SECOND EDITION

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SOLUTIONS MANUAL AND SUPPLEMENTARY MATERIALS FOR

ECONOMETRIC ANALYSIS OF CROSS SECTION AND PANEL DATA

Instructor's Solutions Manual
for
Econometric Analysis of Cross Section and Panel Data, second edition

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Contents

Preface

This manual contains the solutions to all of the problems in the second edition of my MIT Press book, *Econometric Analysis of Cross Section and Panel Data*. In addition to the problems printed in the text, I have included some "bonus problems" along with their solutions. Several of these problems I left out due to space constraints and others occured to me since the book was published. I have a collection of other problems, with solutions, that I have used over the past 10 years for problem sets, takehome exams, and in class exams. I am happy to provide these to instructors who have adopted the book for a course.

I solved the empirical examples using various versions of Stata, ranging from 8.0 through 11.0. I have included the Stata commands and output directly in the text. No doubt there are Stata users and users of other software packages who will, at least in some cases, see more efficient or more elegant ways to compute estimates and test statistics.

Some of the solutions are fairly long. In addition to filling in all or most of the algebraic steps, I have tried to offer commentary about why a particular problem is interesting, why I solved the problem the way I did, or which conclusions would change if we varied some of the assumptions. Several of the problems offer what appear to be novel solutions to situations that can arise in actual empirical work.

My progress in finishing this manual was slowed by a health problem in spring and summer of 2010. Fortunately, several graduate students came to my aid by either working through some problems or organizing the overall effort. I would like to thank Do Won Kwak, Cuicui Lu, Myoung-Jin Keay, Shenwu Sheng, Iraj Rahmani, and Monthien Satimanon for their able assistance.

I would appreciate learning about any mistakes in the solutions and also receiving

suggestions for how to make the answers more transparent. Of course I will gladly entertain suggestions for how the text can be improved, too. I can be reached via email at wooldri1@msu.edu.

Solutions to Chapter 2 Problems

2.1. a. Simple partial differentiation gives

$$\frac{\partial \mathrm{E}(y|x_1,x_2)}{\partial x_1} = \beta_1 + \beta_4 x_2$$

and

$$\frac{\partial \mathrm{E}(y|x_1,x_2)}{\partial x_2} = \beta_2 + 2\beta_3 x_2 + \beta_4 x_1$$

b. By definition, $\mathrm{E}(u|x_1x_2) = 0$. Because x_2^2 and x_1x_2 are functions of (x_1,x_2), it does not matter whether or not we also condition on them: $\mathrm{E}(u|x_1,x_{2,}x_2^2,x_1x_2) = 0$.

c. All we can say about $\mathrm{Var}(u|x_{1,}x_2)$ is that it is nonnegative for all x_1 and x_2: $\mathrm{E}(u|x_1,x_2) = 0$ in no way restricts $\mathrm{Var}(u|x_1,x_2)$.

2.2. a. Because $\partial \mathrm{E}(y|x)/\partial x = \delta_1 + 2\delta_2(x - \mu)$, the marginal effect of x on $\mathrm{E}(y|x)$ is a linear function of x. If δ_2 is negative then the marginal effect is less than δ_1 when x is above its mean. If, for example, $\delta_1 > 0$ and $\delta_2 < 0$, the marginal effect will eventually be negative for x far enough above μ. (Whether the values for x such that $\partial \mathrm{E}(y|x)/\partial x < 0$ represents an interesting segment of the population is a different matter.)

b. Because $\partial \mathrm{E}(y|x)/\partial x$ is a function of x, we take the expectation of $\partial \mathrm{E}(y|x)/\partial x$ over the distribution of x: $\mathrm{E}[\partial \mathrm{E}(y|x)/\partial x] = \mathrm{E}[\delta_1 + 2\delta_2(x - \mu)] = \delta_1 + 2\delta_2 \mathrm{E}[(x - \mu)] = \delta_1$.

c. One way to do this part is to apply Property LP.5 from Appendix 2A. We have

$$\begin{aligned} \mathrm{L}(y|1,x) &= \mathrm{L}[\mathrm{E}(y|x)] = \delta_0 + \delta_1 \mathrm{L}[(x - \mu)|1,x] + \delta_2 \mathrm{L}[(x - \mu)^2|1,x] \\ &= \delta_0 + \delta_1(x - \mu) + \delta_2(\gamma_0 + \gamma_1 x), \end{aligned}$$

because $\mathrm{L}[(x - \mu)|1,x] = x - \mu$ and $\gamma_0 + \gamma_1 x$ is the linear projection of $(x - \mu)^2$ on x. By assumption, $(x - \mu)^2$ and x are uncorrelated, and so $\gamma_1 = 0$. It follows that

$$\mathrm{L}(y|x) = (\delta_0 - \delta_1\mu + \delta_2\gamma_0) + \delta_1 x$$

2.3. a. $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + u$, where u has a zero mean given x_1 and x_2: $\mathrm{E}(u|x_1,x_2) = 0$. We can say nothing further about u.

b. $\partial \mathrm{E}(y|x_1,x_2)/\partial x_1 = \beta_1 + \beta_3 x_2$. Because $\mathrm{E}(x_2) = 0$, $\beta_1 = \mathrm{E}[\partial \mathrm{E}(y|x_1,x_2)/\partial x_1]$, that is, β_1 is the average partial effect of x_1 on $\mathrm{E}(y|x_1,x_2)/\partial x_1$. Similarly, $\beta_2 = \mathrm{E}[\partial \mathrm{E}(y|x_1,x_2)/\partial x_2]$.

c. If x_1 and x_2 are independent with zero mean then $\mathrm{E}(x_1x_2) = \mathrm{E}(x_1)\mathrm{E}(x_2) = 0$. Further, the covariance between x_1x_2 and x_1 is $\mathrm{E}(x_1x_2 \cdot x_1) = \mathrm{E}(x_1^2x_2) = \mathrm{E}(x_1^2)\mathrm{E}(x_2)$ (by independence) $= 0$. A similar argument shows that the covariance between x_1x_2 and x_2 is zero. But then the linear projection of x_1x_2 onto $(1,x_1,x_2)$ is identically zero. Now just use the law of iterated projections (Property LP.5 in Appendix 2A):

$$\begin{aligned}\mathrm{L}(y|1,x_1,x_2) &= \mathrm{L}(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2|1,x_1,x_2) \\ &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 \mathrm{L}(x_1x_2|1,x_1,x_2) \\ &= \beta_0 + \beta_1 x_1 + \beta_2 x_2.\end{aligned}$$

d. Equation (2.47) is more useful because it allows us to compute the partial effects of x_1 and x_2 at *any* values of x_1 and x_2. Under the assumptions we have made, the linear projection in (2.48) does have as its slope coefficients on x_1 and x_2 the partial effects at the population average values of x_1 and x_2 – zero in both cases – but it does not allow us to obtain the partial effects at any other values of x_1 and x_2. Incidentally, the main conclusions of this problem go through if we allow x_1 and x_2 to have nonzero population means.

2.4. By assumption,

$$\mathrm{E}(u|\mathbf{x},v) = \delta_0 + \mathbf{x}\boldsymbol{\delta} + \rho_1 v$$

for some scalars δ_0, ρ_1 and a column vector $\boldsymbol{\delta}$. Now, it suffices to show that $\delta_0 = 0$ and $\boldsymbol{\delta} = \mathbf{0}$. One way to do this is to use LP.7 in Appendix 2A, and in particular, equation (2.56). This says

that $(\delta_0, \boldsymbol{\delta}')'$ can be obtained by first projecting $(1, \mathbf{x})$ onto v, and obtaining the population residual, $\mathbf{r}$. Then, project u onto $\mathbf{r}$. Now, since v has zero mean and is uncorrelated with $\mathbf{x}$, the first step projection does nothing: $\mathbf{r} = (1, \mathbf{x})$. Thus, projecting u onto $\mathbf{r}$ is just projecting u onto $(1, \mathbf{x})$. Since u has zero mean and is uncorrelated with $\mathbf{x}$, this projection is identically zero, which means that $\delta_0 = 0$ and $\boldsymbol{\delta} = 0$.

2.5. By definition and the zero conditional mean assumptions, $\text{Var}(u_1|\mathbf{x}, \mathbf{z}) = \text{Var}(y|\mathbf{x}, \mathbf{z})$ and $\text{Var}(u_2|\mathbf{x}) = \text{Var}(y|\mathbf{x})$. By assumption, these are constant and necessarily equal to $\sigma_1^2 \equiv \text{Var}(u_1)$ and $\boldsymbol{\sigma}_2^2 \equiv \text{Var}(u_2)$, respectively. But then Property CV.4 implies that $\sigma_2^2 \geq \sigma_1^2$. This simple conclusion means that, when error variances are constant, the error variance falls as more explanatory variables are conditioned on.

2.6. a. By linearity of the linear projection,

$$\text{L}(q|1, \mathbf{x}) = \text{L}(q^*|1, \mathbf{x}) + \text{L}(e|1, \mathbf{x}) = \text{L}(q^*|1, \mathbf{x}),$$

where the last inequality follows because $\text{L}(e|1, \mathbf{x}) = 0$ when $\text{E}(e) = 0$ and $\text{E}(\mathbf{x}'e) = \mathbf{0}$. Therefore, the parameters in the linear projection of q onto $(1,\mathbf{x})$ are the same as the linear projection of q^* onto $(1,\mathbf{x})$. This fact is useful for studying equations with measurement error in the explained or explanatory variables.

b. $r = q - \text{L}(q|1, \mathbf{x}) = (q^* + e) - \text{L}(q|1, \mathbf{x}) = (q^* + e) - \text{L}(q^*|1, \mathbf{x})$ (from part a)

$= [q^* - \text{L}(q^*|1, \mathbf{x}) + e = r^* + e$.

2.7. Write the equation in error form as

$$y = g(\mathbf{x}) + \mathbf{z}\boldsymbol{\beta} + u$$
$$\text{E}(u|\mathbf{x}, \mathbf{z}) = 0.$$

Take the expected value of the first equation conditional only on $\mathbf{x}$:

$$\mathrm{E}(y|\mathbf{x}) = g(\mathbf{x}) + [\mathrm{E}(\mathbf{z}|\mathbf{x})]\boldsymbol{\beta}$$

and subtract this from the first equation to get

$$y - \mathrm{E}(y|\mathbf{x}) = [\mathbf{z} - \mathrm{E}(\mathbf{z}|\mathbf{x})]\boldsymbol{\beta} + u$$

or

$$\tilde{y} = \tilde{\mathbf{z}}\boldsymbol{\beta} + u$$

Because $\tilde{\mathbf{z}}$ is a function of $(\mathbf{x}, \mathbf{z})$, $\mathrm{E}(u|\tilde{\mathbf{z}}) = 0$ [since $\mathrm{E}(u|\mathbf{x}, \mathbf{z}) = 0$], and so $\mathrm{E}(\tilde{y}|\tilde{\mathbf{z}}) = \tilde{\mathbf{z}}\boldsymbol{\beta}$.

This basic result is fundamental in the literature on estimating *partial linear models*. First, one estimates $\mathrm{E}(y|\mathbf{x})$ and $\mathrm{E}(\mathbf{z}|\mathbf{x})$ using very flexible methods (typically, *nonparametric methods*). Then, after obtaining residuals of the form $\tilde{y}_i \equiv y_i - \hat{\mathrm{E}}(y_i|\mathbf{x}_i)$ and $\tilde{\mathbf{z}}_i \equiv \mathbf{z}_i - \hat{\mathrm{E}}(\mathbf{z}_i|\mathbf{x}_i)$, $\boldsymbol{\beta}$ is estimated from an OLS regression $\tilde{y}_i$ on $\tilde{\mathbf{z}}_i, i = 1, \ldots, N$. Under general conditions, this kind of nonparametric partialling-out procedure leads to a $\sqrt{N}$-consistent, asymptotically normal estimator of $\boldsymbol{\beta}$. See Robinson (1988) and Powell (1994).

In the case where $\mathrm{E}(y|\mathbf{x})$ and the elements of $\mathrm{E}(\mathbf{z}|\mathbf{x})$ are approximated as linear functions of a common set of functions, say $\{h_1(\mathbf{x}), \ldots, h_Q(\mathbf{x})\}$, the partialling out is equivalent to estimating a linear model

$$y = \alpha_0 + \alpha_1 h_1(\mathbf{x}) + \ldots + \alpha_Q h_Q(\mathbf{x}) + \mathbf{x}\boldsymbol{\beta} + error$$

by OLS.

2.8. a. By exponentiation we can write $y = \exp[g(\mathbf{x}) + u] = \exp[g(\mathbf{x})]\exp(u)$. It follows that

$$\mathrm{E}(y|\mathbf{x}) = \exp[g(\mathbf{x})]\mathrm{E}[\exp(u)|\mathbf{x}] = \exp[g(\mathbf{x})]a(\mathbf{x})$$

Using the product rule gives

$$\frac{\partial \mathrm{E}(y|\mathbf{x})}{\partial x_j} = \frac{\partial g(\mathbf{x})}{\partial x_j}\exp[g(\mathbf{x})]a(\mathbf{x}) + \exp[g(\mathbf{x})]\frac{\partial a(\mathbf{x})}{\partial x_j}$$
$$= \frac{\partial g(\mathbf{x})}{\partial x_j}\mathrm{E}(y|\mathbf{x}) + \mathrm{E}(y|\mathbf{x})\frac{\partial a(\mathbf{x})}{\partial x_j} \cdot \frac{1}{a(\mathbf{x})}$$

Therefore,

$$\frac{\partial \mathrm{E}(y|\mathbf{x})}{\partial x_j} \cdot \frac{x_j}{\mathrm{E}(y|\mathbf{x})} = \frac{\partial g(\mathbf{x})}{\partial x_j} \cdot x_j + \frac{\partial a(\mathbf{x})}{\partial x_j} \cdot \frac{x_j}{a(\mathbf{x})}$$

We can establish this relationship more simply by assuming $\mathrm{E}(y|\mathbf{x}) > 0$ for all $\mathbf{x}$ and using equation (2.10).

b. Write $z_j \equiv \log(x_j)$ so $x_j = \exp(z_j)$. Then, using the chain rule,

$$\frac{\partial g(\mathbf{x})}{\partial \log(x_j)} = \frac{\partial g(\mathbf{x})}{\partial z_j} = \frac{\partial g(\mathbf{x})}{\partial x_j} \cdot \frac{\partial x_j}{\partial z_j} = \frac{\partial g(\mathbf{x})}{\partial x_j} \cdot \exp(z_j) = \frac{\partial g(\mathbf{x})}{\partial x_j} \cdot x_j$$

c. From $\log(y) = g(\mathbf{x}) + u$ and $\mathrm{E}(u|\mathbf{x}) = 0$ we have $\mathrm{E}[\log(y)|\mathbf{x})] = g(\mathbf{x})$. Therefore, using (2.11), the elasticity would be simply

$$\frac{\partial g(\mathbf{x})}{\partial \log(x_j)} = \frac{\partial g(\mathbf{x})}{\partial x_j} \cdot x_j$$

which, compared with the definition based on $\mathrm{E}(y|\mathbf{x})$, omits the elasticity of $a(\mathbf{x})$ with respect to x_j.

2.9. This is easily shown by using iterated expectations:

$$\mathrm{E}(\mathbf{x}'y) = \mathrm{E}[\mathrm{E}(\mathbf{x}'y|\mathbf{x})] = \mathrm{E}[\mathbf{x}'\mathrm{E}(y|\mathbf{x})] = \mathrm{E}[\mathbf{x}'\mu(\mathbf{x})]$$

Therefore,

$$\boldsymbol{\delta} = [\mathrm{E}(\mathbf{x}'\mathbf{x})]^{-1}\mathrm{E}(\mathbf{x}'y) = [\mathrm{E}(\mathbf{x}'\mathbf{x})]^{-1}\mathrm{E}[\mathbf{x}'\mu(\mathbf{x})]$$

and the latter equation is the vector of parameters in the linear projection of $\mu(\mathbf{x})$ on $\mathbf{x}$.

2.10. a. As given in the hint, we can always write

$$\mathrm{E}(y|\mathbf{x},s) = (1-s)\cdot\mu_0(\mathbf{x}) + s\cdot\mu_1(\mathbf{x})$$

Now condition only on s and use iterated expectations:

$$\begin{aligned}\mathrm{E}(y|s) &= \mathrm{E}[\mathrm{E}(y|\mathbf{x},s)|s] = \mathrm{E}[(1-s)\cdot\mu_0(\mathbf{x}) + s\cdot\mu_1(\mathbf{x})|s] \\ &= (1-s)\mathrm{E}[\mu_0(\mathbf{x})|s] + s\mathrm{E}[\mu_1(\mathbf{x})|s]\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}(y|s=1) &= \mathrm{E}[\mu_1(\mathbf{x})|s=1] \\ \mathrm{E}(y|s=0) &= \mathrm{E}[\mu_0(\mathbf{x})|s=0]\end{aligned}$$

and so, by adding and subtracting $\mathrm{E}[\mu_0(\mathbf{x})|s=1]$, we get

$$\begin{aligned}\mathrm{E}(y|s=1) - \mathrm{E}(y|s=0) &= \mathrm{E}[\mu_1(\mathbf{x})|s=1] - \mathrm{E}[\mu_0(\mathbf{x})|s=0] \\ &= \{\mathrm{E}[\mu_1(\mathbf{x})|s=1] - \mathrm{E}[\mu_0(\mathbf{x})|s=1]\} + \{\mathrm{E}[\mu_0(\mathbf{x})|s=1] - \mathrm{E}[\mu_0(\mathbf{x})|s=0]\}\end{aligned}$$

b. Use part a and linearity of the conditional means:

$$\begin{aligned}\mathrm{E}(y|s=1) - \mathrm{E}(y|s=0) &= [\mathrm{E}(\mathbf{x}|s=1)\boldsymbol{\beta}_1 - \mathrm{E}(\mathbf{x}|s=1)\boldsymbol{\beta}_0] + [\mathrm{E}(\mathbf{x}|s=1)\boldsymbol{\beta}_0 - \mathrm{E}(\mathbf{x}|s=0)\boldsymbol{\beta}_0] \\ &= \mathrm{E}(\mathbf{x}|s=1)\cdot(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) + [\mathrm{E}(\mathbf{x}|s=1) - \mathrm{E}(\mathbf{x}|s=0)]\cdot\boldsymbol{\beta}_0\end{aligned}$$

This decomposition attributes the difference in the unconditional means, $\mathrm{E}(y|s=1) - \mathrm{E}(y|s=0)$, to two pieces. The first part is due to differences in the regression parameters, $\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0$ – where we evaluate the difference at the average of the covariates from the $s=1$ subpopulation. The second part is due to a difference in means of the covariates from the two subpopulations – where we apply the regression coefficients from the $s=0$ subpopulation. If, for example, the two regression functions are the same – that is, $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_0$ – then any difference in the subpopulation means $\mathrm{E}(y|s=0)$ and $\mathrm{E}(y|s=1)$ is due to a difference in averages of the covariates across the subpopulations. If the covariate means are the same – that is, $\mathrm{E}(\mathbf{x}|s=1) = \mathrm{E}(\mathbf{x}|s=0)$ – then $\mathrm{E}(y|s=0)$ and $\mathrm{E}(y|s=0)$ can still differ if

$\beta_1 \neq \beta_0$. In many applications, both pieces in $\mathrm{E}(y|s=1) - \mathrm{E}(y|s=0)$ are present.

Incidentally, the approach in this problem is not the only interesting way to decompose $\mathrm{E}(y|s=1) - \mathrm{E}(y|s=0)$. See, for example, T.E. Elder, J.H. Goddeeris, and S.J. Haider, "Unexplained Gaps and Oaxaca–Blinder Decompositions," *Labour Economics*, 2010.

Solutions to Chapter 3 Problems

3.1. To prove Lemma 3.1, we must show that for all $\varepsilon > 0$, there exists $b_\varepsilon < \infty$ and an integer N_ε such that $\mathrm{P}[|x_N| \geq b_\varepsilon] < \boldsymbol{\varepsilon}$, all $N \geq N_\varepsilon$. We use the following fact: since $x_N \xrightarrow{p} a$, for any $\varepsilon > 0$ there exists an integer N_ε such that $\mathrm{P}[|x_N - a| > 1] < \varepsilon$ for all $N \geq N_\varepsilon$. [The existence of N_ε is implied by Definition 3.3(1).] But $|x_N| = |x_N - a + a| \leq |x_N - a| + |a|$ (by the triangle inequality), and so $|x_N| - |a| \leq |x_N - a|$. It follows that $\mathrm{P}[|x_N| - |a| > 1] \leq \mathrm{P}[|x_N - a| > 1]$. Therefore, in Definition 3.3(3) we can take $b_\varepsilon \equiv |a| + 1$ (irrespective of the value of ε) and then the existence of N_ε follows from Definition 3.3(1).

3.2. Each element of the $K \times 1$ vector $\mathbf{Z}_N'\mathbf{x}_N$ is the sum of J terms of the form $Z_{Nji}x_{Nj}$. Because $Z_{Nji} = o_p(1)$ and $x_{Nj} = O_p(1)$, each term in the sum is $o_p(1)$ from Lemma 3.2(4). By Lemma 3.2(1), the sum of $o_p(1)$ terms is $o_p(1)$.

3.3. This follows immediately from Lemma 3.1 because $\mathbf{g}(\mathbf{x}_N) \xrightarrow{p} \mathbf{g}(\mathbf{c})$.

3.4. Both parts follow from the continuous mapping theorem and basic properties of the normal distribution.

a. The function defined by $\mathbf{g}(\mathbf{z}) = \mathbf{A}'\mathbf{z}$ is clearly continuous. Further, if $\mathbf{z} \sim \text{Normal}(\mathbf{0}, \mathbf{V})$ then $\mathbf{A}'\mathbf{z} \sim \text{Normal}(\mathbf{0}, \mathbf{A}'\mathbf{V}\mathbf{A})$. By the continuous mapping theorem, $\mathbf{A}'\mathbf{z}_N \xrightarrow{d} \mathbf{A}'\mathbf{z} \sim \text{Normal}(\mathbf{0}, \mathbf{A}'\mathbf{V}\mathbf{A})$.

b. Because $\mathbf{V}$ is nonsingular, the function $g(\mathbf{z}) = \mathbf{z}'\mathbf{V}^{-1}\mathbf{z}$ is continuous. But if $\mathbf{z} \sim \text{Normal}(\mathbf{0}, \mathbf{V})$, $\mathbf{z}'\mathbf{V}^{-1}\mathbf{z} \sim \chi^2_K$. So $\mathbf{z}_N'\mathbf{V}^{-1}\mathbf{z}_N \xrightarrow{d} \mathbf{z}'\mathbf{V}^{-1}\mathbf{z} \sim \chi^2_K$.

3.5. a. Because $\text{Var}(\bar{y}_N) = \sigma^2/N$, $\text{Var}[\sqrt{N}(\bar{y}_N - \mu)] = N(\sigma^2/N) = \sigma^2$.

b. By the CLT, $\sqrt{N}(\bar{y}_N - \mu) \overset{a}{\sim} \text{Normal}(0, \sigma^2)$, and so $\text{Avar}[\sqrt{N}(\bar{y}_N - \mu)] = \sigma^2$.

c. We obtain $\text{Avar}(\bar{y}_N)$ by dividing $\text{Avar}[\sqrt{N}(\bar{y}_N - \mu)]$ by N. Therefore, $\text{Avar}(\bar{y}_N) = \sigma^2/N$.

As expected, this coincides with the actual variance of $\bar{y}_N$.

d. The asymptotic standard deviation of $\bar{y}_N$ is the square root of its asymptotic variance, or $\sigma/\sqrt{N}$.

e. To obtain the asymptotic standard error of $\bar{y}_N$, we need a consistent estimator of σ. Typically, the unbiased estimator of σ^2 is used: $\hat{\sigma}^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{y}_N)^2$, and then $\hat{\sigma}$ is the positive square root. The asymptotic standard error of $\bar{y}_N$ is simply $\hat{\sigma}/\sqrt{N}$.

3.6. From Definition 3.4, we need to show that for any $0 \leq c < 1/2, N^c(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}) = o_p(1)$. But

$$N^c(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}) = N^{[c-(1/2)]}\sqrt{N}(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}) = N^{[c-(1/2)]} \cdot O_p(1).$$

Because $c < 1/2$, $N^{[c-(1/2)]} = o(1)$, and so $N^c(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}) = o(1) \cdot O_p(1) = o_p(1)$.

3.7. a. For $\theta > 0$ the natural logarithm is a continuous function, and so $\text{plim}[\log(\hat{\theta})] = \log[\text{plim}(\hat{\theta})] = \log(\theta) = \gamma$.

b. We use the delta method to find $\text{Avar}[\sqrt{N}(\hat{\gamma} - \gamma)]$. In the scalar case, if $\hat{\gamma} = g(\hat{\theta})$ then $\text{Avar}[\sqrt{N}(\hat{\gamma} - \gamma)] = [dg(\theta)/d\theta]^2\text{Avar}[\sqrt{N}(\hat{\theta} - \theta)]$. When $g(\theta) = \log(\theta)$ – which is, of course, continuously differentiable – $\text{Avar}[\sqrt{N}(\hat{\gamma} - \gamma)] = (1/\theta)^2\text{Avar}[\sqrt{N}(\hat{\theta} - \theta)]$.

c. In the scalar case, the asymptotic standard error of $\hat{\gamma}$ is generally $|dg(\hat{\theta})/d\theta| \cdot \text{se}(\hat{\theta})$. Therefore, for $g(\theta) = \log(\theta), \text{se}(\hat{\gamma}) = \text{se}(\hat{\theta})/(\hat{\theta})$. When $\hat{\theta} = 4$ and $\text{se}(\hat{\theta}) = 2, \hat{\gamma} = \log(4) \approx 1.39$ and $\text{se}(\hat{\gamma}) = 1/2$.

d. The asymptotic t statistic for testing $H_0 : \theta = 1$ is $(\hat{\theta} - 1)/\text{se}(\hat{\theta}) = 3/2 = 1.5$.

e. Because $\gamma = \log(\theta)$, the null of interest can also be stated as $H_0 : \gamma = 0$. The t statistic based on $\hat{\gamma}$ is about $1.39/(.5) = 2.78$. This leads to a very strong rejection of H_0, whereas the t statistic based on $\hat{\theta}$ is, at best, marginally significant. The lesson is that, using the Wald test,

we can change the outcome of hypotheses tests by using nonlinear transformations.

3.8 a. This follows by Slutsky's Theorem since the function $g(\theta_1, \theta_2) \equiv \theta_1/\theta_2$ is continuous at all points in $\mathbb{R}^2$ where $\theta_2 \neq 0$: $\text{plim}(\hat{\theta}_1/\hat{\theta}_2) = [\text{plim}(\hat{\theta}_1)/\text{plim}(\hat{\theta}_2)] = \theta_1/\theta_2$.

b. To find $\text{Avar}(\hat{\gamma})$ we need to find $\nabla_{\boldsymbol{\theta}} g(\boldsymbol{\theta})$, where $g(\theta_1, \theta_2) = \theta_1/\theta_2$. But $\nabla_{\boldsymbol{\theta}} g(\boldsymbol{\theta}) = (1/\theta_2, -\theta_1/\theta_2^2)$, and so $\text{Avar}(\hat{\gamma}) = (1/\theta_2 - \theta_1/\theta_2^2)[\text{Avar}(\hat{\boldsymbol{\theta}})](1/\theta_2 - \theta_1/\theta_2^2)'$.

c. If $\hat{\boldsymbol{\theta}} = (-1.5, .5)'$ then $\nabla_{\boldsymbol{\theta}} g(\hat{\boldsymbol{\theta}}) = (2, 6)$. Therefore, $\widehat{\text{Avar}}(\hat{\gamma}) = (2,6)[\widehat{\text{Avar}}(\hat{\boldsymbol{\theta}})](2,6)' = 66.4$. Taking the square root gives $\text{se}(\hat{\gamma}) \approx 8.15$.

3.9. By the delta method,

$$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})] = \mathbf{G}(\boldsymbol{\theta})\mathbf{V}_1\mathbf{G}(\boldsymbol{\theta})', \ \text{Avar}[\sqrt{N}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})] = \mathbf{G}(\boldsymbol{\theta})\mathbf{V}_2\mathbf{G}(\boldsymbol{\theta})',$$

where $\mathbf{G}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} \mathbf{g}(\boldsymbol{\theta})$ is $Q \times P$. Therefore,

$$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})] - \text{Avar}[\sqrt{N}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})] = \mathbf{G}(\boldsymbol{\theta})(\mathbf{V}_2 - \mathbf{V}_1)\mathbf{G}(\boldsymbol{\theta})'.$$

By assumption, $\mathbf{V}_2 - \mathbf{V}_1$ is positive semi-definite, and therefore $\mathbf{G}(\boldsymbol{\theta})(\mathbf{V}_2 - \mathbf{V}_1)\mathbf{G}(\boldsymbol{\theta})'$ is p.s.d. This complete the proof.

3.10. By assumption, $\sigma^2 = \text{E}(w_i^2) = \text{Var}(w_i) < \infty$. Because of the i.i.d. assumption,

$$\text{Var}(x_N) = (N^{-1/2})^2 N\sigma^2 = \sigma^2.$$

Now, Chebyshev's inequality gives that for any $b_\varepsilon > 0$,

$$\text{P}[|x_N| \geq b_\varepsilon] \leq \frac{\text{Var}(X_N)}{b_\varepsilon^2} = \frac{\sigma^2}{b_\varepsilon^2}$$

Therefore, in the definition of $O_p(1)$, for any $\varepsilon > 0$ choose $b_\varepsilon = \sigma/\sqrt{\varepsilon}$ and $N_\varepsilon = 1$ and then $\text{P}[|x_N| \geq b_\varepsilon] \leq \varepsilon$ for all $N \geq N_\varepsilon$.

3.11. a. Let $x_N = N^{-1}\sum_{i=1}^{N}(w_i - \mu_i)$ so that

$$\mathrm{Var}(x_N) = N^{-2}\sum_{i=1}^{N}\mathrm{Var}(w_i) = N^{-2}\sum_{i=1}^{N}\sigma_i^2$$

By Chebyshev's inequality, for any $\varepsilon > 0$,

$$\mathrm{P}[|x_N| > \varepsilon] \leq \frac{\mathrm{Var}(x_N)}{\varepsilon^2} = \frac{N^{-2}\sum_{i=1}^{N}\sigma_i^2}{\varepsilon^2}$$

It follows that $\mathrm{P}[|x_N| > \varepsilon] \to 0$ as $N \to \infty$ if $N^{-2}\sum_{i=1}^{N}\sigma_i^2 \to 0$ as $N \to \infty$.

b. If $\sigma_i^2 < b < \infty$ for all i – that is, the sequence of variances is bounded – then

$$N^{-2}\sum_{i=1}^{N}\sigma_i^2 \leq b/N \to 0 \text{ as } N \to \infty.$$

Thus, uniformly bounded variances is sufficient for i.n.i.d. sequences to satisfy the WLLN.

Solutions to Chapter 4 Problems

4.1. a. Exponentiating equation (4.49) gives

$$\begin{aligned} wage &= \exp(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma} + u) \\ &= \exp(u)\exp(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}). \end{aligned}$$

Therefore,

$$\mathrm{E}(wage|\mathbf{x}) = \mathrm{E}[\exp(u)|\mathbf{x}]\exp(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}),$$

where $\mathbf{x}$ denotes all explanatory variables. Now, if u and $\mathbf{x}$ are independent thenE$[exp(u)|\mathbf{x}]$ =E$[\exp(u)]$ $= \delta_0$, say. Therefore

$$\mathrm{E}(wage|\mathbf{x}) = \delta_0 \exp(\beta_0 + \beta_1 married + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}).$$

If we set $married = 1$ and $married = 0$ in this expecation (keeping all else equal) and find the proportionate increase we get

$$\frac{\delta_0 \exp(\beta_0 + \beta_1 + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma}) - \delta_0 \exp(\beta_0 + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma})}{\delta_0 \exp(\beta_0 + \beta_2 educ + \mathbf{z}\boldsymbol{\gamma})} = \exp(\beta_1) - 1.$$

Thus, the percentage difference is $100 \cdot [\exp(\beta_1) - 1]$.

b. Since $\theta_1 = 100 \cdot [\exp(\beta_1) - 1] = g(\beta_1)$, we need the derivative of g with respect to β_1: $dg/d\beta_1 = 100 \cdot \exp(\beta_1)$. The asymptotic standard error of $\hat{\theta}_1$ using the delta method is obtained as the absolute value of $d\hat{g}/d\beta_1$ times se$(\hat{\beta}_1)$:

$$\mathrm{se}(\hat{\theta}_1) = 100 \cdot [exp(\hat{\beta}_1)] \cdot \mathrm{se}(\hat{\beta}_1).$$

c. We can evaluate the conditional expectation in part a at two levels of education, say $educ_0$ and $educ_1$, all else fixed. The proportionate change in expected wage from $educ_0$ to $educ_1$ is

$$[\exp(\beta_2 educ_1) - \exp(\beta_2 educ_0)]/\exp(\beta_2 educ_0) = \exp[\beta_2(educ_1 - educ_0)] - 1 = \exp(\beta_2 \Delta educ) - 1.$$

Using the same arguments in part b, $\hat{\theta}_2 = 100 \cdot [\exp(\beta_2 \Delta educ) - 1]$ and

$$\text{se}(\hat{\theta}_2) = 100 \cdot |\Delta educ| \exp(\hat{\beta}_2 \Delta educ) \text{se}(\hat{\beta}_2).$$

d. For the estimated version of equation (4.29), $\hat{\beta}_1 = .199$, $\text{se}(\hat{\beta}_1) = .039$, $\hat{\beta}_2 = .065$, and $\text{se}(\hat{\beta}_2) = .006$. Therefore, $\hat{\theta}_1 = 22.01$ and $\text{se}(\hat{\theta}_1) = 4.76$. For $\hat{\theta}_2$ we set $\Delta educ = 4$. Then $\hat{\theta}_2 = 29.7$ and $\text{se}(\hat{\theta}_2) = 3.11$.

4.2. a. For each i we have, by OLS.2, $\text{E}(u_i|\mathbf{X}) = 0$. By independence across i and Property CE.5, $\text{E}(u_i|\mathbf{X}) = \text{E}(u_i|\mathbf{x}_i)$ because $(u_i, \mathbf{x}_i)$ is independent of the explanatory variables for all other observations. Letting $\mathbf{U}$ be the $N \times 1$ vector of all errors, this implies $\text{E}(\mathbf{U}|\mathbf{X}) = \mathbf{0}$. But $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{U}$ and so

$$\text{E}(\hat{\boldsymbol{\beta}}|\mathbf{X}) = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\text{E}(\mathbf{U}|\mathbf{X}) = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \cdot \mathbf{0} = \boldsymbol{\beta}.$$

b. From the expression for $\hat{\boldsymbol{\beta}}$ in part a we have

$$\text{Var}(\hat{\boldsymbol{\beta}}|\mathbf{X}) = \text{Var}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{U}|\mathbf{X}] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\text{Var}(\mathbf{U}|\mathbf{X})\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}.$$

Now, because $\text{E}(\mathbf{U}|\mathbf{X}) = \mathbf{0}$, $\text{Var}(\mathbf{U}|\mathbf{X}) = \text{E}(\mathbf{U}\mathbf{U}'|\mathbf{X})$. For the diagonal terms, $\text{E}(u_i^2|\mathbf{X}) = \text{E}(u_i^2|\mathbf{x}_i) = \text{Var}(u_i|\mathbf{x}_i) = \sigma^2$, where the least equality is the homoskedasticity assumption. For the covariance terms, we must show that $\text{E}(u_i u_h|\mathbf{X}) = 0$ for all $i \neq h, i, h = 1, \ldots, N$. Again using Property CE.5, $\text{E}(u_i u_h|\mathbf{X}) = \text{E}(u_i u_h|\mathbf{x}_i, \mathbf{x}_h)$ and $\text{E}(u_i|\mathbf{x}_i, u_h, \mathbf{x}_h) = \text{E}(u_i|\mathbf{x}_i) = 0$. But then $\text{E}(u_i u_h|\mathbf{x}_i, u_h, \mathbf{x}_h) = \text{E}(u_i|\mathbf{x}_i, u_h, \mathbf{x}_h)u_h = 0$. It follows immediately by iterated expectations that conditioning on the smaller set also yields a zero conditional mean: $\text{E}(u_i u_h|\mathbf{x}_i, \mathbf{x}_h) = 0$. This completes the proof.

4.3. a. Not in general. The conditional variance can always be written as $\text{Var}(u|\mathbf{x}) = \text{E}(u^2|\mathbf{x}) - [\text{E}(u|\mathbf{x})]^2$; if $\text{E}(u|\mathbf{x}) \neq 0$, then $\text{E}(u^2|\mathbf{x}) \neq \text{Var}(u|\mathbf{x})$.

b. It could be that $\text{E}(\mathbf{x}'u) = 0$, in which case OLS is consistent, and $\text{Var}(u|\mathbf{x})$ is constant.

But, generally, the usual standard errors would not be valid unless $\mathrm{E}(u|\mathbf{x}) = 0$ because it is $\mathrm{E}(u^2|\mathbf{x})$ that should be constant.

4.4. For each i, $\hat{u}_i = y_i - \mathbf{x}_i\hat{\boldsymbol{\beta}} = u_i - \mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$, and so $\hat{u}_i^2 = u_i^2 - 2u_i\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + [\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2$. Therefore, we can write

$$N^{-1}\sum_{i=1}^{N}\hat{u}_1^2\mathbf{x}_i'\mathbf{x}_i = N^{-1}\sum_{i=1}^{N}u_1^2\mathbf{x}_i'\mathbf{x}_i - 2N^{-1}\sum_{i=1}^{N}[u_i\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]\mathbf{x}_i'\mathbf{x}_i + N^{-1}\sum_{i=1}^{N}[\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2\mathbf{x}_i'\mathbf{x}_i.$$

Dropping the "-2", the second term can be written as the sum of K terms of the form

$$N^{-1}\sum_{i=1}^{N}[u_ix_{ij}(\hat{\beta}_j - \beta_j)]\mathbf{x}_i'\mathbf{x}_i = (\hat{\beta}_j - \beta_j)N^{-1}\sum_{i=1}^{N}(u_ix_{ij})\mathbf{x}_i'\mathbf{x}_i = o_p(1)\cdot O_p(1),$$

where we have used $\hat{\beta}_j - \beta_j = o_p(1)$ and $N^{-1}\sum_{i=1}^{N}(u_ix_{ij})\mathbf{x}_i'\mathbf{x}_i = O_p(1)$ whenever $\mathrm{E}[|u_ix_{ij}x_{ih}x_{ik}|] < \infty$ for all j, h, and k (as would just be assumed). Similarly, the third term can be written as the sum of K^2 terms of the form

$$(\hat{\beta}_j - \beta_j)(\hat{\beta}_h - \beta_h)N^{-1}\sum_{i=1}^{N}(x_{ij}x_{ih})\mathbf{x}_i'\mathbf{x}_i = o_p(1)\cdot o_p(1)\cdot O_p(1) = o_p(1),$$

where we have used $N^{-1}\sum_{i=1}^{N}(x_{ij}x_{ih})\mathbf{x}_i'\mathbf{x}_i = O_p(1)$ whenever $\mathrm{E}[|x_{ij}x_{ih}x_{ik}x_{im}|] < \infty$ for all j, h, k, and m. We have shown that $N^{-1}\sum_{i=1}^{N}\hat{u}_i^2\mathbf{x}_i'\mathbf{x}_i = N^{-1}\sum_{i=1}^{N}u_i^2\mathbf{x}_i'\mathbf{x}_i + o_p(1)$, which is what we wanted to show.

4.5. Write equation (4.50) as $\mathrm{E}(y|\mathbf{w}) = \mathbf{w}\boldsymbol{\delta}$, where $\mathbf{w} = (\mathbf{x}, z)$. Since $\mathrm{Var}(y|\mathbf{w}) = \sigma^2$, it follows by Theorem 4.2 that $\mathrm{Avar}\sqrt{N}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})$ is $\sigma^2[\mathrm{E}(\mathbf{w}'\mathbf{w})]^{-1}$, where $\hat{\boldsymbol{\delta}} = (\hat{\boldsymbol{\beta}}', \hat{\gamma})'$. Importantly, because $\mathrm{E}(\mathbf{x}'z) = \mathbf{0}$, $\mathrm{E}(\mathbf{w}'\mathbf{w})$ is block diagonal, with upper block $\mathrm{E}(\mathbf{x}'\mathbf{x})$ and lower block $\mathrm{E}(z^2)$. Inverting $\mathrm{E}(\mathbf{w}'\mathbf{w})$ and focusing on the upper $K \times K$ block gives

$$\mathrm{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \sigma^2[\mathrm{E}(\mathbf{x}'\mathbf{x})]^{-1}.$$

Next, we need to find $\text{Avar}\sqrt{N}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})$. It is helpful to write $y = \mathbf{x}\boldsymbol{\beta} + v$ where $v = \gamma z + u$ and $u \equiv y - \text{E}(y|\mathbf{x}, z)$. Because $\text{E}(\mathbf{x}'z) = 0$ and $\text{E}(\mathbf{x}'u) = 0$, $\text{E}(\mathbf{x}'v) = 0$. Further, $\text{E}(v^2|\mathbf{x}) = \gamma^2\text{E}(z^2|\mathbf{x}) + \text{E}(u^2|\mathbf{x}) + 2\gamma\text{E}(zu|\mathbf{x}) = \gamma^2\text{E}(z^2|\mathbf{x}) + \sigma^2$, where we use $\text{E}(zu|\mathbf{x}, z) = z\text{E}(u|\mathbf{x}, z) = 0$ and $\text{E}(u^2|\mathbf{x}, z) = \text{Var}(y|\mathbf{x}, z) = \sigma^2$. Unless $\text{E}(z^2|\mathbf{x})$ is constant, the equation $y = \mathbf{x}\beta + v$ generally violates the homoskedasticity assumption OLS.3. So, without further assumptions,

$$\text{Avar}\sqrt{N}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) = [\text{E}(\mathbf{x}'\mathbf{x})]^{-1}\text{E}(v^2\mathbf{x}'\mathbf{x})[\text{E}(\mathbf{x}'\mathbf{x})]^{-1}.$$

Now we can show $\text{Avar}\sqrt{N}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is positive semi-definite by writing

$$\begin{aligned}\text{Avar}\sqrt{N}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) &= [\text{E}(\mathbf{x}'\mathbf{x})]^{-1}\text{E}(v^2\mathbf{x}'\mathbf{x})[\text{E}(\mathbf{x}'\mathbf{x})]^{-1} - \sigma^2[\text{E}(\mathbf{x}'\mathbf{x})]^{-1} \\ &= [\text{E}(\mathbf{x}'\mathbf{x})]^{-1}\text{E}(v^2\mathbf{x}'\mathbf{x})[\text{E}(\mathbf{x}'\mathbf{x})]^{-1} - \sigma^2[\text{E}(\mathbf{x}'\mathbf{x})]^{-1}\text{E}(\mathbf{x}'\mathbf{x})[\text{E}(\mathbf{x}'\mathbf{x})]^{-1} \\ &= [\text{E}(\mathbf{x}'\mathbf{x})]^{-1}[\text{E}(v^2\mathbf{x}'\mathbf{x}) - \sigma^2\text{E}(\mathbf{x}'\mathbf{x})][\text{E}(\mathbf{x}'\mathbf{x})]^{-1}\end{aligned}$$

Because $[\text{E}(\mathbf{x}'\mathbf{x})]^{-1}$ is positive definite, it suffices to show that $\text{E}(v^2\mathbf{x}'\mathbf{x}) - \sigma^2[\text{E}(\mathbf{x}'\mathbf{x})$ is p.s.d. To this end, let $h(\mathbf{x}) \equiv \text{E}(z^2|\mathbf{x})$. Then by the law of iterated expectations, $\text{E}(v^2\mathbf{x}'\mathbf{x}) = \text{E}[\text{E}(v^2|\mathbf{x})\mathbf{x}'\mathbf{x}] = \gamma^2\text{E}[h(\mathbf{x})\mathbf{x}'\mathbf{x}] + \sigma^2\text{E}(\mathbf{x}'\mathbf{x})$. Therefore, $\text{E}(v^2\mathbf{x}'\mathbf{x}) - \sigma^2\text{E}(\mathbf{x}'\mathbf{x}) = \gamma^2\text{E}[h(\mathbf{x})\mathbf{x}'\mathbf{x}]$, which, when $\gamma \neq 0$, is actually a positive definite matrix except by fluke. In particular, if $\text{E}(z^2|\mathbf{x}) = \text{E}(z^2) = \eta^2 > 0$ (in which case $y = \mathbf{x}\boldsymbol{\beta} + v$ satisfies the homoskedasticity assumption OLS.3), $\text{E}(v^2\mathbf{x}'\mathbf{x}) - \sigma^2\text{E}(\mathbf{x}'\mathbf{x}) = \gamma^2\eta^2\text{E}(\mathbf{x}'\mathbf{x})$, which is positive definite.

4.6. Because *nonwhite* is determined at birth, we do not have to worry about *nonwhite* being determined simultaneously with any kind of response variable. Measurement error is certainly a possibility, as a binary indicator for being Caucasian is a very crude way to measure race. Still, many studies hope to isolate systematic differences between those classified as white versus other races, in which case a binary indicator might be a good proxy. Of course, it

is always possible that people are misclassified in survey data. But an important point is that measurement error in *nonwhite* would not follow the classical errors-in-variables assumption. For example, if the issue is simply recording the incorrect entry, then the true indicator, $nonwhite^*$, is also binary. Then, there are four possible outcomes: $nonwhite^* = 1$ and $nonwhite = 1$; $nonwhite^* = 0$ and $nonwhite = 1$; $nonwhite^* = 1$ and $nonwhite = 0$; $nonwhite^* = 0$ and $nonwhite = 0$. In the first and last cases, no error is made. Generally, it makes no sense to write $nonwhite = nonwhite^* + e$, where e is a mean–zero measurement error that is independent of $nonwhite^*$.

Probably in applications that seek to estimate a race effect, we would be most concerned about omitted variables. While race is determined at birth, it is not independent of other factors that generally affect economic and social outcomes. For example, we would want to include family income and wealth in an equation to test for discrimination in loan applications. If we cannot, and race is correlated with income and wealth, then an attempt to test for discrimination can fail. Many other applications could suffer from endogeneity caused by omitted variables. In looking at crime rates by race, we also need to control for family background characteristics.

4.7. a. One important omitted factor in u is family income: students that come from wealthier families tend to do better in school, other things equal. Family income and PC ownership are positively correlated because the probability of owning a PC increases with family income. Another factor in u is quality of high school. This may also be correlated with *PC*: a student who had more exposure with computers in high school may be more likely to own a computer.

b. $\hat{\beta}_3$ is *likely* to have an upward bias because of the positive correlation between u and *PC*,

but it is not clear cut because of the other explanatory variables in the equation. If we write the linear projection

$$u = \delta_0 + \delta_1 hsGPA + \delta_2 SAT + \delta_3 PC + r$$

then the bias is upward if δ_3 is greater than zero. This measures the partial correlation between u (say, family income) and PC, and it is likely to be positive.

c. If data on family income can be collected then it can be included in the equation. If family income is not available sometimes level of parents' education is. Another possibility is to use average house value in each student's home zip code, as zip code is often part of school records. Proxies for high school quality might be faculty–student ratios, expenditure per student, average teacher salary, and so on.

4.8. a. $\partial \mathrm{E}(y|x_1,x_2)/\partial x_1 = \beta_1 + \beta_3 x_2$. Taking the expected value of this equation with respect to the distribution of x_2 gives $\alpha_1 \equiv \beta_1 + \beta_3 \mu_2$. Similarly, $\partial \mathrm{E}(y|x_1,x_2)/\partial x_2 = \beta_2 + \beta_3 x_1 + 2\beta_4 x_2$, and its expected value is $\alpha_2 \equiv \beta_2 + \beta_3 \mu_1 + 2\beta_4 \mu_2$.

b. One way to write $\mathrm{E}(y|x_1,x_2)$ is

$$\mathrm{E}(y|x_1,x_2) = \delta_0 + \alpha_1 x_1 + \alpha_2 x_2 + \beta_3 (x_1 - \mu_1)(x_2 - \mu_2) + \beta_4 (x_2 - \mu_2)^2,$$

where $\delta_0 = \beta_0 + \beta_3 \mu_1 \mu_2 - \beta_4 \mu_2^2$ (as can be verified by matching the intercepts in the two equations).

c. Regress y_i on $1, x_{i1}, x_{i2}, (x_{i1} - \mu_1)(x_{i2} - \mu_2), (x_{i2} - \mu_2)^2, i = 1, 2, \ldots, N$. If we do not know μ_1 and μ_2, we can estimate these using the sample averages, $\bar{x}_1$ and $\bar{x}_2$.

d. The following Stata session can be used to answer this part:

```
. sum educ exper

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        educ |       935    13.46845    2.196654          9         18
       exper |       935    11.56364    4.374586          1         23
```

```
. gen educ0exper0 = (educ - 13.47)*(exper - 11.56)

. gen exper0sq = (exper - 11.56)^2

. reg lwage educ exper  educ0exper0 exper0sq

      Source |       SS       df       MS              Number of obs =     935
-------------+------------------------------           F(  4,   930) =   36.94
       Model |  22.7093743     4  5.67734357            Prob > F      =  0.0000
    Residual |  142.946909   930  .153706354           R-squared     =  0.1371
-------------+------------------------------           Adj R-squared =  0.1334
       Total |  165.656283   934  .177362188           Root MSE      =  .39205

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .0837981   .0069787    12.01   0.000     .0701022     .097494
       exper |   .0223954   .0034481     6.49   0.000     .0156284    .0291624
 educ0exper0 |   .0045485   .0017652     2.58   0.010     .0010843    .0080127
    exper0sq |   .0009943    .000653     1.52   0.128    -.0002872    .0022758
       _cons |   5.392285   .1207342    44.66   0.000     5.155342    5.629228
------------------------------------------------------------------------------

. gen educexper = educ*exper

. gen expersq = exper^2

. reg lwage educ exper  educexper expersq

      Source |       SS       df       MS              Number of obs =     935
-------------+------------------------------           F(  4,   930) =   36.94
       Model |  22.7093743     4  5.67734357            Prob > F      =  0.0000
    Residual |  142.946909   930  .153706354           R-squared     =  0.1371
-------------+------------------------------           Adj R-squared =  0.1334
       Total |  165.656283   934  .177362188           Root MSE      =  .39205

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .0312176   .0193142     1.62   0.106    -.0066869    .0691221
       exper |  -.0618608   .0331851    -1.86   0.063    -.1269872    .0032656
   educexper |   .0045485   .0017652     2.58   0.010     .0010843    .0080127
     expersq |   .0009943    .000653     1.52   0.128    -.0002872    .0022758
       _cons |   6.233415   .3044512    20.47   0.000     5.635924    6.830906
------------------------------------------------------------------------------
```

In the equation where *educ* and *exper* are both demeaned before creating the interaction and the squared terms, the coefficients on *educ* and *exper* seem reasonable. For example, the coefficient on *educ* means that, at the average level of experience, the return to another year of education is about 8.4%. As experience increases above its average value, the return to

education also increases (by .45 percentage points for each year of experience above 11.56). In the model containing *educ* • *exper* and *exper*^{2}, the coefficient on *educ* is the return to education when *exper* = 0 – not an especially interesting segment of the population, and certainly not representative of the men in the sample.(Notice that the standard error of $\hat{\beta}_{educ}$ in the second regression is almost three times the standard error in the first regression. This difference illustrates that we can estimate the marginal effect at the average values of the covariates much more precisely than at extreme values of the covariates.) The coefficient on *exper* in the first regression is the return to another year of experience at the average values of both *educ* and *exper*. So, for a man with about 13.5 years of education and 11.6 years of experience, another year of experience is estimated to be worth about 2.2%. In the second regression, where *educ* and *exper* are not first demeaned, the coefficient on *exper* is the return to the first year of experience for a man with no schooling. This is not an interesting part of the U.S. population, and, in a sample where the lowest completed grade is ninth, we have no hope of estimating such an effect, anyway. The negative, large coefficient on *exper* in the second regression is puzzling only when we forget what it actually estimates. Note that the standard error on $\hat{\beta}_{exper}$ in the second regression is about 10 times as large as the standard error in the first regression.

4.9. a. Just subtract $\log(y_{-1})$ from both sides and define $\Delta\log(y) = \log(y) - \log(y_{-1})$:

$$\Delta\log(y) = \beta_0 + \mathbf{x}\boldsymbol{\beta} + (\alpha_1 - 1)\log(y_{-1}) + u.$$

Clearly, the intercept and slope estimates on **x** will be the same. The coefficient on $\log(y_{-1})$ becomes $\alpha_1 - 1$.

b. For simplicity, let $w = \log(y)$ and $w_{-1} = \log(y_{-1})$. Then the population slope coefficient in a simple regression is always $\alpha_1 = \text{Cov}(w_{-1}, w)/\text{Var}(w_{-1})$. By assumption,

$\text{Var}(w) = \text{Var}(w_{-1})$, which means we can write $\alpha_1 = \text{Cov}(w_{-1}, w)/(\sigma_{w_{-1}}\sigma_w)$, where $\sigma_{w_{-1}} = sd(w_{-1})$ and $\sigma_w = sd(w)$. But $\text{Corr}(w_{-1}, w) = \text{Cov}(w_{-1}, w)/(\sigma_{w_{-1}}\sigma_w)$, and since a correlation coefficient is always between −1 and 1, the result follows.

4.10. Write the linear projection of x_K^* onto the other explanatory variables as $x_K^* = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_{K-1} x_{K-1} + r_K^*$. Now, because $x_K = x_K^* + e_K$,

$$\begin{aligned} \text{L}(x_K|1, x_1, \ldots, x_{K-1}) &= \text{L}(x_K^*|1, x_1, \ldots, x_{K-1}) + \text{L}(e_K|1, x_1, \ldots, x_{K-1}) \\ &= \text{L}(x_K^*|1, x_1, \ldots, x_{K-1}) \end{aligned}$$

because e_K has zero mean and is uncorrelated with $x_1, \ldots, x_{K-1}$ [and so $\text{L}(e_K|1, x_1, \ldots, x_{K-1}) = 0$]. But the linear projection error r_K is

$$r_K \equiv x_K - \text{L}(x_K|1, x_1, \ldots, x_{K-1}) = [x_K^* - \text{L}(x_K^*|1, x_1, \ldots, x_{K-1})] + e_K = r_K^* + e_K.$$

Now we can use the two-step projection formula: the coefficient on x_K in $\text{L}(y|1, x_1, \ldots, x_K)$ is the coefficient in $\text{L}(y|r_K)$, say π_1. But

$$\pi_1 = \text{Cov}(r_K, y)/\text{Var}(r_K) = \beta_K \text{Cov}(r_K^*, x_K^*)/\text{Var}(r_K)$$

since e_K is uncorrelated with $x_1, \ldots, x_{K-1}, x_K^*$, and v by assumption and r_K^* is uncorrelated with $x_1, \ldots, x_{K-1}$, by definition. Now $\text{Cov}(r_K^*, x_K^*) = \text{Var}(r_K^*)$ and $\text{Var}(r_K) = \text{Var}(r_K^*) + \text{Var}(e_K)$ [because $\text{Cov}(r_K^*, e_K) = 0$]. Therefore π_1 is given by equation (4.47), which is what we wanted to show.

4.11. Here is some Stata output obtained to answer this question:

```
. reg lwage exper tenure married south urban black educ iq kww

      Source |       SS       df       MS              Number of obs =     935
-------------+------------------------------           F(  9,   925) =   37.28
       Model |  44.0967944     9  4.89964382            Prob > F      =  0.0000
    Residual |  121.559489   925  .131415664            R-squared     =  0.2662
-------------+------------------------------           Adj R-squared =  0.2591
       Total |  165.656283   934  .177362188            Root MSE      =  .36251

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
```

```
-------------+----------------------------------------------------------------
       exper |   .0127522   .0032308     3.95   0.000     .0064117    .0190927
      tenure |   .0109248   .0024457     4.47   0.000      .006125    .0157246
     married |   .1921449   .0389094     4.94   0.000     .1157839    .2685059
       south |  -.0820295   .0262222    -3.13   0.002    -.1334913   -.0305676
       urban |   .1758226   .0269095     6.53   0.000     .1230118    .2286334
       black |  -.1303995   .0399014    -3.27   0.001    -.2087073   -.0520917
        educ |   .0498375    .007262     6.86   0.000     .0355856    .0640893
          iq |   .0031183   .0010128     3.08   0.002     .0011306    .0051059
         kww |    .003826   .0018521     2.07   0.039     .0001911    .0074608
       _cons |   5.175644    .127776    40.51   0.000     4.924879    5.426408
------------------------------------------------------------------------------

. test iq kww

 ( 1)  iq = 0
 ( 2)  kww = 0

       F(  2,   925) =    8.59
            Prob > F =    0.0002
```

a. The estimated return to education using both *IQ* and *KWW* as proxies for ability is about 5%. When we used no proxy the estimated return was about 6.5%, and with only *IQ* as a proxy it was about 5.4%. Thus, we have an even lower estimated return to education, but it is still practically nontrivial and statistically very significant.

b. We can see from the *t* statistics that these variables are going to be jointly significant. The *F* test verifies this, with *p*-value = .0002.

c. The wage differential between nonblacks and blacks does not disappear. Blacks are estimated to earn about 13% less than nonblacks, holding other factors in the regression fixed.

d. Adding the interaction terms described in the problem gives the following results:

```
. sum iq kww

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          iq |       935    101.2824    15.05264         50        145
         kww |       935    35.74439    7.638788         12         56

. gen educiq0 = educ*(iq - 100)

. gen educkww0 = educ*(kww - 35.74)

. reg lwage exper tenure married south urban black educ iq kww educiq0 educkww0

      Source |       SS       df       MS              Number of obs =     935
```

```
-------------+------------------------------           F( 11,   923) =   31.48
       Model |  45.1916886    11  4.10833533           Prob > F      =  0.0000
    Residual |  120.464595   923  .130514187           R-squared     =  0.2728
-------------+------------------------------           Adj R-squared =  0.2641
       Total |  165.656283   934  .177362188           Root MSE      =  .36127

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       exper |   .0121544   .0032358     3.76   0.000      .005804    .0185047
      tenure |   .0107206   .0024383     4.40   0.000     .0059353     .015506
     married |   .1978269   .0388272     5.10   0.000     .1216271    .2740267
       south |  -.0807609   .0261374    -3.09   0.002    -.1320565   -.0294652
       urban |    .178431    .026871     6.64   0.000     .1256957    .2311664
       black |  -.1381481   .0399615    -3.46   0.001    -.2165741   -.0597221
        educ |   .0452316   .0076472     5.91   0.000     .0302235    .0602396
          iq |   .0048228   .0057333     0.84   0.400     -.006429    .0160745
         kww |  -.0248007   .0107382    -2.31   0.021    -.0458749   -.0037266
     educiq0 |  -.0001138   .0004228    -0.27   0.788    -.0009436    .0007161
    educkww0 |    .002161   .0007957     2.72   0.007     .0005994    .0037227
       _cons |   6.080005   .5610875    10.84   0.000     4.978849     7.18116
------------------------------------------------------------------------------

. test  educiq0 educkww0

 ( 1)  educiq0 = 0
 ( 2)  educkww0 = 0

       F(  2,   923) =    4.19
            Prob > F =    0.0154
```

The interaction *educkww0* is statistically significant, and the two interactions are jointly significant at the 2% signifiance level. The estimated return to education at the average values of *IQ* and *KWW* (in the population and sample, respectively) is somewhat smaller now: about 4.5%. Further, as *KWW* increases above its mean, the return to education increases. For example, if *KWW* is about one standard deviation (7.64) above its mean, the return to education is about $.045 + .0022(7.6) = .06172$, or about 6.2%. So "knowledge of the world of work" interacts positively with education levels.

4.12. Here is the Stata output when *union* is added to both equations:

```
. reg lscrap grant union if d88

      Source |       SS       df       MS              Number of obs =      54
-------------+------------------------------           F(  2,    51) =    1.16
       Model |  4.59902319     2  2.29951159           Prob > F      =  0.3204
    Residual |  100.763637    51  1.97575759           R-squared     =  0.0436
```

```
-------------+------------------------------           Adj R-squared =  0.0061
       Total |   105.36266    53  1.98797472           Root MSE      =  1.4056

------------------------------------------------------------------------------
      lscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       grant |  -.0276192   .4043649    -0.07   0.946    -.8394156    .7841772
       union |   .6222888   .4096347     1.52   0.135    -.2000873    1.444665
       _cons |   .2307292   .2648551     0.87   0.388    -.3009896     .762448
------------------------------------------------------------------------------

. reg lscrap grant union lscrap_1 if d88

      Source |       SS       df       MS              Number of obs =      54
-------------+------------------------------           F(  3,    50) =  122.33
       Model |  92.7289733     3  30.9096578           Prob > F      =  0.0000
    Residual |  12.6336868    50  .252673735           R-squared     =  0.8801
-------------+------------------------------           Adj R-squared =  0.8729
       Total |   105.36266    53  1.98797472           Root MSE      =  .50267

------------------------------------------------------------------------------
      lscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       grant |  -.2851103   .1452619    -1.96   0.055    -.5768775    .0066568
       union |   .2580653   .1477832     1.75   0.087    -.0387659    .5548965
    lscrap_1 |   .8210298    .043962    18.68   0.000     .7327295      .90933
       _cons |  -.0477754   .0958824    -0.50   0.620    -.2403608      .14481
------------------------------------------------------------------------------
```

The basic story does not change: initially, the grant is estimated to have essentially no effect, but adding $\log(scrap_{-1})$ gives the grant a strong effect that is marginally statistically significant. Interestingly, unionized firms are estimated to have larger scrap rates; over 25% more in the second equation. The effect is significant at the 10% level.

4.13. a. Using the 90 counties for 1987 gives

```
. reg lcrmrte lprbarr lprbconv lprbpris lavgsen if d87

      Source |       SS       df       MS              Number of obs =      90
-------------+------------------------------           F(  4,    85) =   15.15
       Model |  11.1549601     4  2.78874002           Prob > F      =  0.0000
    Residual |  15.6447379    85   .18405574           R-squared     =  0.4162
-------------+------------------------------           Adj R-squared =  0.3888
       Total |   26.799698    89  .301120202           Root MSE      =  .42902

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lprbarr |  -.7239696   .1153163    -6.28   0.000    -.9532493   -.4946898
    lprbconv |  -.4725112   .0831078    -5.69   0.000    -.6377519   -.3072706
    lprbpris |   .1596698   .2064441     0.77   0.441    -.2507964     .570136
     lavgsen |   .0764213   .1634732     0.47   0.641    -.2486073    .4014499
```

```
       _cons |  -4.867922   .4315307   -11.28   0.000    -5.725921   -4.009923
------------------------------------------------------------------------------
```

Because of the log-log functional form, all coefficients are elasticities. The elasticities of crime with respect to the arrest and conviction probabilities are the sign we expect, and both are practically and statistically significant. The elasticities with respect to the probability of serving a prison term and the average sentence length are positive but are statistically insignificant.

b. To add the previous year's crime rate we first generate the first lag of *lcrmrte*:

```
. xtset county year
       panel variable:  county (strongly balanced)
        time variable:  year, 81 to 87
                delta:  1 unit

. gen lcrmrte_1 = L.lcrmrte
(90 missing values generated)

. reg lcrmrte lprbarr lprbconv lprbpris lavgsen lcrmrte_1 if d87

      Source |       SS       df       MS              Number of obs =      90
-------------+------------------------------           F(  5,    84) =  113.90
       Model |  23.3549731     5  4.67099462           Prob > F      =  0.0000
    Residual |   3.4447249    84   .04100863           R-squared     =  0.8715
-------------+------------------------------           Adj R-squared =  0.8638
       Total |   26.799698    89  .301120202           Root MSE      =  .20251

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lprbarr |  -.1850424   .0627624    -2.95   0.004    -.3098523   -.0602325
    lprbconv |  -.0386768   .0465999    -0.83   0.409    -.1313457    .0539921
    lprbpris |  -.1266874   .0988505    -1.28   0.204    -.3232625    .0698876
     lavgsen |  -.1520228   .0782915    -1.94   0.056    -.3077141    .0036684
   lcrmrte_1 |   .7798129   .0452114    17.25   0.000     .6899051    .8697208
       _cons |  -.7666256   .3130986    -2.45   0.016    -1.389257   -.1439946
------------------------------------------------------------------------------
```

There are some notable changes in the coefficients on the original variables. The elasticities with respect to *prbarr* and *prbconv* are much smaller now, but still have signs predicted by a deterrent-effect story. The conviction probability is no longer statistically significant. Adding the lagged crime rate changes the signs of the elasticities with respect to *prbpris* and *avgsen*, and the latter is almost statistically significant at the 5% level against a

two-sided alternative (p-value = .056). Not surprisingly, the elasticity with respect to the lagged crime rate is large and very statistically significant. (The elasticity is also statistically less than unity.)

c. Adding the logs of the nine wage variables gives the following:

```
. reg lcrmrte lprbarr lprbconv lprbpris lavgsen lcrmrte_1 lwcon- lwloc if d87

      Source |       SS       df       MS              Number of obs =      90
-------------+------------------------------           F( 14,    75) =   43.81
       Model |  23.8798774    14  1.70570553           Prob > F      =  0.0000
    Residual |  2.91982063    75  .038930942           R-squared     =  0.8911
-------------+------------------------------           Adj R-squared =  0.8707
       Total |   26.799698    89  .301120202           Root MSE      =  .19731

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     lprbarr |  -.1725122   .0659533    -2.62   0.011    -.3038978   -.0411265
    lprbconv |  -.0683639    .049728    -1.37   0.173    -.1674273    .0306994
    lprbpris |  -.2155553   .1024014    -2.11   0.039    -.4195493   -.0115614
     lavgsen |  -.1960546   .0844647    -2.32   0.023     -.364317   -.0277923
   lcrmrte_1 |   .7453414   .0530331    14.05   0.000     .6396942    .8509887
       lwcon |  -.2850008   .1775178    -1.61   0.113    -.6386344    .0686327
       lwtuc |   .0641312    .134327     0.48   0.634    -.2034619    .3317244
       lwtrd |    .253707   .2317449     1.09   0.277    -.2079525    .7153665
       lwfir |  -.0835258   .1964974    -0.43   0.672    -.4749687    .3079171
       lwser |   .1127542   .0847427     1.33   0.187    -.0560619    .2815703
       lwmfg |   .0987371   .1186099     0.83   0.408    -.1375459    .3350201
       lwfed |   .3361278   .2453134     1.37   0.175    -.1525615    .8248172
       lwsta |   .0395089   .2072112     0.19   0.849    -.3732769    .4522947
       lwloc |  -.0369855   .3291546    -0.11   0.911    -.6926951    .6187241
       _cons |  -3.792525   1.957472    -1.94   0.056    -7.692009    .1069593
------------------------------------------------------------------------------

. testparm lwcon-lwloc

 ( 1)  lwcon = 0
 ( 2)  lwtuc = 0
 ( 3)  lwtrd = 0
 ( 4)  lwfir = 0
 ( 5)  lwser = 0
 ( 6)  lwmfg = 0
 ( 7)  lwfed = 0
 ( 8)  lwsta = 0
 ( 9)  lwloc = 0

       F(  9,    75) =    1.50
            Prob > F =    0.1643
```

The nine wage variables are jointly insignificant even at the 15% level. Plus, the elasticities

are not consistently positive or negative. The two largest elasticities – which also have the largest absolute *t* statistics – have the opposite sign. These are with respect to the wage in construction ($-.285$) and the wage for federal employees ($.336$).

d. The following Stata output gives the heteroskedasiticity-robust *F* statistic:

```
. qui reg lcrmrte lprbarr lprbconv lprbpris lavgsen lcrmrte_1 lwcon- lwloc if

. testparm lwcon-lwloc

 ( 1)  lwcon = 0
 ( 2)  lwtuc = 0
 ( 3)  lwtrd = 0
 ( 4)  lwfir = 0
 ( 5)  lwser = 0
 ( 6)  lwmfg = 0
 ( 7)  lwfed = 0
 ( 8)  lwsta = 0
 ( 9)  lwloc = 0

       F(  9,    75) =    2.19
            Prob > F =    0.0319
```

Therefore, we would reject the null at the 5% signifiance level. But we might hesitate to rely on asymptotic theory – which the heteroskedasticity-robust test requires – with $N = 90$ and $K = 15$ parameters to estimate. (This heteroskedasticity-robust *F* statistic is the heteroskedasticity-robust Wald statistic divided by the number of restrictions being tested, which is nine in this example. The division by the number of restrictions turns the asymptotic chi-square statistic into one can be treated as having roughly an *F* distribution.)

4.14. a. Before doing the regression, it is helpful to know some summary statistics for the variables of primary interest:

```
. sum stndfnl atndrte

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     stndfnl |       680    .0296589    .9894611  -3.308824   2.783613
     atndrte |       680    81.70956    17.04699       6.25        100
```

Because the final exam score has been standardized, it has close to a zero mean and its

standard deviation is close to one. The values are not closer to zero and one, respectively, because the standardization was done with a larger data set that included students with missing values on other key variables. It might make sense to redefine the standardized test score using the mean and standard deviation in the sample of 680, but the effect should be minor.

The regression that controls only for year in school in addition to attendance rate is as follows:

```
. reg stndfnl atndrte frosh soph

      Source |       SS       df       MS              Number of obs =     680
-------------+------------------------------           F(  3,   676) =    6.74
       Model |  19.3023776     3  6.43412588            Prob > F      =  0.0002
    Residual |   645.46119   676  .954824246            R-squared     =  0.0290
-------------+------------------------------           Adj R-squared =  0.0247
       Total |  664.763568   679  .979033237            Root MSE      =  .97715

------------------------------------------------------------------------------
     stndfnl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     atndrte |   .0081634   .0022031     3.71   0.000     .0038376    .0124892
       frosh |  -.2898943   .1157244    -2.51   0.012    -.5171168   -.0626719
        soph |  -.1184456   .0990267    -1.20   0.232    -.3128824    .0759913
       _cons |  -.5017308    .196314    -2.56   0.011    -.8871893   -.1162724
------------------------------------------------------------------------------
```

If *atndrte* increases by 10 percentage points (say, from 75 to 85), the standardized test score is estimated to increase by about .082 standard deviations.

b. Certainly there is a potential for self-selection. The better students may also be the ones attending lecture more regularly. So the positive effect of the attendance rate simply might capture the fact that better students tend to do better on exams. It is unlikely that controlling just for year in college (*frosh* and *soph*) solves the endogeneity of *atndrete*.

c. Adding *priGPA* and *ACT* gives

```
 reg stndfnl atndrte frosh soph priGPA ACT

      Source |       SS       df       MS              Number of obs =     680
-------------+------------------------------           F(  5,   674) =   34.93
       Model |  136.801957     5  27.3603913            Prob > F      =  0.0000
    Residual |  527.961611   674  .783325833            R-squared     =  0.2058
-------------+------------------------------           Adj R-squared =  0.1999
```

```
       Total |  664.763568   679  .979033237           Root MSE      =  .88506

------------------------------------------------------------------------------
     stndfnl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     atndrte |   .0052248   .0023844     2.19   0.029      .000543    .0099065
       frosh |  -.0494692   .1078903    -0.46   0.647    -.2613108    .1623724
        soph |  -.1596475   .0897716    -1.78   0.076    -.3359132    .0166181
      priGPA |   .4265845   .0819203     5.21   0.000     .2657348    .5874343
         ACT |   .0844119   .0111677     7.56   0.000     .0624843    .1063395
       _cons |  -3.297342    .308831   -10.68   0.000    -3.903729   -2.690956
------------------------------------------------------------------------------
```

The effect of *atndrte* has fallen, which is what we expect if we think better, smarter students also attend lectures more frequently. The estimate now is that a 10 percentage point increase in *atndrte* increases the standardized test score by .052 standard deviations; the effect is statistically significant at the usual 5% level against a two-sided alternative, but the *t* statistic is much lower than in part a. The strong positive effects of prior GPA and ACT score are also expected.

d. Controlling for *priGPA* and *ACT* causes the sophomore effect (relative to students in year three and beyond) to get slightly larger in magnitude and more statistically significant. These data are for a course taught in the second term, so each *frosh* student does have a prior GPA – his or her GPA for the first semester in college. Adding *priGPA* in particular causes the "freshman effect" to essentially disappear. This is not too surprising because the average prior GPA for first-year students is notably less than the overall average *priGPA*.

e. Here is the Stata session for adding squares in the proxy variables. Because we are not interested in the effects of the proxies, we do not demean them before creating the squared terms:

```
. gen priGPAsq = priGPA^2

. gen ACTsq = ACT^2

. reg stndfnl atndrte frosh soph priGPA ACT  priGPAsq ACTsq

      Source |       SS       df       MS              Number of obs =     680
-------------+------------------------------           F(  7,   672) =   28.94
```

```
      Model |  153.974309     7  21.9963299           Prob > F      =  0.0000
   Residual |  510.789259   672  .760103064           R-squared     =  0.2316
-------------+------------------------------           Adj R-squared =  0.2236
      Total |  664.763568   679  .979033237           Root MSE      =  .87184

------------------------------------------------------------------------------
     stndfnl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     atndrte |   .0062317   .0023583     2.64   0.008     .0016011    .0108623
       frosh |  -.1053368   .1069747    -0.98   0.325    -.3153817    .1047081
        soph |  -.1807289   .0886354    -2.04   0.042    -.3547647   -.0066932
      priGPA |   -1.52614   .4739715    -3.22   0.001    -2.456783   -.5954966
         ACT |  -.1124331    .098172    -1.15   0.253    -.3051938    .0803276
    priGPAsq |   .3682176   .0889847     4.14   0.000     .1934961    .5429391
       ACTsq |   .0041821   .0021689     1.93   0.054    -.0000766    .0084408
       _cons |   1.384812   1.239361     1.12   0.264    -1.048674    3.818298
------------------------------------------------------------------------------
```

Adding the squared terms – one of which is very significant, the other of which is marginally significant – actually increases the attendance rate effect. And it does so while slightly reducing the standard error on *atndrte*, resulting in a *t* statistic that is notably more significant than in part c.

f. Adding the squared attendance rate is not warranted, as it is very insignificant:

```
. gen atndrtesq = atndrte^2

. reg stndfnl atndrte frosh soph priGPA ACT  priGPAsq ACTsq  atndrtesq

      Source |       SS       df       MS              Number of obs =     680
-------------+------------------------------           F(  8,   671) =   25.28
       Model |  153.975323     8  19.2469154           Prob > F      =  0.0000
    Residual |  510.788245   671  .761234344           R-squared     =  0.2316
-------------+------------------------------           Adj R-squared =  0.2225
       Total |  664.763568   679  .979033237           Root MSE      =  .87249

------------------------------------------------------------------------------
     stndfnl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     atndrte |   .0058425   .0109203     0.54   0.593    -.0155996    .0272847
       frosh |  -.1053656   .1070572    -0.98   0.325    -.3155729    .1048418
        soph |  -.1808403   .0887539    -2.04   0.042     -.355109   -.0065716
      priGPA |  -1.524803    .475737    -3.21   0.001    -2.458915   -.5906902
         ACT |  -.1123423   .0982764    -1.14   0.253    -.3053087     .080624
    priGPAsq |   .3679124   .0894427     4.11   0.000      .192291    .5435337
       ACTsq |   .0041802   .0021712     1.93   0.055    -.0000829    .0084433
    atndrtesq |   2.87e-06   .0000787     0.04   0.971    -.0001517    .0001574
       _cons |   1.394292   1.267186     1.10   0.272    -1.093835     3.88242
------------------------------------------------------------------------------
```

The very large increase in the standard error on *atndrte* suggest that *atndrte* and $atndrte^2$

are highly collinear. In fact, their sample correlation is about .983. Importantly, the coefficient on *atndrte* now has an uninteresting interpretation: it measures the partial effect of *atndrte* starting from *atndrte* = 0. The lowest attendance rate in the sample is 6.25, with the vast majority of students (94.3%) attending 50 percent or more of the lectures. If the quadratic term were significant, we might want to center *atndrte* about its mean or median before creating the square. Or, a more sophisticated functional form might be called for. It may be better to define several intervals for *atndrrte* and include dummy variables for those intervals.

4.15. a. Because each x_j has finite second moment, $\text{Var}(\mathbf{x}\boldsymbol{\beta}) < \infty$. Since $\text{Var}(u) < \infty$, $\text{Cov}(\mathbf{x}\boldsymbol{\beta}, u)$ is well–defined. But each x_j is uncorrelated with u, so $\text{Cov}(\mathbf{x}\boldsymbol{\beta}, u) = 0$. Therefore, $\text{Var}(y) = \text{Var}(\mathbf{x}\boldsymbol{\beta}) + \text{Var}(u)$ or $\sigma_y^2 = \text{Var}(\mathbf{x}\boldsymbol{\beta}) + \sigma_u^2$.

b. This is nonsense when we view $\mathbf{x}_i$ as a random draw along with y_i. The statement "$\text{Var}(u_i) = \sigma^2 = \text{Var}(y_i)$ for all i" assumes that the regressors are nonrandom (or $\boldsymbol{\beta} = \mathbf{0}$, which is not a very interesting case). This is another example of how the assumption of nonrandom regressors can lead to counterintuitive conclusions. Suppose that an element of the error term, say z, which is uncorrelated with each x_j, suddenly becomes observed. When we add z to the regressor list, the error changes, and so does the error variance. In the vast majority of economic applications, it makes no sense to think we have access to the entire set of factors that one would ever want to control for, and so we should allow for error variances to change across different sets of explanatory variables that we might use for the same response variable. We avoid trouble by focusing on joint distributions in the population.

c. Write $R^2 = 1 - \text{SSR/SST} = 1 - (\text{SSR}/N)/(\text{SST}/N)$. Therefore,

$\text{plim}(R^2) = 1 - \text{plim}[(\text{SSR}/N)/(\text{SST}/N)] = 1 - [\text{plim}(\text{SSR}/N)]/[\text{plim}(\text{SST}/N)] = 1 - \sigma_u^2/\sigma_y^2 =$

where we use the fact that SSR/N is a consistent estimator of $\boldsymbol{\sigma}_u^2$ and SST/N is a consistent

estimator of σ_y^2.

d. The derivation in part c assumed nothing about Var(u|**x**). The population R-squared depends on only the *unconditional* variances of u and y. Therefore, regardless of the nature of heteroskedasticity in Var(u|**x**), the usual R–squared consistently estimates the population R-squared. Neither R-squared nor the adjusted R-squared has desirable finite-sample properties, such as unbiasedness, so the only analysis we can given in any generality involves asymptotics. The statement in the problem is simply wrong.

4.16. a. The proof is fairly similar to that for random sampling. First, note that the assumptions $N^{-1}\sum_{i=1}^{N}[\mathbf{x}_i'\mathbf{x}_i - \mathrm{E}(\mathbf{x}_i'\mathbf{x}_i)] \xrightarrow{p} \mathbf{0}$ – which is how the WLLN is stated for i.n.i.d. sequences – and $N^{-1}\sum_{i=1}^{N}\mathrm{E}(\mathbf{x}_i'\mathbf{x}_i) \to \mathbf{A}$ – which is not crucial but is pretty harmless and simplifies the proof – imply

$$N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i \xrightarrow{p} \mathbf{A}$$

In addition, $\mathrm{E}(\mathbf{x}_i'u_i) = \mathbf{0}$ and the assumption that $N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'u_i$ satisfies the law of large numbers imply

$$N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'u_i \xrightarrow{p} \mathbf{0}.$$

We are also given that **A** is positive definite, which means $\mathbf{X}'\mathbf{X}/N$ is invertible with probability approaching one and $(\mathbf{X}'\mathbf{X}/N)^{-1} \xrightarrow{p} \mathbf{A}^{-1}$. Therefore,

$$\begin{aligned}
\text{plim}_{N\to\infty}(\hat{\boldsymbol{\beta}}) &= \boldsymbol{\beta} + \text{plim}\left[\left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right] \\
&= \boldsymbol{\beta} + \text{plim}\left[\left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\right]\text{plim}\left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right) \\
&= \boldsymbol{\beta} + \mathbf{A}^{-1}\cdot\mathbf{0} = \boldsymbol{\beta}.
\end{aligned}$$

b. Because $N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i \xrightarrow{d} \text{Normal}(\mathbf{0},\mathbf{B})$, the sequence $N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i$ is $O_p(1)$. We already used in part a that

$$\left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} - \mathbf{A}^{-1} = o_p(1)$$

Now, as in the i.i.d. case, write

$$\begin{aligned}
\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) &= \left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right) \\
&= \left[\left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} - \mathbf{A}^{-1}\right]\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right) + \mathbf{A}^{-1}\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right) \\
&= o_p(1)\cdot O_p(1) + \mathbf{A}^{-1}\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right) \\
&\xrightarrow{d} \text{Normal}(\mathbf{0},\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})
\end{aligned}$$

where we use the assumption $N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i \xrightarrow{d} \text{Normal}(\mathbf{0},\mathbf{B})$. The asymptotic variance of $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ has the usual sandwich form, $\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$.

c. We already know that

$$N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i \xrightarrow{p} \mathbf{A}.$$

Further, by the WLLN and the assumption that $\mathbf{B}_N \to \mathbf{B}$,

$$N^{-1}\sum_{i=1}^{N} u_i^2\mathbf{x}_i'\mathbf{x}_i \xrightarrow{p} \mathbf{B}$$

The hard part – just as with the i.i.d. case – is to show that replacing the u_i with the OLS residuals, $\hat{u}_i$, does not affect consistency. Nevertheless, under general assumptions it follows that

$$N^{-1}\sum_{i=1}^{N} \hat{u}_i^2\mathbf{x}_i'\mathbf{x}_i \xrightarrow{p} \mathbf{B}$$

Naturally, we can use the same degrees-of-freedom adjustment as in the i.i.d. case: replace N^{-1} with $(N-K)^{-1}$.

d. The point of this exercise is that we are led to exactly the same heteroskedasticity-robust estimator whether we assume i.i.d. observations or i.n.i.d. observations. In particular, even if unconditional variances are constant – as they must be in the i.i.d. case – we still might need heteroskedasticity-robust standard errors. In the i.n.i.d. case, the robust variance matrix estimator allows for changing unconditional variances as well as conditional variances that depend on $\mathbf{x}_i$.

4.17. We know that, in general,

$$\text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = [\text{E}(\mathbf{x}'\mathbf{x})]^{-1}\text{E}(u^2\mathbf{x}'\mathbf{x})[\text{E}(\mathbf{x}'\mathbf{x})]^{-1}.$$

Now we just apply iterated expecations to the matrix in the middle:

$$\text{E}(u^2\mathbf{x}'\mathbf{x}) = \text{E}[\text{E}(u^2\mathbf{x}'\mathbf{x}|\mathbf{x})] = \text{E}[\text{E}(u^2|\mathbf{x})\mathbf{x}'\mathbf{x}] = \text{E}[h(\mathbf{x})\mathbf{x}'\mathbf{x}]$$

4.18. a. This is a fairly common misconception – or at least misstatement. Recall that the distribution of any random draw, u_i, is the population distribution of u. But, of course, the population distribution of u is what it is; it does not change with the sample size. In fact, it has

nothing to do with the sample size. Therefore, the random draws on u_i have the same distribution regardless of N. A correct statement is that the standardized average of the errors, $N^{-1/2}\sum_{i=1}^{N} u_i = \sqrt{N}\,\bar{u}$ approaches normality as $N \to \infty$. This is a much different statement. (In regression analysis, we use the fact that $N^{-1/2}\sum_{i=1}^{N} \mathbf{x}_i' u_i$ generally converges to a multivariate normal distribution, which implies the convergence of $N^{-1/2}\sum_{i=1}^{N} u_i$ to normality when $\mathbf{x}_i$ contains unity.)

b. It is tempting but incorrect to think that a single squared OLS residual can consistently estimate a conditional mean, $\mathrm{E}(u_i^2|\mathbf{x}_i) \equiv h(\mathbf{x}_i)$, but there is no sense in which this statement is true. It is not even clear what we would mean by it, but we can make some headway by writing $\hat{u}_i^2 = u_i^2 - 2u_i\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + [\mathbf{x}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]^2$. Now, we can conclude $\hat{u}_i^2 - u_i^2 \xrightarrow{p} 0$ and $N \to \infty$ because $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$. But remember $u_i^2 = h(\mathbf{x}_i) + v_i$ where $\mathrm{E}(v_i|\mathbf{x}_i) = 0$. There is no sense in which u_i^2 is a consistent estimator of $h(\mathbf{x}_i)$; they do not even depend on the sample size N.

It was the view that we needed $\hat{u}_i^2$ to be a good estimate of $\mathrm{E}(u_i^2|\mathbf{x}_i)$ that possibly held up progress on heteroskedasticity-consistent covariance matrices. Fortunately, all we need to consistent estmate is the population mean

$$\mathbf{B} = \mathrm{E}(u^2\mathbf{x}'\mathbf{x}),$$

for which the obvious consistent (and unbiased) estimator is

$$N^{-1}\sum_{i=1}^{N} u_i^2\mathbf{x}_i'\mathbf{x}_i.$$

The rest is demonstrating the replacing the implicit $\boldsymbol{\beta}$ with $\hat{\boldsymbol{\beta}}$ preserves consistency (not unbiasedness). As we know, this requires some tricky algebra with $o_p(1)$ and $O_p(1)$, but the work is not too onerous.

Solutions to Chapter 5 Problems

5.1. Define $\mathbf{x}_1 \equiv (\mathbf{z}_1, y_2)$ and $x_2 \equiv \hat{v}_2$, and let $\hat{\boldsymbol{\beta}} \equiv (\hat{\boldsymbol{\beta}}_1', \hat{\rho}_1)'$ be OLS estimator from (5.52), where $\hat{\boldsymbol{\beta}}_1 = (\hat{\boldsymbol{\delta}}_1', \hat{\alpha}_1)$. Using the hint, $\hat{\beta}_1$can also be obtained by partitioned regression:

(i) Regress $\mathbf{x}_1$ onto $\hat{v}_2$ and save the residuals, say $\ddot{\mathbf{x}}_1$.
(ii) Regress y_1 onto $\ddot{\mathbf{x}}_1$.

But when we regress $\mathbf{z}_1$onto $\hat{v}_2$ the residuals are just $\mathbf{z}_1$ because $\hat{v}_2$ is orthogonal in sample to $\mathbf{z}$. (More precisely, $\sum_{i=1}^{N} \mathbf{z}_{i1}'\hat{v}_{i2} = \mathbf{0}$.) Further, because we can write $y_2 = \hat{y}_2 + \hat{v}_2$, where $\hat{y}_2$ and $\hat{v}_2$ are orthogonal in sample, the residuals from regressing y_2 onto $\hat{v}_2$ are simply the first stage fitted values, $\hat{y}_2$. In other words, $\ddot{\mathbf{x}}_1 = (\mathbf{z}_1, \hat{y}_2)$. But the 2SLS estimator of $\boldsymbol{\beta}_1$ is obtained exactly from the OLS regression y_1 on $\mathbf{z}_1, \hat{y}_2$.

5.2. a. Unobserved factors that tend to make an individual healthier also tend to make that person exercise more. For example, if *health* is a cardiovascular measure, people with a history of heart problems are probably less likely to exercise. Unobserved factors such as prior health or family history are contained in u_1, and so we are worried about correlation between *exercise* and u_1. Self-selection into exercising predicts that the benefits of exercising will be, on average, overestimated. Ideally, the amount of exercise could be randomized across a sample of people, but this can be difficult.

b. If people do *not* systematically choose the location of their homes and jobs relative to health clubs based on unobserved health characteristics, then it is reasonable to believe that *disthome* and *distwork* are uncorrelated with u_1. But the location of health clubs is not necessarily exogenous. Clubs may tend to be built near neighborhoods where residents have higher income and wealth, on average, and these factors can certainly affect overall health. It

may make sense to choose residents from neighborhoods with very similar characteristics but where one neighborhood is located near a health club.

c. The reduced form for *exercise* is

$$exercise = \pi_0 + \pi_1 age + \pi_2 weight + \pi_3 height + \pi_4 male + \pi_5 work + \pi_6 disthome + \pi_7 distwork + u_1,$$

For identification we need at least one of π_6 and π_7 to be different from zero. this assumption can fail if the amount that people exercise is not systematically related to distances to the nearest health club.

d. An F test of $H_0 : \pi_6 = 0, \pi_7 = 0$ is the simplest way to test the identification assumption in part c. As usual, it would be a good idea to compute a heteroskedasticity-robust version.

5.3. a. There may be unobserved health factors correlated with smoking behavior that affect infant birth weight. For example, women who smoke during pregnancy may, on average, drink more coffee or alcohol, or eat less nutritious meals.

b. Basic economics says that *packs* should be negatively correlated with cigarette price, although the correlation might be small (especially because price is aggregated at the state level). At first glance it seems that cigarette price should be exogenous in equation (5.54), but we must be a little careful. One component of cigarette price is the state tax on cigarettes. States that have lower taxes on cigarettes may also have lower quality of health care, on average. Quality of health care is in *u*, and so maybe cigarette price fails the exogeneity requirement for an IV.

c. OLS is followed by 2SLS (IV, in this case):

```
. reg lbwght male parity lfaminc packs

      Source |       SS       df       MS              Number of obs =    1388
-------------+------------------------------           F(  4,  1383) =   12.55
       Model |  1.76664363     4  .441660908            Prob > F      =  0.0000
```

```
    Residual |    48.65369  1383  .035179819           R-squared     =  0.0350
-------------+------------------------------           Adj R-squared =  0.0322
       Total |  50.4203336  1387  .036352079           Root MSE      =  .18756

------------------------------------------------------------------------------
      lbwght |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        male |   .0262407   .0100894     2.60   0.009     .0064486    .0460328
      parity |   .0147292   .0056646     2.60   0.009     .0036171    .0258414
     lfaminc |   .0180498   .0055837     3.23   0.001     .0070964    .0290032
       packs |  -.0837281   .0171209    -4.89   0.000    -.1173139   -.0501423
       _cons |   4.675618   .0218813   213.68   0.000     4.632694    4.718542
------------------------------------------------------------------------------

. ivreg lbwght male parity lfaminc (packs = cigprice)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =    1388
-------------+------------------------------           F(  4,  1383) =    2.39
       Model |  -91.350027     4 -22.8375067           Prob > F      =  0.0490
    Residual |  141.770361  1383  .102509299           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  50.4203336  1387  .036352079           Root MSE      =  .32017

------------------------------------------------------------------------------
      lbwght |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       packs |   .7971063   1.086275     0.73   0.463    -1.333819    2.928031
        male |   .0298205    .017779     1.68   0.094    -.0050562    .0646972
      parity |  -.0012391   .0219322    -0.06   0.955     -.044263    .0417848
     lfaminc |    .063646   .0570128     1.12   0.264    -.0481949    .1754869
       _cons |   4.467861   .2588289    17.26   0.000     3.960122    4.975601
------------------------------------------------------------------------------
Instrumented:  packs
Instruments:   male parity lfaminc cigprice
------------------------------------------------------------------------------
```

The difference between OLS and IV in the estimated effect of *packs* on *bwght* is huge. With the OLS estimate, one more pack of cigarettes is estimated to reduce *bwght* by about 8.4%, and is statistically significant. The IV estimate has the opposite sign, is huge in magnitude, and is not statistically significant. The sign and size of the smoking effect are not realistic.

d. We can see the problem with IV by estimating the reduced form for *packs*.

```
. reg packs male parity lfaminc cigprice

      Source |       SS       df       MS              Number of obs =    1388
-------------+------------------------------           F(  4,  1383) =   10.86
```

```
       Model |  3.76705108     4   .94176277           Prob > F      =  0.0000
    Residual |  119.929078  1383  .086716615           R-squared     =  0.0305
-------------+------------------------------           Adj R-squared =  0.0276
       Total |  123.696129  1387  .089182501           Root MSE      =  .29448

------------------------------------------------------------------------------
       packs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        male |  -.0047261   .0158539    -0.30   0.766    -.0358264    .0263742
      parity |   .0181491   .0088802     2.04   0.041     .0007291    .0355692
     lfaminc |  -.0526374   .0086991    -6.05   0.000    -.0697023   -.0355724
    cigprice |    .000777   .0007763     1.00   0.317    -.0007459    .0022999
       _cons |   .1374075   .1040005     1.32   0.187    -.0666084    .3414234
------------------------------------------------------------------------------
```

The reduced form estimates show that *cigprice* does not significantly affect *packs*. In fact, the coefficient on *cigprice* does not have the sign we expect. Thus, *cigprice* fails as an IV for *packs* because *cigprice* is not partially correlated with *packs* with a sensible sign for the correlation. This is separate from the problem that *cigprice* may not truly be exogenous in the birth weight equation.

5.4. a. Here are the OLS results:

```
. reg lwage educ exper expersq black south smsa reg661-reg668 smsa66

      Source |       SS       df       MS              Number of obs =    3010
-------------+------------------------------           F( 15,  2994) =   85.48
       Model |  177.695591    15  11.8463727           Prob > F      =  0.0000
    Residual |  414.946054  2994  .138592536           R-squared     =  0.2998
-------------+------------------------------           Adj R-squared =  0.2963
       Total |  592.641645  3009  .196956346           Root MSE      =  .37228

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |   .0746933   .0034983    21.35   0.000     .0678339    .0815527
       exper |    .084832   .0066242    12.81   0.000     .0718435    .0978205
     expersq |   -.002287   .0003166    -7.22   0.000    -.0029079   -.0016662
       black |  -.1990123   .0182483   -10.91   0.000    -.2347927   -.1632318
       south |   -.147955   .0259799    -5.69   0.000    -.1988952   -.0970148
        smsa |   .1363845   .0201005     6.79   0.000     .0969724    .1757967
      reg661 |  -.1185698   .0388301    -3.05   0.002     -.194706   -.0424335
      reg662 |  -.0222026   .0282575    -0.79   0.432    -.0776088    .0332036
      reg663 |   .0259703   .0273644     0.95   0.343    -.0276846    .0796251
      reg664 |  -.0634942   .0356803    -1.78   0.075    -.1334546    .0064662
      reg665 |   .0094551   .0361174     0.26   0.794    -.0613623    .0802725
      reg666 |   .0219476   .0400984     0.55   0.584    -.0566755    .1005708
      reg667 |  -.0005887   .0393793    -0.01   0.988     -.077802    .0766245
      reg668 |  -.1750058   .0463394    -3.78   0.000     -.265866   -.0841456
      smsa66 |   .0262417   .0194477     1.35   0.177    -.0118905    .0643739
       _cons |   4.739377   .0715282    66.26   0.000     4.599127    4.879626
```

```
------------------------------------------------------------------------------
```

The estimated return to education is about 7.5%, with a very large *t* statistic. These reproduce the estimates from Table 2, Column (2) in Card (1995).

b. The reduced form for *educ* is

```
. reg educ exper expersq black south smsa reg661-reg668 smsa66 nearc4

      Source |       SS       df       MS              Number of obs =    3010
-------------+------------------------------           F( 15,  2994) =  182.13
       Model |  10287.6179    15  685.841194           Prob > F      =  0.0000
    Residual |  11274.4622  2994  3.76568542           R-squared     =  0.4771
-------------+------------------------------           Adj R-squared =  0.4745
       Total |  21562.0801  3009  7.16586243           Root MSE      =  1.9405

------------------------------------------------------------------------------
        educ |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       exper |  -.4125334   .0336996   -12.24   0.000    -.4786101   -.3464566
     expersq |   .0008686   .0016504     0.53   0.599    -.0023674    .0041046
       black |  -.9355287   .0937348    -9.98   0.000     -1.11932   -.7517377
       south |  -.0516126   .1354284    -0.38   0.703    -.3171548    .2139296
        smsa |   .4021825   .1048112     3.84   0.000     .1966732    .6076918
      reg661 |   -.210271   .2024568    -1.04   0.299    -.6072395    .1866975
      reg662 |  -.2889073   .1473395    -1.96   0.050    -.5778042   -.0000105
      reg663 |  -.2382099   .1426357    -1.67   0.095    -.5178838    .0414639
      reg664 |   -.093089   .1859827    -0.50   0.617    -.4577559    .2715779
      reg665 |  -.4828875   .1881872    -2.57   0.010    -.8518767   -.1138982
      reg666 |  -.5130857   .2096352    -2.45   0.014    -.9241293   -.1020421
      reg667 |  -.4270887   .2056208    -2.08   0.038    -.8302611   -.0239163
      reg668 |   .3136204   .2416739     1.30   0.194    -.1602434    .7874841
      smsa66 |   .0254805   .1057692     0.24   0.810    -.1819071    .2328682
      nearc4 |   .3198989   .0878638     3.64   0.000     .1476194    .4921785
       _cons |   16.84852   .2111222    79.80   0.000     16.43456    17.26248
------------------------------------------------------------------------------
```

The important coefficient is on *nearc*4. Statistically, *educ* and *nearc*4 are partially correlated, and in a way that makes sense: holding other factors in the reduced form fixed, someone living near a four-year college at age 16 has, on average, almost one-third a year more education than a person not near a four-year college at age 16. This is not trivial a effect, so *nearc*4 passes the requirement that it is partially correlated with *educ*.

c. Here are the IV estimates:

```
. ivreg lwage exper expersq black south smsa reg661-reg668 smsa66 (educ = nearc4
```

```
Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =    3010
-------------+------------------------------           F( 15,  2994) =   51.01
       Model |  141.146813    15  9.40978752           Prob > F      =  0.0000
    Residual |  451.494832  2994  .150799877           R-squared     =  0.2382
-------------+------------------------------           Adj R-squared =  0.2343
       Total |  592.641645  3009  .196956346           Root MSE      =  .38833

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |   .1315038   .0549637     2.39   0.017     .0237335    .2392742
       exper |   .1082711   .0236586     4.58   0.000     .0618824    .1546598
     expersq |  -.0023349   .0003335    -7.00   0.000    -.0029888    -.001681
       black |  -.1467757   .0538999    -2.72   0.007    -.2524603   -.0410912
       south |  -.1446715   .0272846    -5.30   0.000      -.19817    -.091173
        smsa |   .1118083    .031662     3.53   0.000     .0497269    .1738898
      reg661 |  -.1078142   .0418137    -2.58   0.010    -.1898007   -.0258278
      reg662 |  -.0070465   .0329073    -0.21   0.830    -.0715696    .0574767
      reg663 |   .0404445   .0317806     1.27   0.203    -.0218694    .1027585
      reg664 |  -.0579172   .0376059    -1.54   0.124    -.1316532    .0158189
      reg665 |   .0384577   .0469387     0.82   0.413    -.0535777     .130493
      reg666 |   .0550887   .0526597     1.05   0.296    -.0481642    .1583416
      reg667 |    .026758   .0488287     0.55   0.584    -.0689832    .1224992
      reg668 |  -.1908912   .0507113    -3.76   0.000    -.2903238   -.0914586
      smsa66 |   .0185311   .0216086     0.86   0.391    -.0238381    .0609003
       _cons |   3.773965    .934947     4.04   0.000     1.940762    5.607169
------------------------------------------------------------------------------
Instrumented:  educ
Instruments:   exper expersq black south smsa reg661 reg662 reg663 reg664
               reg665 reg666 reg667 reg668 smsa66 nearc4
------------------------------------------------------------------------------
```

The estimated return to education has increased to about 13.2%, but notice how wide the 95% confidence interval is: 2.4% to 23.9%. By contrast, the OLS confidence interval is about 6.8% to 8.2%, which is much tighter. Of course, OLS could be inconsistent, in which case a tighter CI is of little value. But the estimated return to education is higher with IV, something that seems a bit counterintuitive.

One possible explanation is that *educ* suffers from classical errors-in-variables. Therefore, while OLS would tend to overestimate the return to schooling because of omitted “ability,” classical measurement error in *educ* leads to an attenuation bias. Measurement error may help explain why the IV estimate is larger, but it is not entirely convincing. It seems unlikely that *educ* satisfies the CEV assumptions. For example, if we think the measurement error is due to

truncation – people are asked about highest grade completed, not actual years of schooling – then *educ* is always less than or equal to $educ^*$. And the measurement error could not be independent of $educ^*$. If we think the mismeasurement is due to is unobserved quality of schooling, it seems likely that quality of schooling – part of the measurement error – is positively correlated with actual amount of schooling. This, too, violates the CEV assumptions. Another possibility for the much higher IV estimate comes out of the recent treatment effect literature, which is covered in Section 21.4. Of course, we must also remember that the point estimates – particularly the IV estimate – are subject to substantial sampling variation. At this point, we do not even know of OLS and IV are statistically different from each other. See Problem 6.1.

d. When *nearc*2 is added to the reduced form of *educ* it has a coefficient (standard error) of .123 (.077), compared with .321 (.089) for *nearc*4. Therefore, *nearc*4 has a much stronger ceteris paribus relationship with *educ*; *nearc*2 is only marginally statistically significant once *nearc*4 has been included. The joint F test gives $F = 7.89$ with p-value $= .004$.

The 2SLS estimate of the return to education becomes about 15.7%, with 95% CI given by 5.4% to 26%. The CI is still very wide.

5.5. Under the null hypothesis that $q$and $\mathbf{z}_2$ are uncorrelated, $\mathbf{z}_1$ and $\mathbf{z}_2$ are exogenous in (5.55) because each is uncorrelated with u_1. Unfortunately, y_2 is correlated with u_1, and so the regression of y_1 on $\mathbf{z}_1, y_2, \mathbf{z}_2$ does not produce a consistent estimator of $\mathbf{0}$ on $\mathbf{z}_2$ even when $\mathrm{E}(\mathbf{z}_2' q) = \mathbf{0}$. We could find that $\hat{\boldsymbol{\psi}}_1$ from this regression is statistically different from zero even when q and $\mathbf{z}_2$ are uncorrelated – in which case we would incorrectly conclude that $\mathbf{z}_2$ is not a valid IV candidate. Or, we might fail to reject $H_0 : \boldsymbol{\psi}_1 = \mathbf{0}$ when $\mathbf{z}_2$and q are correlated – in

which case we incorrectly conclude that the elements in $\mathbf{z}_2$ are valid as instruments.

The point of this exercise is that one cannot simply add instrumental variable candidates in the structural equation and then test for significance of these variables using OLS estimation. This is the sense in which identification cannot be tested: we cannot test whether all of the IV candidates are uncorrelated with q. With a single endogenous variable, we must take a stand that at least one element of $\mathbf{z}_2$ is uncorrelated with q.

5.6. a. By definition, the reduced form is the linear projection

$$\mathrm{L}(q_1|1, \mathbf{x}, q_2) = \pi_0 + \mathbf{x}\boldsymbol{\pi}_1 + \pi_2 q_2,$$

and we want to show that $\boldsymbol{\pi}_1 = \mathbf{0}$ when q_2 is uncorrelated with $\mathbf{x}$. Now, because q_2 is a linear function of q and a_2 , and a_2 is uncorrelated with $\mathbf{x}$, q_2 is uncorrelated with $\mathbf{x}$ if and only if q is uncorrelated with $\mathbf{x}$. Assuming then that q and $\mathbf{x}$ are uncorrelated, q_1 is also uncorrelated with $\mathbf{x}$. A basic fact about linear projections is that, because q_1 and q_2 are each uncorrelated with the vector $\mathbf{x}$, $\boldsymbol{\pi}_1 = \mathbf{0}$. This claim follows from Property LP.7: $\boldsymbol{\pi}_1$ can be obtained by first projecting $\mathbf{x}$ on 1, q_2 and obtaining the population residuals, say $\mathbf{r}$. Then, project q_1 onto r. But because $\mathbf{x}$ and q_2 are orthogonal, $\mathbf{r} = \mathbf{x} - \boldsymbol{\mu}_\mathbf{x}$. Projecting q_1 on $(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})$ just gives the zero vector because $\mathrm{E}[(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})' q_1] = \mathbf{0}$. Therefore, $\boldsymbol{\pi}_1 = \mathbf{0}$.

b. If q_2 and $\mathbf{x}$ are correlated then $\boldsymbol{\pi}_1 \neq \mathbf{0}$, and $\mathbf{x}$ appears in the reduced form for q_1. It is not realistic to assume that q_2 and $\mathbf{x}$ are uncorrelated. Under the multiple indicator assumptions, assuming $\mathbf{x}$ and q_2 are uncorrelated is the same as assuming q and $\mathbf{x}$ are uncorrelated. If we believe q and $\mathbf{x}$ are uncorrelated then there is no need to collect indicators on q to consistently estimate $\boldsymbol{\beta}$: we could simply put q into the error term and estimate β from an OLS regression of y on 1, $\mathbf{x}$. (Of course, if q and $\mathbf{x}$ are uncorrelated we could, in general, gain efficiency for

estimating $\boldsymbol{\beta}$ by including q as an extra regressor.)

5.7. a. If we plug $q = (1/\delta_1)q_1 - (1/\delta_1)a_1$ into equation (5.45) we get

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_K x_K + \eta_1 q_1 + v - \eta_1 a_1, \tag{5.56}$$

where $\eta_1 \equiv (1/\delta_1)$. Now, because the z_h are redundant in (5.45), they are uncorrelated with the structural error, v (by definition of redundancy). Further, we have assumed that the z_h are uncorrelated with a_1. Since each x_j is also uncorrelated with $v - \eta_1 a_1$ we can estimate (5.56) by 2SLS using instruments $(1, x_1, \ldots, x_K, z_1, z_2, \ldots, z_M)$ to get consistent of the β_j and η_1.

Given all of the zero correlation assumptions, what we need for identification is that at least one of the z_h appears in the reduced form for q_1 . More formally, in the linear projection

$$q_1 = \pi_0 + \pi_1 x_1 + \ldots + \pi_K x_K + \pi_{K+1} z_1 + \ldots + \pi_{K+M} z_M + r_1,$$

at least one of $\pi_{K+1}, \ldots, \pi_{K+M}$ must be different from zero.

b. We need family background variables to be redundant in the *log*(*wage*) equation once ability (and other factors, such as *educ* and *exper*), have been controlled for. The idea here is that family background may influence ability but should have no partial effect on *log*(*wage*) once ability has been accounted for. For the rank condition to hold, we need family background variables to be correlated with the indicator, q_1 say *IQ*, once the x_j have been netted out. This is likely to be true if we think that family background and ability are (partially) correlated.

c. Applying the procedure to the data set in NLS80.RAW gives the following results:

```
. ivreg lwage exper tenure educ married south urban black (iq = meduc feduc sibs

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     722
-------------+------------------------------           F(  8,   713) =   25.81
       Model |  19.6029198     8  2.45036497           Prob > F      =  0.0000
```

```
    Residual |  107.208996   713  .150363248           R-squared     =  0.1546
-------------+------------------------------           Adj R-squared =  0.1451
       Total |  126.811916   721  .175883378           Root MSE      =  .38777

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          iq |   .0154368   .0077077     2.00   0.046     .0003044    .0305692
       exper |   .0162185   .0040076     4.05   0.000     .0083503    .0240867
      tenure |   .0076754   .0030956     2.48   0.013     .0015979    .0137529
        educ |   .0161809   .0261982     0.62   0.537     -.035254    .0676158
     married |   .1901012   .0467592     4.07   0.000     .0982991    .2819033
       south |   -.047992   .0367425    -1.31   0.192    -.1201284    .0241444
       urban |   .1869376   .0327986     5.70   0.000     .1225442    .2513311
       black |   .0400269   .1138678     0.35   0.725    -.1835294    .2635832
       _cons |   4.471616    .468913     9.54   0.000        3.551    5.392231
------------------------------------------------------------------------------
Instrumented:  iq
Instruments:   exper tenure educ married south urban black meduc feduc sibs
------------------------------------------------------------------------------

. ivreg lwage exper tenure educ married south urban black (kww = meduc feduc

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     722
-------------+------------------------------           F(  8,   713) =   25.70
       Model |   19.820304     8    2.477538           Prob > F      =  0.0000
    Residual |  106.991612   713  .150058361           R-squared     =  0.1563
-------------+------------------------------           Adj R-squared =  0.1468
       Total |  126.811916   721  .175883378           Root MSE      =  .38737

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         kww |   .0249441   .0150576     1.66   0.098    -.0046184    .0545067
       exper |   .0068682   .0067471     1.02   0.309    -.0063783    .0201147
      tenure |   .0051145   .0037739     1.36   0.176    -.0022947    .0125238
        educ |   .0260808   .0255051     1.02   0.307    -.0239933    .0761549
     married |   .1605273   .0529759     3.03   0.003     .0565198    .2645347
       south |   -.091887   .0322147    -2.85   0.004    -.1551341   -.0286399
       urban |   .1484003   .0411598     3.61   0.000     .0675914    .2292093
       black |  -.0424452   .0893695    -0.47   0.635    -.2179041    .1330137
       _cons |   5.217818   .1627592    32.06   0.000     4.898273    5.537362
------------------------------------------------------------------------------
Instrumented:  kww
Instruments:   exper tenure educ married south urban black meduc feduc sibs
------------------------------------------------------------------------------
```

Even though there are 935 men in the sample, only 722 are used for the estimation because data are missing on *meduc* and *feduc*.

The return to education is estimated to be small and insignificant whether *IQ* or *KWW* used is used as the indicator. This could be because family background variables do not satisfy the

appropriate redundancy condition, or they might be correlated with a_1. (In both first-stage regressions, the F statistic for joint significance of *meduc*,*feduc* and *sibs* have p-values below .002, so it seems the family background variables have some partial correlation with the ability indicators.)

5.8. a. Plug in the indicator q_1 for q and the measurement x_K for x_K^*, being sure to keep track of the errors:

$$\begin{aligned} y &= \gamma_0 + \beta_1 x_1 + \ldots + \beta_K x_K + \gamma_1 q_1 + v - \beta_K e_K + \gamma_1 a_1, \\ &\equiv \gamma_0 + \beta_1 x_1 + \ldots + \beta_K x_K + \gamma_1 q_1 + u \end{aligned}$$

where $\gamma_1 = 1/\delta_1$ Now, if the variables $z_1, \ldots, z_M$ are redundant in the structural equation (so they are uncorrelated with v), and uncorrelated with the measurement error e_K and the indicator error a_1 we can use these as IVs for x_K and q_1 in 2SLS. We need $M \geq 2$ because we have two explanatory variables, x_q and q_1, that are possibly correlated with the composite error u.

b. The Stata results are:

```
. ivreg lwage exper tenure married south urban black (educ iq = kww meduc feduc

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     722
-------------+------------------------------           F(  8,   713) =   18.74
       Model | -.295429993     8 -.036928749           Prob > F      =  0.0000
    Residual |  127.107346   713  .178271172           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  126.811916   721  .175883378           Root MSE      =  .42222

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |   .1646904   .1132659     1.45   0.146    -.0576843    .3870651
          iq |  -.0102736   .0200124    -0.51   0.608    -.0495638    .0290166
       exper |   .0313987   .0122537     2.56   0.011      .007341    .0554564
      tenure |   .0070476   .0033717     2.09   0.037     .0004279    .0136672
     married |   .2133365   .0535285     3.99   0.000     .1082442    .3184289
       south |  -.0941667   .0506389    -1.86   0.063    -.1935859    .0052525
       urban |   .1680721   .0384337     4.37   0.000     .0926152    .2435289
       black |  -.2345713   .2247568    -1.04   0.297    -.6758356    .2066929
```

```
       _cons |   4.932962   .4870124    10.13   0.000     3.976812    5.889112
------------------------------------------------------------------------------
Instrumented:  educ iq
Instruments:   exper tenure married south urban black kww meduc feduc sibs
------------------------------------------------------------------------------
```

The estimated return to education is very large, but imprecisely estimated. The 95% confidence interval is very wide, and easily includes zero. Interestingly, the coefficient on *iq* is actually negative, and not statistically different from zero. The large IV estimate of the return to education and the insignificant ability indicator lend some support to the idea that omitted ability is less of a problem than schooling measurement error in the standard log(*wage*) model estimated by OLS. But the evidence is not very convincing given the very wide confidence interval for the *educ* coefficient.

5.9. Define $\theta_4 = \beta_4 - \beta_3$, so that $\beta_4 = \beta_3 + \theta_4$. Plugging this expression into the equation and rearranging gives

$$\begin{aligned}\log(wage) &= \beta_0 + \beta_1 exper + \beta_2 exper^2 + \beta_3(twoyr + fouryr) + \theta_4 fouryr + u \\ &= \beta_0 + \beta_1 exper + \beta_2 exper^2 + \beta_3 totcoll + \theta_4 fouryr + u,\end{aligned}$$

where $totcoll = twoyr + fouryr$. Now, just estimate the latter equation by 2*SLS* using *exper*, $exper^2$, *dist*2*yr* and *dist*4*yr* as the full set of instruments. We can use the *t* statistic on $\hat{\theta}_4$ to test $H_0 : \theta_4 = 0$ against $H_1 : \theta_4 > 0$.

5.10. a. For $\hat{\beta}_1$, the lower right hand element in the general formula (5.24) with $\mathbf{x} = (1, x)$ and $\mathbf{z} = (1, z)$ is

$$\sigma^2[\text{Cov}(z,x)^2/\text{Var}(z)].$$

Alternatively, you can derive this formula directly by writing

$$\sqrt{N}(\hat{\beta}_1 - \beta_1) = \left(N^{-1} \sum_{i=1}^{N} (z_i - \bar{z})(x_i - \bar{x}) \right)^{-1} \left[N^{-1/2} \sum_{i=1}^{N} (z_i - \bar{z}) u_i \right]$$

Now, $\rho_{zx}^2 = [\text{Cov}(z,x)]^2/(\sigma_z^2\sigma_x^2)$, so simple algebra shows that the asymptotic variance is $\sigma^2/(\rho_{zx}^2\sigma_x^2)$. The asymptotic variance for the OLS estimator is σ^2/σ_x^2. Thus, the difference is the presence of ρ_{zx}^2 in the denominator of the IV asymptotic variance.

b. Naturally, as the error variance σ^2 increases so does the the asymptotic variance of the IV estimator. More variance in x in the population is better for estimating β_1: as σ_x^2 increases the asymptotic variance decreases. These effects are identical to the findings for OLS. A larger correlation between z and x reduces the asymptotic variance of the IV estimator. As $\rho_{zx} \to 0$ the asymptotic variance increases without bound. This illustrates why an instrument that is only weakly correlated with x can lead to very imprecise IV estimators.

