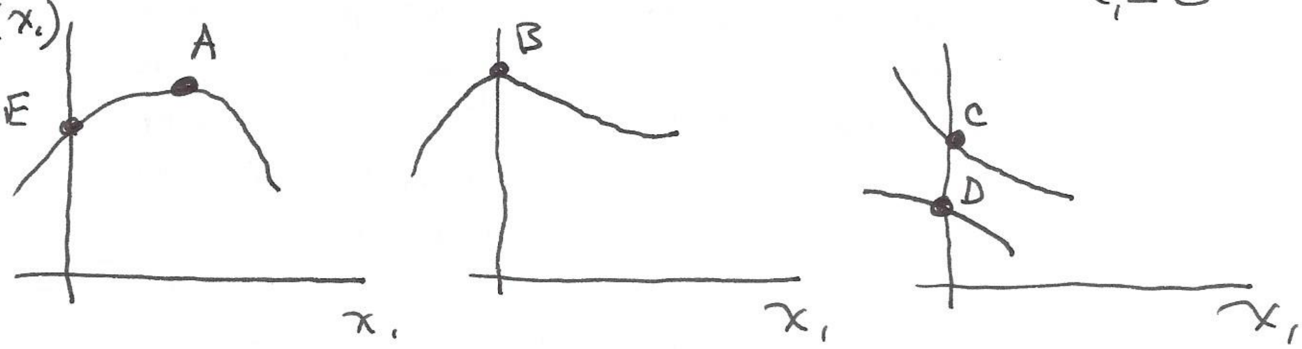
Chiang 21.2, Prob 1

Consider first Figure 21.3 and eqns 21.4 – 21.6.

Fig 21.3 MAX $\pi = f(x_1)$ s.t. $x_1 \geq 0$



INTERIOR SOLUTION | MAX @ BOUNDARY | BOUNDARY SOLUTION

(21.4) $f'(x_1) = 0$ and $x_1 > 0$ [A]

(21.5) $f'(x_1) = 0$ and $x_1 = 0$ [B]

(21.6) $f'(x_1) < 0$ and $x_1 = 0$ [C and D]

Consolodate to

$f'(x_1) \leq 0$ — Marginal Condition

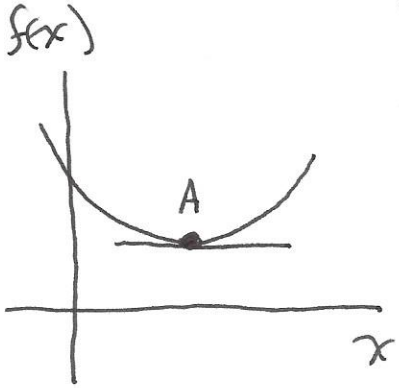
$x_1 \geq 0$ — Non Negativity Condition

$x_1 f'(x_1) = 0$ — Complementary Slackness Condition

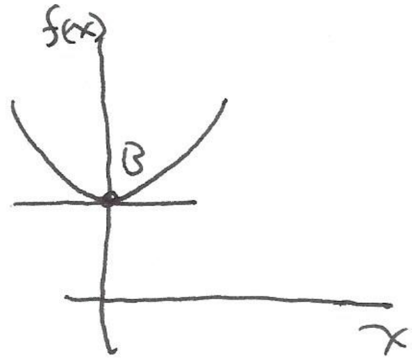
The Corresponding Diagrams for a Minimization Problem

$$\text{Minimize} \quad \pi = f(x)$$

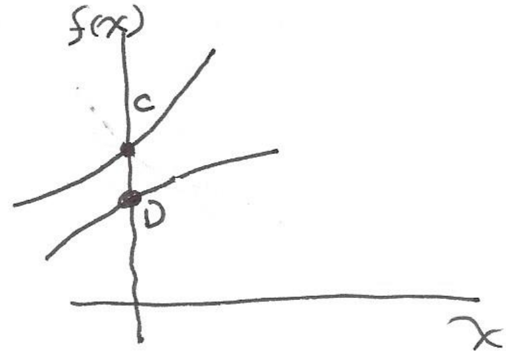
$$\text{s.t.} \quad x \geq 0$$



Interior Solution



Min @ Boundary



Boundary Solution

(21.4′) $\quad f'(x) = 0$ and $x > 0$ $\quad$ [A]

