitute:]
   $(p_2 - p_1) w_H \geq e \Rightarrow w_H \geq \frac{e}{p_2 - p_1}$ (IC)
   $p_2 R_H + (1 - p_2)(R_L) - (I - A) = p_2 w_H$ (PC)

3. Re-writing in terms of A to get:

$$A \geq \bar{A} = \frac{e p_2}{p_2 - p_1} + [I - p_2 R_H - (1 - p_2) R_L]$$

- Intuition
  - What does this say? My wealth has to be $\geq$ some critical number
  - Reexamining the first best:

$$p_2 R_H + (1 - p_2) R_L - e - I \geq 0$$

    * Can be true and yet

$$\bar{A} > 0$$

  [Algebra:
  FB:$-e \geq -p_2 R_H - (1 - p_2) R_L + I$
  SB: $A - e\left(\frac{p_2}{p_2 - p_1}\right) \geq -p_2 R_H - (1 - p_2) R_L + I$

        * Can see that $\left(\frac{p_2}{p_2-p_1}\right) < 1$, so e is being scaled down compared to FB. Hence A needs to be positive to compensate.

    - Size of project matters

$$\frac{ep_1}{p_2 - p_1} > 0 \text{ if marginal project}$$

    If large project, then I>>0, then $\bar{A}$ could be negative. So you're actually raising more than I.

Another way of thinking about this:

- Pledgable income of project
    - What's the most I can get from you?

$$p_2 R_H + (1 - p_2) R_L - \frac{ep_2}{p_2 - p_1}$$

- First term from first best, second term from second best. Why?
    - In first best, pledgable income is: $p_2 R_H + (1 - p_2) R_L - e$
    - I need to be left with some skin in the game
        * That's got to be above e
    - Pay me nothing if R_L, pay me something if R_H
        * If I get $\frac{e}{p_2-p_1}$ if R_H.
        * Otherwise:

$$\frac{ep_2}{p_2 - p_1} - e = \frac{ep_1}{p_2 - p_1} > 0$$

            · Since Entreprenuer needs utility at least 0 or would walk away.

If we could push w_2 below 0, would ease borrowing constraint. And OH jokes that public beatings as way of solving this.

3.3 Incomplete Contracting-Type Model

3.3.1 Concise

$\tilde{R}_1 = \{ {}^{R_1 \text{ prob } p}_{0 \text{ prob } 1-p}$

At date 1, either $\begin{array}{ccc} E \text{ pays } 0 & \rightarrow & \text{continue with prob } y_0 \\ E \text{ pays } D & \rightarrow & \text{continue with prob } 1 \end{array}$

1. Optimal contract solves:

$$\max_{D,y_0} p\left(R_1 - D + R_2\right) + (1-p)\, y_0 R_2$$

$$\text{s.t. } D = (1-y_0)\, R_2 \quad (\text{IC},E)$$

$$pD + (1-p)(1-y_0)\, L = I \quad (\text{PC},S)$$

- **(PC), (IC) are binding.**

$\left[(\text{IC},E) \text{ from } \boxed{R_1 - D + R_2 \geq R_1 + y_0 R_2}\right]$

2. Compare to FB: Solve for I

   (a) Substitute (IC,E) into (PC,S)

   $$p(1-y_0)\, R_2 + (1-p)(1-y_0)\, L = I$$

   (b) Solving for $1 - y_0$:

   $$1 - y_0 = \frac{I}{pR_2 + (1-p)\, L} = \text{Prob of liq if } R_1 = 0$$

   (c) Apply that $1 - y_0 \leq 1$:

   $$1 \geq 1 - y_0 = \frac{I}{pR_2 + (1-p)\, L}$$

   (d) Solve for I.

      i. Necessary & Sufficient Condition for project to be financed is:

      $$I \leq pR_2 + (1-p)\, L < R_2 < pR_1 + R_2 = FB$$

      ii. To few projects in the 2nd best.

3.3.2 More Detail

Holmstrom-Tirole has become workhorse, but doesn't capture many things. Importance of control and collateral missing.
Lit: Aghion-Bolton (control) 1992, Bolton-Schertstein (1990,96), Hart-Moore (1994,98)

- Punishment for failure to pay is asset seizure
- Threat can persuade E to disgorge project returns

Investment $\quad\quad\quad \tilde{R}_1 = \{ {R_1 \text{ prob } p \atop 0 \text{ prob } 1-p} \quad\quad R_2$

|————————————|——————————————|
0 $\quad\quad\quad\quad\quad\quad\quad\quad$ 1 $\quad\quad\quad\quad\quad\quad\quad\quad$ 2

- Set Up
  - No liquidation value at date 2
  - No discounting
  - Risk neutrality
  - At $t = 1$, liquidate for L?
    * $R_2 > L$
    * $I > L$
  - Symmetric info
  - $\tilde{R}_1$ not verifyable ("obervable but not verifyable")
  - E can divert cash flows
    * This is what makes problem interesting. Thing that makes him want to hand over to investor is the threat of liquidation

  Investor can get nothing at date 2

- **Form of Contract**

$$\text{At date 1, either} \begin{array}{ccc} E \text{ pays } 0 & \rightarrow & \text{continue with prob } y_0 \\ E \text{ pays } D & \rightarrow & \text{continue with prob } 1 \end{array}$$

  - Since can show that $y_1 = 1$.
  - Assume R_1 "large"
  - If liquidation, investor gets it all.

- **Incentive Contraint** (IC) for E:

$$R_1 - D + R_2 \geq R_1 + y_0 R_2$$

  - Can write as $D \leq (1 - y_0) R_2$

- **Optimal contract solves:**

$$\max_{D, y_0} p (R_1 - D + R_2) + (1 - p) y_0 R_2$$

$$\text{s.t. } D \leq (1 - y_0) R_2 \quad (\text{IC}, E)$$

$$pD + (1 - p)(1 - y_0) L \geq I \quad (\text{PC}, I)$$

- Show that **(PC), (IC) are binding:**
  - First we notice that (PC) is binding. Because otherwise we could reduce D.
  - (IC) is also binding: otherwise, we could increase y_0 by ε [$\Delta y_0 = \varepsilon$], $\Delta D = \frac{\varepsilon L(1-p)}{p}$
    * $\Delta E's$ payoff: $-\varepsilon L (1 - p)(R_2 - L) > 0$
      · We've made the probability of continuation a little bit higher, increased the debt a bit.

- **Re-write constraints:**
  1. (IC) holds with equality:

$$D = (1 - y_0) R_2$$

  2. Subsitute for D into (PC), which holds with equality:

$$p (1 - y_0) R_2 + (1 - p)(1 - y_0) L = I$$

  3. Re-write (PC), solving for $1 - y_0$:

$$1 - y_0 = \frac{I}{pR_2 + (1 - p) L} = \text{Prob of liq if } R_1 = 0 \quad (\text{PC})$$

     (a) This is for optimal contract; both constraints are satisfied.
     (b) We assume: $D = (1 - y_0) R_2 = \frac{IR_2}{pR_2 + (1-p)L} < R_1$
  4. **Apply that** $1 - y_0 \leq 1$

$$1 \geq 1 - y_0 = \frac{I}{pR_2 + (1 - p) L}$$

5. Solve for I.

   (a) Necessary & Sufficient Condition for project to be financed is:

$$I \leq pR_2 + (1-p)\,L$$

$$< R_2 < pR_1 + R_2 = FB$$

– Compare to the first-best rule:

$$I < pR_1 + R_2$$

– To few projects in the 2nd best.

– Also inefficient less