5.11. Following the hint, let y_2^0 be the linear projection of y_2 on $\mathbf{z}_2$, let a_2 be the projection error, and assume that λ_2 is known. (The results on generated regressors in Section 6.1.1 show that the argument carries over to the case when λ_2 is estimated.) Plugging in $y_2 = y_2^0 + a_2$ gives

$$y_1 = \mathbf{z}_1\delta_1 + \alpha_1 y_2^0 + \alpha_1 a_2 + u_1.$$

Effectively, we regress y_1 on $\mathbf{z}_1, y_2^0$. The key consistency condition is that each explanatory is orthogonal to the composite error, $\alpha_1 a_2 + u_1$. By assumption, $\text{E}(\mathbf{z}_1' u_1) = \mathbf{0}$. Further, $\text{E}(y_2^0 a_2) = 0$ by construction. The problem is that, in general, $\text{E}(\mathbf{z}_1' a_2) \neq \mathbf{0}$ because $\mathbf{z}_1$was not included in the linear projection for y_2. Therefore, OLS will be inconsistent for all parameters in general. Contrast this conclusion with 2SLS when y_2^* is the projection on $\mathbf{z}_1$ and $\mathbf{z}_2$:

$$\begin{aligned} y_2 &= y_2^* + r_2 = \mathbf{z}\boldsymbol{\pi}_2 + r_2 \\ \text{E}(\mathbf{z}' r_2) &= \mathbf{0} \end{aligned}$$

The second step regression (assuming that $\boldsymbol{\pi}_2$ is known) is

$$y_1 = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2^* + \alpha_1 r_2 + u_1.$$

By construction r_2 is uncorrelated with $\mathbf{z}$, and so $\mathrm{E}(\mathbf{z}_1' r_2) = \mathbf{0}$ and $\mathrm{E}(y_2^* r_2) = 0$.

The lesson is that one must be very careful if manually carrying out 2SLS by explicitly doing the first- and second- stage regressions: all exogenous variables must be included in the first stage.

5.12. This problem is essentially proven by the hint. Given the description of $\mathbf{\Pi}$, the only way the K columns of $\mathbf{\Pi}$ can be linearly dependent is if the last column can be written as a linear combination of the first $K-1$ columns. This is true if and only if each $\boldsymbol{\theta}_j$ is zero. Thus, if at least one $\boldsymbol{\theta}_j$ is different from zero, $\mathrm{rank}(\mathbf{\Pi}) = K$.

5.13. a. In a simple regression model with a single IV, the IV estimate of the slope can be written as

$$\begin{aligned}\hat{\beta}_1 &= \left(\sum_{i=1}^{N}(z_i - \bar{z})(y_i - \bar{y})\right) \Big/ \left(\sum_{i=1}^{N}(z_i - \bar{z})(x_i - \bar{x})\right) \\ &= \left(\sum_{i=1}^{N} z_i(y_i - \bar{y})\right) \Big/ \left(\sum_{i=1}^{N} z_i(x_i - \bar{x})\right).\end{aligned}$$

Now the numerator can be written as

$\sum_{i=1}^{N} z_i(y_i - \bar{y}) = \sum_{i=1}^{N} z_i y_i - \left(\sum_{i=1}^{N} z_i\right)\bar{y} = N_1\bar{y}_1 - N_1\bar{y} = N_1(\bar{y}_1 - \bar{y})$ where $N_1 = \sum_{i=1}^{N} z_i$ is the number of observations in the sample with $z_i = 1$ and $\bar{y}_1$ is the average of the y_i over the observations with $z_i = 1$. Next, write $\bar{y}$ as a weighted average:

$$\bar{y} = (N_0/N)\bar{y}_0 + (N_1/N)\bar{y}$$

where the notation should be clear. Straightforward algebra shows that

$$\begin{aligned}\bar{y}_1 - \bar{y} &= [(N - N_1)/N]\bar{y} - (N_0/N)\bar{y}_0 \\ &= (N_0/N)(\bar{y}_1 - \bar{y}_0).\end{aligned}$$

Therefore, the numerator of the IV estimate is $(N_0 N_1/N)(\bar{y}_1 - \bar{y}_0)$. The same argument shows

that the denominator is $(N_0N_1/N)(\bar{x}_1 - x_0)$. Taking the ratio proves the result.

b. If x is also binary – representing some "treatment" – $\bar{x}_1$ is the fraction of observations receiving treatment when $z_i = 1$ and $\bar{x}_0$ is the fraction receiving treatment when $z_i = 0$. Suppose $x_i = 1$ if person i participates in a job training program, and let $z_i = 1$ if person i is eligible for participation in the program. Then $\bar{x}_1$ is the fraction of people participating in the program out of those made eligible, and $\bar{x}_0$ is the fraction of people participating who are not eligible. (When eligibility is necessary for participation, $\bar{x}_0 = 0$.) Generally, $\bar{x}_1 - \bar{x}_0$ is the difference in participation rates when $z = 1$ and $z = 0$. So the difference in the mean response between the $z = 1$ and $z = 0$ groups gets divided by the difference in participation rates across the two groups.

5.14. a. Taking the linear projection of (5.1) under the assumption that $(x_1, \ldots, x_{K-1}, z_i, \ldots, z_M)$ are uncorrelated with u gives

$$\begin{aligned} \mathrm{L}(y|\mathbf{z}) &= \beta_0 + \beta_1 x_1 + \ldots + \beta_{K-1} x_{K-1} + \beta_K \mathrm{L}(x_K|\mathbf{z}) + \mathrm{L}(u|\mathbf{z}) \\ &= \beta_0 + \beta_1 x_1 + \ldots + \beta_{K-1} x_{K-1} + \beta_K x_K^* \end{aligned}$$

because $\mathrm{L}(u|\mathbf{z}) = 0$.

b. By the law of iterated projections,

$$\mathrm{L}(y|1, x_1, \ldots, x_{K-1}, x_K^*) = \beta_0 + \beta_1 x_1 + \ldots + \beta_{K-1} x_{K-1} + \beta_K x_K^*.$$

Consistency of OLS for the β_j from the regression y on $1, x_1, \ldots, x_{K-1}, x_K^*$ follows immediately from our treatment of OLS from Chapter 4: OLS consistently estimates the parameters in a linear projection provided there is not perfect collinearity in $(1, x_1, \ldots, x_{K-1}, x_K^*)$.

c. I should have said explicitly to assume $\mathrm{E}(\mathbf{z}'\mathbf{z})$ is nonsingular – that is, 2SLS.2a holds. Then, x_K^* is not a perfect linear combination of $(x_1, \ldots, x_{K-1})$ if and only if at least one element

of $z_1, \ldots, z_M$ has nonzero coefficient in $\mathrm{L}(x_K|1, x_1, \ldots x_{K-1}, z_1, \ldots, z_M)$. In the model with a single endogenous explanatory variable, we know this condition is equivalent to Assumption 2SLS.2b, the standard rank condition.

5.15. In $\mathrm{L}(\mathbf{x}|\mathbf{z}) = \mathbf{z}\boldsymbol{\Pi}$ we can write

$$\boldsymbol{\Pi} = \begin{pmatrix} \boldsymbol{\Pi}_{11} & \mathbf{0} \\ \boldsymbol{\Pi}_{12} & \mathbf{I}_{K_2} \end{pmatrix},$$

where $\mathbf{I}_{K_2}$ is the $K_2 \times K_2$ identity matrix, $\mathbf{0}$ is the $L_1 \times K_2$ zero matrix, $\boldsymbol{\Pi}_{11}$ is $L_1 \times K_1$, and $\boldsymbol{\Pi}_{12}$ is $K_2 \times K_1$. As in Problem 5.12, the rank condition holds if and only if rank $(\boldsymbol{\Pi}) = K$.

a. If for some x_j, the vector $\mathbf{z}_1$ does not appear in $\mathrm{L}(x_j|\mathbf{z})$, then $\boldsymbol{\Pi}_{11}$has a column which is entirely zeros. Then that column of $\boldsymbol{\Pi}$ can be written as a linear combination of the last K_2 columns of $\boldsymbol{\Pi}$ – because any $K_2 \times 1$ vector in $\boldsymbol{\Pi}_{12}$ can be written as a linear combination of the columns of $\mathbf{I}_{K_2}$. This implies rank$(\boldsymbol{\Pi}) < K$. Therefore, a necessary condition for the rank condition is that no columns of $\boldsymbol{\Pi}_{11}$ be exactly zero, which means that at least one z_h must appear in the reduced form of each x_j, $j = 1, \ldots, K_1$.

b. Suppose $K_1 = 2$ and $L_1 = 2$, where z_1 appears in the reduced form from both x_1 and x_2, but z_2 appears in neither reduced form. Then the 2×2 matrix $\boldsymbol{\Pi}_{11}$ has zeros in its second row, which means that the second row of $\boldsymbol{\Pi}$ is all zeros. In that case, it cannot have rank K. Intuitively, while we began with two instruments, only one of them turned out to be partially correlated with x_1 and x_2.

c. Without loss of generality, assume that z_j appears in the reduced form for x_j; we can simply reorder the elements of $\mathbf{z}_1$ to ensure this is the case. Then $\boldsymbol{\Pi}_{11}$ is a $K_1 \times K_1$ diagonal matrix with nonzero diagonal elements. Looking at

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{\Pi}_{11} & \mathbf{0} \\ \mathbf{\Pi}_{12} & \mathbf{I}_{K_2} \end{pmatrix}$$

we see that if $\mathbf{\Pi}_{11}$ is diagonal with all nonzero diagonals then $\mathbf{\Pi}$ is lower triangular with all diagonal elements nonzero. Therefore, rank $\mathbf{\Pi} = K$.

5.16. a. The discussion below equation (5.24) implies directly that

$$\text{Avar}\sqrt{N}(\tilde{\beta} - \beta) = \sigma_u^2/\text{Var}(w^*)$$

because there are no other explanatory variables – exogenous or endogenous – in the equation. Remember, the expression

$$\sigma_u^2[\text{E}(\mathbf{x}^{*\prime}\mathbf{x}^*)]^{-1}$$

has the same form as that for OLS but with $\mathbf{x}^*$ replacing $\mathbf{x}$. So any algebra derived for OLS can be applied to 2SLS.

b. We can write

$$v = u - \mathbf{h}\boldsymbol{\gamma},$$

so if $\text{E}(\mathbf{g}'u) = \mathbf{0}$ and $\text{E}(\mathbf{g}'\mathbf{h}) = \mathbf{0}$ then

$$\text{E}(\mathbf{g}'v) = \text{E}(\mathbf{g}'u) - \text{E}(\mathbf{g}'\mathbf{h})\boldsymbol{\gamma} = \mathbf{0}.$$

c. For the hint here to be entirely correct, I should have stated that $\text{E}(w) = 0$. As we will see, when w has a nonzero mean, $\check{r}$ differs from w^* by an additive constant [which, of course, implies $\text{Var}(\check{r}) = \text{Var}(w^*)$].

Again using the discussion following equation (5.24),

$$\text{Avar}\sqrt{N}(\hat{\beta} - \beta) = \sigma_v^2/\text{Var}(\check{r}),$$

where $\sigma_v^2 = \text{Var}(v)$, $\check{r}$ is the population residual from the regression $\check{w}$ on $1, \mathbf{h}$, and $\check{w}$ are the

population fitted values from the linear projection of w on $\mathbf{g}$, $\mathbf{h}$.

Because $\mathrm{E}(\mathbf{g}'\mathbf{h}) = \mathbf{0}$, we can write

$$\check{w} = \mathbf{g}\boldsymbol{\pi}_1 + \mathbf{h}\boldsymbol{\pi}_2$$

where

$$\begin{aligned}\boldsymbol{\pi}_1 &= [\mathrm{E}(\mathbf{g}'\mathbf{g})]^{-1}\mathrm{E}(\mathbf{g}'w)\\ \boldsymbol{\pi}_2 &= [\mathrm{E}(\mathbf{h}'\mathbf{h})]^{-1}\mathrm{E}(\mathbf{h}'w).\end{aligned}$$

Note that

$$w^* = \mathrm{L}(w|\mathbf{g}) = \mathbf{g}\boldsymbol{\pi}_1.$$

Next,

$$\begin{aligned}\mathrm{L}(\check{w}|1,\mathbf{h}) &= \mathrm{L}(\mathbf{g}\boldsymbol{\pi}_1 + \mathbf{h}\boldsymbol{\pi}_2|1,\mathbf{h}) = \mathrm{L}(\mathbf{g}|1,\mathbf{h})\boldsymbol{\pi}_1 + \mathbf{h}\boldsymbol{\pi}_2\\ &= \mathrm{L}(\mathbf{g}|1)\boldsymbol{\pi}_1 + \mathbf{h}\boldsymbol{\pi}_2\end{aligned}$$

because $\mathrm{E}(\mathbf{h}) = \mathbf{0}$ and $\mathrm{E}(\mathbf{g}'\mathbf{h}) = \mathbf{0}$ are assumed. Now $\mathrm{L}(\mathbf{g}|1) = \mathrm{E}(\mathbf{g})$, and so

$$\mathrm{L}(\check{w}|1,\mathbf{h}) = \eta_1 + \mathbf{h}\boldsymbol{\pi}_2$$

where $\eta_1 = \boldsymbol{\mu}_\mathbf{g}\boldsymbol{\pi}_1$. Therefore,

$$\begin{aligned}\check{r} &= \check{w} - \mathrm{L}(\check{w}|1,\mathbf{h}) = (\mathbf{g}\boldsymbol{\pi}_1 + \mathbf{h}\boldsymbol{\pi}_2) - (\eta_1 + \mathbf{h}\boldsymbol{\pi}_2) = -\eta_1 + \mathbf{g}\boldsymbol{\pi}_1\\ &= -\eta_1 + w^*\end{aligned}$$

It follows that $\mathrm{Var}(\check{r}) = \mathrm{Var}(w^*)$ and so we have shown

$$\mathrm{Avar}\sqrt{N}(\hat{\beta} - \beta) = \sigma_v^2/\mathrm{Var}(w^*),$$

d. Because $\mathrm{E}(\mathbf{h}'v) = \mathbf{0}$ by definition, we have

$$\sigma_u^2 = \mathrm{Var}(\mathbf{h}\boldsymbol{\gamma}) + \sigma_u^2 = \boldsymbol{\gamma}'\boldsymbol{\Sigma}_h\boldsymbol{\gamma} + \sigma_v^2 \geq \sigma_v^2,$$

with strict inequality if $\boldsymbol{\Sigma}_h$ is positive definite and $\boldsymbol{\gamma} \neq \mathbf{0}$ (and even in some cases where

$\boldsymbol{\Sigma}_h \equiv \text{Var}(\mathbf{h})$ is not positive definite). This means that, asymptotically, we generally get a smaller asymptotic variance for estimate $\boldsymbol{\beta}$ by including exogenous variables that are uncorrelated with the instruments $\mathbf{g}$:

$$\begin{aligned}\text{Avar}\sqrt{N}(\tilde{\beta}-\beta)-\text{Avar}\sqrt{N}(\hat{\beta}-\beta) &= \frac{\sigma_u^2}{\text{Var}(w^*)}-\frac{\sigma_v^2}{\text{Var}(w^*)} \\ &= \frac{\boldsymbol{\gamma}'\boldsymbol{\Sigma}_h\boldsymbol{\gamma}}{\text{Var}(w^*)} \geq 0.\end{aligned}$$

Solutions to Chapter 6 Problems

6.1. a. Here is abbreviated Stata output for testing the null hypothesis that *educ* is exogenous:

```
. use card

. qui reg educ nearc4 nearc2 exper expersq black south smsa reg661-reg668
      smsa66

. predict v2hat, resid

. reg lwage educ exper expersq black south smsa reg661-reg668 smsa66 v2hat

      Source |       SS       df       MS              Number of obs =    3010
-------------+------------------------------           F( 16,  2993) =   80.37
       Model |  178.100803    16  11.1313002           Prob > F      =  0.0000
    Residual |  414.540842  2993  .138503455           R-squared     =  0.3005
-------------+------------------------------           Adj R-squared =  0.2968
       Total |  592.641645  3009  .196956346           Root MSE      =  .37216

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |   .1570594   .0482814     3.25   0.001     .0623912    .2517275
       exper |   .1188149   .0209423     5.67   0.000     .0777521    .1598776
     expersq |  -.0023565   .0003191    -7.38   0.000    -.0029822   -.0017308
       black |  -.1232778   .0478882    -2.57   0.010    -.2171749   -.0293806
       south |  -.1431945   .0261202    -5.48   0.000    -.1944098   -.0919791
        smsa |    .100753   .0289435     3.48   0.001     .0440018    .1575042
      reg661 |   -.102976   .0398738    -2.58   0.010    -.1811588   -.0247932
      reg662 |  -.0002286   .0310325    -0.01   0.994    -.0610759    .0606186
      reg663 |   .0469556   .0299809     1.57   0.117    -.0118296    .1057408
      reg664 |  -.0554084   .0359807    -1.54   0.124    -.1259578    .0151411
      reg665 |   .0515041   .0436804     1.18   0.238    -.0341426    .1371509
      reg666 |   .0699968   .0489487     1.43   0.153    -.0259797    .1659734
      reg667 |   .0390596   .0456842     0.85   0.393     -.050516    .1286352
      reg668 |  -.1980371   .0482417    -4.11   0.000    -.2926273   -.1034468
      smsa66 |   .0150626   .0205106     0.73   0.463    -.0251538    .0552789
       v2hat |  -.0828005   .0484086    -1.71   0.087     -.177718    .0121169
       _cons |   3.339687    .821434     4.07   0.000     1.729054    4.950319
------------------------------------------------------------------------------
```

The t statistic on $\hat{v}_2$ is -1.71, which is not significant at the 5% level against a two-sided alternative. The negative correlation between u_1 and *educ* is essentially the same finding that the 2SLS estimated return to education is larger than the OLS estimate. In any case, I would call this marginal evidence that *educ* is endogenous. The quandary is that the OLS and 2SLS

point estimates are quite different.

b. To test the single over indentifying restriction we obtain the 2SLS residuals:

```
. qui reg lwage educ exper expersq black south smsa reg661-reg668 smsa66
      (nearc4 nearc2 exper expersq black south smsa reg661-reg668 smsa66)

. predict uhat1, resid

. qui reg u1hat exper expersq black south smsa reg661-reg668 smsa66 nearc4
      nearc2

. di e(r2)
.00041467

. di 3010*e(r2)
1.2481535

. di chiprob(1,3010*e(r2))
.26390545
```

The test statistic is the sample size times the R-squared from this regression, or about 1.25. The *p*-value, obtained from χ_1^2 distribution, is about .264, so the instruments pass the over identification test.

6.2. We first obtain the reduced form residuals, $\hat{v}_{21}$ and $\hat{v}_{22}$, for *educ* and *IQ*, respectively. The regression output is suppressed:

```
. qui reg educ exper tenure married south urban black kww meduc feduc sibs

. predict v21hat, resid
(213 missing values generated)

. qui reg iq exper tenure married south urban black kww meduc feduc sibs

. predict v22hat, resid
(213 missing values generated)

. qui reg lwage exper tenure married south urban black educ iq v21hat v22hat

. test v21hat v22hat

 ( 1)  v21hat = 0
 ( 2)  v22hat = 0

       F(  2,   711) =    4.20
            Prob > F =    0.0153
```

The *p*-value of the joint *F* test, which is justified asymptotically, is .0153. Therefore, the

test finds fairly strong evidence for endogeneity of at least one of *educ* and *IQ*, although this conclusion relies on the instruments being truly exogenous. If you look back at Problem 5.8, this IV solution did not seem to work very well. So we still do not know what should be treated as exogenous in this method.

6.3. a. We need prices to satisfy two requirements. First, *calories* and *protein* must be partially correlated with prices of food. While this is easy to test separately by estimating the two reduced forms, the rank condition could still be violated. (Problem 5.15c contains a sufficient condition for the rank condition to hold.) In addition, we must also assume prices are exogenous in the productivity equation. Ideally, prices vary because of things like transportation costs that are not systematically related to regional variations in individual productivity. A potential problem is that prices reflect food quality and that features of the food other than calories and protein appear in the disturbance u_1.

b. Since there are two endogenous explanatory variables we need at least two prices.

c. We would first estimate the two reduced forms for *calories* and *protein* by regressing each on a constant, *exper*, *exper*^{2}, *educ*, and the M prices, p_1, ..., p_MWe obtain the residuals, $\hat{v}_{21}$ and $\hat{v}_{22}$. Then we would run the regression log(*produc*) on 1,*exper*, *exper*^{2}, *educ*, $\hat{v}_{21}$, $\hat{v}_{22}$ and do a joint significance test on $\hat{v}_{21}$ and $\hat{v}_{22}$. We could use a standard F test or use a heteroskedasticity-robust test.

6.4.a. Since $y = \mathbf{x}\boldsymbol{\beta} + q + v$ it follows that

$$\mathrm{E}(y|\mathbf{x}) = \mathbf{x}\boldsymbol{\beta} + \mathrm{E}(q|\mathbf{x}) + \mathrm{E}(v|\mathbf{x}) = \mathbf{x}\boldsymbol{\beta} + \mathbf{x}\boldsymbol{\delta} = \mathbf{x}(\boldsymbol{\beta} + \boldsymbol{\delta}) \equiv \mathbf{x}\boldsymbol{\gamma}.$$

Since $\mathrm{E}(y|\mathbf{x})$ is linear in $\mathbf{x}$ there is no functional form misspecification in this conditional expectation. Therefore, no functional form test will detect correlation between q and $\mathbf{x}$, no matter how strong it is: $\boldsymbol{\delta}$ can be anything.

b. Since $E(v|\mathbf{x},q) = 0, \text{Var}(v|\mathbf{x},q) = E(v^2|\mathbf{x},q) = \sigma_v^2 = E(v^2|\mathbf{x}) = \text{Var}(v|\mathbf{x})$. Therefore, $\text{Var}(y|\mathbf{x}) = \text{Var}(q + v|\mathbf{x}) = \text{Var}(q|\mathbf{x}) + \text{Var}(v|\mathbf{x}) + 2E(qv|\mathbf{x})$, where we use $\text{Cov}(q, v|\mathbf{x}) = E(qv|\mathbf{x})$ because $E(v|\mathbf{x}) = 0$. Now

$$E(qv|\mathbf{x}) = E[E(qv|\mathbf{x},q)|\mathbf{x}] = E[qE(v|\mathbf{x},q)|\mathbf{x}] = E[q \cdot 0|\mathbf{x}] = 0.$$

Therefore, $\text{Var}(y|\mathbf{x}) = \text{Var}(q|\mathbf{x}) + \text{Var}(v|\mathbf{x}) = \sigma_q^2 + \sigma_v^2$, so that y is conditionally homoskedastic. But if $E(y|\mathbf{x}) = \mathbf{x}\boldsymbol{\gamma}$ and $\text{Var}(y|\mathbf{x})$ is constant, a test for heteroskedasticity will always have a limiting chi-square distribution. It will have no power for detecting omitted variables.

c. Since $E(u^2|\mathbf{x}) = \text{Var}(u|\mathbf{x}) + [E(u|\mathbf{x})]^2$ and $\text{Var}(u|\mathbf{x})$ is constant, $E(u^2|\mathbf{x})$ is constant if and only if $E[(u|\mathbf{x})]^2$ is constant. If $E(u|\mathbf{x}) \neq E(u)$ then $E(u|\mathbf{x})$ is not constant, so $[E(u|\mathbf{x})]^2$ generally will be a function of $\mathbf{x}$. So $E(u^2|\mathbf{x})$ depends on $\mathbf{x}$, which means that u^2 can be correlated with functions of $\mathbf{x}$, say $\mathbf{h}(\mathbf{x})$. It follows that regression tests of the form (6.36) can be expected, at least in some cases, to detect "heteroskedasticity". (If the goal is to determine when heteroskedasticity-robust inference is called for, the regression-based tests do the right thing.)

6.5. a. For simplicity, absorb the intercept in $\mathbf{x}$, so $y = \mathbf{x}\boldsymbol{\beta} + u, E(u|\mathbf{x}) = 0, \text{Var}(u|\mathbf{x}) = \sigma^2$. In these tests, $\hat{\boldsymbol{\sigma}}^2$ is implicitly SSR/N – there is no degrees of freedom adjustment. (In any case, the *df* adjustment makes no difference asymptotically.) So $\hat{u}_i^2 - \hat{\sigma}^2$ has a zero sample average, which means that

$$N^{-1/2} \sum_{i=1}^{N} (\mathbf{h}_i - \boldsymbol{\mu}_h)'(\hat{u}_i^2 - \hat{\boldsymbol{\sigma}}^2) = N^{-1/2} \sum_{i=1}^{N} \mathbf{h}_i'(\hat{u}_i^2 - \hat{\sigma}^2).$$

Next, $N^{-1/2} \sum_{i=1}^{N} (\mathbf{h}_i - \boldsymbol{\mu}_h)' = O_p(1)$ by the central limit theorem and $\hat{\sigma}^2 - \sigma^2 = o_p(1)$. So

$N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\boldsymbol{\mu}_h)'(\hat{\sigma}^2-\sigma^2)=O_p(1)\cdot o_p(1)=o_p(1)$. Therefore, so far we have

$$N^{-1/2}\sum_{i=1}^{N}\mathbf{h}_i'(\hat{u}_i^2-\hat{\boldsymbol{\sigma}}^2)=N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\mu_h)'(\hat{u}_i^2-\hat{\sigma}^2)+o_p(1).$$

We are done with this part if we show

$N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\mu_h)'\hat{u}_i^2=N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\mu_h)'\hat{u}_i^2+o_p(1)$. Now, as in Problem 4.4, we can write $\hat{u}_i^2=u_i^2-2u_i\mathbf{x}_i(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})+[\mathbf{x}_i(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2$, so

$$\begin{aligned}N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\boldsymbol{\mu}_h)'\hat{u}_i^2=\;&N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\boldsymbol{\mu}_h)'\hat{u}_i^2\\&-2\left[N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\boldsymbol{\mu}_h)'\mathbf{x}_i\right](\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})\\&+\left[N^{-1/2}\sum_{i=1}^{N}(\mathbf{h}_i-\boldsymbol{\mu}_h)'(\mathbf{x}_i\otimes\mathbf{x}_i)\right]\{vec[(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})']\},\end{aligned}\tag{6.62}$$

where the expression for the third term follows from $[\mathbf{x}_i(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]^2=\mathbf{x}_i(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\mathbf{x}_i'=(\mathbf{x}_i\otimes\mathbf{x}_i)\cdot\text{vec}[(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})']$. Dropping the "–2" the second term can be written as $\left(N^{-1}\sum_{i=1}^{N}u_i(\mathbf{h}_i-\boldsymbol{\mu}_h)'\mathbf{x}_i\right)\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})=o_p(1)\cdot O_p(1)$ because $\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})=O_p(1)$ and, under $\text{E}(u_i|\mathbf{x}_i)=0,\text{E}[u_i(\mathbf{h}_i-\mu_h)'\mathbf{x}_i]=0$; the law-of-large-numbers implies that the sample average is $o_p(1)$. The third term can be written as $N^{-1/2}\left[N^{-1}\sum_{i=1}^{N}(\mathbf{h}_i-\boldsymbol{\mu}_h)'(\mathbf{x}_i\otimes\mathbf{x}_i)\right]\{\text{vec}[\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})']\}=N^{-1/2}\cdot O_p(1)\cdot O_p(1)$, where we again use the fact that sample averages are $O_p(1)$ by the law of large numbers and $\text{vec}[\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})']=O_p(1)$. We have shown that the last two terms in (6.62) are $o_p(1)$, which proves part a.

b. By part a, the asymptotic variance of $N^{-1/2}\sum_{i=1}^{N}\mathbf{h}_i'(\hat{u}_i^2-\sigma^2)$ is $\text{Var}[(\mathbf{h}_i-\boldsymbol{\mu}_h)'(u_i^2-\sigma^2)]=\text{E}[(u_i^2-\sigma^2)^2(\mathbf{h}_i-\boldsymbol{\mu}_h)'(\mathbf{h}_i-\boldsymbol{\mu}_h)]$. Now

$(u_i^2 - \sigma^2)^2 = u_i^4 - 2u_i^2\sigma^2 + \sigma^4$. Under the null, $\mathrm{E}(u_i^2|\mathbf{x}_i) = \mathrm{Var}(u_i|\mathbf{x}_i) = \sigma^2$ [since $\mathrm{E}(u_i|\mathbf{x}_i) = 0$ is assumed] and therefore, when we add (6.37), $\mathrm{E}[(u_i^2 - \sigma^2)^2|\mathbf{x}_i] = \kappa^2 - \sigma^4 \equiv \eta^2$. A standard iterated expectations argument gives

$\mathrm{E}[(u_i^2 - \sigma^2)^2(\mathbf{h}_i - \boldsymbol{\mu}_h)'(\mathbf{h}_i - \boldsymbol{\mu}_h)] = \mathrm{E}\{\mathrm{E}[(u_i^2 - \sigma^2)^2(\mathbf{h}_i - \boldsymbol{\mu}_h)'(\mathbf{h}_i - \boldsymbol{\mu}_h)]|\mathbf{x}_i\} = \mathrm{E}\{\mathrm{E}[(u_i^2 - \sigma^2)^2|$

which is what we wanted to show. (Whether we carry out the calculation for a random draw i or for random variables representing the population is a matter of taste.)

c. From part b and Lemma 3.8, the following statistic has an asymptotic χ_Q^2 distribution:

$$\left[N^{-1/2} \sum_{i=1}^{N} (\hat{u}_i^2 - \hat{\sigma}^2)\mathbf{h}_i \right] \{\eta^2 \mathrm{E}[(\mathbf{h}_i - \boldsymbol{\mu}_h)'(\mathbf{h}_i - \boldsymbol{\mu}_h)]\}^{-1} \left[N^{-1/2} \sum_{i=1}^{N} \mathbf{h}_i'(\hat{u}_i^2 - \hat{\sigma}^2) \right].$$

Using again the fact that $\sum_{i=1}^{N}(\hat{u}_i^2 - \hat{\sigma}^2) = 0$, we can replace $\mathbf{h}_i$ with $\mathbf{h}_i - \bar{\mathbf{h}}$ in the two vectors forming the quadratic form. Then, again by Lemma 3.8, we can replace the matrix in the quadratic form with a consistent estimator, which is

$$\hat{\eta}^2 \left[N^{-1} \sum_{i=1}^{N} (\mathbf{h}_i - \bar{\mathbf{h}})'(\mathbf{h}_i - \bar{\mathbf{h}}) \right],$$

where $\hat{\eta}^2 = N^{-1}\sum_{i=1}^{N}(\hat{u}_i^2 - \hat{\sigma}^2)^2$. The computable statistic, after simple algebra, can be written as

$$\hat{\eta}^{-2} \left(\sum_{i=1}^{N} (\hat{u}_i^2 - \hat{\sigma}^2)(\mathbf{h}_i - \bar{\mathbf{h}}) \right) \left(\sum_{i=1}^{N} (\mathbf{h}_i - \bar{\mathbf{h}})'(\mathbf{h}_i - \bar{\mathbf{h}}) \right)^{-1} \left(\sum_{i=1}^{N} (\mathbf{h}_i - \bar{\mathbf{h}})'(\hat{u}_i^2 - \hat{\sigma}^2) \right).$$

Now $\hat{\eta}^2$ is just the total sum of squares of the $\hat{u}_i^2$ divided by N. The numerator of the statistic is simply the explained sum of squares from the regression $\hat{u}_i^2$ on $1, \mathbf{h}_i$, $i = 1, \ldots, N$. Therefore, the test statistic is N times the usual (centered) R-squared from the regression $\hat{u}_i^2$ on $1, \mathbf{h}_1, i = 1, \ldots, N$, or NR_c^2.

d. Without assumption (6.37) we need to estimate $\mathrm{E}[(u_i^2 - \sigma^2)^2(\mathbf{h}_i - \boldsymbol{\mu}_h)'(\mathbf{h}_i - \boldsymbol{\mu}_h)]$ generally. Hopefully, the approach is by now pretty clear. We replace the population expected value with the sample average and replace any unknown parameters – $\beta, \sigma^2,$ and $\boldsymbol{\mu}_h$ in this case – with their consistent estimators (under H_0). So a generally consistent estimator of $\text{Avar}\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{h}_i'(\hat{u}_i^2 - \hat{\sigma}^2)\right)$ is

$$N^{-1}\sum_{i=1}^{N}(\hat{u}_i^2 - \hat{\sigma}^2)^2(\mathbf{h}_i - \bar{\mathbf{h}})'(\mathbf{h}_i - \bar{\mathbf{h}}),$$

and the test statistic robust to heterokurtosis can be written as

$$\left(\sum_{i=1}^{N}(\hat{u}_i^2 - \hat{\sigma}^2)(\mathbf{h}_i - \bar{\mathbf{h}})\right)\left(\sum_{i=1}^{N}(\hat{u}_i^2 - \hat{\sigma}^2)^2(\mathbf{h}_i - \bar{\mathbf{h}})'(\mathbf{h}_i - \bar{\mathbf{h}})\right)^{-1}$$
$$\cdot\left(\sum_{i=1}^{N}(\mathbf{h}_i - \bar{\mathbf{h}})'(\hat{u}_i^2 - \hat{\sigma}^2)\right),$$

which is easily seen to be the explained sum of squares from the regression of 1 on $(\hat{u}_i^2 - \hat{\sigma}^2)(\mathbf{h}_i - \bar{\mathbf{h}}), i = 1, \ldots, N$ (without an intercept). Since the total sum of squares, without demeaning, of unity is simply N, the statistic is equivalent to $N - SSR_0$, where SSR_0 is the sum of squared residuals.

6.6. Here is my Stata session using the data NLS80.RAW:

```
. qui reg lwage exper tenure married south urban black educ

. predict lwageh
(option xb assumed; fitted values)

. gen lwagehsq = lwageh^2

. predict uhat, resid

. gen uhatsq = uhat^2

. reg uhatsq lwageh lwagehsq

      Source |       SS       df       MS              Number of obs =     935
```

```
-------------+------------------------------           F(  2,   932) =    2.43
       Model |  .288948987     2  .144474493           Prob > F      =  0.0883
    Residual |  55.3447136   932   .05938274           R-squared     =  0.0052
-------------+------------------------------           Adj R-squared =  0.0031
       Total |  55.6336626   934  .059564949           Root MSE      =  .24369

------------------------------------------------------------------------------
      uhatsq |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      lwageh |   3.027285   1.880375     1.61   0.108    -.6629745    6.717544
    lwagehsq |  -.2280088   .1390444    -1.64   0.101    -.5008853    .0448677
       _cons |  -9.901227   6.353656    -1.56   0.119    -22.37036    2.567902
------------------------------------------------------------------------------
```

An asymptotically valid test for heteroskedasticity is just the F statistic for joint significance of $\hat{y}$ and $\hat{y}^2$, and this yields $p - value = .088$ (although this version maitains Assumption (6.37) under the null, along with homoskedasticity). Thus, there is only modest evidence of heteroskedasticity. It could be ignored or heteroskedasticity-robust standard errors and test statistics can be used.

6.7. a. The simple regression results are:

```
. use hprice

. reg lprice ldist if y81

      Source |       SS       df       MS              Number of obs =     142
-------------+------------------------------           F(  1,   140) =   30.79
       Model |  3.86426989     1  3.86426989           Prob > F      =  0.0000
    Residual |  17.5730845   140  .125522032           R-squared     =  0.1803
-------------+------------------------------           Adj R-squared =  0.1744
       Total |  21.4373543   141  .152037974           Root MSE      =  .35429

------------------------------------------------------------------------------
      lprice |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       ldist |   .3648752   .0657613     5.55   0.000     .2348615    .4948889
       _cons |   8.047158   .6462419    12.45   0.000     6.769503    9.324813
------------------------------------------------------------------------------
```

This regression suggests a strong link between housing price and distance from the incinerator (as distance increases, so does housing price). The elasticity is $.365$ and the t statistic is 5.55. However, this is not a good causal regression: the incinerator may have been put near homes with lower values to begin with. If so, we would expect the positive

relationship found in the simple regression even if the new incinerator had no effect on housing prices.

b. The parameter δ_3 should be positive: after the incinerator is built a house should be worth relatively more the farther it is from the incinerator. Here is the Stata session:

```
. gen y81ldist = y81*ldist

. reg lprice y81 ldist y81ldist

      Source |       SS       df       MS              Number of obs =     321
-------------+------------------------------           F(  3,   317) =   69.22
       Model |  24.3172548     3  8.10575159           Prob > F      =  0.0000
    Residual |  37.1217306   317  .117103251           R-squared     =  0.3958
-------------+------------------------------           Adj R-squared =  0.3901
       Total |  61.4389853   320  .191996829           Root MSE      =   .3422

------------------------------------------------------------------------------
      lprice |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         y81 |  -.0113101   .8050622    -0.01   0.989     -1.59525     1.57263
       ldist |    .316689   .0515323     6.15   0.000     .2153005    .4180775
    y81ldist |   .0481862   .0817929     0.59   0.556    -.1127394    .2091117
       _cons |   8.058468   .5084358    15.85   0.000     7.058133    9.058803
------------------------------------------------------------------------------
```

The coefficient on *ldist* reveals the shortcoming of the regression in part a. This coefficient measures the relationship between *lprice* and *ldist* in 1978, before the incinerator was even being rumored. The effect of the incinerator is given by the coefficient on the interaction, *y*81*ldist*. While the direction of the effect is as expected, it is not especially large, and it is statistically insignificant, anyway. Therefore, at this point, we cannot reject the null hypothesis that building the incinerator had no effect on housing prices.

c. Adding the variables listed in the problem gives

```
. reg lprice y81 ldist y81ldist lintst lintstsq larea lland age agesq rooms
      baths

      Source |       SS       df       MS              Number of obs =     321
-------------+------------------------------           F( 11,   309) =  108.04
       Model |  48.7611143    11  4.43282858           Prob > F      =  0.0000
    Residual |   12.677871   309  .041028709           R-squared     =  0.7937
-------------+------------------------------           Adj R-squared =  0.7863
       Total |  61.4389853   320  .191996829           Root MSE      =  .20256
```

```
------------------------------------------------------------------------------
      lprice |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         y81 |   -.229847   .4877198    -0.47   0.638    -1.189519    .7298249
       ldist |   .0866424   .0517205     1.68   0.095    -.0151265    .1884113
    y81ldist |   .0617759   .0495705     1.25   0.214    -.0357625    .1593143
       lintst |   .9633332   .3262647     2.95   0.003     .3213517    1.605315
    lintstsq |  -.0591504   .0187723    -3.15   0.002     -.096088   -.0222128
       larea |   .3548562   .0512328     6.93   0.000     .2540468    .4556655
       lland |    .109999   .0248165     4.43   0.000     .0611683    .1588297
         age |  -.0073939   .0014108    -5.24   0.000    -.0101699   -.0046178
       agesq |   .0000315   8.69e-06     3.63   0.000     .0000144    .0000486
       rooms |   .0469214   .0171015     2.74   0.006     .0132713    .0805715
       baths |   .0958867    .027479     3.49   0.001      .041817    .1499564
       _cons |   2.305525   1.774032     1.30   0.195    -1.185185    5.796236
------------------------------------------------------------------------------
```

The incinerator effect is now larger (the elasticity is about .062) and the *t* statistic is larger, but the *p*-value for the interaction term is still fairly large, .214. Against a one-sided alternative, the *p*-value is .107, so it is almost significant at the 10% level. Still, using these two years of data and controlling for the listed factors, the evidence that housing prices were adversely affected by the new incinerator is somewhat weak.

6.8. a. The following is my Stata session:

```
. use fertil1

. gen agesq = age^2

. reg kids educ age agesq black east northcen west farm othrural town smcity
      y74-y84

      Source |       SS       df       MS              Number of obs =    1129
-------------+------------------------------           F( 17,  1111) =    9.72
       Model |  399.610888    17  23.5065228           Prob > F      =  0.0000
    Residual |  2685.89841  1111  2.41755033           R-squared     =  0.1295
-------------+------------------------------           Adj R-squared =  0.1162
       Total |   3085.5093  1128  2.73538059           Root MSE      =  1.5548

------------------------------------------------------------------------------
        kids |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |  -.1284268   .0183486    -7.00   0.000    -.1644286    -.092425
         age |   .5321346   .1383863     3.85   0.000     .2606065    .8036626
       agesq |   -.005804   .0015643    -3.71   0.000    -.0088733   -.0027347
       black |   1.075658   .1735356     6.20   0.000     .7351631    1.416152
        east |    .217324   .1327878     1.64   0.102    -.0432192    .4778672
    northcen |    .363114   .1208969     3.00   0.003      .125902    .6003261
        west |   .1976032   .1669134     1.18   0.237    -.1298978    .5251041
        farm |  -.0525575    .14719    -0.36   0.721    -.3413592    .2362443
```

```
    othrural |  -.1628537    .175442    -0.93   0.353    -.5070887    .1813814
        town |   .0843532    .124531     0.68   0.498    -.1599893    .3286957
      smcity |   .2118791    .160296     1.32   0.187    -.1026379    .5263961
         y74 |   .2681825    .172716     1.55   0.121    -.0707039    .6070689
         y76 |  -.0973795   .1790456    -0.54   0.587     -.448685    .2539261
         y78 |  -.0686665   .1816837    -0.38   0.706    -.4251483    .2878154
         y80 |  -.0713053   .1827707    -0.39   0.697      -.42992    .2873093
         y82 |  -.5224842   .1724361    -3.03   0.003    -.8608214    -.184147
         y84 |  -.5451661   .1745162    -3.12   0.002    -.8875846   -.2027477
       _cons |  -7.742457   3.051767    -2.54   0.011    -13.73033   -1.754579
------------------------------------------------------------------------------
```

The estimate says that a women with about eight more years of education has about one fewer child (gotten from $.128(8) = 1.024$), other factors fixed. The coefficient is very statistically significant. Also, there has been a notable secular decline in fertility over this period: on average, with other factors held fixed, a women in 1984 had about half a child less ($.545$) than a similar woman in 1972, the base year. The effect is also statistically significant with p-value $= .002$.

b. Estimating the reduced form for *educ* gives

```
. reg educ age agesq black east northcen west farm othrural town smcity
      y74-y84 meduc feduc

      Source |       SS       df       MS              Number of obs =    1129
-------------+------------------------------           F( 18,  1110) =   24.82
       Model |  2256.26171    18  125.347873            Prob > F      =  0.0000
    Residual |  5606.85432  1110  5.05122011            R-squared     =  0.2869
-------------+------------------------------           Adj R-squared =  0.2754
       Total |  7863.11603  1128  6.97084755            Root MSE      =  2.2475

------------------------------------------------------------------------------
        educ |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.2243687   .2000013    -1.12   0.262     -.616792    .1680546
       agesq |   .0025664   .0022605     1.14   0.256     -.001869    .0070018
       black |   .3667819   .2522869     1.45   0.146    -.1282311     .861795
        east |   .2488042   .1920135     1.30   0.195    -.1279462    .6255546
    northcen |   .0913945   .1757744     0.52   0.603    -.2534931    .4362821
        west |   .1010676   .2422408     0.42   0.677    -.3742339    .5763691
        farm |  -.3792615   .2143864    -1.77   0.077    -.7999099    .0413869
    othrural |   -.560814   .2551196    -2.20   0.028    -1.061385    -.060243
        town |   .0616337   .1807832     0.34   0.733    -.2930816     .416349
      smcity |   .0806634   .2317387     0.35   0.728    -.3740319    .5353588
         y74 |   .0060993    .249827     0.02   0.981    -.4840872    .4962858
         y76 |   .1239104   .2587922     0.48   0.632    -.3838667    .6316874
         y78 |   .2077861   .2627738     0.79   0.429    -.3078033    .7233755
         y80 |   .3828911   .2642433     1.45   0.148    -.1355816    .9013638
         y82 |   .5820401   .2492372     2.34   0.020     .0930108    1.071069
```

```
        y84 |   .4250429   .2529006     1.68   0.093    -.0711741      .92126
      meduc |   .1723015   .0221964     7.76   0.000     .1287499    .2158531
      feduc |   .2074188   .0254604     8.15   0.000     .1574629    .2573747
      _cons |   13.63334   4.396773     3.10   0.002     5.006421    22.26027
------------------------------------------------------------------------------

. test meduc feduc

 ( 1)  meduc = 0
 ( 2)  feduc = 0

       F(  2,  1110) =  155.79
            Prob > F =    0.0000
```

The joint F test shows that *educ* is significantly partially correlated with *meduc* and *feduc*; the t statistics also show this clearly. If we make the test robust to heteroskedasticity of unknown form, the F statistic drops to 131.37 but the p-value is still zero to four decimal places.

To test the null that *educ* is exogenous, we need to reduced form residuals and then include them in the OLS regression. I suppress the output here:

```
. predict v2hat, resid

. reg kids educ age agesq black east northcen west farm othrural town smcity
      y74-y84 v2hat

      Source |       SS       df       MS              Number of obs =    1129
-------------+------------------------------           F( 18,  1110) =    9.21
       Model |  400.802376    18  22.2667987           Prob > F      =  0.0000
    Residual |  2684.70692  1110  2.41865489           R-squared     =  0.1299
-------------+------------------------------           Adj R-squared =  0.1158
       Total |   3085.5093  1128  2.73538059           Root MSE      =  1.5552

------------------------------------------------------------------------------
        kids |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |  -.1527395   .0392012    -3.90   0.000    -.2296562   -.0758227
         age |   .5235536   .1389568     3.77   0.000     .2509059    .7962013
       agesq |   -.005716   .0015697    -3.64   0.000    -.0087959   -.0026362
       black |   1.072952    .173618     6.18   0.000     .7322958    1.413609
        east |   .2285554   .1337787     1.71   0.088    -.0339322     .491043
    northcen |   .3744188   .1219925     3.07   0.002     .1350569    .6137807
        west |   .2076398   .1675628     1.24   0.216    -.1211357    .5364153
        farm |  -.0770015   .1512869    -0.51   0.611     -.373842    .2198389
    othrural |  -.1952451   .1814491    -1.08   0.282    -.5512671    .1607769
        town |     .08181   .1246122     0.66   0.512     -.162692    .3263119
      smcity |   .2124996    .160335     1.33   0.185    -.1020943    .5270936
         y74 |   .2721292    .172847     1.57   0.116    -.0670145    .6112729
         y76 |  -.0945483   .1791319    -0.53   0.598    -.4460236    .2569269
```

```
         y78 |  -.0572543   .1824512    -0.31   0.754    -.4152424    .3007337
         y80 |   -.053248   .1846139    -0.29   0.773    -.4154795    .3089836
         y82 |  -.4962149   .1764897    -2.81   0.005     -.842506   -.1499238
         y84 |  -.5213604   .1778207    -2.93   0.003    -.8702631   -.1724578
       v2hat |   .0311374   .0443634     0.70   0.483    -.0559081    .1181829
       _cons |  -7.241244   3.134883    -2.31   0.021    -13.39221    -1.09028
------------------------------------------------------------------------------
```

The *t* statistic on *v2hat* is .702, so there is little evidence that *educ* is endogenous in the equation. Still, we can see if 2SLS produces very different estimates:

```
. ivreg kids age agesq black east northcen west farm othrural town smcity
        y74-y84 (educ = meduc feduc)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =    1129
-------------+------------------------------           F( 17,  1111) =    7.72
       Model |   395.36632    17  23.2568424           Prob > F      =  0.0000
    Residual |  2690.14298  1111  2.42137082           R-squared     =  0.1281
-------------+------------------------------           Adj R-squared =  0.1148
       Total |   3085.5093  1128  2.73538059           Root MSE      =  1.5561

------------------------------------------------------------------------------
        kids |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |  -.1527395   .0392232    -3.89   0.000    -.2296993   -.0757796
         age |   .5235536   .1390348     3.77   0.000     .2507532     .796354
       agesq |   -.005716   .0015705    -3.64   0.000    -.0087976   -.0026345
       black |   1.072952   .1737155     6.18   0.000      .732105      1.4138
        east |   .2285554   .1338537     1.71   0.088    -.0340792    .4911901
    northcen |   .3744188    .122061     3.07   0.002     .1349228    .6139148
        west |   .2076398   .1676568     1.24   0.216    -.1213199    .5365995
        farm |  -.0770015   .1513718    -0.51   0.611    -.3740083    .2200053
    othrural |  -.1952451    .181551    -1.08   0.282    -.5514666    .1609764
        town |     .08181   .1246821     0.66   0.512     -.162829    .3264489
      smcity |   .2124996    .160425     1.32   0.186    -.1022706    .5272698
         y74 |   .2721292    .172944     1.57   0.116    -.0672045    .6114629
         y76 |  -.0945483   .1792324    -0.53   0.598    -.4462205    .2571239
         y78 |  -.0572543   .1825536    -0.31   0.754     -.415443    .3009343
         y80 |   -.053248   .1847175    -0.29   0.773    -.4156825    .3091865
         y82 |  -.4962149   .1765888    -2.81   0.005       -.8427   -.1497297
         y84 |  -.5213604   .1779205    -2.93   0.003    -.8704586   -.1722623
       _cons |  -7.241244   3.136642    -2.31   0.021    -13.39565   -1.086834
------------------------------------------------------------------------------
Instrumented:  educ
Instruments:   age agesq black east northcen west farm othrural town smcity
               y74 y76 y78 y80 y82 y84 meduc feduc
------------------------------------------------------------------------------
```

The estimated coefficient on *educ* is larger in magnitude than before, but the test for endogeneity shows that we can reasonably attribute the difference between OLS and 2SLS to sampling error.

c. Since there is little evidence that *educ* is endogenous, we could just use OLS. I did it both ways. First, I just added interactions $y74 \cdot educ, y76 \cdot educ, \ldots, y84 \cdot educ$ to the model in part a and used OLS. Some of the interactions, particularly in the last two years, are marginally significant and negative, showing that the effect of education has become stronger over time. But the joint F test for the interaction terms yields $p - value = .180$, and so we do not reject the model without the interactions. Still, the possibility that the link between fertility and education has become stronger over time is deserves attention, especially using more recent data.

To estimate the full model by 2SLS, I obtained instruments by interacting all year dummies with both *meduc* and *feduc*. The Stata command is then

```
. ivreg kids age agesq black east northcen west farm othrural town smcity y74
      (educ y74educ-y84educ = meduc feduc y74meduc-y84feduc )

. test y74educ y76educ y78educ y80educ y82educ y84educ
```

Qualitatively, the results are similar to the OLS estimates. The $p - value$ for the joint F test on the interactions is $.205$ – again, this has asymptotic justification under Assumption 2SLS.3, the homoskedasticity assumption – so again there is no strong evidence favoring including of the interactions of year dummies and education.

6.9. a. The Stata results are

```
. use injury

. reg ldurat afchnge highearn afhigh male married head-construc if ky

      Source |       SS       df       MS              Number of obs =    5349
-------------+------------------------------           F( 14,  5334) =   16.37
       Model |  358.441793    14  25.6029852           Prob > F      =  0.0000
    Residual |  8341.41206  5334  1.56381928           R-squared     =  0.0412
-------------+------------------------------           Adj R-squared =  0.0387
       Total |  8699.85385  5348  1.62674904           Root MSE      =  1.2505

------------------------------------------------------------------------------
      ldurat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     afchnge |   .0106274   .0449167     0.24   0.813    -.0774276    .0986824
```

```
    highearn |   .1757598   .0517462     3.40   0.001     .0743161    .2772035
      afhigh |   .2308768   .0695248     3.32   0.001     .0945798    .3671738
        male |  -.0979407   .0445498    -2.20   0.028    -.1852766   -.0106049
     married |   .1220995   .0391228     3.12   0.002     .0454027    .1987962
        head |  -.5139003   .1292776    -3.98   0.000    -.7673372   -.2604634
        neck |   .2699126   .1614899     1.67   0.095    -.0466737    .5864988
      upextr |   -.178539   .1011794    -1.76   0.078     -.376892    .0198141
       trunk |   .1264514   .1090163     1.16   0.246    -.0872651     .340168
     lowback |  -.0085967   .1015267    -0.08   0.933    -.2076305    .1904371
     lowextr |  -.1202911   .1023262    -1.18   0.240    -.3208922    .0803101
      occdis |   .2727118    .210769     1.29   0.196    -.1404816    .6859052
       manuf |  -.1606709   .0409038    -3.93   0.000    -.2408591   -.0804827
    construc |   .1101967   .0518063     2.13   0.033     .0086352    .2117581
       _cons |   1.245922   .1061677    11.74   0.000      1.03779    1.454054
------------------------------------------------------------------------------
```

The estimated coefficient on the interaction term is actually higher now – .231 – than in equation (6.54), and it has a large5 *t* statistic (3.32 compare with 2.78). Adding the other explanatory variables only slightly increased the standard error on the interaction term.

b. The small *R*-squared, on the order of 4.1%, or 3.9% if we used the adjusted *R*-squared, means that we do not explain much of the variation in time on workers compensation using the variables included in the regression. This is often the case in the social sciences: it is very difficult to include the multitude of factors that can affect something like *durat*. The low *R*-squared means that making predictions of log(*durat*) would be very difficult given the factors we have included in the regression: the variation in the unobservables pretty much swamps the explained variation. However, the low *R*-squared does not mean we have a biased or inconsistent estimator of the effect of the policy change. Provided the Kentucky policy change provides a good natural experiment, the OLS estimator is consistent. With over 5,000 observations, we can get a reasonably precise estimate of the effect, although the 95% confidence interval is pretty wide.

c. Using the data for Michigan to estimate the basic regression gives

```
. reg ldurat afchnge highearn afhigh if mi

      Source |       SS       df       MS              Number of obs =    1524
-------------+------------------------------           F(  3,  1520) =    6.05
       Model |  34.3850177     3  11.4616726           Prob > F      =  0.0004
```

```
    Residual |  2879.96981  1520  1.89471698          R-squared     =  0.0118
-------------+-----------------------------          Adj R-squared =  0.0098
       Total |  2914.35483  1523  1.91356194          Root MSE      =  1.3765

------------------------------------------------------------------------------
      ldurat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     afchnge |   .0973808   .0847879     1.15   0.251    -.0689329    .2636945
    highearn |   .1691388   .1055676     1.60   0.109    -.0379348    .3762124
      afhigh |   .1919906   .1541699     1.25   0.213    -.1104176    .4943988
       _cons |   1.412737   .0567172    24.91   0.000     1.301485    1.523989
------------------------------------------------------------------------------
```

The coefficient on the interaction term, .192, is remarkably similar to that for Kentucky. Unfortunately, because of the many fewer observations, the *t* statistic is insignificant at the 10% level against a one-sided alternative. Asymptotic theory roughly predicts that the standard error for Michigan will be about $(5,626/1,524)^{1/2} \approx 1.92$ larger than that for Kentucky (assuming the same error variance and same fraction of observations in the different groups). In fact, the ratio of standard errors is about 2.23. The difference precision in the KY and MI cases shows the importance of a large sample size for this kind of policy analysis.

6.10. a. As suggested by the hint, we can write $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = N^{-1/2}\sum_{i=1}^{N}\mathbf{A}^{-1}\mathbf{z}_i'u_i,$ where $\mathbf{A} \equiv \mathrm{E}(\mathbf{z}'\mathbf{z})$, plus a term we can ignore by the asymptotic equivalence lemma. Further, $\sqrt{N}(\bar{\mathbf{x}} - \mu) = N^{-1/2}\sum_{i=1}^{N}(\mathbf{x}_i - \boldsymbol{\mu}).$ When we stack these two representations, we see that the asymptotic covariance between $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ and $\sqrt{N}(\bar{\mathbf{x}} - \mu)$ is $\mathrm{E}[\mathbf{A}^{-1}\mathbf{z}_i'u_i(\mathbf{x}_i - \boldsymbol{\mu})] = \mathbf{A}^{-1}\mathrm{E}[u_i\mathbf{z}_i'(\mathbf{x}_i - \boldsymbol{\mu})]$. Because $\mathrm{E}(u_i|\mathbf{x}_i) = \mathbf{0}$, the standard iterated expectations argument shows that $\mathrm{E}[u_i\mathbf{z}_i'(\mathbf{x}_i - \boldsymbol{\mu})] = \mathbf{0}$ because $\mathbf{z}_i$ is a function of $\mathbf{x}_i$. This completes the proof.

b. While the delta method leads to the same place, it is not needed because of linearity of $\hat{\boldsymbol{\beta}}$ in the data. We can write $\hat{\alpha}_1 = \hat{\beta}_1 + \hat{\beta}_3\bar{x}_2 = \hat{\beta}_1 + \hat{\beta}_3\mu_2 + \hat{\beta}_3(\bar{x}_2 - \mu_2) \equiv \tilde{\alpha}_1 + \hat{\beta}_3(\bar{x}_2 - \mu_2)$, and so $\sqrt{N}(\hat{\alpha}_1 - \alpha_1) = \sqrt{N}(\tilde{\alpha}_1 - \alpha_1) + \hat{\beta}_3[\sqrt{N}(\bar{x}_2 - \mu_2)]$. Now $\hat{\beta}_3[\sqrt{N}(\bar{x}_2 - \mu_2)] = \beta_3[\sqrt{N}(\bar{x}_2 - \mu_2)] + o_p(1)$ because $\hat{\beta}_3 - \beta_3 = o_p(1)$ and

$\sqrt{N}(\bar{x}_2 - \mu_2) = O_p(1)$. So we have

$$\sqrt{N}(\hat{\alpha}_1 - \alpha_1) = \sqrt{N}(\tilde{\alpha}_1 - \alpha_1) + \beta_3[\sqrt{N}(\bar{x}_2 - \mu_2)] + o_p(1).$$

By part a, we know that $\sqrt{N}(\hat{\beta} - \beta)$ and $\sqrt{N}(\bar{x}_2 - \mu_2)$ are asymptotically jointly normal and asymptotically independent (uncorrelated). Because $\sqrt{N}(\tilde{\alpha}_1 - \alpha_1)$ is just a deterministic linear combination of $\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ it follows that $\sqrt{N}(\tilde{\alpha}_1 - \alpha_1)$ and $\sqrt{N}(\bar{x}_2 - \mu_2)$ are asymptotically uncorrelated. Therefore,.

$$\begin{aligned}\text{Avar}[\sqrt{N}(\hat{\alpha}_1 - \alpha_1)] &= \text{Avar}[\sqrt{N}(\tilde{\alpha}_1 - \alpha_1)] + \beta_3^2\text{Avar}[\sqrt{N}(\bar{x}_2 - \mu_2)] \\ &= \text{Avar}[\sqrt{N}(\tilde{\alpha}_1 - \alpha_1)] + \beta_3^2\sigma_2^2,\end{aligned}$$

where $\sigma_2^2 = \text{Var}(x_2)$. Therefore, by the convention introduced in Section 3.5, we write

$$\text{Avar}(\hat{\alpha}_1) = \text{Avar}(\tilde{\alpha}_1) + \beta_3^2(\sigma_2^2/N),$$

which is what we wanted to show.

c. As stated in the hint, the standard error we get from the regression in Problem 4.8d is really $\text{se}(\tilde{\alpha}_1)$, as it does not account for the sampling variation in $\bar{x}_2$. So

$$\text{se}(\hat{\alpha}_1) = \{[\text{se}(\tilde{\alpha}_1)]^2 + \hat{\beta}_3^2(\hat{\sigma}_2^2/N)\}^{1/2} = \{[\text{se}(\tilde{\alpha}_1)]^2 + \hat{\beta}_3^2[\text{se}(\bar{x}_2)]^2\}^{1/2}$$

since $\text{se}(\bar{x}_2) = \sigma^2/\sqrt{N}$.

d. The standard error reported for the education variable in Problem 4.8d, $\text{se}(\hat{\alpha}_1)$, is about .00698, the coefficient on the interaction term ($\hat{\beta}_3$) is about .00455, and the sample standard deviation of *exper* is about 4.375. Plugging these numbers into the formula from part c gives $\text{se}(\hat{\alpha}_1) = [(.00698)^2 + (.00455)^2(4.375)^2/935]^{1/2} \approx .00701$. For practical purposes, this is not much bigger than .00698: the effect of accounting for estimation of the population mean of *exper* is very modest.

6.11. The following is Stata output for answering the first three parts:

```
. use cps78_85

. reg lwage y85 educ y85educ exper expersq union female y85fem

      Source |       SS       df       MS              Number of obs =    1084
-------------+------------------------------           F(  8,  1075) =   99.80
       Model |  135.992074     8  16.9990092           Prob > F      =  0.0000
    Residual |  183.099094  1075  .170324738           R-squared     =  0.4262
-------------+------------------------------           Adj R-squared =  0.4219
       Total |  319.091167  1083   .29463635           Root MSE      =   .4127

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         y85 |   .1178062   .1237817     0.95   0.341     -.125075    .3606874
        educ |   .0747209   .0066764    11.19   0.000     .0616206    .0878212
     y85educ |   .0184605   .0093542     1.97   0.049      .000106     .036815
       exper |   .0295843   .0035673     8.29   0.000     .0225846     .036584
     expersq |  -.0003994   .0000775    -5.15   0.000    -.0005516   -.0002473
       union |   .2021319   .0302945     6.67   0.000     .1426888    .2615749
      female |  -.3167086   .0366215    -8.65   0.000    -.3885663    -.244851
      y85fem |    .085052    .051309     1.66   0.098    -.0156251     .185729
       _cons |   .4589329   .0934485     4.91   0.000     .2755707     .642295
------------------------------------------------------------------------------
```

a. The return to another year of education increased by about $.0185$, or 1.85 percentage points, between 1978 and 1985. The t statistic on $y85educ$ is 1.97, which is marginally significant at the 5% level against a two-sided alternative.

b. The coefficient on $y85fem$ is positive and shows that the estimated gender gap declined by about 8.5 percentage points. It is still very large, with the gender difference in *lwage* in 1985 estimated at about $-.232$. The t statistic on $y85fem$ is only significant at about the 10% level against a two-sided alternative. Still, this is suggestive of some closing of wage differentials between women and men at given levels of education and workforce experience.

c. Only the coefficient on $y85$ changes if wages are measured in 1978 dollars. In fact, you can check that when 1978 wages are used, the coefficient on $y85$ becomes about $-.383 = .118 - \log(1.65) \approx .118 - .501$.

d. To answer this question, I just took the squared OLS residuals and regressed those on the year dummy, $y85$. The coefficient is about $.042$ with a standard error of about $.022$, which

gives a t statistic of about 1.91. So there is some evidence that the variance of the unexplained part of log wages (or even log real wages) has increased over time.

e. As the equation is written in the problem, the coefficient δ_0 is the growth in nominal wages for a male with no years of education! For a male with 12 years of education, we want $\theta_0 \equiv \delta_0 + 12\delta_1$.

Many packages have simple commands that deliver standard errors and tests for linear combinations. But a general way to obtain the standard error for $\hat{\theta}_0 \equiv \hat{\delta}_0 + 12\hat{\delta}_1$ is to replace $y85 \cdot educ$ with $y85 \cdot (educ - 12)$ and reestimate the equation. Simple algebra shows that, in the new equation, θ_0 is the coefficient on *educ*. In Stata we have

```
. gen y85educ_12 = y85*(educ - 12)

. reg lwage y85 educ y85educ_12 exper expersq union female y85fem

      Source |       SS       df       MS              Number of obs =    1084
-------------+------------------------------           F(  8,  1075) =   99.80
       Model |  135.992074     8  16.9990092           Prob > F      =  0.0000
    Residual |  183.099094  1075  .170324738           R-squared     =  0.4262
-------------+------------------------------           Adj R-squared =  0.4219
       Total |  319.091167  1083   .29463635           Root MSE      =   .4127

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y85 |   .3393326   .0340099     9.98   0.000     .2725993    .4060659
        educ |   .0747209   .0066764    11.19   0.000     .0616206    .0878212
  y85educ_12 |   .0184605   .0093542     1.97   0.049      .000106     .036815
       exper |   .0295843   .0035673     8.29   0.000     .0225846     .036584
     expersq |  -.0003994   .0000775    -5.15   0.000    -.0005516   -.0002473
       union |   .2021319   .0302945     6.67   0.000     .1426888    .2615749
      female |  -.3167086   .0366215    -8.65   0.000    -.3885663    -.244851
      y85fem |    .085052    .051309     1.66   0.098    -.0156251     .185729
       _cons |   .4589329   .0934485     4.91   0.000     .2755707     .642295
------------------------------------------------------------------------------
```

So the growth in nominal wages for a man with $educ = 12$ is about .339, or 33.9%. [We could use the more accurate estimate, .404, obtained from $\exp(.339) - 1 = .404$.] The 95% confidence interval goes from about 27.3 to 40.6.

Stata users can verify that the command

```
. lincom y85 + 12*y85educ
```

after estimation of the original equation delivers the same estimate and inference.

6.12. Under the assumptions listed, $\mathrm{E}(\mathbf{x}'u) = 0, \mathrm{E}(\mathbf{z}'u) = 0,$ and the rank conditions hold for OLS and 2SLS, so we can write

$$\sqrt{N}(\hat{\boldsymbol{\beta}}_{2SLS} - \boldsymbol{\beta}) = \mathbf{A}_*^{-1}\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i^{*\prime}u_i\right) + o_p(1),$$

$$\sqrt{N}(\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta}) = \mathbf{A}^{-1}\left(N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_i'u_i\right) + o_p(1)$$

where $\mathbf{A} = \mathrm{E}(\mathbf{x}_i'\mathbf{x}_i)$, $\mathbf{A}_* = \mathrm{E}(\mathbf{x}_i^{*\prime}\mathbf{x}_i^*)$, and $\mathbf{x}_i^* = \mathbf{z}_i\boldsymbol{\Pi}$. Further, because of the homoskedasticity assumptions, $\mathrm{E}(u_i^2\mathbf{x}_i'\mathbf{x}_i) = \sigma^2\mathbf{A}$, $\mathrm{E}(u_i^2\mathbf{x}_i^{*\prime}\mathbf{x}_i) = \sigma^2\mathbf{A}_*$, and $\mathrm{E}(u_i^2\mathbf{x}_i^{*\prime}\mathbf{x}_i) = \sigma^2\mathrm{E}(\mathbf{x}_i^{*\prime}\mathbf{x}_i)$. But we know from Chapter 5 that $\mathrm{E}(\mathbf{x}_i^{*\prime}\mathbf{x}_i) = \mathbf{A}_*$. Next, we can stack the above equations to obtain that OLS and 2SLS, when appropriately centered and scaled, are jointly asymptotically normal with variance-covariance matrix

$$\begin{pmatrix} \mathbf{V}_1 & \mathbf{C} \\ \mathbf{C}' & \mathbf{V}_2 \end{pmatrix},$$

where $\mathbf{V}_1 = \mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}}_{2SLS} - \boldsymbol{\beta})]$, $\mathbf{V}_2 = \mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta})]$, and $\mathbf{C} = \mathbf{A}_*^{-1}\mathrm{E}(u_i^2\mathbf{x}_i^{*\prime}\mathbf{x}_i)\mathbf{A}^{-1} = \sigma^2\mathbf{A}^{-1}$. Therefore, we can write the asymptotic variance matrix of both estimators as

$$\sigma^2\begin{pmatrix} \mathbf{A}_*^{-1} & \mathbf{A}^{-1} \\ \mathbf{A}^{-1} & \mathbf{A}^{-1} \end{pmatrix}.$$

Now, the asymptotic variance of any linear combination is easy to obtain. In particular, the asymptotic variance of $\sqrt{N}(\hat{\boldsymbol{\beta}}_{2SLS} - \boldsymbol{\beta}) - \sqrt{N}(\hat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta})$ is simply $\sigma^2(\mathbf{A}_*^{-1} + \mathbf{A}^{-1} - \mathbf{A}^{-1} - \mathbf{A}^{-1}) = \sigma^2\mathbf{A}_*^{-1} - \sigma^2\mathbf{A}^{-1}$, which is the difference in the asymptotic

variances, as we wanted to show.

6.13. This is a simple application of the law of iterated expectations. The statement of the problem should add the requirement $\rho_1 \neq 0$. By the LIE,

$$\mathrm{E}(u_1|\mathbf{z}) = \mathrm{E}[\mathrm{E}(u_1|\mathbf{z}, v_2)|\mathbf{z}] = \mathrm{E}(\rho_1 v_2|\mathbf{z}) = \rho_1 \mathrm{E}(v_2|\mathbf{z})$$

and so if $\mathrm{E}(u_1|\mathbf{z}) = 0$ then $\mathrm{E}(v_2|\mathbf{z}) = 0$, too.

6.14. a. First, y_2 is a function of $(\mathbf{z}, v_2)$, and so, from the structural equation,

$$\mathrm{E}(y_1|\mathbf{z}, v_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \mathbf{g}(y_2)\boldsymbol{\alpha}_1 + \mathrm{E}(u_1|\mathbf{z}, v_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \mathbf{g}(y_2)\boldsymbol{\alpha}_1 + \mathrm{E}(u_1|v_2),$$

where

$$\mathrm{E}(u_1|\mathbf{z}, v_2) = \mathrm{E}(u_1|v_2)$$

follows because (u_1, v_2) is independent of $\mathbf{z}$. (Note that, in general, it is not enough to assume that u_1 and v_2 are separately independent of $\mathbf{z}$; joint independence is needed.)

b. If $\mathrm{E}(u_1|v_2) = \rho_1 v_2$ then, under the previous assumptions,

$$\mathrm{E}(y_1|\mathbf{z}, v_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \mathbf{g}(y_2)\boldsymbol{\alpha}_1 + \rho_1 v_2.$$

Therefore, in the first step, we would run OLS of y_{i2} on $\mathbf{z}_i, i = 1, \ldots, N,$ and obtain the OLS residuals, $\hat{v}_{i2}$. In the second step, we would regress y_{i1} on $\mathbf{z}_{i1}, \mathbf{g}(y_{i2}), \hat{v}_{i2}, i = 1, \ldots, N$. By the usual two-step estimation results, all coefficients are $\sqrt{N}$-consistent and asymptotically normal for the corresponding population parameter. The interesting thing about this method is that, if $G_1 > 1$ we have more than one endogenous explanatory variable – $g_1(y_1), \ldots, g_{G_1}(y_2)$ – but adding a single regressor, $\hat{v}_{i2}$, cleans up the endogeneity. This occurs because all endogenous regressors are a function of y_2, and we have assumed y_2 is an additive function of $\mathbf{z}$ and an independent error, which pretty much restricts y_2 to be continuous. (We can easily replace the

linear function $\mathbf{z}\boldsymbol{\pi}_2$ with known nonlinear functions of $\mathbf{z}$.)

As specific examples, the second stage regression might be

$$y_{i1} \text{ or } \mathbf{z}_{i1}, y_{i2}, y_{i2}^2, y_{i2}^3, \hat{v}_{i2},\ i = 1, \ldots N$$

or

$$y_{i1} \text{ or } \mathbf{z}_{i1}, 1[a_1 < y_{i2} \leq a_2], \ldots, 1[a_{m-1} < y_{i2} \leq a_m], 1[y_{i2} > a_M], \hat{v}_{i2}, i = 1, \ldots, N.$$

In the latter case, dummies for whether y_{i2} falls into one of the intervals $(-\infty, a_1], (a_1, a_2], \ldots, (a_{M-1}, a_M], (a_M, \infty)$ appear in the structural model.

c. If $\rho_1 = 0$, no adjustment is needed to the asymptotic variance, so we can use the usual t statistic on $\hat{v}_{i2}$ as a test of endogeneity of y_2, where the null is exogeneity: $H_0 : \rho_1 = 0$. Actually, nothing guarantees that $\text{Var}(y_1|\mathbf{z}, v_2)$ does not depend on v_2 – and, under weaker assumptions, it could also depend on $\mathbf{z}$ – so there is a good case for making the test robust to heteroskedasticity.

d. The estimating equation becomes

$$\text{E}(y_1|\mathbf{z}, v_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \mathbf{g}(y_2)\boldsymbol{\alpha}_1 + \rho_1 v_2 + \xi_1(v_2^2 - \tau_2^2)$$

and now, to implement a two-step control function procedure, we obtain $\hat{\tau}_2^2$, the usual OLS error variance estimate, along with $\hat{\boldsymbol{\pi}}_2$. The residuals are constructed as before, $\hat{v}_{i2} = y_{i2} - \mathbf{z}_i\hat{\boldsymbol{\pi}}_2$. The second-step regression is now

$$y_{i1} \text{ on } \mathbf{z}_{i1},\ \mathbf{g}(y_{i2}),\ \hat{v}_{i2},\ (\hat{v}_{i2}^2 - \hat{\tau}_2^2),\ i = 1, \ldots, N$$

Now we can use a heteroskedasticity-robust Wald test of joint significance of $\hat{v}_{i2}$ and $(\hat{v}_{i2}^2 - \hat{\tau}_2^2)$. Under the null $H_0 : \rho_1 = 0, \xi_1 = 0$, we do not have to adjust the statistic for the first-step estimation.

e. We would use traditional 2SLS, where we need at least one IV for each $\mathbf{g}_j(y_2)$. Methods

for coming up with such IVs are discussed in Section 9.5. Briefly, they will be nonlinear functions of $\mathbf{z}$, which is why $\mathrm{E}(u_1|\mathbf{z}) = 0$ should be assumed. Generally, we add enough nonlinear functions, say $\mathbf{h}(\mathbf{z})$, to the original instrument list $\mathbf{z}$. So, do 2SLS of y_1 on $\mathbf{z}_1, \mathbf{g}_2$ using IVs $[\mathbf{z}, \mathbf{h}(\mathbf{z})]$. 2SLS will be more robust than the method described in part b because the reduced form for y_2 is not restricted in any way, and we need not assume u_1 is independent of $\mathbf{z}$.

6.15. a. Because $y_2 = \mathbf{z}\boldsymbol{\pi}_2 + v_2$, we can find $\mathrm{E}(y_1|\mathbf{z}, v_2)$ or $\mathrm{E}(y_1|\mathbf{z}, y_2)$; they are the same. Now

$$\begin{aligned}\mathrm{E}(y_1|\mathbf{z}, v_2) &= \mathbf{z}_1\boldsymbol{\delta}_1 + \mathbf{g}(\mathbf{z}_1, y_2)\boldsymbol{\alpha}_1 + \mathbf{g}(\mathbf{z}_1, y_2)\mathrm{E}(\mathbf{v}_1|\mathbf{z}, v_2) + \mathrm{E}(u_1|\mathbf{z}, v_2) \\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \mathbf{g}(\mathbf{z}_1, y_2)\boldsymbol{\alpha}_1 + \mathbf{g}(\mathbf{z}_1, y_2)v_2\boldsymbol{\theta}_1 + \rho_1 v_2\end{aligned}$$

b. The first step is to regress y_{i2} on $\mathbf{z}_i$ and get the residuals, $\hat{v}_{i2}$. Second, run the regression

$$y_{i1} \text{ on } \mathbf{z}_{i1},\ \mathbf{g}(\mathbf{z}_{i1}, y_{i2}),\ \mathbf{g}(\mathbf{z}_{i1}, y_{i2})\hat{v}_{i2},\ \hat{v}_{i2}$$

which means that $\hat{v}_{i2}$ appears by itself and interacted with all elements of $\mathbf{g}(\mathbf{z}_{i1}, y_{i2})$.

c. The null is $H_0 : \boldsymbol{\theta}_1 = \mathbf{0}, \rho_1 = 0$, which means we can compute a heteroskedasticity-robust Wald test of joint significance of $\mathbf{g}(\mathbf{z}_{i1}, y_{i2})\hat{v}_{i2}$ and $\hat{v}_{i2}$.

d. For the specific model give, the second-step regression is

$$y_{i1} \text{ on } \mathbf{z}_{i1},\ y_{i2},\ y_{i2}^2,\ \mathbf{z}_{i1}y_{i2},\ y_{i2}\hat{v}_{i2},\ \hat{v}_{i2},\ i = 1, \ldots, N.$$

In other words, $\hat{v}_{i2}$ appears by itself and interacted with y_{i2}, as in Garen (1984).

Solutions to Chapter 7 Problems

7.1. Write (with probability approaching one)

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \left(N^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{X}_i \right)^{-1} \left(N^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{u}_i \right).$$

From Assumption SOLS. 2, the weak law of large numbers, and Slutsky's Theorem,

$$\text{plim} \left(N^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{X}_i \right)^{-1} = \mathbf{A}^{-1}.$$

Further, under SOLS.1, the WLLN implies that plim $\left(N^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{u}_i \right) = \mathbf{0}$. Thus,

$$\text{plim}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} + \text{plim} \left(N^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{X}_i \right)^{-1} \cdot \text{plim} \left(N^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{u}_i \right) = \boldsymbol{\beta} + \mathbf{A}^{-1} \cdot \mathbf{0} = \boldsymbol{\beta}.$$

7.2. a. Under SOLS. 1 and SOLS.2, Theorem 7.2 implies that $\text{Avar}(\hat{\boldsymbol{\beta}}_{OLS}) = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}/N$, where $\mathbf{A} = \text{E}(\mathbf{X}_i'\mathbf{X}_i)$ and $\mathbf{B} = \text{E}(\mathbf{X}_i \mathbf{u}_i \mathbf{u}_i' \mathbf{X}_i)$. But we have assumed that $\text{E}(\mathbf{X}_i \mathbf{u}_i \mathbf{u}_i' \mathbf{X}_i) = \text{E}(\mathbf{X}_i' \boldsymbol{\Omega} \mathbf{X}_i)$, which proves the assertion. Effectively, this is what we can expect for the asymptotic variance of OLS under the system version of homoskedasticity. [Note that Assumption SGLS. 3 and $\text{E}(\mathbf{X}_i \mathbf{u}_i \mathbf{u}_i' \mathbf{X}_i) = \text{E}(\mathbf{X}_i' \boldsymbol{\Omega} \mathbf{X}_i)$ are not the same, but both are implied by condition (7.53). There are other cases where they reduce to the same assumption, such as in a SUR model when $\boldsymbol{\Omega}$ is diagonal.]

b. The estimator in (7.28) is always valid. An estimator that uses the structure of $\text{Avar}\left(\hat{\boldsymbol{\beta}}_{SOLS}\right)$ obtained in part a is obtained as follows. Let $\hat{\boldsymbol{\Omega}} = N^{-1} \sum_{i=1}^{N} \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i'$, where the $\hat{\mathbf{u}}_i$ are the $G \times 1$ system OLS residuals. Then

$$\widehat{\text{Avar}}\left(\hat{\boldsymbol{\beta}}_{SOLS}\right) = \left(\sum_{i=1}^{N} \mathbf{X}_i' \mathbf{X}_i \right)^{-1} \left(\sum_{i=1}^{N} \mathbf{X}_i' \hat{\boldsymbol{\Omega}} \mathbf{X}_i \right) \left(\sum_{i=1}^{N} \mathbf{X}_i' \mathbf{X}_i \right)^{-1}$$

is a valid estimator provided the homoskedasticity assumption holds.

c. Using the hint and dropping the division by N on the right hand side, we have

$$[\text{Avar}(\hat{\boldsymbol{\beta}}_{FGLS})]^{-1} - [\text{Avar}(\hat{\boldsymbol{\beta}}_{SOLS})]^{-1} = \text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i) - \text{E}(\mathbf{X}_i'\mathbf{X}_i)[\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}\mathbf{X}_i)]^{-1}\text{E}(\mathbf{X}_i'\mathbf{X}_i).$$

Define $\mathbf{Z}_i \equiv \boldsymbol{\Omega}^{-1/2}\mathbf{X}_i$ and $\mathbf{W}_i \equiv \boldsymbol{\Omega}^{1/2}\mathbf{X}_i$. Then the difference can be written as

$$\text{E}(\mathbf{Z}_i'\mathbf{Z}_i) - \text{E}(\mathbf{Z}_i'\mathbf{W}_i)[\text{E}(\mathbf{W}_i'\mathbf{W}_i)]^{-1}\text{E}(\mathbf{W}_i'\mathbf{Z}_i).$$

Now, define $\mathbf{R}_i \equiv \mathbf{Z}_i - \mathbf{W}_i\boldsymbol{\Pi},$ where $\boldsymbol{\Pi} \equiv [\text{E}(\mathbf{W}_i'\mathbf{W}_i)]^{-1}\text{E}(\mathbf{W}_i'\mathbf{Z}_i)$; $\mathbf{R}_i$ is the $G \times K$ matrix of population residuals from the linear projection of $\mathbf{Z}_i$ on $\mathbf{W}_i$. Straightforward multiplication shows that

$$\text{E}(\mathbf{Z}_i'\mathbf{Z}_i) - \text{E}(\mathbf{Z}_i'\mathbf{W}_\mathbf{i})[\text{E}(\mathbf{W}_i'\mathbf{W}_i)]^{-1}\text{E}(\mathbf{W}_i'\mathbf{Z}_i) = \text{E}(\mathbf{R}_i'\mathbf{R}_i),$$

which is necessarily positive semi-definite. We have shown that if (7.53) holds along with SGLS.1 and the rank conditions for SGLS and SOLS, then FGLS is more efficient than OLS.

d. If $\boldsymbol{\Omega} = \sigma^2\mathbf{I}_G$,

$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}}_{SOLS} - \boldsymbol{\beta})] = [\text{E}(\mathbf{X}_i'\mathbf{X}_i)]^{-1}\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}\mathbf{X}_i)[\text{E}(\mathbf{X}_i'\mathbf{X}_i)]^{-1} = \sigma^2[\text{E}(\mathbf{X}_i'\mathbf{X}_i)]^{-1}$ and

$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}}_{SOLS} - \boldsymbol{\beta})] = [\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i)]^{-1} = [\text{E}(\mathbf{X}_i'(\sigma^2\mathbf{I}_G)^{-1}\mathbf{X}_i)]^{-1} = \sigma^2[\text{E}(\mathbf{X}_i'\mathbf{X}_i)]^{-1}$.

e. This statement is true provided we consider only asymptotic efficiency under the assumption that SGLS.1 holds. In other words, under SGLS.1, the standard rank conditions, and $\text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{X}_i) = \boldsymbol{\Omega}$, there is nothing to lose asymptotically by using FGLS. Of course, SOLS is more robust in that it only requires SOLS.1 for consistency (and asymptotic normality). Small sample properties are another issue because it is difficult to characterize the exact properties of FGLS under general conditions.

7.3. a. Since OLS equation-by-equation is the same as GLS when $\boldsymbol{\Omega}$ is diagonal, it suffices to show that the GLS estimators for different equations are asymptotically uncorrelated. This

follows if the asymptotic variance matrix is block diagonal (see Section 3.5), where the blocking is by the parameter vector for each equation. To establish block diagonality, we use the result from Theorem 7.4: under SGLS. 1, SGLS.2, and SGLS.3,

$$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = [\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i)]^{-1}.$$

Now, we can use the special form of $\mathbf{X}_i$ for SUR (see Example 7.1), the fact that $\boldsymbol{\Omega}^{-1}$ is diagonal, and SGLS.3. In the SUR model with diagonal $\boldsymbol{\Omega}$, SGLS.3 implies that $\text{E}(u_{ig}^2\mathbf{x}_{ig}'\mathbf{x}_{ig}) = \sigma_g^2\text{E}(\mathbf{x}_{ig}'\mathbf{x}_{ig})$ for all $g = 1, \ldots, G,$ and

$$\text{E}(u_{ig}u_{ih}\mathbf{x}_{ig}'\mathbf{x}_{ih}) = \text{E}(u_{ig}u_{ih})\text{E}(\mathbf{x}_{ig}'\mathbf{x}_{ih}) = \mathbf{0}, \text{ all } g \neq h.$$

Therefore, we have

$$\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i) = \begin{pmatrix} \sigma_1^{-2}\text{E}(\mathbf{x}_{i1}'\mathbf{x}_{il}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_G^{-2}\text{E}(\mathbf{x}_{iG}'\mathbf{x}_{iG}) \end{pmatrix}.$$

When this matrix is inverted, it is also block diagonal. This shows that $\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]$ is block diagonal, and therefore the $\sqrt{N}(\hat{\boldsymbol{\beta}}_g - \boldsymbol{\beta}_g)$ are asymptotically uncorrelated.

b. To test any linear hypothesis, we can either construct the Wald Statistic or we can use the weighted sum of squared residuals form of the statistic as in (7.56) or (7.57). For the restricted SSR we must estimate the model with the restriction $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ imposed. See Problem 7.6 for one way to impose general linear restrictions.

c. Actually, for the conclusion to hold about asymptotic equivalence, we need to assume SGLS.1 along with SOLS.2 and SGLS.2. When $\boldsymbol{\Omega}$ is diagonal in a SUR system, system OLS and GLS are the same. Under SGLS.1 and SGLS.2, GLS and FGLS are asymptotically equivalent (regardless of the structure of $\boldsymbol{\Omega}$) whether or not SGLS.3 holds. Now if

$\hat{\boldsymbol{\beta}}_{SOLS} = \hat{\boldsymbol{\beta}}_{GLS}$ and $\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{FGLS} - \hat{\boldsymbol{\beta}}_{GLS}\right) = o_p(1)$, then $\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{SOLS} - \hat{\boldsymbol{\beta}}_{FGLS}\right) = o_p(1)$. Thus, when $\boldsymbol{\Omega}$ is diagonal, OLS and FGLS are asymptotically equivalent under the exogeneity assumption SGLS.1, even if $\hat{\boldsymbol{\Omega}}$ is estimated in an unrestricted fashion and even if the system homoskedasticity assumption SGLS.3 does not hold.

If only SOLS.1 holds, we cannot conclude $\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{FGLS} - \hat{\boldsymbol{\beta}}_{GLS}\right) = o_p(1)$, and so $\sqrt{N}\left(\hat{\boldsymbol{\beta}}_{SOLS} - \hat{\boldsymbol{\beta}}_{FGLS}\right)$ is not generally $o_p(1)$. It is true that FGLS is still consistent under SOLS.1 because its plim is

$$\begin{pmatrix} \sigma_1^{-2}\mathrm{E}(\mathbf{x}_{i1}'\mathbf{x}_{il}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_G^{-2}\mathrm{E}(\mathbf{x}_{iG}'\mathbf{x}_{iG}) \end{pmatrix}^{-1} \begin{pmatrix} \sigma_1^{-2}\mathrm{E}(\mathbf{x}_{i1}'u_{i1}) \\ \vdots \\ \sigma_G^{-2}\mathrm{E}(\mathbf{x}_{iG}'u_{iG}) \end{pmatrix}$$

and $\mathrm{E}(\mathbf{x}_{ig}'u_{ig}) = \mathbf{0}$, $g = 1, \ldots, G$.

7.4. To make the notation align with the text, use $\check{\boldsymbol{\beta}}$ to denote the SOLS estimator, and let $\check{\mathbf{u}}_i$ denote the $G \times 1$ vector of SOLS residuals that are used in obtaining $\hat{\boldsymbol{\Omega}}$. Then it suffices to show that

$$N^{-1/2}\sum_{i=1}^{N}\check{\mathbf{u}}_i\check{\mathbf{u}}_i' = N^{-1/2}\sum_{i=1}^{N}\mathbf{u}_i\mathbf{u}_i + o_p(1), \tag{7.82}$$

and this follows if, when we sum across N and divide by $\sqrt{N}$, the last three terms in (7.42) are $o_p(1)$. Since the third term is the transpose of the second it suffices to consider only the second and fourth terms. Now

$$\begin{aligned} N^{-1/2}\sum_{i=1}^{N}\mathrm{vec}\left[\mathbf{u}_i\left(\check{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)'\mathbf{X}_i'\right] &= N^{-1/2}\sum_{i=1}^{N}(\mathbf{X}_i \otimes \mathbf{u}_i)\cdot\left(\check{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \\ &= \left[N^{-1}\sum_{i=1}^{N}(\mathbf{X}_i \otimes \mathbf{u}_i)\right]\sqrt{N}\left(\check{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = o_p(1)\cdot O_p(1) = o_p(1). \end{aligned}$$

Also,

$$N^{-1/2}\sum_{i=1}^{N}\text{vec}\left[\mathbf{X}_i\left(\hat{\check{\boldsymbol{\beta}}}-\beta\right)\left(\hat{\check{\boldsymbol{\beta}}}-\beta\right)'\mathbf{X}_i'\right]=\left[N^{-1}\sum_{i-1}^{N}(\mathbf{X}_i\otimes\mathbf{X}_i)\right]\text{vec}\left\{\sqrt{N}\left(\check{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)\sqrt{N}\left(\check{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)'\right\}/\sqrt{N}$$
$$=O_p(1)\cdot O_p(1)\cdot N^{-1/2}=o_p(1).$$

Together, these imply $N^{-1/2}\sum_{i=1}^{N}\check{\mathbf{u}}_i\check{\mathbf{u}}_i'=N^{-1/2}\sum_{i=1}^{N}\mathbf{u}_i\mathbf{u}_i+o_p(1)$ and so

$$N^{-1/2}\sum_{i=1}^{N}(\check{\mathbf{u}}_i\check{\mathbf{u}}_i'-\boldsymbol{\Omega})=N^{-1/2}\sum_{i=1}^{N}(\mathbf{u}_i\mathbf{u}_i-\boldsymbol{\Omega})+o_p(1).$$

7.5. This is easy with the hint. Note that

$$\left(\hat{\boldsymbol{\Omega}}^{-1}\otimes\left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)\right)^{-1}=\hat{\boldsymbol{\Omega}}\otimes\left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}.$$

Therefore,

$$\hat{\boldsymbol{\beta}}=\left(\hat{\boldsymbol{\Omega}}\otimes\left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\right)\left(\hat{\boldsymbol{\Omega}}^{-1}\otimes\mathbf{I}_K\right)\begin{pmatrix}\sum_{i=1}^{N}\mathbf{x}_i'y_{i1}\\ \vdots\\ \sum_{i=1}^{N}\mathbf{x}_i'y_{iG}\end{pmatrix}=\left(\mathbf{I}_G\otimes\left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\right)\begin{pmatrix}\sum_{i=1}^{N}\mathbf{x}_i'y_{i1}\\ \vdots\\ \sum_{i=1}^{N}\mathbf{x}_i'y_{iG}\end{pmatrix}.$$

$$=\begin{pmatrix}\left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} & \mathbf{0} & \vdots\\ \vdots & \mathbf{0} & \ddots & \mathbf{0}\\ \mathbf{0} & \cdots & \mathbf{0} & \left(\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\end{pmatrix}\begin{pmatrix}\sum_{i=1}^{N}\mathbf{x}_i'y_{i1}\\ \sum_{i=1}^{N}\mathbf{x}_i'y_{i2}\\ \\ \sum_{i=1}^{N}\mathbf{x}_i'y_{iG}\end{pmatrix}=\begin{pmatrix}\check{\boldsymbol{\beta}}_1\\ \check{\boldsymbol{\beta}}_2\\ \vdots\\ \check{\boldsymbol{\beta}}_G\end{pmatrix}$$

where $\check{\boldsymbol{\beta}}_g$ is the OLS estimator for equation g.

7.6. The model for a random draw from the population is $\mathbf{y}_i=\mathbf{X}_i\boldsymbol{\beta}+\mathbf{u}_i$, which can be written as

$$\mathbf{y}_i=\mathbf{X}_{i1}\boldsymbol{\beta}_1+\mathbf{X}_{i2}\boldsymbol{\beta}_2+\mathbf{u}_i,$$

where the partition of $\mathbf{X}_i$ is defined in the problem. Now, if $\boldsymbol{\beta}_1 = \mathbf{R}_1^{-1}(\mathbf{r} - \mathbf{R}_2\boldsymbol{\beta}_2)$, we just plug this into the previous equation:

$$\begin{aligned}\mathbf{y}_i &= \mathbf{X}_{i1}\boldsymbol{\beta}_1 + \mathbf{X}_{i2}\boldsymbol{\beta}_2 + \mathbf{u}_i = \mathbf{X}_{i1}\mathbf{R}_1^{-1}(\mathbf{r} - \mathbf{R}_2\boldsymbol{\beta}_2) + \mathbf{X}_{i2}\boldsymbol{\beta}_2 + \mathbf{u}_i \\ &= \mathbf{X}_{i1}\mathbf{R}_1^{-1}\mathbf{r} + (\mathbf{X}_{i2} - \mathbf{X}_{i1}\mathbf{R}_2)\boldsymbol{\beta}_2 + \mathbf{u}_i.\end{aligned}$$

Bringing $\mathbf{X}_{i1}\mathbf{R}_1^{-1}\mathbf{r}$ to the left hand side gives

$$\mathbf{y}_i - \mathbf{X}_{i1}\mathbf{R}_1^{-1}\mathbf{r} = (\mathbf{X}_{i2} - \mathbf{X}_{i1}\mathbf{R}_2)\boldsymbol{\beta}_2 + \mathbf{u}_i.$$

If we define $\tilde{\mathbf{y}}_i = \mathbf{y}_i - \mathbf{X}_{i1}\mathbf{R}_1^{-1}\mathbf{r}$ and $\tilde{\mathbf{X}}_{i2} \equiv \mathbf{X}_{i2} - \mathbf{X}_{i1}\mathbf{R}_2$, then we get the desired equation:

$$\tilde{\mathbf{y}}_i = \tilde{\mathbf{X}}_{i2}\boldsymbol{\beta}_2 + \mathbf{u}_i.$$

(Note that $\tilde{\mathbf{y}}_i$ and $\tilde{\mathbf{X}}_{i2}$ are functions of the data for observation i and the known matrices $\mathbf{R}_1, \mathbf{R}_2,$ and the known vector $\mathbf{r}$.)

This general result is very convenient for computing the weighted SSR form of the F statistic (under SGL.3). Let $\hat{\boldsymbol{\Omega}}$ denote the estimate of Ω based on estimation of the unconstrained system; typically, $\hat{\boldsymbol{\Omega}} = N^{-1}\sum_{i=1}^{N}\check{\mathbf{u}}_i\check{\mathbf{u}}_i'$ where $\check{\mathbf{u}}_i$ are the system OLS residuals. Using this matrix, we estimate $\mathbf{y}_i = \mathbf{X}_{i1}\boldsymbol{\beta}_1 + \mathbf{X}_{i2}\boldsymbol{\beta}_2 + \mathbf{u}_i$ and then $\tilde{\mathbf{y}}_i = \tilde{\mathbf{X}}_{i2}\boldsymbol{\beta}_2 + \mathbf{u}_i$ by FGLS using $\hat{\boldsymbol{\Omega}}$. Let $\hat{\mathbf{u}}_i$ denote the FGLS residuals from the unrestricted model and let $\tilde{\mathbf{u}}_i = \tilde{\mathbf{y}}_i - \tilde{\mathbf{X}}_{i2}\tilde{\boldsymbol{\beta}}_2$ denote the restricted FGLS residuals, where $\tilde{\boldsymbol{\beta}}_2$ is the FGLS estimator from the restricted estimation. Then the F statistic computed from (7.57) has an approximate $\mathcal{F}_{Q,NG-K}$ distribution under H_0 (assuming SGLS.1, SGLS.2, and SGLS.3 hold).

7.7. a. First, the diagonal elements of $\boldsymbol{\Omega}$ are easily found because $\mathrm{E}(u_{it}^2) = \mathrm{E}[\mathrm{E}(u_{it}^2|\mathbf{x}_{it})] = \sigma_t^2$ by iterated expectations. Now, consider $\mathrm{E}(u_{it}u_{is})$, and take $s < t$ without loss of generality. Under $\mathrm{E}(u_{it}|\mathbf{x}_{it}, u_{i,t-1}, \ldots) = 0$, $\mathrm{E}(u_{it}|u_{is}) = 0$ because u_{is} is a subset of the larger conditioning set. Applying LIE again we have

$$\mathrm{E}(u_{it}u_{is}) = \mathrm{E}[\mathrm{E}(u_{it}u_{is}|u_{is})] = \mathrm{E}[\mathrm{E}(u_{it}|u_{is})u_{is})] = 0.$$

So

$$\mathbf{\Omega} = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_T^2 \end{pmatrix}.$$

b. The GLS estimator is

$$\begin{aligned} \boldsymbol{\beta}^* &\equiv \left(\sum_{i=1}^{N} \mathbf{X}_i' \mathbf{\Omega}^{-1} \mathbf{X}_i \right)^{-1} \left(\sum_{i=1}^{N} \mathbf{X}_i' \mathbf{\Omega}^{-1} \mathbf{y}_i \right) \\ &= \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \sigma_t^{-2} \mathbf{x}_{it}' \mathbf{x}_{it} \right)^{-1} \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \sigma_t^{-2} \mathbf{x}_{it}' y_{it} \right), \end{aligned}$$

which is a weighted least squares estimator with every observation for time period t weighted by σ_t^{-2}, the inverse of the variance.

c. If, say, $y_{it} = \beta_0 + \beta_1 y_{i,t-1} + u_{it}$, then y_{it} is clearly correlated with u_{it}, which says that $\mathbf{x}_{i,t+1} = y_{it}$ is correlated with u_{it}. Thus, SGLS.1 cannot hold. Generally, SGLS.1 fails to hold whenever there is feedback from y_{it} to $\mathbf{x}_{is}, s > t$. Nevertheless, because $\mathbf{\Omega}^{-1}$ is diagonal, $\mathbf{X}_i' \mathbf{\Omega}^{-1} \mathbf{u}_i = \sum_{t=1}^{T} \mathbf{x}_{it}' \sigma_t^{-2} u_{it}$, and so

$$\mathrm{E}(\mathbf{X}_i' \mathbf{\Omega}^{-1} \mathbf{u}_i) - \sum_{t=1}^{T} \sigma_t^{-2} \mathrm{E}(\mathbf{x}_{it}' u_{it}) = \mathbf{0},$$

where we use $\mathrm{E}(\mathbf{x}_{it}' u_{it}) = 0$ under $\mathrm{E}(u_{it}|\mathbf{x}_{it}, u_{i,t-1}, \ldots) = 0$. It follows that the GLS estimator is GLS is consistent in this case without SGLS.1.

d. First, since $\mathbf{\Omega}^{-1}$ is diagonal, $\mathbf{X}_i' \mathbf{\Omega}^{-1} = (\sigma_1^{-2} \mathbf{x}_{il}', \sigma_2^{-2} \mathbf{x}_{i2}', \ldots, \sigma_T^{-2} \mathbf{x}_{iT}')'$, and so

$$\mathrm{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i) = \sum_{t=1}^{T}\sum_{s=1}^{T}\sigma_t^{-2}\sigma_s^{-2}\mathrm{E}(u_{it}u_{is}\mathbf{x}_{it}'\mathbf{x}_{is}).$$

First consider the terms for $s \neq t$. Under $\mathrm{E}(u_{it}|\mathbf{x}_{it}, u_{i,t-1}, \dots) = 0$, $\mathrm{E}(u_{it}|\mathbf{x}_{it}, u_{is}, \mathbf{x}_{is}) = 0$ for $s < t$, and so by the LIE, $\mathrm{E}(u_{it}u_{is}\mathbf{x}_{it}'\mathbf{x}_{is}) = 0$, all $t \neq s$. Next, for each t,

$$\begin{aligned}\mathrm{E}(u_{it}^2\mathbf{x}_{it}'\mathbf{x}_{it}) &= \mathrm{E}[\mathrm{E}(u_{it}^2\mathbf{x}_{it}'\mathbf{x}_{it}|\mathbf{x}_{it})] = \mathrm{E}[\mathrm{E}(u_{it}^2|\mathbf{x}_{it})\mathbf{x}_{it}'\mathbf{x}_{it})] \\ &= \mathrm{E}(\sigma_t^2\mathbf{x}_{it}'\mathbf{x}_{it}) = \sigma_t^2\mathrm{E}(\mathbf{x}_{it}'\mathbf{x}_{it}),\ t = 1, 2, \dots, T.\end{aligned}$$

It follows that

$$\mathrm{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i) = \sum_{t=1}^{T}\sigma_t^{-2}\mathrm{E}(\mathbf{x}_{it}'\mathbf{x}_{it}) = \mathrm{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i).$$

e. First, run pooled OLS across all i and t and let $\check{u}_{it}$ denote the pooled OLS residuals. Then, for each t, define

$$\hat{\sigma}_t^2 = N^{-1}\sum_{i=1}^{N}\check{u}_{it}^2$$

(We might replace N with $N - K$ as a degree-of-freedom adjustment.) By standard arguments, $\hat{\sigma}_t^2 \xrightarrow{p} \sigma_t^2$ as $N \to \infty$.

f. What we need to show is that replacing the σ_t^2 with the $\hat{\sigma}_t^2$ does not affect the $\sqrt{N}$-asymptotic distribution of the FGLS estimator. We know this generally under SGLS.1, but we have relaxed that assumption. To show it holds in the current setting we need to show

$$\begin{aligned}N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{\sigma}_t^{-2}\mathbf{x}_{it}'\mathbf{x}_{it} &= N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\sigma_t^{-2}\mathbf{x}_{it}'\mathbf{x}_{it} + o_p(1) \\ N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{\sigma}_t^{-2}\mathbf{x}_{it}'u_{it} &= N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sigma_t^{-2}\mathbf{x}_{it}'u_{it} + o_p(1).\end{aligned}$$

The first follows from the consistency of each $\hat{\sigma}_t^2$ using standard arguments we have used

before. The second requirement follows from

$$N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{\sigma}_t^{-2}\mathbf{x}_{it}'u_{it} - N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sigma_t^{-2}\mathbf{x}_{it}'u_{it} = \sum_{t=1}^{T}\left[N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_{it}'u_{it}\right](\hat{\sigma}_t^{-2}-\sigma_t^{-2})$$
$$= \sum_{t=1}^{T} O_p(1)\cdot o_p(1) = o_p(1)$$

because $N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_{it}'u_{it}$ satisfies the CLT under Under $\mathrm{E}(u_{it}|\mathbf{x}_{it}, u_{i,t-1}, \ldots) = 0$ and second moment assumptions.

So now we know all inference is as if we are applying pooled OLS to

$$(y_{it}/\sigma_t) = (\mathbf{x}_{it}/\sigma_t)\boldsymbol{\beta} + e_{it},\ t = 1, 2, \ldots, T$$

where this equation satisfies POLS.1, POLS.2, and POLS.3. Thus, we can use the usual statistics – standard errors, confidence intervals, *t* and *F* statistics – from the regression

$$(y_{it}/\hat{\sigma}_t) = (\mathbf{x}_{it}/\hat{\sigma}_t),\ t = 1, \ldots, T; i = 1, \ldots, N.$$

For *F* testing, note that the $\hat{\sigma}_t^2$ should be obtained using the pooled OLS residuals for the unrestricted model.

g. If $\sigma_t^2 = \sigma^2$ for all $t = 1, \ldots, T,$ inference is very easy because with weighted least squares method reduces to pooled OLS. Thus, we can use the standard errors and test statistics reported by a standard OLS regression pooled across *i* and *t*.

7.8. Here is some Stata output:

```
. use fringe
. gen hrvac = vacdays/annhrs
. gen hrsick = sicklve/annhrs
. gen hrins = insur/annhrs
. gen hrpens = pension/annhrs
. sureg (hrearn hrvac hrsick hrins hrpens = educ exper expersq tenure
```

```
        tenuresq union south nrtheast nrthcen married white male), corr

Seemingly unrelated regression
------------------------------------------------------------------------------
Equation           Obs  Parms        RMSE    "R-sq"       chi2        P
------------------------------------------------------------------------------
hrearn             616     12      4.3089    0.2051     158.93   0.0000
hrvac              616     12    .1389899    0.3550     339.01   0.0000
hrsick             616     12     .056924    0.2695     227.23   0.0000
hrins              616     12    .1573797    0.3891     392.27   0.0000
hrpens             616     12    .2500388    0.3413     319.16   0.0000
------------------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
hrearn       |
        educ |   .4588139    .068393     6.71   0.000     .3247662    .5928617
       exper |  -.0758428   .0567371    -1.34   0.181    -.1870455    .0353598
     expersq |   .0039945   .0011655     3.43   0.001     .0017102    .0062787
      tenure |   .1100846   .0829207     1.33   0.184     -.052437    .2726062
    tenuresq |  -.0050706   .0032422    -1.56   0.118    -.0114252    .0012839
       union |   .8079933   .4034789     2.00   0.045     .0171892    1.598797
       south |  -.4566222   .5458508    -0.84   0.403     -1.52647    .6132258
    nrtheast |  -1.150759   .5993283    -1.92   0.055     -2.32542    .0239032
     nrthcen |  -.6362663   .5501462    -1.16   0.247    -1.714533    .4420005
     married |   .6423882   .4133664     1.55   0.120     -.167795    1.452571
       white |   1.140891   .6054474     1.88   0.060    -.0457639    2.327546
        male |   1.784702   .3937853     4.53   0.000     1.012897    2.556507
       _cons |  -2.632127   1.215291    -2.17   0.030    -5.014054   -.2501997
-------------+----------------------------------------------------------------
hrvac        |
        educ |   .0201829   .0022061     9.15   0.000      .015859    .0245068
       exper |   .0066493   .0018301     3.63   0.000     .0030623    .0102363
     expersq |  -.0001492   .0000376    -3.97   0.000    -.0002229   -.0000755
      tenure |    .012386   .0026747     4.63   0.000     .0071436    .0176284
    tenuresq |  -.0002155   .0001046    -2.06   0.039    -.0004205   -.0000106
       union |   .0637464   .0130148     4.90   0.000     .0382378    .0892549
       south |  -.0179005   .0176072    -1.02   0.309      -.05241     .016609
    nrtheast |  -.0169824   .0193322    -0.88   0.380    -.0548728    .0209081
     nrthcen |   .0002511   .0177458     0.01   0.989      -.03453    .0350321
     married |   .0227586   .0133337     1.71   0.088    -.0033751    .0488923
       white |   .0084869   .0195296     0.43   0.664    -.0297905    .0467642
        male |   .0569525   .0127021     4.48   0.000     .0320568    .0818482
       _cons |  -.1842348    .039201    -4.70   0.000    -.2610674   -.1074022
-------------+----------------------------------------------------------------
hrsick       |
        educ |   .0096054   .0009035    10.63   0.000     .0078346    .0113763
       exper |    .002145   .0007495     2.86   0.004     .0006759    .0036141
     expersq |  -.0000383   .0000154    -2.48   0.013    -.0000684   -8.08e-06
      tenure |   .0050021   .0010954     4.57   0.000      .002855    .0071491
    tenuresq |  -.0001391   .0000428    -3.25   0.001    -.0002231   -.0000552
       union |  -.0046655   .0053303    -0.88   0.381    -.0151127    .0057816
       south |   -.011942   .0072111    -1.66   0.098    -.0260755    .0021916
    nrtheast |  -.0026651   .0079176    -0.34   0.736    -.0181833    .0128531
     nrthcen |  -.0222014   .0072679    -3.05   0.002    -.0364462   -.0079567
     married |   .0038338   .0054609     0.70   0.483    -.0068694     .014537
       white |   .0038635   .0079984     0.48   0.629    -.0118132    .0195401
        male |   .0042538   .0052022     0.82   0.414    -.0059423      .01445
```

```
       _cons |  -.0937606    .016055    -5.84   0.000    -.1252278   -.0622935
-------------+----------------------------------------------------------------
hrins        |
        educ |   .0080042    .002498     3.20   0.001     .0031082    .0129002
       exper |   .0054052   .0020723     2.61   0.009     .0013436    .0094668
     expersq |  -.0001266   .0000426    -2.97   0.003      -.00021   -.0000431
      tenure |   .0116978   .0030286     3.86   0.000     .0057618    .0176338
    tenuresq |  -.0002466   .0001184    -2.08   0.037    -.0004787   -.0000146
       union |   .1441536   .0147368     9.78   0.000       .11527    .1730372
       south |   .0196786   .0199368     0.99   0.324    -.0193969    .0587541
    nrtheast |  -.0052563   .0218901    -0.24   0.810    -.0481601    .0376474
     nrthcen |   .0242515   .0200937     1.21   0.227    -.0151315    .0636345
     married |   .0365441   .0150979     2.42   0.016     .0069527    .0661355
       white |   .0378883   .0221136     1.71   0.087    -.0054535    .0812301
        male |   .1120058   .0143827     7.79   0.000     .0838161    .1401955
       _cons |  -.1180824   .0443877    -2.66   0.008    -.2050807   -.0310841
-------------+----------------------------------------------------------------
hrpens       |
        educ |   .0390226   .0039687     9.83   0.000      .031244    .0468012
       exper |   .0083791   .0032924     2.55   0.011     .0019262    .0148321
     expersq |  -.0001595   .0000676    -2.36   0.018    -.0002921    -.000027
      tenure |   .0243758   .0048118     5.07   0.000     .0149449    .0338067
    tenuresq |  -.0005597   .0001881    -2.97   0.003    -.0009284   -.0001909
       union |   .1621404   .0234133     6.93   0.000     .1162513    .2080296
       south |  -.0130816   .0316749    -0.41   0.680    -.0751632        .049
    nrtheast |  -.0323117   .0347781    -0.93   0.353    -.1004755    .0358521
     nrthcen |  -.0408177   .0319241    -1.28   0.201    -.1033878    .0217525
     married |  -.0051755    .023987    -0.22   0.829    -.0521892    .0418381
       white |   .0395839   .0351332     1.13   0.260    -.0292758    .1084437
        male |   .0952459   .0228508     4.17   0.000     .0504592    .1400325
       _cons |  -.4928338   .0705215    -6.99   0.000    -.6310534   -.3546143
------------------------------------------------------------------------------

Correlation matrix of residuals:

             hrearn   hrvac  hrsick   hrins  hrpens
hrearn       1.0000
 hrvac       0.2719  1.0000
hrsick       0.2541  0.5762  1.0000
 hrins       0.2609  0.6701  0.2922  1.0000
hrpens       0.2786  0.7070  0.4569  0.6345  1.0000

Breusch-Pagan test of independence: chi2(10) =  1393.265, Pr = 0.0000

. test married

 ( 1)  [hrearn]married = 0
 ( 2)  [hrvac]married = 0
 ( 3)  [hrsick]married = 0
 ( 4)  [hrins]married = 0
 ( 5)  [hrpens]married = 0

           chi2(  5) =   14.48
         Prob > chi2 =    0.0128

. lincom [hrpens]educ - [hrins]educ

 ( 1)  - [hrins]educ + [hrpens]educ = 0
```

```
------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         (1) |   .0310184   .0030676    10.11   0.000      .025006    .0370308
------------------------------------------------------------------------------
```

The first test shows that there is some evidence that marital status affects at least one of the five forms of compensation. In fact, it has the largest economic effect on hourly earnings: .642, but its t statistic is only about 1.54. The most statistically significant effect is on *hrins*: .037 with $t = 2.42$. It is marginally significant and positive for *hrvac* as well.

The `lincom` command tests whether another year of education has the same effect on *hrpens* and *hrins*. The t statistic is 10.11 and the p-value is effectively zero. The estimate in the *hrpens* equation (with standard error) is .039 (.004) while the estimate in the *hrins* equation is .008 (.003). Thus, each is positive and statistically significant, and they are significantly different from one another.

All of the standard errors and statistics reported above assume that SGLS.3 holds, so that there can be no system heteroskedasticity. This is unlikely to hold in this example.

7.9. The Stata session follows, including a test for serial correlation before computing the fully robust standard errors:

```
. use jtrain1

. xtset fcode year
       panel variable:  fcode (strongly balanced)
        time variable:  year, 1987 to 1989
                delta:  1 unit

. reg lscrap d89 grant grant_1 lscrap_1 if year != 1987

      Source |       SS       df       MS              Number of obs =     108
-------------+------------------------------           F(  4,   103) =  153.67
       Model |  186.376973     4  46.5942432           Prob > F      =  0.0000
    Residual |  31.2296502   103  .303200488           R-squared     =  0.8565
-------------+------------------------------           Adj R-squared =  0.8509
       Total |  217.606623   107  2.03370676           Root MSE      =  .55064

------------------------------------------------------------------------------
      lscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
```

```
         d89 |  -.1153893   .1199127    -0.96   0.338    -.3532078    .1224292
       grant |  -.1723924   .1257443    -1.37   0.173    -.4217765    .0769918
     grant_1 |  -.1073226   .1610378    -0.67   0.507     -.426703    .2120579
    lscrap_1 |   .8808216   .0357963    24.61   0.000      .809828    .9518152
       _cons |  -.0371354   .0883283    -0.42   0.675    -.2123137     .138043
------------------------------------------------------------------------------
```

The estimated effect of *grant*, and its lag, are now the expected sign (if we think the job training program should reduce the scrap rate), but neither is strongly statistically significant. The variable *grant* would be if we use a 10% significance level and a one-sided test. The results are certainly different from when we omit the lag of log(*scrap*).