(21.5′) $\quad f'(x) = 0$ and $x = 0$ $\quad$ [B]

(21.6′) $\quad f'(x) > 0$ and $x = 0$ $\quad$ [C and D]

Consolodate to

(21.7′) $\quad f'(x) \geq 0 \qquad x \geq 0 \qquad x \cdot f'(x) = 0$

Marginal Condition $\quad$ Non-Negativity Condition $\quad$ Complementary Slackness Condition

Chiang, Exercise 21.2, Prob 4

To ease notation let $X = [x_1, x_2, \cdots, x_N]^T$
and let $Y = [y_1, \cdots, y_M]^T$.

Prob 21.2 $\quad \min \; C = f(X)$

$$\text{s.t.} \quad g^1(X) \geq r_1$$
$$\vdots$$
$$g^M(X) \geq r_M$$
$$\text{and} \quad x_j \geq 0 \qquad j = 1, 2, \cdots, N$$

Lagrangian

$$Z(X,Y) = f(X) - \sum_{i=1}^{M} y_i [g^i(X) - r_i] \qquad (1)$$

Cf. Chiang (21.17)

Marginal Conditions

$$\frac{\partial Z}{\partial x_j} = \frac{\partial f(X)}{\partial x_j} - \sum_{i=1}^{M} y_i \frac{\partial g^i(X)}{\partial x_j} \geq 0; \quad j = 1, \ldots, N; \quad (2)$$

↘

MARGINAL CONDITIONS (CONTINUED)

$$\frac{\partial Z}{\partial y_i} = -g^i(x) + r_i \le 0; \quad i = 1, \ldots, M \qquad (3)$$

NOTE THAT (3) yields $-g^i(x) \le -r_i$

or $\boxed{g^i(x) \ge r_i \qquad (3')}$

NON-NEGATIVITY CONDITIONS

$$\boxed{\begin{array}{lll} x_j \ge 0; & j = 1, \ldots, N & (4) \\ y_i \ge 0; & i = 1, \ldots, M & (5) \end{array}}$$

COMPLEMENTARY SLACKNESS CONDITIONS

$$\frac{\partial Z}{\partial x_j} \cdot x_j = \left[\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{M} y_i \frac{\partial g^i(x)}{\partial x_j}\right] x_j = 0 \qquad (6)$$

$$j = 1, \ldots, N$$

$$\frac{\partial Z}{\partial y_i} \cdot y_i = \left[-g^i(x) + r_i\right] y_i = 0; \quad i = 1, \ldots, M; \qquad (7)$$

NOTE (7) CAN BE WRITTEN AS

$$\boxed{y_i \left[g^i(x) - r_i\right] = 0 \qquad (7')}$$

Chiang, Exercise 21.2, Prob 5

Consider again the general minimization problem, 21.2 on p 717 of Chiang.

Let $X = [x_1, \cdots, x_N]^T$ and $Y = [y_1, \cdots, y_M]^T$

$$\text{Minimize} \quad C = f(X)$$

$$(21.2) \qquad \text{s.t.} \quad g^1(X) \ge r_1$$
$$\vdots$$
$$g^M(X) \ge r_M$$

$$\text{and} \quad x_j \ge 0 \qquad j = 1, \cdots, N$$

Convert to maximization problem by multiplying the objective function and each of the $(g^i \ge r_i)$ constraints by -1.

$$\text{Maximize} \quad -C = -f(X)$$

$$\text{Subject to} \quad -g^1(X) \le -r_1$$
$$\vdots$$
$$-g^M(X) \le -r_M$$

$$\text{and} \quad x_j \ge 0; \quad j = 1, \cdots, N$$

Formulate Lagrangian

$$Z(x,y) = -f(x) - \sum_{i=1}^{M} y_i\left[-g^i(x) + r_i\right] \qquad (1)$$

Marginal Conditions for Maximization (cf. 21.18)

$$\frac{\partial Z}{\partial x_j} = -\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{M} y_i\left[\frac{-\partial g^i(x)}{\partial x_j}\right] \leq 0; \quad j = 1, \ldots, N; \qquad (2)$$

$$\frac{\partial Z}{\partial y_i} = -\left[-g^i(x) + r_i\right] \geq 0; \quad i = 1, \ldots, M; \qquad (3)$$