Now test for $AR(1)$ serial correlation:

```
. gen uhat_1 = l.uhat
(417 missing values generated)

. reg lscrap grant grant_1 lscrap_1 uhat_1 if d89

      Source |       SS       df       MS              Number of obs =      54
-------------+------------------------------           F(  4,    49) =   73.47
       Model |  94.4746525     4  23.6186631           Prob > F      =  0.0000
    Residual |  15.7530202    49  .321490208           R-squared     =  0.8571
-------------+------------------------------           Adj R-squared =  0.8454
       Total |  110.227673    53  2.07976741           Root MSE      =    .567

------------------------------------------------------------------------------
      lscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       grant |   .0165089    .215732     0.08   0.939    -.4170208    .4500385
     grant_1 |  -.0276544   .1746251    -0.16   0.875    -.3785767    .3232679
    lscrap_1 |   .9204706   .0571831    16.10   0.000     .8055569    1.035384
      uhat_1 |   .2790328   .1576739     1.77   0.083    -.0378247    .5958904
       _cons |   -.232525   .1146314    -2.03   0.048    -.4628854   -.0021646
------------------------------------------------------------------------------
```

The estimate of ρ is about .28, and it is marginally significant with $t = 1.77$. (Note we are relying on asymptotics with $N = 54$.) One could probably make a case for ignoring the serial correlation. But it is easy enough to obtain the serial-correlation and heteroskedasticity-robust standard errors:

```
. reg lscrap d89 grant grant_1 lscrap_1 if year != 1987, robust cluster(fcode

Linear regression                                       Number of obs =     108
                                                        F(  4,    53) =   77.24
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.8565
```

```
                                                      Root MSE      =  .55064

                                 (Std. Err. adjusted for 54 clusters in fcode
------------------------------------------------------------------------------
             |               Robust
      lscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d89 |  -.1153893   .1145118    -1.01   0.318    -.3450708    .1142922
       grant |  -.1723924   .1188807    -1.45   0.153    -.4108369    .0660522
     grant_1 |  -.1073226   .1790052    -0.60   0.551    -.4663616    .2517165
    lscrap_1 |   .8808216   .0645344    13.65   0.000     .7513821    1.010261
       _cons |  -.0371354   .0893147    -0.42   0.679     -.216278    .1420073
------------------------------------------------------------------------------
```

The robust standard errors for *grant* and $grant_{-1}$ are actually smaller than the usual ones, but each is still not statistically significant at the 5% level against a one-sided alternative. In addition, they are not jointly significant, as the *p*-value is about .33:

```
. test grant grant_1

 ( 1)  grant = 0
 ( 2)  grant_1 = 0

       F(  2,    53) =    1.14
            Prob > F =    0.3266
```

7.10. The Stata results are:

```
. use gpa

. reg trmgpa spring cumgpa crsgpa frstsem season sat verbmath hsperc hssize
      black female

      Source |       SS       df       MS              Number of obs =     732
-------------+------------------------------           F( 11,   720) =   70.64
       Model |  218.156689    11  19.8324263           Prob > F      =  0.0000
    Residual |  202.140267   720  .280750371           R-squared     =  0.5191
-------------+------------------------------           Adj R-squared =  0.5117
       Total |  420.296956   731  .574961636           Root MSE      =  .52986

------------------------------------------------------------------------------
      trmgpa |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      spring |  -.0121568   .0464813    -0.26   0.794    -.1034118    .0790983
      cumgpa |   .3146158   .0404916     7.77   0.000     .2351201    .3941115
      crsgpa |   .9840371   .0960343    10.25   0.000     .7954964    1.172578
     frstsem |   .7691192   .1204162     6.39   0.000     .5327104    1.005528
      season |  -.0462625   .0470985    -0.98   0.326    -.1387292    .0462042
         sat |   .0014097   .0001464     9.63   0.000     .0011223    .0016972
    verbmath |   -.112616   .1306157    -0.86   0.389    -.3690491    .1438171
      hsperc |  -.0066014   .0010195    -6.48   0.000    -.0086029   -.0045998
      hssize |  -.0000576   .0000994    -0.58   0.562    -.0002527    .0001375
       black |  -.2312855   .0543347    -4.26   0.000    -.3379589   -.1246122
      female |   .2855528   .0509641     5.60   0.000     .1854967    .3856089
```

```
       _cons |  -2.067599   .3381007    -6.12   0.000    -2.731381   -1.403818
------------------------------------------------------------------------------

. reg trmgpa spring cumgpa crsgpa frstsem season sat verbmath hsperc hssize
      black female, robust cluster(id)

Linear regression                                      Number of obs =     732
                                                       F( 11,   365) =   71.31
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.5191
                                                       Root MSE      =  .52986

                                    (Std. Err. adjusted for 366 clusters in id
------------------------------------------------------------------------------
             |               Robust
      trmgpa |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      spring |  -.0121568   .0395519    -0.31   0.759     -.089935    .0656215
      cumgpa |   .3146158   .0514364     6.12   0.000     .2134669    .4157647
      crsgpa |   .9840371     .09182    10.72   0.000     .8034745      1.1646
     frstsem |   .7691192   .1437178     5.35   0.000     .4865003    1.051738
      season |  -.0462625   .0431631    -1.07   0.285     -.131142     .038617
         sat |   .0014097   .0001743     8.09   0.000      .001067    .0017525
    verbmath |   -.112616   .1495196    -0.75   0.452    -.4066441    .1814121
      hsperc |  -.0066014   .0011954    -5.52   0.000    -.0089522   -.0042506
      hssize |  -.0000576   .0001066    -0.54   0.589    -.0002673    .0001521
       black |  -.2312855   .0695278    -3.33   0.001     -.368011   -.0945601
      female |   .2855528   .0511767     5.58   0.000     .1849146     .386191
       _cons |  -2.067599   .3327336    -6.21   0.000    -2.721915   -1.413284
------------------------------------------------------------------------------
```

Some of the fully robust standard errors are actually smaller than the corresponding nonrobust standard error, although the one on *cumgpa* is quite a bit larger, and drops the t statistic from 10.25 to 6.12. No variable that was statistically significant based on the usual t statistic becomes statistically insignificant, although the length of some confidence intervals change. The t statistics for the key variable, *season*, are similarly and show *season* is not statistically significant.

7.11. a. The following Stata output should be self-explanatory. There is clearly strong positive serial correlation in the errors of the static model ($\hat{\rho} = .792$, $t_{\hat{\rho}} = 28.84$) and the fully robust standard errors are much larger than the nonrobust ones. Not, for example, that the t statistic on the log of the conviction probability, *lprbconv* goes from -20.69 to -7.75.

```
. use cornwell
```

```
. xtset county year
       panel variable:  county (strongly balanced)
        time variable:  year, 81 to 87
                delta:  1 unit

. reg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87

      Source |       SS       df       MS              Number of obs =     630
-------------+------------------------------           F( 11,   618) =   74.49
       Model |  117.644669    11  10.6949699           Prob > F      =  0.0000
    Residual |   88.735673   618  .143585231           R-squared     =  0.5700
-------------+------------------------------           Adj R-squared =  0.5624
       Total |  206.380342   629  .328108652           Root MSE      =  .37893

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lprbarr |  -.7195033   .0367657   -19.57   0.000    -.7917042   -.6473024
    lprbconv |  -.5456589   .0263683   -20.69   0.000    -.5974413   -.4938765
    lprbpris |   .2475521   .0672268     3.68   0.000     .1155314    .3795728
     lavgsen |  -.0867575   .0579205    -1.50   0.135    -.2005023    .0269872
      lpolpc |   .3659886   .0300252    12.19   0.000     .3070248    .4249525
         d82 |   .0051371    .057931     0.09   0.929    -.1086284    .1189026
         d83 |   -.043503   .0576243    -0.75   0.451    -.1566662    .0696601
         d84 |  -.1087542    .057923    -1.88   0.061     -.222504    .0049957
         d85 |  -.0780454   .0583244    -1.34   0.181    -.1925835    .0364928
         d86 |  -.0420791   .0578218    -0.73   0.467      -.15563    .0714719
         d87 |  -.0270426    .056899    -0.48   0.635    -.1387815    .0846963
       _cons |  -2.082293   .2516253    -8.28   0.000    -2.576438   -1.588149
------------------------------------------------------------------------------

. predict uhat, resid

. gen uhat_1 = l.uhat
(90 missing values generated)

. reg uhat uhat_1

      Source |       SS       df       MS              Number of obs =     540
-------------+------------------------------           F(  1,   538) =  831.46
       Model |  46.6680407     1  46.6680407           Prob > F      =  0.0000
    Residual |  30.1968286   538  .056127934           R-squared     =  0.6071
-------------+------------------------------           Adj R-squared =  0.6064
       Total |  76.8648693   539  .142606437           Root MSE      =  .23691

------------------------------------------------------------------------------
        uhat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      uhat_1 |   .7918085     .02746    28.84   0.000     .7378666    .8457504
       _cons |   1.74e-10   .0101951     0.00   1.000    -.0200271    .0200271
------------------------------------------------------------------------------

. reg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87, cluster(county

Linear regression                                       Number of obs =     630
                                                        F( 11,    89) =   37.19
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.5700
                                                        Root MSE      =  .37893
```

```
                                (Std. Err. adjusted for 90 clusters in county
------------------------------------------------------------------------------
             |               Robust
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     lprbarr |  -.7195033   .1095979    -6.56   0.000    -.9372719   -.5017347
    lprbconv |  -.5456589   .0704368    -7.75   0.000    -.6856152   -.4057025
    lprbpris |   .2475521   .1088453     2.27   0.025     .0312787    .4638255
     lavgsen |  -.0867575   .1130321    -0.77   0.445    -.3113499    .1378348
      lpolpc |   .3659886    .121078     3.02   0.003     .1254092    .6065681
         d82 |   .0051371   .0367296     0.14   0.889    -.0678439    .0781181
         d83 |   -.043503    .033643    -1.29   0.199    -.1103509    .0233448
         d84 |  -.1087542   .0391758    -2.78   0.007    -.1865956   -.0309127
         d85 |  -.0780454   .0385625    -2.02   0.046    -.1546683   -.0014224
         d86 |  -.0420791   .0428788    -0.98   0.329    -.1272783    .0431201
         d87 |  -.0270426   .0381447    -0.71   0.480    -.1028353    .0487502
       _cons |  -2.082293   .8647054    -2.41   0.018    -3.800445   -.3641423
------------------------------------------------------------------------------

. drop uhat uhat_1
```

b. We lose the first year, 1981, when we add the lag of log(*crmrte*):

```
. gen lcrmrte_1 = l.lcrmrte
(90 missing values generated)

. reg lcrmrte lcrmrte_1 lprbarr lprbconv lprbpris lavgsen lpolpc d83-d87

      Source |       SS       df       MS              Number of obs =     540
-------------+------------------------------           F( 11,   528) =  464.68
       Model |  163.287174    11  14.8442885           Prob > F      =  0.0000
    Residual |  16.8670945   528  .031945255           R-squared     =  0.9064
-------------+------------------------------           Adj R-squared =  0.9044
       Total |  180.154268   539  .334237975           Root MSE      =  .17873

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
   lcrmrte_1 |   .8263047   .0190806    43.31   0.000     .7888214    .8637879
     lprbarr |  -.1668349   .0229405    -7.27   0.000    -.2119007   -.1217691
    lprbconv |  -.1285118   .0165096    -7.78   0.000    -.1609444   -.0960793
    lprbpris |  -.0107492   .0345003    -0.31   0.755     -.078524    .0570255
     lavgsen |  -.1152298    .030387    -3.79   0.000     -.174924   -.0555355
      lpolpc |    .101492   .0164261     6.18   0.000     .0692234    .1337606
         d83 |  -.0649438   .0267299    -2.43   0.015    -.1174537   -.0124338
         d84 |  -.0536882   .0267623    -2.01   0.045    -.1062619   -.0011145
         d85 |  -.0085982   .0268172    -0.32   0.749    -.0612797    .0440833
         d86 |   .0420159    .026896     1.56   0.119    -.0108203    .0948522
         d87 |   .0671272   .0271816     2.47   0.014     .0137298    .1205245
       _cons |  -.0304828   .1324195    -0.23   0.818    -.2906166     .229651
------------------------------------------------------------------------------
```

Not surprisingly, the coefficient on the lagged crime rate is very large and statistically significant. Further, including it makes all other coefficients much smaller in magnitude. The

variable log(*prbpris*) now has a negative sign, although it is insignificant. Adding the lagged crime rate does not change the positive coefficient on the size of the police force: it is smaller but now even more statistically significant.

c. There is little evidence of serial correlation in the model with a lagged dependent variable. The coefficient on $\hat{u}_{t-1}$ is small and statistically insignificant:

```
. predict uhat, resid
(90 missing values generated)

. gen uhat_1 = l.uhat
(180 missing values generated)

. reg lcrmrte lcrmrte_1 lprbarr lprbconv lprbpris lavgsen lpolpc d84-d87
      uhat_1

      Source |       SS       df       MS              Number of obs =     450
-------------+------------------------------           F( 11,   438) =  370.77
       Model |  138.488359    11  12.5898508           Prob > F      =  0.0000
    Residual |  14.8729012   438  .033956395           R-squared     =  0.9030
-------------+------------------------------           Adj R-squared =  0.9006
       Total |   153.36126   449  .341561826           Root MSE      =  .18427

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
   lcrmrte_1 |    .829714   .0248121    33.44   0.000     .7809485    .8784796
     lprbarr |  -.1576381   .0278786    -5.65   0.000    -.2124305   -.1028457
    lprbconv |  -.1293032   .0191735    -6.74   0.000    -.1669868   -.0916197
    lprbpris |  -.0040031   .0395191    -0.10   0.919    -.0816738    .0736675
     lavgsen |  -.1241479    .034481    -3.60   0.000    -.1919166   -.0563791
      lpolpc |   .1107055   .0187613     5.90   0.000     .0738323    .1475788
         d84 |   .0103772   .0277393     0.37   0.709    -.0441415    .0648959
         d85 |   .0557956   .0277577     2.01   0.045     .0012407    .1103505
         d86 |    .107831   .0277087     3.89   0.000     .0533724    .1622895
         d87 |   .1333345   .0279635     4.77   0.000     .0783751    .1882938
      uhat_1 |  -.0592978   .0601101    -0.99   0.324     -.177438    .0588423
       _cons |   .0126059   .1524765     0.08   0.934    -.2870706    .3122823
------------------------------------------------------------------------------
```

d. None of the log(*wage*) variables is statistically significant, and the magnitudes are pretty small in all cases. The *p*-value for the joint test, made fully robust, is .33, which means the log(*wage*) variables are jointly insignificant, too. (Plus, the different signs on the wage variables is hard to explain, except to conclude that each is estimated with substantial sampling

error.)

```
. reg lcrmrte lcrmrte_1 lprbarr lprbconv lprbpris lavgsen lpolpc d83-d87
      lwcon-lwloc, cluster(county)

Linear regression                                       Number of obs =     540
                                                        F( 20,    89) =  398.63
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.9077
                                                        Root MSE      =  .17895

                                (Std. Err. adjusted for 90 clusters in county
------------------------------------------------------------------------------
             |               Robust
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
   lcrmrte_1 |   .8087768   .0406432    19.90   0.000     .7280195     .889534
     lprbarr |  -.1746053   .0495539    -3.52   0.001    -.2730678   -.0761428
    lprbconv |  -.1337714   .0289031    -4.63   0.000    -.1912012   -.0763415
    lprbpris |  -.0195318   .0404094    -0.48   0.630    -.0998243    .0607608
     lavgsen |  -.1108926   .0455404    -2.44   0.017    -.2013804   -.0204049
      lpolpc |   .1050704   .0575404     1.83   0.071    -.0092612     .219402
         d83 |  -.0729231   .0293628    -2.48   0.015    -.1312664   -.0145799
         d84 |  -.0652494   .0226239    -2.88   0.005    -.1102026   -.0202962
         d85 |  -.0258059   .0413435    -0.62   0.534    -.1079545    .0563428
         d86 |   .0263763   .0393741     0.67   0.505    -.0518591    .1046118
         d87 |   .0465632   .0441727     1.05   0.295     -.041207    .1343334
       lwcon |  -.0283133   .0272813    -1.04   0.302    -.0825207     .025894
       lwtuc |  -.0034567   .0208431    -0.17   0.869    -.0448715    .0379582
       lwtrd |   .0121236   .0496718     0.24   0.808    -.0865733    .1108205
       lwfir |   .0296003   .0184296     1.61   0.112    -.0070189    .0662195
       lwser |    .012903   .0269695     0.48   0.634    -.0406847    .0664908
       lwmfg |  -.0409046   .0508117    -0.81   0.423    -.1418664    .0600573
       lwfed |   .1070534   .0760639     1.41   0.163     -.044084    .2581908
       lwsta |  -.0903894   .0548237    -1.65   0.103     -.199323    .0185442
       lwloc |   .0961124   .1355681     0.71   0.480    -.1732585    .3654833
       _cons |  -.6438061   .7958054    -0.81   0.421    -2.225055    .9374423
------------------------------------------------------------------------------

. testparm lwcon-lwloc

 ( 1)  lwcon = 0
 ( 2)  lwtuc = 0
 ( 3)  lwtrd = 0
 ( 4)  lwfir = 0
 ( 5)  lwser = 0
 ( 6)  lwmfg = 0
 ( 7)  lwfed = 0
 ( 8)  lwsta = 0
 ( 9)  lwloc = 0

       F(  9,    89) =    1.15
            Prob > F =    0.3338
```

7.12. Wealth at the beginning of the year cannot be strictly exogenous in a savings

equation: if saving increases unexpectedly this year – so that the disturbance in year t is positive – beginning of year wealth is higher next year. This is analogous to Example 7.8, where cumulative grade point average at the start of the semester cannot be strictly exogenous in an equation to explain current-term GPA.

7.13. a. The Stata output is below. Married men are estimated to have a scoring average about 1.2 points higher, and assists are $.42$ higher. The coefficient in the *rebounds* equation is $-.24$, but it is not statistically significant. The coefficient in the *assists* equation is significant at the 5% level against a two-sided alternative (p-value = $.048$).

```
. use nbasal

. sureg (points rebounds assists =  age exper expersq coll guard forward black

Seemingly unrelated regression
------------------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"       chi2        P
------------------------------------------------------------------------------
points            282      8    5.352116    0.1750      59.80   0.0000
rebounds          282      8    2.375338    0.3123     128.07   0.0000
assists           282      8     1.64516    0.3727     167.51   0.0000
------------------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
points       |
         age |  -1.214936     .27656    -4.39   0.000    -1.756984   -.6728889
       exper |   2.261943   .3759275     6.02   0.000     1.525138    2.998747
     expersq |  -.0649631   .0204961    -3.17   0.002    -.1051347   -.0247915
        coll |  -1.011535   .4026169    -2.51   0.012    -1.800649   -.2224201
       guard |   1.997013   .9380542     2.13   0.033     .1584603    3.835565
     forward |   1.348821   .9390513     1.44   0.151    -.4916863    3.189327
       black |   1.476842   .8171698     1.81   0.071    -.1247815    3.078465
        marr |   1.236043    .696663     1.77   0.076    -.1293911    2.601478
       _cons |    34.8283   6.771391     5.14   0.000     21.55662    48.09998
-------------+----------------------------------------------------------------
rebounds     |
         age |  -.2818077   .1227409    -2.30   0.022    -.5223754     -.04124
       exper |    .830967   .1668415     4.98   0.000     .5039637     1.15797
     expersq |  -.0344878   .0090964    -3.79   0.000    -.0523165   -.0166591
        coll |  -.3689707   .1786866    -2.06   0.039      -.71919   -.0187514
       guard |  -2.727081   .4163206    -6.55   0.000    -3.543054   -1.911107
     forward |   .0896382   .4167631     0.22   0.830    -.7272024    .9064789
       black |   1.003824   .3626705     2.77   0.006     .2930028    1.714645
        marr |  -.2406585    .309188    -0.78   0.436    -.8466559    .3653389
       _cons |   10.87601   3.005231     3.62   0.000     4.985864    16.76615
```

```
-------------+----------------------------------------------------------------
assists      |
         age |  -.3013925   .0850104    -3.55   0.000    -.4680097   -.1347752
       exper |   .6633331   .1155545     5.74   0.000     .4368506    .8898157
     expersq |  -.0222961   .0063002    -3.54   0.000    -.0346442    -.009948
        coll |  -.1894703   .1237584    -1.53   0.126    -.4320323    .0530916
       guard |   2.478626   .2883437     8.60   0.000     1.913482    3.043769
     forward |   .4804238   .2886502     1.66   0.096    -.0853202    1.046168
       black |  -.1528242   .2511857    -0.61   0.543     -.645139    .3394907
        marr |   .4236511   .2141437     1.98   0.048     .0039371     .843365
       _cons |   7.501437   2.081423     3.60   0.000     3.421922    11.58095
------------------------------------------------------------------------------
```

b. The Stata `test` command gives

```
. test marr

 ( 1)  [points]marr = 0
 ( 2)  [rebounds]marr = 0
 ( 3)  [assists]marr = 0

           chi2(  3) =   12.02
         Prob > chi2 =    0.0073
```

The rejection is very strong, presumably coming mainly from the points and assists equations. Rather than thinking being married causes a basketball player to be more productive, it could be that the more productive players – at least when it comes to points and assists – are more likely to be married.

7.14. Let $\hat{\boldsymbol{\beta}}$ be the estimator that uses $\hat{\boldsymbol{\Omega}}$ and let $\check{\boldsymbol{\beta}}$ be the estimator that uses $\hat{\boldsymbol{\Lambda}}$. Because SGLS.1 to SGLS.3 hold,

$$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = [\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i)]^{-1}.$$

Further, we know from the general result for FGLS,

$$\text{Avar}[\sqrt{N}(\check{\boldsymbol{\beta}} - \boldsymbol{\beta})] = [\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)]^{-1}\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)[\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)]^{-1}.$$

Now, because $\text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{X}_i) = \boldsymbol{\Omega}$, it follows that

$$\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i) = \text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)$$

by a simple iterated expectations argument. So, we have to show that

$$[\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i)]^{-1}\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{\Omega}\mathbf{\Lambda}^{-1}\mathbf{X}_i)[\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i)]^{-1} - [\mathrm{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i)]^{-1}$$

is positive semi-definite. We use the standard trick of showing that

$$\mathrm{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i) - \mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i)[\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{\Omega}\mathbf{\Lambda}^{-1}\mathbf{X}_i)]^{-1}\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i)$$

is positive semi-definite. To this end, define $\mathbf{Z}_i \equiv \mathbf{\Omega}^{-1/2}\mathbf{X}_i$ and $\mathbf{W}_i \equiv \mathbf{\Omega}^{-1/2}\mathbf{\Lambda}^{-1}\mathbf{X}_i$. Then straightforward algebra shows that the difference above can be written as $\mathrm{E}(\mathbf{Z}_i'\mathbf{Z}_i) - \mathrm{E}(\mathbf{Z}_i'\mathbf{W}_i)[\mathrm{E}(\mathbf{W}_i'\mathbf{W}_i)]^{-1}\mathrm{E}(\mathbf{W}_i'\mathbf{Z}_i)$ which is easily seen to be $\mathrm{E}(\mathbf{R}_i'\mathbf{R}_i)$, where $\mathbf{R}_i$ is the $G \times K$ matrix of population residuals from the regression of $\mathbf{Z}_i$ on $\mathbf{W}_i$: $\mathbf{R}_i = \mathbf{Z}_i - \mathbf{W}_i\mathbf{\Pi}$ where $\mathbf{\Pi} = [\mathrm{E}(\mathbf{W}_i'\mathbf{W}_i)]^{-1}\mathrm{E}(\mathbf{W}_i'\mathbf{Z}_i)$. Matrices of the form $\mathrm{E}(\mathbf{R}_i'\mathbf{R}_i)$ are always positive semi-definite because for a nonrandom vector $\mathbf{a}$, $\mathbf{a}'\mathrm{E}(\mathbf{R}_i'\mathbf{R}_i)\mathbf{a} = \mathrm{E}[(\mathbf{a}\mathbf{R}_i)'(\mathbf{a}\mathbf{R}_i)] \geq 0$.

7.15. Let $\hat{\boldsymbol{\delta}} = (\hat{\boldsymbol{\beta}}', \hat{\boldsymbol{\gamma}}')'$ be the FGLS estimator from the full model. Then, because SGLS.1 through SGLS.3 hold, we know

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})] = [\mathrm{E}(\mathbf{W}_i'\mathbf{\Omega}^{-1}\mathbf{W}_i)]^{-1}$$

where $\mathbf{W}_i = (\mathbf{X}_i, \mathbf{Z}_i)$. Using partitioned matrix multiplication,

$$\mathrm{E}(\mathbf{W}_i'\mathbf{\Omega}^{-1}\mathbf{W}_i) = \begin{pmatrix} \mathrm{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i) & \mathrm{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{Z}_i) \\ \mathrm{E}(\mathbf{Z}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i) & \mathrm{E}(\mathbf{Z}_i'\mathbf{\Omega}^{-1}\mathbf{Z}_i) \end{pmatrix}.$$

Further, because $\mathrm{E}(\mathbf{X}_i \otimes \mathbf{Z}_i) = \mathbf{0}$, it follows that $\mathrm{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{Z}_i) = \mathbf{0}$. Therefore, $\mathrm{E}(\mathbf{W}_i'\mathbf{\Omega}^{-1}\mathbf{W}_i)$ is block diagonal and is equal to

$$\begin{pmatrix} \mathrm{E}(\mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i) & \mathbf{0} \\ \mathbf{0} & \mathrm{E}(\mathbf{Z}_i'\mathbf{\Omega}^{-1}\mathbf{Z}_i) \end{pmatrix}$$

Inverting this matrix gives

$$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\delta}}-\boldsymbol{\delta})] = \begin{pmatrix} [\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i)]^{-1} & \mathbf{0} \\ \mathbf{0} & [\text{E}(\mathbf{Z}_i'\boldsymbol{\Omega}^{-1}\mathbf{Z}_i)]^{-1} \end{pmatrix}$$

and $\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]$ is the upper left hand block:

$$\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})] = [\text{E}(\mathbf{X}_i'\boldsymbol{\Omega}^{-1}\mathbf{X}_i)]^{-1}.$$

Now let $\tilde{\boldsymbol{\beta}}$ be the FGLS estimator from $\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{v}_i$. We know this estimator is consistent for $\boldsymbol{\beta}$ because $\mathbf{v}_i = \mathbf{Z}_i\boldsymbol{\gamma} + \mathbf{u}_i$, and so

$$\text{E}(\mathbf{X}_i \otimes \mathbf{v}_i) = \mathbf{0}$$

because $\text{E}(\mathbf{X}_i \otimes \mathbf{Z}_i) = \mathbf{0}$ and $\text{E}(\mathbf{u}_i|\mathbf{X}_i, \mathbf{Z}_i) = \mathbf{0}$. Now, FGLS of $\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{v}_i$ using a consistent estimator of $\boldsymbol{\Lambda} = \text{E}(\mathbf{v}_i\mathbf{v}_i)$ generally has asymptotic variance

$$[\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)]^{-1}\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{v}_i\mathbf{v}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)[\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)]^{-1}$$

Let $\mathbf{r}_i = \mathbf{Z}_i\boldsymbol{\gamma}$ so that we can write

$$\mathbf{v}_i\mathbf{v}_i' = (\mathbf{r}_i + \mathbf{u}_i)(\mathbf{r}_i + \mathbf{u}_i)' = \mathbf{r}_i\mathbf{r}_i' + \mathbf{r}_i\mathbf{u}_i' + \mathbf{u}_i'\mathbf{r}_i + \mathbf{u}_i\mathbf{u}_i'.$$

Now $\text{E}(\mathbf{r}_i\mathbf{u}_i'|\mathbf{X}_i) = \mathbf{0}$ because $\text{E}(\mathbf{u}_i|\mathbf{X}_i, \mathbf{Z}_i) = \mathbf{0}$ and $\mathbf{r}_i$ is a function of $\mathbf{Z}_i$. Therefore,

$$\text{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{X}_i) = \text{E}(\mathbf{r}_i\mathbf{r}_i'|\mathbf{X}_i) + \text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{X}_i) = \text{E}(\mathbf{r}_i\mathbf{r}_i'|\mathbf{X}_i) + \boldsymbol{\Omega}.$$

Using iterated expectations,

$$\begin{aligned} \text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{v}_i\mathbf{v}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i) &= \text{E}[\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{v}_i\mathbf{v}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i|\mathbf{X}_i)] = \text{E}[\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\text{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{X}_i)\boldsymbol{\Lambda}^{-1}\mathbf{X}_i] \\ &= \text{E}[\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\text{E}(\mathbf{r}_i\mathbf{r}_i'|\mathbf{X}_i)\boldsymbol{\Lambda}^{-1}\mathbf{X}_i] + \text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\mathbf{X}_i) \\ &= \text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{r}_i\mathbf{r}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i) + \text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\mathbf{X}_i) \end{aligned}$$

We have shown that

$$\begin{aligned} \text{Avar}[\sqrt{N}(\tilde{\boldsymbol{\beta}}-\boldsymbol{\beta})] &= \mathbf{A}_2^{-1}\{\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{r}_i\mathbf{r}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i) + \text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)\}\mathbf{A}_2^{-1} \\ &= \mathbf{A}_2^{-1}\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\mathbf{r}_i\mathbf{r}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)\mathbf{A}_2^{-1} + \mathbf{A}_2^{-1}\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)\mathbf{A}_2^{-1} \\ &\equiv \mathbf{C}_2 + \mathbf{A}_2^{-1}\text{E}(\mathbf{X}_i'\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)\mathbf{A}_2^{-1} \end{aligned}$$

where $\mathbf{A}_2 \equiv \mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i)$ and $\mathbf{C}_2 \equiv \mathbf{A}_2^{-1}\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{r}_i\mathbf{r}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i)\mathbf{A}_2^{-1}$. Note that $\mathbf{C}_2$ is positive semi-definite.

Now, Problem 7.14 established that

$$\mathbf{A}_2^{-1}\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{\Omega}\mathbf{\Lambda}^{-1}\mathbf{X}_i)\mathbf{A}_2^{-1} - \mathbf{A}_1^{-1}$$

is positive semi-definite. Therefore,

$$\mathrm{Avar}[\sqrt{N}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})] - \mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = \mathbf{C}_2 + [\mathbf{A}_2^{-1}\mathrm{E}(\mathbf{X}_i'\mathbf{\Lambda}^{-1}\mathbf{\Omega}\mathbf{\Lambda}^{-1}\mathbf{X}_i)\mathbf{A}_2^{-1} - \mathbf{A}_1^{-1}]$$

and each matrix is positive-semi-definite. We have shown the result.

Interestingly, the proof shows that the asymptotic inefficiency of $\tilde{\boldsymbol{\beta}}$ has two sources. First, we have omitted variables that are uncorrelated with $\mathbf{X}_i$. The second piece is due to using the wrong variance matrix, $\mathbf{\Lambda}$. If we could effectively use $\mathbf{\Omega}$ in obtaining the estimator with $\mathbf{Z}_i$ omitted – which we can in principle if we observe $\mathbf{Z}_i$ – then the only source of inefficiency would be due to omitting $\mathbf{Z}_i$ (as happens in the single-equation case).

Solutions to Chapter 8 Problems

8.1. Letting $Q(\mathbf{b})$ denote the objective function in equation (8.27), it follows from multivariable calculus that

$$\frac{\partial Q(\mathbf{b})'}{\partial \mathbf{b}} = -2\left(\sum_{i=1}^{N} \mathbf{Z}_i'\mathbf{X}_i\right)' \hat{\mathbf{W}}\left(\sum_{i=1}^{N} \mathbf{Z}_i'(\mathbf{y}_i - \mathbf{X}_i\mathbf{b})\right).$$

Evaluating the derivative at the solution $\hat{\boldsymbol{\beta}}$ gives

$$\left(\sum_{i=1}^{N} \mathbf{Z}_i'\mathbf{X}_i\right)' \hat{\mathbf{W}}\left(\sum_{i=1}^{N} \mathbf{Z}_i'(\mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}})\right) = \mathbf{0}.$$

In terms of full data matrices, we can write, after simple algebra,

$$(\mathbf{X}'\mathbf{Z}\hat{\mathbf{W}}\mathbf{Z}'\mathbf{X})\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{Z}\hat{\mathbf{W}}\mathbf{Z}'\mathbf{Y}).$$

Solving for $\hat{\boldsymbol{\beta}}$ gives (8.28).

8.2. a. We can apply general GMM theory to obtain consistency and $\sqrt{N}$ asymptotic normality of the 3SLS estimator (GMM version). The four assumptions given in the problem are sufficient for SIV.1 to SIV.3, where $\hat{\mathbf{W}} = \left(N^{-1}\sum_{i=1}^{N} \mathbf{Z}_i'\hat{\boldsymbol{\Omega}}\mathbf{Z}_i\right)^{-1}$ and $\mathbf{W} \equiv [\mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Omega}\mathbf{Z}_i)]^{-1} = \mathrm{plim}(\hat{\mathbf{W}})$. (This assumes plim $\hat{\boldsymbol{\Omega}} = \boldsymbol{\Omega} \equiv \mathrm{E}(\mathbf{u}_i\mathbf{u}_i')$, something that holds quite generally.) However, without SIV.5, 3SLS is not necessarily an asymptotically efficient GMM estimator.

b. The asymptotic variance of the 3SLS estimator is given in equation (8.29) with the choice of $\mathbf{W}$ in part a:

$$\mathrm{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}}_{3SLS} - \boldsymbol{\beta}) = (\mathbf{C}'\mathbf{W}\mathbf{C})^{-1}(\mathbf{C}'\mathbf{W}\boldsymbol{\Lambda}\mathbf{W}\mathbf{C})(\mathbf{C}'\mathbf{W}\mathbf{C})^{-1},$$

where $\boldsymbol{\Lambda} \equiv \mathrm{E}(\mathbf{Z}_i'\mathbf{u}_i\mathbf{u}_i'\mathbf{Z}_i)$, as in the text. (Note this expression collapses to $(\mathbf{C}'\mathbf{W}\mathbf{C})^{-1}$ when

$\mathbf{\Lambda} = \mathbf{W}^{-1}$, as happens under SIV.5.)

c. A consistent estimator of Avar$\sqrt{N}(\hat{\boldsymbol{\beta}}_{3SLS} - \boldsymbol{\beta})$ is given in equation (8.31) with $\hat{\mathbf{\Lambda}} \equiv N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\mathbf{Z}_i)$ and $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}_{3SLS}$ the 3SLS residuals:

$$[(\mathbf{X}'\mathbf{Z}/N)\hat{\mathbf{W}}(\mathbf{Z}'\mathbf{X}/N)]^{-1}(\mathbf{X}'\mathbf{Z}/N)\hat{\mathbf{W}}\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\mathbf{Z}_i\right)\hat{\mathbf{W}}(\mathbf{Z}'\mathbf{X}/N)[(\mathbf{X}'\mathbf{Z}/N)\hat{\mathbf{W}}(\mathbf{Z}'\mathbf{X}/N)]^{-1}.$$

The estimator of Avar$(\hat{\boldsymbol{\beta}}_{3SLS})$ is simply this expression divided by N. Even though the formula looks complicated, it can be programmed fairly easily in a matrix-based language. Of course, if we doubt SIV.5 in the first place, we would probably use the more general minimum chi-square estimator, as it is asymptotically more efficient. (If we were going to obtain the robust variance matrix estimate for 3SLS anyway, it is no harder to obtain the minimum chi-square estimate and its asymptotic variance estimate.)

8.3. First, we can always write $\mathbf{x}$ as its linear projection plus an error: $\mathbf{x} = \mathbf{x}^* + \mathbf{e}$, where $\mathbf{x}^* = \mathbf{z}\mathbf{\Pi}$ and $\mathrm{E}(\mathbf{z}'\mathbf{e}) = \mathbf{0}$. Therefore, $\mathrm{E}(\mathbf{z}'\mathbf{x}) = \mathrm{E}(\mathbf{z}'\mathbf{x}^*)$, which verifies the first part of the hint. To verify the second step, let $\mathbf{h} \equiv \mathbf{h}(\mathbf{z})$, and write the linear projection as

$$\mathrm{L}(\mathbf{y}|\mathbf{z},\mathbf{h}) = \mathbf{z}\mathbf{\Pi}_1 + \mathbf{h}\mathbf{\Pi}_2$$

where $\mathbf{\Pi}_1$ is $M \times K$ and $\mathbf{\Pi}_2$ is $Q \times K$. Then we must show that $\mathbf{\Pi}_2 = \mathbf{0}$. But, from the two-step projection theorem (see Property LP.7 in Chapter 2),

$$\mathbf{\Pi}_2 = [\mathrm{E}(\mathbf{s}'\mathbf{s})]^{-1}\mathrm{E}(\mathbf{s}'\mathbf{r}), \text{ where } \mathbf{s} \equiv \mathbf{h} - \mathrm{L}(\mathbf{h}|\mathbf{z}) \text{ and } \mathbf{r} \equiv \mathbf{x} - \mathrm{L}(\mathbf{x}|\mathbf{z}).$$

Now, by the assumption that $\mathrm{E}(\mathbf{x}|\mathbf{z}) = \mathrm{L}(\mathbf{x}|\mathbf{z}), \mathbf{r}$ is also equal to $\mathbf{x} - \mathrm{E}(\mathbf{x}|\mathbf{z})$. Therefore, $\mathrm{E}(\mathbf{r}|\mathbf{z}) = 0$, and so $\mathbf{r}$ is uncorrelated with all functions of $\mathbf{z}$. But $\mathbf{s}$ is simply a function of $\mathbf{z}$ since $\mathbf{h} \equiv \mathbf{h}(\mathbf{z})$. Therefore, $\mathrm{E}(\mathbf{s}'\mathbf{r}) = \mathbf{0}$, and this shows that $\Pi_2 = \mathbf{0}$.

8.4.a. For the system in (8.12), we show that, for each g, rank $\mathrm{E}[(\mathbf{z},\mathbf{h})'\mathbf{x}_g]$ = rank $\mathrm{E}(\mathbf{z}'\mathbf{x}_g)$

for any function $\mathbf{h}=\mathbf{h}(\mathbf{z})$. Now, by Problem 8.3, $\mathrm{L}(\mathbf{x}_g|\mathbf{z},\mathbf{h}) = \mathrm{L}(\mathbf{x}_g|\mathbf{z}) = \mathbf{z}\boldsymbol{\Pi}_1$ when $\mathrm{E}(\mathbf{x}_g|\mathbf{z})$ is linear in $\mathbf{z}$ and $\mathbf{h}$ is any function of $\mathbf{z}$. As in Problem 8.3, $\mathrm{E}(\mathbf{z}'\mathbf{x}_g) = \mathrm{E}(\mathbf{z}'\mathbf{x}_g^*) = \mathrm{E}(\mathbf{z}'\mathbf{z})\boldsymbol{\Pi}_1$. Also, if we let $\mathbf{e}_g=\mathbf{x}_g - \mathbf{x}_g^*$, then $\mathrm{E}(\mathbf{h}'\mathbf{e}_g) = \mathbf{0}$, and so $\mathrm{E}[(\mathbf{z},\mathbf{h})'\mathbf{x}_g] = \mathrm{E}[(\mathbf{z},\mathbf{h})'\mathbf{x}_g^*] = \mathrm{E}[(\mathbf{z},\mathbf{h})'\mathbf{z}]\boldsymbol{\Pi}_1$. But rank $\mathrm{E}[(\mathbf{z},\mathbf{h})'z]$ = rank $\mathrm{E}(\mathbf{z}'\mathbf{z})$, which means that rank $\mathrm{E}[(\mathbf{z},\mathbf{h})'\mathbf{z}]\boldsymbol{\Pi}_1$ = rank $\mathrm{E}[(\mathbf{z}'\mathbf{z})\boldsymbol{\Pi}_1$. We have shown that rank $\mathrm{E}[(\mathbf{z},\mathbf{h})'\mathbf{x}_g]$ = rank $\mathrm{E}[(\mathbf{z}'\mathbf{x}_g)$, which means adding $\mathbf{h}$ to the instrument list does not help satisfy the rank condition.

b. If $\mathrm{E}(\mathbf{x}_g|\mathbf{z})$ is nonlinear in $\mathbf{z}$, then $\mathrm{L}(\mathbf{x}_g|\mathbf{z},\mathbf{h})$ will generally depend on $\mathbf{h}$. This can certainly help in satisfying the rank condition. For example, if $K_g < M$ (the dimension of $\mathbf{z}$) then the order condition fails for equating g using instruments $\mathbf{z}$. But we can add nonlinear functions of $\mathbf{z}$ to the instrument list that are partially correlated with $\mathbf{x}_g$ and satisfy the order and rank condition. We use this fact in Section 9.5.

8.5. This follows directly from the hint. Straightforward matrix algebra shows that $(\mathbf{C}'\boldsymbol{\Lambda}^{-1}\mathbf{C})-(\mathbf{C}'\mathbf{W}\mathbf{C})(\mathbf{C}'\mathbf{W}\boldsymbol{\Lambda}\mathbf{W}\mathbf{C})^{-1}(\mathbf{C}'\mathbf{W}\mathbf{C})$ can be written as

$$\mathbf{C}'\boldsymbol{\Lambda}^{-1/2}[\mathbf{I}_L - \mathbf{D}(\mathbf{D}'\mathbf{D})^{-1}\mathbf{D}']\boldsymbol{\Lambda}^{-1/2}\mathbf{C},$$

where $\mathbf{D} \equiv \boldsymbol{\Lambda}^{1/2}\mathbf{W}\mathbf{C}$. Since this is a matrix quadratic form in the $L \times L$ symmetric, idempotent matrix $\mathbf{I}_L - \mathbf{D}(\mathbf{D}'\mathbf{D})^{-1}\mathbf{D}'$, it is necessarily itself positive semi-definite.

8.6. a. First, $\boldsymbol{\Omega}^{-1}\mathbf{u}_i = (\sigma^{11}u_{i1} + \sigma^{12}u_{i2}, \sigma^{12}u_{i1} + \sigma^{22}u_{i2})'$. Therefore,

$$\begin{aligned}\mathbf{Z}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i &= \begin{pmatrix} \mathbf{z}_{i1}' & 0 \\ 0 & \mathbf{z}_{i2}' \end{pmatrix}(\sigma^{11}u_{i1} + \sigma^{12}u_{i2}, \sigma^{12}u_{i1} + \sigma^{22}u_{i2})' \\ &= \begin{pmatrix} \mathbf{z}_{i1}'(\sigma^{11}u_{i1} + \sigma^{12}u_{i2}) \\ \mathbf{z}_{i2}'(\sigma^{12}u_{i1} + \sigma^{22}u_{i2}) \end{pmatrix}.\end{aligned}$$

The expected value of this vector depends on $\mathrm{E}(\mathbf{z}_{i1}'u_{i1})$, $\mathrm{E}(\mathbf{z}_{i1}'u_{i2})$, $\mathrm{E}(\mathbf{z}_{i2}'u_{i1})$, and $\mathrm{E}(\mathbf{z}_{i2}'u_{i2})$. If

$\mathrm{E}(\mathbf{z}_{i1}'u_{i2}) \neq \mathbf{0}$ or $\mathrm{E}(\mathbf{z}_{i2}'u_{i1}) \neq \mathbf{0}$ then $\mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i) \neq \mathbf{0}$ except by fluke. In fact, if $\mathrm{E}(\mathbf{z}_{i1}'u_{i1}) = \mathbf{0}$, $\mathrm{E}(\mathbf{z}_{i2}'u_{i2}) = \mathbf{0}$, and $\sigma^{12} \neq 0$ then $\mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i) \neq \mathbf{0}$ if $\mathrm{E}(\mathbf{z}_{i1}'u_{i2}) \neq \mathbf{0}$ or $\mathrm{E}(\mathbf{z}_{i2}'u_{i1}) \neq \mathbf{0}$.

b. When $\sigma_{12} = 0, \sigma^{12} = 0,$ in which case $\mathrm{E}(\mathbf{z}_{i1}'u_{i1}) = \mathbf{0}$ and $\mathrm{E}(\mathbf{z}_{i2}'u_{i2}) = \mathbf{0}$ imply $\mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i) = \mathbf{0}$.

c. If the same instruments are valid in each equation – so $\mathrm{E}(\mathbf{z}_i'u_{i1}) = \mathrm{E}(\mathbf{z}_i'u_{i2}) = \mathbf{0}$ – then $\mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Omega}^{-1}\mathbf{u}_i) = \mathbf{0}$ without restrictions on $\boldsymbol{\Omega}$.

8.7. When $\hat{\boldsymbol{\Omega}}$ is diagonal and $\mathbf{z}_i$ has the form in (8.15), $\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\boldsymbol{\Omega}}\mathbf{Z}_i = \mathbf{Z}'(\mathbf{I}_N \otimes \hat{\boldsymbol{\Omega}})\mathbf{Z}$ is a block diagonal matrix with g^{th} block $\hat{\sigma}_g^2\left(\sum_{i=1}^{N}\mathbf{z}_{ig}'\mathbf{z}_{ig}\right) \equiv \hat{\sigma}_g^2\mathbf{z}_g'\mathbf{z}$, where $\mathbf{Z}_g$ denotes the $N \times L_g$ observation matrix of instruments for the g^{th} equation. Further, $\mathbf{Z}'\mathbf{X}$ is block diagonal with g^{th} block $\mathbf{Z}_g'\mathbf{X}_g$. Using these facts, it is now straightforward to show that the 3SLS estimator consists of $[\mathbf{X}_g'\mathbf{Z}_g(\mathbf{Z}_g'\mathbf{Z}_g)^{-1}\mathbf{Z}_g'\mathbf{X}_g]^{-1}\mathbf{X}_g'\mathbf{Z}_g(\mathbf{Z}_g'\mathbf{Z}_g)^{-1}\mathbf{Z}_g'\mathbf{Y}_g$ stacked from $g = 1, \ldots, G.$ This is just the system 2SLS estimator or, equivalently, 2SLS equation-by-equation.

8.8. a. With $\mathbf{Z}_1 = (\mathbf{z}_{i1}', \mathbf{z}_{i2}', \ldots, \mathbf{z}_{iT}')'$ and $\mathbf{X}_i = (\mathbf{x}_{i1}', \mathbf{x}_{i2}', \ldots, \mathbf{x}_{iT}')'$,

$$\mathbf{Z}_i'\mathbf{Z}_i = \sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it},\ \mathbf{Z}_i'\mathbf{X}_i = \sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}, \text{ and } \mathbf{Z}_i'\mathbf{y}_i = \sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{y}_{it}.$$

Summing over all i gives

$$\mathbf{Z}'\mathbf{Z} = \sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it},\ \mathbf{Z}'\mathbf{X} = \sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}, \text{ and } \mathbf{Z}'\mathbf{Y} = \sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'y_{it}.$$

b. rank $\mathrm{E}\left(\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}\right) = K$.

c. Let $\hat{\mathbf{u}}_i$ be the $T \times 1$ vector of pooled 2SLS residuals, $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}$. Then we just use (8.31) with $\hat{\mathbf{W}} = (\mathbf{Z}'\mathbf{Z}/N)^{-1}$ and $\hat{\boldsymbol{\Lambda}} = N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\mathbf{Z}_i$, cancelling N everywhere:

$$[(\mathbf{X}'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{X})]^{-1}(\mathbf{X}'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\left(\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\mathbf{Z}_i\right)\cdot(\mathbf{Z}'\mathbf{Z}^{-1})(\mathbf{Z}'\mathbf{X})[(\mathbf{X}'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{X})]^{-1}. \quad (8.67)$$

d. Using reasoning almost identical to Problem 7.7, (8.65) implies that, for $s < t$,

$$\begin{aligned}\mathrm{E}(u_{it}u_{is}\mathbf{z}_{it}'\mathbf{z}_{is}) &= \mathrm{E}[\mathrm{E}(u_{it}u_{is}\mathbf{z}_{it}'\mathbf{z}_{is}|\mathbf{z}_{it},u_{is},\mathbf{z}_{is})]\\ &= \mathrm{E}[\mathrm{E}(u_{it}|\mathbf{z}_{it}',u_{is},\mathbf{z}_{is})u_{is},\mathbf{z}_{it}',\mathbf{z}_{is}]\\ &= \mathrm{E}[0\cdot u_{is}\mathbf{z}_{it}',\mathbf{z}_{is}] = \mathbf{0}\end{aligned}$$

because $\mathrm{E}(u_{it}|\mathbf{z}_{it},u_{is},\mathbf{z}_{is}) = 0$ for $s < t$. A similar argument works for $t > s$. So for all $t \neq s$,

$$\mathrm{E}(u_{it}u_{is}\mathbf{z}_{it}',\mathbf{z}_{is}) = 0.$$

Similarly, (8.66) and iterated expectations implies that

$$\begin{aligned}\mathrm{E}(u_{it}^2\mathbf{z}_{it}'\mathbf{z}_{it}) &= \mathrm{E}[\mathrm{E}(u_{it}^2\mathbf{z}_{it}'\mathbf{z}_{it}|\mathbf{z}_{it})]\\ &= \mathrm{E}[\mathrm{E}(u_{it}^2|\mathbf{z}_{it})\mathbf{z}_{it}'\mathbf{z}_{it}] = \sigma^2\mathrm{E}[(\mathbf{z}_{it}'\mathbf{z}_{it}),\ t = 1,\ldots T.\end{aligned}$$

Together, these results imply that

$$\mathrm{Var}(\mathbf{z}_i'\mathbf{u}_i) = \sigma^2\sum_{t=1}^{T}\mathrm{E}[(\mathbf{z}_{it}'\mathbf{z}_{it})].$$

A consistent estimator of this matrix is $\hat{\sigma}^2(\mathbf{Z}'\mathbf{Z}/N)$, where $\hat{\sigma}^2 = 1/(NT)\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{u}_{it}^2$, by the usual law-of-large-numbers arguments. A degrees of freedom adjustment replaces NT with $NT - K$. Replacing $\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\mathbf{Z}_i$ in (8.67) with $\hat{\sigma}^2(\mathbf{Z}'\mathbf{Z})$ [since $\hat{\sigma}^2(\mathbf{Z}'\mathbf{Z}/N)$ can play the role of $\hat{\mathbf{\Lambda}}$ under the maintained assumptions] and cancelling gives the estimated asymptotic variance of $\hat{\boldsymbol{\beta}}$ as

$$\hat{\sigma}^2[(\mathbf{X}'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{X})]^{-1}.$$

This is exactly the variance estimator that would be computed from the pooled 2SLS estimation. This means that the usual 2SLS standard errors and test statistics are

asymptotically valid.

e. If the unconditional variance changes across t, the simplest approach is to weight the variables in each time period by $1/\hat{\sigma}_t$, where $\hat{\sigma}_t^2$ is a consistent estimator of $\sigma_t^2 = \text{Var}(u_{it})$. A consistent estimator of $\hat{\sigma}_t^2$ is

$$\hat{\sigma}_t^2 = N^{-1} \sum_{i=1}^{N} \hat{u}_{it}^2.$$

Now, apply pooled 2SLS to the equation

$$(y_{it}/\hat{\sigma}_t) = (\mathbf{x}_{it}/\hat{\sigma}_t)\boldsymbol{\beta} + error_{it}$$

using instruments $\mathbf{z}_{it}/\hat{\sigma}_t$. The usual statistics from this procedure are asymptotically valid: it can be shown that it has the same $\sqrt{N}$-asymptotic distribution as a if we knew the σ_t^2. This estimator is a generalized instrumental variables (GIV) estimator except it is consistent under the contemporaneous exogeneity assumption only. It turns out to be identical to the GMM estimator that uses weighting matrix $\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{\sigma}_t^2\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}$ – the optimal weighting matrix under the assumptions in the problem. See Im, Ahn, Schmidt, and Wooldridge (1999, Section 2) for discussion of a more general result.

8.9 The optimal instruments are given in Theorem 8.5, with $G = 1$:

$$\mathbf{z}_i^* = [\omega(\mathbf{z}_i)]^{-1}\text{E}(\mathbf{x}_i|\mathbf{z}_i),\ \omega(\mathbf{z}_i) = \text{E}(u_i^2|\mathbf{z}_i).$$

If $\text{E}(u_i^2|\mathbf{z}_i) = \sigma^2$ and $\text{E}(\mathbf{x}_i|\mathbf{z}_i) = \mathbf{z}_i\boldsymbol{\Pi}$, then the optimal instruments are $\sigma^{-2}\,\mathbf{z}_i\boldsymbol{\Pi}$. The constant multiple σ^{-2} clearly has no effect on the optimal IV estimator, so the optimal instruments are $\mathbf{z}_i\boldsymbol{\Pi}$. These are the optimal IVs underlying 2SLS, except that $\boldsymbol{\Pi}$ is replaced with its $\sqrt{N}$-consistent OLS estimator. The 2SLS estimator has the same asymptotic variance whether $\boldsymbol{\Pi}$ or $\hat{\boldsymbol{\Pi}}$ is used, and so 2SLS is asymptotically efficient.

If $\mathrm{E}(u|\mathbf{x}) = 0$ and $\mathrm{E}(u^2|\mathbf{x}) = \sigma^2$, the optimal instruments are $\sigma^{-2}\mathrm{E}(\mathbf{x}|\mathbf{x}) = \sigma^{-2}\mathbf{x}$, and this leads to the OLS estimator.

8.10.a. Write $u_{it} = \rho u_{i,t-1} + e_{it}$, and plug into $y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + u_{it}$ to get

$$y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + \rho u_{i,t-1} + e_{it}, t = 2, \ldots, T.$$

Under the assumption

$$\mathrm{E}(u_{it}|\mathbf{z}_{it}, u_{it,t-1}, \mathbf{x}_{i,t-1}, \mathbf{z}_{i,t-1}, \mathbf{x}_{i,t-2}, \ldots, u_{i1}, \mathbf{x}_{i1}, \mathbf{z}_{i1}) = 0 \tag{8.68}$$

the previous assumption satisfies the dynamic completeness assumption when $\rho = 0$. If we assume that $\mathrm{E}(u_{it}^2|\mathbf{z}_{it}, u_{i,t-1})$ is constant under H_0, then it satisfies the requisite homoskedasticity assumption as well. As shown in Problem 8.8, pooled 2SLS estimation of this equation using instruments $(\mathbf{z}_{it}, u_{i,t-1})$ results in valid test statistics.

Now we apply the results from Section 6.1.3: when $\rho = 0$, replacing $u_{i,t-1}$ with the initial 2SLS residuals $\hat{u}_{i,t-1}$ has no effect as N gets large, provided that (8.68) holds. Thus, we can estimate

$$y_{it} = \mathbf{x}_{it}\beta + \rho\hat{u}_{i,t-1} + error_{it}, t = 2, \ldots, T,$$

by pooled 2SLS using instruments $(\mathbf{z}_{it}, \hat{u}_{i,t-1})$, and obtain the usual t statistic for $\hat{\rho}$.

b. If $\mathrm{E}(u_{it}^2|\mathbf{z}_{it}, u_{i,t-1})$ is not constant, we can use the usual heteroskedasticity-robust t statistic from pooled 2SLS for $\hat{\rho}$. This allows for dynamic forms of heteroskedasticity, such as ARCH and GARCH, as well as static forms of heteroskedasticity.

8.11. a. This is a simple application of Theorem 8.5 when $G = 1$. Without the i subscript, $\mathbf{x}_1 = (\mathbf{z}_1, y_2)$ and so $\mathrm{E}(\mathbf{x}_1|\mathbf{z}) = [\mathbf{z}_1, \mathrm{E}(y_2|\mathbf{z})]$. Further, $\Omega(\mathbf{z}) = \mathrm{Var}(u_1|\mathbf{z}) = \sigma_1^2$. It follows that the optimal instruments are $(1/\sigma_1^2)[\mathbf{z}_i, \mathrm{E}(y_2|\mathbf{z})]$. Dropping the division by σ_1^2 clearly does not affect the optimal instruments.

b. If y_2 is binary then $\mathrm{E}(y_2|\mathbf{z}) = \mathrm{P}(y_2 = 1|\mathbf{z}) = F(\mathbf{z})$, and so the optimal IVs are $[\mathbf{z}_1, F(\mathbf{z})]$.

8.12. a. As long as $\mathrm{E}(\mathbf{Z}'\mathbf{u}) = \mathbf{0}$ holds the estimator is consistent. After all, it is a GMM estimator with a particular weighting matrix that satisfies all of the GMM regularity conditions.

b. Unless the optimal weighting matrix $\hat{\mathbf{W}}$ consistently estimates $[\mathrm{Var}(\mathbf{Z}_i'\mathbf{u}_i)]^{-1}$, the statistic fails to be asymptotically chi-square.

c. Since $\hat{\mathbf{\Omega}}$ and $\hat{\mathbf{\Lambda}}$ converge to the same constant matrix $\mathbf{\Lambda} = \mathbf{\Omega}$, there is no difference in asymptotic efficiency (at least using the usual $\sqrt{N}$-asymptotic distribution).

8.13. a. The optimal instrumental variable is

$$\mathbf{z}^* = [\mathrm{E}(u_1^2|\mathbf{z})]^{-1} \cdot \mathrm{E}[\mathbf{z}_1, y_2, \mathbf{z}_1 y_2|\mathbf{z}] = (\sigma_1^2)^{-1}[\mathbf{z}_1, \mathrm{E}(y_2|\mathbf{z}), \mathbf{z}_1\mathrm{E}(y_2|\mathbf{z})] = (\sigma_1^2)^{-1}[\mathbf{z}_1, \mathbf{z}\boldsymbol{\pi}_2, \mathbf{z}_1(\mathbf{z}\boldsymbol{\pi}_2)].$$

b. The coefficients $\boldsymbol{\pi}_2$ can be estimated by running an OLS of y_2 on $\mathbf{z}$. Since the inverse of the variance is a scalar that does not depend on $\mathbf{z}$, it cancels out in the IV estimation. Thus we can operationalize the optimal IV estimator by using $[\mathbf{z}_1, \mathbf{z}\hat{\boldsymbol{\pi}}_2, \mathbf{z}_1(\mathbf{z}\hat{\boldsymbol{\pi}}_2)]$ as the IVs. The estimator as the same $\sqrt{N}$-asymptotic distribution as if we knew $\boldsymbol{\pi}_2$.

8.14. a. With $\mathbf{y}_{it2} = \mathbf{z}_{it}\mathbf{\Pi}_2 + \mathbf{v}_{it2}$ and $\mathrm{E}(\mathbf{z}_{it}'u_{it1}) = \mathbf{0}, t = 1, \ldots, T$ maintained, $\mathrm{E}(\mathbf{y}_{it2}'u_{it1}) = \mathbf{0}$ is the same as $\mathrm{E}(\mathbf{y}_{it2}'u_{it1}) = \mathbf{0}$. We can always write the linear projection of u_{it1} onto $\mathbf{v}_{it2}$ as

$$u_{it1} = \mathbf{v}_{it2}\boldsymbol{\rho}_1 + e_{it1}$$
$$\mathrm{E}(\mathbf{v}_{it2}'e_{it1}) = \mathbf{0},\ t = 1, \ldots, T$$

where we assume that the coefficients $\boldsymbol{\rho}_1$ do not change over time. Thus, we can write the extended equation

$$y_{it1} = \eta_{t1} + \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \mathbf{y}_{it2}\boldsymbol{\alpha}_2 + \mathbf{v}_{it2}\boldsymbol{\rho}_1 + e_{it1},\ t = 1, \ldots, T$$

Now the control function procedure is clear. (1) Estimate the reduced form $\mathbf{y}_{it2} = \mathbf{z}_{it}\mathbf{\Pi}_2 + \mathbf{v}_{it2}$

by pooled OLS (equation-by-equation if necessary when $\mathbf{y}_{it2}$ is a vector) and obtain the residuals, $\hat{\mathbf{v}}_{it2}$. (2) Run the pooled OLS regression

$$y_{it1} \text{ on } 1,\ d2_t, ..., dT_t, \mathbf{z}_{it1},\ \mathbf{y}_{it2}, \hat{\mathbf{v}}_{it2},\ t = 1,\ldots,T; i = 1,\ldots,N.$$

and use a fully robust Wald test of $\mathrm{H}_0 : \boldsymbol{\rho}_1 = \mathbf{0}$. The test has G_1 degrees of freedom in the chi-square distribution, or one can use an F approximation by dividing the chi-square statistic by G_1.

b. Extending the discussion in the text around equation (6.32), partition $\mathbf{z}_{it2} = (\mathbf{g}_{it2}, \mathbf{h}_{it2})$ where $\mathbf{g}_{it2}$ is $1 \times G_1$ (the same dimension as $\mathbf{y}_{it1}$) and $\mathbf{h}_{it2}$ is $1 \times Q_1$. Obtain the fitted values $\hat{\mathbf{y}}_{it2}$ from the first-stage regressions. Then, obtain the residuals, $\hat{\mathbf{r}}_{it2}$ from the pooled OLS regression

$$\mathbf{h}_{it2} \text{ on } \mathbf{z}_{it1}, \hat{\mathbf{y}}_{it2},\ t = 1,\ldots,T; i = 1,\ldots,N.$$

Let $\hat{u}_{it1}$ be the P2SLS residuals. Then run the pooled OLS regression

$$\hat{u}_{it1} \text{ on } \hat{\mathbf{r}}_{it2},\ t = 1,\ldots,T; i = 1,\ldots,N,$$

and test the $\hat{\mathbf{r}}_{it2}$ for joint significance. A fully robust Wald test is most appropriate, and its limiting distribution under the null that all elements of $\mathbf{z}_{it}$ are exogenous is $\chi^2_{Q_1}$.

8.15. a. The coefficient shows that a higher fare reduces passenger demand for flights. The estimated elasticity is $-.565$, which is fairly large. Even the fully robust 95% confidence interval is pretty narrow, from $-.696$ to $-.434$. Incidentally, the standard error that is robust only to heteroskedasticity and not serial correlation is about .0364, which is actually slightly smaller than the usual OLS standard error. So it is important to use the fully robust version.

```
.
. use airfare

. xtset id year
       panel variable:  id (strongly balanced)
        time variable:  year, 1997 to 2000
                delta:  1 unit
```

```
. reg lpassen y98 y99 y00 lfare ldist ldistsq

      Source |       SS       df       MS              Number of obs =    4596
-------------+------------------------------           F(  6,  4589) =   52.48
       Model |  230.557732     6  38.4262887            Prob > F      =  0.0000
    Residual |  3360.12968  4589  .732213921            R-squared     =  0.0642
-------------+------------------------------           Adj R-squared =  0.0630
       Total |  3590.68741  4595  .781433605            Root MSE      =  .85569

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y98 |   .0321212   .0357118     0.90   0.368    -.0378911    .1021335
         y99 |    .081651    .035724     2.29   0.022     .0116148    .1516873
         y00 |   .1380369   .0358761     3.85   0.000     .0677024    .2083713
       lfare |  -.5647711   .0369644   -15.28   0.000    -.6372392   -.4923031
       ldist |   -1.54939   .3265076    -4.75   0.000    -2.189502   -.9092778
     ldistsq |   .1227088   .0247935     4.95   0.000     .0741017     .171316
       _cons |   13.65144   1.094166    12.48   0.000     11.50635    15.79653
------------------------------------------------------------------------------

. reg lpassen y98 y99 y00 lfare ldist ldistsq, cluster(id)

Linear regression                               Number of obs =    4596
                                                F(  6,  1148) =   34.95
                                                Prob > F      =  0.0000
                                                R-squared     =  0.0642
                                                Root MSE      =  .85569

                                  (Std. Err. adjusted for 1149 clusters in id
------------------------------------------------------------------------------
             |               Robust
     lpassen |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y98 |   .0321212   .0050262     6.39   0.000     .0222597    .0419827
         y99 |    .081651   .0073679    11.08   0.000     .0671949    .0961072
         y00 |   .1380369   .0104857    13.16   0.000     .1174636    .1586101
       lfare |  -.5647711   .0667107    -8.47   0.000    -.6956597   -.4338826
       ldist |   -1.54939     .69818    -2.22   0.027    -2.919242    -.179538
     ldistsq |   .1227088   .0524034     2.34   0.019     .0198916    .2255261
       _cons |   13.65144   2.316661     5.89   0.000     9.106074     18.1968
------------------------------------------------------------------------------
```

b. I use the test that allows the explanatory variables to be non-strictly exogenous. The estimate of ρ is essentially one. In a pure time series context we would have to worry how this amount of persistence in the errors affects inference. Here, inference is standard because it is with fixed T and $N \to \infty$. But the "unit root" in $\{u_{it} : t = 1, \ldots, T\}$ is of some concern because it calls into question whether there is a meaningful relationship between passenger demand and airfares. If the error term rarely returns to its mean (which we can take to be zero), in what

sense is do movements in airfare over time cause movements in passenger demand?

```
. predict uhat, resid

. gen uhat_1 = l.uhat
(1149 missing values generated)

. reg lpassen y99 y00 lfare ldist ldistsq uhat_1, robust

Linear regression                                      Number of obs =    3447
                                                       F(  6,  3440) = 7168.14
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.9647
                                                       Root MSE      =   .1684

------------------------------------------------------------------------------
             |               Robust
     lpassen |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y99 |   .0502195   .0065875     7.62   0.000     .0373036    .0631354
         y00 |   .1105098   .0072252    15.30   0.000     .0963437     .124676
       lfare |   -.628955   .0095767   -65.68   0.000    -.6477315   -.6101784
       ldist |  -1.549142   .0726222   -21.33   0.000    -1.691528   -1.406755
     ldistsq |   .1269054   .0055092    23.04   0.000     .1161037    .1377071
      uhat_1 |   1.005428   .0062555   160.73   0.000     .9931627    1.017693
       _cons |   13.81801   .2389316    57.83   0.000     13.34955    14.28647
------------------------------------------------------------------------------
```

c. The coefficient on $concen_{it}$ is .360 and the *t*-statistic that accounts for heteroskedasticity and serial correlation is 6.15. Therefore, the partial correlation between *lfare* and *concen* is enough to implement an IV procedure.

```
. reg lfare y98 y99 y00 ldist ldistsq concen, cluster(id)

Linear regression                                      Number of obs =    4596
                                                       F(  6,  1148) =  205.63
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.4062
                                                       Root MSE      =  .33651

                                  (Std. Err. adjusted for 1149 clusters in id
------------------------------------------------------------------------------
             |               Robust
       lfare |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y98 |   .0211244   .0041474     5.09   0.000     .0129871    .0292617
         y99 |   .0378496   .0051795     7.31   0.000     .0276872     .048012
         y00 |     .09987   .0056469    17.69   0.000     .0887906    .1109493
       ldist |  -.9016004   .2719464    -3.32   0.001    -1.435168   -.3680328
     ldistsq |   .1030196   .0201602     5.11   0.000     .0634647    .1425745
      concen |   .3601203    .058556     6.15   0.000     .2452315    .4750092
       _cons |   6.209258   .9117551     6.81   0.000     4.420364    7.998151
------------------------------------------------------------------------------
```

d. The IV estimates are given below. The estimated elasticity is huge, -1.78. This seems very large. The fully robust standard error is about twice as large as the usual OLS standard error, and the fully robust 95% confidence interval is -2.71 to $-.84$, which is very wide, but it excludes the point estimate from pooled OLS ($-.5.65$).

```
. ivreg lpassen y98 y99 y00 ldist ldistsq (lfare=concen)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =    4596
-------------+------------------------------           F(  6,  4589) =   20.45
       Model | -556.334915     6 -92.7224858           Prob > F      =  0.0000
    Residual |  4147.02233  4589  .903687586           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  3590.68741  4595  .781433605           Root MSE      =  .95062

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lfare |  -1.776549   .2358788    -7.53   0.000    -2.238985   -1.314113
         y98 |   .0616171   .0400745     1.54   0.124    -.0169481    .1401824
         y99 |   .1241675   .0405153     3.06   0.002      .044738    .2035971
         y00 |   .2542695   .0456607     5.57   0.000     .1647525    .3437865
       ldist |  -2.498972   .4058371    -6.16   0.000    -3.294607   -1.703336
     ldistsq |   .2314932   .0345468     6.70   0.000     .1637648    .2992216
       _cons |   21.21249   1.891586    11.21   0.000     17.50407     24.9209
------------------------------------------------------------------------------
Instrumented:  lfare
Instruments:   y98 y99 y00 ldist ldistsq concen
------------------------------------------------------------------------------

. ivreg lpassen y98 y99 y00 ldist ldistsq (lfare=concen), cluster(id)

Instrumental variables (2SLS) regression              Number of obs =    4596
                                                      F(  6,  1148) =   28.02
                                                      Prob > F      =  0.0000
                                                      R-squared     =
                                                      Root MSE      =  .95062

                                  (Std. Err. adjusted for 1149 clusters in id
------------------------------------------------------------------------------
             |               Robust
     lpassen |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lfare |  -1.776549   .4753368    -3.74   0.000    -2.709175   -.8439226
         y98 |   .0616171   .0131531     4.68   0.000     .0358103    .0874239
         y99 |   .1241675   .0183335     6.77   0.000     .0881967    .1601384
         y00 |   .2542695   .0458027     5.55   0.000      .164403    .3441359
       ldist |  -2.498972    .831401    -3.01   0.003    -4.130207   -.8677356
     ldistsq |   .2314932   .0705247     3.28   0.001     .0931215    .3698649
       _cons |   21.21249   3.860659     5.49   0.000     13.63775    28.78722
```

```
------------------------------------------------------------------------------
Instrumented:  lfare
Instruments:   y98 y99 y00 ldist ldistsq concen
------------------------------------------------------------------------------
```

e. To compute the asymptotic standard error of $\sqrt{N}(\hat{\beta}_{1,P2SLS} - \hat{\beta}_{1,POLS})$ using the traditional Hausman approach, we have to maintain enough assumptions so that POLS is relatively efficient under the null. Letting

$$\mathbf{w}_{it} = (1, y98_t, y99_t, y00_t, lfare_{it}, ldist_i, ldist_i^2, concen_{it})$$

we would have to assume, under H_0,

$$\begin{aligned} \mathrm{E}(\mathbf{w}_{it}' u_{it1}) &= \mathbf{0}, \; t = 1, \ldots, T \\ \mathrm{E}(u_{it1}^2 | \mathbf{w}_{it}) &= \sigma^2, \; t = 1, \ldots, T \\ \mathrm{E}(u_{it1} u_{ir1} | \mathbf{w}_{it}, \mathbf{w}_{ir}) &= 0, \; r \neq t. \end{aligned}$$

The first assumption must be maintained under the null for the test to make sense. The second assumption – homoskedasticity – can never be guaranteed, and so it is always a good idea to make tests robust to heteroskedaticity. The current application is a static equation, and so the assumption of no serial correlation is especially strong. In fact, from part b we already have good evidence that there is substantial serial correlation in the errors (although this test maintains contemporaneous exogeneity of $lfare_{it}$, along with the distance variables).

f. The Stata commands are given below. The fully robust t statistic on $\hat{v}_{it2}$ is 2.92, which is a strong rejection of the null that $lfare_{it}$ is (contemporaneousl) exogenous – assuming that $concen_{it}$ is contemporaneously exogenous.

```
. qui reg lfare y98 y99 y00 ldist ldistsq concen

. predict v2hat, resid

. reg lpassen y98 y99 y00 lfare ldist ldistsq v2hat, cluster(id)

Linear regression                               Number of obs =    4596
                                                F(  7,  1148) =   31.50
                                                Prob > F      =  0.0000
                                                R-squared     =  0.0711
```

```
                                                      Root MSE      =  .85265

                                  (Std. Err. adjusted for 1149 clusters in id
------------------------------------------------------------------------------
             |               Robust
     lpassen |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y98 |   .0616171   .0112127     5.50   0.000     .0396175    .0836167
         y99 |   .1241675   .0160906     7.72   0.000     .0925973    .1557378
         y00 |   .2542695    .040158     6.33   0.000     .1754782    .3330608
       lfare |  -1.776549   .4197937    -4.23   0.000    -2.600198   -.9528999
       ldist |  -2.498972    .767078    -3.26   0.001    -4.004004   -.9939395
     ldistsq |   .2314932   .0640361     3.62   0.000     .1058524    .3571341
       v2hat |   1.249653   .4273322     2.92   0.004     .4112137    2.088093
       _cons |   21.21249    3.46901     6.11   0.000     14.40618     28.0188
------------------------------------------------------------------------------
```

8.16 (**Bonus Question**). Consider the GIV estimator with incorrect restrictions imposed on the estimator of $\mathbf{\Omega}$. That is, in (8.47) use $\hat{\mathbf{\Lambda}}$ in place of $\mathbf{\Omega}$ with $\hat{\mathbf{\Lambda}} \xrightarrow{p} \mathbf{\Lambda} \neq \mathbf{\Omega}$.

a. If Assumption GIV.1 holds, that is $\mathrm{E}(\mathbf{Z}_i \otimes \mathbf{u}_i) = \mathbf{0}$, argue that the GIV estimator is still consistent under an appropriate rank condition (and state the rank condition).

b. Argue that, under the assumptions of part a, the GIV estimator that uses $\hat{\mathbf{\Lambda}}$ is $\sqrt{N}$ –asymptotically equivalent to the (infeasible) GIV estimator that uses $\mathbf{\Lambda}$.

c. If you insist on using $\hat{\mathbf{\Lambda}}$ but want to guard against inappropriate inference, what would you do?

Solution

a. From equation (8.47), and applying the law of large numbers, the key orthogonality condition for consistency is

$$\mathrm{E}(\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{u}_i) = \mathbf{0}$$

because $\hat{\mathbf{\Lambda}} \xrightarrow{p} \mathbf{\Lambda}$. But if Assumption GIV.1 holds, any linear combination of $\mathbf{Z}_i$ is uncorrelated with $\mathbf{u}_i$, including $\mathbf{\Lambda}^{-1}\mathbf{Z}_i$. There are two parts to the rank condition, with the first being the most important:

$$\begin{aligned} \text{rank } \mathrm{E}(\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{Z}_i) &= L \\ \text{rank } \mathrm{E}(\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i) &= K \end{aligned}$$

b. This follows the same line of reasoning that we used for FGLS in Chapter 7 can be used. First, using the same trick with the Kronecker product,

$$\begin{aligned} N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i &= N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{Z}_i + o_p(1) \xrightarrow{p} \mathrm{E}(\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{Z}_i) \\ N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{X}_i &= N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i + o_p(1) \xrightarrow{p} \mathrm{E}(\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{X}_i) \end{aligned}$$

Second,

$$N^{-1/2}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{u}_i - N^{-1/2}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{\Lambda}^{-1}\mathbf{u}_i = N^{-1/2}\sum_{i=1}^{N}(\mathbf{u}_i\otimes\mathbf{Z}_i)'\text{vec}(\hat{\mathbf{\Lambda}}^{-1}-\mathbf{\Lambda}^{-1})$$
$$= O_p(1)\cdot o_p(1) = o_p(1).$$

Combining these asymptotic equivalances shows that replacing $\mathbf{\Lambda}$ with the consistent estimator $\hat{\mathbf{\Lambda}}$ does not affect the $\sqrt{N}$-limiting distribution of the GIV estimator.

c. Use a full robust asymptotic variance matrix estimator. Write

$$\widehat{\text{Avar}}[\sqrt{N}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})] = \hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1}$$

where

$$\hat{\mathbf{A}} = \left(N^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i\right)\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{X}_i\right)$$
$$\hat{\mathbf{B}} = \left(N^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i\right)\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i\right)$$
$$\cdot\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{Z}_i\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\mathbf{\Lambda}}^{-1}\mathbf{X}_i\right)$$

where $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}$ are the GIV residuals. This asymptotic variance matrix estimator allows $\text{E}(\mathbf{u}_i\mathbf{u}_i') \neq \mathbf{\Lambda}$ as well as system heteroskedasticity, that is, $\text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{Z}_i) \neq \text{E}(\mathbf{u}_i\mathbf{u}_i')$. Of course, we get $\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}})$ as $\hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1}/N$, whereby all of the divisions by N disappear.

8.17 (**Bonus Question**). Consider a panel data model with contemporaneously exogenous instruments $\mathbf{z}_{it}$:

$$y_{it1} = \mathbf{x}_{it}\boldsymbol{\beta} + u_{it},$$
$$\text{E}(\mathbf{z}_{it}'u_{it}) = \mathbf{0},\ t = 1,\ldots,T,$$

where $\mathbf{x}_{it}$ is $1\times K$ and $\mathbf{z}_{it}$ is $1\times L$ for all t, $L \geq K$.

a. If we maintain the assumptions

ASSUMPTION P2SLS.1: $\mathrm{E}(\mathbf{z}_{it}'u_{it}) = 0,\ t = 1,\ldots,T$

ASSUMPTION P2SLS.2: (a) rank $\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{z}_{it}) = L$; (b) rank $\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{x}_{it}) = K$,

argue that the pooled 2SLS (P2SLS) estimator is generally consistent (as always with T fixed, $N \to \infty$, and random sampling across i.

b. Explain how to estimate the asymptotic variance matrix of the P2SLS estimator under the assumptions in part a.

c. Suppose we add the assumption

ASSUMPTION P2SLS.3: (a) $\mathrm{E}(u_{it}^2\mathbf{z}_{it}'\mathbf{z}_{it}) = \sigma^2\mathrm{E}(\mathbf{z}_{it}'\mathbf{z}_{it}),\ t = 1,\ldots,T$; (b) $\mathrm{E}(u_{it}u_{ir}\mathbf{z}_{it}'\mathbf{z}_{is}) = \mathbf{0}$, $t \neq r$.

Argue that the usual 2SLS variance matrix estimator that assumes homoskedasticity and ignores the time series component is valid.

d. What would you do if Assumption P2SLS.3(b) holds but not necessarily P2SLS.3(a)?

Solution

a. Using the general formula for the S2SLS estimator, we can write the P2SLS estimator (with probability approaching one) as

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= \left[\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{x}_{it}'\mathbf{z}_{it}\right)\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}\right)\right]^{-1} \\ &\quad\cdot\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{x}_{it}'\mathbf{z}_{it}\right)\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'y_{it}\right) \\ &= \boldsymbol{\beta} + \left[\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{x}_{it}'\mathbf{z}_{it}\right)\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}\right)\right]^{-1} \\ &\quad\cdot\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{x}_{it}'\mathbf{z}_{it}\right)\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'u_{it}\right)\end{aligned}$$

Notice how the law of large numbers implies

$$N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it} \xrightarrow{p} \sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{z}_{it})$$
$$N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it} \xrightarrow{p} \sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{x}_{it})$$

and the rank condition states that these matrices of ranks L and K, respectively. Therefore, the plim can pass through all inverses. We also apply the WLLN and Assumption P2SLS.1 to get

$$N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'u_{it} \xrightarrow{p} \sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'u_{it}) = \mathbf{0}.$$

Now we just pass the plim through using Slutsky's Theorem to get $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$.

b. We have

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$$

where

$$\mathbf{A} = \left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{x}_{it}'\mathbf{z}_{it})\right)\left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{z}_{it})\right)^{-1}\left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{x}_{it})\right)$$
$$\mathbf{B} = \left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{x}_{it}'\mathbf{z}_{it})\right)\left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{z}_{it})\right)^{-1}\left(\sum_{t=1}^{T}\sum_{r=1}^{T}\mathrm{E}(u_{it}u_{ir}\mathbf{z}_{it}'\mathbf{z}_{ir})\right)$$
$$\cdot\left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{z}_{it})\right)^{-1}\left(\sum_{t=1}^{T}\mathrm{E}(\mathbf{z}_{it}'\mathbf{x}_{it})\right)$$

We can consistently estimate each of these matrices:

$$\hat{\mathbf{A}} = \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{x}_{it}' \mathbf{z}_{it} \right) \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{z}_{it}' \mathbf{z}_{it} \right)^{-1} \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{z}_{it}' \mathbf{x}_{it} \right)$$

$$\hat{\mathbf{B}} = \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{x}_{it}' \mathbf{z}_{it} \right) \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{z}_{it}' \mathbf{z}_{it} \right)^{-1} \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{r=1}^{T} \hat{u}_{it} \hat{u}_{ir} \mathbf{z}_{it}' \mathbf{z}_{ir} \right)$$
$$\cdot \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{z}_{it}' \mathbf{z}_{it} \right)^{-1} \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathbf{z}_{it}' \mathbf{x}_{it} \right)$$

where $\hat{u}_{it} = y_{it} - \mathbf{x}_{it}\hat{\boldsymbol{\beta}}$ are the P2SLS residuals.

c. With Assumption P2SLS.3,

$$\sum_{t=1}^{T} \sum_{r=1}^{T} \mathrm{E}(u_{it} u_{ir} \mathbf{z}_{it}' \mathbf{z}_{ir}) = \sum_{t=1}^{T} \mathrm{E}(u_{it}^2 \mathbf{z}_{it}' \mathbf{z}_{it}) = \sigma^2 \sum_{t=1}^{T} \mathrm{E}(\mathbf{z}_{it}' \mathbf{z}_{it}),$$

where the first equality follows from $\mathrm{E}(u_{it} u_{ir} \mathbf{z}_{it}' \mathbf{z}_{is}) = \mathbf{0}$, $t \neq r$, and the second follows from $\mathrm{E}(u_{it}^2 \mathbf{z}_{it}' \mathbf{z}_{it}) = \sigma^2 \mathrm{E}(\mathbf{z}_{it}' \mathbf{z}_{it})$, $t = 1, \ldots, T$. Therefore,

$$\mathbf{B} = \sigma^2 \left(\sum_{t=1}^{T} \mathrm{E}(\mathbf{x}_{it}' \mathbf{z}_{it}) \right) \left(\sum_{t=1}^{T} \mathrm{E}(\mathbf{z}_{it}' \mathbf{z}_{it}) \right)^{-1} \left(\sum_{t=1}^{T} \mathrm{E}(\mathbf{z}_{it}' \mathbf{x}_{it}) \right) = \sigma^2 \mathbf{A},$$

and so

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] = \sigma^2 \mathbf{A}^{-1}$$

When we use $\hat{\mathbf{A}}$ from part b and a consistent estimator of σ^2 (with optional but standard degrees-of-freedom adjustment),

$$\hat{\sigma}^2 = \frac{1}{NT - K} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{u}_{it}^2,$$

then we get

$$\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2\left[\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{x}_{it}'\mathbf{z}_{it}\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}\right)\right]^{-1},$$

which is exactly the standard formula for 2SLS treating the panel data set as one long cross section.

d. We need to make the variance matrix robust to heteroskedasticity only. So

$$\hat{\mathbf{B}} = \left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{x}_{it}'\mathbf{z}_{it}\right)\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{u}_{it}^2\mathbf{z}_{it}'\mathbf{z}_{it}\right)$$
$$\cdot\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{z}_{it}\right)^{-1}\left(N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{z}_{it}'\mathbf{x}_{it}\right).$$

The resulting $\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}})$ is exactly what would be computed by treating the panel data set as one long cross section with inference robust to heteroskedasticity.

8.18 (**Bonus Question**). Consider the panel data model

$$y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + u_{it}, t = 1,\ldots,T,$$

where $\mathbf{x}_{it}$ is a $1 \times K$ vector and the instruments at time t are $\mathbf{z}_{it}$, a $1 \times L$ vector for all t. Suppose the instruments are strictly exogenous in the sense that

$$\text{E}(u_{it}|\mathbf{z}_{i1},\mathbf{z}_{i2},\ldots,\mathbf{z}_{iT}) = \text{E}(u_{it}|\mathbf{z}_i) = \mathbf{0}, t = 1,\ldots,T.$$

Assume that $\text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{z}_i) = \text{E}(\mathbf{u}_i\mathbf{u}_i') = \boldsymbol{\Omega}$, where $\mathbf{z}_i$ is the vector all all exogenous variables in all time periods. Further, assume that $\boldsymbol{\Omega}$ has the AR(1) form:

$$\boldsymbol{\Omega} = \sigma_e^2\begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \rho \\ \rho^{T-1} & \rho^{T-2} & \cdots & \rho & 1 \end{pmatrix} \equiv \sigma_e^2\boldsymbol{\Psi}$$

where $u_{it} = \rho u_{i,t-1} + e_{it}, t = 1, \ldots, T.$

a. If $\mathbf{Z}_i' = (\mathbf{z}_{i1}', \ldots, \mathbf{z}_{iT}')$, find the matrix of transformed instruments, $\boldsymbol{\Psi}^{-1/2}\mathbf{Z}_i$.

b. Describe how to implement the GIV estimator as a particular pooled 2SLS estimation when $\boldsymbol{\Omega}$ has the AR(1) structure.

c. If you think the AR(1) model might be incorrect, or the system homoskedasticity assumption does not hold, propose a simple method for obtaining valid standard errors and test statistics.

Solution

From Section 7.8.6, we know that when $\boldsymbol{\Psi}$ has the AR(1) structure given above,

$$\boldsymbol{\Psi}^{-1/2}\mathbf{Z}_i = \begin{pmatrix} (1-\rho^2)^{1/2}\mathbf{z}_{i1} \\ \mathbf{z}_{i2} - \rho\mathbf{z}_{i1} \\ \vdots \\ \mathbf{z}_{iT} - \rho\mathbf{z}_{i,T-1} \end{pmatrix}$$

so that, for $t \geq 2$, the transformation results in quasi-difference. For $t = 1$, the transformation ensures that the transformed errors will have common varianec for all $t = 1, \ldots, T$.

b. We need to estimate ρ, so we would use pooled 2SLS to get residuals, say $\check{u}_{it}$. Then, estimate ρ from the pooled OLS regression

$$\check{u}_{it} \text{ on } \check{u}_{i,t-1},\ t = 2, \ldots, T;\ i = 1, \ldots, N.$$

The GIV transformed equation is

$$\begin{aligned} (1-\rho^2)^{1/2} y_{i1} &= (1-\rho^2)^{1/2}\mathbf{x}_{i1}\boldsymbol{\beta} + (1-\rho^2)^{1/2} u_{i1} \\ y_{it} - \rho y_{i,t-1} &= (\mathbf{x}_{it} - \rho\mathbf{x}_{i,t-1})\boldsymbol{\beta} + u_{it} - \rho u_{i,t-1},\ t = 2, \ldots, T. \end{aligned}$$

The GIV estimator is obtained by replacing ρ with $\hat{\rho}$ and estimating

$$\tilde{y}_{it} = \tilde{\mathbf{x}}_{it}\boldsymbol{\beta} + error_{it},\ t = 1, \ldots, T;\ i = 1, \ldots, N$$

using IVs $\tilde{\mathbf{z}}_{it}$, where

$$\tilde{\mathbf{z}}_{i1} = (1 - \hat{\rho}^2)^{1/2}\mathbf{z}_{i1}$$
$$\tilde{\mathbf{z}}_{it} = \mathbf{z}_{it} - \hat{\rho}\mathbf{z}_{i,t-1}, \; t = 2, \ldots, T$$

and where similar definitions hold for $\tilde{y}_{it}$ and $\tilde{\mathbf{x}}_{it}$. As always, the estimation of ρ has no effect on the $\sqrt{N}$-asymptotic distribution under the strict exogeneity assumption on the IVs. The usual P2SLS statistics from the estimation on the transformed variables are asymptotically valid.

c. If we have misspecified $\text{Var}(\mathbf{u}_i|\mathbf{z}_i)$ then we should make the P2SLS inference from part b fully robust – to heteroskedasticity and serial correlation. In other words, the transformed errors

$$e_{i1} = (1 - \rho^2)^{1/2}u_{i1}$$
$$e_{it} = u_{it} - \rho u_{i,t-1}, \; t = 2, \ldots, T$$

will have serial correlation if the AR(1) model is incorrect, and such errors can always have heteroskedsticity if $\{u_{it}\}$ does. We know that the GIV estimator that uses an incorrect variance structure is still consistent and $\sqrt{N}$-asymptotically normal. We might get a more efficient estimator assuming a simple AR(1) structure than using P2SLS on the original: accounting for the serial correlation at all might be better than ignoring it in estimation. This is the same motivation underlying the generalized estimation equations literature when the explanatory variables are strictly exogenous.

Solutions to Chapter 9 Problems

9.1. a. No. What causal inference could one draw from this? We may be interested in the tradeoff between wages and benefits, but then either of these can be taken as the dependent variable and estimation of either equation would be by OLS. Of course, if we have omitted some important factors, or have a measurement error problem, OLS could be inconsistent for estimating the tradeoff. But there is no simultaneity problem: wages and benefits are jointly determined, but there is no sense in which an equation for wage and another for benefits satisfy the autonomy requirement.

b. Yes. We can certainly think of an exogenous change in law enforcement expenditures causing a reduction in crime, and we are certainly interested in such counterfactuals. If we could do the appropriate experiment, where expenditures are assigned randomly across cities, then we could estimate the crime equation by OLS. The simultaneous equations model recognizes that cities choose law enforcement expenditures in part based on what they expect the crime rate to be. An SEM is a convenient way to allow expenditures to depend on unobservables (to the econometrician) that affect crime.

c. No. These are both choice variables of the firm, and the parameters in a two-equation system modeling one in terms of the other, and vice versa, have no economic meaning. If we want to know how a change in the price of foreign technology affects foreign technology (FT) purchases, why would we want to hold fixed R&D spending? Clearly FT purchases and R&D spending are simultaneously chosen, but we should use a two-equation SUR setup where neither is an explanatory variable in the other's equation.

d. Yes. We can be interested in the causal effect of alcohol consumption on productivity, and therefore on wage. One's hourly wage is determined by productivity, and other factors;

alcohol consumption is determined by individual choice, where one factor is income.

e. No. These are choice variables by the same household. It makes no sense to think about how exogenous changes in one would affect the other. Further, suppose that we look at the effects of changes in local property tax rates. We would not want to hold fixed family saving and then measure the effect of changing property taxes on housing expenditures. When the property tax changes, a family will generally adjust expenditure in all categories. A SUR system with property tax as an explanatory variable is the appropriate strategy.

f. No. These are both chosen by the firm, presumably to maximize profits. It makes no sense to hold advertising expenditures fixed while looking at how other variables affect price markup.

g. Yes. The outcome variables – quantity demanded and advertising expenditures – are determined by different economic agents. It makes sense to model quantity demanded as a function of advertising expenditures – reflecting that more exposure to the public can affect demand – and at the same time recognize that how much a firm spends on advertising can be determined by how much of the product it can sell.

h. Yes. The rate of HIV infection is determined by many factors, with condom usage being one. We can easily imagine being interested in the effects of making condoms more available on the incidence of HIV. The second equation, which models demand for condoms as a function of HIV incidence, captures the idea that more people might use condoms as the risk of HIV infection increases. Each equation stands on its own.

9.2. a. Write the system as

$$\begin{pmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mathbf{z}_{(1)}\boldsymbol{\delta}_{(1)} + u_1 \\ \mathbf{z}_{(2)}\boldsymbol{\delta}_{(2)} + u_2 \end{pmatrix}.$$

Unique solutions for y_1 and y_2 exist only if the matrix premultiplying $(y_1, y_2)'$ is nonsingular. But its determinant is $1 - \gamma_1\gamma_2$, so a necessary and sufficient condition for the reduced forms to exist is $\gamma_1\gamma_2 \neq 1$.

b. The rank condition holds for the first equation if and only if $\mathbf{z}_{(2)}$ contains an element not in $\mathbf{z}_{(1)}$ and the coefficient in $\boldsymbol{\delta}_{(2)}$ on that variable is not zero. Similarly, the rank condition holds for the second equation if and only if $\mathbf{z}_{(1)}$ contains an element not in $\mathbf{z}_{(2)}$ and the coefficient in $\boldsymbol{\delta}_{(1)}$ on that variable is not zero.

9.3. a. We can apply part b of Problem 9.2. First, the only variable excluded from the *support* equation is the variable *mremarr*; since the *support* equation contains one endogenous variable, this equation is identified if and only if $\delta_{21} \neq 0$. This ensures that there is an exogenous variable shifting the mother's reaction function that does not also shift the father's reaction function.

The *visits* equation is identified if and only if at least one of *finc* and *fremarr* actually appears in the *support* equation; that is, we need $\delta_{11} \neq 0$ or $\delta_{13} \neq 0$.

b. Each equation can be estimated by 2SLS using instruments $1, finc, fremarr, dist, mremarr$.

c. First, obtain the reduced form for *visits* :

$$visits = \boldsymbol{\pi}_{20} + \boldsymbol{\pi}_{21} finc + \boldsymbol{\pi}_{22} fremarr + \boldsymbol{\pi}_{23} dist + \boldsymbol{\pi}_{24} mremarr + v_2.$$

Estimate this equation by OLS, and save the residuals, $\hat{v}_2$. Then, run the OLS regression

$$support \text{ on } 1, visits, finc, fremarr, dist, \hat{v}_2$$

and do a (heteroskedasticity-robust) *t* test that the coefficient on $\hat{v}_2$ is zero. If this test rejects we conclude that *visits* is in fact endogenous in the *support* equation.

d. There is one overidentifying restriction in the *visits* equation, assuming that δ_{11} and δ_{12} are both different from zero. Assuming homoskedasticity of u_2, the easiest way to test the overidentifying restriction is to first estimate the *visits* equation by 2SLS. as in part b. Let $\hat{u}_2$ be the 2SLS residuals. Then, run the auxiliary regression

$$\hat{u}_2 \text{ on } 1, finc, fremarr, dist, mremarr;$$

the sample size times the usual *R*-squared from this regression is distributed asymptotically as χ_1^2 under the null hypothesis that all instruments are exogenous.

A heteroskedasticity-robust test is also easy to obtain. Let $\widehat{support}$ denote the fitted values from the reduced form regression for *support*. Next, regress *finc* (or *fremarr*) on $\widehat{support}, mremarr, dist$, and save the residuals, say $\hat{r}_1$. Then, run the simple regression (without intercept) of $\hat{u}_2$ on $\hat{r}_1$ and use the heteroskedasticity-robust *t* statistic on $\hat{r}_1$. (Note that no intercept is needed in this final regression, but including one is harmless.)

9.4. a. Because the third equation contains no right hand side endogenous variables, a reduced form exists for the system if and only if the first two equations can be solved for y_1 and y_2 as functions of $y_3, z_1, z_2, z_3, u_1,$ and u_2. But this is equivalent to asking when the system

$$\begin{pmatrix} 1 & -\gamma_{12} \\ 1 & -\gamma_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

has a unique solution in y_1 and y_2. This matrix is nonsingular if and only if $\gamma_{12} \neq \gamma_{22}$. This implies that the 3×3 matrix $\mathbf{\Gamma}$ in the general SEM notation is nonsingular.

b. The third equation satisfies the rank condition because it includes no right-hand-side endogenous variables. The first equation fails the order condition because there are no excluded exogenous variables in it, but there is one included endogenous variable. This means

it fails the rank condition also. The second equation is just identified according to the order condition because it contains two endogenous variables and also excludes two exogenous variables. To examine the rank condition, write the second equation as $\mathbf{y}\boldsymbol{\gamma}_2 + \mathbf{z}\boldsymbol{\delta}_2 + u_2 = 0$, where $\boldsymbol{\gamma}_2 = (-1, \gamma_{22}, \gamma_{23})'$ and $\boldsymbol{\delta}_2 = (\delta_{21}, 0, 0)'$. Write $\boldsymbol{\beta}_2 = (-1, \gamma_{22}, \gamma_{23}, \delta_{21}, \delta_{22}, \delta_{23})'$ as the vector of parameters for the second equation with only the normalization $\gamma_{21} = -1$ imposed. Then, the restrictions $\delta_{22} = 0$ and $\delta_{23} = 0$ can be written as $\mathbf{R}_2\boldsymbol{\beta}_2 = \mathbf{0}$, where

$$\mathbf{R}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now letting $\mathbf{B}$ be the 6×3 matrix of all parameters, and imposing all exclusion restrictions in the system,

$$\mathbf{R}_2\mathbf{B} = \begin{pmatrix} \delta_{12} & 0 & \delta_{32} \\ \delta_{13} & 0 & \delta_{33} \end{pmatrix}.$$

The rank condition requires this matrix have rank equal to two. Provided the vector $(\delta_{32}, \delta_{33})'$ is not a multiple of $(\delta_{12}, \delta_{13})'$, or $\delta_{12}\delta_{33} \neq \delta_{13}\delta_{32}$, the rank condition is satisfied.

9.5. a. Let $\boldsymbol{\beta}_1$ denote the 7×1 vector of parameters in the first equation with only the normalization restriction imposed:

$$\boldsymbol{\beta}_1' = (-1, \gamma_{12}, \gamma_{13}, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}).$$

The restrictions $\delta_{12} = 0$ and $\delta_{13} + \delta_{14} = 1$ are obtained by choosing

$$\mathbf{R}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Because $\mathbf{R}_1$ has two rows, and $G - 1 = 2$, the order condition is satisfied. Now we need to check the rank condition. Letting $\mathbf{B}$ denote the 7×3 matrix of all structural parameters with

only the three normalizations, straightforward matrix multiplication gives

$$\mathbf{R}_1\mathbf{B} = \begin{pmatrix} \delta_{12} & \delta_{22} & \delta_{32} \\ \delta_{13} + \delta_{14} - 1 & \delta_{23} + \delta_{24} - \gamma_{21} & \delta_{33} + \delta_{34} - \gamma_{31} \end{pmatrix}.$$

By definition of the constraints on the first equation, the first column of $\mathbf{R}_1\mathbf{B}$ is zero. Next, we use the constraints in the remainder of the system to get the expression for $\mathbf{R}_1\mathbf{B}$ with all information imposed. But $\gamma_{23} = 0, \delta_{22} = 0, \delta_{23} = 0, \delta_{24} = 0, \gamma_{31} = 0,$ and $\gamma_{32} = 0$, and so $\mathbf{R}_1\mathbf{B}$ becomes

$$\mathbf{R}_1\mathbf{B} = \begin{pmatrix} 0 & 0 & \delta_{32} \\ 0 & -\gamma_{21} & \delta_{33} + \delta_{34} - \gamma_{31} \end{pmatrix}.$$

Identification requires $\gamma_{21} \neq 0$ and $\delta_{32} \neq 0$.

b. It is easy to see how to estimate the first equation under the given assumptions. Set $\delta_{14} = 1 - \delta_{13}$ and plug into the equation. After simple algebra we get

$$y_1 - z_4 = \gamma_{12}y_2 + \gamma_{13}y_3 + \delta_{11}z_1 + \delta_{13}(z_3 - z_4) + u_1.$$

This equation can be estimated by 2SLS using instruments (z_1, z_2, z_3, z_4). Note that, if we just count instruments, there are just enough instruments to estimate this equation.

9.6. a. If $\gamma_{13} = 0$ then the two equations constitute a linear SEM. In that case, the first equation is identified if and only if $\delta_{23} \neq 0$ and the second equation is identified if and only if $\delta_{12} \neq 0$.

b. If we plug the second equation into the first we obtain

$$\begin{aligned}(1 - \gamma_{12}\gamma_{21} - \gamma_{13}\gamma_{21}z_1)y_1 = {} & (\delta_{10}\gamma_{12}\delta_{20}) + (\gamma_{12}\delta_{21} + \gamma_{13}\delta_{20} + \delta_{11})z_1 \\ & + \gamma_{13}\delta_{21}z_1^2 + \delta_{12}z_2 + \gamma_{12}\delta_{23}z_3 + \gamma_{13}\delta_{23}z_1z_3 + u_1 + (\gamma_{12} + \gamma_{13}z_1)u_2.\end{aligned}$$

This can be solved for y_1 provided $(1 - \gamma_{12}\gamma_{21} - \gamma_{13}\gamma_{21}z_1) \neq 0$. Given the solution for y_1, we

can use the second equation to get y_2. Note that both are nonlinear in z_1 unless $\gamma_{13} = 0$.

c. Since $\mathrm{E}(u_1|\mathbf{z}) = \mathrm{E}(u_2|\mathbf{z}) = 0$, we can use part (b) to get

$$\mathrm{E}(y_1|z_1,z_2,z_3) = [(\delta_{10} + \gamma_{12}\delta_{20}) + (\gamma_{12}\delta_{21} + \gamma_{13}\delta_{20} + \delta_{11})z_1 + \gamma_{13}\delta_{21}z_1^2 + \delta_{12}z_2 + \gamma_{12}\delta_{23}z_3 + \gamma_{13}\delta_{23}z_1z_3]/(1 - \gamma_{12}\gamma_{21} - \gamma_{13}\gamma_{21}z_1).$$

Again, this is a nonlinear function of the exogenous variables appearing in the system unless $\gamma_{13} = 0$. If $\gamma_{21} = 0$, $\mathrm{E}(y_1|z_1,z_2,z_3)$ becomes linear in z_2 and quadratic in z_1 and z_3.

d. If $\gamma_{13} = 0$, we saw in part a that the first equation is identified. If we include $\gamma_{13}y_2z_1$ in the model, we need at least one instrument for it. But regardless of the value of γ_{13}, terms z_1^2 and z_1z_3 – as well as many other nonlinear functions of $\mathbf{z}$ – are partially correlated with y_2z_1. In other words, the linear projection of y_2z_1 onto $1,z_1,z_2,z_3,z_1^2$ and z_1z_3 will – except by fluke – depend on at least one of the last two terms. In any case, we can test this using OLS with $y_{i2}z_{i1}$ as the dependent variarbale and a heteroskedasticity-robust test of two exclusion restrictions. Identification of the second equation is no problem, as z_3 is always available as an IV for y_2. To enhance efficiency when $\gamma_{13} \neq 0$, we could add z_1^2 and z_1z_3 (say) to the instrument list.

e. We could use IVs $(1,z_1,z_2,z_3,z_1^2,z_1z_3)$ in estimating the equation

$$y_1 = \delta_{10} + \gamma_{12}y_2 + \gamma_{13}y_2z_1 + \delta_{11}z_1 + \delta_{12}z_2 + u_1$$

by 2SLS, which implies a single overidentifying restriction. We can add other IVs – z_2^2, z_3^2, z_1z_2, and z_2z_3 seem natural – or even reciprocals, such as $1/z_1$ (or $1/(1 + |z_1|)$ if z_1 can equal zero).

f. We can use the instruments in part e for *both* equations. With a large sample size we might expand the list of IVs as discussed in part e.

g. Technically, the parameters in the first equation can be consistently estimated if $\gamma_{13} \neq 0$ because $\mathrm{E}(y_2|\mathbf{z})$ is a nonlinear function of $\mathbf{z}$, and so z_1^2, z_1z_2, and other nonlinear functions

would generally be partially correlated with y_2 and $y_2 z_1$. But, if $\gamma_{13} = 0$ also, $\mathrm{E}(y_2|\mathbf{z})$ is linear in z_1 and z_2, and additional nonlinear functions are not partially correlated with y_2; thus, there is no instrument for y_2. Since the equation is not identified when $\gamma_{13} = 0$ (and $\delta_{23} = 0$), $\mathrm{H}_0 : \gamma_{13} = 0$ cannot be tested.

9.7. a. Because *alcohol* and *educ* are endogenous in the first equation, we need at least two elements in $(\mathbf{z}_{(2)}, \mathbf{z}_{(3)})$ that are not also in $\mathbf{z}_{(1)}$. Ideally, we have a least one such element in $\mathbf{z}_{(2)}$ and at least one such element in $\mathbf{z}_{(3)}$.

b. Let $\mathbf{z}$ denote all nonredundant exogenous variables in the system. Then use these as instruments in a 2SLS analysis.

c. The matrix of instruments for each *i* is

$$\mathbf{z}_1 = \begin{pmatrix} \mathbf{z}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{z}_i, educ_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{z}_i \end{pmatrix}.$$

d. $\mathbf{z}_{(3)} = \mathbf{z}$. That is, we should not make any exclusion restrictions in the reduced form for *educ*.

9.8. a. I interact *nearc*4 with experience and its quadratic, and the race indicator. The Stata output follows.

```
. use card

. gen educsq = educ^2

. gen nearc4exper = nearc4*exper

. gen nearc4expersq = nearc4*expersq

. gen nearc4black = nearc4*black

. reg educsq exper expersq black south smsa reg661-reg668 smsa66 nearc4
      nearc4exper nearc4expersq nearc4black, robust

Linear regression                                      Number of obs =    3010
```

```
                                                F( 18,  2991) =  233.34
                                                Prob > F      =  0.0000
                                                R-squared     =  0.4505
                                                Root MSE      =  52.172

------------------------------------------------------------------------------
             |               Robust
      educsq |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       exper |  -18.01791   1.229128   -14.66   0.000    -20.42793   -15.60789
     expersq |   .3700966    .058167     6.36   0.000     .2560452    .4841479
       black |  -21.04009   3.569591    -5.89   0.000    -28.03919   -14.04098
       south |  -.5738389   3.973465    -0.14   0.885     -8.36484    7.217162
        smsa |   10.38892   3.036816     3.42   0.001     4.434463    16.34338
      reg661 |  -6.175308   5.574484    -1.11   0.268    -17.10552    4.754903
      reg662 |  -6.092379   4.254714    -1.43   0.152    -14.43484    2.250083
      reg663 |  -6.193772   4.010618    -1.54   0.123    -14.05762    1.670077
      reg664 |  -3.413348   5.069994    -0.67   0.501    -13.35438    6.527681
      reg665 |  -12.31649   5.439968    -2.26   0.024    -22.98295   -1.650031
      reg666 |  -13.27102   5.693005    -2.33   0.020    -24.43362    -2.10842
      reg667 |  -10.83381   5.814901    -1.86   0.063    -22.23542     .567801
      reg668 |   8.427749   6.627727     1.27   0.204    -4.567616    21.42312
      smsa66 |  -.4621454   3.058084    -0.15   0.880    -6.458307    5.534016
      nearc4 |  -12.25914   7.012394    -1.75   0.081    -26.00874    1.490464
 nearc4exper |   4.192304    1.55785     2.69   0.007     1.137738     7.24687
nearc4expe~q |  -.1623635   .0753242    -2.16   0.031     -.310056    -.014671
 nearc4black |  -4.789202   4.247869    -1.13   0.260    -13.11824     3.53984
       _cons |    307.212   6.617862    46.42   0.000     294.2359     320.188
------------------------------------------------------------------------------

. test nearc4exper nearc4expersq nearc4black

 ( 1)  nearc4exper = 0
 ( 2)  nearc4expersq = 0
 ( 3)  nearc4black = 0

       F(  3,  2991) =    3.72
            Prob > F =    0.0110

. ivreg lwage exper expersq black south smsa reg661-reg668 smsa66
        (educ educsq = nearc4 nearc4exper nearc4expersq nearc4black)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =    3010
-------------+------------------------------           F( 16,  2993) =   45.92
       Model |  116.731381    16  7.29571132           Prob > F      =  0.0000
    Residual |  475.910264  2993  .159007773           R-squared     =  0.1970
-------------+------------------------------           Adj R-squared =  0.1927
       Total |  592.641645  3009  .196956346           Root MSE      =  .39876

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .3161298   .1457578     2.17   0.030     .0303342    .6019254
      educsq |  -.0066592   .0058401    -1.14   0.254    -.0181103    .0047918
       exper |   .0840117   .0361077     2.33   0.020     .0132132    .1548101
     expersq |  -.0007825   .0014221    -0.55   0.582    -.0035709    .0020058
       black |  -.1360751   .0455727    -2.99   0.003    -.2254322   -.0467181
```

```
       south |   -.141488   .0279775    -5.06   0.000    -.1963451   -.0866308
        smsa |   .1072011   .0290324     3.69   0.000     .0502755    .1641267
      reg661 |  -.1098848   .0428194    -2.57   0.010    -.1938432   -.0259264
      reg662 |   .0036271   .0325364     0.11   0.911    -.0601688    .0674231
      reg663 |   .0428246   .0315082     1.36   0.174    -.0189554    .1046045
      reg664 |  -.0639842   .0391843    -1.63   0.103    -.1408151    .0128468
      reg665 |   .0480365   .0445934     1.08   0.281    -.0394003    .1354734
      reg666 |   .0672512   .0498043     1.35   0.177    -.0304028    .1649052
      reg667 |   .0347783   .0471451     0.74   0.461    -.0576617    .1272183
      reg668 |  -.1933844   .0512395    -3.77   0.000    -.2938526   -.0929161
      smsa66 |   .0089666   .0222745     0.40   0.687    -.0347083    .0526414
       _cons |   2.610889   .9706341     2.69   0.007     .7077116    4.514067
------------------------------------------------------------------------------
Instrumented:  educ educsq
Instruments:   exper expersq black south smsa reg661 reg662 reg663 reg664
               reg665 reg666 reg667 reg668 smsa66 nearc4 nearc4exper
               nearc4expersq nearc4black
------------------------------------------------------------------------------
```

The heteroskedasticity-robust Wald test, reported in the form of an F statistic, shows that the three interaction terms are partially correlated with $educ^2$: the p-value = .011. (Whether the partial correlation is strong enough is a reasonable concern.)

The 2SLS estimate of β_{educ^2} is $-.0067$ with $t = -1.14$. Without stronger evidence, we can safely leave $educ^2$ out of the wage equation

b. If $E(u_2|\mathbf{z}) = 0$, as we would typically assume, than any function of $\mathbf{z}$ is uncorrelated with u_1, including interactions of the form $black \cdot z_j$ for any exogenous variable z_j. Such interactions are likely to be correlated with $black \cdot educ$ if z_j is correlated with $educ$.

c. The 2SLS estimates, first using $black \cdot nearc4$ as the IV for $black \cdot educ$ and then using $black \cdot \widehat{educ}$ as the IV, are given by the Stata output. The heteroskedasticity-robust standard errors are computed. The standard error using $black \cdot \widehat{educ}$ as the IV is much smaller than the standard error using $black \cdot nearc4$ as the IV. (The point estimate is also substantially higher.)

```
. ivreg lwage exper expersq black south smsa reg661-reg668 smsa66
        (educ blackeduc = nearc4 nearc4black), robust

Instrumental variables (2SLS) regression              Number of obs =    3010
                                                      F( 16,  2993) =   52.35
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.2435
                                                      Root MSE      =  .38702
```

```
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .1273557   .0561622     2.27   0.023     .0172352    .2374762
   blackeduc |   .0109036   .0399278     0.27   0.785    -.0673851    .0891923
       exper |   .1059116   .0249463     4.25   0.000     .0569979    .1548253
     expersq |  -.0022406   .0004902    -4.57   0.000    -.0032017   -.0012794
       black |   -.282765   .5012131    -0.56   0.573    -1.265522    .6999922
       south |  -.1424762   .0298942    -4.77   0.000    -.2010914    -.083861
        smsa |   .1111555   .0310592     3.58   0.000      .050256    .1720551
      reg661 |  -.1103479   .0418554    -2.64   0.008    -.1924161   -.0282797
      reg662 |  -.0081783   .0339196    -0.24   0.809    -.0746863    .0583298
      reg663 |   .0382413   .0335008     1.14   0.254    -.0274456    .1039283
      reg664 |  -.0600379   .0398032    -1.51   0.132    -.1380824    .0180066
      reg665 |   .0337805   .0519109     0.65   0.515    -.0680042    .1355652
      reg666 |   .0498975   .0559569     0.89   0.373    -.0598204    .1596155
      reg667 |   .0216942   .0528376     0.41   0.681    -.0819075    .1252959
      reg668 |  -.1908353   .0506182    -3.77   0.000    -.2900853   -.0915853
      smsa66 |   .0180009   .0205709     0.88   0.382    -.0223337    .0583356
       _cons |    3.84499   .9545666     4.03   0.000     1.973317    5.716663
------------------------------------------------------------------------------
Instrumented:  educ blackeduc
Instruments:   exper expersq black south smsa reg661 reg662 reg663 reg664
               reg665 reg666 reg667 reg668 smsa66 nearc4 nearc4black
------------------------------------------------------------------------------

. ivreg lwage exper expersq black south smsa reg661-reg668 smsa66
        (educ blackeduc = nearc4 blackeduchat), robust

Instrumental variables (2SLS) regression              Number of obs =    3010
                                                      F( 16,  2993) =   52.52
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.2501
                                                      Root MSE      =  .38535

------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .1178141   .0554036     2.13   0.034     .0091811     .226447
   blackeduc |    .035984   .0105707     3.40   0.001     .0152573    .0567106
       exper |   .1004843   .0241951     4.15   0.000     .0530436     .147925
     expersq |  -.0020235   .0003597    -5.63   0.000    -.0027288   -.0013183
       black |  -.5955669   .1587782    -3.75   0.000    -.9068923   -.2842415
       south |  -.1374265   .0294259    -4.67   0.000    -.1951236   -.0797294
        smsa |   .1096541   .0306748     3.57   0.000     .0495083    .1697998
      reg661 |  -.1161759   .0409317    -2.84   0.005     -.196433   -.0359189
      reg662 |  -.0107817   .0335743    -0.32   0.748    -.0766127    .0550494
      reg663 |   .0331736   .0326007     1.02   0.309    -.0307484    .0970955
      reg664 |   -.064916   .0388398    -1.67   0.095    -.1410715    .0112395
      reg665 |    .023022   .0505787     0.46   0.649    -.0761506    .1221946
      reg666 |   .0379568   .0534653     0.71   0.478    -.0668757    .1427892
      reg667 |   .0100466   .0513629     0.20   0.845    -.0906637    .1107568
      reg668 |  -.1907066   .0502527    -3.79   0.000    -.2892399   -.0921733
      smsa66 |   .0167814   .0203639     0.82   0.410    -.0231472    .0567101
       _cons |    4.00836   .9416251     4.26   0.000     2.162062    5.854658
------------------------------------------------------------------------------
```

```
Instrumented:  educ blackeduc
Instruments:   exper expersq black south smsa reg661 reg662 reg663 reg664
               reg665 reg666 reg667 reg668 smsa66 nearc4 blackeduchat
------------------------------------------------------------------------------
```

d. Suppose $\mathrm{E}(educ|\mathbf{z}) = \mathbf{z}\boldsymbol{\pi}_2$ and $\mathrm{Var}(u_1|\mathbf{z}) = \sigma_1^2$. Then by Theorem 8.5, the optimal IVs for *educ* and *black* • *educ* are

$$\sigma_1^{-2}\mathrm{E}(educ|\mathbf{z}) = \sigma_1^{-2}\mathbf{z}\boldsymbol{\pi}_2$$
$$\sigma_1^{-2}\mathrm{E}(black \cdot educ|\mathbf{z}) = \sigma_1^{-2}black \cdot \mathrm{E}(educ|\mathbf{z}) = \sigma_1^{-2}black \cdot (\mathbf{z}\boldsymbol{\pi}_2).$$

We can drop the constant σ_1^{-2}, and so the optimal IVs can be taken to be

$$[\mathbf{z}_1, \mathbf{z}\boldsymbol{\pi}_2, black \cdot (\mathbf{z}\boldsymbol{\pi}_2)].$$

When we operationalize this procedure, we can use

$$[\mathbf{z}_{i1}, \mathbf{z}_i\hat{\boldsymbol{\pi}}_2, black_i \cdot (\mathbf{z}_i\hat{\boldsymbol{\pi}}_2)] = [\mathbf{z}_{i1}, \widehat{educ}_i, black_i \cdot \widehat{educ}_i]$$

as the optimal IVs. Nothing is lost asymptotically by including $black_i \cdot \widehat{educ}_i$ in the reduced form for $educ_i$ along with $\mathbf{z}_i$. So using 2SLS with IVs $[\mathbf{z}_i, black_i \cdot \widehat{educ}_i]$ produces the asymptotically efficient IV estimator.

9.9. a. The Stata output for 3SLS estimation of (9.28) and (9.29), along with the output for 2SLS on each equation, is given below. For coefficients that are statistically significant, the 3SLS and 2SLS are reasonably close. For coefficients estimated imprecisely, there are some differences between 2SLS and 3SLS, but these are not unexpected. Generally, the 3SLS standard errors are smaller (but recall that none of the standard errors are robust to heteroskedasticity).

```
. reg3 (hours lwage educ age kidslt6 kidsge6 nwifeinc)
       (lwage hours educ exper expersq)

Three-stage least-squares regression
------------------------------------------------------------------------------
Equation          Obs  Parms        RMSE    "R-sq"       chi2         P
------------------------------------------------------------------------------
hours             428      6    1368.362   -2.1145      34.54    0.0000
```

```
lwage             428      4    .6892584    0.0895      79.87   0.0000
------------------------------------------------------------------------------

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
hours        |
       lwage |   1676.933    431.169     3.89   0.000     831.8577    2522.009
        educ |  -205.0267   51.84729    -3.95   0.000    -306.6455   -103.4078
         age |  -12.28121   8.261529    -1.49   0.137    -28.47351    3.911094
     kidslt6 |  -200.5673   134.2685    -1.49   0.135    -463.7287    62.59414
     kidsge6 |  -48.63986   35.95137    -1.35   0.176    -119.1032    21.82352
    nwifeinc |   .3678943   3.451518     0.11   0.915    -6.396957    7.132745
       _cons |   2504.799   535.8919     4.67   0.000      1454.47    3555.128
-------------+----------------------------------------------------------------
lwage        |
       hours |    .000201   .0002109     0.95   0.340    -.0002123    .0006143
        educ |   .1129699   .0151452     7.46   0.000     .0832858    .1426539
       exper |   .0208906   .0142782     1.46   0.143    -.0070942    .0488753
     expersq |  -.0002943   .0002614    -1.13   0.260    -.0008066     .000218
       _cons |  -.7051103   .3045904    -2.31   0.021    -1.302097   -.1081241
------------------------------------------------------------------------------
Endogenous variables:  hours lwage
Exogenous variables:   educ age kidslt6 kidsge6 nwifeinc exper expersq
------------------------------------------------------------------------------

. ivreg hours educ age kidslt6 kidsge6 nwifeinc (lwage = exper expersq)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     428
-------------+------------------------------           F(  6,   421) =    3.41
       Model |  -456272250     6   -76045375           Prob > F      =  0.0027
    Residual |   713583270   421  1694972.14           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |   257311020   427   602601.92           Root MSE      =  1301.

------------------------------------------------------------------------------
       hours |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lwage |   1544.819   480.7387     3.21   0.001     599.8713    2489.766
        educ |   -177.449    58.1426    -3.05   0.002    -291.7349   -63.16302
         age |  -10.78409   9.577347    -1.13   0.261    -29.60946    8.041289
     kidslt6 |  -210.8339    176.934    -1.19   0.234    -558.6179    136.9501
     kidsge6 |  -47.55708   56.91786    -0.84   0.404    -159.4357     64.3215
    nwifeinc |  -9.249121   6.481116    -1.43   0.154     -21.9885    3.490256
       _cons |   2432.198   594.1719     4.09   0.000     1264.285    3600.111
------------------------------------------------------------------------------
Instrumented:  lwage
Instruments:   educ age kidslt6 kidsge6 nwifeinc exper expersq
------------------------------------------------------------------------------

. ivreg lwage educ exper expersq (hours = age kidslt6 kidsge6 nwifeinc)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     428
-------------+------------------------------           F(  4,   423) =   18.80
       Model |  24.8336445     4  6.20841113           Prob > F      =  0.0000
```

```
    Residual |  198.493796   423  .469252474           R-squared     =  0.1112
-------------+------------------------------           Adj R-squared =  0.1028
       Total |  223.327441   427  .523015084           Root MSE      =  .68502

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       hours |   .0001608   .0002154     0.75   0.456    -.0002626    .0005842
        educ |   .1111175   .0153319     7.25   0.000     .0809814    .1412536
       exper |    .032646    .018061     1.81   0.071    -.0028545    .0681465
     expersq |  -.0006765   .0004426    -1.53   0.127    -.0015466    .0001935
       _cons |    -.69279   .3066002    -2.26   0.024     -1.29544   -.0901403
------------------------------------------------------------------------------
Instrumented:  hours
Instruments:   educ exper expersq age kidslt6 kidsge6 nwifeinc
------------------------------------------------------------------------------
```

More efficient GMM estimators can be obtained, along with robust standard errors. The weighting matrix is the optimal one that allows for system heteroskedasticity. The (valid) standard errors from the GMM estimation are quite a bit larger than the usual 3SLS standard errors. The coefficient estimates change some, but the magnitudes are similar and all qualitative conclusions hold.

```
. gmm (hours - {b1}*lwage - {b2}*educ - {b3}*age - {b4}*kidslt6
             - {b5}*kidsge6 - {b6}*nwifeinc - {b7})
      (lwage - {b8}*hour s - {b9}*educ - {b10}*age - {b11}*exper
             - {b12}*expersq - {b13}),
      instruments(educ age kidslt6 kidsge6 nwifeinc exper
                  expersq) winitial(identity)

Step 1
Iteration 0:   GMM criterion Q(b) =  1.339e+11
Iteration 1:   GMM criterion Q(b) =  350.92722
Iteration 2:   GMM criterion Q(b) =  350.92722

Step 2
Iteration 0:   GMM criterion Q(b) =  .08810078
Iteration 1:   GMM criterion Q(b) =  .00317682
Iteration 2:   GMM criterion Q(b) =  .00317682

GMM estimation

Number of parameters =  13
Number of moments    =  16
Initial weight matrix: Identity                     Number of obs  =     428
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         /b1 |    1606.63   618.7605     2.60   0.009      393.882    2819.379
```

```
         /b2 |  -179.4037   69.66358    -2.58   0.010    -315.9418   -42.86564
         /b3 |  -10.29206   10.88031    -0.95   0.344    -31.61708    11.03295
         /b4 |  -238.2421   212.7398    -1.12   0.263    -655.2044    178.7202
         /b5 |  -46.23438    59.7293    -0.77   0.439    -163.3017    70.83289
         /b6 |  -10.39225   5.576519    -1.86   0.062    -21.32203    .5375285
         /b7 |   2385.333   630.2282     3.78   0.000     1150.108    3620.558
         /b8 |   .0002704   .0003012     0.90   0.369      -.00032    .0008608
         /b9 |   .1153863   .0158272     7.29   0.000     .0843655     .146407
        /b10 |   .0031491     .00654     0.48   0.630    -.0096691    .0159673
        /b11 |   .0336343   .0246234     1.37   0.172    -.0146266    .0818952
        /b12 |  -.0008599   .0006453    -1.33   0.183    -.0021247     .000405
        /b13 |  -.9910496   .5850041    -1.69   0.090    -2.137637    .1555373
------------------------------------------------------------------------------
Instruments for equation 1: educ age kidslt6 kidsge6 nwifeinc exper expersq
                            _cons
Instruments for equation 2: educ age kidslt6 kidsge6 nwifeinc exper expersq
                            _cons
```

b. Using γ as coefficients on endogenous variables, the three-equation system can be expressed as

$$hours = \gamma_{12} lwage + \gamma_{13} educ + \delta_{11} + \delta_{12} age + \delta_{13} kidslt6 + \delta_{14} kidsge6 + \delta_{15} nwifeinc + u_1$$

$$lwage = \gamma_{21} hours + \gamma_{23} educ + \delta_{21} + \delta_{12} age + \delta_{22} exper + \delta_{23} exper^2 + u_2$$

$$educ = \delta_{31} + \delta_{32} age + \delta_{33} kidslt6 + \delta_{34} kidsge6 + \delta_{35} nwifeinc + \delta_{36} exper + \delta_{37} exper^2 + \delta_{38} motheduc + \delta_{39} fatheduc + \delta_{3,10} huseduc + u_3$$

The IVs for the first equation are all appearing in the third equation, plus *educ*. For the second and third equations, the valid IVs are all variables appearing in the third equation (which is also a reduced form for *educ*).

The following Stata output produces the GMM estimates using the optimal weighting matrix, along with the valid standard errors.

```
. gmm (hours - {b1}*lwage - {b2}*educ - {b3}*age - {b4}*kidslt6 -
        {b5}*kidsge6 - {b6}*nwifeinc - {b7})
      (lwage - {b8}*hours - {b9}*educ - {b10}*age - {b11}*exper
         - {b12}*expersq - {b13})
      (educ - {b14}*age - {b15}*kidslt6 - {b16}*kidsge6 - {b17}*nwifeinc
          - {b18}*exper - {b19}*expersq - {b20}*motheduc - {b21}*fatheduc
          - {b22}*huseduc - {b23}),
      instruments(age kidslt6 kidsge6 nwifeinc exper expersq
                  motheduc fatheduc huseduc)
      instruments(1: educ) winitial(identity)

Step 1
Iteration 0:   GMM criterion Q(b) =  1.344e+11
Iteration 1:   GMM criterion Q(b) =  116344.22
```

```
Iteration 2:   GMM criterion Q(b) =  116344.22

Step 2
Iteration 0:   GMM criterion Q(b) =  .22854932
Iteration 1:   GMM criterion Q(b) =   .0195166
Iteration 2:   GMM criterion Q(b) =   .0195166

GMM estimation

Number of parameters =  23
Number of moments    =  31
Initial weight matrix: Identity                     Number of obs  =     428
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         /b1 |   1195.924   383.1797     3.12   0.002     444.9052    1946.942
         /b2 |  -140.7006   46.54044    -3.02   0.003    -231.9181   -49.48298
         /b3 |  -9.160326   8.687135    -1.05   0.292     -26.1868    7.866146
         /b4 |  -298.5054    169.207    -1.76   0.078    -630.1449    33.13422
         /b5 |  -72.19858   43.44575    -1.66   0.097    -157.3507    12.95353
         /b6 |  -6.128558     3.9233    -1.56   0.118    -13.81809    1.560969
         /b7 |   2297.594   511.7623     4.49   0.000     1294.559     3300.63
         /b8 |   .0002455   .0002852     0.86   0.389    -.0003134    .0008044
         /b9 |   .1011445   .0233025     4.34   0.000     .0554724    .1468167
        /b10 |   .0007376   .0061579     0.12   0.905    -.0113317    .0128069
        /b11 |    .031704   .0194377     1.63   0.103    -.0063933    .0698013
        /b12 |   -.000695   .0003921    -1.77   0.076    -.0014635    .0000735
        /b13 |  -.6841501   .6119956    -1.12   0.264    -1.883639    .5153392
        /b14 |  -.0061529   .0135359    -0.45   0.649    -.0326827    .0203769
        /b15 |   .5229541    .209776     2.49   0.013     .1118006    .9341075
        /b16 |  -.1128699    .070775    -1.59   0.111    -.2515864    .0258465
        /b17 |   .0273278   .0093602     2.92   0.004     .0089821    .0456735
        /b18 |   .0284576   .0333103     0.85   0.393    -.0368295    .0937447
        /b19 |  -.0001177   .0010755    -0.11   0.913    -.0022256    .0019902
        /b20 |   .1226775   .0302991     4.05   0.000     .0632923    .1820627
        /b21 |   .0976753   .0284494     3.43   0.001     .0419155     .153435
        /b22 |   .3352722   .0357583     9.38   0.000     .2651873    .4053572
        /b23 |   5.858645   .7413707     7.90   0.000     4.405585    7.311704
------------------------------------------------------------------------------
Instruments for equation 1: educ age kidslt6 kidsge6 nwifeinc exper expersq
                            motheduc fatheduc huseduc _cons
Instruments for equation 2: age kidslt6 kidsge6 nwifeinc exper expersq
                            motheduc fatheduc huseduc _cons
Instruments for equation 3: age kidslt6 kidsge6 nwifeinc exper expersq
                            motheduc fatheduc huseduc _cons
```

9.10. a. No. 2SLS estimation of the first equation uses all nonredundant elements of $\mathbf{z}_1$ and $\mathbf{z}_2$ – call these $\mathbf{z}$ – in the first stage regression for y_2. Therefore, the exclusion restrictions in the second equation are not imposed.

b. No, except in the extremely rare case where the covariance between the structural errors is estimated to be zero. (If we impose a zero covariance, then the 2SLS estimates and 3SLS estimates will be the same.) Effectively, each equation – including the second – is overidentified.

c. This just follows from the first two parts. Because 2SLS puts no restrictions on the reduced form for y_2, whereas 3SLS assumes only $\mathbf{z}_2$ appears in the reduced form for y_2, 2SLS will be more robust for estimating the parameters in the first equation.

9.11. a. Because z_2 and z_3 are both omitted from the first equation, we just need $\delta_{22} \neq 0$ or $\delta_{23} \neq 0$. The second equation is identified if and only if $\delta_{11} \neq 0$.

b. After substitution and straightforward algebra, it can be seen that $\pi_{11} = \delta_{11}/(1 - \gamma_{12}\gamma_{21})$.

c. We can estimate the system by 3SLS; for the second equation , this is identical to 2SLS since it is just identified. Or, we could just use 2SLS on each equation. Given $\hat{\delta}_{11}, \hat{\gamma}_{12},$ and $\hat{\gamma}_{21}$, we would form $\hat{\pi}_{11} = \hat{\delta}_{11}/(1 - \hat{\gamma}_{12}\hat{\gamma}_{21})$.

d. Whether we estimate the parameters by 2SLS or 3SLS, we will generally inconsistently estimate δ_{11} and γ_{12}. (We are estimating the second equation by 2SLS so we will still consistently estimate γ_{21} provided we have not misspecified this equation.) So our estimate of $\boldsymbol{\pi}_{11} = \partial \mathrm{E}(y_2|\mathbf{z})/\partial \mathbf{z}_1$ will be inconsistent in any case.

e. We can just estimate the reduced form $\mathrm{E}(y_2|z_1, z_2, z_3)$ by ordinary least squares.

f. Consistency of OLS for $\boldsymbol{\pi}_{11}$does not hinge on the validity of the exclusion restrictions in the structural model, whereas using an SEM does. Of course, if the SEM is correctly specified, we obtain a more efficient estimator of the reduced form parameters by imposing the restrictions in estimating $\boldsymbol{\pi}_{11}$.

9.12. a. Generally, $\mathrm{E}(y_2^2|\mathbf{z}) = \mathrm{Var}(y_2|\mathbf{z}) + [\mathrm{E}(y_2|\mathbf{z})]^2$; when $\gamma_{13} = 0$ and u_1 and u_2 are

homoskedastic, $\text{Var}(y_2|\mathbf{z})$ is constant, say τ_2^2. (This is easily seen from the reduced form for y_2, which is linear when $\gamma_{13} = 0$.) Therefore, $\text{E}(y_2^2|\mathbf{z}) = \tau_2^2 + (\pi_{20} + \mathbf{z}\boldsymbol{\pi}_2)^2$.

b. We do not really need to use part a; in fact, it turns out to be a red herring for this problem. Since $\gamma_{13} = 0$,

$$\begin{aligned}\text{E}(y_1|\mathbf{z}) &= \delta_{10} + \gamma_{12}\text{E}(y_2|\mathbf{z}) + \mathbf{z}_1\boldsymbol{\delta}_1 + \text{E}(u_1|\mathbf{z}) \\ &= \delta_{10} + \gamma_{12}\text{E}(y_2|\mathbf{z}) + \mathbf{z}_1\boldsymbol{\delta}_1.\end{aligned}$$

c. When $\gamma_{13} = 0$, *any* nonlinear function of $\mathbf{z}$, including $(\boldsymbol{\pi}_{20} + \mathbf{z}\boldsymbol{\pi}_2)^2$, has zero coefficient in $\text{E}(y_1|\mathbf{z}) = \delta_{10} + \gamma_{12}(\pi_{20} + \mathbf{z}\boldsymbol{\pi}_2) + \mathbf{z}_1\boldsymbol{\delta}_1$. Plus, if $\gamma_{13} = 0$, then the parameters π_{20} and $\boldsymbol{\pi}_2$ are consistently estimated from the first stage regression y_{i2} on $1, \mathbf{z}_1, i = 1, \ldots, N$. Therefore, the regression y_{i1} on $1, (\hat{\pi}_{20} + \mathbf{z}_i\hat{\boldsymbol{\pi}}_2), (\hat{\pi}_{20} + \mathbf{z}_i\hat{\boldsymbol{\pi}}_2)^2, \mathbf{z}_{i1}, i = 1, \ldots, N$ consistently estimates δ_{10}, γ_{12}, 0, and $\boldsymbol{\delta}_1$, respectively. But this is just the regression y_{i1} on $1, \hat{y}_{i2}, (\hat{y}_{i2})^2, \mathbf{z}_{i1}, i = 1, \ldots, N$.

d. Because $\text{E}(u_1|\mathbf{z}) = 0$ and $\text{Var}(u_1|\mathbf{z}) = \sigma_1^2$, we can immediately apply Theorem 8.5 to conclude that the optimal IVS for estimating the first equation are $[1, \text{E}(y_2|\mathbf{z}), \text{E}(y_2^2|\mathbf{z}), \mathbf{z}_1]/\sigma_1^2$, and we can drop the division by σ_1^2. But, if $\gamma_{13} = 0$, then $\text{E}(y_2|\mathbf{z})$ is linear in $\mathbf{z}$ and, from part a, $\text{E}(y_2^2|\mathbf{z}) = \boldsymbol{\tau}_2^2 + [\text{E}(y_2|\mathbf{z})]^2$. So the optimal IVs are a linear combination of $\{1, \text{E}(y_2|\mathbf{z}), [\text{E}(y_2|\mathbf{z})]^2, \mathbf{z}_1\}$, which means they are a linear combination of $\{1, \mathbf{z}, [\text{E}(y_2|\mathbf{z})]^2\}$. We never do worse asymptotically by using more IVs, so we can use $\{1, \mathbf{z}, [\text{E}(y_2|\mathbf{z})]^2\}$ as an optimal set. Why would we use this larger set instead of $\{1, \text{E}(y_2|\mathbf{z}), [\text{E}(y_2|\mathbf{z})]^2, \mathbf{z}_1\}$? For one, the larger set will generally yield overidentifying restrictions. In addition, if $\gamma_{13} \neq 0$, we will generally be better off using more instruments: $\mathbf{z}$ rather than only $\text{L}(y_2|1, \mathbf{z})$.

e. The estimates below are similar to those reported in Section 9.5.2, where we just added $educ^2, age^2$, and $nwifeinc^2$ to the IV list and using 2SLS with $lwage = \log(wage)$ and $lwagesq = [\log(wage)]^2$ as endogenous explanatory variables. In particular, the coefficient on

lwagesq is still statistically insignificant. The standard errors reported here are robust to heteroskedasticity (unlike in the text).

```
. gen lwagesq = lwage^2

. qui reg lwage educ age kidslt6 kidsge6 nwifeinc exper expersq

. predict lwagehat
(option xb assumed; fitted values)

. gen lwagehatsq = lwagehat^2

. ivreg hours (lwage lwagesq = exper expersq lwagehatsq)
               educ age kidslt6 kidsge6 nwifeinc, robust

Instrumental variables (2SLS) regression              Number of obs =     428
                                                      F(  7,   420) =    2.88
                                                      Prob > F      =  0.0059
                                                      R-squared     =
                                                      Root MSE      =  1177.

------------------------------------------------------------------------------
             |               Robust
       hours |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lwage |   1846.902   856.1346     2.16   0.032     164.0599    3529.745
     lwagesq |    -373.16   401.9586    -0.93   0.354    -1163.261    416.9412
        educ |  -103.2347   71.99564    -1.43   0.152    -244.7513    38.28199
         age |  -9.425115   8.848798    -1.07   0.287    -26.81856    7.968333
     kidslt6 |  -187.0236   177.1703    -1.06   0.292    -535.2747    161.2274
     kidsge6 |  -55.70163   45.86915    -1.21   0.225    -145.8633    34.46007
    nwifeinc |    -7.5979   4.491138    -1.69   0.091    -16.42581    1.230009
       _cons |   1775.847   881.2631     2.02   0.045     43.61095    3508.082
------------------------------------------------------------------------------
Instrumented:  lwage lwagesq
Instruments:   educ age kidslt6 kidsge6 nwifeinc exper expersq lwagehatsq
------------------------------------------------------------------------------

. gen educsq = educ^2

. gen agesq = age^2

. gen nwifeincsq = nwifeinc^2

. ivreg hours (lwage lwagesq = exper expersq educsq agesq nwifeincsq)
               educ age kidslt6 kidsge6 nwifeinc, robust

Instrumental variables (2SLS) regression              Number of obs =     428
                                                      F(  7,   420) =    3.53
                                                      Prob > F      =  0.0011
                                                      R-squared     =
                                                      Root MSE      =    1161

------------------------------------------------------------------------------
             |               Robust
```

```
       hours |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lwage |    1873.62   792.3005     2.36   0.018     316.2521    3430.989
     lwagesq |  -437.2911   313.7658    -1.39   0.164    -1054.038    179.4559
        educ |   -87.8511   50.01226    -1.76   0.080    -186.1566    10.45442
         age |  -9.142303   8.512639    -1.07   0.283    -25.87499    7.590381
     kidslt6 |  -185.0554   179.9036    -1.03   0.304    -538.6789    168.5682
     kidsge6 |  -58.18949    44.6767    -1.30   0.193    -146.0073     29.6283
    nwifeinc |  -7.233422   4.037903    -1.79   0.074    -15.17044    .7035957
       _cons |   1657.926   671.9361     2.47   0.014     337.1489    2978.702
------------------------------------------------------------------------------
Instrumented:  lwage lwagesq
Instruments:   educ age kidslt6 kidsge6 nwifeinc exper expersq educsq agesq
               nwifeincsq
------------------------------------------------------------------------------
```

9.13. a. The first equation is identified if, and only if, $\delta_{22} \neq 0$ (the rank condition).

b. Here is the Stata output:

```
. use openness

. reg open lpcinc lland, robust

Linear regression                               Number of obs =     114
                                                F(  2,   111) =   22.22
                                                Prob > F      =  0.0000
                                                R-squared     =  0.4487
                                                Root MSE      =  17.796

------------------------------------------------------------------------------
             |               Robust
        open |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      lpcinc |   .5464812   1.436115     0.38   0.704    -2.299276    3.392238
       lland |  -7.567103   1.141798    -6.63   0.000    -9.829652   -5.304554
       _cons |   117.0845   18.24808     6.42   0.000     80.92473    153.2443
------------------------------------------------------------------------------
```

With $t = -6.63$, we can conclude that is shows that log(*land*) is very statistically significant in the reduced form for *open*. The negative coefficient implies that smaller countries are more "open."

c. Here is the Stata output. First 2SLS, the OLS, both with heteroskedasticity-robust standard errors.

```
. ivreg inf (open = lland) lpcinc, robust

Instrumental variables (2SLS) regression        Number of obs =     114
                                                F(  2,   111) =    2.53
                                                Prob > F      =  0.0844
```

```
                                                      R-squared     =  0.0309
                                                      Root MSE      =  23.836

------------------------------------------------------------------------------
             |               Robust
         inf |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        open |  -.3374871   .1524489    -2.21   0.029    -.6395748   -.0353994
      lpcinc |   .3758247   1.378542     0.27   0.786    -2.355848    3.107497
       _cons |   26.89934    10.9199     2.46   0.015     5.260821    48.53785
------------------------------------------------------------------------------
Instrumented:  open
Instruments:   lpcinc lland
------------------------------------------------------------------------------

. reg inf open lpcinc, robust

Linear regression                                     Number of obs =     114
                                                      F(  2,   111) =    3.84
                                                      Prob > F      =  0.0243
                                                      R-squared     =  0.0453
                                                      Root MSE      =  23.658

------------------------------------------------------------------------------
             |               Robust
         inf |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        open |  -.2150695   .0794571    -2.71   0.008    -.3725191   -.0576199
      lpcinc |   .0175683   1.278747     0.01   0.989    -2.516354     2.55149
       _cons |   25.10403    9.99078     2.51   0.013     5.306636    44.90143
------------------------------------------------------------------------------
```

The IV estimate is larger in magnitude – by more than 50% – but its standard error is almost twice as large as the OLS standard error. There is some but not overwhelming evidence that *open* is actually endogenous. The variable-addition Hausman test, made robust to heteroskedasticity, has $t = 1.48$. With $N = 114$, we might not expect very strong evidence.

d. If we add $\gamma_{13} open^2$ to the equation, we need an IV for it. Since log(*land*) is partially correlated with *open*, $[\log(land)]^2$ is a natural candidate. A regression of $open^2$ on $\log(land), [\log(land)]^2$, and log(*pcinc*) gives a heteroskedasticity-robust *t* statistic on $[\log(land)]^2$ of about 2. This is borderline, but we will go ahead. The Stata output for 2SLS is

```
. ivreg inf (open opensq = lland llandsq) lpcinc, robust

Instrumental variables (2SLS) regression              Number of obs =     114
                                                      F(  3,   110) =    1.44
                                                      Prob > F      =  0.2350
```

```
                                                      R-squared     =
                                                      Root MSE      =    24.

------------------------------------------------------------------------------
             |               Robust
         inf |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        open |  -1.198637   .7139934    -1.68   0.096    -2.613604    .2163303
      opensq |   .0075781   .0053779     1.41   0.162    -.0030796    .0182358
      lpcinc |   .5066092   1.490845     0.34   0.735    -2.447896    3.461114
       _cons |   43.17124   18.37223     2.35   0.021     6.761785     79.5807
------------------------------------------------------------------------------
Instrumented:  open opensq
Instruments:   lpcinc lland llandsq
------------------------------------------------------------------------------
```

The squared term indicates that the impact of *open* on *inf* diminishes; the estimate would be significant at about the 8.1% level against a one-sided alternative.

e. Here is the Stata output for implementing the method described in the problem:

```
. qui reg open lpcinc lland

. predict openhat
(option xb assumed; fitted values)

. gen openhatsq = openhat^2

. reg inf openhat openhatsq lpcinc, robust

Linear regression                                     Number of obs =     114
                                                      F(  3,   110) =    2.52
                                                      Prob > F      =  0.0620
                                                      R-squared     =  0.0575
                                                      Root MSE      =  23.612

------------------------------------------------------------------------------
             |               Robust
         inf |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     openhat |  -.8648092   .5762007    -1.50   0.136    -2.006704    .2770854
   openhatsq |   .0060502   .0050906     1.19   0.237    -.0040383    .0161387
      lpcinc |   .0412172   1.293368     0.03   0.975    -2.521935    2.604369
       _cons |   39.17831   15.99614     2.45   0.016     7.477717     70.8789
------------------------------------------------------------------------------
```

Qualitatively, the results are similar to the appropriate IV method from part d, but the coefficient on *openhat* is quite a bit smaller in magnitude using the "forbidden regression." If $\gamma_{13} = 0$, $\mathrm{E}(open|lpcinc, lland)$ is linear, and $\mathrm{Var}(open|lpcinc, lland)$ is constant then, as shown

in Problem 9.12, both methods are consistent. But the forbidden regression implemented in this part is unnecessary, less robust, and we cannot trust the standard errors, anyway.

Incidentally, using *openhatsq* as an IV, rather than a regressor, gives very similar estimates to using *llandsq* as an IV for *opensq*.

9.14. a. Agree. In equation (9.13), the reduced form variance matrix, $\mathbf{\Lambda} = \mathrm{E}(\mathbf{v}'\mathbf{v})$ is always identified. Now the structural variance matrix can be written as $\mathbf{\Sigma} = \mathbf{\Gamma}'\mathbf{\Lambda}\mathbf{\Gamma}$, so if $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ are both identified, so is $\mathbf{\Sigma}$.

b. Disagree. In many cases a linear version of the model is not identified because there are not enough instruments. In that case, identification of the more general model hinges on the model actually having nonlinearities, a tenuous situation.

c. Disagree. $\mathrm{E}(\mathbf{u}|\mathbf{z}) = \mathbf{0}$ implies that z_h is uncorrelated with u_g for all $h = 1,\ldots,L$ and $g = 1,\ldots,G$. This kind of orthogonality, along with the rank condition, is sufficient for the GMM, traditional, or GIV versions of 3SLS to be consistent. We need not restrict $\mathrm{Var}(\mathbf{u}|\mathbf{z})$ for consistency, and robust inference is easily obtained.

d. Disagree. Even true SEMs can have other problems that cause endogeneity, name, omitted variables and measurement error. In these cases, some variables may be valid instruments in one equation but not other equations.

e. Disagree. Control function approaches generally require more assumptions in order to be consistent. Take equations (9.70) and (9.71) as an example. The CF approach proposed there basically requires assumption (9.72), which can be very restrictive, especially if y_2 or y_3 exhibit discreteness. By contrast, we can use an IV approach that specifies $\mathbf{z}$ and nonlinear functions of $\mathbf{z}$ as instruments in directly estimating (9.70) (by 2SLS or GMM). When they are consistent, we might expect CF approaches to be more efficient asymptotically.

9.15. a. The model with $\alpha_{13} = 0$, $\boldsymbol{\gamma}_{11} = \mathbf{0}$, and $\boldsymbol{\gamma}_{12} = \mathbf{0}$ is

$$y_1 = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_{11}y_2 + \alpha_{12}y_3 + u_3$$

and we maintain that this equation is identified. Necessary is that $L \geq L_1 + 2$, as is assumed in the problem. The rank condition is more complicated, and we assume it holds. In other words, we assume we have enough relevant instruments for y_2 and y_3. If (9.71) also holds, and say $\mathrm{E}(u_2|\mathbf{z}) = \mathrm{E}(u_3|\mathbf{z}) = 0$, then

$$\begin{aligned}\mathrm{E}(y_2y_3|\mathbf{z}) &= (\mathbf{z}\boldsymbol{\beta}_2)(\mathbf{z}\boldsymbol{\beta}_3) + \mathrm{E}(u_2u_3|\mathbf{z}) \\ \mathrm{E}(y_2\mathbf{z}_1|\mathbf{z}) &= (\mathbf{z}\boldsymbol{\beta}_2)\mathbf{z}_1 \\ \mathrm{E}(y_3\mathbf{z}_1|\mathbf{z}) &= (\mathbf{z}\boldsymbol{\beta}_3)\mathbf{z}_1\end{aligned}$$

which means that squares and cross products in $\mathbf{z}$ are natural instruments for the interaction terms. Note that there are plenty of these interaction terms, and they are likely to provide enough variation because the linear version of the model is identified. For example, if z_L has zero coefficient in the reduced form for y_3, then $z_L\mathbf{z}_1$ appears in $\mathrm{E}(y_2\mathbf{z}_1|\mathbf{z})$ but not in $\mathrm{E}(y_3\mathbf{z}_1|\mathbf{z})$. Further, suppose $\mathrm{E}(u_2u_3|\mathbf{z})$ is actually constant. then squares of elements in $\mathbf{z}_2$, where $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$, appear in $\mathrm{E}(y_2y_3|\mathbf{z})$ but not the other expectations. If $\mathrm{E}(u_2u_3|\mathbf{z})$ is not constant, it would cancel out with those squares only by fluke. Further, even if we only take (9.71) to be linear projections, that would only helps our cause because if these are not conditional expectations then even more nonlinear functions of $\mathbf{z}$ would be useful as instruments.

b. There are no overidentification restrictions. We have the same number of instruments as explanatory variables. That is one drawback to using fitted values as IVs: there are no ovidentification restrictions to test.

c. There are $L - L_1 - 2$ overidentification restrictions. The L_1 subvector $\mathbf{z}_1$ acts as its own instruments, and we need two more elements of $\mathbf{z}$ to instrument for y_2 and y_3. The rest of the

explanatory variables are taking care of by the interaction terms.

d. We can get many overidentification restrictions by using as IVs the nonredundant elements of $(\mathbf{z}_i, \mathbf{z}_i \otimes \mathbf{z}_i)$ – that is, the levels, squares, and cross products of $\mathbf{z}_i$. For example, if $L = 5$, $L_1 = 2$, and $\mathbf{z}$ and $\mathbf{z}_1$ include a constant, then there are $5 + 4 + 6 = 15$ instruments and $2 + 2 + 2 + 2 = 8$ explanatory variables.

e. To solve this problem, we also need to assume $\mathrm{E}(u_2 u_3|\mathbf{z})$ is constant, and to explicitly state that $\mathbf{z}_1$ includes a constant. By Theorem 8.5, the optimal IVs are then given in part a, along with $\mathbf{z}_1$, because $\mathrm{Var}(u_1|\mathbf{z}) = \sigma_1^2$. If $\mathrm{E}(u_2 u_3|\mathbf{z})$ is constant then $(\mathbf{z}\boldsymbol{\beta}_2)(\mathbf{z}\boldsymbol{\beta}_3) + \mathrm{E}(u_2 u_3|\mathbf{z}) = (\mathbf{z}\boldsymbol{\beta}_2)(\mathbf{z}\boldsymbol{\beta}_3) + \textit{constant}$. We operationalize the IVs by replacing $\boldsymbol{\beta}_2$ and $\boldsymbol{\beta}_3$ with the estimators from the first-stage regressions.

The estimators from parts c and d would also be asymptotically efficient because linear combinations of the instruments in those two parts are the optimal instruments. Asymptotically, it does not hurt to use redundant instruments; asymptotically, the optimal linear combination will be picked out via the first-stage regression.

Solutions to Chapter 10 Problems

10.1. a. Because investment is likely to be affected by macroeconomic factors, it is important to allow for these by including separate time intercepts; this is done by using $T - 1$ time period dummies.

b. Putting the unobserved effect c_i in the equation is a simple way to account for time-constant features of a county that affect investment and might also be correlated with the tax variable. Something like "average" county economic climate, which affects investment, could easily be correlated with tax rates because tax rates are, at least to a certain extent, selected by state and local officials. If only a cross section were available, we would have to find an instrument for the tax variable that is uncorrelated with c_i and correlated with the tax rate. This is often a difficult task.

c. Standard investment theories suggest that, ceteris paribus, larger marginal tax rates decrease investment.

d. I would start with a fixed effects analysis to allow arbitrary correlation between all time-varying explanatory variables and c_i. (Actually, doing pooled OLS is a useful initial exercise; these results can be compared with those from an FE analysis). Such an analysis assumes strict exogeneity of z_{it}, tax_{it}, and $disaster_{it}$ in the sense that these are uncorrelated with the errors u_{is} for all t and s.

I have no strong intuition for the likely serial correlation properties of the $\{u_{it}\}$. These might have little serial correlation because we have allowed for c_i, in which case I would use standard fixed effects. However, it seems more likely that the u_{it} are positively autocorrelated, in which case I might use first differencing instead. In either case, I would compute the fully robust standard errors along with the usual ones. In either case we can test for serial correlation

in $\{u_{it}\}$.

e. If tax_{it} and $disaster_{it}$ do not have lagged effects on investment, then the only possible violation of the strict exogeneity assumption is if future values of these variables are correlated with u_{it}. It seems reasonable not to worry whether future natural disasters are determined by past investment. On the other hand, state officials might look at the levels of past investment in determining future tax policy, especially if there is a target level of tax revenue the officials are trying to achieve. This could be similar to setting property tax rates: sometimes property tax rates are set depending on recent housing values because a larger base means a smaller rate can achieve the same amount of revenue. Given that we allow tax_{it} to be correlated with c_i, feedback might not be much of a problem. But it cannot be ruled out ahead of time.

10.2. a. θ_2, δ_2, and $\boldsymbol{\gamma}$ can be consistently estimated (assuming all elements of $\mathbf{z}_{it}$ are time-varying). The first period intercept, θ_1, and the coefficient on $female_1$, δ_1, cannot be estimated.

b. Everything else equal, θ_2 measures the growth in wage for men over the period. This is because, if we set $female_i = 0$ and $\mathbf{z}_{i1} = \mathbf{z}_{i2}$, the change in log wage is, on average, θ_2 (set $d2_1 = 0$ and $d2_2 = 1$). We can think of this as being the growth in wage rates (for males) due to aggregate factors in the economy. The parameter δ_2 measures the *difference* in wage growth rates between women and men, all else equal. If $\delta_2 = 0$ then, for men and women with the same characteristics, average wage growth is the same.

c. Write

$$\log(wage_{i1}) = \theta_1 + \mathbf{z}_{i1}\boldsymbol{\gamma} + \delta_1 female_i + c_i + u_{i1}$$
$$\log(wage_{i2}) = \theta_1 + \theta_2 + \mathbf{z}_{i2}\boldsymbol{\gamma} + \delta_1 female_i + \delta_2 female + c_i + u_{i2},$$

where I have used the fact that $d2_1 = 0$ and $d2_2 = 1$. Subtracting the first equation from the

second gives

$$\Delta \log(wage_i) = \theta_2 + \Delta \mathbf{z}_i \boldsymbol{\gamma} + \delta_2 female_i + \Delta u_i.$$

This equation shows explicitly that the growth in wages depends on $\Delta \mathbf{z}_i$ and gender. If $\mathbf{z}_{i1} = \mathbf{z}_{i2}$ then $\Delta \mathbf{z}_i = \mathbf{0}$, and the growth in wage for men is θ_2 and that for women is $\theta_2 + \delta_2$, just as above. This shows that we can allow for c_i and still test for a gender differential in the growth of wages. But we cannot say anything about the wage differential between men and women for a given year.

10.3. a. Let $\overline{\mathbf{x}}_i = (\mathbf{x}_{i1} + \mathbf{x}_{i2})/2$, $\overline{y}_i = (y_{i1} + y_{i2})/2$, $\ddot{\mathbf{x}}_{i1} = \mathbf{x}_{i1} - \overline{\mathbf{x}}_i$, $\ddot{\mathbf{x}}_{i2} = \mathbf{x}_{i2} - \overline{\mathbf{x}}_i$, and similarly for $\ddot{y}_{i1}$ and $\ddot{y}_{i2}$. For $T = 2$ the fixed effects estimator can be written as

$$\hat{\boldsymbol{\beta}}_{FE} = \left[\sum_{i=1}^{N} (\ddot{\mathbf{x}}_{i1}' \ddot{\mathbf{x}}_{i1} + \ddot{\mathbf{x}}_{i2}' \ddot{\mathbf{x}}_{i2}) \right]^{-1} \left[\sum_{i=1}^{N} (\ddot{\mathbf{x}}_{i1}' \mathbf{y}_{i1} + \ddot{\mathbf{x}}_{i2}' \mathbf{y}_{i2}) \right].$$

Now, by simple algebra,

$$\begin{aligned}
\ddot{\mathbf{x}}_{i1} &= (\mathbf{x}_{i1} - \mathbf{x}_{i2})/2 = -\Delta \mathbf{x}_i/2 \\
\ddot{\mathbf{x}}_{i2} &= (\mathbf{x}_{i2} - \mathbf{x}_{i1})/2 = \Delta \mathbf{x}_i/2 \\
\ddot{y}_{i1} &= (y_{i1} - y_{i2})/2 = -\Delta y_i/2 \\
\ddot{y}_{i2} &= (y_{i2} - y_{i1})/2 = \Delta y_i/2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\ddot{\mathbf{x}}_{i1}' \ddot{\mathbf{x}}_{i1} + \ddot{\mathbf{x}}_{i2}' \ddot{\mathbf{x}}_{i2} &= \Delta \mathbf{x}_i' \Delta \mathbf{x}_i/4 + \Delta \mathbf{x}_i' \Delta \mathbf{x}_i/4 = \Delta \mathbf{x}_i' \Delta \mathbf{x}_i/2 \\
\ddot{\mathbf{x}}_{i1}' \ddot{y}_{i1} + \ddot{\mathbf{x}}_{i2}' \ddot{y}_{i2} &= \Delta \mathbf{x}_i' \Delta y_i/4 + \Delta \mathbf{x}_i' \Delta y_i/4 = \Delta \mathbf{x}_i' \Delta y_i/2
\end{aligned}$$

and so

$$\widehat{\boldsymbol{\beta}}_{FE} = \left(\sum_{i=1}^{N} \Delta \mathbf{x}_i' \Delta \mathbf{x}_i / 2 \right)^{-1} \left(\sum_{i=1}^{N} \Delta \mathbf{x}_i' \Delta y_i / 2 \right)$$
$$= \left(\sum_{i=1}^{N} \Delta \mathbf{x}_i' \Delta \mathbf{x}_i \right)^{-1} \left(\sum_{i=1}^{N} \Delta \mathbf{x}_i' \Delta y_i \right) = \widehat{\boldsymbol{\beta}}_{FD}.$$

b. Let $\hat{u}_{i1} = \ddot{y}_{i1} - \ddot{\mathbf{x}}_{i1}\widehat{\boldsymbol{\beta}}_{FE}$ and $\hat{u}_{i2} = \ddot{y}_{i2} - \ddot{\mathbf{x}}_{i2}\widehat{\boldsymbol{\beta}}_{FE}$ be the fixed effects residuals for the two time periods for cross section observation i. Since $\widehat{\boldsymbol{\beta}}_{FE} = \widehat{\boldsymbol{\beta}}_{FD}$, and using the representations above, we have

$$\hat{u}_{i1} = -\Delta y_i/2 - (-\Delta \mathbf{x}_i/2)\widehat{\boldsymbol{\beta}}_{FD} = -(\Delta y_i - \Delta \mathbf{x}_i \widehat{\boldsymbol{\beta}}_{FD})/2 \equiv -\hat{e}_i/2$$
$$\hat{u}_{i2} = \Delta y_i/2 - (\Delta \mathbf{x}_i/2)\widehat{\boldsymbol{\beta}}_{FD} = (\Delta y_i - \Delta \mathbf{x}_i \widehat{\boldsymbol{\beta}}_{FD})/2 \equiv \hat{e}_i/2,$$

where $\hat{e}_i \equiv \Delta y_i - \Delta \mathbf{x}_i \widehat{\boldsymbol{\beta}}_{FD}$ are the first difference residuals, $i = 1, 2, \ldots, N.$ Therefore,

$$\sum_{i=1}^{N} (\hat{u}_{i1}^2 + \hat{u}_{i2}^2) = (1/2) \sum_{i=1}^{N} \hat{e}_i^2.$$

This shows that the sum of squared residuals from the fixed effects regression is exactly one half the sum of squared residuals from the first difference regression. Since we know the variance estimate for fixed effects is the *SSR* divided by $N - K$ (when $T = 2$), and the variance estimate for first difference is the *SSR* divided by $N - K$, the error variance from fixed effects is always half the size as the error variance for first difference estimation, that is, $\hat{\sigma}_u^2 = \hat{\sigma}_e^2/2$ (contrary to what the problem asks you to show). What I wanted you to show is that the variance matrix estimates of $\widehat{\boldsymbol{\beta}}_{FE}$ and $\widehat{\boldsymbol{\beta}}_{FD}$ are identical. This is easy since the variance matrix estimate for fixed effects is

$$\hat{\sigma}_u^2 \left[\sum_{i=1}^{N} (\ddot{\mathbf{x}}_{i1}' \ddot{\mathbf{x}}_{i1} + \ddot{\mathbf{x}}_{i2}' \ddot{\mathbf{x}}_{i2}) \right]^{-1} = (\hat{\sigma}_e^2/2) \left(\sum_{i=1}^{N} \Delta \mathbf{x}_i' \Delta \mathbf{x}_i / 2 \right)^{-1} = \hat{\sigma}_e^2 \left(\sum_{i=1}^{N} \Delta \mathbf{x}_i' \Delta \mathbf{x}_i \right)^{-1},$$

which is the variance matrix estimator for FD estimator. Thus, the standard errors, and the fact all other test statistics (*F* statistics) will be numerically identical using the two approaches.

10.4. a. Including the aggregate time effect, $d2_t$, can be very important. Without it, we must assume that any change in average y over the two time periods is due to the program, and not to external factors. For example, if y_{it} is the unemployment rate for city i at time t, and $prog_{it}$ denotes a job creation program, we want to be sure that we account for the fact that the general economy may have worsened or improved over the period. If $d2_t$ is omitted, and $\theta_2 < 0$ (an improving economy, since unemployment has fallen), we might attribute a decrease in unemployment to the job creation program, when in fact it had nothing to do with it. For general T, each time period should have its own intercept (otherwise the analysis is not entirely convincing).

b. The presence of c_i allows program participation to be correlated with unobserved individual heterogeneity, something crucial in contexts where the experimental group is not randomly assigned. Two examples are when individuals "self-select" into the program and when program administrators target specific groups that may benefit more or less from the program.

c. If we first difference the equation, use the fact that $prog_{i1} = 0$ for all i, $d2_1 = 0$, and $d2_2 = 1$, we get

$$y_{i2} + y_{i1} = \theta_2 + \delta_1 prog_{i2} + u_{i2} - u_{i1},$$

or

$$\Delta y_i = \theta_2 + \delta_1 prog_{i2} + \Delta u_i.$$

Now, the *FE* (and *FD*) estimates of θ_2 and δ_1 are just the OLS estimators from this equation

(on cross section data). From basic two-variable regression with a dummy independent variable, $\hat{\theta}_2$ is the average value of Δy over the group with $prog_{12} = 0$ – that is, the control group. Also, $\hat{\theta}_2$ and $\hat{\delta}_1$ is the average value of Δy over the group with $prog_{i1} = 1$ – that is, the treatment group. Thus, as asserted, we have

$$\hat{\theta}_2 = \overline{\Delta y}_{control},\ \hat{\delta}_1 = \overline{\Delta y}_{treat} - \overline{\Delta y}_{control}.$$

If we did not include the $d2_t$, $\hat{\delta}_1 = \overline{\Delta y}_{treat}$, the average change of the treated group. The demonstrates the claim in part b that without the aggregate time effect any change in the average value of y for the treated group is attributed to the program. Differencing and averaging over the treated group allows program participation to depend on time-constant unobservables affecting the level of y, but that does not account for external factors that affect y for everyone.

d. In general, for T time periods we have

$$y_{it} = \theta_1 + \theta_2 d2_t + \theta_3 d3_t + \ldots + \theta_T dT_t + \delta_1 prog_{it} + c_i + u_{it};$$

that is, we have separate year intercepts, an unobserved effect c_i, and the program indicator.

e. First, the model from part d is more flexible because it allows any sequence of program participation. Equation (10.89), when extended to $T > 2$, applies only when treatment is ongoing. In addition, (10.89) is restrictive in terms of aggregate time effects: it assumes that any aggregate time effects correspond to the start of the program only. It is better to use the unobserved effects model from part d, and estimate it using either *FE* or *FD*.

10.5. a. Write $\mathbf{v}_i\mathbf{v}_i' = c_i^2\mathbf{j}_T\mathbf{j}_T' + \mathbf{u}_i\mathbf{u}_i' + \mathbf{j}_T(c_i\mathbf{u}_i') + (c_i\mathbf{u}_i')\mathbf{j}_T'$. Under RE.1, $\mathrm{E}(\mathbf{u}_i|\mathbf{x}_i, c_i) = \mathbf{0}$, which implies that $\mathrm{E}(c_i\mathbf{u}_i'|\mathbf{x}_i) = \mathbf{0}$ by iterated expectations. Under RE.3a, $\mathrm{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i, c_i) = \sigma_u^2\mathbf{I}_T$, which implies that $\mathrm{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i) = \sigma_u^2\mathbf{I}_T$ (again, by iterated expectations). Therefore,

$$\mathrm{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i) = \mathrm{E}(c_i^2|\mathbf{x}_i)\mathbf{j}_T\mathbf{j}_T' + \mathrm{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i) = h(\mathbf{x}_i)\mathbf{j}_T\mathbf{j}_T' + \sigma_u^2\mathbf{I}_T,$$

where $h(\mathbf{x}_i) \equiv \mathrm{Var}(c_i|\mathbf{x}_i) = \mathrm{E}(c_i^2|\mathbf{x}_i)$ (by RE.1b). This shows that the conditional variance matrix of $\mathbf{v}_i$ given $\mathbf{x}_i$ has the same covariance for all $t \neq s$, $h(\mathbf{x}_i)$, and the same variance for all t, $h(\mathbf{x}_i) + \sigma_u^2$. Therefore, while the variances and covariances depend on $\mathbf{x}_i$, they do not depend on time.

b. The RE estimator is still consistent and $\sqrt{N}$ –asymptotically normal without Assumption RE.3b, but the usual random effects variance estimator of $\mathrm{Avar}(\hat{\boldsymbol{\beta}}_{RE})$ is no longer valid because $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{x}_i)$ does not have the form (10.30) (because it depends on $\mathbf{x}_i$). The robust variance matrix estimator given in (7.52) should be used in obtaining standard errors and Wald statistics.

10.6. a. By stacking the formulas for the *FD* and *FE* estimators, and using standard asymptotic arguments, we have, under *FE*.1 and the rank conditions,

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) = \mathbf{G}^{-1}\left(N^{-1}\sum_{i=1}^{N}\mathbf{s}_i\right) + o_p(1),$$

where $\mathbf{G}$ is the $2K \times 2K$ block diagonal matrix with blocks $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, and $\mathbf{s}_i$ is the $2K \times 1$ vector

$$\mathbf{s}_i \equiv \begin{pmatrix} \Delta\mathbf{X}_i'\Delta\mathbf{u}_i \\ \ddot{\mathbf{X}}_i'\ddot{\mathbf{u}}_i \end{pmatrix}.$$

b. Let $\Delta\hat{\mathbf{u}}_i$ denote the $(T-1) \times 1$ vector of *FD* residuals, and let $\hat{\ddot{\mathbf{u}}}_i$ denote the $T \times 1$ vector of *FE* residuals. Plugging these into the formula for $\mathbf{s}_i$ givens $\hat{\mathbf{s}}_i$. Let $\hat{\mathbf{D}} = N^{-1}\sum_{i=1}^{N}\hat{\mathbf{s}}_i\hat{\mathbf{s}}_i'$, and define $\hat{\mathbf{G}}$ by replacing $\mathbf{A}_1$ and $\mathbf{A}_2$ with their obvious consistent estimators. Then $\widehat{\mathrm{Avar}}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})] = \hat{\mathbf{G}}^{-1}\hat{\mathbf{D}}\hat{\mathbf{G}}^{-1}$ is a consistent estimator of $\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})]$.

c. Let $\mathbf{R}$ be the $K \times 2K$ partitioned matrix $\mathbf{R} = [\mathbf{I}_K \mid -\mathbf{I}_K]$. Then the null hypothesis imposed by the Hausman test is $H_0 : \mathbf{R\theta} = \mathbf{0}$. We can form a Wald-type statistic,

$$H = (\mathbf{R}\hat{\boldsymbol{\theta}})'[\mathbf{R}\hat{\mathbf{G}}^{-1}\hat{\mathbf{D}}\hat{\mathbf{G}}^{-1}\mathbf{R}']^{-1}(\mathbf{R}\hat{\boldsymbol{\theta}}).$$

Under FE.1 and the rank conditions for *FD* and *FE*, H has a limiting x_K^2 distribution. The statistic requires no particular second moment assumptions of the kind in FE.3. Note that $\mathbf{R}\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\beta}}_{FD} - \hat{\boldsymbol{\beta}}_{FE}$.

10.7. a. The random effects estimates are given below. The coefficient on *season* is $-.044$, which means that being in season is estimated to reduce an athlete's term GPA by $.044$ points. The nonrobust t statistic is only -1.12.

```
. use gpa

. xtset id term
       panel variable:  id (strongly balanced)
        time variable:  term, 8808 to 8901, but with gaps
                delta:  1 unit

. xtreg trmgpa spring crsgpa frstsem season sat verbmath hsperc hssize black

Random-effects GLS regression                   Number of obs      =       732
Group variable: id                              Number of groups   =       366

R-sq:  within  = 0.2067                         Obs per group: min =
       between = 0.5390                                        avg =       2.
       overall = 0.4785                                        max =

Random effects u_i ~Gaussian                    Wald chi2(10)      =    512.77
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
      trmgpa |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      spring |  -.0606536   .0371605    -1.63   0.103    -.1334868    .0121797
      crsgpa |   1.082365   .0930877    11.63   0.000     .8999166    1.264814
     frstsem |   .0029948   .0599542     0.05   0.960    -.1145132    .1205028
      season |  -.0440992   .0392381    -1.12   0.261    -.1210044     .032806
         sat |   .0017052   .0001771     9.63   0.000     .0013582    .0020523
    verbmath |    -.15752     .16351    -0.96   0.335    -.4779937    .1629538
      hsperc |  -.0084622   .0012426    -6.81   0.000    -.0108977   -.0060268
      hssize |  -.0000775   .0001248    -0.62   0.534     -.000322     .000167
       black |  -.2348189   .0681573    -3.45   0.001    -.3684048   -.1012331
      female |    .358153   .0612948     5.84   0.000     .2380173    .4782886
       _cons |   -1.73492   .3566599    -4.86   0.000     -2.43396   -1.035879
-------------+----------------------------------------------------------------
```

```
     sigma_u |  .37185442
     sigma_e |  .40882825
         rho |   .4527451   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

b. Below are the fixed effects estimates with nonrobust standard errors. The time-constant variables have been dropped. The coefficient on *season* is now larger in magnitude, $-.057$, and it is more statistically significant with $t = -1.37$.

```
. xtreg trmgpa spring crsgpa frstsem season, fe

Fixed-effects (within) regression               Number of obs      =       732
Group variable: id                              Number of groups   =       366

R-sq:  within  = 0.2069                         Obs per group: min =
       between = 0.0333                                        avg =       2.
       overall = 0.0613                                        max =

                                                F(4,362)           =     23.61
corr(u_i, Xb)  = -0.0893                        Prob > F           =    0.0000

------------------------------------------------------------------------------
      trmgpa |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      spring |  -.0657817   .0391404    -1.68   0.094    -.1427528    .0111895
      crsgpa |   1.140688   .1186538     9.61   0.000     .9073505    1.374025
     frstsem |   .0128523   .0688364     0.19   0.852    -.1225172    .1482218
      season |  -.0566454   .0414748    -1.37   0.173    -.1382072    .0249165
       _cons |  -.7708055   .3305004    -2.33   0.020    -1.420747   -.1208636
-------------+----------------------------------------------------------------
     sigma_u |  .67913296
     sigma_e |  .40882825
         rho |  .73400603   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(365, 362) =     5.40            Prob > F = 0.0000
```

c. The following Stata output gives the nonrobust and fully robust regression-based Hausman test. Whether we test the three time averages, *crsgpabar*, *frstsembar*, and *seasonbar*, or just *seasonbar*, the *p*-value is large (.068 for the joint nonrobust test, .337 for the single nonrobust test). And the findings do not depend on using a robust test: the *p*-values are a little smaller but not close to being significant.

For comparision, the traditional way of computing the Hausman statistic – directly forming the quadratic form in the FE and RE estimates is included at the end, computed two different ways. The first uses the difference in the estimated variance matrices, and the value of the

statistic, 1.81, is very close to the nonrobust, regression-based statistic, 1.83. But the degrees of freedom reported by Stata are incorrect: it should be three, not four. Thus, the *p*-value reported by Stata using the `hausman` command is too large. This will be the case whenever aggregate time variables – most commonly, time period dummies – are included among the coefficients to test.

If we impose that the RE estimate of the variance σ_u^2 is used to estimate both the FE and RE asymptotic variances, Stata then recognizes that the variance matrix has rank three rather than rank four. (The same is true if we use the FE estimate of σ_u^2 in both places.) The *p*-value in this case agrees very closely with that for the nonrobust, regression-based test (and both statistics are 1.83 rounded to two decimal places.)

```
. egen crsgpabar = mean(crsgpa), by(id)

. egen frstsembar = mean(frstsem), by(id)

. egen seasonbar = mean(season), by(id)

. xtreg trmgpa spring crsgpa frstsem season sat verbmath hsperc hssize black
        female crsgpabar frstsembar seasonbar, re

Random-effects GLS regression                   Number of obs      =       732
Group variable: id                              Number of groups   =       366

R-sq:  within  = 0.2069                         Obs per group: min =
       between = 0.5408                                        avg =       2.
       overall = 0.4802                                        max =

Random effects u_i ~Gaussian                    Wald chi2(13)      =    513.77
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
      trmgpa |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      spring |  -.0657817   .0391404    -1.68   0.093    -.1424954    .0109321
      crsgpa |   1.140688   .1186538     9.61   0.000     .9081308    1.373245
     frstsem |   .0128523   .0688364     0.19   0.852    -.1220646    .1477692
      season |  -.0566454   .0414748    -1.37   0.172    -.1379345    .0246438
         sat |   .0016681   .0001804     9.24   0.000     .0013145    .0020218
    verbmath |  -.1316461   .1654748    -0.80   0.426    -.4559708    .1926785
      hsperc |  -.0084655   .0012554    -6.74   0.000    -.0109259    -.006005
      hssize |  -.0000783    .000125    -0.63   0.531    -.0003232    .0001666
       black |  -.2447934   .0686106    -3.57   0.000    -.3792676   -.1103192
      female |   .3357016   .0711808     4.72   0.000     .1961898    .4752134
```

```
   crsgpabar |  -.1861551   .2011254    -0.93   0.355    -.5803537    .2080434
  frstsembar |   -.078244   .1461014    -0.54   0.592    -.3645975    .2081095
   seasonbar |   .1243006   .1293555     0.96   0.337    -.1292315    .3778326
       _cons |  -1.423761   .5183296    -2.75   0.006    -2.439668   -.4078539
-------------+----------------------------------------------------------------
     sigma_u |  .37185442
     sigma_e |  .40882825
         rho |   .4527451   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. test crsgpabar frstsembar seasonbar

 ( 1)  crsgpabar = 0
 ( 2)  frstsembar = 0
 ( 3)  seasonbar = 0

           chi2(  3) =    1.83
         Prob > chi2 =    0.6084

. test  seasonbar

 ( 1)  seasonbar = 0

           chi2(  1) =    0.92
         Prob > chi2 =    0.3366


. xtreg trmgpa spring crsgpa frstsem season sat verbmath hsperc hssize black
        female crsgpabar frstsembar seasonbar , re cluster(id)

Random-effects GLS regression                   Number of obs      =       732
Group variable: id                              Number of groups   =       366

R-sq:  within  = 0.2069                         Obs per group: min =
       between = 0.5408                                        avg =       2.
       overall = 0.4802                                        max =

Random effects u_i ~Gaussian                    Wald chi2(13)      =    629.75
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                                    (Std. Err. adjusted for 366 clusters in id
------------------------------------------------------------------------------
             |               Robust
      trmgpa |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      spring |  -.0657817   .0394865    -1.67   0.096    -.1431737    .0116104
      crsgpa |   1.140688   .1317893     8.66   0.000     .8823856     1.39899
     frstsem |   .0128523   .0684334     0.19   0.851    -.1212746    .1469793
      season |  -.0566454   .0411639    -1.38   0.169    -.1373251    .0240344
         sat |   .0016681   .0001848     9.03   0.000     .0013059    .0020304
    verbmath |  -.1316461    .166478    -0.79   0.429    -.4579371    .1946448
      hsperc |  -.0084655   .0013131    -6.45   0.000    -.0110391   -.0058918
      hssize |  -.0000783   .0001172    -0.67   0.504     -.000308    .0001514
       black |  -.2447934    .075569    -3.24   0.001     -.392906   -.0966808
      female |   .3357016    .067753     4.95   0.000     .2029081    .4684951
   crsgpabar |  -.1861551   .1956503    -0.95   0.341    -.5696227    .1973125
  frstsembar |   -.078244   .1465886    -0.53   0.594    -.3655525    .2090644
   seasonbar |   .1243006   .1342238     0.93   0.354    -.1387732    .3873743
       _cons |  -1.423761   .4571037    -3.11   0.002    -2.319668   -.5278545
```

```
-------------+----------------------------------------------------------------
     sigma_u |  .37185442
     sigma_e |  .40882825
         rho |   .4527451   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. test crsgpabar frstsembar seasonbar

 ( 1)  crsgpabar = 0
 ( 2)  frstsembar = 0
 ( 3)  seasonbar = 0

           chi2(  3) =    1.95
         Prob > chi2 =    0.5829

. test  seasonbar

 ( 1)  seasonbar = 0

           chi2(  1) =    0.86
         Prob > chi2 =    0.3544

. * The traditional Hausman test that incorrectly includes the coefficients
. * on "spring" (the time dummy) among those being tested.

. qui xtreg trmgpa spring crsgpa frstsem season, fe

. estimates store fe

. qui xtreg trmgpa spring crsgpa frstsem season sat verbmath hsperc hssize
      black female, re

. estimates store re

. hausman fe re

                ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |       fe           re         Difference          S.E.
-------------+----------------------------------------------------------------
      spring |   -.0657817    -.0606536       -.0051281         .012291
      crsgpa |    1.140688     1.082365        .0583227        .0735758
     frstsem |    .0128523     .0029948        .0098575        .0338223
      season |   -.0566454    -.0440992       -.0125462        .0134363
------------------------------------------------------------------------------
                           b = consistent under Ho and Ha; obtained from xtreg
            B = inconsistent under Ha, efficient under Ho; obtained from xtreg

    Test:  Ho:  difference in coefficients not systematic

                  chi2(4) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =        1.81
                Prob>chi2 =      0.7702

. qui xtreg trmgpa spring crsgpa frstsem season, fe

. estimates store fe

. qui xtreg trmgpa spring crsgpa frstsem season sat verbmath hsperc hssize black
```

```
. estimates store re

. hausman fe re, sigmamore

Note: the rank of the differenced variance matrix (3) does not equal the
      number of coefficients being tested (4); be sure
      this is what you expect, or there may be problems computing the test.
      Examine the output of your estimators for anything unexpected and
      possibly consider scaling your variables so that the coefficients are
      on a similar scale.

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |       fe           re         Difference          S.E.
-------------+----------------------------------------------------------------
      spring |   -.0657817    -.0606536       -.0051281        .0121895
      crsgpa |    1.140688     1.082365        .0583227        .0734205
     frstsem |    .0128523     .0029948        .0098575        .0337085
      season |   -.0566454    -.0440992       -.0125462         .013332
------------------------------------------------------------------------------
                           b = consistent under Ho and Ha; obtained from xtreg
            B = inconsistent under Ha, efficient under Ho; obtained from xtreg

    Test:  Ho:  difference in coefficients not systematic

                  chi2(3) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =        1.83
                Prob>chi2 =      0.6077
                (V_b-V_B is not positive definite)
```

10.8. a. The Stata output is below. The coefficients on the lagged "clear-up" percentages are very close in magnitude. For example, if the first lag is 10 percentage points higher, the crime rate is estimated to fall by about 18.5 percent, a very large effect. The estimate of ρ in the AR(1) serial correlation test is $.574$ and $t = 5.82$, so there is very strong evidence of serial correlation.

```
. use norway

. xtset district year, delta(6)
       panel variable:  district (strongly balanced)
        time variable:  year, 72 to 78
                delta:  6 units

. reg lcrime d78 clrprc_1 clrprc_2

      Source |       SS       df       MS              Number of obs =     106
-------------+------------------------------           F(  3,   102) =   30.27
       Model |  18.7948264     3  6.26494214           Prob > F      =  0.0000
    Residual |  21.1114968   102  .206975459           R-squared     =  0.4710
-------------+------------------------------           Adj R-squared =  0.4554
```

```
       Total |  39.9063233   105  .380060222           Root MSE      =  .45495

------------------------------------------------------------------------------
      lcrime |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d78 |  -.0547246   .0944947    -0.58   0.564    -.2421544    .1327051
    clrprc_1 |  -.0184955   .0053035    -3.49   0.001    -.0290149    -.007976
    clrprc_2 |  -.0173881   .0054376    -3.20   0.002    -.0281735   -.0066026
       _cons |    4.18122   .1878879    22.25   0.000     3.808545    4.553894
------------------------------------------------------------------------------

. predict vhat, resid

. gen vhat_1 = l.vhat
(53 missing values generated)

. reg vhat vhat_1

      Source |       SS       df       MS              Number of obs =      53
-------------+------------------------------           F(  1,    51) =   33.85
       Model |   3.8092697     1   3.8092697           Prob > F      =  0.0000
    Residual |  5.73894345    51  .112528303           R-squared     =  0.3990
-------------+------------------------------           Adj R-squared =  0.3872
       Total |  9.54821315    52  .183619484           Root MSE      =  .33545

------------------------------------------------------------------------------
        vhat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      vhat_1 |   .5739582   .0986485     5.82   0.000     .3759132    .7720033
       _cons |  -3.01e-09   .0460779    -0.00   1.000    -.0925053    .0925053
------------------------------------------------------------------------------
```

b. The fixed effects estimates are given below. The coefficient on *clrprc*_1 falls dramatically in magnitude, and becomes statistically insignificant. The coefficient on *clrprc*_2 falls somewhat but is still practically large and statistically significant.

To obtain the heteroskedasticity-robust standard error for FE, we must use the FD estimation (which is the same as FE because $T = 2$) in order to make the calculation simple. Stock and Watson (2008, *Econometrica*) show that just applying the usual heteroskedasticity-robust standard error using pooled regression on the time-demeaned data does not produce valid standard errors. The reason is simple: as shown in the text, the time demeaning induces serial correlation in the errors. Of course, one can always use the fully robust standard errors, which allow for any kind of serial correlation in the original errors and

any kind of heteroskedasticity. In this example, obtaining the heteroskedasticity-robust standard errors has little effect on inference.

```
. xtreg lcrime d78 clrprc_1 clrprc_2, fe

Fixed-effects (within) regression               Number of obs      =       106
Group variable: district                        Number of groups   =        53

R-sq:  within  = 0.4209                         Obs per group: min =
       between = 0.4798                                        avg =       2.
       overall = 0.4234                                        max =

                                                F(3,50)            =     12.12
corr(u_i, Xb)  = 0.3645                         Prob > F           =    0.0000

------------------------------------------------------------------------------
      lcrime |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d78 |   .0856556   .0637825     1.34   0.185    -.0424553    .2137665
    clrprc_1 |  -.0040475   .0047199    -0.86   0.395    -.0135276    .0054326
    clrprc_2 |  -.0131966   .0051946    -2.54   0.014    -.0236302   -.0027629
       _cons |   3.350995   .2324736    14.41   0.000     2.884058    3.817932
-------------+----------------------------------------------------------------
     sigma_u |  .47140473
     sigma_e |   .2436645
         rho |  .78915666   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(52, 50) =     5.88                 Prob > F = 0.0000

. reg clcrime cclrprc_1 cclrprc_2, robust

Linear regression                                      Number of obs =      53
                                                       F(  2,    50) =    4.78
                                                       Prob > F      =  0.0126
                                                       R-squared     =  0.1933
                                                       Root MSE      =  .34459

------------------------------------------------------------------------------
             |               Robust
     clcrime |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
   cclrprc_1 |  -.0040475   .0042659    -0.95   0.347    -.0126158    .0045207
   cclrprc_2 |  -.0131966   .0047286    -2.79   0.007    -.0226942    -.003699
       _cons |   .0856556   .0554876     1.54   0.129    -.0257945    .1971057
------------------------------------------------------------------------------
```

c. I use the FD regression to easily allow for heteroskedasticity. The two-sided *p*-value is .183. Because we do not reject $H_0 : \beta_1 = \beta_2$ at even the 15% level, we might justify estimating a model with $\beta_1 = \beta_2$ (and the pooled OLS results suggest it, too). The variable *avgclr* is the average of *clrprc*_1and *clrprc*_2, and so we can use it as the only explanatory

variable. Imposing the restriction gives a large estimated effect – a 10 percentage point increase in the average clear-up rate decreases crime by about 16.7% – and the heteroskedasticity-robust *t* statistic is –2.89.

```
. qui reg clcrime cclrprc_1 cclrprc_2, robust

. lincom cclrprc_1 - cclrprc_2

 ( 1)  cclrprc_1 - cclrprc_2 = 0

------------------------------------------------------------------------------
     clcrime |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         (1) |    .009149   .0067729     1.35   0.183    -.0044548    .0227529
------------------------------------------------------------------------------

. reg clcrime cavgclr, robust

Linear regression                                      Number of obs =      53
                                                       F(  1,    51) =    8.38
                                                       Prob > F      =  0.0056
                                                       R-squared     =  0.1747
                                                       Root MSE      =  .34511

------------------------------------------------------------------------------
             |               Robust
     clcrime |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     cavgclr |  -.0166511   .0057529    -2.89   0.006    -.0282006   -.0051016
       _cons |   .0993289   .0554764     1.79   0.079    -.0120446    .2107024
------------------------------------------------------------------------------

. reg clcrime cavgclr

      Source |       SS       df       MS              Number of obs =      53
-------------+------------------------------           F(  1,    51) =   10.80
       Model |  1.28607105     1  1.28607105           Prob > F      =  0.0018
    Residual |  6.07411496    51  .119100293           R-squared     =  0.1747
-------------+------------------------------           Adj R-squared =  0.1586
       Total |  7.36018601    52  .141542039           Root MSE      =  .34511

------------------------------------------------------------------------------
     clcrime |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     cavgclr |  -.0166511   .0050672    -3.29   0.002    -.0268239   -.0064783
       _cons |   .0993289   .0625916     1.59   0.119    -.0263289    .2249867
------------------------------------------------------------------------------
```

10.9. a. The RE and FE estimates, with fully robust standard errors for each, are given below. The variable-addition Hausman test is obtained by adding time averages of all variables

except the year dummies; all other explanatory variables change across *i* and *t*. For comparison, the traditional nonrobust Hausman statistic is computed. This version uses the RE estimate of σ_u^2 in estimating both the RE and FE asymptotic variances and properly computes the degrees of freedom (which is five for this application).

While there are differences in the RE and FE estimates, the signs are the same and the magnitudes are similar. The fully robust Hausman test gives a strong statistical rejection of the RE assumption that county heterogeneity is uncorrelated with the criminal justice variables. Therefore, for magnitudes, we should prefer the FE estimates. (Remember, though, that both RE and FE maintain strict exogeneity conditional on the heterogeneity.) The nonrobust Hausman test gives a substantially larger statistic, 78.79 compared with 60.53, but the conclusion is the same.

```
. xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87, re
        cluster(county)

Random-effects GLS regression                   Number of obs      =       630
Group variable: county                          Number of groups   =        90

R-sq:  within  = 0.4287                         Obs per group: min =
       between = 0.4533                                        avg =       7.
       overall = 0.4454                                        max =

Random effects u_i ~Gaussian                    Wald chi2(11)      =    156.83
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                                (Std. Err. adjusted for 90 clusters in county
------------------------------------------------------------------------------
             |               Robust
     lcrmrte |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lprbarr |  -.4252097   .0629147    -6.76   0.000    -.5485202   -.3018993
    lprbconv |  -.3271464   .0499587    -6.55   0.000    -.4250636   -.2292292
    lprbpris |  -.1793507   .0457547    -3.92   0.000    -.2690283   -.0896731
     lavgsen |  -.0083696   .0322377    -0.26   0.795    -.0715543    .0548152
      lpolpc |   .4294148   .0878659     4.89   0.000     .2572007    .6016288
         d82 |   .0137442   .0164857     0.83   0.404    -.0185671    .0460556
         d83 |   -.075388   .0194832    -3.87   0.000    -.1135743   -.0372017
         d84 |  -.1130975   .0217025    -5.21   0.000    -.1556335   -.0705614
         d85 |  -.1057261   .0254587    -4.15   0.000    -.1556242   -.0558279
         d86 |  -.0795307   .0239141    -3.33   0.001    -.1264014   -.0326599
         d87 |  -.0424581   .0246408    -1.72   0.085    -.0907531     .005837
       _cons |  -1.672632   .5678872    -2.95   0.003    -2.785671   -.5595939
```

```
-------------+----------------------------------------------------------------
     sigma_u |  .30032934
     sigma_e |  .13871215
         rho |  .82418424   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87, fe
       cluster(county)

Fixed-effects (within) regression               Number of obs      =       630
Group variable: county                          Number of groups   =        90

R-sq:  within  = 0.4342                         Obs per group: min =
       between = 0.4066                                        avg =       7.
       overall = 0.4042                                        max =

                                                F(11,89)           =     11.49
corr(u_i, Xb)  = 0.2068                         Prob > F           =    0.0000

                                (Std. Err. adjusted for 90 clusters in county
------------------------------------------------------------------------------
             |               Robust
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     lprbarr |  -.3597944   .0594678    -6.05   0.000    -.4779557   -.2416332
    lprbconv |  -.2858733    .051522    -5.55   0.000    -.3882464   -.1835001
    lprbpris |  -.1827812   .0452811    -4.04   0.000    -.2727538   -.0928085
     lavgsen |  -.0044879   .0333499    -0.13   0.893    -.0707535    .0617777
      lpolpc |   .4241142   .0849052     5.00   0.000     .2554095     .592819
         d82 |   .0125802   .0160066     0.79   0.434    -.0192246     .044385
         d83 |  -.0792813   .0195639    -4.05   0.000    -.1181544   -.0404081
         d84 |  -.1177281   .0217118    -5.42   0.000     -.160869   -.0745872
         d85 |  -.1119561   .0256583    -4.36   0.000    -.1629386   -.0609736
         d86 |  -.0818268   .0236276    -3.46   0.001    -.1287745   -.0348792
         d87 |  -.0404704   .0241765    -1.67   0.098    -.0885087    .0075678
       _cons |  -1.604135   .5102062    -3.14   0.002    -2.617904   -.5903664
-------------+----------------------------------------------------------------
     sigma_u |  .43487416
     sigma_e |  .13871215
         rho |  .90765322   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. egen lprbatb = mean(lprbarr), by(county)

. egen lprbctb = mean(lprbconv), by(county)

. egen lprbptb = mean(lprbpris), by(county)

. egen lavgtb = mean(lavgsen), by(county)

. egen lpoltb = mean(lpolpc), by(county)

. qui xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87  lprbatb
           lprbctb lprbptb lavgtb lpoltb, re cluster(county)

. test  lprbatb lprbctb lprbptb lavgtb lpoltb

 ( 1)  lprbatb = 0
 ( 2)  lprbctb = 0
```

```
 ( 3)  lprbptb = 0
 ( 4)  lavgtb = 0
 ( 5)  lpoltb = 0

           chi2(  5) =   60.53
         Prob > chi2 =    0.0000


. qui xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87, fe

. estimates store fe

. qui xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87, re

. estimates store re

. hausman fe re, sigmamore

Note: the rank of the differenced variance matrix (3) does not equal the
      number of coefficients being tested (4); be sure
      this is what you expect, or there may be problems computing the test.
      Examine the output of your estimators for anything unexpected and
      possibly consider scaling your variables so that the coefficients are
      on a similar scale.

                 ---- Coefficients ----
             |      (b)          (B)            (b-B)     sqrt(diag(V_b-V_B))
             |       fe           re         Difference          S.E.
-------------+----------------------------------------------------------------
     lprbarr |   -.3597944    -.4252097        .0654153        .0133827
    lprbconv |   -.2858733    -.3271464        .0412731        .0084853
    lprbpris |   -.1827812    -.1793507       -.0034305        .0065028
     lavgsen |   -.0044879    -.0083696        .0038816        .0037031
      lpolpc |    .4241142     .4294148       -.0053005        .0103217
         d82 |    .0125802     .0137442        -.001164        .0010763
         d83 |   -.0792813     -.075388       -.0038933        .0008668
         d84 |   -.1177281    -.1130975       -.0046306        .0013163
         d85 |   -.1119561    -.1057261         -.00623        .0014304
         d86 |   -.0818268    -.0795307       -.0022962        .0007719
         d87 |   -.0404704    -.0424581        .0019876         .001219
------------------------------------------------------------------------------
                           b = consistent under Ho and Ha; obtained from xtreg
            B = inconsistent under Ha, efficient under Ho; obtained from xtreg

    Test:  Ho:  difference in coefficients not systematic

                  chi2(5) = (b-B)'[(V_b-V_B)^(-1)](b-B)
                          =       78.79
                Prob>chi2 =      0.0000
                (V_b-V_B is not positive definite)
```

b. Below is the Stata output for fixed effects with the nine wage variables; the inference is fully robust. The log wage variables are jointly significant at the 1.2% significance level. Unfortunately, one of the most significant wage variables, *lwtuc*, has a positive, statistically

significant coefficient. It is difficult to give that a causal interpretation. The coefficient on the manufacturing wage implies that a ceteris paribus 10% increase in manufacturing wage reduces the crime rate by about three percent.

```
. xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87  lwcon lwtuc
         lwtrd lwfir lwser lwmfg lwfed lwsta lwloc, fe cluster(county)

Fixed-effects (within) regression               Number of obs      =       630
Group variable: county                          Number of groups   =        90

R-sq:  within  = 0.4575                         Obs per group: min =
       between = 0.2518                                        avg =       7.
       overall = 0.2687                                        max =

                                                F(20,89)           =     13.39
corr(u_i, Xb)  = 0.0804                         Prob > F           =    0.0000

                                (Std. Err. adjusted for 90 clusters in county
------------------------------------------------------------------------------
             |               Robust
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     lprbarr |  -.3563515   .0615202    -5.79   0.000    -.4785909   -.2341121
    lprbconv |  -.2859539   .0507647    -5.63   0.000    -.3868224   -.1850855
    lprbpris |  -.1751355   .0457628    -3.83   0.000    -.2660652   -.0842058
     lavgsen |  -.0028739   .0333994    -0.09   0.932    -.0692379      .06349
      lpolpc |      .4229   .0822172     5.14   0.000     .2595362    .5862639
         d82 |   .0188915   .0221161     0.85   0.395    -.0250527    .0628357
         d83 |   -.055286   .0306016    -1.81   0.074    -.1160906    .0055187
         d84 |  -.0615162   .0406029    -1.52   0.133    -.1421934    .0191609
         d85 |  -.0397115   .0603405    -0.66   0.512    -.1596068    .0801837
         d86 |  -.0001133   .0720231    -0.00   0.999    -.1432217    .1429952
         d87 |   .0537042   .0847749     0.63   0.528    -.1147418    .2221501
       lwcon |  -.0345448   .0253345    -1.36   0.176    -.0848839    .0157943
       lwtuc |   .0459747   .0161107     2.85   0.005      .013963    .0779864
       lwtrd |  -.0201766   .0313131    -0.64   0.521    -.0823951    .0420418
       lwfir |  -.0035445   .0130093    -0.27   0.786    -.0293937    .0223047
       lwser |   .0101264   .0202248     0.50   0.618    -.0300599    .0503128
       lwmfg |  -.3005691   .1063746    -2.83   0.006    -.5119331    -.089205
       lwfed |  -.3331226   .2245785    -1.48   0.142    -.7793554    .1131101
       lwsta |   .0215209   .1051755     0.20   0.838    -.1874606    .2305023
       lwloc |   .1810215   .1629903     1.11   0.270    -.1428368    .5048797
       _cons |   .8931726   1.479937     0.60   0.548     -2.04743    3.833775
-------------+----------------------------------------------------------------
     sigma_u |  .47756823
     sigma_e |  .13700505
         rho |  .92395784   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. testparm lwcon-lwloc

 ( 1)  lwcon = 0
 ( 2)  lwtuc = 0
 ( 3)  lwtrd = 0
```

```
 ( 4)  lwfir = 0
 ( 5)  lwser = 0
 ( 6)  lwmfg = 0
 ( 7)  lwfed = 0
 ( 8)  lwsta = 0
 ( 9)  lwloc = 0

       F(  9,    89) =    2.54
            Prob > F =    0.0121
```

c. First, we need to compute the changes in log wages. Then, we just use pooled OLS. Rather than difference the year dummies we just include dummies for 1983 through 1987. Both the usual and full robust standard errors are computed and compared with those from FE.

The nonrobust FD and FE standard errors are similar, and often very different from the comparable robust standard errors. In fact, in many cases the robust standard errors are double or more than the nonrobust ones, although some nonrobust ones are actually smaller. The wage variables generally have much smaller coefficients when FD is used, but they are still jointly significant using a robust test.

```
. gen clwcon = lwcon - lwcon[_n-1] if year > 81
(90 missing values generated)

. gen clwtuc = lwtuc - lwtuc[_n-1] if year > 81
(90 missing values generated)

. gen clwtrd = lwtrd - lwtrd[_n-1] if year > 81
(90 missing values generated)

. gen clwfir = lwfir - lwfir[_n-1] if year > 81
(90 missing values generated)

. gen clwser = lwser - lwser[_n-1] if year > 81
(90 missing values generated)

. gen clwmfg = lwmfg - lwmfg[_n-1] if year > 81
(90 missing values generated)

. gen clwfed = lwfed - lwfed[_n-1] if year > 81
(90 missing values generated)

. gen clwsta = lwsta - lwsta[_n-1] if year > 81
(90 missing values generated)

. gen clwloc = lwloc - lwloc[_n-1] if year > 81
(90 missing values generated)
```

```
. reg clcrmrte clprbarr clprbcon clprbpri clavgsen clpolpc clwcon-clwloc
      d83-d87

      Source |       SS       df       MS              Number of obs =     540
-------------+------------------------------           F( 19,   520) =   21.90
       Model |  9.86742162    19   .51933798            Prob > F      =  0.0000
    Residual |  12.3293822   520   .02371035            R-squared     =  0.4445
-------------+------------------------------           Adj R-squared =  0.4242
       Total |  22.1968038   539  .041181454            Root MSE      =  .15398

------------------------------------------------------------------------------
    clcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    clprbarr |  -.3230993   .0300195   -10.76   0.000    -.3820737   -.2641248
    clprbcon |  -.2402885   .0182474   -13.17   0.000    -.2761362   -.2044407
    clprbpri |  -.1693859     .02617    -6.47   0.000    -.2207978    -.117974
    clavgsen |  -.0156167   .0224126    -0.70   0.486    -.0596469    .0284136
     clpolpc |   .3977221    .026987    14.74   0.000     .3447051     .450739
      clwcon |  -.0442368   .0304142    -1.45   0.146    -.1039865     .015513
      clwtuc |   .0253997   .0142093     1.79   0.074     -.002515    .0533144
      clwtrd |  -.0290309   .0307907    -0.94   0.346    -.0895203    .0314586
      clwfir |    .009122   .0212318     0.43   0.668    -.0325886    .0508326
      clwser |   .0219549   .0144342     1.52   0.129    -.0064016    .0503113
      clwmfg |  -.1402482   .1019317    -1.38   0.169    -.3404967    .0600003
      clwfed |   .0174221   .1716065     0.10   0.919     -.319705    .3545493
      clwsta |  -.0517891   .0957109    -0.54   0.589    -.2398166    .1362385
      clwloc |  -.0305153   .1021028    -0.30   0.765       -.2311    .1700694
         d83 |  -.1108653   .0268105    -4.14   0.000    -.1635355   -.0581951
         d84 |  -.0374103    .024533    -1.52   0.128    -.0856063    .0107856
         d85 |  -.0005856    .024078    -0.02   0.981    -.0478877    .0467164
         d86 |   .0314757   .0245099     1.28   0.200    -.0166749    .0796262
         d87 |   .0388632   .0247819     1.57   0.117    -.0098218    .0875482
       _cons |   .0198522   .0206974     0.96   0.338    -.0208086     .060513
------------------------------------------------------------------------------

. reg clcrmrte clprbarr clprbcon clprbpri clavgsen clpolpc clwcon-clwloc
      d83-d87, cluster(county)

Linear regression                                       Number of obs =     540
                                                        F( 19,    89) =   13.66
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.4445
                                                        Root MSE      =  .15398

                                  (Std. Err. adjusted for 90 clusters in county
------------------------------------------------------------------------------
             |               Robust
    clcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    clprbarr |  -.3230993   .0584771    -5.53   0.000    -.4392919   -.2069066
    clprbcon |  -.2402885   .0403223    -5.96   0.000     -.320408   -.1601689
    clprbpri |  -.1693859   .0459288    -3.69   0.000    -.2606455   -.0781263
    clavgsen |  -.0156167   .0267541    -0.58   0.561    -.0687765    .0375432
     clpolpc |   .3977221   .1038642     3.83   0.000     .1913461     .604098
      clwcon |  -.0442368   .0165835    -2.67   0.009    -.0771879   -.0112856
      clwtuc |   .0253997   .0123845     2.05   0.043      .000792    .0500075
      clwtrd |  -.0290309   .0180398    -1.61   0.111    -.0648755    .0068138
      clwfir |    .009122    .006921     1.32   0.191    -.0046299    .0228739
      clwser |   .0219549   .0180754     1.21   0.228    -.0139606    .0578703
```

```
      clwmfg |  -.1402482   .1190279    -1.18   0.242    -.3767541    .0962578
      clwfed |   .0174221      .1326     0.13   0.896    -.2460511    .2808954
      clwsta |  -.0517891   .0674058    -0.77   0.444     -.185723    .0821449
      clwloc |  -.0305153   .1269012    -0.24   0.811    -.2826652    .2216346
         d83 |  -.1108653   .0270368    -4.10   0.000    -.1645868   -.0571437
         d84 |  -.0374103   .0237018    -1.58   0.118    -.0845052    .0096845
         d85 |  -.0005856   .0256369    -0.02   0.982    -.0515257    .0503544
         d86 |   .0314757   .0214193     1.47   0.145     -.011084    .0740353
         d87 |   .0388632   .0263357     1.48   0.144    -.0134653    .0911917
       _cons |   .0198522   .0180545     1.10   0.274    -.0160217    .0557261
------------------------------------------------------------------------------

. test clwcon clwtuc clwtrd clwfir clwser clwmfg clwfed clwsta clwloc

 ( 1)  clwcon = 0
 ( 2)  clwtuc = 0
 ( 3)  clwtrd = 0
 ( 4)  clwfir = 0
 ( 5)  clwser = 0
 ( 6)  clwmfg = 0
 ( 7)  clwfed = 0
 ( 8)  clwsta = 0
 ( 9)  clwloc = 0

       F(  9,    89) =    2.38
            Prob > F =    0.0184

. xtreg lcrmrte lprbarr lprbconv lprbpris lavgsen lpolpc d82-d87  lwcon lwtuc
        lwtrd lwfir lwser lwmfg lwfed lwsta lwloc, fe

Fixed-effects (within) regression               Number of obs      =       630
Group variable: county                          Number of groups   =        90

R-sq:  within  = 0.4575                         Obs per group: min =
       between = 0.2518                                        avg =       7.
       overall = 0.2687                                        max =

                                                F(20,520)          =     21.92
corr(u_i, Xb)  = 0.0804                         Prob > F           =    0.0000

------------------------------------------------------------------------------
     lcrmrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     lprbarr |  -.3563515   .0321591   -11.08   0.000    -.4195292   -.2931738
    lprbconv |  -.2859539   .0210513   -13.58   0.000    -.3273099   -.2445979
    lprbpris |  -.1751355   .0323403    -5.42   0.000    -.2386693   -.1116017
     lavgsen |  -.0028739   .0262108    -0.11   0.913     -.054366    .0486181
      lpolpc |      .4229   .0263942    16.02   0.000     .3710476    .4747524
         d82 |   .0188915   .0251244     0.75   0.452    -.0304662    .0682492
         d83 |   -.055286   .0330287    -1.67   0.095    -.1201721    .0096001
         d84 |  -.0615162   .0410805    -1.50   0.135    -.1422204    .0191879
         d85 |  -.0397115   .0561635    -0.71   0.480    -.1500468    .0706237
         d86 |  -.0001133   .0680124    -0.00   0.999    -.1337262    .1334996
         d87 |   .0537042   .0798953     0.67   0.502    -.1032532    .2106615
       lwcon |  -.0345448   .0391616    -0.88   0.378    -.1114792    .0423896
       lwtuc |   .0459747    .019034     2.42   0.016     .0085817    .0833677
       lwtrd |  -.0201766   .0406073    -0.50   0.619    -.0999511    .0595979
       lwfir |  -.0035445    .028333    -0.13   0.900    -.0592058    .0521168
       lwser |   .0101264   .0191915     0.53   0.598     -.027576    .0478289
```

```
       lwmfg |  -.3005691   .1094068    -2.75   0.006    -.5155028   -.0856354
       lwfed |  -.3331226    .176448    -1.89   0.060    -.6797612     .013516
       lwsta |   .0215209   .1130648     0.19   0.849    -.2005991    .2436409
       lwloc |   .1810215   .1180643     1.53   0.126    -.0509203    .4129632
       _cons |   .8931726   1.424067     0.63   0.531     -1.90446    3.690805
-------------+----------------------------------------------------------------
     sigma_u |  .47756823
     sigma_e |  .13700505
         rho |  .92395784   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(89, 520) =    39.12             Prob > F = 0.0000
```

d. There is strong evidence of negative serial correlation in the FD equation, suggesting that if the idiosyncratic errors follow an AR(1) process, the coefficient is less than unity.

```
. qui reg clcrmrte clprbarr clprbcon clprbpri clavgsen clpolpc clwcon-clwloc
      d83-d87

. predict ehat, resid
(90 missing values generated)

. gen ehat_1 = l.ehat
(180 missing values generated)

. reg ehat ehat_1

      Source |       SS       df       MS              Number of obs =     450
-------------+------------------------------           F(  1,   448) =   21.29
       Model |  .490534556     1  .490534556           Prob > F      =  0.0000
    Residual |  10.3219221   448  .023040005           R-squared     =  0.0454
-------------+------------------------------           Adj R-squared =  0.0432
       Total |  10.8124566   449  .024081195           Root MSE      =  .15179

------------------------------------------------------------------------------
        ehat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      ehat_1 |   -.222258   .0481686    -4.61   0.000    -.3169225   -.1275936
       _cons |   5.97e-10   .0071554     0.00   1.000    -.0140624    .0140624
------------------------------------------------------------------------------
```

10.10. a. To allow for different intercepts in the original model we can include a year dummy for 1993 in the FD equation. (The three years of data are 1987, 1990, and 1993.) There is no evidence of serial correlation in the FD errors, $e_{it} = u_{it} - u_{i,t-1}$, as the coefficient on $\hat{e}_{i,t-1}$ is puny and so is its t statistic. It appears that a random walk for $\{u_{it} : t = 1,2,3\}$ is a reasonably characterization, although concluding this with $T = 3$ is tenuous.

```
. use murder

. xtset id year
```

```
       panel variable:  id (strongly balanced)
        time variable:  year, 87 to 93, but with gaps
                delta:  1 unit

. reg cmrdrte d93 cexec cunem

      Source |       SS       df       MS              Number of obs =     102
-------------+------------------------------           F(  3,    98) =    0.84
       Model |  46.7620386     3  15.5873462            Prob > F      =  0.4736
    Residual |  1812.28688    98  18.4927233            R-squared     =  0.0252
-------------+------------------------------           Adj R-squared = -0.0047
       Total |  1859.04892   101   18.406425           Root MSE      =  4.3003

------------------------------------------------------------------------------
     cmrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d93 |  -1.296717   1.016118    -1.28   0.205    -3.313171    .7197368
       cexec |  -.1150682   .1473871    -0.78   0.437     -.407553    .1774167
       cunem |   .1630854   .3079049     0.53   0.598     -.447942    .7741127
       _cons |    1.51099   .6608967     2.29   0.024     .1994622    2.822518
------------------------------------------------------------------------------

. predict ehat, resid
(51 missing values generated)

. gen ehat_1 = l.ehat
(102 missing values generated)

. reg ehat ehat_1

      Source |       SS       df       MS              Number of obs =      51
-------------+------------------------------           F(  1,    49) =    0.06
       Model |  .075953071     1  .075953071            Prob > F      =  0.8016
    Residual |  58.3045094    49  1.18988795            R-squared     =  0.0013
-------------+------------------------------           Adj R-squared = -0.0191
       Total |  58.3804625    50  1.16760925           Root MSE      =  1.0908

------------------------------------------------------------------------------
        ehat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      ehat_1 |   .0065807   .0260465     0.25   0.802    -.0457618    .0589231
       _cons |  -9.10e-10   .1527453    -0.00   1.000    -.3069532    .3069532
------------------------------------------------------------------------------
```

b. To make all of the FE and FD estimates comparable, the year dummies are differenced along with the other variables in the FD estimation, and no constant is included. (The *R*-squared for the FD equation is computed using the usual total sum of squares, but the FE and FD *R*-squareds are not directly comparable.) The FE and FD coefficient estimates are similar but, especially for the execution variable, the FD standard error is much smaller. Because these are fully robust it is sensible to compare them. Because we found no serial correlation in the

FD errors, it makes sense that the FD estimator is more efficient than FE (whose idiosyncratic errors appear to follow a random walk).

```
. gen cd90 = d90 - d90[_n-1] if year > 87
(51 missing values generated)

. gen cd93 = d93 - d93[_n-1] if year > 87
(51 missing values generated)

. reg cmrdrte cd90 cd93 cexec cunem, nocons tsscons cluster(id)

Linear regression                                       Number of obs =     102
                                                        F(  4,    50) =    2.95
                                                        Prob > F      =  0.0291
                                                        R-squared     =  0.0252
                                                        Root MSE      =  4.3003

                                    (Std. Err. adjusted for 51 clusters in id
------------------------------------------------------------------------------
             |               Robust
     cmrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        cd90 |    1.51099   1.056408     1.43   0.159    -.6108675    3.632848
        cd93 |   1.725264   .8603626     2.01   0.050    -.0028256    3.453353
       cexec |  -.1150682   .0386021    -2.98   0.004    -.1926027   -.0375337
       cunem |   .1630854   .2998749     0.54   0.589     -.439231    .7654018
------------------------------------------------------------------------------

. xtreg mrdrte d90 d93 exec unem, fe cluster(id)

Fixed-effects (within) regression               Number of obs      =       153
Group variable: id                              Number of groups   =        51

R-sq:  within  = 0.0734                         Obs per group: min =
       between = 0.0037                                        avg =       3.
       overall = 0.0108                                        max =

                                                F(4,50)            =      1.80
corr(u_i, Xb)  = 0.0010                         Prob > F           =    0.1443

                                    (Std. Err. adjusted for 51 clusters in id
------------------------------------------------------------------------------
             |               Robust
      mrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d90 |   1.556215   1.119004     1.39   0.170    -.6913706      3.8038
         d93 |   1.733242   .8685105     2.00   0.051    -.0112126    3.477697
        exec |  -.1383231   .0805733    -1.72   0.092    -.3001593    .0235132
        unem |   .2213158    .374899     0.59   0.558    -.5316909    .9743225
       _cons |   5.822104   2.814864     2.07   0.044     .1682823    11.47593
-------------+----------------------------------------------------------------
     sigma_u |  8.7527226
     sigma_e |  3.5214244
         rho |  .86068589   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

c. The explanatory variable $exec_{it}$ might fail strict exogeneity if states increase future executions in response to current positive shocks to the murder rate. Given the relatively short stretch of time, feedback from murder rates to future executions may not be much of a concern, as the judicial process in capital cases tends to move slowly. (Of course, if it were sped up because of an increase in murder rates, that could violate strict exogeneity.) With a longer time series we could add $exec_{i,t+1}$ (and even values from further in the future) and estimate the equation by FE or FD, testing $exec_{i,t+1}$ for statistical significance.

10.11. a. The key coefficient is β_1. Because AFDC participation gives women access to better nutrition and prenatal care, we hope that AFDC participation causes the percent of low-weight births to fall. This only makes sense witih a ceteris paribus thought experiment, holding fixed economic and other variables, such as demographic variables and quality of other kinds of health care. A reasonable expectation is $\beta_2 < 0$: more physicians means relatively fewer low-weight births. The variable *bedspc* is another proxy for health-care availability, and we expect $\beta_3 < 0$. Higher per capita income should lead to lower *lowbrth*, too ($\beta_4 < 0$). The effect of population on a per capita variable is ambiguous, especially because it is total population and not population density.

b. The Stata output follows. Both the usual and fully robust standard errors are computed. The standard errors robust to serial correlation (and heteroskedasticity) are, as expected, somewhat larger. (If you test for AR(1) serial correlation in the composite error, v_{it} it is very strong. In fact, the estimated ρ is slightly above one). Only the per capita income variable is statistically significant. The estimate implies that a 10 percent rise in per capita income is associated with a .25 percentage point fall in the percent of low-weight births.

```
. reg lowbrth d90 afdcprc lphypc lbedspc lpcinc lpopul
```

```
      Source |       SS       df       MS              Number of obs =     100
-------------+------------------------------           F(  6,    93) =    5.19
       Model |  33.7710894     6   5.6285149           Prob > F      =  0.0001
    Residual |  100.834005    93  1.08423661           R-squared     =  0.2509
-------------+------------------------------           Adj R-squared =  0.2026
       Total |  134.605095    99  1.35964742           Root MSE      =  1.0413

------------------------------------------------------------------------------
     lowbrth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d90 |   .5797136   .2761244     2.10   0.038     .0313853    1.128042
     afdcprc |   .0955932   .0921802     1.04   0.302    -.0874584    .2786448
      lphypc |   .3080648     .71546     0.43   0.668    -1.112697    1.728827
     lbedspc |   .2790041   .5130275     0.54   0.588    -.7397668    1.297775
      lpcinc |  -2.494685   .9783021    -2.55   0.012      -4.4374   -.5519711
      lpopul |    .739284   .7023191     1.05   0.295    -.6553826    2.133951
       _cons |   26.57786   7.158022     3.71   0.000     12.36344    40.79227
------------------------------------------------------------------------------

. reg lowbrth d90 afdcprc lphypc lbedspc lpcinc lpopul, cluster(state)

Linear regression                                       Number of obs =     100
                                                        F(  6,    49) =    4.73
                                                        Prob > F      =  0.0007
                                                        R-squared     =  0.2509
                                                        Root MSE      =  1.0413

                                   (Std. Err. adjusted for 50 clusters in state
------------------------------------------------------------------------------
             |               Robust
     lowbrth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d90 |   .5797136   .2214303     2.62   0.012     .1347327    1.024694
     afdcprc |   .0955932   .1199883     0.80   0.429    -.1455324    .3367188
      lphypc |   .3080648   .9063342     0.34   0.735    -1.513282    2.129411
     lbedspc |   .2790041   .7853754     0.36   0.724    -1.299267    1.857275
      lpcinc |  -2.494685   1.203901    -2.07   0.044    -4.914014   -.0753567
      lpopul |    .739284   .9041915     0.82   0.418    -1.077757    2.556325
       _cons |   26.57786    9.29106     2.86   0.006     7.906773    45.24894
------------------------------------------------------------------------------
```

c. The FD (equivalently, FE) estimates are given below. The heteroskedasticity-robust standard error for the AFDC variable is actually smaller. In any case, removing the state unobserved effect changes the sign on the AFDC participation variable, and it is marginally statistically significant. Oddly, physicians-per-capita now has a positive, significant effect on percent of low-weight births. The hospital beds-per-capita variable has the expected negative effect.

```
. reg clowbrth cafdcprc clphypc clbedspc clpcinc clpopul

      Source |       SS       df       MS              Number of obs =      50
-------------+------------------------------           F(  5,    44) =    2.53
       Model |  .861531934     5  .172306387           Prob > F      =  0.0428
    Residual |  3.00026764    44  .068187901           R-squared     =  0.2231
-------------+------------------------------           Adj R-squared =  0.1348
       Total |  3.86179958    49  .078812236           Root MSE      =  .26113

------------------------------------------------------------------------------
    clowbrth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    cafdcprc |  -.1760763   .0903733    -1.95   0.058    -.3582116     .006059
     clphypc |   5.894509   2.816689     2.09   0.042     .2178452    11.57117
    clbedspc |  -1.576195   .8852111    -1.78   0.082    -3.360221    .2078308
     clpcinc |  -.8455268   1.356773    -0.62   0.536    -3.579924     1.88887
     clpopul |   3.441116   2.872175     1.20   0.237    -2.347372    9.229604
       _cons |   .1060158   .3090664     0.34   0.733    -.5168667    .7288983
------------------------------------------------------------------------------

. reg clowbrth cafdcprc clphypc clbedspc clpcinc clpopul, robust

Linear regression                                       Number of obs =      50
                                                        F(  5,    44) =    1.97
                                                        Prob > F      =  0.1024
                                                        R-squared     =  0.2231
                                                        Root MSE      =  .26113

------------------------------------------------------------------------------
             |               Robust
    clowbrth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    cafdcprc |  -.1760763   .0767568    -2.29   0.027    -.3307695    -.021383
     clphypc |   5.894509   3.098646     1.90   0.064    -.3504018    12.13942
    clbedspc |  -1.576195   1.236188    -1.28   0.209    -4.067567    .9151775
     clpcinc |  -.8455268   1.484034    -0.57   0.572      -3.8364    2.145346
     clpopul |   3.441116   2.687705     1.28   0.207    -1.975596    8.857829
       _cons |   .1060158   .3675668     0.29   0.774    -.6347664    .8467981
------------------------------------------------------------------------------
```

d. Adding a quadratic in *afdcprc* yields a diminishing impact of AFDC participation. The turning point in the quadratic is at about $afdcprc = 6.4$, and only four states have AFDC participation rates above 6.4 percent. So, the largest effect is at low AFDC participation rates and the effect is negative until $afdcprc = 6.4$. It is not clear this makes sense: if AFDC participation increases then more women in living in poverty get better prenatal care for their children. But the quadratic is not statistically significant at the usual levels and we could safely drop it.

```
. reg clowbrth cafdcprc cafdcpsq clphypc clbedspc clpcinc clpopul, robust

Linear regression                                      Number of obs =      50
                                                       F(  6,    43) =    2.07
                                                       Prob > F      =  0.0762
                                                       R-squared     =  0.2499
                                                       Root MSE      =  .25956

------------------------------------------------------------------------------
             |               Robust
    clowbrth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    cafdcprc |  -.5035049   .2612029    -1.93   0.061    -1.030271     .023261
    cafdcpsq |   .0396094   .0317531     1.25   0.219    -.0244267    .1036456
     clphypc |   6.620885   3.448026     1.92   0.061     -.332723    13.57449
    clbedspc |  -1.407963   1.344117    -1.05   0.301    -4.118634    1.302707
     clpcinc |  -.9987865   1.541609    -0.65   0.521    -4.107738    2.110165
     clpopul |   4.429026   2.925156     1.51   0.137    -1.470113    10.32817
       _cons |   .1245915    .386679     0.32   0.749     -.655221    .9044041
------------------------------------------------------------------------------

. di abs(_b[cafdcprc]/(2*_b[cafdcpsq]))
6.3558685


. sum afdcprc if d90

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     afdcprc |        50    4.162976    1.317277   1.688183   7.358795

. count if afdcprc >= 6.4 & d90
    4
```

10.12. a. Even if c_i is uncorrelated with $\mathbf{x}_{it}$ for all t, the usual OLS standard errors do not account for the serial correlation in $v_{it} = c_i + u_{it}$. You can see that the fully robust standard errors are substantially larger than the usual ones, in some cases more than double.

```
. use wagepan

. reg lwage educ black hisp exper expersq married union d81-d87, cluster(nr)

Linear regression                                      Number of obs =    4360
                                                       F( 14,   544) =   47.10
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.1893
                                                       Root MSE      =  .48033

                                   (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
```

```
        educ |   .0913498   .0110822     8.24   0.000     .0695807    .1131189
       black |  -.1392342   .0505238    -2.76   0.006    -.2384798   -.0399887
        hisp |   .0160195   .0390781     0.41   0.682     -.060743     .092782
       exper |   .0672345   .0195958     3.43   0.001     .0287417    .1057273
     expersq |  -.0024117   .0010252    -2.35   0.019    -.0044255   -.0003979
     married |   .1082529    .026034     4.16   0.000     .0571135    .1593924
       union |   .1824613   .0274435     6.65   0.000     .1285531    .2363695
         d81 |     .05832    .028228     2.07   0.039     .0028707    .1137693
         d82 |   .0627744   .0369735     1.70   0.090    -.0098538    .1354027
         d83 |   .0620117    .046248     1.34   0.181    -.0288348    .1528583
         d84 |   .0904672    .057988     1.56   0.119    -.0234407     .204375
         d85 |   .1092463   .0668474     1.63   0.103    -.0220644     .240557
         d86 |   .1419596   .0762348     1.86   0.063     -.007791    .2917102
         d87 |   .1738334   .0852056     2.04   0.042     .0064611    .3412057
       _cons |   .0920558   .1609365     0.57   0.568    -.2240773    .4081888
------------------------------------------------------------------------------

. reg lwage educ black hisp exper expersq married union d81-d87

      Source |       SS       df       MS              Number of obs =    4360
-------------+------------------------------           F( 14,  4345) =   72.46
       Model |  234.048277    14  16.7177341           Prob > F      =  0.0000
    Residual |  1002.48136  4345  .230720682           R-squared     =  0.1893
-------------+------------------------------           Adj R-squared =  0.1867
       Total |  1236.52964  4359  .283672779           Root MSE      =  .48033

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .0913498   .0052374    17.44   0.000     .0810819    .1016177
       black |  -.1392342   .0235796    -5.90   0.000    -.1854622   -.0930062
        hisp |   .0160195   .0207971     0.77   0.441    -.0247535    .0567925
       exper |   .0672345   .0136948     4.91   0.000     .0403856    .0940834
     expersq |  -.0024117     .00082    -2.94   0.003    -.0040192   -.0008042
     married |   .1082529   .0156894     6.90   0.000     .0774937    .1390122
       union |   .1824613   .0171568    10.63   0.000     .1488253    .2160973
         d81 |     .05832   .0303536     1.92   0.055    -.0011886    .1178286
         d82 |   .0627744   .0332141     1.89   0.059    -.0023421    .1278909
         d83 |   .0620117   .0366601     1.69   0.091    -.0098608    .1338843
         d84 |   .0904672   .0400907     2.26   0.024      .011869    .1690654
         d85 |   .1092463   .0433525     2.52   0.012     .0242533    .1942393
         d86 |   .1419596    .046423     3.06   0.002     .0509469    .2329723
         d87 |   .1738334    .049433     3.52   0.000     .0769194    .2707474
       _cons |   .0920558   .0782701     1.18   0.240    -.0613935    .2455051
------------------------------------------------------------------------------
```

b. The random effects estimates on the time-constant variables are similar to the pooled OLS estimates. The coefficients on the quadratic in experience for RE show an initially stronger effect of experience, but with the slope diminishing more rapidly. There are important differences in the variables that chance across individual and time; they are notably lower for random effects. The random effects marriage premium is about 6.4%, while the pooled OLS

estimate is about 10.8%. For union status, the random effects estimate is 10.6% compared with a pooled OLS estimate of 18.2%.

Note that the RE standard errors for the coefficients on the time-constant explanatory variables are similar to the fully robust POLS standard errors. However, the RE standard errors for *married* and *union* are substantially smaller than the robust POLS standard errors, suggestive of the relative efficiency of RE. To be fair, we should compute the fully robust standard errors for RE. As shown below, these are somewhat larger than the usual RE standard errors, but for the *married* and *union* still not nearly as large as the robust standard errors for POLS. An important conclusion is that, even though RE might not be the asymptotically efficient FGLS estimator, it appears to be more efficient than POLS, at least for the time-varying explanatory variables.

```
. xtreg lwage educ black hisp exper expersq married union d81-d87, re

Random-effects GLS regression                   Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1799                         Obs per group: min =
       between = 0.1860                                        avg =       8.
       overall = 0.1830                                        max =

Random effects u_i ~Gaussian                    Wald chi2(14)      =    957.77
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .0918763   .0106597     8.62   0.000     .0709836    .1127689
       black |  -.1393767   .0477228    -2.92   0.003    -.2329117   -.0458417
        hisp |   .0217317   .0426063     0.51   0.610    -.0617751    .1052385
       exper |   .1057545   .0153668     6.88   0.000     .0756361    .1358729
     expersq |  -.0047239   .0006895    -6.85   0.000    -.0060753   -.0033726
     married |    .063986   .0167742     3.81   0.000     .0311091    .0968629
       union |   .1061344   .0178539     5.94   0.000     .0711415    .1411273
          d81 |    .040462   .0246946     1.64   0.101    -.0079385    .0888626
          d82 |   .0309212   .0323416     0.96   0.339    -.0324672    .0943096
          d83 |   .0202806    .041582     0.49   0.626    -.0612186    .1017798
          d84 |   .0431187   .0513163     0.84   0.401    -.0574595    .1436969
          d85 |   .0578155   .0612323     0.94   0.345    -.0621977    .1778286
          d86 |   .0919476   .0712293     1.29   0.197    -.0476592    .2315544
          d87 |   .1349289   .0813135     1.66   0.097    -.0244427    .2943005
        _cons |   .0235864   .1506683     0.16   0.876     -.271718    .3188907
```

```
-------------+----------------------------------------------------------------
     sigma_u |  .32460315
     sigma_e |  .35099001
         rho |  .46100216   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lwage educ black hisp exper expersq married union d81-d87, re
        cluster(nr)

Random-effects GLS regression                   Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1799                         Obs per group: min =
       between = 0.1860                                        avg =       8.
       overall = 0.1830                                        max =

Random effects u_i ~Gaussian                    Wald chi2(14)      =    610.97
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                                  (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |   .0918763   .0111455     8.24   0.000     .0700315    .1137211
       black |  -.1393767   .0509251    -2.74   0.006    -.2391882   -.0395653
        hisp |   .0217317   .0399157     0.54   0.586    -.0565015     .099965
       exper |   .1057545    .016379     6.46   0.000     .0736522    .1378568
     expersq |  -.0047239   .0007917    -5.97   0.000    -.0062756   -.0031723
     married |    .063986   .0189722     3.37   0.001     .0268013    .1011708
       union |   .1061344    .020844     5.09   0.000      .065281    .1469879
         d81 |    .040462   .0275684     1.47   0.142    -.0135711    .0944951
         d82 |   .0309212   .0350705     0.88   0.378    -.0378158    .0996581
         d83 |   .0202806    .043861     0.46   0.644    -.0656853    .1062466
         d84 |   .0431187   .0555848     0.78   0.438    -.0658254    .1520628
         d85 |   .0578155   .0645584     0.90   0.370    -.0687167    .1843476
         d86 |   .0919476   .0747028     1.23   0.218    -.0544671    .2383623
         d87 |   .1349289   .0848618     1.59   0.112    -.0313971    .3012549
       _cons |   .0235864   .1599577     0.15   0.883     -.289925    .3370977
-------------+----------------------------------------------------------------
     sigma_u |  .32460315
     sigma_e |  .35099001
         rho |  .46100216   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

c. The variable $exper_{it}$ is redundant because everyone in the sample works every year, so $exper_{i,t+1} = exper_{it} + 1$, $t = 1, \ldots, 7$, for all i. The effects of the initial levels of experience, $exper_{i1}$, cannot be distinguished from c_i because we are allowing $exper_{i1}$ to be correlated with c_i. Then, because each experience variable follows the same linear time trend, the effects cannot be separated from the aggregate time effects (year dummies).

TheFE estimates follow. The marriage and union premiums fall even more, although both are still statistically significant and economically relevant. The fully robust standard errors are somewhat larger than the usual FE standard errors.