Non-Negativity Conditions

$$x_j \geq 0; \qquad j = 1, \ldots, N; \qquad (4)$$

$$y_j \geq 0; \qquad i = 1, \ldots, M; \qquad (5)$$

Complementary Slackness Conditions

$$\frac{\partial Z}{\partial x_j} \cdot x_j = 0 \qquad j = 1, \ldots, N \qquad (6)$$

$$\frac{\partial Z}{\partial y_i} \cdot y_i = 0 \qquad i = 1, \ldots, M \qquad (7)$$

↘

Are these results, eqns (2) - (7) here, consistent with the results obtained in the preceding problem?

(a) If we multiply eqn (2) here through by -1 we obtain eqn (2) of the preceding problem.

(b) Eqn (3) here gives $g^i(x) - r_i \geq 0$ or

$$g^i(x) \geq r_i \qquad (3')$$

which is the same as (3') from the preceding problem.

(c) Note that eqns (4), (5), (6) and (7) here are the same as eqns (4), (5), (6) and (7) from the preceding problem.

So, <u>yes</u>, they are consistent.

Chiang, Exercise 21.6, Prob 1

UTILITY MAXIMIZATION

$$\text{Maximize } U = U(x) \qquad x = [x_1, \ldots, x_N]^T$$

$$(21.29) \qquad \text{subject to } P_1 x_1 + \cdots + P_N x_N \leq B$$

$$\text{and } x_1 \geq 0, \ldots, x_N \geq 0$$

(a) Form the appropriate Lagrangian

(b) Write out the Kuhn-Tucker conditions for this problem

(a) Lagrangian

$$L(x,\lambda) = U(x) - \lambda[P_1 x_1 + \cdots + P_N x_N - B] \qquad (1)$$

(b) K-T Conditions for $i = 1, \ldots, N$.

Marginal Conditions	Non-Negativity Conditions	Complementary Slackness Conditions
$\frac{\partial L}{\partial x_i} \leq 0$	$x_i \geq 0$	$x_i \cdot \frac{\partial L}{\partial x_i} = 0$
$\frac{\partial L}{\partial \lambda} \geq 0$ or $P_1 x_1 + \cdots + P_N x_N \leq B$	$\lambda \geq 0$	$\lambda \cdot \frac{\partial L}{\partial \lambda} = 0$ or $\lambda[P_1 x_1 + \cdots + P_N x_N - B] = 0$

S+B Exercise 16.6

Quadratic form, $Q(X) = X^T A X$ where $X_{(N\times 1)}$, $A_{(N\times N)}$.

Subject to constraint, BX where $B_{(M\times N)}$

Bordered Matrix ("Hessian")

$$H = \begin{bmatrix} 0_{(M\times M)} & B_{(M\times N)} \\ B^T_{(N\times M)} & A_{(N\times N)} \end{bmatrix}$$

a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $N=2$

$Bx = x_1 + x_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $M=1$

$$H = \begin{bmatrix} 0_{(1\times 1)} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$\det H = (1+1) - (1-1) = 2 > 0.$ $(-1)^N = (-1)^2 = 1 > 0$

$\boxed{\text{sign}[\det H] = \text{sign}(-1)^N}$ ✓

$N-M$ principle minors: $N-M=1$ So evaluate $\det H$ only.

$\boxed{\text{sign}[\det H] \neq \text{sign}(-1)^M = \text{sign}(-1)^1 = \text{sign}(-1)}$

Q is <u>Negative Definite</u>

c.) $Q(x) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$

or $Q(x) = x^T A x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad N = 3$

s.t. $Bx = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad M = 2$

$$H = \begin{bmatrix} O_{(2\times 2)} & B_{(2\times 3)} \\ B^T_{(3\times 2)} & A_{(3\times 3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{bmatrix}$$

$\det H = 16 > 0.$ $\quad (-1)^N = (-1)^3 = -1 < 0$

$$\boxed{\text{Sign}[\det H] \neq \text{Sign}(-1)^N}$$

N-M principle minors: $N - M = 1$ So evaluate det H only

$$\boxed{\text{Sign}[\det H] = \text{Sign}(-1)^M = \text{Sign}(-1)^2 = \text{Sign}(1) > 0}$$