```
. xtreg lwage expersq married union d81-d87, fe

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1806                         Obs per group: min =
       between = 0.0286                                        avg =       8.
       overall = 0.0888                                        max =

                                                F(10,3805)         =     83.85
corr(u_i, Xb)  = -0.1222                        Prob > F           =    0.0000

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     expersq |  -.0051855   .0007044    -7.36   0.000    -.0065666   -.0038044
     married |   .0466804   .0183104     2.55   0.011     .0107811    .0825796
       union |   .0800019   .0193103     4.14   0.000     .0421423    .1178614
         d81 |   .1511912   .0219489     6.89   0.000     .1081584     .194224
         d82 |   .2529709   .0244185    10.36   0.000     .2050963    .3008454
         d83 |   .3544437   .0292419    12.12   0.000     .2971125    .4117749
         d84 |   .4901148   .0362266    13.53   0.000     .4190894    .5611402
         d85 |   .6174823   .0452435    13.65   0.000     .5287784    .7061861
         d86 |   .7654966   .0561277    13.64   0.000     .6554532    .8755399
         d87 |   .9250249   .0687731    13.45   0.000     .7901893    1.059861
       _cons |   1.426019   .0183415    77.75   0.000     1.390058    1.461979
-------------+----------------------------------------------------------------
     sigma_u |  .39176195
     sigma_e |  .35099001
         rho |  .55472817   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(544, 3805) =     9.16           Prob > F = 0.0000

. xtreg lwage expersq married union d81-d87, fe cluster(nr)

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1806                         Obs per group: min =
       between = 0.0286                                        avg =       8.
       overall = 0.0888                                        max =

                                                F(10,544)          =     46.59
corr(u_i, Xb)  = -0.1222                        Prob > F           =    0.0000

                                    (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
```

```
     expersq |  -.0051855   .0008102    -6.40   0.000    -.0067771   -.0035939
     married |   .0466804   .0210038     2.22   0.027     .0054218    .0879389
       union |   .0800019   .0227431     3.52   0.000     .0353268    .1246769
          d81 |   .1511912   .0255648     5.91   0.000     .1009733    .2014091
          d82 |   .2529709   .0286624     8.83   0.000     .1966684    .3092733
          d83 |   .3544437   .0348608    10.17   0.000     .2859655     .422922
          d84 |   .4901148   .0454581    10.78   0.000     .4008199    .5794097
          d85 |   .6174823   .0568088    10.87   0.000     .5058908    .7290737
          d86 |   .7654966    .071244    10.74   0.000     .6255495    .9054436
          d87 |   .9250249   .0840563    11.00   0.000     .7599103     1.09014
        _cons |   1.426019   .0209824    67.96   0.000     1.384802    1.467235
-------------+----------------------------------------------------------------
     sigma_u |  .39176195
     sigma_e |  .35099001
         rho |  .55472817   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

d. The following Stata session adds the year dummy-education interaction terms. There is no evidence that the return to education has changed over time for the population represented by these men. The *p*-value for the joint robust test is about .89.

```
. gen d81educ = d81*educ

. gen d82educ = d82*educ

. gen d83educ = d83*educ

. gen d84educ = d84*educ

. gen d85educ = d85*educ

. gen d86educ = d86*educ

. gen d87educ = d87*educ

. xtreg lwage expersq married union d81-d87 d81educ-d87educ, fe cluster(nr)

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1814                         Obs per group: min =
       between = 0.0211                                        avg =       8.
       overall = 0.0784                                        max =

                                                F(17,544)          =     28.33
corr(u_i, Xb)  = -0.1732                        Prob > F           =    0.0000

                                  (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     expersq |  -.0060437   .0010323    -5.85   0.000    -.0080715   -.0040159
     married |   .0474337   .0210293     2.26   0.024      .006125    .0887423
```

```
       union |   .0789759    .022762     3.47   0.001     .0342638     .123688
          d81 |   .0984201   .1463954     0.67   0.502    -.1891495    .3859897
          d82 |   .2472016   .1490668     1.66   0.098    -.0456155    .5400186
          d83 |    .408813   .1716953     2.38   0.018      .071546     .74608
          d84 |   .6399247   .1873708     3.42   0.001     .2718659    1.007984
          d85 |   .7729397   .2090195     3.70   0.000     .3623554    1.183524
          d86 |   .9699322   .2463734     3.94   0.000     .4859724    1.453892
          d87 |   1.188777   .2580167     4.61   0.000     .6819456    1.695608
     d81educ |   .0049906   .0122858     0.41   0.685    -.0191429    .0291241
     d82educ |    .001651    .012194     0.14   0.892    -.0223021     .025604
     d83educ |  -.0026621   .0136788    -0.19   0.846    -.0295319    .0242076
     d84educ |  -.0098257   .0146869    -0.67   0.504    -.0386757    .0190243
     d85educ |  -.0092145   .0151166    -0.61   0.542    -.0389085    .0204796
     d86educ |  -.0121382   .0168594    -0.72   0.472    -.0452558    .0209794
     d87educ |  -.0157892   .0163557    -0.97   0.335    -.0479172    .0163389
       _cons |   1.436283   .0227125    63.24   0.000     1.391668    1.480897
-------------+----------------------------------------------------------------
     sigma_u |  .39876325
     sigma_e |   .3511451
         rho |  .56324361   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. testparm d81educ-d87educ

 ( 1)  d81educ = 0
 ( 2)  d82educ = 0
 ( 3)  d83educ = 0
 ( 4)  d84educ = 0
 ( 5)  d85educ = 0
 ( 6)  d86educ = 0
 ( 7)  d87educ = 0

       F(  7,   544) =    0.43
            Prob > F =    0.8851
```

e. First, I created the lead variable, and then included it in the FE estimation with fully robust inference. As you can see, *unionp*1 is statistically significant with p-value = .029, and its coefficient is not small. It seems $union_{it}$ fails the strict exogeneity assumption, and we possibly should use an IV approach as described in Chapter 11. (However, coming up with instruments is not trivial.)

```
. gen unionp1 = union[_n+1] if year < 1987
(545 missing values generated)

. xtreg lwage expersq married union unionp1 d81-d86, fe cluster(nr)

Fixed-effects (within) regression               Number of obs      =      3815
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1474                         Obs per group: min =
       between = 0.0305                                        avg =       7.
```

```
       overall = 0.0744                                        max =

                                                F(10,544)          =     31.29
corr(u_i, Xb)  = -0.1262                        Prob > F           =    0.0000

                                  (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     expersq |  -.0054448   .0009786    -5.56   0.000    -.0073671   -.0035226
     married |   .0448778   .0235662     1.90   0.057    -.0014141    .0911697
       union |   .0763554   .0236392     3.23   0.001     .0299202    .1227906
     unionp1 |   .0497356   .0227497     2.19   0.029     .0050477    .0944236
         d81 |   .1528275   .0257236     5.94   0.000     .1022977    .2033573
         d82 |   .2576486   .0304975     8.45   0.000     .1977413    .3175558
         d83 |   .3618296   .0384587     9.41   0.000     .2862839    .4373754
         d84 |   .5023642   .0517471     9.71   0.000     .4007155    .6040129
         d85 |   .6342402    .065288     9.71   0.000     .5059928    .7624876
         d86 |   .7841312   .0826431     9.49   0.000     .6217924    .9464699
       _cons |   1.417924   .0225168    62.97   0.000     1.373693    1.462154
-------------+----------------------------------------------------------------
     sigma_u |  .39716048
     sigma_e |  .35740734
         rho |   .5525375   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

f. The Stata output shows that the union premium for Hispanic men is well below that of non-black men: about 13 percentage points lower. The difference is statistically significant, too. The estimated union "premium" is actually about –3.5% for Hispanics, although it is not statistically different from zero. The estimated wage premium for black men is about 7.1 percentage points higher than the base group, but the difference is not statistically significant.

```
. gen black_union = black*union

. gen hisp_union = hisp*union

. xtreg lwage expersq married union black_union hisp_union d81-d87, fe

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1830                         Obs per group: min =
       between = 0.0267                                        avg =       8.
       overall = 0.0871                                        max =

                                                F(12,3803)         =     70.99
corr(u_i, Xb)  = -0.1360                        Prob > F           =    0.0000

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
```

```
     expersq |   -.005308   .0007048    -7.53   0.000    -.0066898   -.0039262
     married |   .0461639   .0182922     2.52   0.012     .0103004    .0820275
       union |   .0957205   .0244326     3.92   0.000     .0478183    .1436227
 black_union |   .0714378   .0532042     1.34   0.179    -.0328737    .1757492
  hisp_union |  -.1302478   .0485409    -2.68   0.007    -.2254166   -.0350791
          d81 |   .1507003   .0219236     6.87   0.000     .1077172    .1936833
          d82 |   .2545937   .0243936    10.44   0.000      .206768    .3024194
          d83 |   .3576139    .029227    12.24   0.000     .3003119     .414916
          d84 |   .4947141   .0362132    13.66   0.000      .423715    .5657132
          d85 |   .6236823   .0452345    13.79   0.000     .5349961    .7123686
          d86 |   .7750896   .0561524    13.80   0.000      .664998    .8851813
          d87 |   .9344805   .0687783    13.59   0.000     .7996347    1.069326
        _cons |    1.42681   .0183207    77.88   0.000     1.390891    1.462729
--------------+----------------------------------------------------------------
     sigma_u |    .393505
     sigma_e |  .35056318
         rho |  .55752062   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(544, 3803) =     9.17                 Prob > F = 0.0000

. xtreg lwage expersq married union black_union hisp_union d81-d87, fe
        cluster(nr)

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1830                         Obs per group: min =
       between = 0.0267                                        avg =       8.
       overall = 0.0871                                        max =

                                                F(12,544)          =     40.16
corr(u_i, Xb)  = -0.1360                        Prob > F           =    0.0000

                                  (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     expersq |   -.005308   .0008095    -6.56   0.000    -.0068982   -.0037178
     married |   .0461639   .0209641     2.20   0.028     .0049834    .0873444
       union |   .0957205   .0304494     3.14   0.002     .0359077    .1555333
 black_union |   .0714378   .0600866     1.19   0.235    -.0465925     .189468
  hisp_union |  -.1302478   .0493283    -2.64   0.009    -.2271451   -.0333505
          d81 |   .1507003    .025519     5.91   0.000     .1005725    .2008281
          d82 |   .2545937   .0286062     8.90   0.000     .1984014    .3107859
          d83 |   .3576139   .0348463    10.26   0.000     .2891641    .4260638
          d84 |   .4947141   .0453929    10.90   0.000     .4055473    .5838809
          d85 |   .6236823    .056721    11.00   0.000     .5122633    .7351014
          d86 |   .7750896    .071142    10.89   0.000      .635343    .9148363
          d87 |   .9344805   .0838788    11.14   0.000     .7697145    1.099247
        _cons |    1.42681   .0209431    68.13   0.000     1.385671    1.467949
--------------+----------------------------------------------------------------
     sigma_u |    .393505
     sigma_e |  .35056318
         rho |  .55752062   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. lincom union + hisp_union
```

```
 ( 1)  union + hisp_union = 0

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         (1) |  -.0345273   .0388508    -0.89   0.375    -.1108432    .0417886
------------------------------------------------------------------------------
```

g. We reject the null hypothesis that $\{union_{it} : t = 1,\ldots,T\}$ is strictly exogenous even when the union premium is allowed to differ by race and ethnicity.

```
. xtreg lwage expersq married union black_union hisp_union d81-d86 unionp1, fe
       cluster(nr)

Fixed-effects (within) regression               Number of obs      =      3815
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1497                         Obs per group: min =
       between = 0.0293                                        avg =       7.
       overall = 0.0735                                        max =

                                                F(12,544)          =     27.21
corr(u_i, Xb)  = -0.1386                        Prob > F           =    0.0000

                                  (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     expersq |  -.0055477   .0009833    -5.64   0.000    -.0074793   -.0036162
     married |    .044659   .0235905     1.89   0.059    -.0016807    .0909986
       union |   .0886305    .031931     2.78   0.006     .0259075    .1513536
 black_union |   .0849246   .0627531     1.35   0.177    -.0383434    .2081926
  hisp_union |  -.1179177   .0525974    -2.24   0.025    -.2212365   -.0145988
         d81 |   .1522945   .0256633     5.93   0.000     .1018831    .2027058
         d82 |   .2589966   .0304368     8.51   0.000     .1992085    .3187846
         d83 |   .3643699   .0385023     9.46   0.000     .2887385    .4400013
         d84 |    .506142   .0518005     9.77   0.000     .4043885    .6078955
         d85 |   .6393639   .0654104     9.77   0.000     .5108761    .7678517
         d86 |   .7921705   .0828197     9.57   0.000     .6294849     .954856
     unionp1 |   .0502016   .0227235     2.21   0.028     .0055649    .0948382
       _cons |   1.418021    .022508    63.00   0.000     1.373808    1.462234
-------------+----------------------------------------------------------------
     sigma_u |  .39854263
     sigma_e |  .35703614
         rho |   .5547681   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

10.13. a. Showing that this procedure is consistent with fixed T as $N \to \infty$ requires some algebra. First, in the sum of squared residuals, we can "concentrate out" the a_i by finding $\hat{a}_i(\mathbf{b})$ as a function of $(\mathbf{x}_i, \mathbf{y}_i)$ and $\mathbf{b}$, substituting back into the sum of squared residuals, and then

minimizing with respect to **b** only. Straightforward algebra gives the first order conditions for each i as

$$\sum_{t=1}^{T}(y_{it} - \hat{a}_i - \mathbf{x}_{it}\mathbf{b})/h_{it} = 0$$

which implies

$$\hat{a}_i(\mathbf{b}) = w_i\left(\sum_{t=1}^{T} y_{it}/h_{it}\right) - w_i\left(\sum_{t=1}^{T}(\mathbf{x}_{it}/h_{it})\right)\mathbf{b} \equiv \bar{y}_i^w - \bar{\mathbf{x}}_i^w\mathbf{b},$$

where $w_i \equiv \left(\sum_{t=1}^{T}(1/h_{it})\right)^{-1} > 0$ and $\bar{y}_i^w \equiv w_i\left(\sum_{t=1}^{T} y_{it}/h_{it}\right)$, and a similar definition holds for $\bar{\mathbf{x}}_i^w$. Note that $\bar{y}_i^w$ and $\bar{\mathbf{x}}_i^w$ are weighted averages with weights w_i/h_{it}, $t = 1, 2, \ldots, T$. If h_{it} equals the same constant for all t, $\bar{y}_i^w$ and $\bar{\mathbf{x}}_i^w$ are simply weighted averages. If h_{it} equals the same constant for all t, $\bar{y}_i^w$ and $\bar{\mathbf{x}}_i^w$ are the usual time averages.

Now we can plug each $\hat{a}_i(\mathbf{b})$ into the SSR to get the problem solved by $\hat{\boldsymbol{\beta}}_{FEWLS}$:

$$\min_{\mathbf{b}\in\mathbb{R}^K} \sum_{i=1}^{N}\sum_{t=1}^{T}[(y_{it} - \bar{y}_i^w) - (\mathbf{x}_{it} - \bar{\mathbf{x}}_i^w)\mathbf{b}]^2/h_{it}.$$

This is just a pooled weighted least squares regression of $(y_{it} - \bar{y}_i^w)$ on $(\mathbf{x}_{it} - \bar{\mathbf{x}}_i^w)$ with weights $1/h_{it}$. Equivalently, define $\tilde{y}_{it} \equiv (y_{it} - \bar{y}_i^w)/\sqrt{h_{it}}$, $\tilde{\mathbf{x}}_{it} \equiv (\mathbf{x}_{it} - \bar{\mathbf{x}}_i^w)/\sqrt{h_{it}}$, all $t = 1, \ldots, T, i = 1, \ldots, N$. Then $\hat{\boldsymbol{\beta}}$ can be expressed in usual pooled OLS form:

$$\hat{\boldsymbol{\beta}}_{FEWLS} = \left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it}\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{\mathbf{x}}_{it}'\tilde{y}_{it}\right) \tag{10.90}$$

Note carefully how the initial y_{it} are weighted by $1/h_{it}$ to obtain $\bar{y}_i^w$, but where the usual $1/\sqrt{h_{it}}$ weighting shows up in the sum of squared residuals on the time-demeaned data (where the demeaning is a weighted average). Given (10.90), we can easily study the asymptotic

($N \to \infty$) properties of $\hat{\boldsymbol{\beta}}$. First, we can write $\bar{y}_i^w = \mathbf{x}_i^{-w}\boldsymbol{\beta} + c_i + \bar{u}_i^w$where $\bar{u}_i^w \equiv w_i\left(\sum_{t=1}^T u_{it}/h_{it}\right)$. Subtracting this equation from $y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + c_i + u_{it}$ for all t gives $\tilde{y}_{it} = \tilde{\mathbf{x}}_{it}\boldsymbol{\beta} + \tilde{u}_{it}$, where $\tilde{u}_{it} \equiv (u_{it} - \bar{u}_i^w)/\sqrt{h_{it}}$. When we plug this in for $\tilde{y}_{it}$ in (10.90) and divide by N in the appropriate places we get

$$\hat{\boldsymbol{\beta}}_{FEWLS} = \boldsymbol{\beta} + \left(N^{-1}\sum_{i=1}^N\sum_{t=1}^T \tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it}\right)\left(N^{-1}\sum_{i=1}^N\sum_{t=1}^T \tilde{\mathbf{x}}_{it}'u_{it}/\sqrt{h_{it}}\right). \tag{10.91}$$

From this last equation we can immediately read off the consistency of $\hat{\boldsymbol{\beta}}_{FEWLS}$ regardless of whether $\text{Var}(u_{it}|\mathbf{x}_i, \mathbf{h}_i, c_i) = \sigma_u^2 h_{it}$. Why? We assumed that $\text{E}(u_{it}|\mathbf{x}_i, \mathbf{h}_i, c_i) = 0,$ which means u_{it} is uncorrelated with any function of $(\mathbf{x}_i, \mathbf{h}_i)$, including $\tilde{\mathbf{x}}_{it}$. Therefore, $\text{E}(\tilde{\mathbf{x}}_{it}'u_{it}) = \mathbf{0},\ t = 1, \ldots, T$ under $\text{E}(u_{it}|\mathbf{x}_i, \mathbf{h}_i, c_i) = 0$ with any restrictions on the conditional second moments of $\{u_{it} : t = 1, \ldots, T\}$. As long as we assume that $\sum_{t=1}^T \text{E}(\tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it})$ has rank K, we can apply the consistency result for pooled OLS to conclude $\text{plim}(\hat{\boldsymbol{\beta}}_{FEWLS}) = \boldsymbol{\beta}$. (We can even show that $\text{E}(\hat{\boldsymbol{\beta}}_{FEWLS}|\mathbf{X}, \mathbf{H}) = \boldsymbol{\beta}$, that is, the FEWLS estimator is conditionally unbiased.)

b. It is clear from (10.91) that $\hat{\boldsymbol{\beta}}_{FEWLS}$ is $\sqrt{N}$-asymptotically normal under mild assumptions because we can write

$$\sqrt{N}(\hat{\boldsymbol{\beta}}_{FEWLS} - \boldsymbol{\beta}) = \left(N^{-1}\sum_{i=1}^N\sum_{t=1}^T \tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it}\right)\left(N^{-1/2}\sum_{i=1}^N\sum_{t=1}^T \tilde{\mathbf{x}}_{it}'u_{it}/\sqrt{h_{it}}\right)$$

The asymptotic variance is generally

$$\text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}}_{FEWLS} - \boldsymbol{\beta}) = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1},$$

where

$$\mathbf{A} \equiv \sum_{t=1}^{T} \mathrm{E}(\tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it})$$

$$\mathbf{B} \equiv \mathrm{E}\left[\left(\sum_{t=1}^{T} \tilde{\mathbf{x}}_{it}' u_{it}/\sqrt{h_{it}}\right)\left(\sum_{t=1}^{T} \tilde{\mathbf{x}}_{it}' u_{it}/\sqrt{h_{it}}\right)'\right].$$

Iif we assume that $\mathrm{Cov}(u_{it}, u_{is}|\mathbf{x}_i, \mathbf{h}_i, c_i) = \mathrm{E}(u_{it}u_{is}|\mathbf{x}_i, \mathbf{h}_i, c_i) = 0,\ t \neq s$ then by a standard iterated expectations argument,

$$\mathrm{E}\left[\left(\tilde{\mathbf{x}}_{it}' u_{it}/\sqrt{h_{it}}\right)\left(\tilde{\mathbf{x}}_{is}' u_{is}/\sqrt{h_{is}}\right)'\right] = \mathrm{E}\left[\left(u_{it}u_{is}\tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{is}/\sqrt{h_{it}h_{is}}\right)\right] = \mathbf{0},\ t \neq s.$$

Further, given the variance assumption $\mathrm{Var}(u_{it}|\mathbf{x}_i, \mathbf{h}_i, c_i) = \mathrm{E}(u_{it}^2|\mathbf{x}_i, \mathbf{h}_i, c_i) = \sigma_u^2 h_{it}$, iterated expectations implies

$$\mathrm{E}\left[\left(\tilde{\mathbf{x}}_{it}' u_{it}/\sqrt{h_{it}}\right)\left(\tilde{\mathbf{x}}_{it}' u_{it}/\sqrt{h_{it}}\right)'\right] = \mathrm{E}(u_{it}^2\tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it}/h_{it}) = \sigma_u^2\mathrm{E}(\tilde{\mathbf{x}}_{it}'\tilde{\mathbf{x}}_{it}/h_{it}).$$

It follows then that

$$\mathbf{B} = \sigma_u^2\mathbf{A}$$

and so

$$\mathrm{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}}_{FEWLS} - \boldsymbol{\beta}) = \sigma_u^2\mathbf{A}^{-1}.$$

c. The same subtleties that arise in estimating σ_u^2 for the usual fixed effects estimator crop up here as well. Assume the zero conditional covariance assumption and correct variance specification in part b. Then the residuals from the pooled OLS regression

$$\tilde{y}_{it} \text{ on } \tilde{\mathbf{x}}_{it},\ t = 1, \ldots, T,\ \ i = 1, \ldots, N, \tag{10.92}$$

say $\hat{\ddot{u}}_{it}$, are estimating $\ddot{u}_{it} = (u_{it} - \bar{u}_i^w)/\sqrt{h_{it}}$ in the sense that we obtain $\hat{\ddot{u}}_{it}$ from $\ddot{u}_{it}$ by replacing $\boldsymbol{\beta}$ with $\hat{\boldsymbol{\beta}}_{FEWLS}$). Now

$$\mathrm{E}(\ddot{u}_{it}^2) = \mathrm{E}[(\ddot{u}_{it}^2/h_{it})] - 2\mathrm{E}[(u_{it}\bar{u}_i^w)/h_{it}] = \mathrm{E}[(\bar{u}_i^w)^2/h_{it})] = \sigma_u^2 - 2\sigma_u^2\mathrm{E}[(w_i/h_{it})] + \sigma_u^2\mathrm{E}[(w_i/h_{it})],$$

where the law of iterated expectations is applied several times, and $\mathrm{E}[(\bar{u}_i^w)^2|\mathbf{x}_i, \mathbf{h}_i] = \sigma_u^2 w_i$ has been used. Therefore, $\mathrm{E}(\ddot{u}_{it}^2) = \sigma_u^2[1 - \mathrm{E}(w_i/h_{it})]$, $t = 1, \ldots, T$, and so

$$\sum_{t=1}^{T} \mathrm{E}[(\ddot{u}_{it}^2) = \sigma_u^2\{T - \mathrm{E}[w_i \cdot \sum_{t=1}^{T}(1/h_{it})]\} \quad = \sigma_u^2(T-1).$$

This contains the usual result for the within transformation as a special case. A consistent estimator of σ_u^2 is SSR/$[N(T-1) - K]$, where SSR is the usual sum of squared residuals from (10.92), and the subtraction of K is optional as a degrees-of-freedom adjustment. The estimator of $\mathrm{Avar}(\hat{\boldsymbol{\beta}}_{FEWLS})$ is then

$$\hat{\sigma}_u^2 \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \tilde{\mathbf{x}}_{it}' \tilde{\mathbf{x}}_{it} \right)^{-1}.$$

d. If we want to allow serial correlation in the $\{u_{it}\}$, or allow $\mathrm{Var}(u_{it}|\mathbf{x}_i, \mathbf{h}_i, c_i) \neq \sigma_u^2 h_{it}$, then we can just apply the robust formula for the pooled OLS regression (10.92). See equation (7.77) in the text.

10.14. a. Because $\mathrm{E}(h_i|\mathbf{z}_i) = 0$, $\mathbf{z}_i\boldsymbol{\gamma}$ and h_i are uncorrelated, so $\mathrm{Var}(c_i) = \mathrm{Var}(\mathbf{z}_i\boldsymbol{\gamma}) + \sigma_h^2 = \boldsymbol{\gamma}'\mathrm{Var}(\mathbf{z}_i)\boldsymbol{\gamma} + \sigma_h^2 \geq \sigma_h^2$. Assuming that $\mathrm{Var}(\mathbf{z}_i)$ is positive definite – which we must to satisfy the RE rank condition – strict inequality holds whenever $\boldsymbol{\gamma} \neq \mathbf{0}$. Of course $\boldsymbol{\gamma}'\mathrm{Var}(\mathbf{z}_i)\boldsymbol{\gamma} > 0$ is possible even if $\mathrm{Var}(\mathbf{z}_i)$ is not positive definite.

b. If we estimate the model by fixed effects, the associated estimate of the variance of the unobserved effect is σ_c^2. If we estimate the model by random effects (with, of course, $\mathbf{z}_i$ included), the variance component is σ_h^2. This makes intuitive sense: with random effects we are able to explicitly control for time-constant variances, and so the $\mathbf{z}_i$ are effectively taken out of c_i with h_i as the remainder.

c. Using equation (10.81), we obtain λ_h and λ_c as follows.

$$\lambda_h = 1 - \{1/[1 + T(\sigma_h^2/\sigma_u^2)]\}^{1/2}$$
$$\lambda_c = 1 - \{1/[1 + T(\sigma_c^2/\sigma_u^2)]\}^{1/2}$$

so

$$\lambda_c - \lambda_h = \{1/[1 + T(\sigma_h^2/\sigma_u^2)]\}^{1/2} - \{1/[1 + T(\sigma_c^2/\sigma_u^2)]\}^{1/2}.$$

Therefore, $\lambda_c - \lambda_h \geq 0$ iff $1/[1 + T(\sigma_h^2/\sigma_u^2)] \geq 1/[1 + T(\sigma_c^2/\sigma_u^2)]$ (because $1 + T(\sigma_h^2/\sigma_u^2)$ and $1 + T(\sigma_c^2/\sigma_u^2)$ are positive). We conclude that

$$\lambda_c - \lambda_h \geq 0 \text{ iff } T(\sigma_c^2/\sigma_u^2) \geq T(\sigma_h^2/\sigma_u^2)$$

which holds because we already showed that $\sigma_c^2 \geq \sigma_h^2$ (often with strict inequality).

d. If we use FE to estimate the heterogeneity variance then we are estimating σ_c^2, which means λ_c is effectively the quasi-time demeaning parameter used in subsequent RE estimation. If we use POLS, then we estimate σ_h^2, which then delivers the appropriate quasi-time demeaning parameter λ_h. Thus, we should use POLS, not FE, as the initial estimator for RE estimation when the model includes time-constant variables.

e. Because Problem 7.15 contains a more general result, a separate proof is not provided here. One need not have an RE structure and, as mentioned in the test, we do not need homoskedasticity of $\text{Var}(c_i|\mathbf{z}_i)$, either.

10.15. a. Because $\mathbf{v}_i$ is independent of $\mathbf{x}_i$, $\bar{v}_i$ is also independent of $\mathbf{x}_i$. Therefore,

$$\text{E}(v_{it}|\mathbf{x}_i, \overline{v}_i) = \text{E}(v_{it}|\overline{v}_i), t = 1, \ldots, T.$$

Because we assume linearity, we know $\text{E}(v_{it}|\bar{v}_i)$ is the linear projection (with intercept zero because $\text{E}(v_{it}) = 0$ for all t):

$$\mathrm{E}(v_{it}|\bar{v}_i) = \frac{\mathrm{Cov}(\bar{v}_i, v_{it})}{\mathrm{Var}(\overline{v}_i)} \cdot \bar{v}_i.$$

Now

$$\mathrm{Cov}(\bar{v}_i, v_{it}) = T^{-1}\mathrm{Cov}\left(\sum_{r=1}^{T} v_{ir}, v_{it}\right) = T^{-1}\left[\mathrm{Var}(v_{it}) + \sum_{r\neq t}^{T} \mathrm{Cov}(v_{ir}, v_{it})\right]$$
$$= T^{-1}[(\sigma_c^2 + \sigma_u^2) + (T-1)\sigma_c^2] = \sigma_c^2 + \sigma_u^2/T$$

Also, because $\bar{v}_i = c_i + \bar{u}_i$, and $\{u_{it} : t = 1, \ldots, T\}$ is serially uncorrelated with constant variance, and each u_{it} is uncorrelated with c_i,

$$\mathrm{Var}(\overline{v}_i) = \mathrm{Var}(c_i) + \mathrm{Var}(\bar{u}_i) = \sigma_c^2 + \sigma_u^2/T = \mathrm{Cov}(\overline{v}_i, v_{it})$$

We have shown that the slope in the population regression of v_{it} on $\bar{v}_i$ is unity, and so

$$\mathrm{E}(v_{it}|\bar{v}_i) = \bar{v}_i.$$

b. Because $\bar{y}_i = \bar{\mathbf{x}}_i\boldsymbol{\beta} + \bar{v}_i$, $\mathrm{E}(v_{it}|\mathbf{x}_i, \bar{v}_i) = \mathrm{E}(v_{it}|\mathbf{x}_i, \bar{y}_i)$, and so

$$\mathrm{E}(y_{it}|\mathbf{x}_i, \bar{y}_i) = \mathrm{E}(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it}|\mathbf{x}_i, \bar{y}_i) = \mathbf{x}_{it}\boldsymbol{\beta} + \mathrm{E}(v_{it}|\mathbf{x}_i, \bar{y}_i) = \mathbf{x}_{it}\boldsymbol{\beta} + \bar{v}_i$$
$$= \mathbf{x}_{it}\boldsymbol{\beta} + (\bar{y}_i - \bar{\mathbf{x}}_i\boldsymbol{\beta}).$$

c. We can rewrite equation in part b as follows.

$$y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + \bar{y}_i - \bar{\mathbf{x}}_i\boldsymbol{\beta} + r_{it}$$
$$\mathrm{E}(r_{it}|\mathbf{x}_i, \bar{y}_i) = 0$$

To impose the coefficient of unity on $\bar{y}_i$ and the common vector on $\mathbf{x}_{it}$ and $\bar{\mathbf{x}}_i$, write

$$y_{it} - \bar{y}_i = (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)\boldsymbol{\beta} + r_{it}, \; t = 1, \ldots, T,$$

which we can estimate consistently by POLS because $\mathrm{E}(r_{it}|\mathbf{x}_i) = 0$ for all t.

d. The RE estimator is not based on $\mathrm{E}(y_{it}|\mathbf{x}_i, \overline{y}_i) = \mathrm{E}(y_{it}|\mathbf{x}_{it}, \bar{y}_i, \bar{\mathbf{x}}_i)$, but on $\mathrm{E}(y_{it}|\mathbf{x}_i) = \mathrm{E}(y_{it}|\mathbf{x}_{it})$. We now see that the FE estimator can be given an interpretion as an

estimator that "controls for" $\bar{y}_i$, along with $\bar{\mathbf{x}}_i$ (even though it does not need to under the RE assumptions).

e. Under Assumptions RE1 to RE3 we can derive

$$\mathrm{L}(v_{it}|\mathbf{x}_i, \bar{v}_i) = \mathrm{L}(v_{it}|\bar{v}_i),\ t = 1, \ldots, T$$

without further assumptions. First, we know that the LP has the form

$$\mathrm{L}(v_{it}|\mathbf{x}_i, \bar{v}_i) = \mathbf{x}_i\boldsymbol{\psi}_t + \rho_t\bar{v}_i \equiv \mathbf{w}_i\boldsymbol{\delta}_t$$

where

$$\boldsymbol{\delta}_t = [\mathrm{E}(\mathbf{w}_i'\mathbf{w}_i)]^{-1}\mathrm{E}(\mathbf{w}_i'v_{it})$$

But $\mathrm{E}(\mathbf{w}_i'\mathbf{w}_i)$ is block diagonal because $\mathrm{E}(\mathbf{x}_i'\bar{v}_i) = \mathbf{0}$. Further, $\mathrm{E}(\mathbf{x}_i'v_{it}) = \mathbf{0}$, and so

$$\boldsymbol{\delta}_t = \begin{pmatrix} \mathbf{0} \\ \rho_t \end{pmatrix}$$

It is easy to see that

$$\rho_t = \frac{\mathrm{E}(\bar{v}_i v_{it})}{\mathrm{E}(\bar{v}_i^2)} = \frac{\mathrm{Cov}(\bar{v}_i, v_{it})}{\mathrm{Var}(\bar{v}_i)} = 1,$$

10.16. a. By independence between $(v_{i1}, v_{i2}, \ldots, v_{iT}, v_{i,T+1})$ and $(\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{iT}, \mathbf{x}_{i,T+1})$, it follows immediately that

$$\mathrm{E}(v_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}, v_{i1}, \ldots, v_{iT}) = \mathrm{E}(v_{i,T+1}|v_{i1}, \ldots, v_{iT}).$$

Because we are assuming that all conditional expectations involving $\{v_{it}\}$ are linear, we know $\mathrm{E}(v_{i,T+1}|v_{i1}, \ldots, v_{iT})$ is a linear function of $(v_{i1}, \ldots, v_{iT})$. The tricky part is to show that $\mathrm{E}(v_{i,T+1}|v_{i1}, \ldots, v_{iT}) = \mathrm{E}(v_{i,T+1}|\bar{v}_i)$. Intuitively it makes sense that the elements in $(v_{i1}, \ldots, v_{iT})$ should get equal weight under the RE variance-covariance structure. One way to verify that each element gets the same weight, and to determine that common weight, is to note that the

vector in the LP, say $\boldsymbol{\rho}_T$, satisfies

$$\begin{pmatrix} \sigma_c^2 + \sigma_u^2 & \sigma_c^2 & \cdots & \sigma_c^2 \\ \sigma_c^2 & \sigma_c^2 + \sigma_u^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma_c^2 \\ \sigma_c^2 & \cdots & \sigma_c^2 & \sigma_c^2 + \sigma_u^2 \end{pmatrix} \boldsymbol{\rho}_T = \begin{pmatrix} \sigma_c^2 \\ \sigma_c^2 \\ \vdots \\ \sigma_c^2 \end{pmatrix}.$$

If we hypothesis that $\boldsymbol{\rho}_T = \eta_T \mathbf{j}_T'$, where $\mathbf{j}_T$ is the $T \times 1$ vector of ones, then

$$(T\sigma_c^2 + \sigma_u^2)\eta_T = \sigma_c^2$$

so

$$\eta_T = \frac{\sigma_c^2}{(T\sigma_c^2 + \sigma_u^2)}$$

This must be the unique solution because the RE variance matrix is assumed to be nonsingular ($\sigma_u^2 > 0$). So we have shown

$$\begin{aligned} \mathrm{E}(v_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}, v_{i1}, \ldots, v_{iT}) &= \left[\frac{\sigma_c^2}{(T\sigma_c^2 + \sigma_u^2)} \right](v_{i1} + v_{i2} + \ldots + v_{iT}) \\ &= \left[\frac{T\sigma_c^2}{(T\sigma_c^2 + \sigma_u^2)} \right]\bar{v}_i \\ &= \left[\frac{\sigma_c^2}{(\sigma_c^2 + \sigma_u^2/T)} \right]\bar{v}_i \end{aligned}$$

which is what the problem asked to show.

b. This is straightforward given part a:

$$\begin{aligned} \mathrm{E}(y_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}, v_{i1}, \ldots, v_{iT}) &= \mathbf{x}_{i,T+1}\boldsymbol{\beta} + \mathrm{E}(v_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}, v_{i1}, \ldots, v_{iT}) \\ &= \mathbf{x}_{i,T+1}\boldsymbol{\beta} + \mathrm{E}(v_{i,T+1}|\bar{v}_i) = \mathbf{x}_{i,T+1}\boldsymbol{\beta} + \left[\frac{\sigma_c^2}{(\sigma_c^2 + \sigma_u^2/T)} \right]\bar{v}_i \end{aligned}$$

c. If we condition only on the history of covariates, and not the past composite errors, we get

$$\mathrm{E}(y_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}) = \mathbf{x}_{i,T+1}\boldsymbol{\beta} + \mathrm{E}(v_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}) = \mathbf{x}_{i,T+1}\boldsymbol{\beta} + 0 = \mathbf{x}_{i,T+1}\boldsymbol{\beta}$$

because we are assuming RE.1 for all $T + 1$ time periods.

d. Forecast errors are given by:

$$\begin{aligned} y_{i,T+1} - \mathrm{E}(y_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1}, v_{i1}, \ldots, v_{iT}) &= (\mathbf{x}_{i,T+1}\boldsymbol{\beta} + v_{i,T+1}) - \left\{\mathbf{x}_{i,T+1}\boldsymbol{\beta} - \left[\frac{\sigma_c^2}{(\sigma_c^2 + \sigma_u^2/T)}\right]\bar{v}_i\right\} \\ &= v_{i,T+1} - \left[\frac{\sigma_c^2}{(\sigma_c^2 + \sigma_u^2/T)}\right]\bar{v}_i \end{aligned}$$

Let $\theta = \sigma_c^2/(\sigma_c^2 + \sigma_u^2/T)$. Then the variance of the forecast error is

$$\begin{aligned} \mathrm{Var}(v_{i,T+1} - \theta\bar{v}_i) &= \mathrm{Var}(v_{i,T+1}) + \theta^2\mathrm{Var}(\bar{v}_i) - 2\theta\mathrm{Cov}(v_{i,T+1}, \bar{v}_i) \\ &= \sigma_c^2 + \sigma_u^2 - 2\theta(\sigma_c^2) + \theta^2(\sigma_c^2 + \sigma_u^2/T) \\ &= \sigma_c^2 + \sigma_u^2 + \frac{\sigma_c^2\sigma_c^2}{(\sigma_c^2 + \sigma_u^2/T)} - 2\frac{\sigma_c^2\sigma_c^2}{(\sigma_c^2 + \sigma_u^2/T)} \\ &= \sigma_c^2 + \sigma_u^2 - \sigma_c^2\theta. \end{aligned}$$

If we use $\mathrm{E}(y_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1})$ to forecast $y_{i,T+1}$ the forecast error is simply $v_{i,T+1} = y_{i,T+1} - \mathrm{E}(y_{i,T+1}|\mathbf{x}_i, \mathbf{x}_{i,T+1})$, which has variance $\sigma_c^2 + \sigma_u^2$. Because $\sigma_c^2 \geq 0$ and $\theta \geq 0$,

$$\mathrm{Var}(v_{i,T+1} - \theta\bar{v}_i) \leq \mathrm{Var}(v_{i,T+1})$$

with strict inequality when $\sigma_c^2 > 0$. Of course, all we have really shown is that $\mathrm{Var}[\mathrm{E}(v_{i,T+1}|\bar{v}_i)] \leq \mathrm{Var}(v_{i,T+1})$, which we already know from general properties of conditional expectations. Using more information in a conditional mean results in a smaller prediction error variance.

e. We can use N cross section observations and the first T time periods to estimate the parameters by random effects. Let $\hat{\boldsymbol{\beta}}_{RE}$ be the RE estimator, and let

$$\hat{\theta} = \frac{\hat{\sigma}_c^2}{(\hat{\sigma}_c^2 + \hat{\sigma}_u^2/T)}.$$

Then

$$\widehat{y}_{i,T+1} = \mathbf{x}_{i,T+1}\hat{\boldsymbol{\beta}}_{RE} + \hat{\theta}\widehat{\overline{v}}_i$$

where

$$\widehat{\overline{v}}_i = \frac{1}{T}\sum_{t=1}^{T}\hat{v}_{it} \text{ and } \hat{v}_{it} = y_{it} - \mathbf{x}_{it}\hat{\boldsymbol{\beta}}_{RE}.$$

10.17. a. By the usual averaging across time, the quasi-time-demeaned equation can be written, for each time period, as

$$\begin{aligned} y_{it} - \lambda\bar{y}_i &= (1-\lambda)\alpha + (\mathbf{d}_t - \lambda\overline{\mathbf{d}})\boldsymbol{\eta} + (\mathbf{w}_{it} - \lambda\overline{\mathbf{w}}_i)\boldsymbol{\delta} + (v_{it} - \lambda\bar{v}_i) \\ &= [(1-\lambda)\alpha + (1-\lambda)\overline{\mathbf{d}}\boldsymbol{\eta}] + (\mathbf{d}_t - \overline{\mathbf{d}})\boldsymbol{\eta} + (\mathbf{w}_{it} - \lambda\overline{\mathbf{w}}_i)\boldsymbol{\delta} + (v_{it} - \lambda\bar{v}) \end{aligned}$$

which is what we wanted to show by letting $\mu = (1-\lambda)\alpha + (1-\lambda)\overline{\mathbf{d}}\boldsymbol{\eta}$.

b. The first part – the $\sqrt{N}$-asymptotic representation of the RE estimator – is just the usual linear representation of a pooled OLS estimator laid out in Chapter 7. It also follows from the discussion of random effects in Section 10.7.2. For the second part, write $v_{it} - \lambda\bar{v}_i = (c_i + u_{it}) - (\lambda c_i + \lambda\bar{u}_i) = (1-\lambda)c_i + u_{it} - \lambda\bar{u}_i$, and plug in:

$$\begin{aligned} \sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}})(v_{it} - \lambda\bar{v}_i) &= \sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}})(1-\lambda)c_i + \sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}})u_{it} - \sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}})\lambda\bar{u}_i \\ &= (1-\lambda)c_i\sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}}) + \sum_{\mathbf{t=1}}^{\mathbf{T}}(\mathbf{d_t} - \overline{\mathbf{d}})u_{it} - (\lambda\bar{u}_i)\sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}}) \\ &= \sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}})u_{it} \end{aligned}$$

because $\sum_{t=1}^{T}(\mathbf{d}_t - \overline{\mathbf{d}}) = \mathbf{0}$.

c. Actually, there is nothing to do here. This is just the usual first-order representation of the fixed effects estimator, which follows from the general pooled OLS results.

d. From part b we can write

$$\mathbf{A}_1\sqrt{N}(\hat{\boldsymbol{\beta}}_{RE}-\boldsymbol{\beta})=\left(N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{g}_{it}'(v_{it}-\lambda\bar{v}_i)\right)+o_p(1)\equiv\left(N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{r}_{it}\right)+o_p(1),$$

where $\mathbf{r}_{it}=[(\mathbf{d}_t-\overline{\mathbf{d}})u_{it},(\mathbf{w}_{it}-\lambda\overline{\mathbf{w}}_i)(v_{it}-\lambda\bar{v}_i)]'$. From part c we can write

$$\mathbf{A}_2\sqrt{N}(\hat{\boldsymbol{\beta}}_{FE}-\boldsymbol{\beta})=\left(N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{h}_{it}'u_{it}\right)+o_p(1)\equiv\left(N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{s}_{it}\right)+o_p(1),$$

where $\mathbf{s}_{it}=[(\mathbf{d}_t-\overline{\mathbf{d}})u_{it},(\mathbf{w}_{it}-\overline{\mathbf{w}}_i)u_{it}]'$. But the first R elements of $\mathbf{r}_{it}$ and $\mathbf{s}_{it}$ are $(\mathbf{d}_t-\overline{\mathbf{d}})'u_{it}$, which implies that

$$\mathbf{A}_1\sqrt{N}(\hat{\boldsymbol{\beta}}_{RE}-\boldsymbol{\beta})-\mathbf{A}_2\sqrt{N}(\hat{\boldsymbol{\beta}}_{FE}-\boldsymbol{\beta})=N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\begin{pmatrix}\mathbf{0}\\(\mathbf{w}_{it}-\lambda\overline{\mathbf{w}}_i)'e_{it}-(\mathbf{w}_{it}-\overline{\mathbf{w}}_i)'u_{it}\end{pmatrix}+o_p(1),$$

where "$\mathbf{0}$" is $R\times 1$ and $e_{it}=v_{it}-\lambda v_{it}$. The second part of the vector is $M\times 1$ and generally satisfies the central limit theorem. Under standard rank assumptions

$\text{Var}[(\mathbf{w}_{it}-\lambda\overline{\mathbf{w}}_i)'e_{it}-(\mathbf{w}_{it}-\overline{\mathbf{w}}_i)'u_{it}]$ has rank M

e. If there were no $\mathbf{w}_{it}$, part d would imply that the limiting distribution of the difference between RE and FE is degenerate. In other words, we cannot compute the Hausman test comparing the FE and RE estimators if the only time-varying covariates are aggregates. (In fact, the FE and RE estimates are numerically identical in this case.) More generally, the variance-covariance matrix of the difference has rank M, not $M+R$ (whether or not we assume RE.3 under H_0). A properly computed Hausman test will have only M degrees-of-freedom, not $M+R$. The regression-based test from equation (10.88) forces one to get the degrees-of-freedom corret, as there is obviously no value in adding $\overline{\mathbf{d}}$, a vector of constants, to the regression.

10.18. a. The Stata results are given below. All estimates are numerically identical.

```
. reg lwage d81-d87

      Source |       SS       df       MS              Number of obs =    4360
-------------+------------------------------           F(  7,  4352) =   50.54
       Model |  92.9668229     7  13.2809747            Prob > F      =  0.0000
    Residual |  1143.56282  4352  .262767192           R-squared     =  0.0752
-------------+------------------------------           Adj R-squared =  0.0737
       Total |  1236.52964  4359  .283672779           Root MSE      =  .51261

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d81 |   .1193902   .0310529     3.84   0.000     .0585107    .1802697
         d82 |   .1781901   .0310529     5.74   0.000     .1173106    .2390696
         d83 |   .2257865   .0310529     7.27   0.000     .1649069     .286666
         d84 |   .2968181   .0310529     9.56   0.000     .2359386    .3576976
         d85 |   .3459333   .0310529    11.14   0.000     .2850538    .4068128
         d86 |   .4062418   .0310529    13.08   0.000     .3453623    .4671213
         d87 |   .4730023   .0310529    15.23   0.000     .4121228    .5338818
       _cons |   1.393477   .0219577    63.46   0.000     1.350429    1.436525
------------------------------------------------------------------------------

. xtreg lwage d81-d87, re

Random-effects GLS regression                   Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.0000                         Obs per group: min =
       between = 0.0000                                        avg =       8.
       overall = 0.0752                                        max =

Random effects u_i ~Gaussian                    Wald chi2(7)       =    738.94
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d81 |   .1193902    .021487     5.56   0.000     .0772765    .1615039
         d82 |   .1781901    .021487     8.29   0.000     .1360764    .2203038
         d83 |   .2257865    .021487    10.51   0.000     .1836728    .2679001
         d84 |   .2968181    .021487    13.81   0.000     .2547044    .3389318
         d85 |   .3459333    .021487    16.10   0.000     .3038196     .388047
         d86 |   .4062418    .021487    18.91   0.000     .3641281    .4483555
         d87 |   .4730023    .021487    22.01   0.000     .4308886     .515116
       _cons |   1.393477   .0219577    63.46   0.000     1.350441    1.436513
-------------+----------------------------------------------------------------
     sigma_u |  .37007665
     sigma_e |  .35469771
         rho |  .52120938   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lwage d81-d87, fe

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545
```

```
R-sq:  within  = 0.1625                         Obs per group: min =
       between =      .                                       avg =       8.
       overall = 0.0752                                       max =

                                                F(7,3808)          =    105.56
corr(u_i, Xb)  = 0.0000                         Prob > F           =    0.0000

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d81 |   .1193902    .021487     5.56   0.000     .0772631    .1615173
         d82 |   .1781901    .021487     8.29   0.000      .136063    .2203172
         d83 |   .2257865    .021487    10.51   0.000     .1836594    .2679135
         d84 |   .2968181    .021487    13.81   0.000      .254691    .3389452
         d85 |   .3459333    .021487    16.10   0.000     .3038063    .3880604
         d86 |   .4062418    .021487    18.91   0.000     .3641147    .4483688
         d87 |   .4730023    .021487    22.01   0.000     .4308753    .5151294
       _cons |   1.393477   .0151936    91.71   0.000     1.363689    1.423265
-------------+----------------------------------------------------------------
     sigma_u |  .39074676
     sigma_e |  .35469771
         rho |  .54824631   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(544, 3808) =     9.71           Prob > F = 0.0000

. reg d.(lwage d81-d87), nocons

      Source |       SS       df       MS              Number of obs =    3815
-------------+------------------------------           F(  7,  3808) =   14.06
       Model |  19.3631642     7  2.76616631           Prob > F      =  0.0000
    Residual |  749.249837  3808  .196756785           R-squared     =  0.0252
-------------+------------------------------           Adj R-squared =  0.0234
       Total |  768.613001  3815  .201471298           Root MSE      =  .44357

------------------------------------------------------------------------------
     D.lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d81 |
         D1. |   .1193902   .0190006     6.28   0.000     .0821379    .1566425
             |
         d82 |
         D1. |   .1781901   .0268709     6.63   0.000     .1255074    .2308728
             |
         d83 |
         D1. |   .2257865     .03291     6.86   0.000     .1612636    .2903093
             |
         d84 |
         D1. |   .2968181   .0380011     7.81   0.000     .2223136    .3713226
             |
         d85 |
         D1. |   .3459333   .0424866     8.14   0.000     .2626347    .4292319
             |
         d86 |
         D1. |   .4062418   .0465417     8.73   0.000     .3149927    .4974908
             |
         d87 |
         D1. |   .4730023   .0502708     9.41   0.000     .3744421    .5715626
------------------------------------------------------------------------------
```

b. The Stata output follows. The POLS and RE estimates are identical on the year dummies and the three time-constant variables. This is a general result: if the model includes only aggregate time effects and individual-specific covariates that have no time variation, POLS = RE (and, in particular, there is no efficiency gain in using RE).

When FE is used, of course the time-constant variables drop out. The estimates on the year dummies are the same as POLS and RE. (Recall that the "constant" reported by FE is the average of the estimated heterogeneity terms. When POLS and RE include time-constant variables the FE "constant" does not equal the intercept from POLS/RE.)

```
. reg lwage d81-d87 educ black hisp

      Source |       SS       df       MS              Number of obs =    4360
-------------+------------------------------           F( 10,  4349) =   73.66
       Model |  179.091659    10  17.9091659           Prob > F      =  0.0000
    Residual |  1057.43798  4349  .243145087           R-squared     =  0.1448
-------------+------------------------------           Adj R-squared =  0.1429
       Total |  1236.52964  4359  .283672779           Root MSE      =   .4931

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d81 |   .1193902    .029871     4.00   0.000     .0608279    .1779526
         d82 |   .1781901    .029871     5.97   0.000     .1196277    .2367524
         d83 |   .2257865    .029871     7.56   0.000     .1672241    .2843488
         d84 |   .2968181    .029871     9.94   0.000     .2382557    .3553804
         d85 |   .3459333    .029871    11.58   0.000      .287371    .4044957
         d86 |   .4062418    .029871    13.60   0.000     .3476794    .4648041
         d87 |   .4730023    .029871    15.83   0.000       .41444    .5315647
        educ |   .0770943   .0043766    17.62   0.000     .0685139    .0856747
       black |  -.1225637   .0237021    -5.17   0.000    -.1690319   -.0760955
        hisp |    .024623   .0213056     1.16   0.248    -.0171468    .0663928
       _cons |   .4966384   .0566686     8.76   0.000     .3855391    .6077377
------------------------------------------------------------------------------

. xtreg lwage d81-d87 educ black hisp, re

Random-effects GLS regression                   Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1625                         Obs per group: min =
       between = 0.1296                                        avg =       8.
       overall = 0.1448                                        max =

Random effects u_i ~Gaussian                    Wald chi2(10)      =    819.51
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000
```

```
------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d81 |   .1193902    .021487     5.56   0.000     .0772765    .1615039
         d82 |   .1781901    .021487     8.29   0.000     .1360764    .2203038
         d83 |   .2257865    .021487    10.51   0.000     .1836728    .2679001
         d84 |   .2968181    .021487    13.81   0.000     .2547044    .3389318
         d85 |   .3459333    .021487    16.10   0.000     .3038196     .388047
         d86 |   .4062418    .021487    18.91   0.000     .3641281    .4483555
         d87 |   .4730023    .021487    22.01   0.000     .4308886     .515116
        educ |   .0770943    .009177     8.40   0.000     .0591076    .0950809
       black |  -.1225637   .0496994    -2.47   0.014    -.2199728   -.0251546
        hisp |    .024623   .0446744     0.55   0.582    -.0629371    .1121831
       _cons |   .4966384   .1122718     4.42   0.000     .2765897    .7166871
-------------+----------------------------------------------------------------
     sigma_u |  .34337144
     sigma_e |  .35469771
         rho |  .48377912   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lwage d81-d87 educ black hisp, fe
note: educ omitted because of collinearity
note: black omitted because of collinearity
note: hisp omitted because of collinearity

Fixed-effects (within) regression               Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1625                         Obs per group: min =
       between =      .                                        avg =       8.
       overall = 0.0752                                        max =

                                                F(7,3808)          =    105.56
corr(u_i, Xb)  = 0.0000                         Prob > F           =    0.0000

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d81 |   .1193902    .021487     5.56   0.000     .0772631    .1615173
         d82 |   .1781901    .021487     8.29   0.000      .136063    .2203172
         d83 |   .2257865    .021487    10.51   0.000     .1836594    .2679135
         d84 |   .2968181    .021487    13.81   0.000      .254691    .3389452
         d85 |   .3459333    .021487    16.10   0.000     .3038063    .3880604
         d86 |   .4062418    .021487    18.91   0.000     .3641147    .4483688
         d87 |   .4730023    .021487    22.01   0.000     .4308753    .5151294
        educ |  (omitted)
       black |  (omitted)
        hisp |  (omitted)
       _cons |   1.393477   .0151936    91.71   0.000     1.363689    1.423265
-------------+----------------------------------------------------------------
     sigma_u |  .39074676
     sigma_e |  .35469771
         rho |  .54824631   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(544, 3808) =     8.45           Prob > F = 0.0000
```

c. The reported standard errors for POLS and RE are not the same. The POLS standard errors assume, in addition to homoskedasticity, no serial correlation in the composite error – in other words, that there is no unobserved heterogeneity. At least the RE standard errors allow for the standard RE structure, which means constant variance and correlations that are the same across all pairs (t,s). This may be too restrictive, but it is less restrictive than the usual OLS standard errors.

d. The fully robust POLS standard errors – that allow any kind of serial correlation and heteroskedasticity – are reported below. We prefer these to the usual RE standard errors because, as noted in part c, the usual RE standard errors impose a special kind of serial correlation. Notice that the fully robust POLS standard errors are not uniformly larger than the usual RE standard errors.

```
. reg lwage d81-d87 educ black hisp, cluster(nr)

Linear regression                                      Number of obs =    4360
                                                       F( 10,   544) =   49.41
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.1448
                                                       Root MSE      =   .4931

                                    (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         d81 |   .1193902   .0244086     4.89   0.000     .0714435    .1673369
         d82 |   .1781901   .0241987     7.36   0.000     .1306558    .2257243
         d83 |   .2257865   .0243796     9.26   0.000     .1778968    .2736761
         d84 |   .2968181   .0271485    10.93   0.000     .2434894    .3501468
         d85 |   .3459333   .0263181    13.14   0.000     .2942358    .3976309
         d86 |   .4062418   .0273064    14.88   0.000     .3526029    .4598807
         d87 |   .4730023    .025996    18.20   0.000     .4219374    .5240672
        educ |   .0770943   .0090198     8.55   0.000     .0593763    .0948122
       black |  -.1225637   .0532662    -2.30   0.022    -.2271964    -.017931
        hisp |    .024623   .0411235     0.60   0.550    -.0561573    .1054033
       _cons |   .4966384   .1097474     4.53   0.000     .2810579    .7122189
------------------------------------------------------------------------------
```

e. The fully robust standard errors for RE are given below. They are numerically identical

to the fully robust POLS standard errors. Because we really have only one estimator – remeber, POLS = RE in this setup – there is one asymptotic variance. While there could be different ways to estimate that asymptotic variance, in this case the estimators are the same, and that is appealing because it means inference does not rely on the particular pre-programmed command.

```
. xtreg lwage d81-d87 educ black hisp, re cluster(nr)

Random-effects GLS regression                   Number of obs      =      4360
Group variable: nr                              Number of groups   =       545

R-sq:  within  = 0.1625                         Obs per group: min =
       between = 0.1296                                        avg =       8.
       overall = 0.1448                                        max =

Random effects u_i ~Gaussian                    Wald chi2(10)      =    494.13
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                                    (Std. Err. adjusted for 545 clusters in nr
------------------------------------------------------------------------------
             |               Robust
       lwage |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d81 |   .1193902   .0244086     4.89   0.000     .0715502    .1672302
         d82 |   .1781901   .0241987     7.36   0.000     .1307616    .2256186
         d83 |   .2257865   .0243796     9.26   0.000     .1780033    .2735696
         d84 |   .2968181   .0271485    10.93   0.000     .2436081    .3500281
         d85 |   .3459333   .0263181    13.14   0.000     .2943508    .3975159
         d86 |   .4062418   .0273064    14.88   0.000     .3527222    .4597613
         d87 |   .4730023    .025996    18.20   0.000      .422051    .5239536
        educ |   .0770943   .0090198     8.55   0.000     .0594157    .0947728
       black |  -.1225637   .0532662    -2.30   0.021    -.2269636   -.0181638
        hisp |    .024623   .0411235     0.60   0.549    -.0559775    .1052236
       _cons |   .4966384   .1097474     4.53   0.000     .2815375    .7117392
-------------+----------------------------------------------------------------
     sigma_u |  .34337144
     sigma_e |  .35469771
         rho |  .48377912   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

Solutions to Chapter 11 Problems

11.1. a. It is important to remember that, any time we put a variable in a regression model (whether we are using cross section or panel data), we are controlling for the effects of that variable on the dependent variable. The whole point of regression analysis is that it allows the explanatory variables to be correlated while estimating ceteris paribus effect of each explanatory variable. Thus, the inclusion of $y_{i,t-1}$ in the equation allows $prog_{it}$ to be correlated with $y_{i,t-1}$, and also recognizes that, due to inertia, y_{it} is often strongly related to $y_{i,t-1}$.

An assumption that implies pooled OLS is consistent is

$$\mathrm{E}(u_{it}|\mathbf{z}_i, \mathbf{x}_{it}, y_{i,t-1}, prog_{it}) = 0, \text{ all } t,$$

which is implied by but is weaker than dynamic completeness. Without additional assumptions, the pooled OLS standard errors and test statistics need to be adjusted for heteroskedasticity and serial correlation (although the latter will not be present under dynamic completeness).

When $y_{i,t-1}$ is added to a regression model in an astructural way, we can think of the goal as being to estimate

$$\mathrm{E}(y_{it}|\mathbf{z}_i, \mathbf{x}_{it}, y_{i,t-1}, prog_{it}),$$

which means that we are controlling for differences in the lagged response when gauging the effect of the program. Of course, we might not have the conditional mean correctly specified; we may be simply estimating a linear projection.

b. As we discussed in Section 7.8.2, this statement is incorrect. Provided our interest is in $\mathrm{E}(y_{it}|\mathbf{z}_i, \mathbf{x}_{it}, y_{i,t-1}, prog_{it})$, we are not especially concerned about serial correlation in the implied errors,

$$u_{it} \equiv y_{it} - \mathrm{E}(y_{it}|\mathbf{z}_i, \mathbf{x}_{it}, y_{i,t-1}, prog_{it}).$$

Nor does serial correlation cause inconsistency in the OLS estimators.

c. Such a model is the standard unobserved effects model:

$$y_{it} = \theta_t + \mathbf{x}_{it}\boldsymbol{\beta} + \delta_1 prog_{it} + c_i + u_{it}, \; t = 1, 2, \ldots, T,$$

where the θ_t are the time effects (that can be treated as parameters). We would probably assume that $(\mathbf{x}_{it}, prog_{it})$ is strictly exogenous; the weakest form of strict exogeneity is that $(\mathbf{x}_{it}, prog_{it})$ is uncorrelated with u_{is} for all t and s. Then we could estimate the equation by fixed effects or first differencing. If the u_{it} are serially uncorrelated, FE is preferred. We could also do a GLS analysis after the fixed effects or first-differencing transformations, but we should have a large N.

d. A model that incorporates features from parts a and c is

$$y_{it} = \theta_t + \mathbf{x}_{it}\beta + \delta_1 prog_{it} + \rho_1 y_{i,t-1} + c_i + u_{it}, \; t = 1, \ldots, T.$$

Now, program participation can depend on unobserved city heterogeneity as well as on lagged y_{it} (we assume that y_{i0} is observed). Fixed effects and first-differencing are both inconsistent and $N \to \infty$ with fixed T.

Assuming that $\mathrm{E}(u_{it}|\mathbf{x}_i, prog_i, y_{i,t-1}, y_{i,t-2}, \ldots, y_{i0}) = 0$, a consistent procedure is obtained by first differencing, to get

$$y_{it} = \Delta\mathbf{x}_{it}\beta + \delta_1 \Delta prog_{it} + \rho_1 \Delta y_{i,t-1} + \Delta u_{it}, \; t = 2, \ldots, T.$$

At time t and $\Delta\mathbf{x}_{it}, \Delta prog_{it}$ can be used as their own instruments, along with $y_{i,t-j}$ for $j \geq 2$. Either pooled 2SLS or a GMM procedure can be used. Past and future values of $\mathbf{x}_{it}$ can also be used as instruments because $\{\mathbf{x}_{it}\}$ is strictly exogenous.

11.2. a. OLS estimation on the first-differenced equation is inconsistent (for all parameters)

if $\text{Cov}(\Delta w_i, \Delta u_i) \neq 0$. Because w_{it} is correlated with u_{it}, for all t we cannot assume that Δw_i and Δu_i are uncorrelated.

b. Because u_{it} is uncorrelated with $\mathbf{z}_{i1}, \mathbf{z}_{i2}$, for $t = 1, 2$, $\Delta u_i = u_{i2} - u_{i1}$ is uncorrelated with $\mathbf{z}_{i1}$ and $\mathbf{z}_{i2}$, and so $(\mathbf{z}_{i1}, \mathbf{z}_{i2})$ are exogenous in the equation

$$\Delta y_i = \Delta \mathbf{z}_i \boldsymbol{\gamma} + \delta \Delta w_i + \Delta u_i$$

The linear projection of Δw_1 on $(\mathbf{z}_{i1}, \mathbf{z}_{i2})$ can be written as

$$\Delta w_i = \mathbf{z}_{i1} \boldsymbol{\pi}_1 + \mathbf{z}_{i2} \boldsymbol{\pi}_2 + r_i, \ \text{E}(\mathbf{z}_{it}' r_i) = 0, \ t = 1, 2.$$

The question is whether the rank condition holds. Rewrite this linear projection in terms of $\Delta \mathbf{z}_i$ and, say, $\mathbf{z}_{i1}$ as

$$\Delta w_i = \mathbf{z}_{i1}(\boldsymbol{\pi}_1 - \boldsymbol{\pi}_2) + (\mathbf{z}_{i2} - \mathbf{z}_{i1})\boldsymbol{\pi}_2 + r_i = \mathbf{z}_{i1} \boldsymbol{\lambda}_1 + \Delta \mathbf{z}_i \boldsymbol{\pi}_2 + r_i,$$

where $\boldsymbol{\lambda}_1 \equiv \boldsymbol{\pi}_1 - \boldsymbol{\pi}_2$. If $\boldsymbol{\lambda}_1 = 0$, that is $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2$, then the reduced from of Δw_i depends only on $\Delta \mathbf{z}_i$. Because $\Delta \mathbf{z}_i$ appears in the equation for Δy_i, there are no instruments for Δw_i. Thus, the change in w_{it} must depend on the level of $\mathbf{z}_{it}$, and not just on the change in $\mathbf{z}_{it}$.

c. With $T \geq 2$ time periods we can write the differenced equation as

$$\Delta y_{it} = \Delta \mathbf{z}_{it} \boldsymbol{\delta} + \gamma \Delta w_{it} + \Delta u_{it}, \ t = 2, \ldots, T.$$

Now, under the assumption that w_{is} is uncorrelated with u_{it} for $s < t$, we have natural instruments for Δw_{it}. At time t, Δu_{it} depends on u_{it} and $u_{1,t-1}$. Thus, valid instruments at time t in the FD equation are $w_{i,t-2}, \ldots, w_{i1}$. We need $T \geq 3$ for an IV procedure to work. With $T = 3$ we have the cross sectional equation

$$\Delta y_{i3} = \Delta \mathbf{z}_{i3} \boldsymbol{\delta} + \gamma \Delta w_{i3} + \Delta u_{i3}$$

and we can instrument for Δw_{i3} with w_{i1} (and possibly $\mathbf{z}_{ir}$ from earlier time periods).

With $T \geq 4$, we can implement an IV estimator by using the simple pooled IV estimator described Section 11.4. Or, we can use the more efficient GMM procedure. Write the $T-1$ time periods as

$$\Delta \mathbf{y}_i = \Delta \mathbf{Z}_i \boldsymbol{\beta} + \gamma \Delta w_i + \Delta u_i,$$

where each data vector or matrix has $T-1$ rows. The matrix that includes all possible instruments of observation i (with $T-1$ rows) is

$$\begin{pmatrix} \mathbf{z}_i & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 & 0 & \cdots & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{z}_i & w_{i1} & \mathbf{0} & \cdots & 0 & 0 & \cdots & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{z}_i & w_{i2} & w_{i1} & 0 & \cdots & \mathbf{0} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & \mathbf{z}_i & w_{i,T-2} \cdots & w_{il} \end{pmatrix}.$$

Putting in the levels of all $\mathbf{z}_{it}$ for instruments in each time period is perhaps using too many overidentifying restrictions. The dimension could be reduced substantially by using only $(\mathbf{z}_{it}, \mathbf{z}_{i,t-1})$ at period t rather than $\mathbf{z}_i$. Further, periods for $t \geq 3$ one would use only $w_{i,t-2}$ and $w_{i,t-3}$ as the IVs.

d. Generally, the IV estimator applied to the time-demeaned equation is inconsistent. This is because $w_{i,t-j}$ is generally correlated with $\ddot{u}_{it}$, as the latter depends on the idiosyncratic errors in all time periods.

11.3. Writing $y_{it} = \beta x_{it} + c_i + u_{it} - \beta r_{it}$, the fixed effects estimator $\hat{\beta}_{FE}$ can be written as

$$\beta + \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_i) \right)^2 \left(N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x}_i)(u_{it} - \bar{u}_i - \beta(r_{it} - \bar{r}_i) \right).$$

Now $x_{it} - \bar{x}_i = (x_{it}^* - \bar{x}_i^*) + (r_{it} - \bar{r}_i)$. Then, because $\mathrm{E}(r_{it} | \mathbf{x}_i^*, c_i) = 0$ for all t, $(x_{it}^* - \bar{x}_i^*)$ and $(r_{it} - \bar{r}_i)$ are uncorrelated, and so

$$\text{Var}(x_{it} - \bar{x}_i) = \text{Var}(x_{it}^* - \bar{x}_i^*) + \text{Var}(r_{it} - \bar{r}_i), \text{ all } t.$$

Similarly, under (11.42), $(x_{it} - \bar{x}_i)$ and $(u_{it} - \bar{u}_i)$ are uncorrelated for all t. Now $\text{E}[(x_{it} - \bar{x}_i)(r_{it} - \bar{r}_i)] = \text{E}[\{(x_{it}^* - \bar{x}_i^*) + (r_{it} - \bar{r}_i)\}(r_{it} - \bar{r}_i)] = \text{Var}(r_{it} - \bar{r}_i)$. By the law of large numbers and the assumption of constant variances across t,

$$N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it} - \bar{x}_i) \xrightarrow{p} \sum_{t=1}^{T}\text{Var}(x_{it} - \bar{x}_i) = T[\text{Var}(x_{it}^* - \bar{x}_i^*) + \text{Var}(r_{it} - \bar{r}_i)]$$

and

$$N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(x_{it} - \bar{x}_i)[u_{it} - \bar{u}_i - \beta(r_{it} - \bar{r}_i)] \xrightarrow{p} -T\beta \cdot \text{Var}(r_{it} - \bar{r}_i).$$

Therefore,

$$\text{plim}\hat{\beta}_{FE} = \beta - \beta\left(\frac{\text{Var}(r_{it} - \bar{r}_i)}{[\text{Var}(x_{it}^* - \bar{x}_i^*) + \text{Var}(r_{it} - \bar{r}_i)]}\right) = \beta\left(1 - \frac{\text{Var}(r_{it} - \bar{r}_i)}{[\text{Var}(x_{it}^* - \bar{x}_i^*) + \text{Var}(r_{it} - \bar{r}_i)]}\right).$$

11.4. a. For each i we can average across t and rearrange to get

$$c_i = \bar{y}_i - \bar{\mathbf{x}}_i\boldsymbol{\beta} - \bar{u}_i.$$

Because $\text{E}(\bar{u}_i) = 0$, $\mu_c \equiv \text{E}(c_i) = \text{E}(\bar{y}_i - \bar{\mathbf{x}}_i\boldsymbol{\beta})$. By the law of large numbers,

$$N^{-1}\sum_{i=1}^{N} c_i = N^{-1}\sum_{i=1}^{N}(\bar{y}_i - \bar{\mathbf{x}}_i\boldsymbol{\beta}) \xrightarrow{p} \mu_c.$$

Now replace $\boldsymbol{\beta}$ with $\hat{\boldsymbol{\beta}}_{FE}$ and call the estimator $\hat{\mu}_c$:

$$\begin{aligned}\hat{\mu}_c &= N^{-1}\sum_{i=1}^{N}(\bar{y}_i - \bar{\mathbf{x}}_i\hat{\boldsymbol{\beta}}_{FE}) = N^{-1}\sum_{i=1}^{N}(\bar{y}_i - \bar{\mathbf{x}}_i\boldsymbol{\beta}) - \left(N^{-1}\sum_{i=1}^{N}\bar{\mathbf{x}}_i\right)(\hat{\boldsymbol{\beta}}_{FE} - \boldsymbol{\beta}) \\ &= N^{-1}\sum_{i=1}^{N} c_i + O_p(1) \cdot o_p(1) = N^{-1}\sum_{i=1}^{N} c_i + o_p(1) \xrightarrow{p} \mu_c,\end{aligned}$$

where we use $N^{-1}\sum_{i=1}^{N}\ddot{\mathbf{x}}_i = O_p(1)$ (by the law of large numbers – see Lemma 3.2) and $\hat{\boldsymbol{\beta}}_{FE} - \boldsymbol{\beta} = o_p(1)$.

b. There is more than one way to estimate μ_g, but a simple approach is to first difference, giving

$$\Delta y_{it} = g_i + \mathbf{x}_{it}\boldsymbol{\beta} + \Delta u_{it}, t = 2, \ldots, T.$$

Then we can estimate $\boldsymbol{\beta}$ by fixed effects on the first differences (using $t = 2, \ldots, T$), and then apply the estimator from part a to the first differenced data. This means we just replace $\ddot{y}_i$ with $\overline{\Delta y}_i$ and $\ddot{\mathbf{x}}_i$ with $\overline{\Delta \mathbf{x}}_i$ everywhere (and the time averages are based on $T - 1$, not T, time periods).

11.5. a. $\mathrm{E}(\mathbf{v}_i|\mathbf{z}_i, \mathbf{x}_i) = \mathbf{Z}_i[\mathrm{E}(\mathbf{a}_i|\mathbf{z}_i, \mathbf{x}_i) - \boldsymbol{\alpha}] + \mathrm{E}(\mathbf{u}_i|\mathbf{z}_i, \mathbf{x}_i) = \mathbf{Z}_i(\boldsymbol{\alpha} - \boldsymbol{\alpha}) + \mathbf{0} = \mathbf{0}$. Next,

$$\begin{aligned}\mathrm{Var}(\mathbf{v}_i|\mathbf{z}_i, \mathbf{x}_i) &= \mathbf{Z}_i\mathrm{Var}(\mathbf{a}_i|\mathbf{z}_i, \mathbf{x}_i)\mathbf{Z}_i' + \mathrm{Var}(\mathbf{u}_i|\mathbf{z}_i, \mathbf{x}_i) + \mathrm{Cov}(\mathbf{a}_i, \mathbf{u}_i|\mathbf{z}_i, \mathbf{x}_i) + \mathrm{Cov}(\mathbf{u}_i, \mathbf{a}_i|\mathbf{z}_i, \mathbf{x}_i) \\ &= \mathbf{Z}_i\mathrm{Var}(a_i|\mathbf{z}_i, \mathbf{x}_i)\mathbf{Z}_i' + \mathrm{Var}(\mathbf{u}_i|\mathbf{z}_i, \mathbf{x}_i)\end{aligned}$$

because $\mathbf{a}_i$ and $\mathbf{u}_i$ are uncorrelated, conditional on $(\mathbf{z}_i, \mathbf{x}_i)$, by Assumption FE.1′ and the usual iterated expectations argument,

$$\mathrm{Var}(\mathbf{v}_i|\mathbf{z}_i, \mathbf{x}_i) = \mathbf{Z}_i\boldsymbol{\Lambda}\mathbf{Z}_i' + \sigma_u^2\mathbf{I}_T.$$

Therefore, under the assumptions given, which shows that the conditional variance depends on $\mathbf{z}_i$. Unlike in the standard random effects model, there is conditional heteroskedasticity.

b. If we use the usual RE analysis, we are applying FGLS to the equation $\mathbf{y}_i = \mathbf{Z}_i\boldsymbol{\alpha} + \mathbf{X}_i\boldsymbol{\beta} + \mathbf{v}_i$, where $\mathbf{v}_i = \mathbf{Z}_i(\mathbf{a}_i - \boldsymbol{\alpha}) + \mathbf{u}_i$. From part a, we know that $\mathrm{E}(\mathbf{v}_i|\mathbf{x}_i, \mathbf{z}_i) = \mathbf{0}$, and so the usual RE estimator is consistent (as $N \to \infty$ for fixed T) and $\sqrt{N}$-asymptotically normal, provided the rank condition, Assumption RE.2, holds. (Remember, a feasible GLS analysis with any $\hat{\boldsymbol{\Omega}}$ will be consistent provided $\hat{\boldsymbol{\Omega}}$ converges in probability to a nonsingular matrix as $N \to \infty$. It need not be the case that $\mathrm{Var}(\mathbf{v}_i|\mathbf{x}_i, \mathbf{z}_i) = \mathrm{plim}(\hat{\boldsymbol{\Omega}})$, or even that

$\text{Var}(\mathbf{v}_i) = \text{plim}(\hat{\boldsymbol{\Omega}})$.

From part a, we know that $\text{Var}(\mathbf{v}_i|\mathbf{x}_i, \mathbf{z}_i)$ depends on $\mathbf{z}_i$ unless we restrict almost all elements of $\boldsymbol{\Lambda}$ to be zero (all but those corresponding to the constant in $\mathbf{z}_{it}$). Therefore, the usual random effects inference – that is, based on the usual RE variance matrix estimator – will be invalid.

c. We can easily make the RE analysis fully robust to an arbitrary $\text{Var}(\mathbf{v}_i|\mathbf{x}_i, \mathbf{z}_i)$, as in equation (7.52). Naturally, we expand the set of explanatory variables to $(\mathbf{z}_{it}, \mathbf{x}_{it})$, and we estimate $\boldsymbol{\alpha}$ along with $\boldsymbol{\beta}$.

11.6. No. Assumption (11.42) maintains strict exogeneity of $\{w_{it}^*\}$ in (11.41), and strict exogeneity clearly fails when $w_{it}^* = y_{i,t-1}^*$.

11.7. When $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}/T$ for all t, we can rearrange (11.6) to get

$$y_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\lambda} + r_{it}, t = 1, 2, \ldots, T.$$

Let $\hat{\boldsymbol{\beta}}$ (along with $\hat{\boldsymbol{\lambda}}$) denote the pooled OLS estimator from this equation. By standard results on partitioned regression [for example, Davidson and MacKinnon (1993, Section 1.4)], $\hat{\boldsymbol{\beta}}$ can be obtained by the following two-step procedure:

(i) Regress $\mathbf{x}_{it}$ on $\bar{\mathbf{x}}_i$ across all t and i, and save the $1 \times K$ vectors of residuals, say $\hat{\mathbf{g}}_{it}$, $t = 1, \ldots, T;\ i = 1, \ldots, N$.

(ii) Regress y_{it} on $\hat{\mathbf{g}}_{it}$ across all t and i. The OLS vector on $\hat{\mathbf{g}}_{it}$ is $\hat{\boldsymbol{\beta}}$.

We want to show that $\hat{\boldsymbol{\beta}}$ is the FE estimator. Given that the FE estimator can be obtained by pooled OLS of y_{it} on $(\mathbf{x}_{it} - \bar{\mathbf{x}}_i)$, it suffices to show that $\hat{\mathbf{g}}_{it} = \mathbf{x}_{it} - \bar{\mathbf{x}}_i$ for all t and i. But,

$$\hat{\mathbf{g}}_{it} = \mathbf{x}_{it} - \bar{\mathbf{x}}_i\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{\mathbf{x}}_i'\bar{\mathbf{x}}_i\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{\mathbf{x}}_i'\mathbf{x}_{it}\right)$$

and

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{\mathbf{x}}_i'\bar{\mathbf{x}}_{it} = \sum_{i=1}^{N}\bar{\mathbf{x}}_i'\sum_{t=1}^{T}\mathbf{x}_{it} = \sum_{i=1}^{N}T\bar{\mathbf{x}}_i'\bar{\mathbf{x}}_{it} = \sum_{i=1}^{N}\sum_{t=1}^{T}\bar{\mathbf{x}}_i'\bar{\mathbf{x}}_i.$$

It follows that $\hat{\mathbf{g}}_{it} = \mathbf{x}_{it} - \bar{\mathbf{x}}_i\mathbf{I}_K = \mathbf{x}_{it} - \bar{\mathbf{x}}_i$. This completes the proof.

11.8. a. This is just a special case of Problem 8.8, where we now apply the results to the FD equation and account for the loss of the first time period. The rank condition is $\text{rank}\left(\sum_{t=2}^{T}\text{E}(\mathbf{z}_{it}'\Delta\mathbf{x}_{it})\right) = K$.

b. Again, Problem 8.8 provides the answer. Letting $e_{it} = \Delta u_{it}, t \geq 2,$ two sufficient conditions are $\text{Var}(e_{it}|\mathbf{z}_{it}) = \sigma_e^2,\ t = 2,\ldots,T$ and $\text{E}(e_{it}|\mathbf{z}_{it}, e_{i,t-1},\ldots,\mathbf{z}_{i2}, e_{i2}) = 0,\ t = 2,\ldots,T.$

c. As in the case of pooled OLS after first differencing, this is only useful (and can only be implemented) when $T \geq 3$. First, estimate equation (11.100) by pooled 2SLS and obtain the residuals, $e_{it}, t = 2,\ldots,T, i = 1,\ldots,N.$ Then, estimate the augmented equation,

$$\Delta y_{it} = \Delta\mathbf{x}_{it}\boldsymbol{\beta} + \rho\hat{e}_{i,t-1} + error_{it}, t = 3,\ldots,T$$

by pooled 2SLS, using IVs $(\mathbf{z}_{it}, \hat{e}_{i,t-1})$. If we strengthen the condition from part b to

$$\text{E}(e_{it}|\mathbf{z}_{it}, \Delta\mathbf{x}_{i,t-1}, e_{i,t-1},\ldots,\mathbf{z}_{i2}, \Delta\mathbf{x}_{i2}, e_{i2}) = 0$$

then, under H_0, the usual t statistic on $\hat{e}_{i,t-1}$ is distributed as asymptotically standard normal, provided we add a dynamic homoskedasticity assumption. See Problem 8.10 for verification in a general IV setting.

11.9. a. We can apply Problem 8.8.b because we are applying pooled 2SLS – this time to the time-demeaned equation. Therefore, the rank condition is

$$\text{rank}\left(\sum_{t=1}^{T}\text{E}(\ddot{\mathbf{z}}_{it}'\ddot{\mathbf{x}}_{it})\right) = K.$$

The rank condition clearly fails if $\mathbf{x}_{it}$ contains any time-constant explanatory variables (across all i, as usual). The condition $\text{rank}\left(\sum_{t=1}^{T} \text{E}(\ddot{\mathbf{z}}_{it}'\ddot{\mathbf{z}}_{it})\right) = L$ also should be assumed, and this rules out time-constant instruments (and perfectly collinear instruments). If the rank condition holds, we can always redefine $\mathbf{z}_{it}$ so that $\sum_{t=2}^{T} \text{E}(\ddot{\mathbf{z}}_{it}'\ddot{\mathbf{z}}_{it})$ has full rank.

b. We can apply the results on GMM estimation in Chapter 8. In particular, in equation (8.25), take $\mathbf{C} = \text{E}(\ddot{\mathbf{Z}}_i'\ddot{\mathbf{X}}_i)$, $\mathbf{W} = [\text{E}(\ddot{\mathbf{Z}}_i'\ddot{\mathbf{Z}}_i)]^{-1}$, $\mathbf{\Lambda} = \text{E}(\mathbf{Z}_i'\ddot{\mathbf{u}}_i\ddot{\mathbf{u}}_i'\ddot{\mathbf{Z}}_i)$.

A key point is that $\ddot{\mathbf{Z}}_i'\ddot{\mathbf{u}}_i = (\mathbf{Q}_T\mathbf{Z}_i)'(\mathbf{Q}_T\mathbf{u}_i) = \mathbf{Z}_i'\mathbf{Q}_T\mathbf{u}_i = \ddot{\mathbf{Z}}_i'\mathbf{u}_i$, where $\mathbf{Q}_T$ is the $T \times T$ time-demeaning matrix defined in Chapter 10. Under Assumption FEIV.3, $\text{E}(\mathbf{u}_i\mathbf{u}_i'|\ddot{\mathbf{Z}}_i) = \sigma_u^2\mathbf{I}_T$ (by the usual iterated expectations argument), and so $\mathbf{\Lambda} = \sigma_u^2\text{E}(\ddot{\mathbf{Z}}_i'\ddot{\mathbf{Z}}_i)$. If we plug these choices of C, W, and $\wedge$ into (8.29) and simplify, we obtain

$$\text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \sigma_u^2\{\text{E}(\ddot{\mathbf{X}}_i'\ddot{\mathbf{Z}}_i)[\text{E}(\ddot{\mathbf{Z}}_i'\ddot{\mathbf{Z}}_i)]^{-1}\text{E}(\ddot{\mathbf{Z}}_i'\ddot{\mathbf{X}}_i)\}^{-1}.$$

c. The argument is very similar to the case of the fixed effects estimator. First, we already showed in Chapter 10 that $\sum_{t=1}^{T} \text{E}(\ddot{u}_{it}^2) = (T-1)\sigma_u^2$. If $\hat{\ddot{u}}_{it} = \ddot{y}_{it} - \ddot{\mathbf{x}}_{it}\hat{\boldsymbol{\beta}}$ are the pooled 2SLS residuals applied to the time-demeaned data, then $[N(T-1)]^{-1}\sum_{i=1}^{N}\sum_{t=1}\hat{\ddot{u}}_{it}^2$ is a consistent estimator of σ_u^2. Typically, $N(T-1)$ would be replaced by $N(T-1)-K$ as a degrees of freedom adjustment.

d. From Problem 5.1 – which is purely algebraic, and so applies directly to pooled 2SLS, even with lots of dummy variables – the 2SLS estimates, including $\hat{\boldsymbol{\beta}}$, can be obtained as follows. First, run the regression $\mathbf{x}_{it}$ on $d1_i, \ldots, dN_i, \mathbf{z}_{it}$ across all t and i, and obtain the residuals, say $\hat{\ddot{\mathbf{r}}}_{it}$. Second, obtain $\hat{c}_1, \ldots, \hat{c}_N$, $\hat{\boldsymbol{\beta}}$ from pooled regression y_{it} on $d1_i, \ldots, dN_i, \mathbf{x}_{it}$, $\hat{\ddot{\mathbf{r}}}_{it}$. Now, by algebra of partial regression, $\hat{\boldsymbol{\beta}}$ and the coefficient on $\hat{\ddot{\mathbf{r}}}_{it}$, say $\hat{\boldsymbol{\delta}}$, from this last regression can be obtained by first partialling out the dummy variables, $d1_i, \ldots, dN_i$. As we

know from Chapter 10, this partialling out is equivalent to time demeaning all variables. Therefore, $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\delta}}$ can be obtained form the pooled regression $\ddot{y}_{it}$ on $\ddot{\mathbf{x}}_{it}, \hat{\ddot{\mathbf{r}}}_{it},$ where we use the fact that the time average of $\hat{\ddot{\mathbf{r}}}_{it}$ for each i is identically zero.

Now consider the 2SLS estimator of $\boldsymbol{\beta}$ from

$$\ddot{y}_{it} = \ddot{\mathbf{x}}_{it}\boldsymbol{\beta} + \ddot{u}_{it} \tag{11.102}$$

using IVs $\ddot{\mathbf{z}}_{it}$. Again appealing to Problem 5.1, the pooled 2SLS estimator can be obtained from regressing $\ddot{\mathbf{x}}_{it}$ on $\ddot{\mathbf{z}}_{it}$ and saving the residuals, say $\hat{\ddot{\mathbf{s}}}_{it},$ and then running the OLS regression $\ddot{y}_{it}$ on $\ddot{\mathbf{x}}_{it}, \hat{\ddot{\mathbf{s}}}_{it}$. By partial regression and the fact that regressing on $d1_i, \ldots, dN_i$ results in time demeaning, $\hat{\ddot{\mathbf{s}}}_{it} = \hat{\ddot{\mathbf{r}}}_{it}$ for all i and t. This proves that the 2SLS estimates of $\boldsymbol{\beta}$ from (11.102) and

$$y_{it} = c_1 d1_i + c_2 d2_i + \ldots + c_N dN_i + \mathbf{x}_{it}\boldsymbol{\beta} + u_{it} \tag{11.103}$$

are identical.

e. By writing down the first order condition for the 2SLS estimates from (11.103) (with dn_i as their own instruments, and $\hat{\mathbf{x}}_{it}$ as the IVs for $\mathbf{x}_{it}$), it is easy to show that $\hat{c}_i = \bar{y}_i - \bar{\mathbf{x}}_i\hat{\boldsymbol{\beta}},$ where $\hat{\boldsymbol{\beta}}$ is the FE2SLS estimator. Therefore, the 2SLS residuals from (11.103) are computed as

$$y_{it} - \hat{c}_i - \mathbf{x}_{it}\hat{\boldsymbol{\beta}} = y_{it} - (\bar{y}_i - \bar{\mathbf{x}}_i\hat{\boldsymbol{\beta}}) - \mathbf{x}_{it}\hat{\boldsymbol{\beta}} = (y_{it} - \bar{y}_i) - (\mathbf{x}_{it} - \bar{\mathbf{x}}_i)\hat{\boldsymbol{\beta}} = \ddot{y}_{it} - \ddot{\mathbf{x}}_{it}\hat{\boldsymbol{\beta}},$$

which are exactly the 2SLS residuals from (11.102). Because the N dummy variables are explicitly included in (11.103), the degrees of freedom in estimating σ_u^2 from part c are properly calculated.

The general, messy estimator in equation (8.31) should be used, where $\mathbf{X}$ and $\mathbf{Z}$ are replaced with $\ddot{\mathbf{X}}$ and $\ddot{\mathbf{Z}},$ respectively, $\hat{\mathbf{W}} = (\ddot{\mathbf{Z}}'\ddot{\mathbf{Z}}/N)^{-1}$, $\hat{\ddot{\mathbf{u}}}_i = \ddot{\mathbf{y}}_i - \ddot{\mathbf{X}}_i\hat{\boldsymbol{\beta}}$, and $\hat{\boldsymbol{\Lambda}} = N^{-1}\sum_{i=1}^{N}\ddot{\mathbf{Z}}_i'\hat{\ddot{\mathbf{u}}}_i\hat{\ddot{\mathbf{u}}}_i'\ddot{\mathbf{Z}}_i$.

11.10. Let $\tilde{\mathbf{a}}_i,\ i = 1, \ldots, N,$ and $\tilde{\boldsymbol{\beta}}$ be the OLS estimates from the pooled OLS regression (11.101). By partial regression, $\tilde{\boldsymbol{\beta}}$ can be obtained by first regressing y_{it} on $d1_i\mathbf{z}_{it},\ d2_i\mathbf{z}_{it},\ \ldots,$ $dN_1\mathbf{z}_{it}$ and obtaining the residuals, $\ddot{y}_{it}$, and likewise for $\ddot{\mathbf{x}}_{it}$. Then, we regress $\ddot{y}_{it}$ on $\ddot{\mathbf{x}}_{it}$, $t = 1, \ldots, T;\ i = 1, \ldots, N$. But regressing on $d1_i\mathbf{z}_{it},\ d2_i\mathbf{z}_{it},\ \ldots,\ dN_i\mathbf{z}_{it}$ across all t and i is the same as regressing on $\mathbf{z}_{it},\ t = 1, \ldots, T,$ for each cross section observation, i. Therefore, we can write

$$\ddot{y}_{it} = y_{it} - \mathbf{z}_{it}[(\mathbf{Z}_i'\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\mathbf{y}_i]$$
$$\ddot{\mathbf{y}}_i = \mathbf{M}_i\mathbf{y}_i$$

where $\mathbf{M}_i = \mathbf{I}_T - \mathbf{Z}_i[(\mathbf{Z}_i'\mathbf{Z}_i)^{-1}\mathbf{Z}_i'$. A similar expression holds for $\ddot{\mathbf{x}}_{it}$. We have shown that regression (11.101) is identical to the pooled OLS regression $\ddot{y}_{it}$ on $\ddot{\mathbf{x}}_{it}$, $t = 1, \ldots, T, i = 1, \ldots, N.$ The residuals from the two regressions are exactly the same by the two-step projection result. The regression in (11.101) results in $NT - NJ - K = N(T - J) - K$ degrees of freedom, which is exactly what we need in (11.76).

11.11. Differencing twice and using the resulting cross section is easily done in most statistical packages. Alternatively, Equivalently, use FE on the FD equation (which is the same as FD on the FD equation).I can use fixed effects on the first differences

The Stata output follows. The estimates from the random growth model are pretty bad: the estimates on the grant variables are of the "wrong" sign, and they are very imprecise.

The joint F test for the 53 different firm intercepts (when we treat the heterogeneity as estimable parameters) is significant at the 5% level(p-value = .033), which does suggest a random growth model is appropriate. (But remember, this statistic is only valid under restrictive assumptions.) It is hard to know what to make of the poor estimates, but it does cast doubt on the standard unobserved effects model without a random growth term.

```
. xtreg clscrap d89 cgrant cgrant_1, fe

Fixed-effects (within) regression               Number of obs      =       108
Group variable: fcode                           Number of groups   =        54

R-sq:  within  = 0.0577                         Obs per group: min =
       between = 0.0476                                        avg =       2.
       overall = 0.0050                                        max =

                                                F(3,51)            =      1.04
corr(u_i, Xb)  = -0.4011                        Prob > F           =    0.3826

------------------------------------------------------------------------------
     clscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         d89 |  -.2377384   .1407362    -1.69   0.097    -.5202783    .0448014
      cgrant |   .1564748   .2632934     0.59   0.555    -.3721088    .6850584
    cgrant_1 |   .6099015   .6343411     0.96   0.341    -.6635913    1.883394
       _cons |  -.2240491    .114748    -1.95   0.056    -.4544153    .0063171
-------------+----------------------------------------------------------------
     sigma_u |  .50956703
     sigma_e |  .49757778
         rho |  .51190251   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(53, 51) =     1.67                 Prob > F = 0.0334
```

11.12. a. Using only the changes from 1990 to 1993 and estimating the first-differenced equation by OLS gives:

```
. reg cmrdrte cexec cunem if d93

      Source |       SS       df       MS              Number of obs =      51
-------------+------------------------------           F(  2,    48) =    2.96
       Model |   6.8879023     2  3.44395115           Prob > F      =  0.0614
    Residual |  55.8724857    48  1.16401012           R-squared     =  0.1097
-------------+------------------------------           Adj R-squared =  0.0727
       Total |   62.760388    50  1.25520776           Root MSE      =  1.0789

------------------------------------------------------------------------------
     cmrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       cexec |  -.1038396   .0434139    -2.39   0.021    -.1911292     -.01655
       cunem |  -.0665914   .1586859    -0.42   0.677    -.3856509     .252468
       _cons |   .4132665   .2093848     1.97   0.054    -.0077298    .8342628
------------------------------------------------------------------------------
```

The coefficient on *cexec* means that one more execution reduces the murder rate by about .10, and the effect is statistically significant.

b. If executions in the future respond to changes in the past murder rate, then *exec* may not be strictly exogenous. If executions more than three years ago have a partial effect on the

murder rate, this would also violate strict exogeneity because, effectively, we do not have enough lags. In principle, we could handle the latter problem by collecting more data and including more lags.

If we assume that only $exec_{it}$ appears in the equation at time t, so that current and past executions are uncorrelated with u_{it}, then we can difference away c_i and apply IV:

$$\Delta mrdrte_{it} = \delta_0 + \beta_1 \Delta exec_{it} + \beta_2 \Delta unem_{it} + \Delta u_{it}.$$

A valid IV for $\Delta exec_{it}$ is $\Delta exec_{i,t-1}$ because, by assumption, $exec_{i,t-1}$ and $exec_{i,t-2}$ are both uncorrelated with u_{it} and $u_{i,t-1}$. This results in a cross section IV estimation.

c. To test the rank condition, we regress $\Delta exec_{it}$ on $1, \Delta exec_{i,t-1}, \Delta unem_{it}$ for 1993, and do a t test on $\Delta exec_{i,t-1}$:

```
. reg cexec cexec_1 cunem if d93

      Source |       SS       df       MS              Number of obs =      51
-------------+------------------------------           F(  2,    48) =   20.09
       Model |  281.429488     2  140.714744            Prob > F      =  0.0000
    Residual |  336.217571    48  7.00453273            R-squared     =  0.4556
-------------+------------------------------           Adj R-squared =  0.4330
       Total |  617.647059    50  12.3529412            Root MSE      =  2.6466

------------------------------------------------------------------------------
       cexec |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     cexec_1 |   -1.08241   .1707822    -6.34   0.000     -1.42579   -.7390289
       cunem |   .0400493   .3892505     0.10   0.918    -.7425912    .8226898
       _cons |   .3139609   .5116532     0.61   0.542    -.7147868    1.342709
------------------------------------------------------------------------------
```

Interestingly, there is a one-for-one negative relationship between the change in lagged executions and the change in current executions. Certainly the rank condition passes.

The IV estimates are below:

```
. reg cmrdrte cexec cunem (cexec_1 cunem) if d93

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =      51
-------------+------------------------------           F(  2,    48) =    1.31
       Model |  6.87925253     2  3.43962627            Prob > F      =  0.2796
```

```
    Residual |  55.8811355    48  1.16419032           R-squared     =  0.1096
-------------+------------------------------           Adj R-squared =  0.0725
       Total |   62.760388    50  1.25520776           Root MSE      =   1.079

------------------------------------------------------------------------------
     cmrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       cexec |  -.1000972   .0643241    -1.56   0.126    -.2294293     .029235
       cunem |  -.0667262   .1587074    -0.42   0.676    -.3858289    .2523764
       _cons |    .410966   .2114237     1.94   0.058    -.0141298    .8360617
------------------------------------------------------------------------------
```

The point estimate on Δ*exec* is essentially the same as the OLS estimate, but, of course, the IV standard error is larger. We can justify the POLS estimator on the FD equation (as the null of exogeneity of Δ*exec* would not be rejected).

d. The following Stata command gives the results without Texas:

```
. reg cmrdrte cexec cunem if (d93==1 & state!= "TX")

      Source |       SS       df       MS              Number of obs =      50
-------------+------------------------------           F(  2,    47) =    0.32
       Model |  .755191109     2  .377595555           Prob > F      =  0.7287
    Residual |  55.7000012    47  1.18510641           R-squared     =  0.0134
-------------+------------------------------           Adj R-squared = -0.0286
       Total |  56.4551923    49  1.15214678           Root MSE      =  1.0886

------------------------------------------------------------------------------
     cmrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       cexec |   -.067471    .104913    -0.64   0.523    -.2785288    .1435868
       cunem |  -.0700316   .1603712    -0.44   0.664    -.3926569    .2525936
       _cons |   .4125226   .2112827     1.95   0.057    -.0125233    .8375686
------------------------------------------------------------------------------

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =      50
-------------+------------------------------           F(  2,    47) =    0.11
       Model | -1.65785462     2 -.828927308           Prob > F      =  0.8939
    Residual |  58.1130469    47  1.23644781           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  56.4551923    49  1.15214678           Root MSE      =   1.112

------------------------------------------------------------------------------
     cmrdrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       cexec |    .082233    .804114     0.10   0.919    -1.535436    1.699902
       cunem |  -.0826635   .1770735    -0.47   0.643    -.4388895    .2735624
       _cons |   .3939505   .2373797     1.66   0.104    -.0835958    .8714968
------------------------------------------------------------------------------
```

The OLS estimate is smaller in magnitude and not statistically significant, while the IV

estimate actually changes sign (but, statistically, is not different from zero). Clearly, including Texas in the estimation has a big impact. It is easy to see why this is the case by listing the change in the murder rates and executions for Texas along with the averages for all states:

```
. list cmrdrte cexec if (d93==1 & state =="TX")

     +-------------------+
     |   cmrdrte   cexec |
     |-------------------|
132. | -2.200001      23 |
     +-------------------+

. sum cmrdrte cexec if d93==1

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     cmrdrte |        51    .2862745    1.120361  -2.200001   3.099998
       cexec |        51    .6470588    3.514675         -3         23
```

Texas has the largest drop in the murder rate from 1990 to 1993, and also the largest increase in the number of executions. This does not necessarily mean Texas is an outlier, but it clearly is an influential observation. And it is clear why including Texas makes for a fairly strong deterrent effect.

11.13. a. The following Stata output estimates the reduced form for $\Delta \log(pris)$ and tests joint significance of *final*1 and *final*2, and also tests equality of the coefficients on *final*1 and *final*2. The latter is actually not very interesting Technically, because we do not reject, we could reduce our instrument to *final*1 + *final*2, but we could always look ex post for restrictions on the parameters in a reduced form.

```
. use prison

. xtset state year
       panel variable:  state (strongly balanced)
        time variable:  year, 80 to 93
                delta:  1 unit

. reg gpris final1 final2 gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17
       cag18_24 cag25_34 y81-y93

      Source |       SS       df       MS              Number of obs =     714
-------------+------------------------------           F( 24,   689) =    5.15
```

```
       Model |  .481041472    24  .020043395           Prob > F      =  0.0000
    Residual |  2.68006631   689  .003889791           R-squared     =  0.1522
-------------+------------------------------           Adj R-squared =  0.1226
       Total |  3.16110778   713  .004433531           Root MSE      =  .06237

------------------------------------------------------------------------------
       gpris |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      final1 |   -.077488   .0259556    -2.99   0.003    -.1284496   -.0265265
      final2 |  -.0529558   .0184078    -2.88   0.004    -.0890979   -.0168136
      gpolpc |  -.0286921   .0440058    -0.65   0.515    -.1150937    .0577094
      gincpc |   .2095521   .1313169     1.60   0.111    -.0482772    .4673815
       cunem |   .1616595   .3111688     0.52   0.604    -.4492935    .7726124
      cblack |  -.0044763   .0262118    -0.17   0.864     -.055941    .0469883
      cmetro |  -1.418389   .7860435    -1.80   0.072    -2.961717    .1249393
     cag0_14 |   2.617307   1.582611     1.65   0.099    -.4900126    5.724627
    cag15_17 |  -1.608738   3.755564    -0.43   0.669    -8.982461    5.764986
    cag18_24 |   .9533678   1.731188     0.55   0.582    -2.445669    4.352405
    cag25_34 |  -1.031684   1.763248    -0.59   0.559    -4.493667      2.4303
         y81 |   .0124113    .013763     0.90   0.367    -.0146111    .0394337
         y82 |   .0773503   .0156924     4.93   0.000     .0465396     .108161
         y83 |   .0767785   .0153929     4.99   0.000     .0465559    .1070011
         y84 |   .0289763   .0176504     1.64   0.101    -.0056787    .0636314
         y85 |   .0279051   .0164176     1.70   0.090    -.0043295    .0601397
         y86 |   .0541489   .0179305     3.02   0.003      .018944    .0893539
         y87 |   .0312716   .0171317     1.83   0.068     -.002365    .0649082
         y88 |    .019245   .0170725     1.13   0.260    -.0142754    .0527654
         y89 |   .0184651   .0172867     1.07   0.286    -.0154759     .052406
         y90 |   .0635926   .0165775     3.84   0.000     .0310442    .0961411
         y91 |   .0263719   .0168913     1.56   0.119    -.0067927    .0595366
         y92 |   .0190481   .0179372     1.06   0.289    -.0161701    .0542663
         y93 |   .0134109   .0189757     0.71   0.480    -.0238461     .050668
       _cons |   .0272013   .0170478     1.60   0.111    -.0062705    .0606731
------------------------------------------------------------------------------

. test final1 final2

 ( 1)  final1 = 0
 ( 2)  final2 = 0

       F(  2,   689) =    8.56
            Prob > F =    0.0002

. test final1 = final2

 ( 1)  final1 - final2 = 0

       F(  1,   689) =    0.60
            Prob > F =    0.4401
```

Jointly, *final*1 and *final*2 are pretty significant. Next, test for serial correlation in $a_{it} \equiv \Delta v_{it}$:

```
. predict ahat, resid

. gen ahat_1 = l.ahat
```

```
(51 missing values generated)

. reg ahat ahat_1

      Source |       SS       df       MS              Number of obs =     663
-------------+------------------------------           F(  1,   661) =   14.33
       Model |  .051681199     1  .051681199           Prob > F      =  0.0002
    Residual |  2.38322468   661  .003605484           R-squared     =  0.0212
-------------+------------------------------           Adj R-squared =  0.0197
       Total |  2.43490588   662  .003678106           Root MSE      =  .06005

------------------------------------------------------------------------------
        ahat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      ahat_1 |   .1426247   .0376713     3.79   0.000     .0686549    .2165945
       _cons |   4.24e-11    .002332     0.00   1.000     -.004579     .004579
------------------------------------------------------------------------------
```

There is strong evidence of positive serial correlation, although the estimated size of the AR(1) coefficient, .143, is not especially large. Still, a fully robust variance matrix should be used for the joint significance test of *final*1 and *final*2. These two IVs are much more significant when the robust variance matrix is used:

```
. qui reg gpris final1 final2 gpolpc gincpc cunem cblack cmetro cag0_14
      cag15_17  cag18_24 cag25_34 y81-y93, cluster(state)

. test final1 final2

 ( 1)  final1 = 0
 ( 2)  final2 = 0

       F(  2,    50) =   18.82
            Prob > F =    0.000
```

b. First, we do pooled 2SLS to obtain the 2SLS residuals, $\hat{e}_{it}$. Then we add the lagged residual to the equation, and use it as its own IV:

```
. ivreg gcriv gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
       cag25_34 y81-y93 (gpris = final1 final2)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     714
-------------+------------------------------           F( 23,   690) =    6.08
       Model | -.696961613    23 -.030302679           Prob > F      =  0.0000
    Residual |  6.28846843   690  .009113722           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  5.59150682   713  .007842226           Root MSE      =  .09547

------------------------------------------------------------------------------
       gcriv |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
```

```
-------------+----------------------------------------------------------------
       gpris |  -1.031956   .3699628    -2.79   0.005    -1.758344   -.3055684
      gpolpc |    .035315   .0674989     0.52   0.601    -.0972128    .1678428
      gincpc |   .9101992   .2143266     4.25   0.000     .4893885     1.33101
       cunem |   .5236958   .4785632     1.09   0.274     -.415919     1.46331
      cblack |  -.0158476   .0401044    -0.40   0.693    -.0945889    .0628937
      cmetro |   -.591517   1.298252    -0.46   0.649    -3.140516    1.957482
     cag0_14 |   3.379384   2.634893     1.28   0.200    -1.793985    8.552753
    cag15_17 |   3.549945   5.766302     0.62   0.538    -7.771659    14.87155
    cag18_24 |   3.358348   2.680839     1.25   0.211    -1.905233    8.621929
    cag25_34 |   2.319993   2.706345     0.86   0.392    -2.993667    7.633652
          y81 |  -.0560732   .0217346    -2.58   0.010    -.0987471   -.0133992
          y82 |   .0284616   .0384773     0.74   0.460     -.047085    .1040082
          y83 |    .024703   .0373965     0.66   0.509    -.0487216    .0981276
          y84 |   .0128703   .0293337     0.44   0.661    -.0447236    .0704643
          y85 |   .0354026   .0275023     1.29   0.198    -.0185956    .0894008
          y86 |   .0921857   .0343884     2.68   0.008     .0246672    .1597042
          y87 |    .004771   .0290145     0.16   0.869    -.0521964    .0617383
          y88 |   .0532706   .0273221     1.95   0.052    -.0003738     .106915
          y89 |   .0430862   .0275204     1.57   0.118    -.0109476    .0971201
          y90 |   .1442652   .0354625     4.07   0.000     .0746379    .2138925
          y91 |   .0618481   .0276502     2.24   0.026     .0075595    .1161366
          y92 |   .0266574   .0285333     0.93   0.350    -.0293651    .0826799
          y93 |   .0222739   .0296099     0.75   0.452    -.0358624    .0804103
        _cons |   .0148377   .0275197     0.54   0.590    -.0391948    .0688702
------------------------------------------------------------------------------
Instrumented:  gpris
Instruments:   gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
               cag25_34 y81 y82 y83 y84 y85 y86 y87 y88 y89 y90 y91 y92 y93
               final1 final2
------------------------------------------------------------------------------

. predict ehat, resid

. gen ehat_1 = l.ehat
(51 missing values generated)

. ivreg gcriv gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
        cag25_34 y81-y93 ehat_1 (gpris = final1 final2)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     663
-------------+------------------------------           F( 23,   639) =    5.14
       Model | -.815873465    23 -.035472759           Prob > F      =  0.0000
    Residual |  5.90425699   639  .009239839           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  5.08838353   662   .00768638           Root MSE      =  .09612

------------------------------------------------------------------------------
       gcriv |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       gpris |  -1.084446   .4071905    -2.66   0.008    -1.884039   -.2848525
      gpolpc |   .0179121   .0719595     0.25   0.804    -.1233935    .1592176
      gincpc |   .7492611   .2421405     3.09   0.002     .2737738    1.224748
       cunem |   .1979701    .515973     0.38   0.701    -.8152375    1.211178
      cblack |  -.0102865   .0424589    -0.24   0.809    -.0936622    .0730893
      cmetro |  -.5272326   1.357715    -0.39   0.698    -3.193354    2.138889
     cag0_14 |   3.284496   3.045539     1.08   0.281    -2.695979     9.26497
```

```
    cag15_17 |    .066451   6.105497     0.01   0.991    -11.92281    12.05571
    cag18_24 |   3.094998   2.830038     1.09   0.275    -2.462301    8.652297
    cag25_34 |   2.716353   2.799581     0.97   0.332    -2.781137    8.213843
         y81 |  -.0782703   .0350721    -2.23   0.026    -.1471409   -.0093998
         y82 |   .0090276   .0225246     0.40   0.689    -.0352036    .0532588
         y83 |  (dropped)
         y84 |  -.0113602   .0314408    -0.36   0.718       -.0731    .0503796
         y85 |    .015744   .0309473     0.51   0.611    -.0450267    .0765148
         y86 |   .0752485    .027649     2.72   0.007     .0209547    .1295424
         y87 |  -.0205808   .0282106    -0.73   0.466    -.0759774    .0348159
         y88 |   .0265964   .0315542     0.84   0.400    -.0353661    .0885589
         y89 |   .0182293   .0327158     0.56   0.578    -.0460142    .0824727
         y90 |   .1275351   .0235386     5.42   0.000     .0813126    .1737575
         y91 |   .0435859   .0315328     1.38   0.167    -.0183346    .1055064
         y92 |   .0121958   .0354112     0.34   0.731    -.0573406    .0817321
         y93 |   .0016107   .0365807     0.04   0.965    -.0702221    .0734435
       ehat_1 |   .0763754   .0456451     1.67   0.095    -.0132571     .166008
        _cons |   .0441747   .0477902     0.92   0.356    -.0496701    .1380195
------------------------------------------------------------------------------
Instrumented:  gpris
Instruments:   gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
               cag25_34 y81 y82 y83 y84 y85 y86 y87 y88 y89 y90 y91 y92 y93
               ehat_1 final1 final2
------------------------------------------------------------------------------
```

There is only marginal evidence of positive serial correlation, and it is practically small, anyway ($\hat{\rho} = .076$).

c. Adding a state effect to the change (FD) equation changes very little. In this example, there seems to be little need for a random growth model. The estimated prison effect becomes a little smaller in magnitude, $-.959$. Here is the Stata output:

```
. xtivreg gcriv gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
          cag25_34 y81-y93 (gpris = final1 final2), fe

Fixed-effects (within) IV regression         Number of obs      =         714
Group variable: state                        Number of groups   =          51

R-sq:  within  =      .                      Obs per group: min =          14
       between = 0.0001                                    avg =        14.
       overall = 0.1298                                    max =          14

                                             Wald chi2(23)      =      179.24
corr(u_i, Xb)  = -0.2529                     Prob > chi2        =      0.0000

------------------------------------------------------------------------------
       gcriv |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       gpris |  -.9592287   .3950366    -2.43   0.015    -1.733486   -.1849713
      gpolpc |     .04445   .0664696     0.67   0.504    -.0858281    .1747281
      gincpc |   1.027161   .2157944     4.76   0.000     .6042122    1.450111
       cunem |   .6560942   .4698359     1.40   0.163    -.2647672    1.576956
      cblack |   .0706601   .1496426     0.47   0.637    -.2226339    .3639542
```

```
      cmetro |   3.229287   4.683812     0.69   0.491    -5.950815    12.40939
     cag0_14 |    1.14119   2.749679     0.42   0.678    -4.248082    6.530462
    cag15_17 |   1.402606   6.330461     0.22   0.825    -11.00487    13.81008
    cag18_24 |   1.169114   2.866042     0.41   0.683    -4.448225    6.786453
    cag25_34 |  -2.089449   3.383237    -0.62   0.537    -8.720471    4.541574
         y81 |  -.0590819   .0230252    -2.57   0.010    -.1042104   -.0139534
         y82 |   .0033116   .0388056     0.09   0.932    -.0727459    .0793691
         y83 |   .0080099   .0378644     0.21   0.832     -.066203    .0822228
         y84 |  -.0019285   .0293861    -0.07   0.948    -.0595243    .0556672
         y85 |   .0220412   .0276807     0.80   0.426     -.032212    .0762945
         y86 |    .075621   .0338898     2.23   0.026     .0091981    .1420438
         y87 |  -.0124835   .0294198    -0.42   0.671    -.0701453    .0451783
         y88 |   .0329977   .0286125     1.15   0.249    -.0230817    .0890771
         y89 |    .018718   .0292666     0.64   0.522    -.0386434    .0760794
         y90 |   .1157811   .0354143     3.27   0.001     .0463703    .1851919
         y91 |   .0378784   .0290414     1.30   0.192    -.0190417    .0947984
         y92 |  -.0006633   .0305014    -0.02   0.983    -.0604449    .0591184
         y93 |  -.0007561   .0317733    -0.02   0.981    -.0630306    .0615184
       _cons |   .0014574   .0296182     0.05   0.961    -.0565932    .0595079
-------------+----------------------------------------------------------------
     sigma_u |  .03039696
     sigma_e |   .0924926
         rho |  .09747751   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F  test that all u_i=0:     F(50,640) =     0.69          Prob > F    = 0.9459
------------------------------------------------------------------------------
Instrumented:   gpris
Instruments:    gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
                cag25_34 y81 y82 y83 y84 y85 y86 y87 y88 y89
                y90 y91 y92 y93 final1 final2
------------------------------------------------------------------------------
```

d. When we use the property crime rate, the estimated elasticity with respect to prison size is substantially smaller, but still negative and marginally significant:

```
. ivreg gcrip gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
        cag25_34 y81-y93 (gpris = final1 final2)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     714
-------------+------------------------------           F( 23,   690) =   22.83
       Model |  1.07170564    23  .046595897           Prob > F      =  0.0000
    Residual |   1.5490539   690  .002245006           R-squared     =  0.4089
-------------+------------------------------           Adj R-squared =  0.3892
       Total |  2.62075954   713   .00367568           Root MSE      =  .04738

------------------------------------------------------------------------------
       gcrip |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       gpris |  -.3285567   .1836195    -1.79   0.074    -.6890768    .0319633
      gpolpc |    .014567    .033501     0.43   0.664     -.051209    .0803431
      gincpc |   .0560822   .1063744     0.53   0.598    -.1527741    .2649385
       cunem |   .8583588   .2375199     3.61   0.000     .3920102    1.324707
      cblack |  -.0507462   .0199046    -2.55   0.011     -.089827   -.0116654
      cmetro |   .0404892   .6443472     0.06   0.950    -1.224627    1.305606
```

```
     cag0_14 |   1.890526   1.307747     1.45   0.149    -.6771151    4.458167
    cag15_17 |   5.699448   2.861925     1.99   0.047     .0803221    11.31857
    cag18_24 |   1.712283   1.330551     1.29   0.199    -.9001312    4.324698
    cag25_34 |   2.027833    1.34321     1.51   0.132    -.6094366    4.665102
          y81 |  -.0771684   .0107873    -7.15   0.000    -.0983483   -.0559886
          y82 |  -.0980884    .019097    -5.14   0.000    -.1355836   -.0605932
          y83 |  -.1093989   .0185606    -5.89   0.000    -.1458409   -.0729569
          y84 |  -.0810119   .0145589    -5.56   0.000    -.1095968   -.0524269
          y85 |   -.031369   .0136499    -2.30   0.022    -.0581693   -.0045687
          y86 |  -.0169451   .0170676    -0.99   0.321    -.0504558    .0165656
          y87 |  -.0310865   .0144005    -2.16   0.031    -.0593605   -.0028125
          y88 |  -.0437643   .0135605    -3.23   0.001    -.0703891   -.0171396
          y89 |  -.0359254   .0136589    -2.63   0.009    -.0627434   -.0091074
          y90 |  -.0298029   .0176007    -1.69   0.091    -.0643603    .0047544
          y91 |  -.0505269   .0137233    -3.68   0.000    -.0774713   -.0235824
          y92 |  -.1024579   .0141616    -7.23   0.000    -.1302629   -.0746529
          y93 |  -.0867254    .014696    -5.90   0.000    -.1155796   -.0578712
        _cons |   .0857682   .0136586     6.28   0.000     .0589509    .1125856
------------------------------------------------------------------------------
Instrumented:  gpris
Instruments:   gpolpc gincpc cunem cblack cmetro cag0_14 cag15_17 cag18_24
               cag25_34 y81 y82 y83 y84 y85 y86 y87 y88 y89 y90 y91 y92 y93
               final1 final2
------------------------------------------------------------------------------
```

The test for serial correlation yields a coefficient on $\hat{e}_{i,t-1}$ of $-.024$ ($t = -.52$), and so we conclude that serial correlation is not an issue.

11.14. a. The fixed effects estimate of the first-difference equations are given below. We have included year dummies without differencing them, since we are not interested in the time effects in the original model:

```
. use ezunem

. xtset city year
       panel variable:  city (strongly balanced)
        time variable:  year, 1980 to 1988
                delta:  1 unit

. gen cezt = d.ezt
(22 missing values generated)

. xtreg guclms cez cezt d82-d88, fe

Fixed-effects (within) regression               Number of obs      =       176
Group variable: city                            Number of groups   =        22

R-sq:  within  = 0.6406                         Obs per group: min =
       between = 0.0094                                        avg =       8.
       overall = 0.6205                                        max =

                                                F(9,145)           =     28.71
corr(u_i, Xb)  = -0.0546                        Prob > F           =    0.0000
```

```
------------------------------------------------------------------------------
      guclms |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         cez |   .1937324   .3448663     0.56   0.575    -.4878818    .8753467
        cezt |  -.0783638   .0679161    -1.15   0.250    -.2125972    .0558697
         d82 |   .7787595   .0675022    11.54   0.000     .6453442    .9121748
         d83 |  -.0331192   .0675022    -0.49   0.624    -.1665345    .1002961
         d84 |  -.0127177   .0713773    -0.18   0.859     -.153792    .1283566
         d85 |   .3616479   .0762138     4.75   0.000     .2110144    .5122814
         d86 |   .3277739   .0742264     4.42   0.000     .1810684    .4744793
         d87 |    .089568   .0742264     1.21   0.230    -.0571375    .2362735
         d88 |   .0185673   .0742264     0.25   0.803    -.1281381    .1652728
       _cons |  -.3216319   .0477312    -6.74   0.000    -.4159708   -.2272931
-------------+----------------------------------------------------------------
     sigma_u |  .05880562
     sigma_e |  .22387933
         rho |  .06454083   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(21, 145) =     0.49                 Prob > F = 0.9712

. test cez cezt

 ( 1)  cez = 0
 ( 2)  cezt = 0

       F(  2,   145) =    3.22
            Prob > F =    0.0428
```

The coefficient $\hat{\delta}_2 = -.078$ gives the difference in annual growth rate due to *EZ* designation. It is not significant at the usual 5% level. Note that this formulation does not give the coefficient $\hat{\delta}_1$ a simple interpretation because zone designation happened either at $t = 5$ (if in 1984) or $t = 6$ (if in 1985). A better formulation centers the linear trend at the time of designation before constructing the interactions:

```
. egen nyrsez =sum(ez), by(city)

. gen ezt0 = 0 if ~ez
(46 missing values generated)

. replace ezt0 = ez*(t-5) if nyrsez == 5
(30 real changes made)

. replace ezt0 = ez*(t-6) if nyrsez == 4
(16 real changes made)

. gen cezt0 = ezt0 - ezt0[_n-1] if year > 1980
(22 missing values generated)

. xtreg guclms cez cezt0 d82-d88, fe

Fixed-effects (within) regression               Number of obs      =       176
```

```
Group variable: city                            Number of groups   =        22

R-sq:  within  = 0.6406                         Obs per group: min =
       between = 0.0025                                        avg =       8.
       overall = 0.6185                                        max =

                                                F(9,145)           =     28.72
corr(u_i, Xb)  = -0.0630                        Prob > F           =    0.0000

------------------------------------------------------------------------------
      guclms |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         cez |  -.2341545   .0924022    -2.53   0.012    -.4167837   -.0515252
       cezt0 |   -.082805   .0715745    -1.16   0.249     -.224269     .058659
         d82 |   .7787595   .0675005    11.54   0.000     .6453475    .9121716
         d83 |  -.0331192   .0675005    -0.49   0.624    -.1665312    .1002928
         d84 |  -.0028809   .0720513    -0.04   0.968    -.1452874    .1395256
         d85 |    .355169   .0740177     4.80   0.000      .208876    .5014621
         d86 |   .3297926   .0749318     4.40   0.000      .181693    .4778922
         d87 |   .0915867   .0749318     1.22   0.224    -.0565129    .2396864
         d88 |   .0205861   .0749318     0.27   0.784    -.1275135    .1686857
       _cons |  -.3216319   .0477301    -6.74   0.000    -.4159685   -.2272953
-------------+----------------------------------------------------------------
     sigma_u |  .06091433
     sigma_e |  .22387389
         rho |  .06893091   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(21, 145) =     0.50             Prob > F = 0.9681
```

Now the coefficient on *cez* is the estimated effect of the *EZ* in the first year of designation, and that gets added to $-.083 \cdot$ (*years since initial designation*). This is easier to read.

b. Setting $\delta_1 = 0$ gives a within *R*-squared of about .640, compared with that of the original model in Example 11.4 of about .637. The difference is minor, and we would probably go with the simpler, basic model in Example 11.4. With more years of data, the trend effect in part a might become significant.

c. Because the general model contains $c_i + g_i t$, we cannot distinguish the effects of a time-constant variable, w_i, or its interaction with a linear time trend – at least if we stay in a fixed effects framework. If we assume c_i and g_i are uncorrelated with ez_{it} we could include w_i and $w_i t$.

d. Yes. Provided $\{e_{it} : t = 1, \ldots, T\}$ has the kind of variation that it does in this data set, $w_i ez_{it}$ is linearly independent from other covariates included in the model. Therefore, we can

estimate η. If we add $h_i ez_{it}$ to the model, where h_i is additional unobserved heterogeneity, then η would not be identified (again, allowing h_i to be correlated with ez_{it}).

11.15. a. We would have to assume that $grant_{it}$ is uncorrelated with the idiosyncratic errors, u_{is}, for all t and s. One way to think of this assumption is that while grant designation may depend on firm heterogeneity c_i, it is not related to idiosyncratic fluctuations in any time period. Further, one must assume the grants have an effect on scrap rates only through their effects on job training – the standard assumption for an instrument.

b. The following simple regression shows that $\Delta hrsemp_{it}$ and $\Delta grant_{it}$ are highly positively correlated, as expected:

```
. reg chrsemp cgrant if d88

      Source |       SS       df       MS              Number of obs =     125
-------------+------------------------------           F(  1,   123) =   79.37
       Model |  18117.5987     1  18117.5987           Prob > F      =  0.0000
    Residual |  28077.3319   123  228.270991           R-squared     =  0.3922
-------------+------------------------------           Adj R-squared =  0.3873
       Total |  46194.9306   124  372.539763           Root MSE      =  15.109

------------------------------------------------------------------------------
     chrsemp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      cgrant |   27.87793   3.129216     8.91   0.000     21.68384    34.07202
       _cons |   .5093234   1.558337     0.33   0.744     -2.57531    3.593956
------------------------------------------------------------------------------
```

Unfortunately, this is on a bigger sample than we can use to estimate the scrap rate equation, because the scrap rate is missing for so many firms. Restricted to that sample, we get:

```
. reg chrsemp cgrant if d88 & clscrap ~= .

      Source |       SS       df       MS              Number of obs =      45
-------------+------------------------------           F(  1,    43) =   22.23
       Model |  6316.65458     1  6316.65458           Prob > F      =  0.0000
    Residual |  12217.3517    43  284.124457           R-squared     =  0.3408
-------------+------------------------------           Adj R-squared =  0.3255
       Total |  18534.0062    44  421.227414           Root MSE      =  16.856

------------------------------------------------------------------------------
     chrsemp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      cgrant |   24.43691   5.182712     4.72   0.000     13.98498    34.88885
       _cons |   1.580598   3.185483     0.50   0.622     -4.84354    8.004737
```

```
------------------------------------------------------------------------------
```

So there is still a pretty strong relationship, but we will be using IV on a small sample ($N = 45$).

c. The IV estimate is:

```
. ivreg clscrap (chrsemp = cgrant) if d88

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =      45
-------------+------------------------------           F(  1,    43) =    3.20
       Model |  .274951237     1  .274951237           Prob > F      =  0.0808
    Residual |  17.0148885    43  .395695081           R-squared     =  0.0159
-------------+------------------------------           Adj R-squared = -0.0070
       Total |  17.2898397    44  .392950903           Root MSE      =  .62904

------------------------------------------------------------------------------
     clscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     chrsemp |  -.0141532   .0079147    -1.79   0.081    -.0301148    .0018084
       _cons |  -.0326684   .1269512    -0.26   0.798    -.2886898     .223353
------------------------------------------------------------------------------
Instrumented:  chrsemp
Instruments:   cgrant
------------------------------------------------------------------------------
```

The estimate says that 10 more hours training per employee would lower the average scrap rate by about 14.2 percent, which is a large economic effect. It is marginally statistically significant (assuming we can trust the asymptotic distribution theory for IV with 45 observations).

d. The OLS estimates is only about $-.0076$ – about half of the IV estimate – with $t = -1.68$.

```
. reg clscrap chrsemp if d88

      Source |       SS       df       MS              Number of obs =      45
-------------+------------------------------           F(  1,    43) =    2.84
       Model |  1.07071245     1  1.07071245           Prob > F      =  0.0993
    Residual |  16.2191273    43  .377189007           R-squared     =  0.0619
-------------+------------------------------           Adj R-squared =  0.0401
       Total |  17.2898397    44  .392950903           Root MSE      =  .61416

------------------------------------------------------------------------------
     clscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     chrsemp |  -.0076007   .0045112    -1.68   0.099    -.0166984    .0014971
```

```
       _cons |  -.1035161    .103736    -1.00   0.324    -.3127197    .1056875
------------------------------------------------------------------------------
```

e. Any effect pretty much disappears using two years of differences (even though you can verify the rank condition easily holds):

```
. ivreg clscrap d89 (chrsemp = cgrant)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =      91
-------------+------------------------------           F(  2,    88) =    0.90
       Model |  .538688387     2  .269344194           Prob > F      =  0.4087
    Residual |  33.2077492    88  .377360787           R-squared     =  0.0160
-------------+------------------------------           Adj R-squared = -0.0064
       Total |  33.7464376    90  .374960418           Root MSE      =   .6143

------------------------------------------------------------------------------
     clscrap |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     chrsemp |  -.0028567   .0030577    -0.93   0.353    -.0089332    .0032198
         d89 |  -.1387379   .1296916    -1.07   0.288    -.3964728    .1189969
       _cons |  -.1548094   .0973592    -1.59   0.115    -.3482902    .0386715
------------------------------------------------------------------------------
Instrumented:  chrsemp
Instruments:   d89 cgrant
------------------------------------------------------------------------------
```

11.16. a. Just use fixed effects, or first differencing. Of course w_i gets eliminated by either transformation.

b. Take the expectation of the structural equation conditional on $(w_i, \mathbf{x}_i, r_i)$:

$$\begin{aligned} \mathrm{E}(y_{it}|w_i, \mathbf{x}_i, r_i) &= \gamma w_i + \mathbf{x}_{it}\boldsymbol{\beta} + \mathrm{E}(c_i|w_i, \mathbf{x}_i, r_i) + \mathrm{E}(u_{it}|w_i, \mathbf{x}_i, r_i) \\ &= \gamma w_i + \mathbf{x}_{it}\boldsymbol{\beta} + \delta_0 + \delta_i r_i + \bar{\mathbf{x}}_i\boldsymbol{\delta}_2. \end{aligned}$$

c. Provided a standard rank condition holds for the explanatory variables, γ is identified because it appears in a conditional expectation containing observable variables: $\mathrm{E}(y_{it}|w_i, \mathbf{x}_i, r_i)$. The pooled OLS estimation

$$y_{it} \text{ on } 1,\ w_i,\ \mathbf{x}_{it},\ r_i,\ \bar{\mathbf{x}}_i,\ t = 1, \ldots, T;\ i = 1, \ldots, N$$

consistently estimates all parameters.

d. Following the hint, we can write

$$y_{it} = \delta_0 + \gamma w_i + \mathbf{x}_{it}\beta + \delta_i r_i + \bar{\mathbf{x}}_i\delta_2 + a_i + u_{it}, t = 1, \ldots, T, \tag{11.104}$$

where $a_i = c_i - \mathrm{E}(c_i|w_i, \mathbf{x}_i, r_i)$. Under the assumptions given, the composite error, $v_{it} \equiv a_i + u_{it}$, is easily shown to have variance-covariance matrix that has the random effect form. In particular, $\mathrm{Var}(v_{it}|w_i, \mathbf{x}_i, r_i) = \sigma_a^2 + \sigma_u^2$ and $\mathrm{Cov}(v_{it}, v_{is}|w_i, \mathbf{x}_i, r_i) = \sigma_a^2$. [The arguments for obtaining these expressions should be familiar. For example, since a_i is a function of $c_i, \mathbf{x}_i,$ and r_i, we can replace c_i with a_i in all of the assumptions concerning the first and second moments of $\{u_{it} : t = 1, \ldots, T\}$. Therefore,

$$\mathrm{E}(a_i u_{it}|w_i, \mathbf{x}_i, a_i, r_i) = a_i \mathrm{E}(u_{it}|w_i, \mathbf{x}_i, a_i, r_i) = 0$$

and so, by iterated expectations,

$$\mathrm{Cov}(a_i, u_{it}|w_i, \mathbf{x}_i, r_i) = \mathrm{E}(a_i u_{it}|w_i, \mathbf{x}_i, r_i) = 0.]$$

We conclude that $\mathrm{Var}(\mathbf{v}_i|w_i, \mathbf{x}_i, r_i) = \mathrm{Var}(\mathbf{v}_i)$ has the random effects form, and so we should just apply the usual random effects estimator to (11.104). This is asymptotically more efficient than the pooled OLS estimator.

11.17. To obtain (11.81), we used (11.80) and the representation $\sqrt{N}(\hat{\boldsymbol{\beta}}_{FE} - \boldsymbol{\beta}) = \mathbf{A}^{-1}\left(N^{-1/2}\sum_{i=1}^{N}\ddot{\mathbf{X}}_i'\mathbf{u}_i\right) + o_p(1)$. Simple algebra and standard properties of $O_p(1)$.and $o_p(1)$.give

$$\begin{aligned}\sqrt{N}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) &= N^{-1/2}\sum_{i=1}^{N}[(\mathbf{Z}_i'\mathbf{Z}_i)^{-1}\mathbf{Z}_i'(\mathbf{y}_i - \mathbf{x}_i\boldsymbol{\beta}) - \boldsymbol{\alpha}] - \left(N^{-1}\sum_{i=1}^{N}[(\mathbf{Z}_i'\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\mathbf{X}_i\right)\sqrt{N}(\hat{\boldsymbol{\beta}}_{FE} - \boldsymbol{\beta}) \\ &= N^{-1/2}\sum_{i=1}^{N}(\mathbf{s}_i - \boldsymbol{\alpha}) - \mathbf{C}\mathbf{A}^{-1}N^{-1/2}\sum_{i=1}^{N}\ddot{\mathbf{X}}_i'\mathbf{u}_i + o_p(1)\end{aligned}$$

where $\mathbf{C} \equiv \mathrm{E}[(\mathbf{Z}_i'\mathbf{Z}_i)^{-1}\mathbf{Z}_i'\mathbf{X}_i]$ and $\mathbf{s}_i \equiv (\mathbf{Z}_i'\mathbf{Z}_i)^{-1}\mathbf{Z}_i'(\mathbf{y}_i - \mathbf{x}_i\boldsymbol{\beta})$. By definition, $\mathrm{E}(\mathbf{s}_i) = \boldsymbol{\alpha}$. By combining terms in the sum we have

$$\sqrt{N}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) = N^{-1/2} \sum_{i=1}^{N} [(\mathbf{s}_i - \boldsymbol{\alpha}) - \mathbf{C}\mathbf{A}^{-1}\ddot{\mathbf{X}}_i'\mathbf{u}_i] + o_p(1),$$

which implies by the central limit theorem and the asymptotic equivalence lemma that $\sqrt{N}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})$ is asymptotically normal with zero mean and variance $\mathrm{E}(\mathbf{r}_i\mathbf{r}_i')$, where $\mathbf{r}_i \equiv (\mathbf{s}_i - \boldsymbol{\alpha}) - \mathbf{C}\mathbf{A}^{-1}\ddot{\mathbf{X}}_i'\mathbf{u}_i$. If we replace $\boldsymbol{\alpha}$, $\mathbf{C}$, $\mathbf{A}$, and $\boldsymbol{\beta}$ with their consistent estimators, we get exactly (11.81) because the $\hat{\ddot{\mathbf{u}}}_i$ are the $T \times 1$ FE residuals.

11.18. a. Using equation (8.47) we have

$$\hat{\boldsymbol{\beta}}_{REIV} = \left[\left(\sum_{i=1}^{N} \mathbf{X}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Z}_i \right) \left(\sum_{i=1}^{N} \mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Z}_i \right)^{-1} \left(\sum_{i=1}^{N} \mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}_i \right) \right]^{-1}$$
$$\cdot \left(\sum_{i=1}^{N} \mathbf{X}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Z}_i \right) \left(\sum_{i=1}^{N} \mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Z}_i \right)^{-1} \left(\sum_{i=1}^{N} \mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{y}_i \right)$$

where $\hat{\boldsymbol{\Omega}}$ has the RE form (and is probably estimated from the pooled 2SLS residuals).

By arguments very similar to that for FGLS, we can show

$$\sqrt{N}(\hat{\boldsymbol{\beta}}_{REIV} - \boldsymbol{\beta}) = \mathbf{A}^{-1}\mathbf{C}'\mathbf{D}^{-1} \left(N^{-1/2} \sum_{i=1}^{N} \mathbf{Z}_i'\boldsymbol{\Lambda}^{-1}\mathbf{u}_i \right) + o_p(1)$$

where

$$\boldsymbol{\Lambda} = \mathrm{plim}(\hat{\boldsymbol{\Omega}})$$
$$\mathbf{C} = \mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Lambda}^{-1}\mathbf{X}_i)$$
$$\mathbf{D} = \mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Lambda}^{-1}\mathbf{Z}_i)$$
$$\mathbf{A} = \mathbf{C}'\mathbf{D}^{-1}\mathbf{C}$$

Note that this formulation recognizes that $\hat{\boldsymbol{\Omega}}$ is not generally consistent for $\mathrm{E}(\mathbf{v}_i\mathbf{v}_i')$. It follows that

$$\sqrt{N}(\hat{\boldsymbol{\beta}}_{REIV} - \boldsymbol{\beta}) \xrightarrow{d} Normal(\mathbf{0}, \mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1})$$

where

$$\mathbf{B} = \mathbf{C}'\mathbf{D}^{-1}\mathrm{E}(\mathbf{Z}_i'\boldsymbol{\Lambda}^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Lambda}^{-1}\mathbf{Z}_i)\mathbf{D}^{-1}\mathbf{C}$$

b. Consistent estimators of **A** and **B** are

$$\hat{\mathbf{A}} = \hat{\mathbf{C}}'\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}$$
$$\hat{\mathbf{B}} = \hat{\mathbf{C}}'\hat{\mathbf{D}}^{-1}\left(N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Z}_i\right)\hat{\mathbf{D}}^{-1}\hat{\mathbf{C}}$$

where

$$\hat{\mathbf{C}} = N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{X}_i$$
$$\hat{\mathbf{D}} = N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\hat{\boldsymbol{\Omega}}^{-1}\mathbf{Z}_i$$
$$\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}}_{REIV}$$

11.19. a. Below is the Stata output. The *concen* variable is positive and statistically significant using both RE and FE estimation of the reduced form, and using fully robust (that is, to any serial correlation and heteroskedasticity) standard errors. The coefficient is somewhat larger for RE compared with FE, and its standard error is somewhat smaller for RE. We conclude that *concen* is suitably partially correlated with *lfare* in order to apply REIV and FEIV.

```
. xtreg lfare concen ldist ldistsq y98 y99 y00, re cluster(id)

Random-effects GLS regression                   Number of obs      =      4596
Group variable: id                              Number of groups   =      1149

R-sq:  within  = 0.1348                         Obs per group: min =
       between = 0.4176                                        avg =       4.
       overall = 0.4030                                        max =

Random effects u_i ~Gaussian                    Wald chi2(7)       = 386792.48
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                                  (Std. Err. adjusted for 1149 clusters in id
------------------------------------------------------------------------------
             |               Robust
       lfare |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
```

```
-------------+----------------------------------------------------------------
      concen |   .2089935   .0422459     4.95   0.000      .126193    .2917939
       ldist |  -.8520921   .2720902    -3.13   0.002    -1.385379   -.3188051
     ldistsq |   .0974604   .0201417     4.84   0.000     .0579833    .1369375
         y98 |   .0224743   .0041461     5.42   0.000      .014348    .0306005
         y99 |   .0366898   .0051318     7.15   0.000     .0266317     .046748
         y00 |    .098212   .0055241    17.78   0.000     .0873849     .109039
       _cons |   6.222005   .9144067     6.80   0.000     4.429801    8.014209
-------------+----------------------------------------------------------------
     sigma_u |  .31933841
     sigma_e |  .10651186
         rho |  .89988885   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lfare concen y98 y99 y00, fe cluster(id)

Fixed-effects (within) regression               Number of obs      =      4596
Group variable: id                              Number of groups   =      1149

R-sq:  within  = 0.1352                         Obs per group: min =
       between = 0.0576                                        avg =       4.
       overall = 0.0083                                        max =

                                                F(4,1148)          =    120.06
corr(u_i, Xb)  = -0.2033                        Prob > F           =    0.0000

                                  (Std. Err. adjusted for 1149 clusters in id
------------------------------------------------------------------------------
             |               Robust
       lfare |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      concen |    .168859   .0494587     3.41   0.001     .0718194    .2658985
         y98 |   .0228328    .004163     5.48   0.000     .0146649    .0310007
         y99 |   .0363819   .0051275     7.10   0.000     .0263215    .0464422
         y00 |   .0977717   .0055054    17.76   0.000     .0869698    .1085735
       _cons |   4.953331   .0296765   166.91   0.000     4.895104    5.011557
-------------+----------------------------------------------------------------
     sigma_u |  .43389176
     sigma_e |  .10651186
         rho |  .94316439   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

b. The REIV estimates without *ldist* and *ldistsq* from Stata are given below. For comparison, the REIV estimates in Example 11.1 are also reported. Dropping the distance variables changes the estimated elasticity to –.654, which is notably larger in magnitude than –.508. This is a good example of how relevant time-constant variables – when they are available – should be controlled for in an RE analysis.

```
. xtivreg lpassen y98 y99 y00 (lfare=concen), re

G2SLS random-effects IV regression              Number of obs      =      4596
Group variable: id                              Number of groups   =      1149
```

```
R-sq:  within  = 0.4327                         Obs per group: min =
       between = 0.0487                                        avg =       4.
       overall = 0.0578                                        max =

                                                Wald chi2(4)       =    219.33
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lfare |  -.6540984   .4019123    -1.63   0.104    -1.441832    .1336351
         y98 |   .0342955    .011701     2.93   0.003      .011362     .057229
         y99 |   .0847852   .0154938     5.47   0.000     .0544178    .1151525
         y00 |    .146605   .0390819     3.75   0.000      .070006    .2232041
       _cons |    9.28363   2.032528     4.57   0.000     5.299949    13.26731
-------------+----------------------------------------------------------------
     sigma_u |  .91384976
     sigma_e |  .16964171
         rho |   .9666879   (fraction of variance due to u_i)
------------------------------------------------------------------------------
Instrumented:   lfare
Instruments:    y98 y99 y00 concen
------------------------------------------------------------------------------

. xtivreg lpassen ldist ldistsq y98 y99 y00 (lfare=concen), re

G2SLS random-effects IV regression              Number of obs      =      4596
Group variable: id                              Number of groups   =      1149

R-sq:  within  = 0.4075                         Obs per group: min =
       between = 0.0542                                        avg =       4.
       overall = 0.0641                                        max =

                                                Wald chi2(6)       =    231.10
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lfare |  -.5078762    .229698    -2.21   0.027     -.958076   -.0576763
       ldist |  -1.504806   .6933147    -2.17   0.030    -2.863678   -.1459338
     ldistsq |   .1176013   .0546255     2.15   0.031     .0105373    .2246652
         y98 |   .0307363   .0086054     3.57   0.000     .0138699    .0476027
         y99 |   .0796548     .01038     7.67   0.000     .0593104    .0999992
         y00 |   .1325795   .0229831     5.77   0.000     .0875335    .1776255
       _cons |   13.29643   2.626949     5.06   0.000     8.147709    18.44516
-------------+----------------------------------------------------------------
     sigma_u |  .94920686
     sigma_e |  .16964171
         rho |  .96904799   (fraction of variance due to u_i)
------------------------------------------------------------------------------
Instrumented:   lfare
Instruments:    ldist ldistsq y98 y99 y00 concen
------------------------------------------------------------------------------
```

c. Now we have three endogenous variables: *lfare*, (*ldist* – μ_1) • *lfare*, and

$(ldist^2 - \mu_2) \cdot lfare$. We can use

$$concen, (ldist - \mu_1) \cdot concen, \text{ and } (ldist^2 - \mu_2) \cdot concen$$

as instruments. In other words, we add the interactions $(ldist - \mu_1) \cdot concen$ and $(ldist^2 - \mu_2) \cdot concen$ as extra IVs to account for the endogenous interactions in the structural model.

In practice, we replace μ_1 and μ_2 with the sample averages.

d. The Stata output below provides the estimates. Something interesting happens here. The REIV and FEIV estimates of the coefficient on *lfare* now much closer to each other, and much larger in magnitude then the estimates in Table 11.1. In particular, the estimated elasticity at the mean of *ldist* and *ldistsq* is about −1 for REIV and FEIV. Interestingly, the REIV and FEIV estimates with the interactions are close to the RE and FE estimates without the interactions.

```
. egen  mu_ldist = mean(ldist)

. gen dmldist = ldist-mu_ldist

. egen  mu_ldistsq = mean(ldistsq)

. gen dmldistsq = ldistsq-mu_ldistsq

. gen ldist_lfare = dmldist*lfare

. gen ldistsq_lfare = dmldistsq*lfare

. gen ldist_concen = dmldist*concen

. gen ldistsq_concen = dmldistsq*concen

. xtivreg lpassen ldist ldistsq y98 y99 y00 (lfare ldist_lfare ldistsq_lfare
         = concen ldist_concen ldistsq_concen), re

G2SLS random-effects IV regression              Number of obs      =      4596
Group variable: id                              Number of groups   =      1149

R-sq:  within  = 0.1319                         Obs per group: min =
       between = 0.0006                                        avg =       4.
       overall = 0.0016                                        max =

                                                Wald chi2(8)       =    180.72
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000
```

```
------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lfare |  -1.048873   .3250545    -3.23   0.001    -1.685969   -.4117783
 ldist_lfare |   29.63707   7.957828     3.72   0.000     14.04001    45.23413
ldistsq_lf~e |  -2.330287    .638173    -3.65   0.000    -3.581083   -1.079491
       ldist |  -157.8477   42.76284    -3.69   0.000    -241.6613   -74.03409
     ldistsq |   12.45005   3.437782     3.62   0.000     5.712121    19.18798
         y98 |   .0319578   .0105546     3.03   0.002     .0112713    .0526444
         y99 |    .080002   .0127579     6.27   0.000     .0549969    .1050071
         y00 |   .1570325    .026578     5.91   0.000     .1049406    .2091244
       _cons |   504.8691   131.4462     3.84   0.000     247.2392     762.499
-------------+----------------------------------------------------------------
     sigma_u |  1.3686882
     sigma_e |  .19436268
         rho |  .98023276   (fraction of variance due to u_i)
------------------------------------------------------------------------------
Instrumented:   lfare ldist_lfare ldistsq_lfare
Instruments:    ldist ldistsq y98 y99 y00 concen ldist_concen ldistsq_concen
------------------------------------------------------------------------------

. xtivreg lpassen y98 y99 y00 (lfare ldist_lfare ldistsq_lfare
          = concen ldist_concen ldistsq_concen), fe

Fixed-effects (within) IV regression         Number of obs      =         4596
Group variable: id                           Number of groups   =         1149

R-sq:  within  =      .                      Obs per group: min =
       between = 0.0016                                     avg =          4.
       overall = 0.0016                                     max =

                                             Wald chi2(6)       =     4.40e+06
corr(u_i, Xb)  = -0.9913                     Prob > chi2        =       0.0000

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       lfare |  -1.011863   .3214187    -3.15   0.002    -1.641832   -.3818937
 ldist_lfare |   24.11579   6.951145     3.47   0.001      10.4918    37.73979
ldistsq_lf~e |  -1.905021   .5593222    -3.41   0.001    -3.001273   -.8087699
         y98 |   .0322146   .0102786     3.13   0.002     .0120689    .0523603
         y99 |    .080772   .0123315     6.55   0.000     .0566026    .1049414
         y00 |    .155485   .0260008     5.98   0.000     .1045244    .2064456
       _cons |   11.33584   1.694321     6.69   0.000     8.015032    14.65665
-------------+----------------------------------------------------------------
     sigma_u |  6.6845875
     sigma_e |  .19436268
         rho |  .99915529   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F  test that all u_i=0:     F(1148,3441) =    72.37       Prob > F    = 0.0000
------------------------------------------------------------------------------
Instrumented:   lfare ldist_lfare ldistsq_lfare
Instruments:    ldist ldistsq y98 y99 y00 concen ldist_concen ldistsq_concen
------------------------------------------------------------------------------
```

e. We can use the command `xtivreg2`, a user-written program for Stata. The 95%

confidence interval for α_1 is $[-2.408, .385]$, which includes zero. The fully robust joint test of the two interaction terms gives p-value $= .101$, so we might be justified in dropping them. The robust standard error

```
. xtivreg2 lpassen y98 y99 y00 (lfare ldist_lfare ldistsq_lfare
          = concen ldist_concen ldistsq_concen), fe cluster(id)

FIXED EFFECTS ESTIMATION
------------------------
Number of groups =      1149                    Obs per group: min =
                                                               avg =       4.
                                                               max =

IV (2SLS) estimation
--------------------

Estimates efficient for homoskedasticity only
Statistics robust to heteroskedasticity and clustering on id

Number of clusters (id) = 1149                          Number of obs =     4596
                                                        F(  6,  1148) =    14.90
                                                        Prob > F      =   0.0000
Total (centered) SS     =  128.0991685                  Centered R2   =  -0.0148
Total (uncentered) SS   =  128.0991685                  Uncentered R2 =  -0.0148
Residual SS             =  129.9901441                  Root MSE      =    .1942

------------------------------------------------------------------------------
             |               Robust
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       lfare |  -1.011863   .7124916    -1.42   0.156    -2.408321     .384595
 ldist_lfare |   24.11579   11.26762     2.14   0.032      2.03166    46.19993
ldistsq_lf~e |  -1.905021   .8941964    -2.13   0.033    -3.657614   -.1524285
         y98 |   .0322146   .0167737     1.92   0.055    -.0006613    .0650905
         y99 |    .080772   .0261059     3.09   0.002     .0296053    .1319387
         y00 |    .155485   .0625692     2.49   0.013     .0328515    .2781184
------------------------------------------------------------------------------
Instrumented:         lfare ldist_lfare ldistsq_lfare
Included instruments: y98 y99 y00
Excluded instruments: concen ldist_concen ldistsq_concen
------------------------------------------------------------------------------

. test ldist_lfare ldistsq_lfare

 ( 1)  ldist_lfare = 0
 ( 2)  ldistsq_lfare = 0

           chi2(  2) =    4.59
         Prob > chi2 =    0.1008
```

f. In general, the estimated elasticities can be obtained from

$$\widehat{\frac{lpassen}{lfare}} = \hat{\alpha}_1 + \hat{\gamma}_1(ldist - \hat{\mu}_1) + \hat{\gamma}_2(ldist^2 - \hat{\mu}_2)$$

for any value of *ldist*. Calculations are given below. For $dist = 500$ the estimated elasticity is about $.047$ with a very small t statistic. For $dist = 1,500$, the estimated elasticity is -1.77 with fully robust $t = -1.55$. So the magnitude of the elasticity increases substantially as the route distance increase, but the estimates contain substantial noise.

```
. sum ldist ldistsq if y00

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       ldist |      1149    6.696482    .6595331   4.553877   7.909857
     ldistsq |      1149    45.27747    8.729749   20.73779   62.56583

. di log(500) -  6.696482
-.4818739

. di (log(500))^2 - 45.27747
-6.6561162

. lincom lfare - .4818739*ldist_lfare - 6.6561162*ldistsq_lfare

 ( 1)  lfare - .4818739*ldist_lfare - 6.656116*ldistsq_lfare = 0

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         (1) |   .0474081   .7405447     0.06   0.949    -1.404033    1.498849
------------------------------------------------------------------------------

. di log(1500) -  6.696482
.61673839

. di (log(1500))^2 - 45.27747
8.2057224

. lincom lfare +.61673839*ldist_lfare +8.2057224*ldistsq_lfare

 ( 1)  lfare + .6167384*ldist_lfare + 8.205722*ldistsq_lfare = 0

------------------------------------------------------------------------------
     lpassen |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         (1) |  -1.770802   1.142114    -1.55   0.121    -4.009305    .4677006
------------------------------------------------------------------------------
```

Solutions to Chapter 12 Problems

12.1. a. Take the conditional expectation of equation (12.4) with respect to $\mathbf{x}$, and use $\mathrm{E}(u|\mathbf{x}) = 0$:

$$\begin{aligned}\mathrm{E}\{[y - m(\mathbf{x},\boldsymbol{\theta})]^2|\mathbf{x}\} &= \mathrm{E}(u^2|\mathbf{x}) + 2[m(\mathbf{x},\boldsymbol{\theta}_o) - m(\mathbf{x},\boldsymbol{\theta})]\mathrm{E}(u|\mathbf{x}) + \mathrm{E}\{[m(\mathbf{x},\boldsymbol{\theta}_o) - m(\mathbf{x},\boldsymbol{\theta})]^2|\mathbf{x}\} \\ &= \mathrm{E}(u^2|\mathbf{x}) + 0 + [m(\mathbf{x},\boldsymbol{\theta}_o) - m(\mathbf{x},\boldsymbol{\theta})]^2 \\ &= \mathrm{E}(u^2|\mathbf{x}) + [m(\mathbf{x},\boldsymbol{\theta}_o) - m(\mathbf{x},\boldsymbol{\theta})]^2.\end{aligned}$$

The first term does not depend on $\boldsymbol{\theta}$ and the second term is clearly minimized at $\boldsymbol{\theta} = \boldsymbol{\theta}_o$ for any $\mathbf{x}$. Therefore, the parameters of a correctly specified conditional mean function minimize the squared error conditional on any value of $\mathbf{x}$.

b. Part a shows that

$$\mathrm{E}\{[y - m(\mathbf{x},\boldsymbol{\theta}_o)]^2|\mathbf{x}\} \leq \mathrm{E}\{[y - m(\mathbf{x},\boldsymbol{\theta})]^2|\mathbf{x}\}, \text{ all } \boldsymbol{\theta} \in \boldsymbol{\Theta}, \text{ all } \mathbf{x} \in \mathcal{X}.$$

If we take the expected value of both sides – with respect the the distribution of $\mathbf{x}$, of course – an apply iterated expectations, we conclude

$$\mathrm{E}\{[y - m(\mathbf{x},\boldsymbol{\theta}_o)]^2\} \leq \mathrm{E}\{[y - m(\mathbf{x},\boldsymbol{\theta})]^2\}, \text{ all } \boldsymbol{\theta} \in \boldsymbol{\Theta}.$$

In other words, if we know $\boldsymbol{\theta}_o$ solves the population minimization problem conditional on any $\mathbf{x}$, then it also solves the unconditional population problem. Of course, conditional on a particular value of $\mathbf{x}$, $\boldsymbol{\theta}_o$ would usually not be the unique solution. (For example, in the linear case $m(\mathbf{x},\boldsymbol{\theta}) = \mathbf{x}\boldsymbol{\theta}$, any $\boldsymbol{\theta}$ such as that $\mathbf{x}(\boldsymbol{\theta}_o - \boldsymbol{\theta}) = 0$ sets $m(\mathbf{x},\boldsymbol{\theta}_o) - m(\mathbf{x},\boldsymbol{\theta})$ to zero.) Uniqueness of $\boldsymbol{\theta}_o$ as a population minimizer is realistic only after we integrate out $\mathbf{x}$ to obtain $\mathrm{E}\{[y - m(\mathbf{x},\boldsymbol{\theta})]^2\}$.

12.2. a. Since $u = y - \mathrm{E}(y|\mathbf{x})$,

$$\mathrm{Var}(y|\mathbf{x}) = \mathrm{Var}(u|\mathbf{x}) = \mathrm{E}(u^2|\mathbf{x})$$

because $E(u|\mathbf{x}) = 0$. So $E(u^2|\mathbf{x}) = \exp(\alpha_o + \mathbf{x}\boldsymbol{\gamma}_o)$.

b. If we knew the $u_i = y_i - m(\mathbf{x}_i, \boldsymbol{\theta}_o)$, then we could do a nonlinear regression of u_i^2 on $\exp(\alpha + \mathbf{x}\boldsymbol{\gamma})$ and just use the asymptotic theory for nonlinear regression. The NLS estimators of α and $\boldsymbol{\gamma}$ would then solve

$$\min_{\alpha,\boldsymbol{\gamma}} \sum_{i=1}^{N} [u_i^2 - \exp(\alpha + \mathbf{x}_i\boldsymbol{\gamma})]^2.$$

The problem is that $\boldsymbol{\theta}_o$ is unknown. When we replace $\boldsymbol{\theta}_o$ with its NLS estimator, $\hat{\boldsymbol{\theta}}$ – that is we replace u_i^2 with $\hat{u}_i^2$, the squared NLS residuals – we are solving the problem

$$\min_{\alpha,\boldsymbol{\gamma}} \sum_{i=1}^{N} \{[y_i - m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})]^2 - \exp(\alpha + \mathbf{x}_i\boldsymbol{\gamma})\}^2.$$

This objective function has the form of a two-step M-estimator in Section 12.4. Since $\hat{\boldsymbol{\theta}}$ is generally consistent for $\boldsymbol{\theta}_o$, the two-step M-estimator is generally consistent for α_o and $\boldsymbol{\gamma}_o$ (under weak regularity and identification conditions). In fact, $\sqrt{N}$-consistency of $\hat{\alpha}$ and $\hat{\boldsymbol{\gamma}}$ holds very generally.

c. We now estimate $\boldsymbol{\theta}_o$ by solving

$$\min_{\theta} \sum_{i=1}^{N} [y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2 / \exp(\hat{\alpha} + \mathbf{x}_i\hat{\boldsymbol{\gamma}}),$$

where $\hat{\alpha}$ and $\hat{\boldsymbol{\gamma}}$ are from part b. The general theory of WNLS under WNLS.1 to WNLS.3 can be applied.

d. Using the definition of v, write $u^2 = \exp(\alpha_o + \mathbf{x}\gamma_o)v^2$. Taking logs gives $\log(u^2) = \alpha_o + \mathbf{x}\boldsymbol{\gamma}_o + \log(v^2)$. Now, if v is independent of $\mathbf{x}$, so is $\log(v^2)$. Therefore, $E[\log(u^2)|\mathbf{x}] = \alpha_o + \mathbf{x}\boldsymbol{\gamma}_o + E[\log(v^2)|\mathbf{x}] = \alpha_o + \mathbf{x}\boldsymbol{\gamma}_o + \kappa_o$, where $\kappa_o \equiv E[\log(v^2)]$. So, if we

could observe the u_i, and OLS regression of $\log(u_i^2)$ on $1, \mathbf{x}_i$ would be consistent for $(\alpha_o + \kappa_o, \boldsymbol{\gamma}_o)$; in fact, it would be unbiased. By two-step estimation theory, consistency still holds if u_i is replaced with $\hat{u}_i$, by essentially the same argument in part b. So, if $m(\mathbf{x}, \boldsymbol{\theta})$ is linear in $\boldsymbol{\theta}$, we can carry out a weighted NLS procedure without ever doing nonlinear estimation.

e. If we have misspecified the variance function – or, for example, we use the approach in part d but v is not independent of $\mathbf{x}$ – then we should use a fully robust variance-covariance matrix in equation (12.60) with $\hat{h}_i = \exp(\hat{\alpha} + \mathbf{x}_i\hat{\boldsymbol{\gamma}})$.

12.3. a. The approximate elasticity is

$$\partial \log[\hat{\mathrm{E}}(y|\mathbf{z})]/\partial \log(z_1) = \partial[\hat{\theta}_1 + \hat{\theta}_2 \log(z_1) + \hat{\theta}_3 z_2]/\partial \log(z_1) = \hat{\theta}_2.$$

b. This is approximated by $100 \cdot \partial \log[\hat{\mathrm{E}}(y|\mathbf{z})]/\partial z_2 = 100 \cdot \hat{\theta}_3$.

c. Since $\partial \hat{\mathrm{E}}(y|\mathbf{z})/\partial z_2 = \exp[\hat{\theta}_1 + \hat{\theta}_2 \log(z_1) + \hat{\theta}_3 z_2 + \hat{\theta}_4 z_2^2] \cdot (\hat{\theta}_3 + 2\hat{\theta}_4 z_2)$, the estimated turning point is $\hat{z}_2^* = \hat{\theta}_3/(-2\hat{\theta}_4)$. This is a consistent estimator of $z_2^* \equiv \theta_3/(-2\theta_4)$.

d. Since $\nabla_{\boldsymbol{\theta}} m(\mathbf{x}, \boldsymbol{\theta}) = \exp(\mathbf{x}_1\boldsymbol{\theta}_1 + \mathbf{x}_2\boldsymbol{\theta}_2)\mathbf{x}$, the gradient of the mean function evaluated under the null is

$$\nabla_{\boldsymbol{\theta}} \tilde{m}_i = \exp(\mathbf{x}_{i1}\tilde{\boldsymbol{\theta}}_1)\mathbf{x}_i \equiv \tilde{m}_i \mathbf{x}_i,$$

where $\tilde{\boldsymbol{\theta}}_1$ is the restricted NLS estimator. From regression (12.72), we can compute the usual LM statistic as NR_u^2 from the regression $\tilde{u}_i$ on $\tilde{m}_i\mathbf{x}_{i1}, \tilde{m}_i\mathbf{x}_{i2}$, $i = 1, \ldots, N$, where $\tilde{u}_i = y_i - \tilde{m}_i$. For the robust test, we first regress $\tilde{m}_i\mathbf{x}_{i2}$ on $\tilde{m}_i\mathbf{x}_{i1}$ and obtain the $1 \times K_2$ residuals, $\tilde{\mathbf{r}}_i$. Then we compute the statistic as in regression (12.75).

12.4. a. Write the objective function as $(1/2)\sum_{i=1}^{N}[y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2/h(\mathbf{x}_i, \hat{\boldsymbol{\gamma}})$. The objective function, for any value of $\boldsymbol{\gamma}$, is

$$q(\mathbf{w}_i, \boldsymbol{\theta}; \boldsymbol{\gamma}) = (1/2)[y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2/h(\mathbf{x}_i, \boldsymbol{\gamma}).$$

Taking the gradient with respect to $\boldsymbol{\theta}$ gives

$$\begin{aligned}\nabla_{\boldsymbol{\theta}} q(w_i, \boldsymbol{\theta}; \boldsymbol{\gamma}) &= -\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta})[y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]/h(\mathbf{x}_i, \boldsymbol{\gamma}) \\ &= -\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}) u_i(\boldsymbol{\theta})/h(\mathbf{x}_i, \boldsymbol{\gamma}).\end{aligned}$$

Taking the transpose gives us the score with respect to $\boldsymbol{\theta}$ for any $\boldsymbol{\theta}$ and any $\boldsymbol{\gamma}$.

b. This follows because, under WNLS.1, $u_i \equiv u_i(\boldsymbol{\theta}_o)$ has a zero mean given $\mathbf{x}_i$:

$$\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma})|\mathbf{x}_i] = -\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \mathrm{E}(u_i|\mathbf{x}_i)/h(\mathbf{x}_i, \boldsymbol{\gamma}) = \mathbf{0};$$

the value of $\boldsymbol{\gamma}$ plays no role.

c. First, the Jacobian of $\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma})$ with respect to $\boldsymbol{\gamma}$ is

$\nabla_{\boldsymbol{\gamma}} \mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}) = \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' u_i \nabla_{\boldsymbol{\gamma}} h(\mathbf{x}_i, \boldsymbol{\gamma})/[h(\mathbf{x}_i, \boldsymbol{\gamma})]^2$. Everything but u_i is a function only of $\mathbf{x}_i$, so

$$\mathrm{E}[\nabla_{\gamma} \mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma})|\mathbf{x}_i] = \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \mathrm{E}(u_i|\mathbf{x}_i) \nabla_{\boldsymbol{\gamma}} h(\mathbf{x}_i, \boldsymbol{\gamma})/[h(\mathbf{x}_i, \boldsymbol{\gamma})]^2 = \mathbf{0}.$$

It follows by the LIE that the unconditional expectation is zero, too. In other words, we have shown that the key condition (12.37) holds (and we did not rely on Assumption WNLS.3).

d. We would just use equation (12.60), which can be written as

$$\widehat{\mathrm{Avar}(\hat{\boldsymbol{\theta}})} = \left(\sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} \check{m}_i' \nabla_{\boldsymbol{\theta}} \check{m}_i \right)^{-1} \left(\sum_{i=1}^{N} \check{u}_i^2 \nabla_{\boldsymbol{\theta}} \check{m}_i' \nabla_{\boldsymbol{\theta}} \check{m}_i \right) \left(\sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} \check{m}_i' \nabla_{\boldsymbol{\theta}} \check{m}_i \right)^{-1},$$

where $\check{u}_i \equiv \hat{u}_i/\hat{h}_i^{1/2}$ and $\nabla_{\boldsymbol{\theta}} \check{m}_i \equiv \nabla_{\boldsymbol{\theta}} \hat{m}_i/\hat{h}_i^{1/2}$ are the standardized residuals and gradient, respectively.

e. Under Assumption WNLS.3 (along with WNLS.1),

$$\mathrm{Var}(y_i|\mathbf{x}_i) = \mathrm{E}(u_i^2|\mathbf{x}_i) = \sigma_o^2 h(\mathbf{x}_i, \boldsymbol{\gamma}_o),$$

and $\hat{\boldsymbol{\gamma}}$ is $\sqrt{N}$-consistent for $\boldsymbol{\gamma}_o$. This ensures that the asymptotic variance of $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)$ does

not depend on that of $\sqrt{N}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$. Further,

$$\begin{aligned}
\mathbf{A}_o &= \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/h(\mathbf{x}_i, \boldsymbol{\gamma}_o)] \\
\mathbf{B}_o &= \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)'] = \mathrm{E}\{u_i^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/[h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]^2\}.
\end{aligned}$$

By iterated expectations and WNLS.3,

$$\begin{aligned}
\mathrm{E}\{u_i^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/[h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]^2\} &= \mathrm{E}\{u_i^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/[h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]^2\} \\
&= \mathrm{E}(\mathrm{E}\{u_i^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/[h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]^2\} | \mathbf{x}_i) \\
&= \mathrm{E}\{\mathrm{E}(u_i^2 | \mathbf{x}_i) \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/[h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]^2\} \\
&= \mathrm{E}\{\sigma_o^2 h(\mathbf{x}_i, \boldsymbol{\gamma}_o) \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/[h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]^2\} \\
&= \sigma_o^2 \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)/h(\mathbf{x}_i, \boldsymbol{\gamma}_o)] \\
&= \sigma_o^2 \mathbf{A}_o.
\end{aligned}$$

Therefore,

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)] = \sigma_o^2 \mathbf{A}_o^{-1}$$

an a consistent estimator is

$$\hat{\sigma}^2 \left[N^{-1} \sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})/h(\mathbf{x}_i, \hat{\boldsymbol{\gamma}}) \right]^{-1}$$

Dividing this expression by N to get $\widehat{\mathrm{Avar}}(\hat{\boldsymbol{\theta}})$ delivers (12.59).

12.5. a. We need the gradient of $m(\mathbf{x}_i, \boldsymbol{\theta}$ evaluated under the null hypothesis. By the chain rule,

$$\begin{aligned}
\nabla_{\boldsymbol{\beta}} m(\mathbf{x}, \boldsymbol{\theta}) &= g[\mathbf{x}\boldsymbol{\beta} + \delta_1(\mathbf{x}\boldsymbol{\beta})^2 + \delta_2(\mathbf{x}\boldsymbol{\beta})^3] \cdot [\mathbf{x} + 2\delta_1(\mathbf{x}\boldsymbol{\beta})^2\mathbf{x} + 3\delta_2(\mathbf{x}\boldsymbol{\beta})^2\mathbf{x}], \\
\nabla_{\boldsymbol{\delta}} m(\mathbf{x}, \boldsymbol{\theta}) &= g[\mathbf{x}\boldsymbol{\beta} + \delta_1(\mathbf{x}\boldsymbol{\beta})^2 + \delta_2(\mathbf{x}\boldsymbol{\beta})^3] \cdot [(\mathbf{x}\boldsymbol{\beta})^2, (\mathbf{x}\boldsymbol{\beta})^3]
\end{aligned}$$

The gradients with $\delta_1 = \delta_2 = 0$ are

$$\begin{aligned}
\nabla_{\boldsymbol{\beta}} m(\mathbf{x}, \boldsymbol{\beta}, \mathbf{0}) &= g(\mathbf{x}\boldsymbol{\beta}) \cdot \mathbf{x} \\
\nabla_{\boldsymbol{\delta}} m(\mathbf{x}, \boldsymbol{\beta}, \mathbf{0}) &= g(\mathbf{x}\boldsymbol{\beta}) \cdot [(\mathbf{x}\boldsymbol{\beta})^2, (\mathbf{x}\boldsymbol{\beta})^3].
\end{aligned}$$

Let $\tilde{\boldsymbol{\beta}}$ denote the NLS estimator with $\delta_1 = \delta_2 = 0$ imposed. Then $\nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \tilde{\boldsymbol{\theta}}) = g(\mathbf{x}_i\tilde{\boldsymbol{\beta}})\mathbf{x}_i$ and $\nabla_{\boldsymbol{\delta}} m(\mathbf{x}_i, \tilde{\boldsymbol{\theta}}) = g(\mathbf{x}_i\tilde{\boldsymbol{\beta}})[(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2, (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3]$. Therefore, the usual LM statistic can be obtained as NR_u^2 from the regression $\tilde{u}_i$ on $\tilde{g}_i\mathbf{x}_i, \tilde{g}_i \cdot (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2, \tilde{g}_i \cdot (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3,$ where $\tilde{g}_i \equiv g(\mathbf{x}_i\tilde{\boldsymbol{\beta}})$. If $G(\cdot)$ is the identity function, $g(\cdot) \equiv 1$, and the auxiliary regression is

$$\tilde{u}_i \text{ on } \mathbf{x}_i, (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2, (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3,$$

which is a versino of RESET.

b. The VAT version of the test is obtained as follows. As with the LM test, first estimate the model under the null and obtain the NLS estimator, $\tilde{\boldsymbol{\beta}}$, as before. Then estimate the auxiliary model with $(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2$ and $(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3$ as explanatory variables. In other words, act as if the mean function is

$$G[\mathbf{x}_i\boldsymbol{\beta} + \boldsymbol{\delta}_1(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2 + \delta_2(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3]$$

and estimate δ_1 and δ_2 along with $\boldsymbol{\beta}$. A joint Wald test, made robust to heteroskedasticity if necessary, of $H_0 : \delta_1 = 0, \delta_2 = 0$ is asymptotically equivalent (has the same asymptotic size and asymptotic power against local alternatives) to the LM test. Given the way modern software works, this often affords some computational simplification (albeit modest). When $G(\cdot)$ is the identify function, this variable addition approach gives the RESET test in its traditional form.

One danger in using the VAT is that it is tempting to use the second-step estimates of δ_1, δ_1, and even $\boldsymbol{\beta}$ as generally valid estimators. But they are not. If the null is false, $\tilde{\boldsymbol{\beta}}$ is inconsistent for $\boldsymbol{\beta}$ (because $\tilde{\boldsymbol{\beta}}$ is imposed with $\delta_1 = 0$, $\delta_2 = 0$) and so the added variables are not correct under the alternative. The VAT should be used only for testing purposes (just like the LM statistic).

12.6. a. The pooled NLS estimator of $\boldsymbol{\theta}_o$ solves

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N}\sum_{t=1}^{T}[y_{it} - m(\mathbf{x}_{it},\boldsymbol{\theta})]^2/2,$$

and so, to put this into the standard M-estimation framework, we can take the objective function for a random draw i to be $q_i(\boldsymbol{\theta}) \equiv q(\mathbf{w}_i,\boldsymbol{\theta}) \equiv \sum_{t=1}^{T}[y_{it} - m(\mathbf{x}_{it},\boldsymbol{\theta})]^2/2$. The score for random draw i is $\mathbf{s}_i(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} q_i(\boldsymbol{\theta}) = -\sum_{t=1}^{T} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta})' u_{it}(\boldsymbol{\theta})$. Without further assumptions, a consistent estimator of $\mathbf{B}_o$ is

$$\hat{\mathbf{B}} = N^{-1}\sum_{i=1}^{N} \mathbf{s}_i(\hat{\boldsymbol{\theta}})\mathbf{s}_i(\hat{\boldsymbol{\theta}})'$$

where $\hat{\boldsymbol{\theta}}$ is the pooled NLS estimator. The Hessian for observation i, which can be computed as the Jacobian of the score, can be written as

$$\mathbf{H}_i(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}}\mathbf{s}_i(\boldsymbol{\theta}) = -\sum_{t=1}^{T} \nabla^2_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta})u_{it}(\boldsymbol{\theta}) + \sum_{t=1}^{T} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta})'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}).$$

When we plug in $\boldsymbol{\theta}_o$ and use the fact that $\mathrm{E}(u_{it}|\mathbf{x}_{it}) = 0$, all $t = 1,\ldots,T$, then

$$\begin{aligned}\mathbf{A}_o \equiv \mathrm{E}[\mathbf{H}_i(\boldsymbol{\theta}_o)] &= -\sum_{t=1}^{T} \mathrm{E}[\nabla^2_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)u_{it}] + \sum_{t=1}^{T} \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)] \\ &= \sum_{t=1}^{T} \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)]\end{aligned}$$

because $\mathrm{E}[\nabla^2_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)u_{it}] = \mathbf{0}, t = 1,\ldots,T$ by iterated expectations. By the usual law of large numbers argument,

$$\hat{\mathbf{A}} \equiv N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}}) \equiv N^{-1}\sum_{i=1}^{N} \hat{\mathbf{A}}_i$$

is a consistent estimator of $\mathbf{A}_o$. Then, we just use the usual sandwich formula in equation

(12.49).

b. As in the hint we show that $\mathbf{B}_o = \sigma_o^2 \mathbf{A}_o$. First, write $\mathbf{s}_i(\boldsymbol{\theta}) = \sum_{t=1}^{T} \mathbf{s}_{it}(\boldsymbol{\theta})$, where $\mathbf{s}_{it}(\boldsymbol{\theta}) \equiv -\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}) u_{it}(\boldsymbol{\theta})$. Under dynamic completeness of the mean, these scores are serially uncorrelated across t (when evaluated at $\boldsymbol{\theta}_o$, of course). The argument is very similar to the linear regression case from Chapter 7.

Let $r < t$ for concreteness. Then

$$\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{ir}(\boldsymbol{\theta}_o)' | \mathbf{x}_{it}, \mathbf{x}_{ir}, u_{ir}] = \mathrm{E}(u_{it} | \mathbf{x}_{it}, \mathbf{x}_{ir}, u_{ir}) u_{ir} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{ir}, \boldsymbol{\theta}_o) = 0$$

because $\mathrm{E}(u_{it} | \mathbf{x}_{it}, u_{i,t-1}, , \mathbf{x}_{i,t-1}, \ldots,) = 0$ and $r < t$. Now apply the LIE to conclude $\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{ir}(\boldsymbol{\theta}_o)'] = \mathbf{0}$. So we have shown that $\mathbf{B}_o = \sum_{t=1}^{T} \mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{it}(\boldsymbol{\theta}_o)']$. But for each t, apply iterated expectations:

$$\begin{aligned} \mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{it}(\boldsymbol{\theta}_o)'] &= \mathrm{E}(u_{it}^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)] \\ &= \mathrm{E}[\mathrm{E}(u_{it}^2 | \mathbf{x}_{it}) \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)] \\ &= \sigma_o^2 \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)] \end{aligned}$$

where the last equality follows because $\mathrm{E}(u_{it}^2 | \mathbf{x}_{it}) = \sigma_o^2$. It follows that

$$\mathbf{B}_o = \sum_{t=1}^{T} \mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{it}(\boldsymbol{\theta}_o)'] = \sigma_o^2 \sum_{t=1}^{T} \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)] = \sigma_o^2 \mathbf{A}_o.$$

Next, the usual two-step estimation argument – see Lemma 12.1 – shows that

$$(NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{u}_{it}^2 \xrightarrow{p} T^{-1} \sum_{t=1}^{T} \mathrm{E}(u_{it}^2) = \sigma_o^2 \text{ as } N \to \infty.$$

The degrees of freedom correction – putting $NT - P$ in place of NT – does not affect consistency. The variance matrix obtained by ignoring the time dimension and assuming homoskedasticity is simply

$$\hat{\sigma}^2\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})'\nabla_{\theta}m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})\right)^{-1},$$

and we just showed that N times this matrix is a consistent estimator of $\text{Avar}\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o)$.

c. As we just saw in part b, $\mathbf{B}_o = \sigma_o^2\mathbf{A}_o$, which means by slightly extending the argument before (12.69) we can use an extension of the LM statistic there. Namely,

$$LM = \left(\sum_{i=1}^{N}\tilde{\mathbf{s}}_i\right)'\left(\tilde{\sigma}^2\tilde{\mathbf{M}}\right)^{-1}\left(\sum_{i=1}^{N}\tilde{\mathbf{s}}_i\right).$$

It is convenient to choose

$$\tilde{\mathbf{M}} = \sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}}),$$

where $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\beta}}', \tilde{\boldsymbol{\delta}}')$. So the LM statistic can be written as

$$\tilde{\sigma}^{-2}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\tilde{u}_{it}\right)$$

where we take

$$\tilde{\sigma}^2 = (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}^2.$$

(It is common not to use the degrees of freedom adjustment when estimating σ_o^2 under the null.). Finally, the LM statistic can be written as

$$\begin{aligned} LM &= \frac{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\tilde{u}_{it}\right)}{(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}^2} \\ &= NT\cdot\frac{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_{\theta}m(\mathbf{x}_{it},\tilde{\boldsymbol{\theta}})'\tilde{u}_{it}\right.}{\sum_{i=1}^{N}\sum_{t=1}^{T}\tilde{u}_{it}^2} \\ &= NTR_u^2 \end{aligned}$$

because the numerator is the explained sum of squares from the pooled OLS regression

$$\tilde{u}_{it} \text{ on } \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \tilde{\boldsymbol{\theta}}),\ t = 1, \ldots, T;\ i = 1, \ldots, N$$

and the numerator is the (uncentered) total sum of squares.

12.7. a. For each i and g, define $u_{ig} \equiv y_{ig} - m(\mathbf{x}_{ig}, \boldsymbol{\theta}_{og})$, so that $\mathrm{E}(u_{ig}|\mathbf{x}_i) = 0,\ g = 1, \ldots, G$. Further, let $\mathbf{u}_i$ be the $G \times 1$ vector containing the u_{ig}. Then $\mathrm{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i) = \mathrm{E}(\mathbf{u}_i\mathbf{u}_i') = \boldsymbol{\Omega}_o$. Let $\check{\mathbf{u}}_i$ be the vector of nonlinear least squares residuals for each observation i. That is, compute the NLS estimates for each equation g and collect the residuals. Then, by standard arguments (apply Lemma 12.1), a consistent estimator of $\boldsymbol{\Omega}_o$ is

$$\hat{\boldsymbol{\Omega}} \equiv N^{-1} \sum_{i=1}^{N} \check{\mathbf{u}}_i \check{\mathbf{u}}_i'$$

because each NLS estimator, $\hat{\boldsymbol{\theta}}_g$ is consistent for $\boldsymbol{\theta}_{og}$ as $N \to \infty$.

b. This part involves several steps, and I will sketch how each one goes. First, let $\boldsymbol{\gamma}$ be the vector of distinct elements of $\boldsymbol{\Omega}$ – the nuisance parameters in the context of two-step M-estimation. Then, the score for observation i is

$$\begin{aligned} \mathbf{s}(\mathbf{w}_i, \boldsymbol{\theta}; \boldsymbol{\gamma}) &= -\nabla_{\boldsymbol{\theta}} \mathbf{m}(\mathbf{x}_i, \boldsymbol{\theta})' \boldsymbol{\Omega}^{-1} \mathbf{u}_i(\boldsymbol{\theta}) \\ &= -[\mathbf{u}_i(\boldsymbol{\theta}) \otimes \nabla_{\boldsymbol{\theta}} \mathbf{m}(\mathbf{x}_i, \boldsymbol{\theta})]' \mathrm{vec}(\boldsymbol{\Omega}^{-1}) \end{aligned}$$

where $\mathbf{m}(\mathbf{x}_i, \boldsymbol{\theta}$ is the $G \times 1$ vector of conditional mean functions. With this expression we can verify condition (12.37), even though the actual derivatives are complicated. It is clear that $\nabla_{\boldsymbol{\gamma}} \mathbf{s}(\mathbf{w}_i, \boldsymbol{\theta}; \boldsymbol{\gamma})$ is a linear combination of $\mathbf{u}_i(\boldsymbol{\theta})$, where the linear combination is a function of $\mathbf{x}_i$ (and the parameter values). Therefore, because $\mathrm{E}(\mathbf{u}_i|\mathbf{x}_i) = \mathbf{0},\ \mathrm{E}[\nabla_{\boldsymbol{\gamma}} \mathbf{s}(\mathbf{w}_i, \boldsymbol{\theta}_o; \boldsymbol{\gamma})|\mathbf{x}_i] = \mathbf{0}$ for any $\boldsymbol{\gamma}$, that is, any $\boldsymbol{\Omega}$. Its unconditional expectation is zero, too, which verifies (12.37). This shows that we do not have to adjust for the first-stage estimation of $\boldsymbol{\Omega}_o$. (Note: This problem assumes

that $\text{Var}(\mathbf{u}_i|\mathbf{x}_i) = \boldsymbol{\Omega}_o$, but it is clear that (12.37) holds without any assumption about $\text{Var}(\mathbf{u}_i|\mathbf{x}_i)$. We just need the estimator we use, $\hat{\boldsymbol{\Omega}}$, to converge to its limit at the usual $\sqrt{N}$ rate.)

Next we obtain $\mathbf{B}_o \equiv \text{E}[\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)']$:

$$\begin{aligned}\text{E}[\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)'] &= \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)] \\ &= \text{E}\{\text{E}[\nabla\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\mathbf{u}_i\mathbf{u}_i'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)|\mathbf{x}_i] \\ &= \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\text{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i)\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)] \\ &= \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\boldsymbol{\Omega}_o\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)] \\ &= \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)].\end{aligned}$$

Next, we have to derive $\mathbf{A}_o \equiv \text{E}[\mathbf{H}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)]$, and show that $\mathbf{B}_o = \mathbf{A}_o$. The Hessian itself is complicated, but its expected value is not. The Jacobian of $\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma})$ with respect to $\boldsymbol{\theta}$ can be written

$$\mathbf{H}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) = \nabla_{\boldsymbol{\theta}}\mathbf{m}(\mathbf{x}_i,\boldsymbol{\theta})'\boldsymbol{\Omega}^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}(\mathbf{x}_i,\boldsymbol{\theta}) + [\mathbf{I}_P \otimes \mathbf{u}_i(\boldsymbol{\theta})']\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta};\boldsymbol{\gamma}),$$

where $\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta};\boldsymbol{\gamma})$ is a $GP \times P$ matrix, where P is the total number of parameters, that involves Jacobians of the rows $\boldsymbol{\Omega}^{-1}\nabla_{\boldsymbol{\theta}}m_i(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$. The key is that $\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta};\boldsymbol{\gamma})$ depends on $\mathbf{x}_i$, not on $\mathbf{y}_i$. So,

$$\begin{aligned}\text{E}[\mathbf{H}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)|\mathbf{x}_i] &= \nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o) + [\mathbf{I}_P \otimes \text{E}(\mathbf{u}_i|\mathbf{x}_i)']\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o) \\ &= \nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o).\end{aligned}$$

Now iterated expectations gives $\mathbf{A}_o = \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)]$. We have verified (12.37) and also that $\mathbf{A}_o = \mathbf{B}_o$. Therefore, from Theorem 12.3,

$$\text{Avar}\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) = \mathbf{A}_o^{-1} = \{\text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)]\}^{-1}.$$

c. As usual, we replace expectations with sample averages and unknown parameters, and divide the result by N to get $\widehat{\text{Avar}}(\hat{\boldsymbol{\theta}})$:

$$\widehat{\text{Avar}}(\hat{\boldsymbol{\theta}}) = \left(N^{-1} \sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\hat{\boldsymbol{\theta}})' \hat{\boldsymbol{\Omega}}^{-1} \nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\hat{\boldsymbol{\theta}}) \right)^{-1} /N$$
$$= \left(\sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\hat{\boldsymbol{\theta}})' \hat{\boldsymbol{\Omega}}^{-1} \nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\hat{\boldsymbol{\theta}}) \right)^{-1}.$$

The estimate $\hat{\boldsymbol{\Omega}}$ can be based on the multivariate NLS residuals or can be updated after the nonlinear SUR estimates have been obtained.

d. First, note that $\nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\boldsymbol{\theta}_o)$ is a block-diagonal matrix that has G rows, with blocks $\nabla_{\boldsymbol{\theta}_g} m_{ig}(\boldsymbol{\theta}_{og})$, a $1 \times P_g$ vector. (I assume that there are no cross-equation restrictions imposed in the nonlinear SUR estimation.) If $\boldsymbol{\Omega}_o$ is diagonal, so is its inverse. Standard matrix multiplication shows that

$$\nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\boldsymbol{\theta}_o)' \boldsymbol{\Omega}_o^{-1} \nabla_{\boldsymbol{\theta}} \mathbf{m}_i(\boldsymbol{\theta}_o) = \begin{pmatrix} \sigma_{o1}^{-2} \nabla_{\boldsymbol{\theta}_1} m_{i1}^{o\prime} \nabla_{\boldsymbol{\theta}_1} m_{i1}^{o} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sigma_{o2}^{-2} \nabla_{\boldsymbol{\theta}_2} m_{i2}^{o\prime} \nabla_{\boldsymbol{\theta}_1} m_{i2}^{o} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \sigma_{oG}^{-2} \nabla_{\boldsymbol{\theta}_G} m_{iG}^{o\prime} \nabla_{\boldsymbol{\theta}_G} m_{iG}^{o} \end{pmatrix}.$$

Taking expectations and inverting the result shows that $\text{Avar}\sqrt{N}(\hat{\boldsymbol{\theta}}_g - \boldsymbol{\theta}_{og}) = \sigma_{og}^2 [\text{E}(\nabla_{\boldsymbol{\theta}_g} m_{ig}^{o\prime} \nabla_{\boldsymbol{\theta}_g} m_{ig}^{o})]^{-1}$, $g = 1, \ldots, G$. (Note also that the nonlinear SUR estimators are asymptotically uncorrelated across equations.) These asymptotic variances are easily seen to be the same as those for nonlinear least squares on each equation.

e. I cannot see a nonlinear analog of Theorem 7.7. The first hint given in Problem 7.5 does not extend readily to nonlinear models, even when the same regressors appear in each equation. The key is that $\mathbf{X}_i$ is replaced with $\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}_o)$. While this $G \times P$ matrix has a block-diagonal form, as described in part d, the blocks are not the same even when the same regressors appear in each equation. In the linear case, $\nabla_{\boldsymbol{\theta}_g} m_g(\mathbf{x}_i, \boldsymbol{\theta}_{og}) = \mathbf{x}_i$ for all g. But, unless

$\boldsymbol{\theta}_{og}$ is the same in all equations – a very restrictive assumption – $\nabla_{\boldsymbol{\theta}_g} m_g(\mathbf{x}_i, \boldsymbol{\theta}_{og})$ varies across g. For example, if $m_g(\mathbf{x}_i, \boldsymbol{\theta}_{og}) = \exp(\mathbf{x}_i \boldsymbol{\theta}_{og})$ then $\nabla_{\boldsymbol{\theta}_g} m_g(\mathbf{x}_i, \boldsymbol{\theta}_{og}) = \exp(\mathbf{x}_i \boldsymbol{\theta}_{og})\mathbf{x}_i$, and the gradients differ across g.

12.8. As stated in the hint, we can use (12.37) and a modfied version of (12.76),

$$N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) = N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\boldsymbol{\theta}_o; \hat{\boldsymbol{\gamma}}) + \mathbf{A}_o \sqrt{N}\,(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) + o_p(1),$$

to show $\sqrt{N}\,(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}) = \mathbf{A}_o^{-1} N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) + o_p(1)$; this is just standard algebra. Under (12.37),

$$N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) = N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \boldsymbol{\gamma}_o) = o_p(1),$$

by a similar mean value expansion used for the unconstrained two-step M-estimator:

$$N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) = N^{-1/2} \sum_{i=1}^{N} \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \boldsymbol{\gamma}_o) + \mathrm{E}[\nabla_{\boldsymbol{\gamma}} \mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)]\, \sqrt{N}\,(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) + o_p(1),$$

and use $\mathrm{E}[\nabla_{\boldsymbol{\gamma}} \mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)] = \mathbf{0}$. Now, the second-order Taylor expansion gives

$$\begin{aligned} \sum_{i=1}^{N} q(\mathbf{w}_i, \tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) - \sum_{i=1}^{N} q(\mathbf{w}_i, \hat{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) &= \sum_{i=1}^{N} \mathbf{s}_i(\hat{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) + (1/2)(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}})' \left(\sum_{i=1}^{N} \ddot{\mathbf{H}}_i \right) (\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}) \\ &= (1/2)(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}})' \left(\sum_{i=1}^{N} \ddot{\mathbf{H}}_i \right) (\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}). \end{aligned}$$

Therefore,

$$\begin{aligned} 2\left(\sum_{i=1}^{N} q(\mathbf{w}_i, \tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) - \sum_{i=1}^{N} q(\mathbf{w}_i, \hat{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) \right) &= [\sqrt{N}\,(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}})]' \mathbf{A}_o [\sqrt{N}\,(\tilde{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}})] + o_p(1) \\ &= \left(N^{-1/2} \sum_{i=1}^{N} \tilde{\mathbf{s}}_i \right) \mathbf{A}_o^{-1} \left(N^{-1/2} \sum_{i=1}^{N} \tilde{\mathbf{s}}_i \right) + o_p(1), \end{aligned}$$

where $\tilde{\mathbf{s}}_i = \mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \boldsymbol{\gamma}_o)$. Again, this shows the asymptotic equivalence of the QLR and LM statistics. To complete the problem, we should verify that the *LM* statistic is not affected by $\hat{\boldsymbol{\gamma}}$ either, but that follows from $N^{-1/2}\sum_{i=1}^{N}\mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) = N^{-1/2}\sum_{i=1}^{N}\mathbf{s}_i(\tilde{\boldsymbol{\theta}}; \boldsymbol{\gamma}_o) + o_p(1)$.

12.9. a. We cannot say anything in general about Med($y|\mathbf{x}$) because

$$\text{Med}(y|\mathbf{x}) = m(\mathbf{x}, \boldsymbol{\beta}_o) + \text{Med}(u|\mathbf{x})$$

and Med($u|\mathbf{x}$) could be a general function of $\mathbf{x}$.

b. If u and $\mathbf{x}$ are independent, then E($u|\mathbf{x}$) and Med($u|\mathbf{x}$) are both constants, say α and δ. Then E($y|\mathbf{x}$) – Med($y|\mathbf{x}$) = $[m(\mathbf{x}, \boldsymbol{\beta}_o) + \alpha] - [m(\mathbf{x}, \boldsymbol{\beta}_o) + \delta] = \alpha - \delta$, which does not depend on $\mathbf{x}$.

c. When u and $\mathbf{x}$ are independent, the partial effects of x_j on the conditional mean and conditional median are the same, and there is no ambiguity about what is "the effect of x_j on y," at least when only the mean and median are under consideration. In this case, we could interpret large differences between LAD and NLS as perhaps indicating an outlier problem. But it could be just that u and $\mathbf{x}$ are not independent, and so the function $m(\mathbf{x}, \boldsymbol{\beta}_o)$ cannot be both the mean and the median (or differ from each of these by a constant).

12.10. The conditional mean function is $m(\mathbf{x}_i, n_i, \boldsymbol{\beta}) = n_i p(\mathbf{x}_i, \boldsymbol{\beta})$. So we would, as usual, minimize the sum of squared residuals, $\sum_{i=1}^{N}[y_i - n_i p(\mathbf{x}_i, \boldsymbol{\beta})]^2$ with respect to $\boldsymbol{\beta}$. This gives the NLS estimator, say $\check{\boldsymbol{\beta}}$. Define the weights as $\hat{h}_i \equiv n_i p(\mathbf{x}_i, \check{\boldsymbol{\beta}})[1 - p(\mathbf{x}_i, \check{\boldsymbol{\beta}})]$. Then the weighted NLS estimator minimizes $\sum_{i=1}^{N}[y_i - n_i p(\mathbf{x}_i, \beta)]^2/\hat{h}_i$.

12.11. a. The key to the derivation is to verify condition (12.37), which is similar to Problem 12.7. In fact, this contains Problem 12.7 as a special case. In particular, write the score (with respect to $\boldsymbol{\theta}$) for observation i as

$$\begin{aligned}\mathbf{s}(\mathbf{w}_i,\boldsymbol{\theta};\boldsymbol{\gamma}) &= -\nabla_{\boldsymbol{\theta}}\mathbf{m}(\mathbf{x}_i,\boldsymbol{\theta})'[\mathbf{W}(\mathbf{x}_i,\boldsymbol{\gamma})]^{-1}\mathbf{u}_i(\boldsymbol{\theta}) \\ &= -[\mathbf{u}_i(\boldsymbol{\theta}) \otimes \nabla_{\boldsymbol{\theta}}\mathbf{m}(\mathbf{x}_i,\boldsymbol{\theta})]'\text{vec}\{[\mathbf{W}_i(\boldsymbol{\gamma})]^{-1}\}.\end{aligned}$$

The Jacobina of $\mathbf{s}(\mathbf{w}_i,\boldsymbol{\theta};\boldsymbol{\gamma})$ with respect to $\boldsymbol{\gamma}$ is generally complicated, but it is clear that $\nabla_{\boldsymbol{\gamma}}\mathbf{s}(\mathbf{w}_i,\boldsymbol{\theta};\boldsymbol{\gamma})$ is a linear combination of $\mathbf{u}_i(\boldsymbol{\theta})$, where the linear combination is a function of $\mathbf{x}_i$ (and the parameter values). Therefore, because $\text{E}(\mathbf{u}_i|\mathbf{x}_i) = \mathbf{0}$, $\text{E}[\nabla_{\boldsymbol{\gamma}}\mathbf{s}(\mathbf{w}_i,\boldsymbol{\theta}_o;\boldsymbol{\gamma})|\mathbf{x}_i] = \mathbf{0}$ for any $\boldsymbol{\gamma}$, which verifies (12.37). Notice that we do not need to assume $\text{Var}(\mathbf{u}_i|\mathbf{x}_i) = \mathbf{W}(\mathbf{x}_i,\boldsymbol{\gamma}_o)$ for some $\boldsymbol{\gamma}_o$.

Without assuming (12.96) there are no simplications for

$$\begin{aligned}\mathbf{B}_o &\equiv \text{E}[\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*)\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*)'] \\ &= \text{E}\{\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}^*)]^{-1}\mathbf{u}_i\mathbf{u}_i'[\mathbf{W}_i(\boldsymbol{\gamma}^*)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)\}\end{aligned}$$

where $\boldsymbol{\gamma}^* = \text{plim}(\hat{\boldsymbol{\gamma}})$. A consistent estimator of $\mathbf{B}_o$ is

$$\hat{\mathbf{B}} = N^{-1}\sum_{i=1}^{N}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\hat{\boldsymbol{\theta}})'[\mathbf{W}_i(\hat{\boldsymbol{\gamma}})]^{-1}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'[\mathbf{W}_i(\hat{\boldsymbol{\gamma}})]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\hat{\boldsymbol{\theta}})$$

where $\hat{\boldsymbol{\theta}}$ is the WMNLS estimator.

We also need to consistently estimate $\mathbf{A}_o \equiv \text{E}[\mathbf{H}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)]$. Again, the argument is similar to that in Problem 12.7, and uses that the mean function is correctly specified. We can write the Hessian as

$$\mathbf{H}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) = \nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta})'[\mathbf{W}_i(\boldsymbol{\gamma})]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}) + [\mathbf{I}_P \otimes \mathbf{u}_i(\boldsymbol{\theta})']\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta};\boldsymbol{\gamma}),$$

where $\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta};\boldsymbol{\gamma})$ is a $GP \times P$ matrix, where P is the total number of parameters, that involves Jacobians of the rows $[\mathbf{W}_i(\boldsymbol{\gamma})]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$. Therefore,

$$\begin{aligned}\text{E}[\mathbf{H}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*)|\mathbf{x}_i] &= \nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}^*)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o) + [\mathbf{I}_P \otimes \text{E}(\mathbf{u}_i|\mathbf{x}_i)']\mathbf{F}(\mathbf{x}_i,\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*) \\ &= \nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}^*)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o),\end{aligned}$$

and so $\mathbf{A}_o = \mathrm{E}[\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}^*)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)]$. A consistent estimator of $\mathbf{A}_o$ is

$$\hat{\mathbf{A}} = N^{-1}\sum_{i=1}^{N}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\hat{\boldsymbol{\theta}})'[\mathbf{W}_i(\hat{\boldsymbol{\gamma}})]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\hat{\boldsymbol{\theta}}).$$

When we form

$$\widehat{\mathrm{Avar}}(\hat{\boldsymbol{\theta}}) = \hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1}/N,$$

simple algebra shows this expression is the same as (12.98).

b. If we assume (12.96) then

$$\begin{aligned}\mathbf{B}_o &= \mathrm{E}(\mathrm{E}\{\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}_o)]^{-1}\mathbf{u}_i\mathbf{u}_i'[\mathbf{W}_i(\boldsymbol{\gamma}_o)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)\}|\mathbf{x}_i)\\ &= \mathrm{E}\{\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}_o)]^{-1}\mathrm{E}(\mathbf{u}_i\mathbf{u}_i'|\mathbf{x}_i)[\mathbf{W}_i(\boldsymbol{\gamma}_o)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)\}|\mathbf{x}_i)\\ &= \mathrm{E}\{\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)'[\mathbf{W}_i(\boldsymbol{\gamma}_o)]^{-1}\mathbf{W}_i(\boldsymbol{\gamma}_o)[\mathbf{W}_i(\boldsymbol{\gamma}_o)]^{-1}\nabla_{\boldsymbol{\theta}}\mathbf{m}_i(\boldsymbol{\theta}_o)\}\\ &= \mathbf{A}_o\end{aligned}$$

and so

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)] = \mathbf{A}_o^{-1}.$$

c. We can apply Problem 12.8 once we have properly chosen the objective function to ensure $\mathbf{B}_o = \mathbf{A}_o$ when (12.96) holds. That objective function, with nuisance parameters $\boldsymbol{\gamma}$, is

$$q(\mathbf{w}_i, \boldsymbol{\theta}; \boldsymbol{\gamma}) = (1/2)[\mathbf{y}_i - \mathbf{m}(\mathbf{x}_i, \boldsymbol{\theta})]'[\mathbf{W}_i(\boldsymbol{\gamma})]^{-1}[\mathbf{y}_i - \mathbf{m}(\mathbf{x}_i, \boldsymbol{\theta})]$$

The division by two ensures

$$\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)'] = \mathrm{E}[\mathbf{H}_i(\boldsymbol{\theta}_o; \boldsymbol{\gamma}_o)]$$

Now, letting $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ be the restricted and unrestricted estimators, respectively – where both use $\hat{\boldsymbol{\gamma}}$ as the nuisance parameter estimator – the QLR statistic is

$$QLR = 2\left(\sum_{i=1}^{N} q(\mathbf{w}_i, \tilde{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}}) - \sum_{i=1}^{N} q(\mathbf{w}_i, \hat{\boldsymbol{\theta}}; \hat{\boldsymbol{\gamma}})\right)$$
$$= \sum_{i=1}^{N} \tilde{\mathbf{u}}_i' \hat{\mathbf{W}}_i^{-1} \tilde{\mathbf{u}}_i - \sum_{i=1}^{N} \hat{\mathbf{u}}_i' \hat{\mathbf{W}}_i^{-1} \hat{\mathbf{u}}_i$$

where $\hat{\mathbf{W}}_i \equiv \mathbf{W}_i(\hat{\boldsymbol{\gamma}})$, $\tilde{\mathbf{u}}_i \equiv \mathbf{y}_i - \mathbf{m}(\mathbf{x}_i, \tilde{\boldsymbol{\theta}})$, and $\hat{\mathbf{u}}_i \equiv \mathbf{y}_i - \mathbf{m}(\mathbf{x}_i, \hat{\boldsymbol{\theta}})$. Under H_0 and standard regularity conditions, $QLR \xrightarrow{d} \chi^2_Q$, where Q is the number of restrictions.

And F-type statistic is obtained as

$$F = \frac{\left(\sum_{i=1}^{N} \tilde{\mathbf{u}}_i' \hat{\mathbf{W}}_i^{-1} \tilde{\mathbf{u}}_i - \sum_{i=1}^{N} \hat{\mathbf{u}}_i' \hat{\mathbf{W}}_i^{-1} \hat{\mathbf{u}}_i\right) \Big/ Q}{\left(\sum_{i=1}^{N} \hat{\mathbf{u}}_i' \hat{\mathbf{W}}_i^{-1} \hat{\mathbf{u}}_i\right) \Big/ (NG - P)}$$

which can be treated as an approximate $\mathcal{F}_{Q,NG-P}$ random variable. Note that under (12.96)

$$\mathrm{E}[\mathbf{u}_i' \mathbf{W}_i(\boldsymbol{\gamma}_o)^{-1} \mathbf{u}_i] = \mathrm{E}\{\mathrm{E}[\mathbf{u}_i' \mathbf{W}_i(\boldsymbol{\gamma}_o)^{-1} \mathbf{u}_i | \mathbf{x}_i]\}$$
$$= \mathrm{E}\{\mathrm{tr}\, \mathrm{E}[\mathbf{W}_i(\boldsymbol{\gamma}_o)^{-1} \mathbf{u}_i \mathbf{u}_i' | \mathbf{x}_i]\} = G$$

because $\mathrm{E}(\mathbf{u}_i \mathbf{u}_i' | \mathbf{x}_i) = \mathbf{W}_i(\boldsymbol{\gamma}_o)$. Therefore,

$$(NG)^{-1} \sum_{i=1}^{N} \hat{\mathbf{u}}_i' \hat{\mathbf{W}}_i^{-1} \hat{\mathbf{u}}_i \xrightarrow{p} 1$$

and using $NG - P$ is a degrees-of-freedom adjustment.

12.12. a. We can appeal to equation (12.41) and the discussion that follows about the scores for the two problems being uncorrelated. We have

$$\mathbf{s}_i(\boldsymbol{\theta}; \boldsymbol{\delta}) = -\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}), \boldsymbol{\theta})' [y_i - m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}), \boldsymbol{\theta})]$$

and we know, because $\mathrm{E}(y_i | \mathbf{x}_i, \mathbf{w}_i) = m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}_o), \boldsymbol{\theta}_o)$,

$$\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\delta}_o) | \mathbf{x}_i, \mathbf{w}_i] = \mathbf{0}.$$

As usual, this means any function of $(\mathbf{x}_i, \mathbf{w}_i)$ is uncorrelated with $\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\delta}_o)$, including $\mathbf{r}(\mathbf{w}_i, \boldsymbol{\delta}_o)$.

It follows that

$$\mathbf{D}_o = \mathbf{B}_o + \mathbf{F}_o \mathrm{E}[\mathbf{r}(\mathbf{w}_i, \boldsymbol{\delta}_o)\mathbf{r}(\mathbf{w}_i, \boldsymbol{\delta}_o)']\mathbf{F}_o'$$

where

$$\begin{aligned}\mathbf{B}_o &= \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\delta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\delta}_o)'] \\ \mathbf{F}_o &= \mathrm{E}[\nabla_{\boldsymbol{\delta}}\mathbf{s}_i(\boldsymbol{\theta}_o; \boldsymbol{\delta}_o)].\end{aligned}$$

the matrix $\mathbf{F}_o \mathrm{E}[\mathbf{r}(\mathbf{w}_i, \boldsymbol{\delta}_o)\mathbf{r}(\mathbf{w}_i, \boldsymbol{\delta}_o)']\mathbf{F}_o'$ is at least p.s.d., and so $\mathbf{D}_o - \mathbf{B}_o$ is p.s.d. The asymptotic variance of the two-step estimator of $\boldsymbol{\theta}_o$ (standardized by $\sqrt{N}$) is

$$\mathbf{A}_o^{-1}\mathbf{D}_o\mathbf{A}_o^{-1}$$

and that of the estimator where $\boldsymbol{\delta}_o$ is known is

$$\mathbf{A}_o^{-1}\mathbf{B}_o\mathbf{A}_o^{-1}.$$

b. To estimate $\mathbf{A}_o$ under correct specification of the mean it is convenient to use

$$\mathbf{A}_o = \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}_o), \boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}_o), \boldsymbol{\theta}_o)]$$

and so

$$\hat{\mathbf{A}} = N^{-1} \sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \hat{\boldsymbol{\delta}}), \hat{\boldsymbol{\theta}})' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \hat{\boldsymbol{\delta}}), \hat{\boldsymbol{\theta}})]$$

Further,

$$\hat{\mathbf{B}} = N^{-1} \sum_{i=1}^{N} \mathbf{s}_i(\hat{\boldsymbol{\theta}}; \hat{\boldsymbol{\delta}})\mathbf{s}_i(\hat{\boldsymbol{\theta}}; \hat{\boldsymbol{\delta}})'].$$

It remains to consistently estimate $\mathbf{F}_o$. But by the product and chain rules,

$$\begin{aligned}\nabla_{\boldsymbol{\delta}}\mathbf{s}_i(\boldsymbol{\theta}; \boldsymbol{\delta}) = &-\nabla_{\boldsymbol{\delta}}\{\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}), \boldsymbol{\theta})'\}[y_i - m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}), \boldsymbol{\theta})] \\ &+ \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}), \boldsymbol{\theta})' \nabla_{\mathbf{v}} m(\mathbf{x}_i, \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}), \boldsymbol{\theta}) \nabla_{\boldsymbol{\delta}} \mathbf{v}(\mathbf{w}_i, \boldsymbol{\delta}).\end{aligned}$$

When we plug in $(\boldsymbol{\theta}_o, \boldsymbol{\delta}_o)$ the first term has zero mean because the conditional mean is

correctly specified – much like the argument for the Hessian. Therefore,

$$\begin{aligned}\mathbf{F}_o &= \mathrm{E}[\nabla_{\boldsymbol{\delta}}\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\delta}_o)] \\ &= \mathrm{E}[\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_i,\mathbf{v}(\mathbf{w}_i,\boldsymbol{\delta}_o),\boldsymbol{\theta}_o)'\nabla_{\mathbf{v}}m(\mathbf{x}_i,\mathbf{v}(\mathbf{w}_i,\boldsymbol{\delta}_o),\boldsymbol{\theta}_o)\nabla_{\boldsymbol{\delta}}\mathbf{v}(\mathbf{w}_i,\boldsymbol{\delta}_o)\end{aligned}$$

and

$$\hat{\mathbf{F}} = N^{-1}\sum_{i=1}^{N}\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_i,\mathbf{v}(\mathbf{w}_i,\hat{\boldsymbol{\delta}}),\hat{\boldsymbol{\theta}})'\nabla_{\mathbf{v}}m(\mathbf{x}_i,\mathbf{v}(\mathbf{w}_i,\hat{\boldsymbol{\delta}}),\hat{\boldsymbol{\theta}})\nabla_{\boldsymbol{\delta}}\mathbf{v}(\mathbf{w}_i,\hat{\boldsymbol{\delta}})$$

is consistent for $\mathbf{F}_o$. Finally, let

$$\hat{\mathbf{C}} = N^{-1}\sum_{i=1}^{N}\mathbf{r}_i(\hat{\boldsymbol{\delta}})\mathbf{r}_i(\hat{\boldsymbol{\delta}})'.$$

Then

$$\widehat{\mathrm{Avar}}[\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o)] = \hat{\mathbf{A}}^{-1}(\hat{\mathbf{B}}+\hat{\mathbf{F}}\hat{\mathbf{C}}\hat{\mathbf{F}}')\hat{\mathbf{A}}^{-1}$$

which, numerically, will always be larger (in the matrix sense) than $\hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1}$.

12.13. a. Strict exogeneity is not needed because the population objective function is

$$(1/2)\sum_{t=1}^{T}\mathrm{E}\{[y_{it}-m(\mathbf{x}_{it},\boldsymbol{\theta})]^2/h(\mathbf{x}_{it},\boldsymbol{\gamma}^*)],$$

and $\boldsymbol{\theta}_o$ minimizes this function provided

$$\mathrm{E}(y_{it}|\mathbf{x}_{it}) = m(\mathbf{x}_{it},\boldsymbol{\theta}_o),\ t=1,\ldots,T.$$

We do not need

$$\mathrm{E}(y_{it}|\mathbf{x}_{i1},\mathbf{x}_{i2},\ldots,\mathbf{x}_{iT}) = m(\mathbf{x}_{it},\boldsymbol{\theta}_o).$$

The proof of the claim that $\boldsymbol{\theta}_o$ is a minimizer of the population objective function could use the score – assuming that $m(\mathbf{x}_t,\cdot)$ is continuously differentiable and $\boldsymbol{\theta}_o \in \mathrm{int}(\boldsymbol{\Theta})$ – and follow Problem 12.4. But we can show directly that, for each $t = 1,\ldots,T$,

$$\mathrm{E}\{[y_{it} - m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)]^2/h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*)] \leq \mathrm{E}\{[y_{it} - m(\mathbf{x}_{it}, \boldsymbol{\theta})]^2/h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*)], \boldsymbol{\theta} \in \boldsymbol{\Theta}$$

and then inequality clearly holds when we sum over t. Identification requires that strict inequality holds for $\boldsymbol{\theta} \neq \boldsymbol{\theta}_o$ when we sum across t.

To establish the above inequality, we follow Problem 12.1. Applied to a given t, we have

$$\mathrm{E}\{[y_{it} - m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)]^2|\mathbf{x}_{it}\} \leq \mathrm{E}\{[y_{it} - m(\mathbf{x}_{it}, \boldsymbol{\theta})]^2|\mathbf{x}_{it}\}$$

for any $\mathbf{x}_{it}$. Because $h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*) > 0$ the inquality continues to hold if we divide each side by $h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*)$. Further, because $h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*)$ is a function of $\mathbf{x}_{it}$, we can bring it inside both conditional expectations:

$$E\left\{\frac{[y_{it} - m(\mathbf{x}_{it}, \boldsymbol{\theta}_o)]^2}{h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*)} \middle| \mathbf{x}_{it}\right\} \leq E\left\{\frac{[y_{it} - m(\mathbf{x}_{it}, \boldsymbol{\theta})]^2}{h(\mathbf{x}_{it}, \boldsymbol{\gamma}^*)} \middle| \mathbf{x}_{it}\right\}$$

and then take the expected value with respect to $\mathbf{x}_{it}$ on both sides.

b. For each t we can use the same argument for WNLS on a single cross section to show

$$\mathrm{E}[\nabla_{\boldsymbol{\gamma}} \mathbf{s}_{it}(\boldsymbol{\theta}_o; \boldsymbol{\gamma})] = \mathbf{0}$$

for any $\boldsymbol{\gamma}$, where

$$\mathbf{s}_{it}(\boldsymbol{\theta}; \boldsymbol{\gamma}) = -\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it}, \boldsymbol{\theta})' u_{it}(\boldsymbol{\theta})/h(\mathbf{x}_{it}, \boldsymbol{\gamma})$$

Because

$$\mathbf{s}_i(\boldsymbol{\theta}; \boldsymbol{\gamma}) = \sum_{t=1}^{T} \mathbf{s}_{it}(\boldsymbol{\theta}; \boldsymbol{\gamma})$$

it follows that condition (12.37) holds, so we can ignore the estimation of $\boldsymbol{\gamma}^*$ in obtaining $\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)]$. But then we just need to estimate

$$\mathbf{B}_o = \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*)\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*)']$$
$$\mathbf{A}_o = \sum_{t=1}^{T} \mathrm{E}[\mathbf{H}_{it}(\boldsymbol{\theta}_o;\boldsymbol{\gamma}^*)] = \sum_{t=1}^{T} \mathrm{E}\{u_{it}^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)/[h(\mathbf{x}_{it},\boldsymbol{\gamma})]^2\}$$

where $u_{it} = y_{it} - m(\mathbf{x}_{it},\boldsymbol{\theta}_o)$. Consistent estimators are

$$\hat{\mathbf{B}} = N^{-1}\sum_{i=1}^{N} \mathbf{s}_i(\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\gamma}})\mathbf{s}_i(\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\gamma}})'] = N^{-1}\sum_{i=1}^{N}\left[\sum_{t=1}^{T} \mathbf{s}_{it}(\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\gamma}})\right]\left[\sum_{t=1}^{T} \mathbf{s}_{it}(\hat{\boldsymbol{\theta}};\hat{\boldsymbol{\gamma}})\right]'$$
$$= N^{-1}\sum_{i=1}^{N}\left[\sum_{t=1}^{T} \hat{u}_{it}\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})'/h(\mathbf{x}_{it},\hat{\boldsymbol{\gamma}})\right]\left[\sum_{t=1}^{T} \hat{u}_{it}\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})/h(\mathbf{x}_{it},\hat{\boldsymbol{\gamma}})\right]$$

and

$$\hat{\mathbf{A}} = N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} \hat{u}_{it}^2 \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})/[h(\mathbf{x}_{it},\hat{\boldsymbol{\gamma}})]^2.$$

Notice how $\hat{\mathbf{B}}$ includes terms involving $\hat{u}_{it}\hat{u}_{ir}$ for $t \neq r$, thereby allowing for serial correlation. Further, terms involving $\hat{u}_{it}^2/[h(\mathbf{x}_{it},\hat{\boldsymbol{\gamma}})]^2$ mean we are not assuming the variance function is correctly specified.

c. For any $\boldsymbol{\gamma}$ we can write

$$\mathbf{s}_{it}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})\mathbf{s}_{ir}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})' = u_{it}u_{ir}\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{it},\boldsymbol{\theta}_o)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_{ir},\boldsymbol{\theta}_o)/[h(\mathbf{x}_{it},\boldsymbol{\gamma})h(\mathbf{x}_{ir},\boldsymbol{\gamma})].$$

Take $r < t$. Then, by dynamic completeness – that is, $\mathrm{E}(y_{it}|\mathbf{x}_{it}, y_{i,t-1}, \mathbf{x}_{i,t-1}, \ldots, y_{i1}, \mathbf{x}_{i1}) = 0$ – $\mathrm{E}(u_{it}|u_{ir}, \mathbf{x}_{it}, \mathbf{x}_{ir}) = 0$,and so

$$\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})|u_{ir}, \mathbf{x}_{it}, \mathbf{x}_{ir}] = \mathbf{0}.$$

Therefore,

$$\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})\mathbf{s}_{ir}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})'|u_{ir}, \mathbf{x}_{it}, \mathbf{x}_{ir}] = \mathbf{0}$$

and so $\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})\mathbf{s}_{ir}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})'] = \mathbf{0}$. It follows that we need not estimate the terms

$\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})\mathbf{s}_{ir}(\boldsymbol{\theta}_o;\boldsymbol{\gamma})']$, and so a consistent estimator of $\mathbf{B}_o$ is

$$N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\hat{u}_{it}^2\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})'\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\hat{\boldsymbol{\theta}})/[h(\mathbf{x}_{it},\hat{\boldsymbol{\gamma}})]^2,$$

and we need not change $\hat{\mathbf{A}}$ because the conditional mean for each t is assumed to be correctly specified.

Remember that, because our analysis is for fixed T and $N \to \infty$, and we are using the usual $\sqrt{N}$-limiting distribution, there is nothing wrong with using the fully robust form even under dynamic completeness. There is a sense that imposing zero correlation in the scores when they are uncorrelated leads to better finite-sample inference, but that is difficult to establish in any generality.

d. Again, we can keep $\hat{\mathbf{A}}$ the same. For $\hat{\mathbf{B}}$ we can use either of the estimators in parts b or c. But if we want to use both dynamic completeness and a correctly specified conditional variance, we can simplify $\hat{\mathbf{B}}$ even further.

$$\begin{aligned}\mathbf{B}_o &= \sum_{t=1}^{T}\mathrm{E}\{u_{it}^2\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)/[h(\mathbf{x}_{it},\boldsymbol{\gamma}_o)]^2\}\\ &= \sum_{t=1}^{T}\mathrm{E}(\mathrm{E}\{u_{it}^2\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)/[h(\mathbf{x}_{it},\boldsymbol{\gamma}_o)]^2|\mathbf{x}_{it}\})\\ &= \sum_{t=1}^{T}\mathrm{E}\{[\sigma_o^2h(\mathbf{x}_{it},\boldsymbol{\gamma}_o)]\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)/[h(\mathbf{x}_{it},\boldsymbol{\gamma}_o)]^2\}\\ &= \sigma_o^2\sum_{t=1}^{T}\mathrm{E}\{\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}}m(\mathbf{x}_{it},\boldsymbol{\theta}_o)/[h(\mathbf{x}_{it},\boldsymbol{\gamma}_o)]\} \quad = \sigma_o^2\mathbf{A}_o.\end{aligned}$$

So

$$\widehat{\mathrm{Avar}}[\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o)] \quad = \hat{\sigma}^2\hat{\mathbf{A}}^{-1}$$

where

$$\hat{\sigma}^2 = (NT - P)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{u}_{it}^2 / h(\mathbf{x}_{it}, \hat{\boldsymbol{\gamma}})$$

is easily shown to be consistent for σ_o^2: by iterated expectations,

$$\mathrm{E}[u_{it}^2 / h(\mathbf{x}_{it}, \boldsymbol{\gamma}_o)] = \sigma_o^2, \, t = 1, \ldots, T.$$

12.14. Write the score evaluated at $\boldsymbol{\theta}_o$ as

$$\begin{aligned} \mathbf{s}_i(\boldsymbol{\theta}_o) &= -\mathbf{x}_i' \{\tau 1[y_i - \mathbf{x}_i \boldsymbol{\theta}_o \geq 0] - (1 - \tau) 1[y_i - \mathbf{x}_i \boldsymbol{\theta}_o < 0]\} \\ &= -\mathbf{x}_i' \{\tau 1[u_i \geq 0] - (1 - \tau) 1[u_i < 0]\} \end{aligned}$$

where $u_i \equiv y_i - \mathbf{x}_i \boldsymbol{\theta}_o$. Therefore,

$$\begin{aligned} \mathbf{s}_i(\boldsymbol{\theta}_o) \mathbf{s}_i(\boldsymbol{\theta}_o)' &= \{\tau 1[u_i \geq 0] - (1 - \tau) 1[u_i < 0]\}^2 \mathbf{x}_i' \mathbf{x}_i \\ &= (\tau^2 1[u_i \geq 0] + (1 - \tau)^2 1[u_i < 0]) \mathbf{x}_i' \mathbf{x}_i \end{aligned}$$

where this expression uses the hint that $1[u_i \geq 0] \cdot 1[u_i < 0] = 0$ and the square of an indicator function is just itself.

Now take the expectation conditional on $\mathbf{x}_i$:

$$\begin{aligned} \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o) \mathbf{s}_i(\boldsymbol{\theta}_o)' | \mathbf{x}_i] &= \{\tau^2 \mathrm{E}(1[u_i \geq 0] | \mathbf{x}_i) + (1 - \tau)^2 \mathrm{E}(1[u_i < 0] | \mathbf{x}_i)\} \mathbf{x}_i' \mathbf{x}_i \\ &= [\tau^2 (1 - \tau) + (1 - \tau)^2 \tau] \mathbf{x}_i' \mathbf{x}_i \\ &= \tau (1 - \tau) \mathbf{x}_i' \mathbf{x}_i, \end{aligned}$$

where we use the fact that $\mathrm{E}(1[u_i < 0] | \mathbf{x}_i) = \mathrm{P}(y_i < \mathbf{x}_i \boldsymbol{\theta}_o | \mathbf{x}_i) = \tau$ – see the discussion below equation (12.110). Now apply iterated expectations to get (12.115).

12.15. a. $\hat{\boldsymbol{\theta}}$ (approximately) solves the first order condition

$$\sum_{i=1}^{N} \sum_{i=1}^{T} -\mathbf{x}_{it}' \{\tau 1[y_{it} - \mathbf{x}_{it} \hat{\boldsymbol{\theta}} \geq 0] - (1 - \tau) 1[y_{it} - \mathbf{x}_{it} \hat{\boldsymbol{\theta}} < 0]\} \equiv \mathbf{0},$$

so the score function for time period t is

$$\mathbf{s}_{it}(\boldsymbol{\theta}) = -\mathbf{x}_{it}'\{\tau 1[y_{it} - \mathbf{x}_{it}\boldsymbol{\theta} \geq 0] - (1-\tau)1[y_{it} - \mathbf{x}_{it}\boldsymbol{\theta} < 0]\}$$

and the score for random draw i is

$$\mathbf{s}_i(\boldsymbol{\theta}) = \sum_{i=1}^{T} \mathbf{s}_{it}(\boldsymbol{\theta}).$$

b. We have to show that the scores $\{\mathbf{s}_{it}(\boldsymbol{\theta}_o) : t = 1, \ldots, T\}$ in part a are serially uncorrelated. Now

$$\mathbf{s}_{it}(\boldsymbol{\theta}_o) = -\mathbf{x}_{it}'\{\tau 1[u_{it} \geq 0] - (1-\tau)1[u_{it} < 0]\}$$

and, under dynamic completeness of the quantile,

$$\begin{aligned} &\mathrm{E}\{\tau 1[u_{it} \geq 0] - (1-\tau)1[u_{it} < 0] | \mathbf{x}_{it}, y_{i,t-1}, \ldots, y_{i1}, \mathbf{x}_{i1}\} \\ &\quad = \mathrm{E}\{\tau 1[u_{it} \geq 0] - (1-\tau)1[u_{it} < 0] | \mathbf{x}_{it}\} \\ &\quad = \tau(1-\tau) - (1-\tau)\tau = 0. \end{aligned}$$

Therefore,

$$\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o) | \mathbf{x}_{it}, y_{i,t-1}, \ldots, y_{i1}, \mathbf{x}_{i1}] = -\mathbf{x}_{it}'\mathrm{E}[\tau 1[u_{it} \geq 0] - (1-\tau)1[u_{it} < 0] | \mathbf{x}_{it}, y_{i,t-1}, \ldots, y_{i1}, \mathbf{x}_{i1}\} = \mathbf{0},$$

and it follows that if $r < t$, $\mathbf{s}_{ir}(\boldsymbol{\theta}_o)$ is uncorrelated with $\mathbf{s}_{it}(\boldsymbol{\theta}_o)$. Therefore,

$$\mathbf{B}_o = \sum_{t=1}^{T} \mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{it}(\boldsymbol{\theta}_o)'] = \tau(1-\tau)\sum_{t=1}^{T} \mathrm{E}(\mathbf{x}_{it}'\mathbf{x}_{it})$$

and a consistent estimator is

$$\hat{\mathbf{B}} = \tau(1-\tau)N^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} \mathbf{x}_{it}'\mathbf{x}_{it}$$

c. This follows in a way similar to the to the cross section case. Now

$$\mathbf{a}(\boldsymbol{\theta}) = \sum_{i=1}^{T} \mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta})]$$

and we need its Jacobian. We use $\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta})] = \mathrm{E}\{\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta})|\mathbf{x}_{it}]\}$ for each t, and then, just as in Section 12.10.2,

$$\nabla_{\boldsymbol{\theta}}\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta})|\mathbf{x}_{it}] = f_{u_t}(\mathbf{x}_{it}(\boldsymbol{\theta} - \boldsymbol{\theta}_o)|\mathbf{x}_{it})\mathbf{x}_{it}'\mathbf{x}_{it}$$

Then

$$\mathbf{A}_o = \sum_{t=1}^{T} \mathrm{E}\{\nabla_{\boldsymbol{\theta}}\mathrm{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)|\mathbf{x}_{it}]\} = \sum_{t=1}^{T} \mathrm{E}[f_{u_t}(0|\mathbf{x}_{it})\mathbf{x}_{it}'\mathbf{x}_{it}].$$

12.16. a. Because $\mathrm{Med}(y_2|\mathbf{z}) = \mathbf{z}\boldsymbol{\pi}_2$, we would use LAD.

b. From

$$y_1 = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + u_1$$

we have

$$\begin{aligned}\mathrm{Med}(y_1|y_2, \mathbf{z}) &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \mathrm{Med}(u_1|y_2, \mathbf{z}) \\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 v_2 \\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1(y_2 - \mathbf{z}\boldsymbol{\pi}_2)\end{aligned}$$

We can use a control function approach but based on LAD. So, in the first stage, estimate $\boldsymbol{\pi}_2$ by LAD and compute, for each i,

$$\hat{v}_{i2} = y_{i2} - \mathbf{z}_i\hat{\boldsymbol{\pi}}_2.$$

Then use LAD in the second stage. Using dummy arguments of optimization,

$$\min_{\mathbf{d}_1, a_1, r_1} \sum_{i=1}^{N} |y_{i1} - \mathbf{z}_{i1}\mathbf{d}_1 - a_1 y_{i2} - r_1\hat{v}_{i2}|$$

to get $\hat{\boldsymbol{\delta}}_1$, $\hat{\alpha}_1$, and $\hat{\rho}_1$. These estimators are generally consistent by the two-step estimation result discussed in Section 12.4.1.

c. It is natural to use the LAD t statistic for $\hat{\rho}_1$ from the control function procedure in part

b. We know from Chapter 6 that if we were using OLS in both stages then we could ignore the first-stage estimation of $\boldsymbol{\pi}_2$ under $H_0 : \rho_1 = 0$. That seems very likely the case here, too, but it does not follow from the results presented in the text (which assume smooth objective functions with nonsingular expected Hessians).

d. As mentioned in part c, an analytical calculation requires an extended set of tools, such as those in Newey and McFadden (1994). A computationally intensive solution is to bootstrap the two-step estimation method (being sure to recompute $\hat{\boldsymbol{\pi}}_2$ with every bootstrap sample in order to account for its sampling distribution).

12.17. a. We use a mean value expansion, similar to the delta method from Chapter 3 but now allowing for the randomness of $\mathbf{w}_i$. By a mean value expansion, we can write

$$N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i,\hat{\boldsymbol{\theta}}) = N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i,\boldsymbol{\theta}_o) + \left(N^{-1}\sum_{i=1}^{N}\ddot{\mathbf{G}}_i\right)\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o),$$

where $\ddot{\mathbf{G}}_i$ is the $M \times P$ Jacobian of $\mathbf{g}(\mathbf{w}_i,\boldsymbol{\theta})$ evaluated at mean values between $\boldsymbol{\theta}_o$ and $\hat{\boldsymbol{\theta}}$. Now, because $\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) \overset{a}{\sim} \text{Normal}(\mathbf{0},\mathbf{A}_o^{-1}\mathbf{B}_o\mathbf{A}_o^{-1})$, it follows that $\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) = O_p(1)$. Further, by Lemma 12.1, $N^{-1}\sum_{i=1}^{N}\ddot{\mathbf{G}}_i \xrightarrow{p} \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{w},\boldsymbol{\theta}_o)] \equiv \mathbf{G}_o$ (the mean values all converge in probability to $\boldsymbol{\theta}_o$). Therefore,

$$\left(N^{-1}\sum_{i=1}^{N}\ddot{\mathbf{G}}_i\right)\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) = \mathbf{G}_o\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) + o_p(1),$$

and so

$$N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i,\hat{\boldsymbol{\theta}}) = N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i,\boldsymbol{\theta}_o) + \mathbf{G}_o\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) + o_p(1).$$

Because $\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o) = -N^{-1/2}\sum_{i=1}^{N}\mathbf{A}_o^{-1}\mathbf{s}_i(\boldsymbol{\theta}_o) = o_p(1)$, we can write

$$\sqrt{N}\hat{\boldsymbol{\delta}} = N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i,\hat{\boldsymbol{\theta}}) = N^{-1/2}\sum_{i=1}^{N}[g(\mathbf{w}_i,\boldsymbol{\theta}_o) - \mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i(\boldsymbol{\theta}_o)] + o_p(1)$$

or, subtracting $\sqrt{N}\boldsymbol{\delta}_o$ from both sides,

$$\sqrt{N}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}_o) = N^{-1/2}\sum_{i=1}^{N}[\mathbf{g}(\mathbf{w}_i,\boldsymbol{\theta}_o) - \boldsymbol{\delta}_o - \mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i(\boldsymbol{\theta}_o)] + o_p(1).$$

Now

$$\begin{aligned}\mathrm{E}[\mathbf{g}(\mathbf{w}_i,\boldsymbol{\theta}_o) - \boldsymbol{\delta}_o - \mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i(\boldsymbol{\theta}_o)] &= \mathrm{E}[\mathbf{g}(\mathbf{w}_i,\boldsymbol{\theta}_o)] - \boldsymbol{\delta}_o - \mathbf{G}_o\mathbf{A}_o^{-1}\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)] \\ &= \boldsymbol{\delta}_o - \boldsymbol{\delta}_o = \mathbf{0}.\end{aligned}$$

Therefore, by the CLT for i.i.d. sequences,

$$\sqrt{N}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}_o) \overset{a}{\sim} \mathrm{Normal}(0,\mathbf{D}_o)$$

where

$$\mathbf{D}_o = \mathrm{Var}(\mathbf{g}_i - \boldsymbol{\delta}_o - \mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i),$$

where hopefully the shorthand is clear. This differs from the usual delta method result because the randomness in $\mathbf{g}_i = \mathbf{g}_i(\boldsymbol{\theta}_o)$ must be accounted for.

b. We assume we have $\hat{\mathbf{A}}$ consistent for $\mathbf{A}_o$. By the usual arguments, $\hat{\mathbf{G}} = N^{-1}\sum_{i=1}^{N}\nabla_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{w}_i,\hat{\boldsymbol{\theta}})$ is consistent for $\mathbf{G}_o$. Then

$$\hat{\mathbf{D}} = N^{-1}\sum_{i=1}^{N}(\hat{\mathbf{g}}_i - \hat{\boldsymbol{\delta}} - \hat{\mathbf{G}}\hat{\mathbf{A}}^{-1}\hat{\mathbf{s}}_i)(\hat{\mathbf{g}}_i - \hat{\boldsymbol{\delta}} - \hat{\mathbf{G}}\hat{\mathbf{A}}^{-1}\hat{\mathbf{s}}_i)'$$

is consistent for $\mathbf{D}_o$, where the "^" denotes evaluation at $\hat{\boldsymbol{\theta}}$.

c. Using the shorthand notation, if $\mathrm{E}(\mathbf{s}_i|\mathbf{x}_i) = \mathbf{0}$ then $\mathbf{g}_i$ is uncorrelated with $\mathbf{s}_i$ because the premise of the problem is that $\mathbf{g}_i$ is a function of $\mathbf{x}_i$. Thefore, $(\mathbf{g}_i - \boldsymbol{\delta}_o)$ is uncorrelated with $\mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i$, which means

$$
\begin{aligned}
\mathbf{D}_o &= \text{Var}(\mathbf{g}_i - \boldsymbol{\delta}_o - \mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i) \\
&= \text{Var}(\mathbf{g}_i - \boldsymbol{\delta}_o) + \text{Var}(\mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{s}_i) \\
&= \text{Var}(\mathbf{g}_i) + \mathbf{G}_o\mathbf{A}_o^{-1}\mathbf{B}_o\mathbf{A}_o^{-1}\mathbf{G}_o' \\
&= \text{Var}(\mathbf{g}_i) + \mathbf{G}_o[\text{Avar}\ \sqrt{N}\,(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)]\mathbf{G}_o',
\end{aligned}
$$

which is what we wanted to show.

Solutions to Chapter 13 Problems

13.1. No. We know that $\boldsymbol{\theta}_o$ solves

$$\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathrm{E}[\log f(\mathbf{y}_i|\mathbf{x}_i;\theta)],$$

where the expectation is over the joint distribution of $(\mathbf{x}_i, \mathbf{y}_i)$. Therefore, because $\exp(\cdot)$ is an increasing function, $\boldsymbol{\theta}_o$ also maximizes $\exp\{\mathrm{E}[\log f(\mathbf{y}_i|\mathbf{x}_i;\boldsymbol{\theta})]\}$ over $\boldsymbol{\Theta}$. The problem is that the expectation and the exponential function cannot be interchanged: $\mathrm{E}[f(\mathbf{y}_i|\mathbf{x}_i;\boldsymbol{\theta})] \neq \exp\{\mathrm{E}[\log f(\mathbf{y}_i|\mathbf{x}_i;\boldsymbol{\theta})]\}$. In fact, Jensen's inequality tells us that

$$\mathrm{E}[f(\mathbf{y}_i|\mathbf{x}_i;\boldsymbol{\theta})] > \exp\{\mathrm{E}[\log f(\mathbf{y}_i|\mathbf{x}_i;\boldsymbol{\theta})]\}$$

13.2. a. Because

$$f(y|\mathbf{x}_i) = (2\pi\sigma_o^2)^{-1/2} \exp[-(y - m(\mathbf{x}_i, \boldsymbol{\beta}_o))^2/(2\sigma_o^2)],$$

it follows that for observation i the log likelihood is

$$\ell_i(\boldsymbol{\beta}, \sigma^2) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}[y_i - m(\mathbf{x}_i, \boldsymbol{\beta})]^2.$$

Only the last of these terms depends on $\boldsymbol{\beta}$. Further, for any $\sigma^2 > 0$, maximizing $\sum_{i=1}^N \ell_i(\boldsymbol{\beta}, \sigma^2)$ with respect to $\boldsymbol{\beta}$ is the same as minimizing

$$\sum_{i=1}^N [y_i - m(\mathbf{x}_i, \boldsymbol{\beta})]^2,$$

which means the MLE $\hat{\boldsymbol{\beta}}$ is the NLS estimator.

b. First,

$$\nabla_{\boldsymbol{\beta}} \ell_i(\boldsymbol{\beta}, \sigma^2) = \nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \boldsymbol{\beta})[y_i - m(\mathbf{x}_i, \boldsymbol{\beta})]/\sigma^2;$$

note that $\nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \boldsymbol{\beta})$ is $1 \times P$. Next,

$$\frac{\partial \ell_i(\boldsymbol{\beta}, \sigma^2)}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}[y_i - m(\mathbf{x}_i, \boldsymbol{\beta})]^2.$$

For notational simplicity, define the residual function $u_i(\boldsymbol{\beta}) \equiv y_i - m(\mathbf{x}_i, \boldsymbol{\beta})$. Then the score is

$$\mathbf{s}_i(\boldsymbol{\theta}) = \begin{pmatrix} \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta})' u_i(\boldsymbol{\beta})/\sigma^2 \\ -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}[u_i(\boldsymbol{\beta})]^2 \end{pmatrix},$$

where $\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}) \equiv \nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \boldsymbol{\beta})$.