<u>Q is Positive Definite</u>

e.) $Q(x) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$ OR

$$Q(x) = X^TAx = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad N = 3$$

$$Bx = x_1 + x_2 - x_3 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad M = 1$$

$$H = \begin{bmatrix} O_{(1\times1)} & B_{(1\times3)} \\ B^T_{(3\times1)} & A_{(3\times3)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & -3 \\ -1 & 0 & -3 & -1 \end{bmatrix}$$

$\text{DeT } H = 4 > 0.$ $\quad (-1)^N = (-1)^3 = -1 < 0.$

$$\boxed{\text{Sign}[\det H] \neq \text{Sign}(-1)^N}$$

$N - M = 2$: Evaluate 1st and Second principle minors

1st principle minor: $\det H = |H| = 4$

2nd principle minor: $|H_3| = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = 3$

$$\boxed{\text{Sign } |H| = \text{Sign } |H_3| > 0}$$

$$\boxed{\text{Sign } (-1)^M = \text{Sign } (-1)^1 = \text{Sign } (-1) < 0}$$

<u>Q is Indefinite</u>

S+B Exercise 18.10

Find the maximizer of $f(x,y) = x^2 + y^2$

subject to $2x + y \leq 2$, $x \geq 0$, $y \geq 0$

Lagrangian

$$L(x,y,\lambda) = x^2 + y^2 - \lambda[2x + y - 2] \qquad (1)$$

Marginal Conditions

$$\frac{\partial L}{\partial x} = 2x - 2\lambda \leq 0 \qquad (2)$$

$$\frac{\partial L}{\partial y} = 2y - \lambda \leq 0 \qquad (3)$$

$$\frac{\partial L}{\partial \lambda} = -(2x + y) + 2 \geq 0 \qquad (4)$$

Complementary Slackness Conditions

$$\frac{\partial L}{\partial x} \cdot x = 2(x - \lambda) \cdot x = 0 \qquad (6)$$

$$\frac{\partial L}{\partial y} \cdot y = (2y - \lambda) y = 0 \qquad (7)$$

$$\frac{\partial L}{\partial \lambda} \cdot \lambda = (-2x - y + 2)\lambda = 0 \qquad (8)$$

Non-Negativity Conditions

$$x \geq 0, \; y \geq 0, \; \lambda \geq 0 \qquad (9)$$

$\boxed{\lambda \geq 0}$

a) Consider $\lambda = 0$

From (6) and (7), if $\lambda = 0$ then $x = 0$ and $y = 0$

$$\boxed{\text{If } x = 0 \text{ and } y = 0 \text{ then } f(x, y) = 0 \qquad (10)}$$

b) Consider $\lambda > 0$

From (8), if $\lambda > 0$ then $\boxed{2x + y = 2 \quad (11)}$

From (11) we CANNOT have $x = 0$ and $y = 0$

(i) Suppose $y > 0$ and $x = 0$. Then (11) gives

$y = 2$ and $\boxed{f(x, y) = 4 \qquad (12)}$

(ii) Suppose $x > 0$ and $y = 0$. Then (11) gives

$x = 1$ and $\boxed{f(x, y) = 1 \qquad (13)}$

(iii) Finally, suppose $x > 0$ and $y > 0$.

From (3) $y = \frac{1}{2}\lambda$

From (2) $x = \lambda$

Thus $\boxed{y = \frac{1}{2}x \quad (14)}$

use (14) in (11)

$$2x + \frac{1}{2}x = 2$$

↓

or $2.5x = 2$

$$x = \frac{2}{2.5}$$

$$\underline{x = \frac{4}{5}} \quad (15)$$

Then (14) gives $\underline{y = \frac{2}{5}}$ (16)

and $\boxed{f(x,y) = \left(\frac{16}{25} + \frac{4}{25}\right) = \frac{20}{25} = \frac{4}{5} \quad (17)}$

C.) Comparing the possibilities, $f(x,y)$ is maximized at $(x,y) = (0, 2)$

S+B Exercise 18.13

A. Since the NDCQ only evaluates constraints that are binding, we first establish that the monotonicity in Simon + Blume [eqn (3) below] guarantees that the constraint [eqn (2) below] holds w/ equality.