Define the errors as $u_i \equiv u_i(\boldsymbol{\beta}_o)$, so that $\mathrm{E}(u_i|\mathbf{x}_i) = 0$ and $\mathrm{E}(u_i^2|\mathbf{x}_i) = \mathrm{Var}(y_i|\mathbf{x}_i) = \sigma_o^2$. Then, since $\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)$ is a function of $\mathbf{x}_i$, it is easily seen that $\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)|\mathbf{x}_i] = \mathbf{0}$. Note that we only use the fact that $\mathrm{E}(y_i|\mathbf{x}_i) = m(\mathbf{x}_i, \boldsymbol{\beta}_o)$ and $\mathrm{Var}(y_i|\mathbf{x}_i) = \sigma_o^2$ in showing this. In other words, only the first two conditional moments of y_i need to be correctly specified; nothing else about the normal distribution is used.

The equation used to obtain $\hat{\sigma}^2$ is

$$\sum_{i=1}^{N}\left(-\frac{1}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}[y_i - m(\mathbf{x}_i, \hat{\boldsymbol{\beta}})]^2\right) = 0,$$

where $\hat{\boldsymbol{\beta}}$ is the nonlinear least squares estimator. Solving gives

$$\hat{\sigma}^2 = N^{-1}\sum_{i=1}^{N} \hat{u}_i^2,$$

where $\hat{u}_i \equiv y_i - m(\mathbf{x}_i, \hat{\boldsymbol{\beta}})$. Thus, the MLE of σ^2 is the sum of squared residuals divided by N. In practice, N is often replaced with $N - P$ as a degrees-of-freedom adjustment, but this makes no difference as $N \to \infty$.

c. The derivations are a bit tedious but fairly straightforward:

$$\mathbf{H}_i(\theta) = \begin{pmatrix} -\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta})' \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta})/\sigma^2 + \nabla^2_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}) u_i(\boldsymbol{\beta})/\sigma^2 & -\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta})' u_i(\boldsymbol{\beta})/\sigma^4 \\ -\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}) u_i(\boldsymbol{\beta})/\sigma^4 & \frac{1}{2\sigma^4} - \frac{1}{\sigma^6}[u_i(\boldsymbol{\beta})]^2 \end{pmatrix},$$

where $\nabla^2_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta})$ is the $P \times P$ Hessian of $m_i(\boldsymbol{\beta})$.

d. From part c and $\mathrm{E}(u_i|\mathbf{x}_i) = 0$, the off-diagonal blocks are zero. Further,

$$\mathrm{E}[\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)/\sigma_o^2 - \nabla^2_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o) u_i/\sigma_o^2 | \mathbf{x}_i] = \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)/\sigma_o^2$$

Because , $\mathrm{E}(u_i^2|\mathbf{x}_i) = \sigma_o^2$,

$$\mathrm{E}\left(\frac{1}{\sigma_o^6} u_i^2 - \frac{1}{2\sigma_o^4} \middle| \mathbf{x}_i \right) = \frac{1}{\sigma_o} - \frac{1}{2\sigma_o^4} = \frac{1}{2\sigma_o^4}.$$

Therefore,

$$-\mathrm{E}[\mathbf{H}_i(\theta_o)|\mathbf{x}_i] = \begin{pmatrix} \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)/\sigma_o^2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{2\sigma_o^4} \end{pmatrix} \tag{13.99}$$

where we again use $\mathrm{E}(u_i|\mathbf{x}_i) = 0$ and $\mathrm{E}(u_i^2|\mathbf{x}_i) = \sigma_o^2$.

e. To show that $-\mathrm{E}[\mathbf{H}_i(\theta_o)|\mathbf{x}_i]$ equals $\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'|\mathbf{x}_i]$, we need to know that, with u_i defined as above, $\mathrm{E}(u_i^3|\mathbf{x}_i) = 0$, which can be used, along with the zero mean and constant conditional variance, to show

$$\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'|\mathbf{x}_i] = \begin{pmatrix} \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)/\sigma_o^2 & \mathbf{0} \\ \mathbf{0} & \mathrm{E}\left[\left(-\frac{1}{2\sigma_o^2} + \frac{1}{2\sigma_o^4} u_i^2 \right)^2 \right] \end{pmatrix}.$$

Further, $\mathrm{E}(u_i^4|\mathbf{x}_i) = 3\sigma_o^4$, and so

$$\mathrm{E}\left[\left(-\frac{1}{2\sigma_o^2} + \frac{1}{2\sigma_o^4} u_i^2 \right)^2 \right] = \frac{1}{4\sigma_o^4} + \frac{3\sigma_o^4}{4\sigma_o^8} - \frac{2\sigma_o^2}{4\sigma_o^6} = \frac{1}{2\sigma_o^4}.$$

Thus, we have shown $-\mathrm{E}[\mathbf{H}_i(\theta_o)|\mathbf{x}_i] = \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'|\mathbf{x}_i]$.

f. From general MLE, we know that $\text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o)$ is the $P \times P$ upper left hand block of $\{\text{E}[\mathbf{A}_i(\boldsymbol{\theta}_o)]\}^{-1}$, where $\mathbf{A}_i(\boldsymbol{\theta}_o)$ is the matrix in (13.99). Because this matrix is block diagonal, it is easily seen that

$$\text{Avar}\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) = \sigma_o^2\{\text{E}[\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)'\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}_o)]\}^{-1},$$

and this is consistently estimated by

$$\hat{\sigma}^2\left(N^{-1}\sum_{i=1}^{N}\nabla_{\boldsymbol{\beta}}\hat{m}_i'\nabla_{\boldsymbol{\beta}}\hat{m}_i\right)^{-1}, \tag{13.100}$$

which means that $\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}})$ is (13.100) divided by N, or

$$\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2\left(\sum_{i=1}^{N}\nabla_{\boldsymbol{\beta}}\hat{m}_i'\nabla_{\boldsymbol{\beta}}\hat{m}_i\right)^{-1}. \tag{13.101}$$

If the model is linear, $\nabla_{\boldsymbol{\beta}}\hat{m}_i = \mathbf{x}_i$, and we obtain exactly the asymptotic variance estimator for the OLS estimator under homoskedasticity.

13.3. a. The conditional log-likelihood for observation i is

$$\ell_i(\boldsymbol{\theta}) = y_i \log[G(\mathbf{x}_i, \boldsymbol{\theta})] + (1 - y_i)\log[1 - G(\mathbf{x}_i, \boldsymbol{\theta})].$$

b. The derivation for the probit case in Example 13.1 extends immediately:

$$\begin{aligned}\mathbf{s}_i(\boldsymbol{\theta}) &= y_i\nabla_{\boldsymbol{\theta}}G(\mathbf{x}_i, \boldsymbol{\theta})'/G(\mathbf{x}_i, \boldsymbol{\theta}) - (1 - y_i)\nabla_{\boldsymbol{\theta}}G(\mathbf{x}_i, \boldsymbol{\theta})'/[1 - G(\mathbf{x}_i, \boldsymbol{\theta})] \\ &= \nabla_{\boldsymbol{\theta}}G(\mathbf{x}_i, \boldsymbol{\theta})'[y_i - G(\mathbf{x}_i, \boldsymbol{\theta})]/\{G(\mathbf{x}_i, \boldsymbol{\theta})[1 - G(\mathbf{x}_i, \boldsymbol{\theta})]\}.\end{aligned}$$

If we plug in $\boldsymbol{\theta}_o$ for $\boldsymbol{\theta}$ and take the expectation conditional on $\mathbf{x}_i$ we get $\text{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)|\mathbf{x}_i] = \mathbf{0}$ because $\text{E}[y_i - G(\mathbf{x}_i, \boldsymbol{\theta}_o)|\mathbf{x}_i] = 0$, and the functions multiplying $y_i - G(\mathbf{x}_i, \boldsymbol{\theta}_o)$ depend only on $\mathbf{x}_i$.

c. We need to evaluate the score and the expected Hessian with respect to the full set of parameters, but then evaluate these at the restricted estimates. Now,

$$\nabla_{\boldsymbol{\theta}} G(\mathbf{x}_i, \boldsymbol{\beta}, \mathbf{0}) = \phi(\mathbf{x}\boldsymbol{\beta})[\mathbf{x}, (\mathbf{x}\boldsymbol{\beta})^2, (\mathbf{x}\boldsymbol{\beta})^3],$$

a $1 \times (K + 2)$ vector. Let $\tilde{\boldsymbol{\beta}}$ denote the probit estimates of $\boldsymbol{\beta}$, obtained under the null. The score for observation i, evaluated under the null estimates, is the $(K + 2) \times 1$ vector

$$\begin{aligned}\mathbf{s}_i(\tilde{\boldsymbol{\theta}}) &= \nabla_{\boldsymbol{\theta}} G(\mathbf{x}_i, \tilde{\boldsymbol{\beta}}, \mathbf{0})'[y_i - \Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})]/\{\Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})[1 - \Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})]\} \\ &= \phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})\tilde{\mathbf{z}}_i'[y_i - \Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})]/\{\Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})[1 - \Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})]\},\end{aligned}$$

where $\tilde{\mathbf{z}}_i \equiv [\mathbf{x}_i, (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2, (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3]$. The negative of the expected Hessian, evaluated under the null, is the $(K + 2) \times (\mathbf{K} + 2)$ matrix

$$\mathbf{A}(\mathbf{x}_i, \tilde{\boldsymbol{\theta}}) = [\phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})]^2\tilde{\mathbf{z}}_i'\tilde{\mathbf{z}}_i/\{\Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})[1 - \Phi(\mathbf{x}_i\tilde{\boldsymbol{\beta}})]\}.$$

These can be plugged into the second expression in equation (13.36) to obtain a nonnegative, well-behaved LM statistic. Simple algebra shows that the statistic can be computed as N times the explained sum of squares from the regression

$$\frac{\tilde{u}_i}{\sqrt{\tilde{\Phi}_i(1 - \tilde{\Phi}_i)}} \text{ on } \frac{\tilde{\phi}_i \cdot \mathbf{x}_i}{\sqrt{\tilde{\Phi}_i(1 - \tilde{\Phi}_i)}}, \frac{\tilde{\phi}_i \cdot (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2}{\sqrt{\tilde{\Phi}_i(1 - \tilde{\Phi}_i)}}, \frac{\tilde{\phi}_i \cdot (\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3}{\sqrt{\tilde{\Phi}_i(1 - \tilde{\Phi}_i)}}, i = 1, \ldots, N$$

where "~" denotes evaluation at $(\tilde{\boldsymbol{\beta}}, \mathbf{0})$ and $\tilde{u}_i = y_i - \tilde{\Phi}_i$. Under H_o, LM is distributed asymptotically as χ^2_2.

d. The variable addition version of the test is to estimate, in a second step, a probit model with response probability of the form

$$\Phi[\mathbf{x}_i\boldsymbol{\beta} + \delta_1(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^2 + \delta_2(\mathbf{x}_i\tilde{\boldsymbol{\beta}})^3]$$

and the compute a Wald test of $H_0 : \delta_1 = \delta_2 = 0$. As we discussed in Problem 12.5 in a related context, this is to be used only as a test. The estimates of δ_1 and δ_2 obtained by inserting $\tilde{\boldsymbol{\beta}}$ into the square and quadratic are generally inconsistent if at least one of δ_1 and δ_2 is different from zero.

13.4. If the density of **y** given **x** is correctly specified then $\mathrm{E}[\mathbf{s}(\mathbf{w},\boldsymbol{\theta}_o)|\mathbf{x}] = \mathbf{0}$. But then

$$\mathrm{E}[a(\mathbf{x},\boldsymbol{\theta}_o)\mathbf{s}(\mathbf{w},\boldsymbol{\theta}_o)|\mathbf{x}] = a(\mathbf{x},\boldsymbol{\theta}_o)\mathrm{E}[\mathbf{s}(\mathbf{w},\boldsymbol{\theta}_o)|\mathbf{x}] = \mathbf{0}$$

which, of course implies an unconditional expectation of zero. The only restriction on $a(\mathbf{x},\boldsymbol{\theta}_o)$ would be to ensure the expected value is well defined (but this is usually just assumed, not verified).

13.5. a. Because $\mathbf{s}_i^g(\boldsymbol{\phi}_o) = [\mathbf{G}(\boldsymbol{\theta}_o)']^{-1}\mathbf{s}_i(\boldsymbol{\theta}_o)$,

$$\begin{aligned}\mathrm{E}[\mathbf{s}_i^g(\boldsymbol{\phi}_o)s_i^g(\phi_o)'|\mathbf{x}_i] &= \mathrm{E}\{[\mathbf{G}(\boldsymbol{\theta}_o)']^{-1}\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'[\mathbf{G}(\boldsymbol{\theta}_o)]^{-1}|\mathbf{x}_i\} \\ &= [\mathbf{G}(\boldsymbol{\theta}_o)]^{-1}\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'|\mathbf{x}_i][\mathbf{G}(\boldsymbol{\theta}_o)]^{-1} \\ &= [\mathbf{G}(\boldsymbol{\theta}_o)']^{-1}\mathbf{A}_i(\boldsymbol{\theta}_o)[\mathbf{G}(\boldsymbol{\theta}_o)]^{-1}.\end{aligned}$$

where the last equality follows from the conditional information matrix equality.

b. In part a, we just replace θ_o with $\tilde{\theta}$ and ϕ_o with $\tilde{\phi}$:

$$\tilde{\mathbf{A}}_i^g = [\mathbf{G}(\tilde{\boldsymbol{\theta}})']^{-1}\mathbf{A}_i(\tilde{\boldsymbol{\theta}})[\mathbf{G}(\tilde{\boldsymbol{\theta}})]^{-1} \equiv \tilde{\mathbf{G}}'^{-1}\tilde{\mathbf{A}}_i\tilde{\mathbf{G}}^{-1}.$$

c. The expected Hessian form of the statistic is given in the second part of equation (13.36), but where it depends on $\tilde{\mathbf{s}}_i^g$ and $\tilde{\mathbf{A}}_i^g$:

$$\begin{aligned}LM_g &= \left(\sum_{i=1}^N \tilde{\mathbf{s}}_i^g\right)'\left(\sum_{i=1}^N \tilde{\mathbf{A}}_i^g\right)^{-1}\left(\sum_{i=1}^N \tilde{\mathbf{s}}_i^g\right) \\ &= \left(\sum_{i=1}^N \tilde{\mathbf{G}}'^{-1}\tilde{\mathbf{s}}_i\right)'\left(\sum_{i=1}^N \tilde{\mathbf{G}}'^{-1}\tilde{\mathbf{A}}_i\tilde{\mathbf{G}}^{-1}\right)^{-1}\left(\sum_{i=1}^N \tilde{\mathbf{G}}'^{-1}\tilde{s}_i\right) \\ &= \left(\sum_{i=1}^N \tilde{s}_i\right)'\tilde{\mathbf{G}}^{-1}\tilde{\mathbf{G}}\left(\sum_{i=1}^N \tilde{\mathbf{A}}_i\right)^{-1}\tilde{\mathbf{G}}'\tilde{\mathbf{G}}^{-1}\left(\sum_{i=1}^N \tilde{s}_i\right) \\ &= \left(\sum_{i=1}^N \tilde{s}_i\right)'\left(\sum_{i=1}^N \tilde{\mathbf{A}}_i\right)^{-1}\left(\sum_{i=1}^N \tilde{s}_i\right) = LM.\end{aligned}$$

13.6. a. No, for two reasons. First, just specifying a distribution of y_{it} given $\mathbf{x}_{it}$ says

nothing, in general, about the distribution of y_{it} given $\mathbf{x}_i \equiv (\mathbf{x}_{i1}, \ldots, \mathbf{x}_{iT})$. We could assume these two are the same, which is the strict exogeneity assumption. But, even under strict exogeneity, we would have to specify something about joint distributions (perhaps via conditional distributions) involving different time periods. We could assume independence (conditional on $\mathbf{x}_i$) or make a dynamic completeness assumption. Either way, without substantially more assumptions, we cannot derive the distribution of $\mathbf{y}_i$ given $\mathbf{x}_i$.

b. This is given in a more general case in equation (18.69) in Chapter 18. It can be derived easily from Example 13.2, which gives $\ell_i(\theta)$ for the cross section case:

$$\ell_i(\boldsymbol{\theta}) = \sum_{t=1}^{T} [y_{it}\mathbf{x}_{it}\boldsymbol{\theta} - \exp(\mathbf{x}_{it}\boldsymbol{\theta})] \equiv \sum_{t=1}^{T} \ell_{it}(\theta).$$

Taking the gradient and transposing gives

$$\mathbf{s}_i(\boldsymbol{\theta}) = \sum_{t=1}^{T} \mathbf{x}_{it}'[y_{it} - \exp(\mathbf{x}_{it}\boldsymbol{\theta})] \equiv \sum_{t=1}^{T} \mathbf{s}_i(\boldsymbol{\theta}).$$

c. First, we need the Hessian for each i, which is easily obtained as $\nabla_{\boldsymbol{\theta}}\mathbf{s}_i(\boldsymbol{\theta})$:

$$\mathbf{H}_i(\boldsymbol{\theta}) = -\sum_{t=1}^{T} \exp(\mathbf{x}_{it}\boldsymbol{\theta})\mathbf{x}_{it}'\mathbf{x}_{it},$$

which, in this example, does not depend on the y_{it} (see Problem 13.12 for the notion of a canonical link function). In particular, $\mathbf{A}_{it}(\boldsymbol{\theta}_o) = -\mathrm{E}[\mathbf{H}_{it}(\boldsymbol{\theta}_o)|\mathbf{x}_{it}] = -\mathbf{H}_{it}(\boldsymbol{\theta}_o)$. Therefore,

$$\hat{\mathbf{A}} = N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \exp(\mathbf{x}_{it}\hat{\boldsymbol{\theta}})\mathbf{x}_{it}'\mathbf{x}_{it},$$

where $\hat{\boldsymbol{\theta}}$ is the partial MLE. Further,

$$\hat{\mathbf{B}} = N^{-1}\sum_{i=1}^{N}\mathbf{s}_i(\hat{\boldsymbol{\theta}})\mathbf{s}_i(\hat{\boldsymbol{\theta}})',$$

and then $\text{Avar}(\hat{\boldsymbol{\theta}})$ is estimated as

$$\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\exp(\mathbf{x}_{it}\hat{\boldsymbol{\theta}})\mathbf{x}_{it}'\mathbf{x}_{it}\right)^{-1}\left(\sum_{i=1}^{N}\mathbf{s}_i(\hat{\boldsymbol{\theta}})\mathbf{s}_i(\hat{\boldsymbol{\theta}})'\right)\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\exp(\mathbf{x}_{it}\hat{\boldsymbol{\theta}})\mathbf{x}_{it}'\mathbf{x}_{it}\right)^{-1}$$

d. If $\text{E}(y_{it}|\mathbf{x}_{it},y_{i,t-1},\mathbf{x}_{i,t-1},\ldots,y_{i1},\mathbf{x}_{i1}) = \text{E}(y_{it}|\mathbf{x}_{it})$ then

$$\text{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)|\mathbf{x}_{it},y_{it,t-1},\mathbf{x}_{i,t-1},\ldots] = \mathbf{x}_{it}'[\text{E}(y_{it}|\mathbf{x}_{it},y_{it,t-1},\mathbf{x}_{i,t-1},\ldots) - \exp(\mathbf{x}_{it}\boldsymbol{\theta}_o)] = \mathbf{0}.$$

As usual, this finding implies that $\mathbf{s}_{it}(\boldsymbol{\theta}_o)$ and $\mathbf{s}_{ir}(\boldsymbol{\theta}_o)$ are uncorrelated, $t \neq r$. Therefore,

$$\mathbf{B}_o = \sum_{t=1}^{T}\text{E}\left[\mathbf{s}_{it}(\boldsymbol{\theta}_o)\mathbf{s}_{it}(\boldsymbol{\theta}_o)'\right] = \sum_{t=1}^{T}\text{E}(u_{it}^2\mathbf{x}_{it}'\mathbf{x}_{it}),$$

where $u_{it} \equiv y_{it} - \text{E}(y_{it}|\mathbf{x}_{it}) = y_{it} - \exp(\mathbf{x}_{it}\boldsymbol{\theta}_o)$. Now, by the Poisson assumption, $\text{E}(u_{it}^2|\mathbf{x}_{it}) = \text{Var}(y_{it}|\mathbf{x}_{it}) = \exp(\mathbf{x}_{it}\boldsymbol{\theta}_o)$. By iterated expectations,

$$\mathbf{B}_o = \sum_{t=1}^{T}\text{E}[\exp(\mathbf{x}_{it}\boldsymbol{\theta}_o)\mathbf{x}_{it}'\mathbf{x}_{it}] = \mathbf{A}_o.$$

(We have really just verified the conditional information matrix equality for each t in the special case of Poisson regression with an exponential mean function.) herefore, we can estimate $\text{Avar}(\hat{\boldsymbol{\theta}})$ as

$$\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\exp(\mathbf{x}_{it}\hat{\boldsymbol{\theta}})\mathbf{x}_{it}'\mathbf{x}_{it}\right)^{-1},$$

which is exactly what we get by using pooled Poisson estimation and ignoring the time dimension.

13.7. a. The joint density is simply $g(y_1|y_2,\mathbf{x};\boldsymbol{\theta}_o)\cdot h(y_2|\mathbf{x};\boldsymbol{\theta}_o)$. The log-likelihood for

observation i is

$$\ell_i(\theta) \equiv \log[g(y_{i1}|y_{i2}, \mathbf{x}_i; \boldsymbol{\theta}_o)] + \log[h(y_{i2}|\mathbf{x}_i; \boldsymbol{\theta}_o)],$$

and we would use this in a standard MLE analysis (conditional on $\mathbf{x}_i$).

b. First, we know that, for all $(y_{i2}, \mathbf{x}_i)$, $\boldsymbol{\theta}_o$ minimizes $\mathrm{E}[\ell_{i1}(\boldsymbol{\theta})|y_{i2}, \mathbf{x}_i]$. Because r_{i2} is a function of $(y_{i2}, \mathbf{x}_i)$,

$$\mathrm{E}[r_{i2}\ell_{i1}(\boldsymbol{\theta})|y_{i2}, \mathbf{x}_i] = r_{i2}\mathrm{E}[\ell_{i1}(\boldsymbol{\theta})|y_{i2}, \mathbf{x}_i];$$

because $r_{12} \geq 0$, $\boldsymbol{\theta}_o$ maximizes $\mathrm{E}[r_{i2}\ell_{i1}(\theta)|y_{i2}, \mathbf{x}_i]$ for all $(y_{i2}, \mathbf{x}_i)$, and therefore $\boldsymbol{\theta}_o$ maximizes $\mathrm{E}[r_{i2}\ell_{i1}(\boldsymbol{\theta})]$ by iterated expectations. Similarly, $\boldsymbol{\theta}_o$ maximizes $\mathrm{E}[\ell_{i1}(\boldsymbol{\theta})]$, and so it follows that $\boldsymbol{\theta}_o$ maximizes $\mathrm{E}[r_{i2}\ell_{i1}(\boldsymbol{\theta}) + \ell_{i2}(\boldsymbol{\theta})]$. For identification, we have to assume or verify uniqueness.

c. The score is

$$\mathbf{s}_i(\boldsymbol{\theta}) = r_{i2}\mathbf{s}_{i1}(\boldsymbol{\theta}) + \mathbf{s}_{i2}(\boldsymbol{\theta}),$$

where $\mathbf{s}_{i1}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}}\ell_{i1}(\boldsymbol{\theta})'$ and $\mathbf{s}_{i2}(\boldsymbol{\theta}) \equiv \nabla_{\boldsymbol{\theta}}\ell_{i2}(\boldsymbol{\theta})'$. Therefore,

$$\begin{aligned}\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'] = \mathrm{E}[r_{i2}\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'] + \mathrm{E}[\mathbf{s}_{i2}(\boldsymbol{\theta}_o)\mathbf{s}_{i2}(\boldsymbol{\theta}_o)'] \\ + \mathrm{E}[r_{i2}\mathbf{s}_{i1}(\boldsymbol{\theta}_o)\mathbf{s}_{i2}(\boldsymbol{\theta}_o)'] + \mathrm{E}[r_{i2}\mathbf{s}_{i2}(\boldsymbol{\theta}_o)\mathbf{s}_{i1}(\boldsymbol{\theta}_o)'].\end{aligned}$$

Now by the usual conditional MLE theory, $\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)|y_{i2}, \mathbf{x}_i] = 0$ and, since r_{i2} and $\mathbf{s}_{i2}(\boldsymbol{\theta})$ are functions of $(y_{i2}, \mathbf{x}_i)$, it follows that $\mathrm{E}[r_{i2}\mathbf{s}_{i1}(\boldsymbol{\theta}_o)\mathbf{s}_{i2}(\boldsymbol{\theta}_o)'|y_{i2}, \mathbf{x}_i] = 0$, and so its transpose also has zero conditional expectation. As usual, this implies zero unconditional expectation. We have shown

$$\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'] = \mathrm{E}[r_{i2}\mathbf{s}_{i1}(\boldsymbol{\theta}_o)\mathbf{s}_{i1}(\boldsymbol{\theta}_o)'] + \mathrm{E}[\mathbf{s}_{i2}(\boldsymbol{\theta}_o)\mathbf{s}_{i2}(\boldsymbol{\theta}_o)'].$$

Now, by the unconditional information matrix equality for the density $h(y_2|\mathbf{x}; \boldsymbol{\theta})$,

$$\mathrm{E}[\mathbf{s}_{i2}(\boldsymbol{\theta}_o)\mathbf{s}_{i2}(\boldsymbol{\theta}_o)'] = -\mathrm{E}[\mathbf{H}_{i2}(\boldsymbol{\theta}_o)],$$

where $\mathbf{H}_{i2}(\boldsymbol{\theta}_o) = \nabla_{\boldsymbol{\theta}}\mathbf{s}_{i2}(\boldsymbol{\theta})$. Further, by the conditional IM equality for the density $g(y_1|y_2, \mathbf{x}; \boldsymbol{\theta})$,

$$\mathrm{E}[\mathbf{s}_{i1}(\boldsymbol{\theta}_o)\mathbf{s}_{i1}(\boldsymbol{\theta}_o)'|y_{i2}, \mathbf{x}_i] = -\mathrm{E}[\mathbf{H}_{i1}(\boldsymbol{\theta}_o)|y_{i2}, \mathbf{x}_i], \tag{13.102}$$

where $\mathbf{H}_{i1}(\boldsymbol{\theta}_o) = \nabla_{\boldsymbol{\theta}}\mathbf{s}_{i1}(\boldsymbol{\theta})$. Since r_{i2} is a function of $(y_{i2}, \mathbf{x}_i)$, we can put r_{i2} inside both expectations in (13.102). Then, by iterated expectations,

$$\mathrm{E}[r_{i2}\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'] = -\mathrm{E}[r_{i2}\mathbf{H}_{i1}(\boldsymbol{\theta}_o)].$$

Combining all the pieces, we have shown that

$$\begin{aligned}\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o)\mathbf{s}_i(\boldsymbol{\theta}_o)'] &= -\mathrm{E}[r_{i2}\mathbf{H}_{i1}(\boldsymbol{\theta}_o)] - \mathrm{E}[\mathbf{H}_{i2}(\boldsymbol{\theta}_o)] \\ &= -\{\mathrm{E}[r_{i2}\nabla_{\boldsymbol{\theta}}\mathbf{s}_{i1}(\boldsymbol{\theta}) + \nabla_{\boldsymbol{\theta}}\mathbf{s}_{i2}(\boldsymbol{\theta})]\} \\ &= -\mathrm{E}[\nabla^2_{\boldsymbol{\theta}}\ell_i(\boldsymbol{\theta}) \equiv -\mathrm{E}[\mathbf{H}_i(\boldsymbol{\theta})].\end{aligned}$$

So we have verified that an unconditional IM equality holds, which means we can estimate the asymptotic variance of $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)$ by estimating $\{-\mathrm{E}[\mathbf{H}_i(\boldsymbol{\theta})]\}^{-1}$.

d. From part c, one consistent estimator of $\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)]$ is

$$N^{-1}\sum_{i=1}^{N}(r_{i2}\hat{\mathbf{H}}_{i1} + \hat{\mathbf{H}}_{i2}),$$

where the notation should be obvious. In some cases it may be simpler to use an expected Hessian form for each piece, which we can obtain by looking for consistent estimators of $-\mathrm{E}[r_{i2}\mathbf{H}_{i1}(\boldsymbol{\theta}_o)]$ and $-\mathrm{E}[\mathbf{H}_{i2}(\boldsymbol{\theta}_o)]$. By definition, $\mathbf{A}_{i2}(\boldsymbol{\theta}_o) \equiv -\mathrm{E}[\mathbf{H}_{i2}(\boldsymbol{\theta}_o)|\mathbf{x}_i]$, and so $\mathrm{E}[\mathbf{A}_{i2}(\boldsymbol{\theta}_o)] \equiv -\mathrm{E}[\mathbf{H}_{i2}(\boldsymbol{\theta}_o)]$. By the usual law of large numbers argument,

$$N^{-1}\sum_{i=1}^{N}\hat{\mathbf{A}}_{i2} \xrightarrow{p} -\mathrm{E}[\mathbf{H}_{i2}(\boldsymbol{\theta}_o)].$$

Similarly, since $\mathbf{A}_{i1}(\boldsymbol{\theta}_o) \equiv -\mathrm{E}[\mathbf{H}_{i1}(\boldsymbol{\theta}_o)|y_{i2}, \mathbf{x}_i]$, and r_{i2} is a function of $(y_{i2}, \mathbf{x}_i)$, it follows that

$\mathrm{E}[r_{i2}\mathbf{A}_{i1}(\boldsymbol{\theta}_o)] = -\mathrm{E}[r_{i2}\mathbf{H}_{i1}(\boldsymbol{\theta}_o)]$. Under general regularity conditions, $N^{-1}\sum_{i=1}^{N} r_{i2}\hat{\mathbf{A}}_{i1}$ consistently estimates $-\mathrm{E}[r_{i2}\mathbf{H}_{i1}(\boldsymbol{\theta}_o)]$. This completes what we needed to show.

Interestingly, even though we do not have a true conditional maximum likelihood problem, we can still use the conditional expectations of the Hessians – but conditioned on different sets of variables, $(y_{i2}, \mathbf{x}_i)$ in one case, and $\mathbf{x}_i$ in the other – to consistently estimate the asymptotic variance of the partial MLE.

e. (**Bonus Question**) Show that if we were able to use the entire random sample, the resulting conditional MLE would be more efficient than the partial MLE based on the selected sample.

Solution

We use a standard fact about positive definite matrices: if $\mathbf{A}$ and $\mathbf{B}$ are $P \times P$ positive definite matrices, then $\mathbf{A} - \mathbf{B}$ is p.s.d. if and only if $\mathbf{B}^{-1} - \mathbf{A}^{-1}$ is p.s.d. Now, as we showed in part d, the asymptotic variance of the partial MLE is $\{\mathrm{E}[r_{i2}\mathbf{A}_{i1}(\boldsymbol{\theta}_o) + \mathbf{A}_{i2}(\boldsymbol{\theta}_o)]\}^{-1}$. If we could use the entire random sample for both terms, the asymptotic variance would be $\{\mathrm{E}[\mathbf{A}_{i1}(\boldsymbol{\theta}_o) + \mathbf{A}_{i2}(\boldsymbol{\theta}_o)]\}^{-1}$. But

$$\mathrm{E}[\mathbf{A}_{i1}(\boldsymbol{\theta}_o) + \mathbf{A}_{i2}(\boldsymbol{\theta}_o)] - \mathrm{E}[r_{i2}\mathbf{A}_{i1}(\boldsymbol{\theta}_o) + \mathbf{A}_{i2}(\boldsymbol{\theta}_o)] = \mathrm{E}[(1 - r_{i2})\mathbf{A}_{i1}(\boldsymbol{\theta}_o)],$$

which is p.s.d. because $\mathbf{A}_{i1}(\boldsymbol{\theta}_o)$ is p.s.d. and $1 - r_{i2} \geq 0$. Intuitively, the larger is $\mathrm{P}(r_{i2} = 1)$ the smaller is the efficiency difference.

13.8. a. This is similar to Problem 12.12 for nonlinear regression; here we are specifying a full conditional distribution. We can use the results in Section 12.4.2:

$$\begin{aligned}\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)] &= \mathbf{A}_o^{-1}(\mathbf{B}_o + \mathbf{F}_o\mathbf{C}_o\mathbf{F}_o')\mathbf{A}_o^{-1} \\ &= \mathbf{A}_o^{-1} + \mathbf{A}_o^{-1}\mathbf{F}_o\mathbf{C}_o\mathbf{F}_o'\mathbf{A}_o^{-1}\end{aligned}$$

where $\mathbf{C}_o = \mathrm{E}[\mathbf{r}_i(\boldsymbol{\gamma}_o)\mathbf{r}_i(\boldsymbol{\gamma}_o)']$. We also use the information matrix equality, $\mathbf{A}_o = \mathbf{B}_o$, where

$$\mathbf{A}_o = \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)] = -\mathrm{E}[\mathbf{H}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)]$$

and

$$\begin{aligned}\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) &= \nabla_{\boldsymbol{\theta}} \log[f(\mathbf{y}_i|\mathbf{x}_i,\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma});\boldsymbol{\theta})]' \\ \mathbf{H}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o) &= \nabla_{\boldsymbol{\theta}}\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma}).\end{aligned}$$

To use the formula we need to characterize $\mathbf{F}_o$. First,

$$\nabla_{\boldsymbol{\gamma}}\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) = \nabla_{\boldsymbol{\gamma}}\{\nabla_{\boldsymbol{\theta}} \log[f(\mathbf{y}_i|\mathbf{x}_i,\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma});\boldsymbol{\theta})]'\},$$

which generally requires using the chain rule to compute. Write

$$\mathbf{k}(\mathbf{y},\mathbf{x},\mathbf{g};\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} \log[f(\mathbf{y}_i|\mathbf{x}_i,\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma});\boldsymbol{\theta})]'.$$

Then

$$\nabla_{\boldsymbol{\gamma}}\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) = \nabla_{\mathbf{g}}\mathbf{k}(\mathbf{y}_i,\mathbf{x}_i,\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma}),\boldsymbol{\theta})\nabla_{\boldsymbol{\gamma}}\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma})$$

and

$$\mathbf{F}_o = \mathrm{E}[\nabla_{\mathbf{g}}\mathbf{k}(\mathbf{y}_i,\mathbf{x}_i,\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma}_o),\boldsymbol{\theta}_o)\nabla_{\boldsymbol{\gamma}}\mathbf{g}(\mathbf{w}_i,\boldsymbol{\gamma}_o)]$$

b. Generally,

$$\widehat{\mathrm{Avar}}[\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)] = \hat{\mathbf{A}}_o^{-1} + \hat{\mathbf{A}}_o^{-1}\hat{\mathbf{F}}_o\hat{\mathbf{C}}_o\hat{\mathbf{F}}_o'\hat{\mathbf{A}}_o^{-1}$$

where $\hat{\mathbf{A}}_o$ is one of the various choices for estimating the information matrix, evaluated at $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\gamma}}$,

$$\hat{\mathbf{C}} = N^{-1}\sum_{i=1}^{N}\mathbf{r}(\mathbf{w}_i,\hat{\boldsymbol{\gamma}})\mathbf{r}(\mathbf{w}_i,\hat{\boldsymbol{\gamma}})'$$

and

$$\hat{\mathbf{F}} = N^{-1}\sum_{i=1}^{N}\nabla_{\mathbf{g}}\mathbf{k}(\mathbf{y}_i,\mathbf{x}_i,\mathbf{g}(\mathbf{w}_i,\hat{\boldsymbol{\gamma}}),\hat{\boldsymbol{\theta}})\nabla_{\boldsymbol{\gamma}}\mathbf{g}(\mathbf{w}_i,\hat{\boldsymbol{\gamma}})].$$

c. It applies directly where the scalar ρ_o plays the role of $\boldsymbol{\gamma}_o$. The score for this problem (with respect to $\boldsymbol{\theta}$) is

$$\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) = \frac{\phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))}{\{\Phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))[1-\Phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))]\}}\begin{pmatrix}\mathbf{x}_i'\\ g_i(\boldsymbol{\gamma})\end{pmatrix}[y_i-\Phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))],$$

where $g_i(\boldsymbol{\gamma}) = h_i - \mathbf{z}_i\boldsymbol{\gamma}$. The full Jacobian of $\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma})$ with respect to $\boldsymbol{\gamma}$ is complicated, but it is easy to see it has the form

$$\begin{aligned}\nabla_{\boldsymbol{\gamma}}\mathbf{s}_i(\boldsymbol{\theta};\boldsymbol{\gamma}) = {}& \mathbf{L}(\mathbf{x}_i,\mathbf{z}_i,h_i;\boldsymbol{\theta},\rho)[y_i-\Phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))]\\ &+\rho\cdot\frac{\phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))}{\{\Phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))[1-\Phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))]\}}\begin{pmatrix}\mathbf{x}_i'\\ g_i(\boldsymbol{\gamma})\end{pmatrix}\mathbf{z}_i\phi(\mathbf{x}_i\boldsymbol{\beta}+\rho g_i(\boldsymbol{\gamma}))\end{aligned}$$

When evaluated at the true values $\boldsymbol{\theta}_o$ and γ_o, the the first term has zero expectation conditional on $(\mathbf{x}_i,\mathbf{z}_i,h_i)$ because

$$\mathrm{E}(y_i|\mathbf{x}_i,\mathbf{z}_i,h_i) = \Phi(\mathbf{x}_i\boldsymbol{\beta}_o+\rho_o g_i(\boldsymbol{\gamma}_o))$$

So $\mathbf{F}_o$ can be estimated by plugging in the estimators and averaging the second term across i.

d. When $\rho_o = 0$, the second term in $\nabla_{\boldsymbol{\gamma}}\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)$ is zero, and so

$$\mathrm{E}[\nabla_{\boldsymbol{\gamma}}\mathbf{s}_i(\boldsymbol{\theta}_o;\boldsymbol{\gamma}_o)|\mathbf{x}_i,\mathbf{z}_i,h_i] = \mathbf{0},$$

which means condition (12.37) holds and $\mathbf{F}_o = \mathbf{0}$. This implies, from part a,

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_o)] = \mathbf{A}_o^{-1}.$$

e. Because ρ_o is an element of $\boldsymbol{\theta}_o$, for testing $H_0 : \rho_o = 0$ we can ignore the fact that $\boldsymbol{\gamma}_o$ has been estimated in the first stage. In other words, when we run probit of y_i on $\mathbf{x}_i$, $\hat{g}_i$,

$i = 1, \ldots, N$, where $\hat{g}_i = h_i - \mathbf{z}_i\hat{\boldsymbol{\gamma}}$, we can use a standard probit t statistic on g_i.

13.9. a. Under the Markov assumption, the joint density of $(y_{i0}, \ldots, y_{iT})$ is given by

$$f_T(y_T|y_{T-1}) \cdot f_{T-1}(y_{T-1}|y_{T-2}) \cdots f_1(y_i|y_0) \cdot f_0(y_0),$$

so we would need to model $f_0(y_0)$ to obtain a model of the joint density.

b. The log likelihood

$$\ell_i(\boldsymbol{\theta}) = \sum_{t=1}^{T} \log[f_t(y_{it}|y_{i,t-1}; \boldsymbol{\theta})]$$

is the conditional log-likelihood for the density of $(y_{i1}, \ldots, y_{iT})$ given y_{i0}, and so the usual theory of conditional maximum likelihood applies. In practice, this is MLE pooled across i and t.

c. Because we have the density of $(y_{i1}, \ldots, y_{iT})$ given y_{i0}, we can use any of the three asymptotic variance estimators implied by the information matrix equality. However, we can also use the simplifications due to dynamic completeness of each conditional density. Let $\mathbf{s}_{it}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} \log[f(y_{it}|y_{i,t-1}; \boldsymbol{\theta}), \mathbf{H}_{it}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}}\mathbf{s}_{it}(\boldsymbol{\theta})$ and $\mathbf{A}_{it}(\boldsymbol{\theta}_o) = -\mathrm{E}[\mathbf{H}_{it}(\boldsymbol{\theta})|y_{i,t-1}], t = 1, \ldots T$. Then $\mathrm{Avar}\sqrt{N}(\hat{\theta} - \theta_o)$ is consistently estimated using the inverse of any of the three matrices in equation (13.50). If we have a canned package that computes a particular MLE, we can just use any of the usual asymptotic variance estimates obtained from the pooled MLE.

13.10. a. Because of conditional independence, the joint density [conditional on $(\mathbf{x}, c)$] is the product of the marginal densities [conditional on $(\mathbf{x}, c)$]:

$$f(y_1, y_2, \ldots, y_G|\mathbf{x}, c) = \Pi_{g=1}^{G} f_g(y_g|\mathbf{x}, c).$$

b. Let $g(y_i, \ldots, y_G|\mathbf{x})$ be the joint density of y_i given $\mathbf{x}_i = \mathbf{x}$. Then

$$g(y_i, \ldots, y_G|\mathbf{x}) = \int_{\mathbb{R}} f(y_i, y_2, \ldots, y_G|\mathbf{x}, \mathfrak{c}) h(\mathfrak{c}|\mathbf{x}) d\mathfrak{c}.$$

c. The density $g(y_i, \ldots, y_G|\mathbf{x})$ is now

$$g(y_i, \ldots, y_G|\mathbf{x};\boldsymbol{\gamma}_o,\boldsymbol{\delta}_o) = \int_{\mathbb{R}} f(y_i, y_2, \ldots, y_G|\mathbf{x}, \mathfrak{c};\boldsymbol{\gamma}_o)h(\mathfrak{c}|\mathbf{x};\boldsymbol{\delta}_o)d\mathfrak{c}$$
$$= \int_{\mathbb{R}} \prod_{g=1}^{G} f(y_g|\mathbf{x}, \mathfrak{c};\boldsymbol{\gamma}_o^g)h(\mathfrak{c}|\mathbf{x};\boldsymbol{\delta}_o)d\mathfrak{c},$$

and so the log likelihood for observation i is

$$\log[g(y_{i1}, \ldots, y_G|\mathbf{x}_i;\boldsymbol{\gamma}_o,\boldsymbol{\delta}_o)]$$
$$= \log\left[\int_{\mathbb{R}} \prod_{g=1}^{G} f(y_{ig}|\mathbf{x}_i, \mathfrak{c};\boldsymbol{\gamma}_o^g)h(\mathfrak{c}|\mathbf{x}_i;\boldsymbol{\delta}_o)d\mathfrak{c}\right].$$

d. This setup has some features in common with a linear SUR model, although here the correlation across equations is assumed to come through a single common component, c. Because of computational issues with general nonlinear models – especially if G is large and some of the models are for qualitative response – one probably needs to restrict the cross equation correlation somehow.

13.11. a. For each $t \geq 1$, the density of y_{it} given $y_{i,t-1} = y_{i,t-1}, y_{i,t-2} = y_{t-2}, , \ldots, y_{i0} = y_0$ and $c_i = c$ is

$$f_t(y_t|y_{t-1}, c) = (2\pi\sigma_e^2)^{-1/2}\exp[-(y_t - \rho y_{t-1} - c)^2/(2\sigma_e^2)].$$

Therefore, the density of $(y_{i1}, \ldots, y_{iT})$ given $y_{i0} = y_o$ and $c_i = c$ is obtained by the product of these densities:

$$\Pi_{t=1}^{T}(2\pi\sigma_e^2)^{-1/2}\exp[-(y_t - \rho y_{t-1} - c)^2/(2\sigma_e^2)].$$

If we plug in the data for observation i and take the log we get

$$\sum_{t=1}^{T}\{-(1/2)\log(\sigma_e^2) - (y_{it} - \rho y_{i,t-1} - c_i)^2/(2\sigma_e^2)\}$$

$$= -(T/2)\log(\sigma_e^2) - \sum_{t=1}^{T}(y_{it} - \rho y_{i,t-1} - c_i)^2/(2\sigma_e^2),$$

where we have dropped the term that does not depend on the parameters.

It is not a good idea to "estimate" the c_i along with the ρ and σ_e^2, as the incidental parameters problem causes inconsistency – severe in some cases – in the estimator of ρ.

b. If we write $c_i = \alpha_0 + \alpha_1 y_{i0} + a_i$, under the maintained assumption, then the density of $(y_{i1}, \ldots, y_{iT})$ given $(y_{i0} = y_0, a_i = a)$ is

$$\prod_{t=1}^{T}(2\pi\sigma_e^2)^{-1/2}\exp[-(y_t - \rho y_{t-1} - \alpha_0 - \alpha_1 y_0 - a)^2/(2\sigma_e^2)].$$

Now, to get the density condition on $y_{i0} = y_0$ only, we integrate this density over the density of a_i given $y_{i0} = y_0$. But a_i and y_{i0} are independent, and $a_i \sim Normal(0, \sigma_a^2)$. So the density of $(y_{i1}, \ldots, y_{iT})$ given $y_{i0} = y_o$ is

$$\int_{-\infty}^{\infty}\left(\prod_{t=1}^{T}(2\pi\sigma_e^2)^{-1/2}\exp[-(y_t - \rho y_{t-1} - \alpha_0 - \alpha_1 y_0 - \mathfrak{a})^2/(2\sigma_e^2)]\right)\sigma_a^{-1}\phi(\mathfrak{a}/\sigma_a)d\mathfrak{a}.$$

If we now plug in the data $(y_{i0}, y_{i1}, \ldots, y_{iT})$ for each i and take the log we get a conditional log-likelihood (conditional on y_{i0}) for each i. We can estimate the parameters by maximizing the sum of the log-likelihoods across i.

c. As before, we can replace c_i with $\alpha_o + \alpha_1 y_{i0} + a_i$. Then, the density of y_{it} given $(y_{i,t-1}, \ldots, y_{i1}, y_{i0}, a_i)$ is

$$\text{Normal}[\rho y_{i,t-1} + \alpha_0 + \alpha_1 y_{i0} + a_i + \delta(\alpha_0 + \alpha_1 y_{i0} + a_i)y_{i,t-1}, \sigma_e^2],$$

$t = 1, \ldots, T$. Using the same argument as in part b, we just integrate out a_i to get the density

of $(y_{i1}, \ldots, y_{iT})$ given $y_{i0} = y_0$:

$$\int_{-\infty}^{\infty} \left(\prod_{t=1}^{T} (2\pi\sigma_e^2)^{-1/2} \exp[-(y_t - \rho y_{t-1} - \alpha_0 - \alpha_1 y_{i0} - \mathfrak{a} - \delta(\alpha_0 + \alpha_1 y_{i0} + \mathfrak{a}) y_{t-1})^2/(2\sigma_e^2)] \right) \sigma_a^{-1} \phi(\mathfrak{a}/\sigma_a) d\mathfrak{a}.$$

Numerically, this could be a difficult MLE problem to solve. Assuming we can get the MLEs, we would estimate $\rho + \delta \mathrm{E}(c_i)$ as $\hat{\rho} + \hat{\delta}(\hat{\alpha}_0 + \hat{\alpha}_1 \bar{y}_0)$, where $\bar{y}_0$ is the cross-sectional average of the initial observation.

d. The log likelihood for observation i, now conditional on $(y_{i0}, \mathbf{z}_i)$, is the log of

$$\int_{-\infty}^{\infty} \left(\prod_{t=1}^{T} (2\pi\sigma_e^2)^{-1/2} \exp[-(y_{it} - \rho y_{i,t-1} - \mathbf{z}_{it}\beta - \alpha_0 - \alpha_1 y_{i0} - \bar{\mathbf{z}}_i - \mathfrak{a})^2/(2\sigma_e^2)] \right) \sigma_a^{-1} \phi(\mathfrak{a}/\sigma_a) d\mathfrak{a}.$$

The assumption that we can put in the time average, $\bar{\mathbf{z}}_i$, to account for correlation between c_i and $(y_{i0}, \mathbf{z}_i)$, may be too strong. It may be better to put in the full vector $\mathbf{z}_i$, although this leads to many more parameters to estimate.

13.12. a. The first order conditions can be written as

$$\sum_{i=1}^{N} x_{ij}[y_i - m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})] = 0, j = 1, \ldots, K.$$

If, say, the first element of $\mathbf{x}_i$ is unity, $x_{i1} \equiv 1$, then the first entry of the FOC is

$$\sum_{i=1}^{N} [y_i - m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})] = 0$$

or

$$\sum_{i=1}^{N} \hat{u}_i = 0.$$

b. For the Bernoulli QLL with mean function $\Lambda(\mathbf{x}\boldsymbol{\theta}) = \exp(\mathbf{x}\boldsymbol{\theta})/[1 + \exp(\mathbf{x}\boldsymbol{\theta})]$ it is easily

seen that the FOC is

$$\sum_{i=1}^{N} \mathbf{x}_i'[y_i - \Lambda(\mathbf{x}_i\hat{\boldsymbol{\theta}})] \ = 0, j = 1, \ldots, K.$$

and so the canonical mean function is the logistic function. Therefore, $g(\mu) = \Lambda^{-1}(\mu)$, and to find $\Lambda^{-1}(\mu)$ we need to solve for z as a function of μ in

$\mu = \exp(z)/[1 + \exp(z)] \ = 1/[\exp(-z) + 1]$. So

$$\exp(-z) = \frac{1}{\mu} - 1 = \frac{(1-\mu)}{\mu}$$

or

$$\exp(z) = \frac{\mu}{(1-\mu)}.$$

Now just take the log to get $z = \log[\mu/(1-\mu)]$.

c. Generally, the FOC for the Poisson QMLE has the form

$$\sum_{i=1}^{N} \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})'[y_i - m(\mathbf{x}_i, \hat{\boldsymbol{\theta}})]/m(\mathbf{x}_i, \hat{\boldsymbol{\theta}}) = \mathbf{0}$$

and, with $m(\mathbf{x}, \boldsymbol{\theta}) = \exp(\mathbf{x}\boldsymbol{\theta})$, we get

$$\sum_{i=1}^{N} \exp(\mathbf{x}_i\hat{\boldsymbol{\theta}})\mathbf{x}_i'[y_i - \exp(\mathbf{x}_i\hat{\boldsymbol{\theta}})]/\exp(\mathbf{x}_i\hat{\boldsymbol{\theta}}) = \mathbf{0},$$

or

$$\sum_{i=1}^{N} \mathbf{x}_i'[y_i - \exp(\mathbf{x}_i\hat{\boldsymbol{\theta}})] \ = \mathbf{0}.$$

So $m(z) = \exp(z)$ is the canonical mean function and its inverse is $g(\mu) = \log(\mu)$.

d. This is a true statement. The score for observation i has the form

$$\mathbf{s}_i(\boldsymbol{\theta}) = \mathbf{x}_i'[y_i - m(\mathbf{x}_i, \boldsymbol{\theta})],$$

and therefore the Hessian, $-\mathbf{x}_i'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta})$ – which has the form $-r(\mathbf{x}_i, \boldsymbol{\theta})\mathbf{x}_i'\mathbf{x}_i$ for some function $r(\cdot) > 0$ – does not depend on y_i. If $\boldsymbol{\theta}^*$ is the plim of $\hat{\boldsymbol{\theta}}$ whether or not the mean is correctly specified, then a consistent estimator of $-\mathrm{E}[\mathbf{H}(\mathbf{x}_i, \boldsymbol{\theta}^*)]$ is

$$N^{-1}\sum_{i=1}^{N} r(\mathbf{x}_i, \hat{\boldsymbol{\theta}})\mathbf{x}_i'\mathbf{x}_i = \mathbf{0}.$$

By contrast, with any other mean (link) function, the Hessian depends on y_i, and the estimators based on $\mathrm{E}[\mathbf{H}(\mathbf{x}_i, \boldsymbol{\theta}_o)|\mathbf{x}_i]$ under the assumption the mean is correctly specified are generally inconsistent.

13.13. In fact, there is nothing special about the QMLE setup for this problem: the conclusion holds for M-estimation. It is instructive to see the general argument.

To prove the result for general M-estimation (whether a minimization or maximization problem), use a mean value expansion and multiply through by $N^{-1/2}$:

$$N^{-1/2}\sum_{i=1}^{N} q(\mathbf{w}_i, \hat{\boldsymbol{\theta}}) = N^{-1/2}\sum_{i=1}^{N} q(\mathbf{w}_i, \boldsymbol{\theta}^*) + \left(N^{-1}\sum_{i=1}^{N} \mathbf{s}_i(\ddot{\boldsymbol{\theta}})'\right)\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*)$$

where $\mathbf{s}_i(\boldsymbol{\theta})$ is the $P \times 1$ score and $\ddot{\boldsymbol{\theta}}$ is on the line segment between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^*$. By Lemma 12.1,

$$N^{-1}\sum_{i=1}^{N} \mathbf{s}_i(\ddot{\boldsymbol{\theta}}) \xrightarrow{p} \mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}^*)]$$

because $\ddot{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}^*$. Under the regularity conditions in Theorem 12.3, $\mathrm{E}[\mathbf{s}_i(\boldsymbol{\theta}_*)] = \mathbf{0}$, and so $N^{-1}\sum_{i=1}^{N} \mathbf{s}_i(\ddot{\boldsymbol{\theta}}) \xrightarrow{p} \mathbf{0}$. We also know $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) = O_p(1)$, and so

$$N^{-1/2}\sum_{i=1}^{N} q(\mathbf{w}_i, \hat{\boldsymbol{\theta}}) = N^{-1/2}\sum_{i=1}^{N} q(\mathbf{w}_i, \boldsymbol{\theta}^*) + o_p(1) \cdot O_p(1) = N^{-1/2}\sum_{i=1}^{N} q(\mathbf{w}_i, \boldsymbol{\theta}^*) + o_p(1).$$

13.14 (Bonus Question). Let $\{f(y_t|\mathbf{x}_t;\theta) : t = 1,\ldots,T\}$ be a sequence of correctly specified densities for y_{it} given $\mathbf{x}_{it}$. That is, assume that there is $\boldsymbol{\theta}_o \in \text{int}(\Theta)$ such that $f(y_t|\mathbf{x}_t;\theta_o)$ is the density of y_{it} given $\mathbf{x}_{it} = \mathbf{x}_t$. Also assume that $\{\mathbf{x}_{it} : t = 1,2,\ldots,T\}$ is strictly exogenous for each t: $\text{D}(y_{it}|\mathbf{x}_{i1},\ldots,\mathbf{x}_{iT}) = \text{D}(y_{it}|\mathbf{x}_{it})$.

a. It is true that, under the standard regularity conditions for partial MLE, that $\text{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)|\mathbf{x}_{i1},\ldots,\mathbf{x}_{iT}] = \mathbf{0}$, where $\mathbf{s}_{it}(\boldsymbol{\theta}_o) = \nabla_{\boldsymbol{\theta}} \log f_t(y_{it}|\mathbf{x}_{it};\boldsymbol{\theta})'$?

b. Under the assumptions given, is $\{\mathbf{s}_{it}(\boldsymbol{\theta}_o) : t = 1,\ldots,T\}$ necessarily serially uncorrelated?

c. Let c_i be "unobserved heterogeneity" for cross section unit i, and assume that, for each t,

$$\text{D}(y_{it}|\mathbf{z}_{i1},\ldots,\mathbf{z}_{iT},c_i) = \text{D}(y_{it}|\mathbf{z}_{it},c_i)$$

In other words, $\{\mathbf{z}_{it} : t = 1,\ldots,T\}$ is strictly exogenous conditional on c_i. Further, assume that

$$\text{D}(c_i|\mathbf{z}_{i1},\ldots,\mathbf{z}_{iT}) = \text{D}(c_i|\bar{\mathbf{z}}_i),$$

where $\bar{\mathbf{z}}_i = T^{-1}(\mathbf{z}_{i1} + \ldots + \mathbf{z}_{iT})$ is the vector of time averages. Assuming that well-behaved, correctly-specified conditional densities are available, how do we choose $\mathbf{x}_{it}$ to make part a applicable?

Solution

a. This is true because, by the general theory for partial MLE, we know that $\text{E}[\mathbf{s}_{it}(\boldsymbol{\theta}_o)|\mathbf{x}_{it}] = \mathbf{0}$, $t = 1,\ldots,T$. But if $\text{D}(y_{it}|\mathbf{x}_{i1},\ldots,\mathbf{x}_{iT}) = \text{D}(y_{it}|\mathbf{x}_{it})$ then, for any function $m_t(y_{it},\mathbf{x}_{it})$, $\text{E}[m_t(y_{it},\mathbf{x}_{it})|\mathbf{x}_{i1},\ldots,\mathbf{x}_{iT}] = \text{E}[m_t(y_{it},\mathbf{x}_{it})|\mathbf{x}_{it}]$, including the score function.

b. No. Strict exogeneity and complete dynamic specification of the conditional density are entirely different. Saying that $\text{D}(y_{it}|\mathbf{x}_{i1},\ldots,\mathbf{x}_{iT})$ does not depend on $\mathbf{x}_{is}$, $s \neq t$, says nothing about whether y_{ir}, $r < t$, appears in $\text{D}(y_{it}|\mathbf{x}_{it},y_{i,t-1},\mathbf{x}_{i,t-1},\ldots,y_{i1},\mathbf{x}_{i1})$. Of course it is possible (if unlikely) for the score to be serially uncorrelated without complete dynamic specification, but

that is still a separate issue from strict exogeneity.

c. We take $\mathbf{x}_{it} = (\mathbf{z}_{it}, \bar{\mathbf{z}}_i)$, $t = 1, \ldots, T$. If $g_t(y_t|\mathbf{z}_t, c; \boldsymbol{\gamma})$ is correctly specified for the density of y_{it} given $(\mathbf{z}_{it} = \mathbf{z}_t, c_i = c)$, and $h(c|\bar{\mathbf{z}}; \boldsymbol{\delta})$ is correctly specified for the density of c_i given $\bar{\mathbf{z}}_i = \bar{\mathbf{z}}$, then the density of y_{it} given $\mathbf{z}_i$ is obtained as

$$f_t(y_t|\mathbf{z}_i; \boldsymbol{\theta}_o) = \int_{\mathfrak{C}} g_t(y_t|\mathbf{z}_{it}, \mathfrak{c}; \boldsymbol{\gamma}_o) h(\mathfrak{c}|\bar{\mathbf{z}}_i; \boldsymbol{\delta}_o) v(d\mathfrak{c})$$

and this clearly depends only on $(\mathbf{z}_{it}, \bar{\mathbf{z}}_i)$. In other words, under the assumptions given,

$$\mathrm{D}(y_{it}|\mathbf{z}_{i1}, \ldots, \mathbf{z}_{iT}) = \mathrm{D}(y_{it}|\mathbf{z}_{it}, \bar{\mathbf{z}}_i), t = 1, \ldots, T$$

which implies

$$\mathrm{D}(y_{it}|\mathbf{x}_{i1}, \ldots, \mathbf{x}_{iT}) = \mathrm{D}(y_{it}|\mathbf{x}_{it}), t = 1, \ldots, T.$$

Incidentally, we have not eliminated the serial dependence in $\{y_{it}\}$ after only conditioning on $(\mathbf{z}_{it}, \bar{\mathbf{z}}_i)$: the part of c_i not explained by $\bar{\mathbf{z}}_i$ affects y_{it} in each time period.

13.15 (**Bonus Question**). Consider the problem of estimating quantiles in a parametric context. In particular, write

$$y = \alpha_o + \mathbf{x}\boldsymbol{\beta}_o + u$$
$$\mathrm{D}(u|\mathbf{x}) = \mathrm{Normal}(0, \sigma_o^2 \exp(2\mathbf{x}\boldsymbol{\gamma}_o))$$

This means that $\alpha_o + \mathbf{x}\boldsymbol{\beta}_o = \mathrm{E}(y|\mathbf{x}) = \mathrm{Med}(y|\mathbf{x})$.

a. For $0 < \tau < 1$ let η_τ be the τ^{th} quantile in the standard normal distribution. (So, for example, $\eta_{.95} = 1.645$.) Find $\mathrm{Quant}_\tau(y|\mathbf{x})$ in terms of η_τ and all of the parameters. When is $\mathrm{Quant}_\tau(y|\mathbf{x})$ a linear function of $\mathbf{x}$?

b. Given a random sample of size N, how would you estimate $\mathrm{Quant}_\tau(y|\mathbf{x})$ for a given τ?

c. Suppose we do not assume normality but use the weaker assumption that $u/[\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)]$

is independent of $\mathbf{x}$. Can we consistently estimate $\text{Quant}_\tau(y|\mathbf{x})$ in this case?

Solution

a. First note that

$$\text{Quant}_\tau(y|\mathbf{x}) = \alpha_o + \mathbf{x}\boldsymbol{\beta}_o + \text{Quant}_\tau(u|\mathbf{x}).$$

Let $r = u/[\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)]$, so that r is independent of $\mathbf{x}$ with a Normal$(0, 1)$ distribution. Because u has a strictly increasing cdf conditional on its quantile $q_\tau(\mathbf{x})$ is the unique value such that

$$\text{P}[u \leq q_\tau(\mathbf{x})|\mathbf{x}] = \tau,$$

or

$$\text{P}\left[\frac{u}{\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)} \leq \frac{q_\tau(\mathbf{x})}{\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)} \middle| \mathbf{x} \right] = \tau$$

or

$$\text{P}\left[r \leq \frac{q_\tau(\mathbf{x})}{\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)} \middle| \mathbf{x} \right] = \tau.$$

Because r is independent of $\mathbf{x}$, its τ^{th} quantile conditonal on $\mathbf{x}$ is η_τ. Therefore, we must have

$$\frac{q_\tau(\mathbf{x})}{\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)} = \eta_\tau$$

or

$$q_\tau(\mathbf{x}) = \eta_\tau \sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o).$$

So we have derived

$$\text{Quant}_\tau(y|\mathbf{x}) = \alpha_o + \mathbf{x}\boldsymbol{\beta}_o + \eta_\tau \sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o).$$

$\text{Quant}_\tau(y|\mathbf{x})$ is linear in $\mathbf{x}$ for $\tau = .5$ because then $\eta_\tau = 0$. It is also linear in $\mathbf{x}$ for any τ if $\boldsymbol{\gamma}_o = \mathbf{0}$, in which case it can be written as

$$\text{Quant}_\tau(y|\mathbf{x}) = \alpha_o + \mathbf{x}\boldsymbol{\beta}_o + \eta_\tau\sigma_o.$$

Of course if $\boldsymbol{\gamma}_o = \mathbf{0}$ then u and $\mathbf{x}$ are independent, and the quantile functions for different τ are parallel lines with different intercepts, $\alpha_o + \eta_\tau\sigma_o$.

b. Because we have specified

$$\text{D}(y|\mathbf{x}) = \text{Normal}(\alpha_o + \mathbf{x}\boldsymbol{\beta}_o, \sigma_o^2 \exp(2\mathbf{x}\boldsymbol{\gamma}_o))$$

we can use maximum likelihood to estimate all parameters, given a random sample of size N. Then

$$\widehat{\text{Quant}}_\tau(y|\mathbf{x}) = \hat{\alpha} + \mathbf{x}\hat{\boldsymbol{\beta}} + \eta_\tau\hat{\sigma}\exp(\mathbf{x}\hat{\boldsymbol{\gamma}}).$$

c. The quantile function still has the form

$$\text{Quant}_\tau(y|\mathbf{x}) = \alpha_o + \mathbf{x}\boldsymbol{\beta}_o + \eta_\tau\sigma_o \exp(\mathbf{x}\boldsymbol{\gamma}_o)$$

but we must treat η_τ as an unknown parameter because we do not know the distribution of r. Note that the distribution of r may be asymmetric. The key restriction is that $\text{D}(r|\mathbf{x})$ does not depend on $\mathbf{x}$.

We know η_τ is the τ^{th} quantile of the random variable r. If we observed $\{r_i : i = 1, \ldots, N\}$ we could estimate η_τ as the τ^{th} sample quantile of the r_i. Instead, we can use the standardized residuals

$$\hat{r}_i = \frac{\hat{u}_i}{\hat{\sigma}\exp(\mathbf{x}_i\hat{\boldsymbol{\gamma}})} = \frac{(y_i - \hat{\alpha} - \mathbf{x}_i\hat{\boldsymbol{\beta}})}{\hat{\sigma}\exp(\mathbf{x}_i\hat{\boldsymbol{\gamma}})}$$

and compute $\hat{\eta}_\tau$ as the τ^{th} sample quantile of $\{\hat{r}_i : i = 1, \ldots, N\}$. Because $\hat{\eta}_\tau$ solves the problem

$$\min_{h_\tau} \sum_{i=1}^{N} c_\tau(h_\tau - \hat{r}_i),$$

where $c_\tau(\cdot)$ is the "check" function defined in Section 12.10, we can conclude $\hat{\eta}_\tau$ is generally consistent using the consistency result for two-step M-estimators in Section 12.4. Of course, we have to have consistent estimators of the other parameters. From the results of Gourieroux, Monfort, and Trognon (1984a), the normal QMLE is generally consistent for α_o, $\boldsymbol{\beta}_o$, σ_o, and $\boldsymbol{\gamma}_o$ even if normality does not hold. (As usual, we would need to use a sandwich covariance matrix estimator for inference on these parameters.) Obtaining a valid standard error for $\hat{\eta}_\tau$, and then getting the joint variance-covariance matrix of all parameter estimators, is challenging. Probably the nonparametric bootstrap is valid.

Solutions to Chapter 14 Problems

14.1. a. The simplest way to estimate (14.35) is by 2SLS, using instruments $(\mathbf{x}_1, \mathbf{x}_2)$. Nonlinear functions of these can be added to the instrument list, and they would generally improve efficiency if $\gamma_2 \neq 1$. If $\mathrm{E}(u_2^2|\mathbf{x}) = \sigma_2^2$, 2SLS using the given list of instruments is the efficient, single equation GMM estimator. If there is heteroskedasticity an optimal weighting matrix that allows heteroskedasticity of unknown form should be used. Finally, one could try to use the optimal instruments derived in section 14.4.3. Even under homoskedasticity, these are difficult, if not impossible, it find analytically if $\gamma_2 \neq 1$.

With $y_2 \geq 0$, equation (14.35) is suspect as a structural equation because it is a linear model, and generally there are outcomes where $\mathbf{x}_2\boldsymbol{\delta}_2 + \gamma_3 y_1 + u_2 < 0$.

b. No. If $\gamma_1 = 0$ the parameter γ_2 does not appear in the model. Of course, if we knew $\gamma_1 = 0$, we would consistently estimate $\boldsymbol{\delta}_1$ by OLS.

c. We can see this by obtaining $\mathrm{E}(y_1|\mathbf{x})$:

$$\begin{aligned}\mathrm{E}(y_1|\mathbf{x}) &= \mathbf{x}_1\boldsymbol{\delta}_1 + \gamma_1\mathrm{E}(y_2^{\gamma_2}|\mathbf{x}) + \mathrm{E}(u_1|\mathbf{x}) \\ &= \mathbf{x}_1\boldsymbol{\delta}_1 + \gamma_1\mathrm{E}(y_2^{\gamma_2}|\mathbf{x}).\end{aligned}$$

Now, when $\gamma_2 \neq 1, \mathrm{E}(y_2^{\gamma_2}|\mathbf{x}) \neq [\mathrm{E}(y_2|\mathbf{x})]^{\gamma_2}$, so we cannot write

$$\mathrm{E}(y_1|\mathbf{x}) = \mathbf{x}_1\boldsymbol{\delta}_1 + \gamma_1(\mathbf{x}\boldsymbol{\delta}_2)^{\gamma_2};$$

in fact, we cannot find $\mathrm{E}(y_1|\mathbf{x})$ without more assumptions. While the regression y_2 on $\mathbf{x}_2$ consistently estimates $\boldsymbol{\delta}_2$, the two-step NLS estimator of y_{i1} on $\mathbf{x}_{i1}, (\mathbf{x}_i\hat{\boldsymbol{\delta}}_2)^{\gamma_2}$ will not be consistent for $\boldsymbol{\delta}_1$ and γ_2. (This is an example of a "forbidden regression," which we discussed in Chapter 9.) When $\gamma_2 = 1$ and we impose this in estimation, we obtain the usual 2SLS estimator.

14.2. a. When $\rho_1 = 1$, we obtain the level-level model, $hours = -\gamma_1 + \mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 wage + u_1$. Using the hint, let $\rho_1 \to 0$ to get $hours = \mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 \log(wage) + u_1$.

b. We cannot use a standard *t* test after estimating the full model (say, by nonlinear 2SLS), because ρ_1 cannot be estimated under H_0. The score test and QLR test also fail because of lack of identification under H_0. What we can do is *fix* a value for ρ_1 – essentially our best guess – and then use a *t* test on $(wage^{\rho_1} - 1)/\rho_1$ after linear 2SLS estimation (or GMM more generally). This need not be a very good test for detecting $\gamma_1 \neq 0$ if our guess for ρ_1 is not close to the actual value. There is a growing literature on testing hypotheses when parameters are not identified under the null.

c. If $\text{Var}(u_1|\mathbf{z}) = \sigma_1^2$, use nonlinear 2SLS, where we would use $\mathbf{z}$ and functions of $\mathbf{z}$ as IVs. If we are not willing to assume homoskedasticity, GMM is generally more efficient.

d. The residual function is $r(\boldsymbol{\theta}) = [hours - \mathbf{z}_1\boldsymbol{\delta}_1 - \gamma_1(wage^{\rho_1} - 1)/\rho_1]$, where $\boldsymbol{\theta} = (\boldsymbol{\delta}_1', \gamma_1, \rho_1)'$. Using the hint the gradient is

$$\nabla_{\boldsymbol{\theta}} r(\boldsymbol{\theta}) = \{-\mathbf{z}_1, -(wage^{\rho_1} - 1)/\rho_1, \gamma_1[(wage^{\rho_1} - 1) - \rho_1 wage^{\rho_1}\log(wage)]/\rho_1^2\}.$$

The score is just the transpose.

e. Estimate $\boldsymbol{\delta}_1$ and γ_1 by 2SLS, or use the GMM estimator that accounts for heteroskedasticity, under the restriction $\boldsymbol{\rho}_1 = 1$. Suppose the instruments are $\mathbf{z}_i$, a $1 \times L$ vector. This is just linear estimation because the model is linear under H_0. Then, taking $\mathbf{Z}_i = \mathbf{z}_i$, and

$$\begin{aligned} r_i(\tilde{\boldsymbol{\theta}}) &= [hours_i - \mathbf{z}_{i1}\tilde{\boldsymbol{\delta}}_1 - \tilde{\gamma}_1(wage_i - 1)] \\ \nabla_{\boldsymbol{\theta}} r_i(\tilde{\boldsymbol{\theta}}) &= (-\mathbf{z}_{i1}, -(wage_i - 1), \tilde{\gamma}_1[(wage_i - 1) - wage_i \log(wage_i)]), \end{aligned}$$

use the score statistic in equation (14.32).

14.3. Let $\mathbf{Z}_i^*$ be the $G \times G$ matrix of optimal instruments in (14.57), where we suppress its

dependence on $\mathbf{x}_i$. Let $\mathbf{Z}_i$ be the $G \times L$ matrix that is a function of $\mathbf{x}_i$ and let $\boldsymbol{\Xi}_o$ the probability limit of the weighting matrix. Then the asymptotic variance of the GMM estimator has the form (14.10) with $\mathbf{G}_o = \mathrm{E}[\mathbf{Z}_i'\mathbf{R}_o(\mathbf{x}_i)]$. So, in (14.48) take $\mathbf{A} \equiv \mathbf{G}_o'\boldsymbol{\Xi}_o\mathbf{G}_o$ and $\mathbf{s}(\mathbf{w}_i) \equiv \mathbf{G}_o'\boldsymbol{\Xi}_o\mathbf{Z}_i'\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)$. The optimal score function is $\mathbf{s}^*(\mathbf{w}_i) \equiv \mathbf{R}_o(\mathbf{x}_i)'\boldsymbol{\Omega}_o(\mathbf{x}_i)^{-1}\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)$. Now we can verify (14.51) with $\rho = 1$:

$$\begin{aligned}\mathrm{E}[\mathbf{s}(\mathbf{w}_1)\mathbf{s}^*(\mathbf{w}_1)'] &= \mathbf{G}_o'\boldsymbol{\Xi}_o\mathrm{E}[\mathbf{Z}_i'\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)'\boldsymbol{\Omega}_o(\mathbf{x}_i)^{-1}\mathbf{R}_o(\mathbf{x}_i)] \\ &= \mathbf{G}_o'\boldsymbol{\Xi}_o\mathrm{E}[\mathbf{Z}_i'\mathrm{E}\{\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)'|\mathbf{x}_i\}\boldsymbol{\Omega}_o(\mathbf{x}_i)^{-1}\mathbf{R}_o(\mathbf{x}_i)] \\ &= \mathbf{G}_o'\boldsymbol{\Xi}_o\mathrm{E}[\mathbf{Z}_i'\boldsymbol{\Omega}_o(\mathbf{x}_i)\boldsymbol{\Omega}_o(\mathbf{x}_i)^{-1}\mathbf{R}_o(\mathbf{x}_i)] \;= \mathbf{G}_o'\boldsymbol{\Xi}_o\mathbf{G}_o = \mathbf{A}.\end{aligned}$$

14.4. a. The residual function for the conditional mean model $\mathrm{E}(y_i|\mathbf{x}) = m(\mathbf{x}_i, \boldsymbol{\beta}_o)$ is $r_i(\boldsymbol{\beta}) \equiv y_i - m(\mathbf{x}_i, \boldsymbol{\beta})$. Then $\Omega_o(\mathbf{x}_i)$ in (14.55) is just a scalar, $\Omega_o(\mathbf{x}_i) = \mathrm{Var}(y_i|\mathbf{x}_i) \equiv \omega_o(\mathbf{x}_i)$. Under WNLS.3, $\omega_o(\mathbf{x}_i) = \sigma_o^2 h(\mathbf{x}_i, \boldsymbol{\gamma}_o)$ for a known function $h(\cdot)$. Further, $\mathbf{R}_o(\mathbf{x}_i) \equiv \mathrm{E}[\nabla_{\boldsymbol{\beta}}\mathbf{r}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i] = -\nabla_{\boldsymbol{\beta}}m(\mathbf{x}_i, \boldsymbol{\beta}_o)$, and so the optimal instruments are $\nabla_{\boldsymbol{\beta}}m(\mathbf{x}_i, \boldsymbol{\beta}_o)/\omega_o(\mathbf{x}_i)$. The asymptotic variance of the efficient IV estimator is obtained from (14.60):

$$\{\mathrm{E}[\nabla_{\boldsymbol{\beta}}m(\mathbf{x}_i, \boldsymbol{\beta}_o)'[\omega_o(\mathbf{x}_i)]^{-1}\nabla_{\boldsymbol{\beta}}m(\mathbf{x}_i, \boldsymbol{\beta}_o)]\}^{-1} = \sigma_o^2\{\mathrm{E}[\nabla_{\boldsymbol{\beta}}m(\mathbf{x}_i, \boldsymbol{\beta}_o)'\nabla_{\boldsymbol{\beta}}m(\mathbf{x}_i, \boldsymbol{\beta}_o)/h(\mathbf{x}_i, \boldsymbol{\gamma}_o)]\}^{-1},$$

which is the asymptotic variance of the WNLS estimator under WNLS.1, WNLS.2, and WNLS.3.

b. If $\mathrm{Var}(y_i|\mathbf{x}_i) = \sigma_o^2$ then NLS achieves the efficiency bound, as it seen by setting $h(\mathbf{x}, \boldsymbol{\gamma}_o) \equiv 1$ in part a.

c. Now let $r_{i1}(\boldsymbol{\theta}) \equiv u_i(\boldsymbol{\beta}) \equiv y_i - m(\mathbf{x}_i, \boldsymbol{\beta})$ and $r_{i2}(\boldsymbol{\beta}, \sigma^2) = [y_i - m(\mathbf{x}_i, \boldsymbol{\beta})]^2 - \sigma^2$. Let $\mathbf{r}_i(\boldsymbol{\theta})$ denote the 2×1 vector obtained by stacking the two residual functions. Then the moment conditions can be written as

$$\mathrm{E}[\mathbf{r}_i(\boldsymbol{\theta}_o)|\mathbf{x}_i] = \mathbf{0},$$

where $\boldsymbol{\theta}_o = (\boldsymbol{\beta}_o', \sigma_o^2)'$. To obtain the efficient IVs, we first need $\mathrm{E}[\nabla_{\boldsymbol{\theta}}\mathbf{r}_i(\boldsymbol{\theta}_o)|\mathbf{x}_i]$. But

$$\nabla_{\boldsymbol{\theta}}\mathbf{r}_i(\boldsymbol{\theta}) = \begin{pmatrix} -\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}) & 0 \\ -2\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}) u_i(\boldsymbol{\beta}) & -1 \end{pmatrix}.$$

Evaluating at $\boldsymbol{\theta}_o$ and using $\mathrm{E}[u_i(\boldsymbol{\beta}_o)|\mathbf{x}_i] = 0$ gives

$$\mathbf{R}_\circ(\mathbf{x}_i) \equiv \nabla_{\boldsymbol{\theta}}\mathbf{r}_i(\boldsymbol{\theta}) = \begin{pmatrix} -\nabla_{\boldsymbol{\beta}} m_i(\boldsymbol{\beta}) & 0 \\ 0 & -1 \end{pmatrix}.$$

We also need

$$\Omega_o(\mathbf{x}_i) = \mathrm{E}[r_i(\boldsymbol{\theta}_o)\mathbf{r}_i(\boldsymbol{\theta}_o)'|\mathbf{x}_i] = \begin{pmatrix} \sigma_o^2 & \mathrm{E}(u_i^3|\mathbf{x}_i) \\ \mathrm{E}(u_i^3|\mathbf{x}_i) & \mathrm{E}(u_i^4|\mathbf{x}_i) - \sigma_o^4 \end{pmatrix}$$

where $u_i \equiv y_i - m(\mathbf{x}_i, \boldsymbol{\beta}_o)$. The optimal IVs are $[\boldsymbol{\Omega}_o(\mathbf{x}_i)]^{-1}\mathbf{R}_o(\mathbf{x}_i)$. If $\mathrm{E}(u_i^3|\mathbf{x}_i) = 0$, as occurs under conditional symmetry of u_i, then the asymptotic variance matrix of the optimal IV estimator is block diagonal, and for $\hat{\boldsymbol{\beta}}$ it is the same as NLS. In other words, adding the moment condition for the homoskedasticity assumption does not improve efficiency over NLS under symmetry, even if $\mathrm{E}(u_i^4|\mathbf{x}_i)$ is not constant. But there is something subtle here. the NLS estimator is efficient in the class of estimators that only uses information on the first two conditional moments. If we use the information $\mathrm{E}(u_i^3|\mathbf{x}_i) = 0$ then, in general, we could do better. But, of course, such an estimator would be less robust than NLS.

If, in addition, $\mathrm{E}(u_i^4|\mathbf{x}_i)$ is constant, then the usual estimator of σ_o^2 based on the sum of squared NLS residuals is efficient (among estimators that only use the first two conditional moments, but it happens that $\mathrm{E}(u_i^3|\mathbf{x}_i) = 0$ and $\mathrm{E}(u_i^4|\mathbf{x}_i)$ is constant).

14.5. We can write the unrestricted linear projection as

$$y_{it} = \pi_{t0} + \mathbf{x}_i\boldsymbol{\pi}_t + v_{it},\ t = 1,2,3$$

where $\boldsymbol{\pi}_t$ is $1 + 3K \times 1$, and then $\boldsymbol{\pi}$ is the $3 + 9K \times 1$ vector obtained by stacking the π_t. Let $\boldsymbol{\theta} = (\psi, \boldsymbol{\lambda}_1', \boldsymbol{\lambda}_2', \boldsymbol{\lambda}_3', \boldsymbol{\beta}')'$. With the restrictions imposed on the $\boldsymbol{\pi}_t$ we have

$$\pi_{t0} = \psi, t = 1,2,3,\ \boldsymbol{\pi}_1 = [(\boldsymbol{\lambda}_1 + \boldsymbol{\beta})', \boldsymbol{\lambda}_2', \boldsymbol{\lambda}_3']'$$
$$\boldsymbol{\pi}_2 = [\boldsymbol{\lambda}_1', (\boldsymbol{\lambda}_2 + \boldsymbol{\beta})', \boldsymbol{\lambda}_3']',\ \boldsymbol{\pi}_3 = [\boldsymbol{\lambda}_1', \boldsymbol{\lambda}_2', (\boldsymbol{\lambda}_3 + \boldsymbol{\beta})']'$$

Therefore, we can write $\boldsymbol{\pi} = \mathbf{H}\boldsymbol{\theta}$ for the $(3 + 9K) \times (1 + 4K)$ matrix $\mathbf{H}$ defined by

$$\mathbf{H} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0} & \mathbf{I}_K \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_K & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{I}_K \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_K & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_K & \mathbf{I}_K \end{pmatrix}.$$

14.6. By this hint, it suffices to show that

$$[\text{Avar}\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)]^{-1} - [\text{Avar}\sqrt{N}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)]^{-1}$$

is p.s.d. This difference is $\mathbf{H}_o'\boldsymbol{\Xi}_o^{-1}\mathbf{H}_o - \mathbf{H}_o'\boldsymbol{\Lambda}_o^{-1}\mathbf{H}_o = \mathbf{H}_o'(\boldsymbol{\Xi}_o^{-1} - \boldsymbol{\Lambda}_o^{-1})\mathbf{H}_o$. This is positive semi-definite if $\boldsymbol{\Xi}_o^{-1} - \boldsymbol{\Lambda}_o^{-1}$ is p.s.d., which again holds by the hint because $\boldsymbol{\Lambda}_o - \boldsymbol{\Xi}_o$is assumed to be p.s.d.

14.7. With $\mathbf{h}(\boldsymbol{\theta}) = \mathbf{H}\theta$, the minimization problem becomes

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^P} (\hat{\boldsymbol{\pi}} - \mathbf{H}\boldsymbol{\theta})'\hat{\boldsymbol{\Xi}}^{-1}(\hat{\boldsymbol{\pi}} - \mathbf{H}\boldsymbol{\theta}),$$

where it is assumed that no restrictions are placed on θ. The first order condition is easily seen to be

$$-2\mathbf{H}'\hat{\boldsymbol{\Xi}}^{-1}(\hat{\boldsymbol{\pi}} - \mathbf{H}\boldsymbol{\theta}) = \mathbf{0} \text{ or } (\mathbf{H}'\hat{\boldsymbol{\Xi}}^{-1}\mathbf{H})\hat{\boldsymbol{\theta}} = \mathbf{H}'\hat{\boldsymbol{\Xi}}^{-1}\hat{\boldsymbol{\pi}}.$$

Therefore, assuming $\mathbf{H}'\hat{\boldsymbol{\Xi}}^{-1}\mathbf{H}$ is nonsingular – which occurs w.p.a.1. when $\mathbf{H}'\boldsymbol{\Xi}_o^{-1}\mathbf{H}$ – is nonsingular – we have $\hat{\boldsymbol{\theta}} = (\mathbf{H}'\hat{\boldsymbol{\Xi}}^{-1}\mathbf{H})^{-1}\mathbf{H}'\hat{\boldsymbol{\Xi}}^{-1}\hat{\boldsymbol{\pi}}$.

14.8. From the efficiency discussion about maximum likelihood in Section 14.4.2, it is no less asymptotically efficient to use the density of $(y_{i0}, y_{i1}, \ldots, y_{iT})$ than to use the conditional distribution $(y_{i1}, \ldots, y_{iT})$ given y_{i0}. The cost of the asymptotic efficiency is that if we misspecify $f_0(y_0; \boldsymbol{\theta})$, then the unconditional MLE will generally be inconsistent for $\boldsymbol{\theta}_o$. The MLE that conditions on y_{i0} is consistent provided we have the densities $f_t(y_t|y_{t-1}; \boldsymbol{\theta})$ correctly specified, $t \geq 1$. As $f_t(y_t|y_{t-1}; \boldsymbol{\theta})$ is the density of interest, we are usually willing to put more effort into testing our specification of it.

14.9. We have to verify equations (14.49) and (14.50) for the random effects and fixed effects estimators with . The choices of $\mathbf{s}_{i1}$, $\mathbf{s}_{i2}$ (with added i subscripts for clarity), $\mathbf{A}_1$, and $\mathbf{A}_2$ are given in the hint. Now, from Chapter 10, we know that $\mathrm{E}(\mathbf{r}_i\mathbf{r}_i'|\mathbf{x}_i) = \sigma_u^2\mathbf{I}_T$ under RE.1, RE.2, and RE.3, where $\mathbf{r}_i = \mathbf{v}_i - \lambda\mathbf{j}_T\bar{v}_i$. Therefore,

$$\mathrm{E}(\mathbf{s}_{i1}\mathbf{s}_{il}') = \mathrm{E}(\check{\mathbf{X}}_i'\mathbf{r}_i\mathbf{r}_i'\check{\mathbf{X}}_i) = \sigma_u^2\mathrm{E}(\check{\mathbf{X}}_i'\check{\mathbf{X}}_i) \equiv \sigma_u^2\mathbf{A}_1$$

by the usual iterated expectations argument. This means that, in (14.49), $\rho \equiv \sigma_u^2$. Now, we just need to verify (14.50) for this choice of ρ. But $\mathbf{s}_{i2}\mathbf{s}_{i1}' = \ddot{\mathbf{X}}_i'\mathbf{u}_i\mathbf{r}_i'\check{\mathbf{X}}_i$ and, as described in the hint,

$$\ddot{\mathbf{X}}_i'\mathbf{r}_i = \ddot{\mathbf{X}}_i'(\mathbf{v}_i - \boldsymbol{\lambda}\mathbf{j}_T\bar{v}_i) = \ddot{\mathbf{X}}_i'\mathbf{v}_i = \ddot{\mathbf{X}}_i'(c_i\jmath_T + \mathbf{u}_i) = \ddot{\mathbf{X}}_i'\mathbf{u}_i.$$

Therefore, $\mathbf{s}_{i2}\mathbf{s}_{i1}' = \ddot{\mathbf{X}}_i'\mathbf{r}_i\mathbf{r}_i'\check{\mathbf{X}}_i$ and so

$$\mathrm{E}(\mathbf{s}_{i2}\mathbf{s}_{i1}'|\mathbf{x}_i) = \ddot{\mathbf{X}}_i'\mathrm{E}(\mathbf{r}_i\mathbf{r}_i'|\mathbf{x}_i)\check{\mathbf{X}}_i = \sigma_u^2\ddot{\mathbf{X}}_i'\check{\mathbf{X}}_i$$

It follows that

$$\mathrm{E}(\mathbf{s}_{i2}\mathbf{s}_{i1}') = \ddot{\mathbf{X}}_i'\mathrm{E}(\mathbf{r}_i\mathbf{r}_i'|\mathbf{x}_i)\check{\mathbf{X}}_i = \sigma_u^2\mathrm{E}(\ddot{\mathbf{X}}_i'\check{\mathbf{X}}_i)$$

Finally, $\ddot{\mathbf{X}}_i'\check{\mathbf{X}}_i = \ddot{\mathbf{X}}_i'(\mathbf{X}_i - \boldsymbol{\lambda}\mathbf{j}_T\bar{\mathbf{x}}_i) = \ddot{\mathbf{X}}_i'\mathbf{X}_i = \ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i$, and so $\mathrm{E}(\mathbf{s}_{i2}\mathbf{s}_{i1}') = \sigma_u^2\mathrm{E}(\ddot{\mathbf{X}}_i'\ddot{\mathbf{X}}_i)$, and this verifies (14.50) with $\rho = \sigma_u^2$.

14.10. a. For each t we have

$$\begin{aligned}\mathrm{E}(v_{it}|\mathbf{x}_i) &= \mathrm{E}(\eta_t c_i + u_{it}|\mathbf{x}_i) = \mathrm{E}(\eta_t c_i|\mathbf{x}_i) + \mathrm{E}(u_{it}|\mathbf{x}_i)\\ &= \eta_t\mathrm{E}(c_i|\mathbf{x}_i) + \mathrm{E}(u_{it}|\mathbf{x}_i)\\ &= \eta_t \cdot 0 + 0 = 0\end{aligned}$$

because $\mathrm{E}(c_i|\mathbf{x}_i) = 0$ and $\mathrm{E}(u_{it}|\mathbf{x}_i) = 0$ (because $\mathrm{E}(u_{it}|\mathbf{x}_i, c_i) = 0$).

b. Under the assumptions – which are the same as Assumptions RE.1 and RE.3 – we know that

$$\begin{aligned}\mathrm{Var}(u_{it}) &= \sigma_u^2,\ t = 1,\ldots,T\\ \mathrm{Cov}(c_i, u_{it}) &= 0,\ t = 1,\ldots,T\\ \mathrm{Cov}(u_{it}, u_{is}) &= 0,\ t \neq s.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Var}(v_{it}) &= \mathrm{Var}(\eta_t c_i + u_{it}) = \eta_t^2\sigma_c^2 + 2\eta_t\mathrm{Cov}(c_i, u_{it}) + \sigma_u^2\\ &= \eta_t^2\sigma_c^2 + \sigma_u^2\end{aligned}$$

and, for $t \neq s$,

$$\begin{aligned}\mathrm{Cov}(v_{it}, v_{is}) &= \mathrm{Cov}(\eta_t c_i + u_{it}, \eta_s c_i + u_{is})\\ &= \mathrm{Cov}(\eta_t c_i, \eta_s c_i) + \mathrm{Cov}(\eta_t c_i, u_{is}) + \mathrm{Cov}(\eta_s c_i, u_{it}) + \mathrm{Cov}(u_{it}, u_{is})\\ &= \eta_t\eta_s\mathrm{Cov}(c_i, c_i) = \eta_t\eta_s\sigma_c^2\end{aligned}$$

c. The usual RE estimator treats the η_t as constant (which can then be normalized to be unity). In other words, it uses a misspecified model for $\mathbf{\Omega} = \text{Var}(\mathbf{v}_i)$. As we discussed in Chapter 10, the RE estimator is still consistent and $\sqrt{N}$-asymptotically normal, and we can conduct inference using a robust variance matrix estimator.

A more efficient estimator is, of course, FGLS with the correct form of the variance-covariance matrix. Write the T time periods for draw i as

$$\begin{aligned} \mathbf{y}_i &= \mathbf{X}_i\boldsymbol{\beta} + \mathbf{v}_i \\ \text{E}(\mathbf{v}_i|\mathbf{x}_i) &= \mathbf{0} \end{aligned}$$

where the t^{th} row of $\mathbf{X}_i$ is $\mathbf{x}_{it}$. The $T \times T$ variance-covariance matrix (which is also the conditional on $\mathbf{x}_i$) is

$$\text{Var}(\mathbf{v}_i) = \begin{pmatrix} \sigma_c^2 + \sigma_u^2 & \eta_2\sigma_c^2 & \cdots & \eta_T\sigma_c^2 \\ \eta_2\sigma_c^2 & \eta_2^2\sigma_c^2 + \sigma_u^2 & \cdots & \eta_2\eta_T\sigma_c^2 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_T\sigma_c^2 & \eta_T\eta_2\sigma_c^2 & \cdots & \eta_T^2\sigma_c^2 + \sigma_u^2 \end{pmatrix},$$

where we impose the normalization $\eta_1 = 1$. The GLS estimator is

$$\hat{\boldsymbol{\beta}}_{GLS} = \left(\sum_{i=1}^{N} \mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{X}_i \right)^{-1} \left(\sum_{i=1}^{N} \mathbf{X}_i'\mathbf{\Omega}^{-1}\mathbf{y}_i \right).$$

Of course, we would have to estimate σ_c^2, σ_u^2, and $\eta_2, \ldots, \eta_T$. One way to approach estimation of the variance-covariance parameters is to write

$$\begin{aligned} v_{it}v_{is} &= \eta_t\eta_s\sigma_c^2 + d_{ts}\sigma_u^2 + r_{its} \\ \text{E}(r_{its}) &= 0 \end{aligned}$$

for all $t, s = 1, 2, \ldots, T$, where d_{ts} is a dummy variable equal to one if $t = s$, and zero otherwise. Then we can estimate the parameters by pooled NLS after replacing $v_{it}v_{is}$ with $\check{v}_{it}\check{v}_{is}$, where the

$\check{v}_{it}$ are perhaps the RE residuals (or they could be the POLS residuals). Note that $\eta_1 \equiv 1$ is imposed. Then we can form $\hat{\mathbf{\Omega}}$ and then use FGLS.

14.11. First estimate initial parameters $\boldsymbol{\pi}_o$ from a set of linear reduced-form equations:

$$\mathbf{y}_i = \mathbf{X}_i\boldsymbol{\pi}_o + \mathbf{u}_i$$

where $\boldsymbol{\pi}_o$ is $K \times 1$ and unrestricted. Then estimate $\boldsymbol{\pi}_o$ by, say, system OLS. Or, if we assumed $\text{Var}(\mathbf{u}_i|\mathbf{x}_i) = \mathbf{\Omega}_o$ then FGLS would be no less asymptotically efficient.

Given $\hat{\boldsymbol{\pi}}$ and $\hat{\mathbf{\Xi}}$ consistent for

$$\mathbf{\Xi}_o = \text{Avar}[\sqrt{N}(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}_o)],$$

the CMD estimator of $\boldsymbol{\theta}_o$, $\hat{\boldsymbol{\theta}}$, solves

$$\min_{\boldsymbol{\theta}\in\mathbf{\Theta}} [\hat{\boldsymbol{\pi}} - \mathbf{g}(\boldsymbol{\theta})]'\hat{\mathbf{\Xi}}^{-1}[\hat{\boldsymbol{\pi}} - \mathbf{g}(\boldsymbol{\theta})],$$

which is algebraically equivalent to a weighted multivariate nonlinear least squares problem where $\hat{\boldsymbol{\pi}}$ plays the role of the $K \times 1$ vector of "dependent variables." As discussed in the case where $\mathbf{g}(\boldsymbol{\theta})$ is linear, the asymptotic analysis of the CMD estimator is different from the standard WMNLS problem: here K is fixed.

After estimation, $\text{Avar}(\hat{\boldsymbol{\theta}})$ is estimated as

$$(\hat{\mathbf{G}}'\hat{\mathbf{\Xi}}^{-1}\hat{\mathbf{G}})^{-1}/N$$

where $\hat{\mathbf{G}} \equiv \nabla_{\boldsymbol{\theta}}\mathbf{g}(\hat{\boldsymbol{\theta}})$.

Solutions to Chapter 15 Problems

15.1. a. Because the regressors are all orthogonal by construction – that is, $dk_i \cdot dm_i = 0$ for $k \neq m$, and all i – the coefficient on dm_i is obtained from the regression y_i on $dm_i, i = 1, \ldots, N$. But this is easily seen to be the fraction of ones in the sample falling into category m (because it is the average of y_i over the observations from category m). Therefore, the fitted value for any i is the cell frequency for the appropriate category. These frequencies are all necessarily in [0,1].

b. The fitted values for each category will be the same. If we drop $d1_i$ but add an overall intercept, the overall intercept is the cell frequency for the first category, and the coefficient on dm_i becomes the difference in cell frequencies between category m and category one (the base category), $m = 2, \ldots, M$.

15.2. a. First, because utility is increasing in both c and q, the budget constraint is binding at the optimum: $c_i + p_i q_i = m_i$. Plugging $c = m_i - p_i q$ into the utility function reduces the problem to

$$\max_{q \geq 0} \; (m_i - p_i q) + a_i \log(1 + q).$$

Define utility as a function of q, as

$$s_i(q) \equiv (m_i - p_i q) + a_i \log(1 + q).$$

Then, for all $q \geq 0$,

$$\frac{ds_i}{dq}(q) = -p_i + \frac{a_i}{1 + q}.$$

The optimal solution is $q_i = 0$ if the marginal utility of charitable giving at $q = 0$ is nonpositive, that is, if

$$\frac{ds_i}{dq}(0) = -p_i + a_i \leq 0 \text{ or } a_i \leq p_i.$$

(This can also be obtained by solving the Kuhn-Tucker conditions.) Thus, for this utility function, a_i can be interpreted as the reservation price above which no charitable contribution will be made; in other words, we have the corner solution $q_i = 0$ whenever the price of charitable giving is too high relative to the marginal utility of charitable giving. On the other hand, if $a_i > p_i$ then an interior solution exists ($q_i > 0$) and necessarily solves the first order condition

$$\frac{ds_i}{dq}(q_i) = -p_i + \frac{a_i}{1 + q_i} \equiv 0$$

or

$$1 + q_i = a_i/p_i.$$

b. By definition of y_i, $y_i = 1$ if and only if $a_i/p_i > 1$ or $\log(a_i/p_i) > 0$. If $a_i = \exp(\mathbf{z}_i\boldsymbol{\gamma} + v_i)$, the condition for $y_i = 1$ is equivalent to $\mathbf{z}_i\boldsymbol{\gamma} + v_i - \log p_i > 0$. Therefore,

$$\begin{aligned} \mathrm{P}(y_i = 1|\mathbf{z}_i, m_i, p_i) &= \mathrm{P}(y_i = 1|\mathbf{z}_i, p_i) \\ &= \mathrm{P}(\mathbf{z}_i\boldsymbol{\gamma} + v_i - \log p_i > 0|\mathbf{z}_i, p_i) = \mathrm{P}[v_i/\sigma > (-\mathbf{z}_i\boldsymbol{\gamma} + \log p_i)/\sigma] \\ &= 1 - G[(-\mathbf{z}_i\boldsymbol{\gamma} + \log p_i)/\sigma] = G[(\mathbf{z}_i\boldsymbol{\gamma} - \log p_i)\sigma], \end{aligned}$$

where the last equality follows by symmetry of the distribution of v_i/σ.

15.3. a. If $\mathrm{P}(y_i = 1|\mathbf{z}_1, z_2) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2)$ then

$$\frac{\partial \mathrm{P}(y = 1|\mathbf{z}_1, z_2)}{\partial z_2} = (\gamma_1 + 2\gamma_2 z_2) \cdot \phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2);$$

for given $\mathbf{z}$, the partial effect is estimated as

$$(\hat{\gamma}_1 + 2\hat{\gamma}_2 z_2) \cdot \phi(\mathbf{z}_1\hat{\boldsymbol{\delta}}_1 + \hat{\gamma}_1 z_2 + \hat{\gamma}_2 z_2^2),$$

where, or course, the estimates are the probit estimates.

b. In the model

$$\mathrm{P}(y_i = 1|\mathbf{z}_i, z_2, d_1) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 d_1 + \gamma_3 z_2 d_1),$$

the partial effect of z_2 is

$$\frac{\partial \mathrm{P}(y = 1|\mathbf{z}_1, z_2, d_1)}{\partial z_2} = (\gamma_1 + \gamma_3 d_1) \cdot \phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 d_1 + \gamma_3 z_2 d_1).$$

The effect of d_1 is measured as the difference in the probabilities at $d_1 = 1$ and $d_1 = 0$:

$$\mathrm{P}(y = 1|\mathbf{z}, d_1 = 1) - \mathrm{P}(y = 1|\mathbf{z}, d_1 = 0) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_2 + (\gamma_1 + \gamma_3)z_2) - \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2).$$

Again, to estimate these effects at given $\mathbf{z}$ and – in the first case, d_1 – we just replace the parameters with their probit estimates, and use average or other interesting values of $\mathbf{z}$.

c. If the estimated partial effect is for particular values of $(\mathbf{z}_1, z_2, d_1)$, for example,

$$(\hat{\gamma}_1 + \hat{\gamma}_3 d_1^o) \cdot \phi(\mathbf{z}_1^o\hat{\boldsymbol{\delta}}_1 + \hat{\gamma}_1 z_2^o + \hat{\gamma}_2 d_1^o + \hat{\gamma}_3 z_2^o d_1^o),$$

then we can apply the delta method from Chapter 3 (and referred to in Part III). Thus, we would require the full variance matrix of the probit estimates as well as the gradient of the expression of interest, such as $(\gamma_1 + 2\gamma_2 z_2) \cdot \phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 + \gamma_2 z_2^2)$, with respect to all probit parameters. Alternatively, the bootstrap would be simply but require a bit more computation.

If we are interested in the average partial effect (APE) of d_1 going from zero to one then we estimate it as

$$N^{-1}\sum_{i=1}^{N}[\Phi(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1 + (\hat{\gamma}_1 + \hat{\gamma}_3)z_{i2} + \hat{\gamma}_2) - \Phi(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1 + \hat{\gamma}_1 z_{i2})],$$

that is, we estimate the effect for each unit *i* and then average these across all *i*. If we want a standard error for this, we would use the extension of the delta method worked out in Problem

12.17 – to account for the averaging as well as estimation of the parameters. The bootstrap can be used, too.

d. (**Bonus Question**) For a fixed value of z_2, say z_2^o, how would you estimate the average partial effect of d_1 on the response probability?

Solution

Now we average out only with respect to $\mathbf{z}_{i1}$:

$$N^{-1}\sum_{i=1}^{N}[\Phi(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1 + (\hat{\gamma}_1 + \hat{\gamma}_3)z_2^o + \hat{\gamma}_2) - \Phi(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1 + \hat{\gamma}_1 z_2^o)].$$

We can then vary z_2^o to see how the effect of changing d_1 from zero to one varies with z_2^o. Again, we can use Problem 12.17 to obtain an asymptotic standard error.

15.4. This is the kind of (nonsense) statement that arises out of failure to distinguish between the underlying latent variable model and the model for $\mathrm{P}(y = 1|\mathbf{x})$. To compare the LPM and probit on equal footing, we must recognize that the LPM assumes $\mathrm{P}(y = 1|\mathbf{x}) = \mathbf{x}\boldsymbol{\gamma}$ while the probit model assumes that $\mathrm{P}(y = 1|\mathbf{x}) = \Phi(\mathbf{x}\boldsymbol{\beta})$. So the substantive difference is purely in the functional forms for the response probabilities. And the probit functional form has some attractive properties compared with the linear model: $\Phi(\mathbf{x}\boldsymbol{\beta})$ is always between zero and one, and the marginal effect of any x_j is diminishing after some point. The LPM and probit models are both approximations to the true response probability, and the LPM has some deficencies for describing the partial effects over a broad range of the covariates.

If one insists on focusing on normality of the latent error in the probit case then one must compare that assumption with with the the corresponding assumption for the LPM. If we specify a latent variable as $y^* = \mathbf{x}\boldsymbol{\gamma} + e$ then the LPM is obtained when e has a uniform distribution over $[-\alpha, \alpha]$ for some constant $0 < \alpha < \infty$. For most purposes, this is much less

plausible than the normality underlying probit.

15.5. a. If $\mathrm{P}(y = 1|\mathbf{z}, q) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 q)$ then

$$\frac{\partial \mathrm{P}(y = 1|\mathbf{z}, q)}{\partial z_2} = \gamma_1 q \cdot \phi(\mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 z_2 q),$$

assuming that z_2 is not functionally related to $\mathbf{z}_1$.

b. Write $y^* = \mathbf{z}_1\boldsymbol{\delta}_1 + r$, where $r = \gamma_1 z_2 q + e$, and e is independent of $(\mathbf{z}, q)$ with a standard normal distribution. Because q is assumed independent of $\mathbf{z}$, $q|\mathbf{z} \sim Normal(0, \gamma_1^2 z_2^2 + 1)$; this follows because $\mathrm{E}(r|\mathbf{z}) = \gamma_1 z_2 E(q|\mathbf{z}) + E(e|\mathbf{z}) = 0$. Also,

$$Var(r|\mathbf{z}) = \gamma_1^2 z_2^2 Var(q|\mathbf{z}) + Var(e|\mathbf{z}) + 2\gamma_1 z_2 Cov(q, e|\mathbf{z}) = \gamma_1^2 z_2^2 + 1$$

because $Cov(q, e|\mathbf{z}) = 0$ by independence between e and $(\mathbf{z}, q)$. Thus, $r/\sqrt{\gamma_1^2 z_2^2 + 1}$ has a standard normal distribution independent of $\mathbf{z}$. It follows that

$$\mathrm{P}(y = 1|\mathbf{z}) = \Phi\left(\mathbf{z}_1\boldsymbol{\delta}_1/\sqrt{\gamma_1^2 z_2^2 + 1}\right). \tag{15.97}$$

c. Because $\mathrm{P}(y = 1|\mathbf{z})$ depends only on γ_1^2, this is what we can estimate along with $\boldsymbol{\delta}_1$. (For example, $\gamma_1 = -2$ and $\gamma_1 = 2$ give exactly the same model for $\mathrm{P}(y = 1|\mathbf{z})$.) This is why we define $\rho_1 = \gamma_1^2$. Testing $H_0 : \rho_1 = 0$ is most easily done using the score or LM test because, under H_0, we have a standard probit model.

Let $\hat{\boldsymbol{\delta}}_1$ denote the probit estimates under the null that $\rho_1 = 0$. Define $\phi_i = \phi(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1)$, $\hat{\Phi}_i = \Phi(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1)$, $\hat{u}_i = y_i - \hat{\Phi}_i$, and $\tilde{u}_i \equiv \hat{u}_i/\sqrt{\hat{\Phi}_i(1 - \hat{\Phi}_i)}$ (the standardized residuals). The gradient of the mean function in (15.97) with respect to $\boldsymbol{\delta}_1$, evaluated under the null estimates, is simply $\hat{\phi}_i \mathbf{z}_{i1}$. The only other quantity needed is the gradient with respect to ρ_1 evaluated at the null estimates. But the partial derivative of (15.97) with respect to ρ_1 is, for each i,

$$- (\mathbf{z}_{i1}\boldsymbol{\delta}_1)(z_{i2}^2/2)(\rho_1 z_{i2}^2 + 1)^{-3/2}\phi\left(\mathbf{z}_{i1}\boldsymbol{\delta}_1/\sqrt{\gamma_1^2 z_2^2 + 1}\right).$$

When we evaluate this at $\rho_1 = 0$ and $\hat{\boldsymbol{\delta}}_1$ we get $-(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1)(z_{i2}^2/2)\hat{\phi}_i$. Then, the score statistic can be obtained as NR_u^2 from the regression

$$\tilde{u}_i \text{ on } \frac{\hat{\phi}_i \mathbf{z}_{i1}}{\sqrt{\hat{\Phi}_i(1-\hat{\Phi}_i)}},\ \frac{(\mathbf{z}_{i1}\hat{\boldsymbol{\delta}}_1)z_{i2}^2\hat{\phi}_i}{\sqrt{\hat{\Phi}_i(1-\hat{\Phi}_i)}};$$

under H_0, $NR_u^2 \overset{a}{\sim} \chi_1^2$.

d. The model can be estimated by MLE using the formulation with ρ_1 in place of γ_1^2. It is not a standard probit estimation but a kind of "heteroskedastic probit."

15.6. a. What we would like to know is that, if we exogenously change the number of cigarettes that someone smokes per day, what effect would this have on the probability of missing work over a three-month period? In other words, we want to infer causality, not just find a correlation between missing work and cigarette smoking.

b. Since people choose whether and how much to smoke, we certainly cannot treat the data as coming from the experiment we have in mind in part a. (That is, we cannot randomly assign people a daily cigarette consumption.) It is possible that smokers are less healthy to begin with, or have other attributes that cause them to miss work more often. Or, it could go the other way: cigarette consumption may be related to personality traits that make people harder workers. In any case, *cigs* might be correlated with the unobservables in the equation.

c. If we start with the model

$$\mathrm{P}(y = 1|\mathbf{z}, cigs, q_1) = \Phi(\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \gamma_1 cigs + q_1), \tag{15.98}$$

but ignore q_1 when it is correlated with *cigs*, we will not consistently estimate anything of interest, whether the model is linear or nonlinear. Thus, we would not be estimating a causal

effect. If q_1 is independent of *cigs*, the probit ignoring q_1 does estimate the average partial effect of another cigarette.

d. No. There are many people in the working population who do not smoke. Thus, the distribution (conditional or unconditional) of *cigs* piles up at zero. Also, since *cigs* takes on integer values, it cannot be normally distributed. But it is really the pile up at zero that is the most serious issue.

e. Use the Rivers-Vuong test. Obtain the residuals, $\hat{r}_2$, from the regression *cigs* on **z**. Then, estimate the probit of y on $\mathbf{z}_1, cigs, \hat{r}_2$ and use a standard t test on $\hat{r}_2$. This does not rely on normality of r_2 (or *cigs*). It does, of course, rely on the probit model being correct for y under H_0.

f. Assuming people will not immediately move out of their state of residence when the state implements no smoking laws in the workplace, and that state of residence is roughly independent of general health in the population, a dummy indicator for whether the person works in a state with a new law can be treated as exogenous and excluded from (15.98). (These situations are often called "natural experiments.") Further, *cigs* is likely to be correlated with the state law indicator because since people will not be able to smoke as much as they otherwise would. Thus, it seems to be a reasonable instrument for *cigs*.

15.7. a. The LPM estimates, with the usual and heteroskedasticity-robust standard errors, are given below. Interesting, the robust standard errors on the non-demographic variables are often notably smaller than the usual standard errors. The statistical significance of the OLS coefficients is the same using either set of standard errors.