Example 18.5

$$\text{MAX} \quad u(x_1, x_2) \qquad (1)$$

$$\text{S.T.} \quad p_1 x_1 + p_2 x_2 \le I \qquad (2)$$

where $x_1 \ge 0$, $x_2 \ge 0$, ~~and~~ where $p_1 > 0, p_2 > 0$

and where $\dfrac{\partial u(x_1, x_2)}{\partial x_1} > 0$ and $\dfrac{\partial u(x_1, x_2)}{\partial x_2} > 0 \qquad (3)$

1.) Lagrangian

$$L(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda[p_1 x_1 + p_2 x_2 - I] \qquad (4)$$

2.) K-T Conditions

$$\frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1} - \lambda p_1 \le 0 \qquad (5a)$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda p_2 \le 0 \qquad (5b)$$

↘

$$\frac{\partial L}{\partial \lambda} \geq 0 \quad \text{so} \quad -[p_1 x_1 + p_2 x_2 - I] \geq 0$$

$$\text{or } p_1 x_1 + p_2 x_2 - I \leq 0$$

$$\text{or } p_1 x_1 + p_2 x_2 \leq I \quad \text{which is (2)}$$

3.) NON NEGATIVITY CONSTRAINTS

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \lambda \geq 0 \qquad (6)$$

4.) COMPLEMENTARY SLACKNESS CONDITION

$$x_1 \frac{\partial L}{\partial x_1} = 0 \qquad (7a)$$

$$x_2 \frac{\partial L}{\partial x_2} = 0 \qquad (7b)$$

$$\lambda \frac{\partial L}{\partial \lambda} = 0. \quad \lambda \frac{\partial L}{\partial \lambda} = \lambda[-(p_1 x_1 + p_2 x_2 - I)] = 0$$

$$\text{So } \lambda[p_1 x_1 + p_2 x_2 - I] = 0 \qquad (7c)$$

$\downarrow$

5.) Assume an interior solution, That is, $x_1 > 0$ and $x_2 > 0$. Note That many standard utility functions ~~have~~ (CES, Cobb-Douglas, log utility) have The property That

$$\lim_{x_i \downarrow 0} \frac{\partial u}{\partial x_i} = +\infty \qquad (8)$$

~~For example~~

Since (5a,b) give

$$\frac{\partial u}{\partial x_i} \leq \lambda p_i \qquad i = 1, 2 \qquad (5c)$$

and Since $p_i > 0$, it follows from (8) and The diminishing marginal utility of consumption ~~That~~ $\left[\text{i.e., } \frac{\partial^2 u}{\partial x_i^2} < 0\right]$ That $x_1 > 0$, and $x_2 > 0$.

6.) Since $x_1 > 0$ and $x_2 > 0$, eqns (7a,b) Require That $\frac{\partial L}{\partial x_1} = 0$ and $\frac{\partial L}{\partial x_2} = 0$. Thus (5a,b) give That

$$\frac{\partial u}{\partial x_i} = \lambda p_i \qquad i = 1, 2 \qquad (9)$$

7.) MONOTONICITY, eqn(3), gives $\frac{\partial u}{\partial x_i} > 0$ and by ASSUMPTION $p_1 > 0$ and $p_2 > 0$. Thus (11) Requires That $\lambda > 0$. Since $\lambda > 0$ The complementary SIACKNESS condition, (7c), requires

$$\boxed{p_1 x_1 + p_2 x_2 = I \qquad (10)}$$

Thus The Budget CONSTRAINT is Binding, That is, holds with equality.

8.) Now Consider The NDCQ. There is one constraint

$$g(x_1, x_2) = p_1 x_1 + p_2 x_2 \leq I \qquad (2')$$

We know from (10) That This constraint is binding So The NDCQ Requires That

$$Dg(x_1, x_2) = \left[\frac{\partial g(x^*)}{\partial x_1} \quad \frac{\partial g(x^*)}{\partial x_2} \right] = \underset{(1 \times 2)}{[p_1, p_2]}$$

be of MAXIMAL RANK. Since $p_1 > 0$ and $p_2 > 0$, $Dg(x_1, x_2)$ is of RANK 1 and The NDCQ is satisfied.