When *pcnv* goes from .25 to .75, the estimated probability of arrest falls by about .077, or 7.7 percentage points.

```
. use grogger

. gen arr86 = 0

. replace arr86 = 1 if narr86 > 0
(755 real changes made)

. reg arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60

      Source |       SS       df       MS              Number of obs =    2725
-------------+------------------------------           F(  8,  2716) =   30.48
       Model |  44.9720916     8  5.62151145           Prob > F      =  0.0000
    Residual |  500.844422  2716  .184405163           R-squared     =  0.0824
-------------+------------------------------           Adj R-squared =  0.0797
       Total |  545.816514  2724   .20037317           Root MSE      =  .42942

------------------------------------------------------------------------------
       arr86 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        pcnv |  -.1543802   .0209336    -7.37   0.000    -.1954275   -.1133329
      avgsen |   .0035024   .0063417     0.55   0.581    -.0089326    .0159374
     tottime |  -.0020613   .0048884    -0.42   0.673    -.0116466     .007524
     ptime86 |  -.0215953   .0044679    -4.83   0.000    -.0303561   -.0128344
       inc86 |  -.0012248    .000127    -9.65   0.000    -.0014738   -.0009759
       black |   .1617183   .0235044     6.88   0.000     .1156299    .2078066
      hispan |   .0892586   .0205592     4.34   0.000     .0489454    .1295718
      born60 |   .0028698   .0171986     0.17   0.867    -.0308539    .0365936
       _cons |   .3609831   .0160927    22.43   0.000      .329428    .3925382
------------------------------------------------------------------------------

. reg arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60, robust

Linear regression                                       Number of obs =    2725
                                                        F(  8,  2716) =   37.59
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.0824
                                                        Root MSE      =  .42942

------------------------------------------------------------------------------
             |               Robust
       arr86 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        pcnv |  -.1543802    .018964    -8.14   0.000    -.1915656   -.1171948
      avgsen |   .0035024   .0058876     0.59   0.552    -.0080423    .0150471
     tottime |  -.0020613   .0042256    -0.49   0.626     -.010347    .0062244
     ptime86 |  -.0215953   .0027532    -7.84   0.000    -.0269938   -.0161967
       inc86 |  -.0012248   .0001141   -10.73   0.000    -.0014487    -.001001
       black |   .1617183   .0255279     6.33   0.000     .1116622    .2117743
      hispan |   .0892586   .0210689     4.24   0.000     .0479459    .1305714
      born60 |   .0028698   .0171596     0.17   0.867    -.0307774     .036517
       _cons |   .3609831   .0167081    21.61   0.000     .3282214    .3937449
------------------------------------------------------------------------------

. di .5*_b[pcnv]
-.0771901
```

b. The robust statistic and its *p*-value are gotten by using the “test” command after

appending “robust” to the regression command. The *p*-values are virtually identical.

```
. qui reg arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60

. test avgsen tottime

 ( 1)  avgsen = 0
 ( 2)  tottime = 0

       F(  2,  2716) =    0.18
            Prob > F =    0.8360

. qui reg arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60, robust

. test avgsen tottime

 ( 1)  avgsen = 0
 ( 2)  tottime = 0

       F(  2,  2716) =    0.18
            Prob > F =    0.8320
```

c. The probit model estimates follow.

```
. probit arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60

Probit regression                                 Number of obs   =       2725
                                                  LR chi2(8)      =     249.09
                                                  Prob > chi2     =     0.0000
Log likelihood = -1483.6406                       Pseudo R2       =     0.0774

------------------------------------------------------------------------------
       arr86 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        pcnv |  -.5529248   .0720778    -7.67   0.000    -.6941947   -.4116549
      avgsen |   .0127395   .0212318     0.60   0.548     -.028874    .0543531
     tottime |  -.0076486   .0168844    -0.45   0.651    -.0407414    .0254442
     ptime86 |  -.0812017    .017963    -4.52   0.000    -.1164085   -.0459949
       inc86 |  -.0046346   .0004777    -9.70   0.000    -.0055709   -.0036983
       black |   .4666076   .0719687     6.48   0.000     .3255516    .6076635
      hispan |   .2911005   .0654027     4.45   0.000     .1629135    .4192875
      born60 |   .0112074   .0556843     0.20   0.840    -.0979318    .1203466
       _cons |  -.3138331   .0512999    -6.12   0.000    -.4143791    -.213287
------------------------------------------------------------------------------
```

Now, we must compute the difference in the normal cdf at the two different values of *pcnv*, *black* = 1, *hispan* = 0, *born*60 = 1, and at the average values of remaining variables.

```
. sum avgsen tottime ptime86 inc86

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      avgsen |      2725    .6322936    3.508031          0       59.2
     tottime |      2725    .8387523    4.607019          0       63.4
```

```
    ptime86 |      2725     .387156    1.950051          0         12
      inc86 |      2725    54.96705    66.62721          0        541

. di normal(_b[_cons] + _b[pcnv]*.75 + _b[avgsen]*.6322936
     + _b[tottime]*.8387523 + _b[ptime86]* .387156 + _b[inc86]*54.96705
     + _b[black] + _b[born60])
    - normal(_b[_cons] + _b[pcnv]*.25 + _b[avgsen]*.6322936
     + _b[tottime]*.8387523 + _b[ptime86]* .387156 + _b[inc86]* 54.96705
     + _b[black] + _b[born60])

-.10166064
```

This last command shows that the probability falls by about .102, which is somewhat larger than the effect obtained from the LPM.

d. To obtain the percent correctly predicted for each outcome, we first generate the predicted values of *arr*86 as described on page 465:

```
. predict PHIhat
(option pr assumed; Pr(arr86))

. gen arr86t = PHIhat >= .5

. tab arr86t arr86

           |         arr86
    arr86t |         0          1 |     Total
-----------+----------------------+----------
         0 |     1,903        677 |     2,580
         1 |        67         78 |       145
-----------+----------------------+----------
     Total |     1,970        755 |     2,725

. di 1903/1970
.96598985

. di 78/755
.10331126

. di (1903 + 78)/2725
.72697248
```

For men who were not arrested, the probit predicts correctly about 96.6% of the time. Unfortunately, for the men who were arrested, the probit is correct only about 10.3% of the time. The overall percent correctly predicted is pretty high – 72.7% – but we cannot very well predict the outcome we would most like to predict.

e. Adding the quadratic terms gives

```
. probit arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60
         pcnvsq pt86sq inc86sq

Probit regression                                 Number of obs   =       2725
                                                  LR chi2(11)     =     336.77
                                                  Prob > chi2     =     0.0000
Log likelihood = -1439.8005                       Pseudo R2       =     0.1047

------------------------------------------------------------------------------
       arr86 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        pcnv |   .2167615   .2604937     0.83   0.405    -.2937968    .7273198
      avgsen |   .0139969   .0244972     0.57   0.568    -.0340166    .0620105
     tottime |  -.0178158   .0199703    -0.89   0.372     -.056957    .0213253
     ptime86 |   .7449712   .1438485     5.18   0.000     .4630333    1.026909
       inc86 |  -.0058786   .0009851    -5.97   0.000    -.0078094   -.0039478
       black |   .4368131   .0733798     5.95   0.000     .2929913     .580635
      hispan |   .2663945    .067082     3.97   0.000     .1349163    .3978727
      born60 |  -.0145223   .0566913    -0.26   0.798    -.1256351    .0965905
      pcnvsq |  -.8570512   .2714575    -3.16   0.002    -1.389098   -.3250042
      pt86sq |  -.1035031   .0224234    -4.62   0.000    -.1474522    -.059554
     inc86sq |   8.75e-06   4.28e-06     2.04   0.041     3.63e-07    .0000171
       _cons |   -.337362   .0562665    -6.00   0.000    -.4476423   -.2270817
------------------------------------------------------------------------------
Note: 51 failures and 0 successes completely determined.

. test pcnvsq pt86sq inc86sq

 ( 1)  pcnvsq = 0
 ( 2)  pt86sq = 0
 ( 3)  inc86sq = 0

           chi2(  3) =   38.54
         Prob > chi2 =    0.0000
```

The quadratics are individually and jointly significant. The quadratic in *pcnv* means that, at low levels of *pcnv*, there is actually a positive relationship between probability of arrest and *pcnv*, which does not make much sense. The turning point is easily found as $.217/(2(.857)) \approx .127$, and there are many cases – 1,265 – where *pcnv* is less than .127.

```
. sum pcnv

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        pcnv |      2725    .3577872     .395192          0          1

. count if pcnv < .127
 1265
```

15.8. a. The following Stata session answers this part. The difference in estimated probabilities of smoking at 16 and 12 years of education is about $-.080$. In other words, for

non-white women at the average family income, women with 16 years of education are, on average, about eight percentage points less likely to smoke.

```
. use bwght

. gen smokes = cigs > 0

. tab smokes

     smokes |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |      1,176       84.73       84.73
          1 |        212       15.27      100.00
------------+-----------------------------------
      Total |      1,388      100.00

. probit smokes motheduc white lfaminc

Probit regression                                 Number of obs   =       1387
                                                  LR chi2(3)      =      92.67
                                                  Prob > chi2     =     0.0000
Log likelihood = -546.76991                       Pseudo R2       =     0.0781

------------------------------------------------------------------------------
      smokes |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    motheduc |  -.1450599   .0207899    -6.98   0.000    -.1858074   -.1043124
       white |   .1896765   .1098805     1.73   0.084    -.0256853    .4050383
     lfaminc |  -.1669109   .0498894    -3.35   0.001    -.2646923   -.0691296
       _cons |   1.126276   .2504611     4.50   0.000     .6353817    1.617171
------------------------------------------------------------------------------

. sum faminc

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      faminc |      1388    29.02666    18.73928         .5         65

. di normal(_b[_cons] + _b[motheduc]*16 + _b[lfaminc]*log(29.02666))
       - normal(_b[_cons] + _b[motheduc]*12 + _b[lfaminc]*log(29.02666))

-.08020112
```

b. The variance *faminc* is probably not exogenous because, at a minium, income is likely correlated with quality of health care. It might also be correlated with unobserved cultural factors that are correlated with smoking.

c. The reduced form equation for *lfaminc* is estimated below. As expected, *fatheduc* has a positive partial effect on *lfaminc*, and the relationship is statistically significant. We need the

residuals from this regression for part d. We lose 196 observations due to missing data on *fatheduc*, and one observation has already been lost due to a missing value for *motheduc*.

```
. reg lfaminc motheduc white fatheduc

      Source |       SS       df       MS              Number of obs =    1191
-------------+------------------------------           F(  3,  1187) =  119.23
       Model |  140.936735     3  46.9789115           Prob > F      =  0.0000
    Residual |  467.690904  1187  .394010871           R-squared     =  0.2316
-------------+------------------------------           Adj R-squared =  0.2296
       Total |  608.627639  1190  .511451797           Root MSE      =   .6277

------------------------------------------------------------------------------
     lfaminc |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    motheduc |   .0709044   .0098338     7.21   0.000     .0516109     .090198
       white |   .3452115    .050418     6.85   0.000     .2462931    .4441298
    fatheduc |   .0616625    .008708     7.08   0.000     .0445777    .0787473
       _cons |   1.241413   .1103648    11.25   0.000     1.024881    1.457945
------------------------------------------------------------------------------

. predict v2hat, resid
(197 missing values generated)
```

d. To test the null of exogeneity, we estimate the probit that includes $\hat{v}_2$:

```
. probit smokes motheduc white lfaminc v2hat

Iteration 0:   log likelihood = -471.77574
Iteration 1:   log likelihood = -432.90303
Iteration 2:   log likelihood =  -432.0639
Iteration 3:   log likelihood = -432.06242
Iteration 4:   log likelihood = -432.06242

Probit regression                                 Number of obs   =       1191
                                                  LR chi2(4)      =      79.43
                                                  Prob > chi2     =     0.0000
Log likelihood = -432.06242                       Pseudo R2       =     0.0842

------------------------------------------------------------------------------
      smokes |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    motheduc |  -.0826247   .0465204    -1.78   0.076     -.173803    .0085536
       white |   .4611075   .1965245     2.35   0.019     .0759265    .8462886
     lfaminc |  -.7622559   .3652949    -2.09   0.037    -1.478221    -.046291
       v2hat |   .6107298   .3708071     1.65   0.100    -.1160387    1.337498
       _cons |    1.98796   .5996374     3.32   0.001     .8126927    3.163228
------------------------------------------------------------------------------
```

There is not strong evidence of endogeneity, but the sign of the coefficient on *v2hat* is what we expect: unobservables that lead to higher income are positively correlated with unobserved

factors affecting birth weight. There is a further problem in that using this test presumes *fatheduc* can be omitted from the birth weight equation. Remember, the test can be interpreted as a test for endogeneity of *lfaminc* only when we maintain that *fatheduc* is exogenous.

Because of the potential endogeneity of this is perhaps not a very good example, but it shows you how to mechanically carry out the tests.

Incidentally, the probit coefficients on *lfaminc* are very different depending on whether we treat it as exogenous or not. This is true even if we use the same samples, as the Stata output below shows. The APE is probably quite different, too. It is hard to know what to do in such cases (which are all too common).

```
. probit smokes motheduc white lfaminc if v2hat != .

Probit regression                                 Number of obs   =       1191
                                                  LR chi2(3)      =      76.72
                                                  Prob > chi2     =     0.0000
Log likelihood = -433.41656                       Pseudo R2       =     0.0813

------------------------------------------------------------------------------
      smokes |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    motheduc |  -.1497589   .0225634    -6.64   0.000    -.1939823   -.1055355
       white |   .2323285    .137875     1.69   0.092    -.0379015    .5025584
     lfaminc |  -.1719479   .0687396    -2.50   0.012     -.306675   -.0372207
       _cons |   1.133026   .2990124     3.79   0.000     .5469727    1.71908
------------------------------------------------------------------------------
```

15.9. a. Let $\mathrm{P}(y = 1|\mathbf{x}) = \mathbf{x}\boldsymbol{\beta}$, where $x_1 = 1$. Then for each i,

$$\ell_i(\boldsymbol{\beta}) = y_i \log(\mathbf{x}_i\boldsymbol{\beta}) + (1 - y_i)\log(1 - \mathbf{x}_i\boldsymbol{\beta}),$$

which is only well-defined for $0 < \mathbf{x}_i\boldsymbol{\beta} < 1$.

b. For any possible estimate $\hat{\boldsymbol{\beta}}$, the log-likelihood function is well-defined only if $0 < \mathbf{x}_i\hat{\boldsymbol{\beta}} < 1$ for all $i = 1, \ldots, N$. Therefore, during the iterations to obtain the MLE, this condition must be checked. It may be impossible to find an estimate that satisfies these inequalities for every observation, especially if N is large.

c. This follows from the KLIC, and the discussion of Vuong's model selection statistic in Section 13.11.2: the true density of y given $\mathbf{x}$ – evaluated at the true values, of course – maximizes the KLIC. Because the MLEs are consistent for the unknown parameters, asymptotically the true density will produce the highest average log-likelihood function. So, just as we can use an R-squared to choose among different functional forms for $\mathrm{E}(y|\mathbf{x})$, we can use values of the log-likelihood to choose among different models for $\mathrm{P}(y = 1|\mathbf{x})$ when y is binary.

15.10. a. There are several possibilities. One is to define $\hat{p}_i = \hat{\mathrm{P}}(y = 1|\mathbf{x}_i)$ – the estimated response probabilities – and obtain the square of the correlation between y_i and $\hat{p}_i$. For the LPM, this is just the usual R-squared. For the general index model, $G(\mathbf{x}_i\hat{\boldsymbol{\beta}})$ is the estimate of $\mathrm{E}(y|\mathbf{x}_i)$, and so it makes sense to compute an analogous goodness-of-fit measure. This measure is always between zero and one.

An alternative is to use the sum of squared residuals form. While this produces the same R-squared measure for the linear model, it does not for nonlinear models.

b. The Stata output below gives the square of the correlation between y_i and the fitted probabilities for the LPM and probit. The LPM R-squared is about .106 and that for probit is higher, about .115. So probit is preferred based on this goodness-of-fit measure, although the improvement is not overwhelming. (It is about an 8.5% increase in the R-squared.)

```
. reg arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60
      pcnvsq pt86sq inc86sq

      Source |       SS       df       MS              Number of obs =    2725
-------------+------------------------------           F( 11,  2713) =   29.27
       Model |  57.8976285    11  5.26342077           Prob > F      =  0.0000
    Residual |  487.918885  2713  .179844779           R-squared     =  0.1061
-------------+------------------------------           Adj R-squared =  0.1025
       Total |  545.816514  2724   .20037317           Root MSE      =  .42408

------------------------------------------------------------------------------
       arr86 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
```

```
-------------+----------------------------------------------------------------
        pcnv |    .075977   .0803402     0.95   0.344    -.0815573    .2335112
      avgsen |   .0012998   .0062692     0.21   0.836    -.0109932    .0135927
     tottime |  -.0022213   .0048287    -0.46   0.646    -.0116897     .007247
     ptime86 |   .1321786   .0230021     5.75   0.000     .0870752     .177282
       inc86 |  -.0018505   .0002737    -6.76   0.000    -.0023872   -.0013139
       black |   .1447942   .0233225     6.21   0.000     .0990627    .1905258
      hispan |   .0803938   .0204959     3.92   0.000     .0402047    .1205829
      born60 |  -.0062993   .0170252    -0.37   0.711     -.039683    .0270843
      pcnvsq |  -.2456865   .0812584    -3.02   0.003    -.4050211   -.0863519
      pt86sq |  -.0139981   .0020109    -6.96   0.000    -.0179411   -.0100551
     inc86sq |   3.31e-06   1.09e-06     3.03   0.002     1.17e-06    5.45e-06
       _cons |    .363352   .0175536    20.70   0.000     .3289323    .3977718
------------------------------------------------------------------------------

. probit arr86 pcnv avgsen tottime ptime86 inc86 black hispan born60
         pcnvsq pt86sq inc86sq

Probit regression                                 Number of obs   =       2725
                                                  LR chi2(11)     =     336.77
                                                  Prob > chi2     =     0.0000
Log likelihood = -1439.8005                       Pseudo R2       =     0.1047

------------------------------------------------------------------------------
       arr86 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        pcnv |   .2167615   .2604937     0.83   0.405    -.2937968    .7273198
      avgsen |   .0139969   .0244972     0.57   0.568    -.0340166    .0620105
     tottime |  -.0178158   .0199703    -0.89   0.372     -.056957    .0213253
     ptime86 |   .7449712   .1438486     5.18   0.000     .4630332    1.026909
       inc86 |  -.0058786   .0009851    -5.97   0.000    -.0078094   -.0039478
       black |   .4368131   .0733798     5.95   0.000     .2929913     .580635
      hispan |   .2663945    .067082     3.97   0.000     .1349163    .3978727
      born60 |  -.0145223   .0566913    -0.26   0.798    -.1256351    .0965905
      pcnvsq |  -.8570512   .2714575    -3.16   0.002    -1.389098   -.3250042
      pt86sq |  -.1035031   .0224234    -4.62   0.000    -.1474522    -.059554
     inc86sq |   8.75e-06   4.28e-06     2.04   0.041     3.63e-07    .0000171
       _cons |   -.337362   .0562665    -6.00   0.000    -.4476423   -.2270817
------------------------------------------------------------------------------
Note: 51 failures and 0 successes completely determined.

. predict PHIhat
(option pr assumed; Pr(arr86))

. corr PHIhat arr86
(obs=2725)

             |   PHIhat    arr86
-------------+------------------
      PHIhat |   1.0000
       arr86 |   0.3396   1.0000

. di .3396^2
.11532816
```

15.11. We really need to make two assumptions. The first is a conditional independence

assumption: given $\mathbf{x}_i = (\mathbf{x}_{i1}, \ldots, \mathbf{x}_{iT}), (y_{i1}, \ldots, y_{iT})$ are independent. This allows us to write

$$f(y_{i1}, \ldots, y_{iT}|\mathbf{x}_i) = f_1(y_1|\mathbf{x}_i) \cdots f_T(y_T|\mathbf{x}_i),$$

that is, the joint density (conditional on $\mathbf{x}_i$) is the product of the marginal densities (each conditional on $\mathbf{x}_i$). The second assumption is a strict exogeneity assumption: $D(y_{iT}|\mathbf{x}_i) = D(y_{iT}|\mathbf{x}_{it}), t = 1, \ldots, T.$ When we add the standard assumption for pooled probit – that $D(y_{iT}|\mathbf{x}_{it})$ follows a probit model – then

$$f(y_1, \ldots, y_T|\mathbf{x}_i) = \prod_{t=1}^{T} [G(\mathbf{x}_{it}\beta)]^{y_t}[1 - G(\mathbf{x}_{it}\beta)]^{1-y_t},$$

and so pooled probit is conditional MLE.

15.12. We can extend the $T = 2$ case used to obtain equation (15.81):

$$\begin{aligned} \mathrm{P}(y_{i1} = 1|\mathbf{x}_i, c_i, n_i = 1) &= \mathrm{P}(y_{i1} = 1, n_i = 1|\mathbf{x}_i, c_i)/P(n_i = 1|\mathbf{x}_i, c_i) \\ &= \mathrm{P}(y_{i1} = 1, y_{i2} = 0, y_{i3} = 0|\mathbf{x}_i, c_i)/\{\mathrm{P}(y_{i1} = 1, y_{i2} = 0, y_{i3} = 0|\mathbf{x}_i, c_i) \\ &\quad + \mathrm{P}(y_{i1} = 0, y_{i2} = 1, y_{i3} = 0|\mathbf{x}_i, c_i) + \mathrm{P}(y_{i1} = 0, y_{i2} = 0, y_{i3} = 1|\mathbf{x}_i, c_i)\} \end{aligned}$$

Now, we just use the conditional independence assumption (across t) and the logistic functional form:

$$\begin{aligned} \mathrm{P}(y_{i1} = 1, y_{i2} = 0, y_{i3} = 0|\mathbf{x}_i, c_i) &= \Lambda(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i)[1 - \Lambda(\mathbf{x}_{i2}\boldsymbol{\beta} + c_i)] \cdot [1 - \Lambda(\mathbf{x}_{i3}\boldsymbol{\beta} + c_i)] \\ \mathrm{P}(y_{i1} = 0, y_{i2} = 1, y_{i3} = 0|\mathbf{x}_i, c_i) &= [1 - \Lambda(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i)]\Lambda(\mathbf{x}_{i2}\boldsymbol{\beta} + c_i) \cdot [1 - \Lambda(\mathbf{x}_{i3}\boldsymbol{\beta} + c_i)] \end{aligned}$$

and

$$\mathrm{P}(y_{i1} = 0, y_{i2} = 0, y_{i3} = 1|\mathbf{x}_i, c_i) = [1 - \Lambda(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i)] \cdot [1 - \Lambda(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i)\Lambda(\mathbf{x}_{i3}\boldsymbol{\beta} + c_i)].$$

Now, the term

$$1/\{[1 + \exp(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i)] \cdot [1 + \exp(\mathbf{x}_{i2}\boldsymbol{\beta} + c_i)] \cdot [1 + \exp(\mathbf{x}_{i3}\boldsymbol{\beta} + c_i)]\}$$

appears multiplicatively in both the numerator and denominator, and so it disappears.

Therefore,

$$\begin{aligned} P(y_{i1} = 1|\mathbf{x}_i, c_i, n_i = 1) &= \exp(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i)/[\exp(\mathbf{x}_{i1}\boldsymbol{\beta} + c_i) + \exp(\mathbf{x}_{i2}\boldsymbol{\beta} + c_i) + \exp(\mathbf{x}_{i3}\boldsymbol{\beta} + c_i)] \\ &= \exp(\mathbf{x}_{i1}\boldsymbol{\beta})/[\exp(\mathbf{x}_{i1}\boldsymbol{\beta}) + \exp(\mathbf{x}_{i2}\boldsymbol{\beta}) + \exp(\mathbf{x}_{i3}\boldsymbol{\beta})]. \end{aligned}$$

Also,

$$P(y_{i2} = 1|\mathbf{x}_i, c_i, n_i = 1) = \exp(\mathbf{x}_{i2}\boldsymbol{\beta})/[\exp(\mathbf{x}_{i1}\boldsymbol{\beta}) + \exp(\mathbf{x}_{i2}\boldsymbol{\beta}) + \exp(\mathbf{x}_{i3}\boldsymbol{\beta})]$$

and

$$P(y_{i3} = 1|\mathbf{x}_i, c_i, n_i = 1) = \exp(\mathbf{x}_{i3}\boldsymbol{\beta})/[\exp(\mathbf{x}_{i1}\boldsymbol{\beta}) + \exp(\mathbf{x}_{i2}\boldsymbol{\beta}) + \exp(\mathbf{x}_{i3}\boldsymbol{\beta})].$$

Incidentally, a consistent estimator of $\boldsymbol{\beta}$ is obtained using only the $n_i = 1$ observations and applying conditional logit, as described in Chapter 16. This approach would be inefficient because it does not use the $n_i = 2$ observations.

A similar argument can be used for the three possible configurations with $n_i = 2$, which leads to the log-likelihood conditional on $(\mathbf{x}_i, n_i)$, where c_i has dropped out. For example,

$$P(y_{i1} = 1, y_{i2} = 1|\mathbf{x}_i, c_i, n_i = 2) = \frac{\exp[(\mathbf{x}_{i1} + \mathbf{x}_{i2})\boldsymbol{\beta}]}{\exp[(\mathbf{x}_{i1} + \mathbf{x}_{i2})\boldsymbol{\beta}] + [\exp(\mathbf{x}_{i1} + \mathbf{x}_{i3})\boldsymbol{\beta}] + \exp[(\mathbf{x}_{i2} + \mathbf{x}_{i3})\boldsymbol{\beta}]}$$

15.13. a. If there are no covariates, there is no point in using any method other than a straight comparison of means – in particular, the difference-in-differences approach described in Section 6.5.2. The estimated probabilities for the treatment and control groups, both before and after the policy change, will be identical to the sample proportions regardless of the model we use.

b. Let $d2$ be a binary indicator for the second time period, and let dB be an indicator for the treatment group. Then a probit model to evaluate the treatment effect is

$$P(y = 1|\mathbf{x}) = \Phi(\delta_0 + \delta_1 d2 + \delta_2 dB + \delta_3 d2 \cdot dB + \mathbf{x}\boldsymbol{\gamma}),$$

where $\mathbf{x}$ is a vector of covariates. We would estimate all parameters from a probit of y on $1, d2, dB, d2 \cdot dB$, and $\mathbf{x}$ using all observations. Once we have the estimates, we need to compute the "difference-in-differences" estimate, which requires either plugging in a value for $\mathbf{x}$, say $\bar{\mathbf{x}}$, or averaging the differences across $\mathbf{x}_i$. In the former case, we have

$$\hat{\tau}_{PAE} \equiv [\Phi(\hat{\delta}_0 + \hat{\delta}_1 + \hat{\delta}_2 + \hat{\delta}_3 + \bar{\mathbf{x}}\hat{\boldsymbol{\gamma}}) - \Phi(\hat{\delta}_0 + \hat{\delta}_2 + \bar{\mathbf{x}}\hat{\boldsymbol{\gamma}})] - [\Phi(\hat{\delta}_0 + \hat{\delta}_1 + \bar{\mathbf{x}}\hat{\boldsymbol{\gamma}}) - \Phi(\hat{\delta}_0 + \bar{\mathbf{x}}\hat{\boldsymbol{\gamma}})],$$

and in the latter we have

$$\hat{\tau}_{APE} \equiv N^{-1}\sum_{i=1}^{N}\{[\Phi(\hat{\delta}_0 + \hat{\delta}_1 + \hat{\delta}_2 + \hat{\delta}_3 + \mathbf{x}_i\hat{\boldsymbol{\gamma}}) - \Phi(\hat{\delta}_0 + \hat{\delta}_2 + \mathbf{x}_i\hat{\boldsymbol{\gamma}})] - [\Phi(\hat{\delta}_0 + \hat{\delta}_1 + \mathbf{x}_i\hat{\boldsymbol{\gamma}}) - \Phi(\hat{\delta}_0 + \mathbf{x}_i\hat{\boldsymbol{\gamma}})]\}.$$

Probably $\hat{\tau}_{APE}$ is preferred as it averages each of the estimated "treatment effects" – see Chapter 21 – across all units.

c. We would have to use the delta method to obtain a valid standard error for either $\hat{\tau}_{PAE}$ or $\hat{\tau}_{APE}$, with the latter using the extension in Problem 12.17.

15.14. a. First plug in for y_2 from (15.40):

$$\begin{aligned} y_1 &= 1[\mathbf{z}_1\boldsymbol{\delta}_1 + y_2\mathbf{z}_2\boldsymbol{\alpha}_1 + u_1 > 0] = 1[\mathbf{z}_1\boldsymbol{\delta}_1 + (\mathbf{z}\boldsymbol{\delta}_2 + v_2)\mathbf{z}_2\boldsymbol{\alpha}_1 + u_1 > 0] \\ &= 1[\mathbf{z}_1\boldsymbol{\delta}_1 + (\mathbf{z}\boldsymbol{\delta}_2)\mathbf{z}_2\boldsymbol{\alpha}_1 + u_1 + v_2\mathbf{z}_2\boldsymbol{\alpha}_1 > 0] \end{aligned}$$

Given the assumptions, $u_1 + v_2\mathbf{z}_2\boldsymbol{\alpha}_1$ has a mean zero normal distribution conditional on $\mathbf{z}$. Its variance is

$$\text{Var}(u_1 + v_2\mathbf{z}_2\boldsymbol{\alpha}_1|\mathbf{z}) = 1 + 2\eta_1\mathbf{z}_2\boldsymbol{\alpha}_1 + \tau_2^2(\mathbf{z}_2\boldsymbol{\alpha}_1)^2$$

where $\eta_1 = \text{Cov}(v_2, u_1)$ and $\tau_2^2 = \text{Var}(v_2)$. So we can write

$$\mathrm{P}(y_1 = 1|\mathbf{z}) = 1 - \Phi\left[\frac{-[\mathbf{z}_1\boldsymbol{\delta}_1 + (\mathbf{z}\boldsymbol{\delta}_2)\mathbf{z}_2\boldsymbol{\alpha}_1]}{\sqrt{1 + 2\eta_1\mathbf{z}_2\boldsymbol{\alpha}_1 + \tau_2^2(\mathbf{z}_2\boldsymbol{\alpha}_1)^2}}\right]$$
$$= \Phi\left[\frac{\mathbf{z}_1\boldsymbol{\delta}_1 + (\mathbf{z}\boldsymbol{\delta}_2)\mathbf{z}_2\boldsymbol{\alpha}_1}{\sqrt{1 + 2\eta_1\mathbf{z}_2\boldsymbol{\alpha}_1 + \tau_2^2(\mathbf{z}_2\boldsymbol{\alpha}_1)^2}}\right]$$

which is a heteroskedastic-probit model (but not with exponential heteroskedasticity in the latent error).

b. This two-step procedure is inconsistent because the response probability $\mathrm{P}(y_1 = 1|\mathbf{z})$ does not have the usual probit form

$$\Phi[\mathbf{z}_1\boldsymbol{\delta}_1 + (\mathbf{z}\boldsymbol{\delta}_2)\mathbf{z}_2\boldsymbol{\alpha}_1].$$

Under the assumptions given, the first-stage estimation of $\boldsymbol{\delta}_2$ is not the problem: OLS is consistent. It is the misspecified functional form in the second stage that causes the problem.

c. A control function method works nicely here. Scaled coefficients are easily estimated and then $\boldsymbol{\delta}_1$ and $\boldsymbol{\alpha}_1$ can be recovered using the same approach in Section 15.7.2. In addition, average partial effects are easily estimated after control function estimation.

Under (15.40), independence, and bivariate normality, we can write u_1 as in equation (15.42) and then substitute:

$$y_1 = 1[\mathbf{z}_1\boldsymbol{\delta}_1 + y_2\mathbf{z}_2\boldsymbol{\alpha}_1 + \theta_1 v_2 + e_1 > 0]$$
$$e_1|\mathbf{z}, y_2, v_2 \sim \text{Normal}(0, 1 - \rho_1^2)$$

Following the same argument in Section 15.7.2 we have

$$\mathrm{P}(y_1 = 1|\mathbf{z}, y_2, v_2) = \Phi(\mathbf{z}_1\boldsymbol{\delta}_{\rho 1} + y_2\mathbf{z}_2\boldsymbol{\alpha}_{\rho 1} + \theta_{\rho 1} v_2)$$

where $\boldsymbol{\delta}_{\rho 1} = \boldsymbol{\delta}_1/(1 - \rho_1^2)^{1/2}$, $\boldsymbol{\alpha}_{\rho 1} = \boldsymbol{\alpha}_1/(1 - \rho_1^2)^{1/2}$, and $\theta_{\rho 1} = \theta_1/(1 - \rho_1^2)^{1/2}$. Therefore, the following two-step CF method – which extends Procedure 15.1 – consistently estimates the

scaled parameters: (i) Regress y_2 on $\mathbf{z}$ and obtain the OLS residuals, $\hat{v}_2$. (ii) Run a probit of y_1 on $\mathbf{z}_1$, $y_2\mathbf{z}_2$, and $\hat{v}_2$.

Letting $\boldsymbol{\beta}_1 = (\boldsymbol{\delta}_1', \boldsymbol{\alpha}_1')'$ and $\boldsymbol{\beta}_{\rho 1} = \boldsymbol{\beta}_1/(1 - \rho_1^2)$, we use exactly the same unscaling of the parameters as before. Namely,

$$\boldsymbol{\beta}_1 = \boldsymbol{\beta}_{\rho 1}/(1 + \theta_{\rho 1}^2 \tau_2^2)^{1/2}$$

where $\tau_2^2 = \text{Var}(v_2)$. The estimator in equation (15.45) can still be used.

The approach to estimating the APEs follows directly from the estimator of the average structural function in equation (15.47). Allowing for the interactions,

$$\widehat{\text{ASF}}(\mathbf{z}_1, y_2) = N^{-1} \sum_{i=1}^{n} \Phi(\mathbf{z}_1 \hat{\boldsymbol{\delta}}_{\rho 1} + y_2 \mathbf{z}_2 \hat{\boldsymbol{\alpha}}_{\rho 1} + \hat{\theta}_{\rho 1} \hat{v}_{i2}).$$

As usual, we can take derivatives or changes with respect to the elements of $(\mathbf{z}_1, y_2)$ to obtain estimated APEs.

15.15. a. This example falls into the situation described below equation (12.41). Namely, the scores from the two optimization problems are uncorrelated. This follows because the first problem – OLS regression of y_{i2} on $\mathbf{z}_i$ – depends only on the random draws $(\mathbf{z}_i, y_{i2})$. In the second stage, we are estimating a model for $f(y_1|y_2, \mathbf{z})$. Letting $\mathbf{s}_i(\boldsymbol{\gamma}_1; \boldsymbol{\delta}_2)$ denote the score for the second-step MLE – with respect to $\boldsymbol{\gamma}_1$ – $\mathbf{s}_i(\boldsymbol{\gamma}_1; \boldsymbol{\delta}_2)$ is uncorrelated with any function of $(\mathbf{z}_i, y_{i2})$ because $\text{E}[\mathbf{s}_i(\boldsymbol{\gamma}_1; \boldsymbol{\delta}_2)|\mathbf{z}_i, y_{i2}] = \mathbf{0}$. (I do not use a separate notation for the true values of the parameters.)

So that we can apply the results from Section 12.4.2 directly, we set the problem up as a minimization problem. Then, from the usual score formula for the probit model, we have

$$\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2) = -\frac{\mathbf{w}_{i1}(\boldsymbol{\delta}_2)'\phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)[y_{i1} - \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)]}{\Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)[1 - \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)]}$$

where $\mathbf{w}_{i1}(\boldsymbol{\delta}_2) = [\mathbf{x}_{i1}, v_{i2}(\boldsymbol{\delta}_2)]$ and $v_{i2}(\boldsymbol{\delta}_2) = y_{i2} - \mathbf{z}_i\boldsymbol{\delta}_2$. The expected Hessian (that is, the expected Jacobian of $\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2)$ with respect to $\boldsymbol{\gamma}_1$) has the usual form for binary response with a correctly specified response probability:

$$\mathbf{A}_1 = \mathrm{E}\left\{\frac{\mathbf{w}_{i1}(\boldsymbol{\delta}_2)'\mathbf{w}_{i1}(\boldsymbol{\delta}_2)[\phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)]^2}{\Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)[1 - \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)]}\right\}$$

Next, we need $\mathbf{F} = \mathrm{E}[\nabla_{\boldsymbol{\delta}_2}\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2)]$. Like the Hessian in the usual binary response model, the Jacobian $\nabla_{\boldsymbol{\delta}_2}\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2)$ is complicated. But its expectation is not. Using the fact that $\mathrm{E}[y_{i1} - \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)|\mathbf{z}_i, y_{i2}] = 0$ it is easy to show

$$\begin{aligned}\mathbf{F} &= \mathrm{E}\left\{\frac{\mathbf{w}_{i1}(\boldsymbol{\delta}_2)'\phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)}{\Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)[1 - \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)]} \cdot \nabla_{\boldsymbol{\delta}_2}\Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)\right\} \\ &= -\theta_{\rho 1}\mathrm{E}\left\{\frac{\mathbf{w}_{i1}(\boldsymbol{\delta}_2)'\mathbf{z}_i\phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)}{\Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)[1 - \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)]}\right\}\end{aligned}$$

Finally, we need a first-order representation for the OLS estimator, $\hat{\boldsymbol{\delta}}_2$:

$$\sqrt{N}(\hat{\boldsymbol{\delta}}_2 - \boldsymbol{\delta}_2) = \mathbf{A}_2^{-1}N^{-1/2}\sum_{i=1}^{N}\mathbf{z}_i'v_{i2} + o_p(1),$$

where $\mathbf{A}_2 \equiv \mathrm{E}(\mathbf{z}_i'\mathbf{z}_i)$. It follows that the matrix in the middle of the sandwich is

$$\begin{aligned}\mathbf{D} &= \mathrm{Var}\{\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2) + \mathbf{F}\mathbf{A}_2^{-1}\mathbf{z}_i'v_{i2}\} \\ &= \mathrm{Var}[\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2)] + \mathbf{F}\mathbf{A}_2^{-1}\mathrm{Var}(\mathbf{z}_i'v_{i2})\mathbf{A}_2^{-1}\mathbf{F}' \\ &= \mathbf{A}_1 + \tau_2^2\mathbf{F}\mathbf{A}_2^{-1}\mathbf{F}'\end{aligned}$$

because $\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2)$ and $\mathbf{z}_i'v_{i2}$ are uncorrelated, $\mathrm{Var}[\mathbf{s}_i(\boldsymbol{\gamma}_1;\boldsymbol{\delta}_2)] = \mathbf{A}_1$ by the information matrix equality, and $\mathrm{E}(v_{i2}^2|\mathbf{z}_i) = \tau_2^2$ under homoskedasticity for v_2. (The results that follow do not rely in any crucial way on $\mathrm{E}(v_{i2}^2|\mathbf{z}_i) = \tau_2^2$; we could just drop that and use the more general

formula.) Therefore,

$$\begin{aligned}\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_1)] &= \mathbf{A}_1^{-1}(\mathbf{A}_1 + \tau_2^2\mathbf{F}\mathbf{A}_2^{-1}\mathbf{F}')\mathbf{A}_1^{-1} \\ &= \mathbf{A}_1^{-1} + \tau_2^2\mathbf{A}_1^{-1}\mathbf{F}\mathbf{A}_2^{-1}\mathbf{F}'\mathbf{A}_1^{-1}.\end{aligned}$$

It is easy to construct consistent estimators of each part using sample averages and plugging in the consistent estimators.

b. If we ignore estimation of $\boldsymbol{\delta}_2$ we act as if $\text{Avar}[\sqrt{N}(\hat{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}_1)]$ is just $\mathbf{A}_1^{-1}$, the inverse of the information matrix from the second stage problem. But the correct matrix differs from $\mathbf{A}_1^{-1}$ by $\tau_2^2\mathbf{A}_1^{-1}\mathbf{F}\mathbf{A}_2^{-1}\mathbf{F}'\mathbf{A}_1^{-1}$, which is positive semi-definite (and usual positive definite if $\theta_{\rho 1} \neq 0$).

c. In Problem 12.17 we can take $\mathbf{g}(\mathbf{w}_i, \boldsymbol{\theta}) = \Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)\boldsymbol{\gamma}_1$, but we have to be careful in choosing the "score" with respect to $\boldsymbol{\theta}$. The same argument as in Problem 12.17 gives us

$$N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i, \hat{\boldsymbol{\theta}}) = N^{-1/2}\sum_{i=1}^{N}\mathbf{g}(\mathbf{w}_i, \boldsymbol{\theta}) + \mathbf{G}\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_p(1)$$

where $\mathbf{G} \equiv \text{E}[\nabla_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{w}_i, \boldsymbol{\theta})]$. For this application,

$$\nabla_{\boldsymbol{\theta}}\mathbf{g}(\mathbf{w}_i, \boldsymbol{\theta}) = [\Phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)\mathbf{I}_{K_1+1} + \phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)\boldsymbol{\gamma}_1\mathbf{w}_{i1}(\boldsymbol{\delta}_2)|\phi(\mathbf{w}_{i1}(\boldsymbol{\delta}_2)\boldsymbol{\gamma}_1)\boldsymbol{\gamma}_1'\nabla_{\boldsymbol{\delta}_2}\mathbf{w}_{i1}(\boldsymbol{\delta}_2)']$$

where K_1 is the dimension of $\mathbf{x}_1$ and $\nabla_{\boldsymbol{\delta}_2}\mathbf{w}_{i1}(\boldsymbol{\delta}_2)' = [\mathbf{0}|-\mathbf{z}_i']$. To get a representation for $\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ we stack the first-order representations obtained in part a:

$$\begin{aligned}\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) &= N^{-1/2}\sum_{i=1}^{N}\begin{pmatrix} -\mathbf{A}_1^{-1}[\mathbf{s}_i(\boldsymbol{\gamma}_1; \boldsymbol{\delta}_2) + \mathbf{F}\mathbf{A}_2^{-1}\mathbf{z}_i'v_{i2}] \\ \mathbf{A}_2^{-1}\mathbf{z}_i'v_{i2}\end{pmatrix} + o_p(1) \\ &\equiv N^{-1/2}\sum_{i=1}^{N}\mathbf{e}_i(\boldsymbol{\theta}) + o_p(1).\end{aligned}$$

Then, from Problem 12.17,

$$\mathbf{C} = \text{Avar}[\sqrt{N}(\hat{\boldsymbol{\eta}}_1 - \boldsymbol{\eta}_1)] = \text{Var}[\mathbf{g}(\mathbf{w}_i, \boldsymbol{\theta}) - \boldsymbol{\eta}_1 - \mathbf{G}\mathbf{e}_i(\boldsymbol{\theta})]$$

d. A consistent estimator of the asymptotic variance in part c is

$$\hat{\mathbf{C}} = N^{-1}\sum_{i=1}^{N}(\hat{\mathbf{g}}_i - \hat{\boldsymbol{\eta}}_1 - \hat{\mathbf{G}}\hat{\mathbf{e}}_i)(\hat{\mathbf{g}}_i - \hat{\boldsymbol{\eta}}_1 - \hat{\mathbf{G}}\hat{\mathbf{e}}_i)'$$

where $\hat{\mathbf{g}}_i = \Phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)\hat{\boldsymbol{\gamma}}_1$,

$$\hat{\mathbf{G}} = N^{-1}\sum_{i=1}^{N}[\Phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)\hat{\boldsymbol{\gamma}}_1\mathbf{I}_{K_1+1} + \phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)\hat{\boldsymbol{\gamma}}_1\hat{\mathbf{w}}_{i1}|\phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)\hat{\boldsymbol{\gamma}}_1'\nabla_{\delta_2}\mathbf{w}_{i1}(\hat{\boldsymbol{\delta}}_2)']$$

and

$$\hat{\mathbf{e}}_i = \begin{pmatrix} -\hat{\mathbf{A}}_1^{-1}[\hat{\mathbf{s}}_i + \hat{\mathbf{F}}\hat{\mathbf{A}}_2^{-1}\mathbf{z}_i'\hat{v}_{i2}] \\ \hat{\mathbf{A}}_2^{-1}\mathbf{z}_i'\hat{v}_{i2} \end{pmatrix}.$$

The score $\hat{\mathbf{s}}_i$ and Hessian $\hat{\mathbf{A}}_1$ are estimated as usual for a probit model (but with minus signs) and

$$\hat{\mathbf{F}} = N^{-1}\sum_{i=1}^{N}\frac{-\theta_{\rho 1}'\hat{\mathbf{w}}_{i1}\mathbf{z}_i\phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)}{\Phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)[1-\Phi(\hat{\mathbf{w}}_{i1}\hat{\boldsymbol{\gamma}}_1)]}.$$

15.16. a. The response probability is

$$p(\mathbf{x}) = 1 - [1 + \exp(\mathbf{x}\boldsymbol{\beta})]^{-\alpha}$$

and, using the chain rule,

$$\frac{\partial p(\mathbf{x})}{\partial x_K} = \alpha\beta_j\exp(\mathbf{x}\boldsymbol{\beta})[1+\exp(\mathbf{x}\boldsymbol{\beta})]^{-\alpha-1} = \frac{\alpha\beta_j\exp(\mathbf{x}\boldsymbol{\beta})}{[1+\exp(\mathbf{x}\boldsymbol{\beta})]^{\alpha+1}}$$

Of course, we get the logit partial effect as a special case when $\alpha = 1$.

b. The log likeihood has the usual form for a binary response. Let $G(\mathbf{x},\boldsymbol{\theta}) = 1 - [1+\exp(\mathbf{x}\boldsymbol{\beta})]^{-\alpha}$, so $1 - G(\mathbf{x},\boldsymbol{\theta}) = [1+\exp(\mathbf{x}\boldsymbol{\beta})]^{-\alpha}$. Without making the distinction between generic and "true" values,

$$\ell_i(\boldsymbol{\beta}, \alpha) = -(1 - y_i)\alpha \log[1 + \exp(\mathbf{x}_i\boldsymbol{\beta})] + y_i \log\{1 - [1 + \exp(\mathbf{x}_i\boldsymbol{\beta})]^{-\alpha}\}.$$

c. The Stata output is given below. Given the estimated value of α, $\hat{\alpha} = 413,553$, the model does not seem well determined. (Remember, the logit model imposes $\alpha = 1$.) The logit estimates are included for comparison. The $\hat{\beta}_j$ are are all the same sign and of roughly the same statistical significance across the two models. The t statistic for $H_0 : \log(\alpha) = 0$ is very small, about $.02$.

```
. scobit inlf nwifeinc educ exper expersq age kidslt6 kidsge6

Skewed logistic regression                      Number of obs     =        753
                                                Zero outcomes     =        325
Log likelihood = -399.5222                      Nonzero outcomes  =        428

------------------------------------------------------------------------------
        inlf |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    nwifeinc |  -.0148532   .0056874    -2.61   0.009    -.0260003   -.0037061
        educ |   .1512102   .0277346     5.45   0.000     .0968514    .2055689
       exper |    .139092    .020757     6.70   0.000     .0984091    .1797749
     expersq |   -.002257   .0006377    -3.54   0.000    -.0035069   -.0010072
         age |  -.0587203   .0089444    -6.57   0.000     -.076251   -.0411897
     kidslt6 |  -.9977924   .1426425    -7.00   0.000    -1.277367   -.7182183
     kidsge6 |   .0257666    .045345     0.57   0.570    -.0631079    .1146411
       _cons |  -13.09326   666.1339    -0.02   0.984    -1318.692    1292.505
-------------+----------------------------------------------------------------
    /lnalpha |   12.93254   666.1327     0.02   0.985    -1292.663    1318.529
-------------+----------------------------------------------------------------
       alpha |   413553.1   2.75e+08                              0
------------------------------------------------------------------------------
Likelihood-ratio test of alpha=1:   chi2(1) =     4.49    Prob > chi2 = 0.0342

Note: likelihood-ratio tests are recommended for inference with scobit models

. logit inlf nwifeinc educ exper expersq age kidslt6 kidsge6

Logistic regression                             Number of obs   =        753
                                                LR chi2(7)      =     226.22
                                                Prob > chi2     =     0.0000
Log likelihood = -401.76515                     Pseudo R2       =     0.2197

------------------------------------------------------------------------------
        inlf |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    nwifeinc |  -.0213452   .0084214    -2.53   0.011    -.0378509   -.0048394
        educ |   .2211704   .0434396     5.09   0.000     .1360303    .3063105
       exper |   .2058695   .0320569     6.42   0.000     .1430391    .2686999
     expersq |  -.0031541   .0010161    -3.10   0.002    -.0051456   -.0011626
         age |  -.0880244    .014573    -6.04   0.000     -.116587   -.0594618
     kidslt6 |  -1.443354   .2035849    -7.09   0.000    -1.842373   -1.044335
```

```
     kidsge6 |   .0601122   .0747897     0.80   0.422     -.086473    .2066974
       _cons |   .4254524   .8603697     0.49   0.621    -1.260841    2.111746
------------------------------------------------------------------------------
```

d. The likelihood ratio statistic for $H_0 : \alpha = 1$, reported in the Stata output, is 4.49 with $p-$value $=.034$. So this statistic rejects the logit model, although it is not an overwhelming rejection. Its $p-$value is certainly much smaller than the Wald test (t test) for $H_0 : \log(\alpha) = 0$.

e. Given the bizarre value for $\hat{\alpha}$ and the modest gain in fit, the skewed logit model does not seem worth the effort. Plus, the Stata output below shows that the correlations of the fitted probabilities and *inlf* are very similar across the two models ($.5196$ for skewed logit, $.5179$ for logit). The average partial effects are similar, too. For *nwifeinc*, the APE for skewed logit is about $-.0041$ for skewed logit and about $-.0038$ for logit. For *kidslt*6, the APEs are $-.274$ (skewed logit) and $-.258$ (logit). It is likely these differences can be attributed to sampling error.

```
. qui scobit inlf nwifeinc educ exper expersq age kidslt6 kidsge6

. predict phat_skewlog
(option pr assumed; Pr(inlf))

. predict xbh_sklog, xb

. gen scale_sklog = e(alpha)*exp(xbh_sklog)/((1 + exp(xbh_sklog))^(1 + e(alpha

. sum scale_sklog

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
 scale_sklog |       753    .2741413    .0891063   .0098302   .3678786

. predict phat_skewlog
(option pr assumed; Pr(inlf))

. qui logit inlf nwifeinc educ exper expersq age kidslt6 kidsge6

. predict phat_log
(option pr assumed; Pr(inlf))

. predict xbh_log, xb

. gen scale_log = exp(xbh_log)/((1 + exp(xbh_log))^2 )

. sum scale_log
```

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
   scale_log |       753    .1785796    .0617942   .0085973        .25

. corr phat_skewlog inlf
(obs=753)

             | phat_s~g     inlf
-------------+------------------
phat_skewlog |   1.0000
        inlf |   0.5196   1.0000


. corr phat_log inlf
(obs=753)

             | phat_log     inlf
-------------+------------------
    phat_log |   1.0000
        inlf |   0.5179   1.0000

. di  .2741413* (-.0148532)
-.00407188

. di  .1785796*(-.0213452)
-.00381182

. di  .2741413* (-.9977924)
-.27353611

. di  .1785796*(-1.443354)
-.25775358
```

15.17. a. We obtain the joint density by the product rule, since we have independence conditional on $(\mathbf{x}, c)$:

$$f(y_1, \ldots, y_G|\mathbf{x}, c; \boldsymbol{\gamma}_o) = f_1(y_1|\mathbf{x}, c; \boldsymbol{\gamma}_o^1) f_2(y_1|\mathbf{x}, c; \boldsymbol{\gamma}_o^2) \cdots f_G(y_G|\mathbf{x}, c; \boldsymbol{\gamma}_o^G).$$

b. The density of $(y_1, \ldots, y_G)$ given $\mathbf{x}$ is obtained by integrating out with respect to the distribution of c given $\mathbf{x}$:

$$g(y_1, \ldots, y_G|\mathbf{x}; \boldsymbol{\gamma}_o) = \int_{-\infty}^{\infty} \left(\prod_{g=1}^{G} f_g(y_g|\mathbf{x}, \mathfrak{c}; \boldsymbol{\gamma}_o^g) \right) h(\mathfrak{c}|\mathbf{x}; \boldsymbol{\delta}_o) d\mathfrak{c},$$

where c is a dummy argument of integration. Because c appears in each $D(y_g|\mathbf{x}, c), y_1, \ldots, y_G$ are dependent without conditioning on c.

c. The log-likelihood for each i is

$$\log\left[\int_{-\infty}^{\infty}\left(\prod_{g=1}^{G} f_g(y_{ig}|\mathbf{x}_i, c; \boldsymbol{\gamma}^g)\right) h(c|\mathbf{x}_i; \boldsymbol{\delta}) dc\right].$$

As expected, this depends only on the observed data, $(\mathbf{x}_i, y_{i1}, \ldots, y_{iG})$, and the unknown parameters.

15.18. a. The probability is the same as if we assume (15.73), that is,

$$\mathrm{P}(y_{it} = 1|\mathbf{x}_i, a_i) = \Phi(\psi + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i),\ t = 1, 2, \ldots, T.$$

The fact that a_i given $\mathbf{x}_i$ is heteroskedastic has no bearing on the distribution conditional on $(\mathbf{x}_i, a_i)$. Only when we "integrate out" a_i does $\mathrm{D}(a_i|\mathbf{x}_i)$ matter.

b. Let $g_t(y|\mathbf{x}_i, a_i; \boldsymbol{\theta}) = [\Phi(\psi + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)]^{y_t}[1 - \Phi(\psi + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)]^{1-y_t}$. Then, by the product and integration rules,

$$f(y_1, \ldots, y_T|\mathbf{x}; \boldsymbol{\theta}) = \left[\int_{-\infty}^{\infty}\left(\prod_{t=1}^{T} g_t(y_t|\mathbf{x}_i, a; \boldsymbol{\theta})\right) h(a|\mathbf{x}_i; \boldsymbol{\delta}) da\right],$$

where $h(\cdot|\mathbf{x}_i, \boldsymbol{\delta})$ is the Normal$[0, \sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda})]$ density. We get the log-likelihood by plugging in the y_{it} and taking the natural log. For each i, the log likelihood depends on $(\mathbf{x}_i, \mathbf{y}_i)$ and the parameters $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$; a_i does not appear.

c. To estimate the APEs we can estimate the average structural function, which in this case is

$$\begin{aligned}\mathrm{ASF}(\mathbf{x}^o) &= \mathrm{E}_{c_i}[\Phi(\psi + \mathbf{x}^o\boldsymbol{\beta} + c_i)] \\ &= \mathrm{E}_{(\mathbf{x}_i, a_i)}[\Phi(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)] \\ &= \mathrm{E}_{\mathbf{x}_i}\{\mathrm{E}[\Phi(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)|\mathbf{x}_i]\}\end{aligned}$$

To compute $\mathrm{E}[\Phi(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)|\bar{\mathbf{x}}_i]$ we use a similar trick as before. It is the same as computing

$$\mathrm{E}(1[\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i + u_{it} > 0]|\mathbf{x}_i)$$

where

$$a_i + u_{it} | \mathbf{x}_i \sim \text{Normal}(0, 1 + \sigma_a^2 \exp(\bar{\mathbf{x}}_i \boldsymbol{\lambda})).$$

because u_{it} is independent of $(a_i, \mathbf{x}_i)$ with a standard normal distribution. Now

$$\begin{aligned} \text{E}(1[\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i + u_{it} > 0] | \mathbf{x}_i) &= \text{P}[a_i + u_{it} > -(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi}) | \mathbf{x}_i] \\ &= \text{P}\left[\frac{a_i + u_{it}}{[1 + \sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda})]^{1/2}} > \frac{-(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi})}{[1 + \sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda})]^{1/2}} \,\middle|\, \mathbf{x}_i \right] \\ &= \Phi\left[\frac{(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi})}{[1 + \sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda})]^{1/2}} \right]. \end{aligned}$$

(Notice this only depends on $\bar{\mathbf{x}}_i$, not on $\mathbf{x}_i$. We could relax that assumption.)

The ASF is therefore,

$$\text{ASF}(\mathbf{x}^o) = \text{E}_{\bar{\mathbf{x}}_i}\left\{ \Phi\left[\frac{(\psi + \mathbf{x}^o\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi})}{[1 + \sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda})]^{1/2}} \right] \right\}$$

and a consistent estimator is obtained by using a sample average and plugging in the maximum likelihood estimators:

$$\widehat{\text{ASF}}(\mathbf{x}^o) = N^{-1} \sum_{i=1}^{N} \Phi\left[\frac{(\hat{\psi} + \mathbf{x}^o\hat{\boldsymbol{\beta}} + \bar{\mathbf{x}}_i\hat{\boldsymbol{\xi}})}{[1 + \hat{\sigma}_a^2 \exp(\bar{\mathbf{x}}_i\hat{\boldsymbol{\lambda}})]^{1/2}} \right].$$

Now take derivatives and changes with respect to $\mathbf{x}^o$ (a placeholder).

15.19. a. The Stata output is below. We need to assume first-order dynamics for the usual standard errors and test statistics to be valid.

```
. tab year

   81 to 87 |      Freq.     Percent        Cum.
------------+-----------------------------------
         81 |      1,738       14.29       14.29
         82 |      1,738       14.29       28.57
         83 |      1,738       14.29       42.86
         84 |      1,738       14.29       57.14
         85 |      1,738       14.29       71.43
         86 |      1,738       14.29       85.71
         87 |      1,738       14.29      100.00
------------+-----------------------------------
```

```
      Total |     12,166      100.00

. tab black if year == 87

=1 if black |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |      1,065       61.28       61.28
          1 |        673       38.72      100.00
------------+-----------------------------------
      Total |      1,738      100.00

. xtset id year
       panel variable:  id (strongly balanced)
        time variable:  year, 81 to 87
                delta:  1 unit

. gen employ_1 = l.employ
(1738 missing values generated)

. probit employ employ_1 if black

Probit regression                                 Number of obs   =       4038
                                                  LR chi2(1)      =    1091.27
                                                  Prob > chi2     =     0.0000
Log likelihood = -2248.0349                       Pseudo R2       =     0.1953

------------------------------------------------------------------------------
      employ |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    employ_1 |   1.389433   .0437182    31.78   0.000     1.303747    1.475119
       _cons |  -.5396127   .0281709   -19.15   0.000    -.5948268   -.4843987
------------------------------------------------------------------------------
```

b. After estimating the previous model, the Stata calculations are below. The difference in employment probabilities this year, based on employment status last year, is about .508.

```
. di normal(_b[_cons])
.29473206

. di normal(_b[_cons]  + _b[employ_1])
.80228758

. di normal(_b[_cons]  + _b[employ_1]) - normal(_b[_cons])
.50755552
```

c. With year dummies, the story is very similar. The estimated state dependence for 1987 is about .472.

```
. probit employ employ_1 y83-y87 if black

Probit regression                                 Number of obs   =       4038
                                                  LR chi2(6)      =    1156.98
                                                  Prob > chi2     =     0.0000
Log likelihood = -2215.1795                       Pseudo R2       =     0.2071
```

```
------------------------------------------------------------------------------
      employ |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    employ_1 |   1.321349   .0453568    29.13   0.000     1.232452    1.410247
         y83 |   .3427664   .0749844     4.57   0.000     .1957997    .4897331
         y84 |   .4586078   .0755742     6.07   0.000     .3104851    .6067305
         y85 |   .5200576   .0767271     6.78   0.000     .3696753    .6704399
         y86 |   .3936516   .0774704     5.08   0.000     .2418125    .5454907
         y87 |   .5292136   .0773031     6.85   0.000     .3777023     .680725
       _cons |  -.8850412   .0556042   -15.92   0.000    -.9940233    -.776059
------------------------------------------------------------------------------

. di normal(_b[_cons]  + _b[y87] + _b[employ_1]) - normal(_b[_cons] + _b[y87]
.4718734
```

d. Below gives one way in Stata to estimate the dynamic unobserved effects model. Compared with not allowing for heterogeneity as in part c, the coefficient on $employ_{-1}$ has fallen: from about 1.321 to about $.899$. In addition, the coefficient on the initial condition is $.566$ and it is very statistically significant. But we cannot know how much the amount of state dependence has changed without computing an average partial effect.

```
. gen employ81 = employ if y81
(10428 missing values generated)

. replace employ81 = employ[_n-1] if y82
(1738 real changes made)

. replace employ81 = employ[_n-2] if y83
(1738 real changes made)

. replace employ81 = employ[_n-3] if y84
(1738 real changes made)

. replace employ81 = employ[_n-4] if y85
(1738 real changes made)

. replace employ81 = employ[_n-5] if y86
(1738 real changes made)

. replace employ81 = employ[_n-6] if y87
(1738 real changes made)

. xtprobit employ employ_1 employ81 y83-y87 if black, re

Random-effects probit regression               Number of obs      =      4038
Group variable: id                             Number of groups   =       673

Random effects u_i ~Gaussian                   Obs per group: min =         6
                                                              avg =       6.
                                                              max =
```

```
                                                Wald chi2(7)       =    677.59
Log likelihood  = -2176.3738                    Prob > chi2        =    0.0000

------------------------------------------------------------------------------
      employ |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    employ_1 |   .8987806   .0677058    13.27   0.000     .7660797    1.031482
    employ81 |   .5662897   .0884941     6.40   0.000     .3928444     .739735
         y83 |   .4339911   .0804064     5.40   0.000     .2763974    .5915847
         y84 |   .6563094   .0841199     7.80   0.000     .4914374    .8211814
         y85 |   .7919805   .0887167     8.93   0.000      .618099     .965862
         y86 |   .6896344    .090158     7.65   0.000     .5129279    .8663409
         y87 |   .8382018    .091054     9.21   0.000     .6597393    1.016664
       _cons |  -1.005103   .0660945   -15.21   0.000    -1.134646   -.8755602
-------------+----------------------------------------------------------------
    /lnsig2u |  -1.178731   .1995372                     -1.569817   -.7876454
-------------+----------------------------------------------------------------
     sigma_u |   .5546791   .0553396                      .4561615    .6744736
         rho |   .2352804   .0359014                      .1722425    .3126745
------------------------------------------------------------------------------
Likelihood-ratio test of rho=0: chibar2(01) =    47.90 Prob >= chibar2 = 0.000
```

e. There is still plenty of evidence of state dependence because of the very statistically significant coefficient on $employ_{-1}$ ($t = 13.27$). The coefficient still seems quite large, but we still need to compute the APE.

The positive coefficient on $employ_{81}$ shows that that c_i and $employ_{i,81}$ are positively correlated. The estimate of σ_a^2 is $(.5546791)^2$, or $\hat{\sigma}_a^2 \approx .308$.

f. The average state dependence, where we average out the distribution of c_i, is estimated as

$$N^{-1}\sum_{i=1}^{N}\left\{\Phi\left[\frac{(\hat{\psi}+\hat{\delta}_{87}+\hat{\rho}+\hat{\xi}y_{i0})}{(1+\hat{\sigma}_a^2)^{1/2}}\right]-\Phi\left[\frac{(\hat{\psi}+\hat{\delta}_{87}+\hat{\xi}y_{i0})}{(1+\hat{\sigma}_a^2)^{1/2}}\right]\right\}$$

where $\hat{\rho}$ is the coefficient on $y_{-1} = employ_{-1}$, $y_{i0} = employ_{i,1981}$, and, in this case, the averaging is done across the black men in the sample. The Stata calculations below (done after the calculations in part d) show the estimated state dependence is about .283, which is much lower than the estimate of .472 from part c (where we ignored heterogeneity). Bootstrapping is a convenient way to obtain a standard error, as was done in Example 15.6.

```
. gen stdep = normal((_b[_cons] + _b[employ_1] + _b[employ81]*employ81
```

```
                + _b[y87])/sqrt(1 + e(sigma_u)^2))
          - normal((_b[_cons] + _b[employ81]*employ81 + _b[y87])
                /sqrt(1 + e(sigma_u)^2)) if black & y87
(11493 missing values generated)

. sum stdep

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       stdep |       673     .283111    .0257298   .2353074   .2969392
```

15.20. (**Bonus Question**) Estimate the CRE probit model report in Table 15.3 using the generalized estimation equation (GEE) approach described in Section 12.9.2, using an exchangeable correlation structure.

a. How do the point estimates compare with the pooled probit estimates in Column (3) of Table 15.3?

b. Does it appear that the GEE approach improves on the efficiency of pooled probit? Explain.

Solution:

a. The Stata output for pooled probit and GEE is given below. The pooled probit estimates replicate the numbers in Table 15.3.

```
. probit lfp kids lhinc kidsbar lhincbar educ black age agesq per2-per5,
        cluster(id)

Iteration 0:   log pseudolikelihood = -17709.021
Iteration 1:   log pseudolikelihood = -16521.245
Iteration 2:   log pseudolikelihood = -16516.437
Iteration 3:   log pseudolikelihood = -16516.436

Probit regression                                 Number of obs   =      28315
                                                  Wald chi2(12)   =     538.09
                                                  Prob > chi2     =     0.0000
Log pseudolikelihood = -16516.436                 Pseudo R2       =     0.0673

                                (Std. Err. adjusted for 5663 clusters in id
------------------------------------------------------------------------------
             |               Robust
         lfp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        kids |  -.1173749   .0269743    -4.35   0.000    -.1702435   -.0645064
       lhinc |  -.0288098    .014344    -2.01   0.045    -.0569234   -.0006961
     kidsbar |  -.0856913   .0311857    -2.75   0.006     -.146814   -.0245685
    lhincbar |  -.2501781   .0352907    -7.09   0.000    -.3193466   -.1810097
        educ |   .0841338   .0067302    12.50   0.000     .0709428    .0973248
       black |   .2030668   .0663945     3.06   0.002     .0729359    .3331976
         age |   .1516424   .0124831    12.15   0.000      .127176    .1761089
       agesq |  -.0020672   .0001553   -13.31   0.000    -.0023717   -.0017628
        per2 |  -.0135701   .0103752    -1.31   0.191    -.0339051    .0067648
        per3 |  -.0331991   .0127197    -2.61   0.009    -.0581293    -.008269
        per4 |  -.0390317   .0136244    -2.86   0.004    -.0657351   -.0123284
        per5 |  -.0552425   .0146067    -3.78   0.000    -.0838711   -.0266139
       _cons |  -.7260562   .2836985    -2.56   0.010    -1.282095   -.1700173
------------------------------------------------------------------------------
```

```
. xtgee lfp kids lhinc kidsbar lhincbar educ black age agesq per2-per5,
        fam(binomial) link(probit) corr(exch) robust

GEE population-averaged model                   Number of obs      =     28315
Group variable:                           id    Number of groups   =      5663
Link:                                 probit    Obs per group: min =
Family:                             binomial                   avg =       5.
Correlation:                   exchangeable                    max =
                                               Wald chi2(12)      =    536.66
Scale parameter:                           1   Prob > chi2        =    0.0000

                                    (Std. Err. adjusted for clustering on id
------------------------------------------------------------------------------
             |             Semirobust
         lfp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        kids |  -.1125361   .0281366    -4.00   0.000    -.1676828   -.0573894
       lhinc |  -.0276543    .014799    -1.87   0.062    -.0566598    .0013511
     kidsbar |  -.0892543   .0323884    -2.76   0.006    -.1527344   -.0257742
    lhincbar |   -.252001   .0360377    -6.99   0.000    -.3226337   -.1813684
        educ |   .0841304   .0066834    12.59   0.000     .0710312    .0972296
       black |    .205611   .0668779     3.07   0.002     .0745328    .3366893
         age |    .152809   .0125434    12.18   0.000     .1282245    .1773936
       agesq |  -.0020781   .0001565   -13.28   0.000    -.0023847   -.0017714
        per2 |  -.0134259   .0103607    -1.30   0.195    -.0337324    .0068807
        per3 |  -.0329993   .0126967    -2.60   0.009    -.0578845   -.0081141
        per4 |  -.0384026   .0136212    -2.82   0.005    -.0650997   -.0117056
        per5 |    -.05451   .0146135    -3.73   0.000     -.083152    -.025868
       _cons |  -.7532503    .285216    -2.64   0.008    -1.312263   -.1942373
------------------------------------------------------------------------------
```

b. Surprisingly, and disappointingly, the GEE approach does not improve the precision of the estimators. In fact, the robust standard errors for GEE are actually slightly above those for pooled probit. This finding is particular puzzling because there is substantial serial correlation in the standardized residuals, written generally after pooled probit estimation as

$$\hat{r}_{it} \equiv \frac{[y_{it} - \Phi(\mathbf{x}_{it}\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{w}}_i\hat{\boldsymbol{\xi}})]}{\{\Phi(\mathbf{x}_{it}\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{w}}_i\hat{\boldsymbol{\xi}})[1 - \Phi(\mathbf{x}_{it}\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{w}}_i\hat{\boldsymbol{\xi}})]\}^{1/2}},$$

where $\bar{\mathbf{w}}_i$ is the time average of variables that change across i and t ($kids_{it}$ and $lhinc_{it}$ in this application). The first-order correlation in the $\{\hat{r}_{it} : t = 2,\ldots,T; i = 1,\ldots,N\}$ is about .83.

```
. qui probit lfp kids lhinc kidsbar lhincbar educ black age agesq per2-per5

. predict phat
(option pr assumed; Pr(lfp))
```

```
. gen rh = (lfp - phat)/sqrt(phat*(1 - phat))

. gen rh_1 = l.rh
(5663 missing values generated)

. corr rh rh_1
(obs=22652)

             |       rh     rh_1
-------------+------------------
          rh |   1.0000
        rh_1 |   0.8315   1.0000
```

c. This is not an answer to a particular question, but serves as an errata for the estimates on Column (4) of Table 15.3. Those estimates were obtained using a version of Stata earlier than 9.0. Using Stata 11.0, a higher value of the log likelihood is found, and the point estimates are different. Note that the estimated value of ρ, which is the pairwise correlation between any of the two composite errors $a_i + e_{it}$, is very large: .95. The estimated scale factor for the coefficients, about .233, is substantially below that in Table 15.3, but the coefficients reported below are substantially higher. I have deleted the details of the numerical iterations.

```
. xtprobit lfp kids lhinc kidsbar lhincbar educ black age agesq per2-per5, re

Random-effects probit regression                Number of obs      =     28315
Group variable: id                              Number of groups   =      5663

Random effects u_i ~Gaussian                    Obs per group: min =         5
                                                               avg =       5.
                                                               max =

                                                Wald chi2(12)      =    623.40
Log likelihood  = -8609.9002                    Prob > chi2        =    0.0000

------------------------------------------------------------------------------
         lfp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        kids |  -.3970102   .0701298    -5.66   0.000     -.534462   -.2595584
       lhinc |  -.1003399   .0469979    -2.13   0.033    -.1924541   -.0082258
     kidsbar |  -.4085664   .0898875    -4.55   0.000    -.5847428   -.2323901
    lhincbar |  -.8941069   .1199703    -7.45   0.000    -1.129244   -.6589695
        educ |   .3189079    .024327    13.11   0.000     .2712279     .366588
       black |   .6388784   .1903525     3.36   0.001     .2657945    1.011962
         age |   .7282057   .0445623    16.34   0.000     .6408651    .8155462
       agesq |  -.0098358   .0005747   -17.11   0.000    -.0109623   -.0087094
        per2 |  -.0451653   .0499429    -0.90   0.366    -.1430516     .052721
        per3 |  -.1247056   .0501522    -2.49   0.013    -.2230022    -.026409
        per4 |  -.1356834   .0500679    -2.71   0.007    -.2338147   -.0375522
        per5 |   -.200357    .049539    -4.04   0.000    -.2974515   -.1032624
       _cons |  -5.359375   1.000514    -5.36   0.000    -7.320346   -3.398404
```

```
-------------+----------------------------------------------------------------
    /lnsig2u |   2.947234   .0435842                      2.861811    3.032657
-------------+----------------------------------------------------------------
     sigma_u |   4.364995   .0951224                      4.182484     4.55547
         rho |   .9501326    .002065                       .945926    .9540279
------------------------------------------------------------------------------
Likelihood-ratio test of rho=0: chibar2(01) =  1.6e+04 Prob >= chibar2 = 0.000

. * Scale factor for coefficients:

. di 1/sqrt(1 + e(sigma_u)^2)
.22331011
```

Solutions to Chapter 16 Problems

16.1. a. The Stata ouput below contains the estimates for 1981 and, for completeness, 1987. Certainly some magnitudes are fairly different. For example, education has a much larger effect in the latter time period. Also, the effect of experience on the log-odds ratios are quite different.

```
. mlogit status educ exper expersq black if y81, base(0)

Multinomial logistic regression                 Number of obs   =       1737
                                                LR chi2(8)      =     720.39
                                                Prob > chi2     =     0.0000
Log likelihood = -1502.9396                     Pseudo R2       =     0.1933

------------------------------------------------------------------------------
      status |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
0            |  (base outcome)
-------------+----------------------------------------------------------------
1            |
        educ |    -.47558   .0466559   -10.19   0.000    -.5670238   -.3841361
       exper |   3.016025   .4513224     6.68   0.000     2.131449    3.900601
     expersq |  -.5953032   .2690175    -2.21   0.027    -1.122568   -.0680386
       black |   .8649358   .1302512     6.64   0.000     .6096481    1.120224
       _cons |   4.138761   .5276112     7.84   0.000     3.104662     5.17286
-------------+----------------------------------------------------------------
2            |
        educ |  -.1019564   .0495931    -2.06   0.040    -.1991571   -.0047558
       exper |   4.101794   .4359451     9.41   0.000     3.247357    4.956231
     expersq |  -.7069626   .2628842    -2.69   0.007    -1.222206   -.1917191
       black |   .0208189   .1436123     0.14   0.885    -.2606561    .3022938
       _cons |  -.0313456   .5828582    -0.05   0.957    -1.173727    1.111035
------------------------------------------------------------------------------

. mlogit status educ exper expersq black if y87, base(0)

Multinomial logistic regression                 Number of obs   =       1717
                                                LR chi2(8)      =     583.72
                                                Prob > chi2     =     0.0000
Log likelihood = -907.85723                     Pseudo R2       =     0.2433

------------------------------------------------------------------------------
      status |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
0            |  (base outcome)
-------------+----------------------------------------------------------------
1            |
        educ |  -.6736313   .0698999    -9.64   0.000    -.8106325     -.53663
       exper |  -.1062149    .173282    -0.61   0.540    -.4458414    .2334116
     expersq |  -.0125152   .0252291    -0.50   0.620    -.0619633     .036933
       black |   .8130166   .3027231     2.69   0.007     .2196902    1.406343
```

```
       _cons |   10.27787   1.133336     9.07   0.000     8.056578    12.49917
-------------+----------------------------------------------------------------
2            |
        educ |  -.3146573   .0651096    -4.83   0.000    -.4422699   -.1870448
       exper |   .8487367   .1569856     5.41   0.000     .5410507    1.156423
     expersq |  -.0773003   .0229217    -3.37   0.001    -.1222261   -.0323746
       black |   .3113612   .2815339     1.11   0.269     -.240435    .8631574
       _cons |   5.543798   1.086409     5.10   0.000     3.414475    7.673121
------------------------------------------------------------------------------
```

b. Just adding year dummies is probably not sufficient, given the findings in part a, but the results are below. Because the model is static and we have panel data, we should use inference robust to arbitrary serial dependence. In this application, the robust standard errors are typically larger but the difference is not huge.

```
. mlogit status educ exper expersq black y82-y87, base(0)

Multinomial logistic regression                 Number of obs   =      12108
                                                LR chi2(20)     =    6409.72
                                                Prob > chi2     =     0.0000
Log likelihood = -8842.6383                     Pseudo R2       =     0.2660

------------------------------------------------------------------------------
      status |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
0            |  (base outcome)
-------------+----------------------------------------------------------------
1            |
        educ |  -.5473739   .0189537   -28.88   0.000    -.5845225   -.5102253
       exper |    .769957   .0633149    12.16   0.000     .6458621    .8940519
     expersq |  -.1153749   .0107134   -10.77   0.000    -.1363729    -.094377
       black |   .8773806   .0656223    13.37   0.000     .7487633    1.005998
         y82 |   .9871298   .0928663    10.63   0.000     .8051152    1.169144
         y83 |   1.383591   .1035337    13.36   0.000     1.180669    1.586514
         y84 |   1.587213    .115548    13.74   0.000     1.360743    1.813683
         y85 |   2.052594   .1307157    15.70   0.000     1.796396    2.308792
         y86 |   2.652847   .1513588    17.53   0.000     2.356189    2.949505
         y87 |   2.727265   .1701085    16.03   0.000     2.393858    3.060671
       _cons |   5.151552   .2282352    22.57   0.000     4.704219    5.598885
-------------+----------------------------------------------------------------
2            |
        educ |  -.2555556   .0182414   -14.01   0.000     -.291308   -.2198032
       exper |   1.823821    .058522    31.16   0.000      1.70912    1.938522
     expersq |   -.195654   .0095781   -20.43   0.000    -.2144267   -.1768813
       black |     .33846   .0649312     5.21   0.000     .2111972    .4657227
         y82 |   .5624964   .0936881     6.00   0.000     .3788712    .7461217
         y83 |   1.225732   .0998516    12.28   0.000     1.030027    1.421438
         y84 |    1.42652   .1095939    13.02   0.000      1.21172     1.64132
         y85 |   1.662994   .1243071    13.38   0.000     1.419357    1.906632
         y86 |   2.029585   .1447257    14.02   0.000     1.745928    2.313242
         y87 |   1.995639   .1622294    12.30   0.000     1.677675    2.313603
       _cons |   1.858323    .225749     8.23   0.000     1.415863    2.300783
------------------------------------------------------------------------------
```

```
. mlogit status educ exper expersq black y82-y87, base(0) cluster(id)

Multinomial logistic regression                 Number of obs   =      12108
                                                Wald chi2(20)   =    2742.09
                                                Prob > chi2     =     0.0000
Log pseudolikelihood = -8842.6383               Pseudo R2       =     0.2660

                                (Std. Err. adjusted for 1738 clusters in id
------------------------------------------------------------------------------
             |               Robust
      status |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
0            |  (base outcome)
-------------+----------------------------------------------------------------
1            |
        educ |  -.5473739   .0200999   -27.23   0.000     -.586769   -.5079789
       exper |    .769957   .0776371     9.92   0.000      .617791     .922123
     expersq |  -.1153749   .0106075   -10.88   0.000    -.1361653   -.0945846
       black |   .8773806   .0855443    10.26   0.000     .7097169    1.045044
         y82 |   .9871298   .0760747    12.98   0.000     .8380261    1.136234
         y83 |   1.383591   .0888752    15.57   0.000     1.209399    1.557784
         y84 |   1.587213   .1050477    15.11   0.000     1.381323    1.793103
         y85 |   2.052594   .1275644    16.09   0.000     1.802572    2.302615
         y86 |   2.652847   .1526831    17.37   0.000     2.353593      2.9521
         y87 |   2.727265   .1666166    16.37   0.000     2.400702    3.053827
       _cons |   5.151552   .2523957    20.41   0.000     4.656866    5.646238
-------------+----------------------------------------------------------------
2            |
        educ |  -.2555556   .0177679   -14.38   0.000      -.29038   -.2207312
       exper |   1.823821   .0731396    24.94   0.000      1.68047    1.967172
     expersq |   -.195654    .010131   -19.31   0.000    -.2155104   -.1757976
       black |     .33846   .0783575     4.32   0.000     .1848821    .4920378
         y82 |   .5624964   .0796845     7.06   0.000     .4063177    .7186751
         y83 |   1.225732   .0897086    13.66   0.000     1.049907    1.401558
         y84 |    1.42652   .1027116    13.89   0.000     1.225209    1.627831
         y85 |   1.662994    .124454    13.36   0.000     1.419069     1.90692
         y86 |   2.029585   .1526669    13.29   0.000     1.730363    2.328807
         y87 |   1.995639   .1636634    12.19   0.000     1.674865    2.316414
       _cons |   1.858323   .2257666     8.23   0.000     1.415829    2.300817
------------------------------------------------------------------------------
```

c. The time dummies have very large *t* statistics, and the robust joint test gives a χ^2_{12} value of 624.28, which implies a zero *p*-value to many decimal places.

d. After obtaining the estimates from part c, the following commands produce the change in the estimated employment probabilities. It is about .021 for 1981, and about .058 for 1987.

```
. di exp([2]_cons + [2]educ*16 + [2]exper*5 + [2]expersq*25 + [2]black)
    /(1 + exp([1]_cons + [1]educ*16 + [1]exper*5 + [1]expersq*25 + [1]black)
       + exp([2]_cons + [2]educ*16 + [2]exper*5 + [2]expersq*25 + [2]black))
.89820453
```

```
. di exp([2]_cons + [2]educ*12 + [2]exper*5 + [2]expersq*25 + [2]black)
   /(1 + exp([1]_cons + [1]educ*12 + [1]exper*5 + [1]expersq*25 + [1]black)
        + exp([2]_cons + [2]educ*12 + [2]exper*5 + [2]expersq*25 + [2]black))
.91903414

. di .91903414 - .89820453
.02082961

. di exp([2]_cons + [2]educ*12 + [2]exper*5 + [2]expersq*25 + [2]black + [2]y87
     /(1 + exp([1]_cons + [1]educ*12 + [1]exper*5 + [1]expersq*25 + [1]black
         + [1]y87)
      + exp([2]_cons + [2]educ*12 + [2]exper*5 + [2]expersq*25 + [2]black
       + [2]y8))
.89646574

. di exp([2]_cons + [2]educ*16 + [2]exper*5 + [2]expersq*25 + [2]black + [2]y87
    /(1 + exp([1]_cons + [1]educ*16 + [1]exper*5 + [1]expersq*25 + [1]black
     + [1]y87)
     + exp([2]_cons + [2]educ*16 + [2]exper*5 + [2]expersq*25 + [2]black
     + [2]y87))
.95454392

. di .95454392 - .89646574
.05807818
```

16.2. a. The following Stata output contains the linear regression results. Because *pctstck* is discrete (taking on the values 0, 50, and 100), it seems likely that heteroskedasticity is present in a linear model. In fact, the robust standard errors are not very different from the usual ones.

```
. use pension

. tab pctstck

0=mstbnds,5 |
0=mixed,100 |
  =mststcks |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |         78       34.51       34.51
         50 |         85       37.61       72.12
        100 |         63       27.88      100.00
------------+-----------------------------------
      Total |        226      100.00

. reg pctstck choice age educ female black married finc25-finc101 wealth89
      prftshr, robust

Linear regression                                      Number of obs =     194
                                                       F( 14,   179) =    2.15
                                                       Prob > F      =  0.0113
                                                       R-squared     =  0.0998
                                                       Root MSE      =  39.134

------------------------------------------------------------------------------
             |               Robust
```

```
     pctstck |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      choice |   12.04773   5.994437     2.01   0.046     .2188713    23.87658
         age |  -1.625967   .8327895    -1.95   0.052    -3.269315    .0173813
        educ |   .7538685   1.172328     0.64   0.521    -1.559493     3.06723
      female |   1.302856   7.148595     0.18   0.856    -12.80351    15.40922
       black |   3.967391   8.974971     0.44   0.659    -13.74297    21.67775
     married |   3.303436   8.369616     0.39   0.694    -13.21237    19.81924
      finc25 |  -18.18567   16.00485    -1.14   0.257    -49.76813    13.39679
      finc35 |  -3.925374   15.86275    -0.25   0.805    -35.22742    27.37668
      finc50 |  -8.128784    15.3762    -0.53   0.598    -38.47072    22.21315
      finc75 |  -17.57921    16.6797    -1.05   0.293    -50.49335    15.33493
     finc100 |   -6.74559    16.7482    -0.40   0.688     -39.7949    26.30372
     finc101 |  -28.34407   16.57814    -1.71   0.089    -61.05781    4.369672
    wealth89 |  -.0026918   .0114136    -0.24   0.814    -.0252142    .0198307
      prftshr |   15.80791   8.107663     1.95   0.053    -.1909844    31.80681
        _cons |   134.1161   58.87288     2.28   0.024      17.9419    250.2902
------------------------------------------------------------------------------
```

b. With relatively few husband-wife pairs – 23 in this application – we do not expect big differences in standard errors, and we do not see them. On the key variable, *choice*, the cluster-robust standard error is only slightly larger. (Incidentally, this part really should not come until Chapter 20.)

```
. reg pctstck choice age educ female black married finc25-finc101 wealth89
      prftshr, cluster(id)

Linear regression                                      Number of obs =     194
                                                       F( 14,   170) =    2.12
                                                       Prob > F      =  0.0128
                                                       R-squared     =  0.0998
                                                       Root MSE      =  39.134

                                    (Std. Err. adjusted for 171 clusters in id
------------------------------------------------------------------------------
             |               Robust
     pctstck |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      choice |   12.04773   6.184085     1.95   0.053    -.1597617    24.25521
         age |  -1.625967   .8192942    -1.98   0.049    -3.243267   -.0086663
        educ |   .7538685     1.1803     0.64   0.524    -1.576064    3.083801
      female |   1.302856   7.000538     0.19   0.853    -12.51632    15.12203
       black |   3.967391   8.711611     0.46   0.649    -13.22948    21.16426
     married |   3.303436   8.624168     0.38   0.702    -13.72082    20.32769
      finc25 |  -18.18567   16.82939    -1.08   0.281    -51.40716    15.03583
      finc35 |  -3.925374   16.17574    -0.24   0.809    -35.85656    28.00581
      finc50 |  -8.128784   15.91447    -0.51   0.610    -39.54421    23.28665
      finc75 |  -17.57921    17.2789    -1.02   0.310    -51.68804    16.52963
     finc100 |   -6.74559   17.24617    -0.39   0.696    -40.78983    27.29865
     finc101 |  -28.34407   17.10783    -1.66   0.099     -62.1152     5.42707
    wealth89 |  -.0026918   .0119309    -0.23   0.822    -.0262435      .02086
      prftshr |   15.80791   8.356266     1.89   0.060    -.6874979    32.30332
        _cons |   134.1161    58.1316     2.31   0.022     19.36333    248.8688
```

```
------------------------------------------------------------------------------

. di _b[_cons] + _b[age]*60 + _b[educ]*12 + _b[female] + _b[finc50] + _b[wealth89
38.374791

. di _b[_cons] + _b[age]*60 + _b[educ]*12 + _b[female] + _b[finc50] + _b[wealth89
50.422517
```

For later use, the predicted *pctstck* for the person described in the problem, with *choice* = 0 is about 38.37. With choice, it is roughly 50.42.

c. The ordered probit estimates follow, including commands that provide the predictions for *pctstck* with and without choice:

```
. oprobit pctstck choice age educ female black married finc25-finc101 wealth89
          prftshr

Iteration 0:   log likelihood = -212.37031
Iteration 1:   log likelihood =  -202.0094
Iteration 2:   log likelihood =  -201.9865
Iteration 3:   log likelihood =  -201.9865

Ordered probit regression                         Number of obs   =        194
                                                  LR chi2(14)     =      20.77
                                                  Prob > chi2     =     0.1077
Log likelihood =  -201.9865                       Pseudo R2       =     0.0489

------------------------------------------------------------------------------
     pctstck |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      choice |    .371171   .1841121     2.02   0.044      .010318    .7320241
         age |  -.0500516   .0226063    -2.21   0.027    -.0943591    -.005744
        educ |   .0261382   .0352561     0.74   0.458    -.0429626    .0952389
      female |   .0455642    .206004     0.22   0.825    -.3581963    .4493246
       black |   .0933923   .2820403     0.33   0.741    -.4593965    .6461811
     married |   .0935981   .2332114     0.40   0.688    -.3634878     .550684
      finc25 |  -.5784299    .423162    -1.37   0.172    -1.407812    .2509524
      finc35 |  -.1346721   .4305242    -0.31   0.754    -.9784841    .7091399
      finc50 |  -.2620401   .4265936    -0.61   0.539    -1.098148    .5740681
      finc75 |  -.5662312   .4780035    -1.18   0.236    -1.503101    .3706385
     finc100 |  -.2278963   .4685942    -0.49   0.627    -1.146324    .6905316
     finc101 |  -.8641109   .5291111    -1.63   0.102     -1.90115    .1729279
    wealth89 |  -.0000956   .0003737    -0.26   0.798    -.0008279    .0006368
     prftshr |   .4817182   .2161233     2.23   0.026     .0581243     .905312
-------------+----------------------------------------------------------------
       /cut1 |  -3.087373   1.623765                     -6.269894    .0951479
       /cut2 |  -2.053553   1.618611                     -5.225972    1.118865
------------------------------------------------------------------------------

. di b[age]*60 + _b[educ]*12 + _b[female] + _b[finc50] + _b[wealth89]*150
-2.9202491

. di normal(_b[/cut2] + 2.9202491) - normal(_b[/cut1] +2.9202491)
.37330935
```

```
. di 1 - normal(_b[/cut2] + 2.9202491)
.19305438

. di 50*.37330935 + 100*.19305438
37.970906

. di _b[age]*60 + _b[educ]*12 + _b[female] + _b[finc50] + _b[wealth89]*150
      + _b[choice]
-2.5490781

. di normal(_b[/cut2] + 2.5490781) - normal(_b[/cut1] + 2.5490781)
.39469838

. di 1 - normal(_b[/cut2] + 2.5490781)
.31011489

. di 50*.39469838 + 100*.31011489
50.746408

. di 50.75 - 37.97
12.78
```

Using ordered probit, the effect of having choice for this person is about 12.8 percentage points more invested in the stock market, which is pretty similar to the 12.1 points obtained with the linear model.

d. We can compute an *R*-squared for the ordered probit model by using the squared correlation between the predicted $pctstck_i$ and the actual. The following Stata session does this, after using the `oprobit` command. The squared correlation for ordered probit is about .097, which is actually slightly below the linear model *R*-squared, .098. The correlation between the fitted values for the linear and OP models is very high: .998.

```
. qui oprobit pctstck choice age educ female black married finc25-finc101 wealth89

. predict p1hat p2hat p3hat
(option pr assumed; predicted probabilities)
(32 missing values generated)

. sum p1hat p2hat p3hat

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       p1hat |       194     .331408    .1327901   .0685269   .8053644
       p2hat |       194    .3701685    .0321855   .1655734   .3947809
       p3hat |       194    .2984236    .1245914   .0290621   .6747374

. gen pctstck_op = 50*p2hat + 100*p3hat
```

```
(32 missing values generated)

. corr pctstck pctstck_op
(obs=194)

             |  pctstck pctstc~p
-------------+------------------
     pctstck |   1.0000
  pctstck_op |   0.3119   1.0000


. di .312^2
.097344

. qui reg pctstck choice age educ female black married finc25-finc101 wealth89

. predict pctstck_lin
(option xb assumed; fitted values)
(32 missing values generated)

. corr pctstck_lin pctstck_op
(obs=194)

             | pctstc~n pctstc~p
-------------+------------------
 pctstck_lin |   1.0000
  pctstck_op |   0.9980   1.0000
```

16.3. a. We can derive the response probabilities from the latent variable formulation in (16.21) and the rule in (16.22).

$$\begin{aligned}\exp(-\mathbf{x}_1\boldsymbol{\delta})y^* &= \exp(-\mathbf{x}_1\boldsymbol{\delta})\mathbf{x}\boldsymbol{\beta} + \exp(-\mathbf{x}_1\boldsymbol{\delta})e \\ &= \exp(-\mathbf{x}_1\boldsymbol{\delta})\mathbf{x}\boldsymbol{\beta} + a\end{aligned}$$

where

$$a|\mathbf{x} \sim Normal(0,1).$$

Now $\alpha_j < y^* \leq \alpha_{j+1}$ if and only if $\exp(-\mathbf{x}_1\boldsymbol{\delta})\alpha_j < \exp(-\mathbf{x}_1\boldsymbol{\delta})y^* \leq \exp(-\mathbf{x}_1\boldsymbol{\delta})\alpha_{j+1}$, and so

$$\begin{aligned}\mathrm{P}(y = j|\mathbf{x}) &= P[\exp(-\mathbf{x}_1\boldsymbol{\delta})\alpha_j < \exp(-\mathbf{x}_1\boldsymbol{\delta})\mathbf{x}\boldsymbol{\beta} + a \leq \exp(-\mathbf{x}_1\boldsymbol{\delta})\alpha_{j+1}|\mathbf{x}] \\ &= P[\exp(-\mathbf{x}_1\boldsymbol{\delta})\alpha_j - \exp(-\mathbf{x}_1\boldsymbol{\delta})\mathbf{x}\boldsymbol{\beta} < a \leq \exp(-\mathbf{x}_1\boldsymbol{\delta})\alpha_{j+1} - \exp(-\mathbf{x}_1\boldsymbol{\delta})\mathbf{x}\boldsymbol{\beta}|\mathbf{x}] \\ &= \Phi[\exp(-\mathbf{x}_1\boldsymbol{\delta})(\alpha_{j+1} - \mathbf{x}\boldsymbol{\beta})] - \Phi[\exp(-\mathbf{x}_1\boldsymbol{\delta})(\alpha_j - \mathbf{x}\boldsymbol{\beta})].\end{aligned}$$

A similar argument holds at $j = 0$ and $j = J$. Therefore, as described in the text, the response probabilities for the heteroskedastic ordered probit are of the same form as usual ordered probit

but with $\alpha_j - \mathbf{x}\boldsymbol{\beta}$ everywhere replaced with $\exp(-\mathbf{x}_1\boldsymbol{\delta})(\alpha_j - \mathbf{x}\boldsymbol{\beta})$.

b. We can obtain a useful VAT by applying the score statistic – just as in the binary probit case. The score of the log likelihood with respect to $(\boldsymbol{\alpha}', \boldsymbol{\beta}')'$, evaluated at $\boldsymbol{\delta} = \mathbf{0}$, is easily seen to just be the usual score for ordered probit. For $0 < j < J$, the score of the response probability with respect to $\boldsymbol{\delta}$ evaluated at $\boldsymbol{\delta} = \mathbf{0}$ is

$$-1[y_i = j]\frac{\mathbf{x}_{i1}'[(\alpha_{j+1} - \mathbf{x}_i\boldsymbol{\beta})\phi(\alpha_{j+1} - \mathbf{x}_i\boldsymbol{\beta}) - (\alpha_j - \mathbf{x}_i\boldsymbol{\beta})\phi(\alpha_j - \mathbf{x}_i\boldsymbol{\beta})]}{\Phi(\alpha_{j+1} - \mathbf{x}_i\boldsymbol{\beta}) - \Phi(\alpha_j - \mathbf{x}_i\boldsymbol{\beta})}.$$

For $j = 0$ and $j = J$ we have

$$-1[y_i = 0]\frac{\mathbf{x}_{i1}'[(\alpha_1 - \mathbf{x}_i\boldsymbol{\beta})\phi(\alpha_1 - \mathbf{x}_i\boldsymbol{\beta})]}{\Phi(\alpha_1 - \mathbf{x}_i\boldsymbol{\beta})}$$
$$1[y_i = J]\frac{\mathbf{x}_{i1}'[(\alpha_J - \mathbf{x}_i\boldsymbol{\beta})\phi(\alpha_J - \mathbf{x}_i\boldsymbol{\beta})]}{1 - \Phi(\alpha_J - \mathbf{x}_i\boldsymbol{\beta})}$$

It is easily seen that these are identical to the scores that would be obtained by adding

$$-\mathbf{x}_{i1}(\alpha_j - \mathbf{x}_i\boldsymbol{\beta})$$

as a set of explanatory variables to the usual OP model and testing their joint significance. In practice, we would replace the α_j and $\boldsymbol{\beta}$ with the MLEs from the original OP problem.

d. The ASF can be written as

$$\begin{aligned}\text{ASF}(\mathbf{x}) &= \text{E}_{e_i}(1[\alpha_1 - \mathbf{x}\boldsymbol{\beta} < e_i \leq \alpha_2 - \mathbf{x}\boldsymbol{\beta}]) \\ &= \text{P}(\alpha_1 - \mathbf{x}\boldsymbol{\beta} < e_i \leq \alpha_2 - \mathbf{x}\boldsymbol{\beta}) \\ &= F_e(\alpha_2 - \mathbf{x}\boldsymbol{\beta}) - F_e(\alpha_1 - \mathbf{x}\boldsymbol{\beta})\end{aligned}$$

where $F_e(\cdot)$ is the cdf of e_i. We do not know F_e because it depends on the distribution of $\mathbf{x}_1$: we have specified $\text{D}(e_i|\mathbf{x}_i) = \text{D}(e_i|\mathbf{x}_{i1})$, not $\text{D}(e_i)$.

e. From iterated expectations we can write

$$\text{ASF}(\mathbf{x}) = \text{E}_{\mathbf{x}_{i1}}\{\text{E}(1[\alpha_1 - \mathbf{x}\boldsymbol{\beta} < e_i \leq \alpha_2 - \mathbf{x}\boldsymbol{\beta}]|\mathbf{x}_{i1})\}$$

and the conditional expectation is a conditional probability:

$$\begin{aligned}\mathrm{E}(1[\alpha_1 - \mathbf{x}\boldsymbol{\beta} < e_i \leq \alpha_2 - \mathbf{x}\boldsymbol{\beta}]|\mathbf{x}_{i1}) &= \mathrm{P}(\alpha_1 - \mathbf{x}\boldsymbol{\beta} < e_i \leq \alpha_2 - \mathbf{x}\boldsymbol{\beta}|\mathbf{x}_{i1}) \\ &= \mathrm{P}[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_1 - \mathbf{x}\boldsymbol{\beta}) < a_i \leq \exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_2 - \mathbf{x}\boldsymbol{\beta})|\mathbf{x}_{i1}] \\ &= \Phi[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_2 - \mathbf{x}\boldsymbol{\beta})] - \Phi[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_1 - \mathbf{x}\boldsymbol{\beta})].\end{aligned}$$

Therefore,

$$\mathrm{ASF}(\mathbf{x}) = \mathrm{E}_{\mathbf{x}_{i1}}\{\Phi[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_2 - \mathbf{x}\boldsymbol{\beta})] - \Phi[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_1 - \mathbf{x}\boldsymbol{\beta})]\}.$$

By the law of large numbers, a consistent estimator is

$$N^{-1}\sum_{i=1}^{N}\{\Phi[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_2 - \mathbf{x}\boldsymbol{\beta})] - \Phi[\exp(-\mathbf{x}_{i1}\boldsymbol{\delta})(\alpha_1 - \mathbf{x}\boldsymbol{\beta})]\}$$

and, by Lemma 12.1, consistency is preserved if we insert the (consistent) MLES:

$$\widehat{\mathrm{ASF}}(\mathbf{x}) = N^{-1}\sum_{i=1}^{N}\{\Phi[\exp(-\mathbf{x}_{i1}\hat{\boldsymbol{\delta}})(\hat{\alpha}_2 - \mathbf{x}\hat{\boldsymbol{\beta}})] - \Phi[\exp(-\mathbf{x}_{i1}\hat{\boldsymbol{\delta}})(\hat{\alpha}_1 - \mathbf{x}\hat{\boldsymbol{\beta}})]\}.$$

The APEs are estimated by taking derivatives or changes with respect to elements of $\mathbf{x}$ in $\widehat{\mathrm{ASF}}(\mathbf{x})$.

16.4. a. The results of the ordered probit estimation using *invest* as the response variable are given below. Every statistic is identical to when *pctstck* is used as the response variable. This is as it should be, as only the order of the outcomes matter – not the magnitudes.

```
. gen invest =  0 if pctstck == 0
(148 missing values generated)

. replace invest = 1 if pctstck == 50
(85 real changes made)

. replace invest = 2 if pctstck == 100
(63 real changes made)

. oprobit invest choice age educ female black married finc25-finc101 wealth89

Ordered probit regression                         Number of obs   =        194
                                                  LR chi2(14)     =      20.77
                                                  Prob > chi2     =     0.1077
```

```
Log likelihood =  -201.9865                     Pseudo R2       =     0.0489

------------------------------------------------------------------------------
      invest |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      choice |    .371171   .1841121     2.02   0.044      .010318    .7320241
         age |  -.0500516   .0226063    -2.21   0.027    -.0943591    -.005744
        educ |   .0261382   .0352561     0.74   0.458    -.0429626    .0952389
      female |   .0455642    .206004     0.22   0.825    -.3581963    .4493246
       black |   .0933923   .2820403     0.33   0.741    -.4593965    .6461811
     married |   .0935981   .2332114     0.40   0.688    -.3634878     .550684
      finc25 |  -.5784299    .423162    -1.37   0.172    -1.407812    .2509524
      finc35 |  -.1346721   .4305242    -0.31   0.754    -.9784841    .7091399
      finc50 |  -.2620401   .4265936    -0.61   0.539    -1.098148    .5740681
      finc75 |  -.5662312   .4780035    -1.18   0.236    -1.503101    .3706385
     finc100 |  -.2278963   .4685942    -0.49   0.627    -1.146324    .6905316
     finc101 |  -.8641109   .5291111    -1.63   0.102     -1.90115    .1729279
    wealth89 |  -.0000956   .0003737    -0.26   0.798    -.0008279    .0006368
     prftshr |   .4817182   .2161233     2.23   0.026     .0581243     .905312
-------------+----------------------------------------------------------------
       /cut1 |  -3.087373   1.623765                     -6.269894    .0951479
       /cut2 |  -2.053553   1.618611                     -5.225972    1.118865
------------------------------------------------------------------------------
```

b. One quantity that would change is the estimated expected value, something pretty obvious because of the rescaling. In particular,

$$\hat{\mathrm{E}}(invest|\mathbf{x}) = \hat{\mathrm{P}}(invest = 1|\mathbf{x}) + 2 \cdot \hat{\mathrm{P}}(invest = 2|\mathbf{x})$$

whereas

$$\begin{aligned}\hat{\mathrm{E}}(pctstck|\mathbf{x}) &= 50 \cdot \hat{\mathrm{P}}(pctstck = 50|\mathbf{x}) + 100 \cdot \hat{\mathrm{P}}(pctstck = 100|\mathbf{x}) \\ &= 50 \cdot \hat{\mathrm{P}}(invest = 1|\mathbf{x}) + 100 \cdot \hat{\mathrm{P}}(invest = 2|\mathbf{x}) \\ &= 50 \cdot \hat{\mathrm{E}}(invest|\mathbf{x}).\end{aligned}$$

Because $pctstck = 50 \cdot invest$, $\mathrm{E}(pctstck|\mathbf{x}) = 50 \cdot \mathrm{E}(invest|\mathbf{x})$.

16.5. a. We have

$$\mathrm{D}(y_2|\mathbf{z}) = \mathrm{Normal}(\mathbf{z}\boldsymbol{\delta}_2, \exp(\mathbf{z}\boldsymbol{\xi}_2))$$

which means we should use maximum likelihood to estimate $\boldsymbol{\delta}_2$ and $\boldsymbol{\xi}_2$. In fact, $\hat{\boldsymbol{\delta}}_2$ is asymptotically equivalent to a weighted least squares estimator using weights $\exp(-\mathbf{z}_i\boldsymbol{\xi}_2)$.

b. By assumption, (u_1, e_2) is independent of $\mathbf{z}$ and so $\mathrm{D}(u_1|e_2, \mathbf{z}) = \mathrm{D}(u_1|e_2)$. Because

(u_1, e_2) is bivariate normal with zero mean, we can always write

$$u_1 = \theta_1 e_2 + e_1$$

where

$$e_1|e_2 \sim \text{Normal}(0, \tau_1^2)$$

where $\tau_1^2 = \sigma_1^2 - \theta_1^2$, where $\sigma_1^2 = \text{Var}(u_1)$. This is necessarily the distribution conditional on $\mathbf{z}$, too.

We can write

$$y_2 = \mathbf{z}\boldsymbol{\delta}_2 + \exp(\mathbf{z}\boldsymbol{\xi}_2/2)e_2,$$

which shows that y_2 is a function of $(\mathbf{z}, e_2)$. Therefore, e_1 is independent of y_2, too.

c. We can use the latent variable formulation in equation (16.30) and insert $u_1 = \theta_1 e_2 + e_1$:

$$\begin{aligned} y_1^* &= \mathbf{z}_1\boldsymbol{\delta}_1 + \gamma_1 y_2 + \theta_1 e_2 + e_1 \\ e_1|\mathbf{z}, y_2 &\sim \text{Normal}(0, \tau_1^2) \end{aligned}$$

To obtain an error with a unit variance, we divide by τ_1:

$$(y_1^*/\tau_1) = \mathbf{z}_1(\boldsymbol{\delta}_1/\tau_1) + (\gamma_1/\tau_1)y_2 + (\theta_1/\tau_1)e_2 + (e_1/\tau_1)$$

and then the cut parameters also get divided by τ_1. For example, $\alpha_j < y_1^* \leq \alpha_{j+1}$ if and only if

$$(\alpha_j/\tau_1) < (y_1^*/\tau_1) \leq (\alpha_{j+1}/\tau_1)$$

Therefore, if we run ordered probit of

$$y_1 \text{ on } \mathbf{z}_1, y_2, e_2$$

we consistently estimate all parameters multiplied by $1/\tau_1$. Of course we do not observe e_2, but we can replace it with estimates because $e_2 = \exp(-\mathbf{z}\boldsymbol{\xi}_2/2)v_2$.

The two-step approach is to estimate $\boldsymbol{\delta}_2$ and $\boldsymbol{\xi}_2$ by the MLE from part a. Then create

$$\hat{v}_{i2} = y_{i2} - \mathbf{z}_i\hat{\boldsymbol{\delta}}_2$$
$$\hat{e}_{i2} = \exp(-\mathbf{z}_i\hat{\boldsymbol{\xi}}_2/2)\hat{v}_{i2}$$

In the second step, estimate the scaled coefficients by OP of

$$y_{i1} \text{ on } \mathbf{z}_{i1}, y_{i2}, \hat{e}_{i2}.$$

Let $\hat{\alpha}_{\tau j}, j = 1,2,\ldots,J$, $\hat{\boldsymbol{\delta}}_{\tau 1}$, $\hat{\gamma}_{\tau 1}$, and $\hat{\theta}_{\tau 1}$ be the scaled coefficients.

Incidentally, a simple test of the null that y_2 is exogenous is the usual MLE *t* statistic for $\hat{\theta}_{\tau 1}$.

d. We can obtain the ASF by averaging out e_2 in response probabilities of the form

$$\Phi(\alpha_{\tau,j+1} - \mathbf{z}_1\boldsymbol{\delta}_{\tau 1} - \gamma_{\tau 1}y_2 - \theta_{\tau 1}e_2) - \Phi(\alpha_{\tau j} - \mathbf{z}_1\boldsymbol{\delta}_{\tau 1} - \gamma_{\tau 1}y_2 - \theta_{\tau 1}e_2)$$

(for $0 < j < J$). A consistent estimator of the ASF is

$$\widehat{\text{ASF}}(\mathbf{z}_1, y_2) = N^{-1}\sum_{i=1}^{N}[\Phi(\hat{\alpha}_{\tau,j+1} - \mathbf{z}_1\hat{\boldsymbol{\delta}}_{\tau 1} - \hat{\gamma}_{\tau 1}y_2 - \hat{\theta}_{\tau 1}\hat{e}_{i2}) - \Phi(\hat{\alpha}_{\tau j} - \mathbf{z}_1\hat{\boldsymbol{\delta}}_{\tau 1} - \hat{\gamma}_{\tau 1}y_2 - \hat{\theta}_{\tau 1}\hat{e}_{i2})].$$

and, as usual, we can compute derivatives or changes with respect to the elements of $(\mathbf{z}_1, y_2)$.

e. Now the normal MLE is just applied to

$$\log(y_2)|\mathbf{z} \sim Normal(\mathbf{z}\boldsymbol{\delta}_2, \exp(\mathbf{z}\boldsymbol{\xi}_2))$$

and $\hat{v}_{i2} = \log(y_{i2}) - \mathbf{z}_i\hat{\boldsymbol{\delta}}_2$.

f. Without a distributional assumption for $\text{D}(u_2|e_2)$, allowing for endogeneity is tricky. We would still assume that (u_1, e_2) is independent of $\mathbf{z}$. We could just *assume* we can write $u_2 = \theta_1 e_2 + e_1$ where

$$\text{D}(e_1|e_2, \mathbf{z}) \sim Normal(0, \tau_1^2).$$

Then the two-step method from part c, with ASF estimated as in part d, applies but where $\hat{\boldsymbol{\delta}}_2$

and $\hat{\boldsymbol{\xi}}_2$ are obtained from a suitable estimation procedure. It could be a several step procedure or, more conveniently, a single step based on the normal quasi-MLE. That is, we act *as if*

$$\mathrm{D}(y_2|\mathbf{z}) = Normal(\exp(\mathbf{z}\boldsymbol{\delta}_2), \exp(\mathbf{z}\boldsymbol{\xi}_2))$$

even though it cannot be literally true. As the results of Gourieroux, Monfort, and Trognon (1984a) show, this estimator is generall consistent and $\sqrt{N}$-asymptotically normal. Then

$$\begin{aligned}\hat{v}_{i2} &= y_{i2} - \exp(\mathbf{z}_i\hat{\boldsymbol{\delta}}_2)\\ \hat{e}_{i2} &= \exp(-\mathbf{z}_i\hat{\boldsymbol{\xi}}_2/2)\hat{v}_{i2}\end{aligned}$$

and the steps in part c can be followed.

A way to make the method more flexible is to add polynomials in $\hat{e}_{i2}$ to the second-stage OP. For example, if we just add a square, the ASF would be estimated as

$$\begin{aligned}\widehat{\mathrm{ASF}}(\mathbf{z}_1, y_2) = N^{-1}\sum_{i=1}^{N}[&\Phi(\hat{\alpha}_{\tau,j+1} - \mathbf{z}_1\hat{\boldsymbol{\delta}}_{\tau 1} - \hat{\gamma}_{\tau 1}y_2 - \hat{\theta}_{\tau 1}\hat{e}_{i2} - \hat{\eta}_{\tau 1}\hat{e}_{i2}^2)\\ &- \Phi(\hat{\alpha}_{\tau j} - \mathbf{z}_1\hat{\boldsymbol{\delta}}_{\tau 1} - \hat{\gamma}_{\tau 1}y_2 - \hat{\theta}_{\tau 1}\hat{e}_{i2} - \hat{\eta}_{\tau 1}\hat{e}_{i2}^2)].\end{aligned}$$

where $\hat{\eta}_{\tau 1}$ is the estimate on the quadratic term.

16.6. This problem is similar to that treated in Papke and Wooldridge (2008) for a binary or fractional response variable. Using the expression for c_{i1} we can write

$$\begin{aligned}y_{it1}^* &= \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \gamma_1 y_{it2} + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + a_{i1} + u_{it1}\\ &\equiv \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \gamma_1 y_{it2} + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + v_{it1}\end{aligned}$$

where $v_{it1} \equiv a_{i1} + u_{it1}$. Now we need to make some joint distributional assumptions concerning v_{it1} and v_{it2}, where

$$y_{it2} = \mathbf{z}_{it}\boldsymbol{\delta}_2 + \psi_2 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_2 + v_{it2}$$

Given the marginal normal distributions assumed in the problem, it is a small step to assuming

$$v_{it1} = \theta_1 v_{it2} + e_{it1}$$

where

$$\mathrm{D}(e_{it1}|v_{it2}, \mathbf{z}_i) = \mathrm{Normal}(0, \tau_1^2).$$

We could allow θ_1, and even τ_1^2, to depend on t.

Now, we can write a control function equation (in latent variable form) as

$$y_{it1}^* = \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \gamma_1 y_{it2} + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + \theta_1 v_{it2} + e_{it1};$$

given the conditional normality assumption for e_{it1}, and so using pooled probit of

$$y_{it1} \text{ on } \mathbf{z}_{it1}, y_{it2}, 1, \bar{\mathbf{z}}_i, v_{it2}, \ t = 1, \ldots, T; i = 1, \ldots, N$$

consistently estimates all parameters – including the cut parameters – multiplied by $1/\tau_1$. The two-step method is then (1) Estimate $\boldsymbol{\delta}_2$, ψ_2, and $\boldsymbol{\xi}_2$ by pooled OLS of

$$y_{it2} \text{ on } \mathbf{z}_{it}, 1, \bar{\mathbf{z}}_i, \ t = 1, \ldots, T; i = 1, \ldots, N.$$

This is equivalent to fixed effects estimation of $\boldsymbol{\delta}_2$. Obtain the residuals, $\hat{v}_{it2}$. (2) Do pooled OP of

$$y_{it1} \text{ on } \mathbf{z}_{it1}, y_{it2}, 1, \bar{\mathbf{z}}_i, \hat{v}_{it2}, \ t = 1, \ldots, T; i = 1, \ldots, N$$

to obtain $\hat{\boldsymbol{\delta}}_{g1}$, $\hat{\gamma}_{g1}$, and so on. A simple extension is to interact $\hat{v}_{it2}$ with time dummies to allow the regression of u_{it1} on v_{it2} to change over time.

b. Define a dummy variable $w_{ij} = 1[y_{i1} = j]$. Then

$$\begin{aligned} w_{ij} &= 1[\alpha_j < y_{i1}^* \leq \alpha_{j1}] \\ &= 1[\alpha_j < \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \gamma_1 y_{it2} + c_{i1} + u_{it1} \leq \alpha_{j+1}]. \end{aligned}$$

The ASF for w_{ij} is obtained by computing the expected value of the right hand side with respect to the unobservable $c_{i1} + u_{it1}$ at specific values $(\mathbf{z}_{t1}, y_{t2})$:

$$\text{ASF}(\mathbf{z}_{t1}, y_{t2}) = E_{r_{it1}}\{1[\alpha_j < \mathbf{z}_{t1}\boldsymbol{\delta}_1 + \gamma_1 y_{t2} + r_{it1} \leq \alpha_{j+1}]\}$$

where $r_{it1} \equiv c_{i1} + u_{it1}$. Note that $r_{it1} = \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + v_{it1}$ and so we can compute the ASF by taking the average over $(\bar{\mathbf{z}}_i, v_{it1})$:

$$\begin{aligned}\text{ASF}(\mathbf{z}_{t1}, y_{t2}) &= E_{(\bar{\mathbf{z}}_i, v_{it1})}\{1[\alpha_j < \mathbf{z}_{t1}\boldsymbol{\delta}_1 + \gamma_1 y_{t2} + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + v_{it1} \leq \alpha_{j+1}]\} \\ &= E_{(\bar{\mathbf{z}}_i, v_{it2}, e_{it1})}\{1[\alpha_j < \mathbf{z}_{t1}\boldsymbol{\delta}_1 + \gamma_1 y_{t2} + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + \theta_1 v_{it2} + e_{it1} \leq \alpha_{j+1}]\}\end{aligned}$$

Now we can apply iterated expectations. First find $\text{E}(\cdot|\bar{\mathbf{z}}_i, v_{it2})$ and then average out $(\bar{\mathbf{z}}_i, v_{it2})$. Now

$$\begin{aligned}&\text{E}(1[\alpha_j < \mathbf{z}_{t1}\boldsymbol{\delta}_1 + \gamma_1 y_{t2} + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + \theta_1 v_{it2} + e_{it1} \leq \alpha_{j+1}]|\bar{\mathbf{z}}_i, v_{it2}) \\ &= \Phi(\alpha_{g,j+1} - \mathbf{z}_{t1}\boldsymbol{\delta}_{g1} - \gamma_{g1} y_{t2} - \psi_{g1} - \bar{\mathbf{z}}_i\boldsymbol{\xi}_{g1} - \theta_{g1} v_{it2}) \\ &\quad - \Phi(\alpha_{g,j} - \mathbf{z}_{t1}\boldsymbol{\delta}_{g1} - \gamma_{g1} y_{t2} - \psi_{g1} - \bar{\mathbf{z}}_i\boldsymbol{\xi}_{g1} - \theta_{g1} v_{it2})\end{aligned}$$

where "g" denotes divided by τ_1. We use the fact that $\text{D}(e_{it1}|v_{it2}, \mathbf{z}_i) = \text{Normal}(0, \tau_1^2)$. It follows now by iterated expectations that

$$\begin{aligned}\text{ASF}(\mathbf{z}_{t1}, y_{t2}) = \text{E}_{(\bar{\mathbf{z}}_i, v_{it2})}&[\Phi(\alpha_{g,j+1} - \mathbf{z}_{t1}\boldsymbol{\delta}_{g1} - \gamma_{g1} y_{t2} - \psi_{g1} - \bar{\mathbf{z}}_i\boldsymbol{\xi}_{g1} - \theta_{g1} v_{it2}) \\ &- \Phi(\alpha_{g,j} - \mathbf{z}_{t1}\boldsymbol{\delta}_{g1} - \gamma_{g1} y_{t2} - \psi_{g1} - \bar{\mathbf{z}}_i\boldsymbol{\xi}_{g1} - \theta_{g1} v_{it2})].\end{aligned}$$

c. To estimate the ASF, we plug in estimates and use a sample average:

$$\begin{aligned}\widehat{\text{ASF}}(\mathbf{z}_{t1}, y_{t2}) = N^{-1}\sum_{i=1}^{n}&[\Phi(\hat{\alpha}_{g,j+1} - \mathbf{z}_{t1}\hat{\boldsymbol{\delta}}_{g1} - \hat{\gamma}_{g1} y_{t2} - \hat{\psi}_{g1} - \bar{\mathbf{z}}_i\hat{\boldsymbol{\xi}}_{g1} - \hat{\theta}_{g1}\hat{v}_{it2}) \\ &- \Phi(\hat{\alpha}_{g,j} - \mathbf{z}_{t1}\hat{\boldsymbol{\delta}}_{g1} - \hat{\gamma}_{g1} y_{t2} - \hat{\psi}_{g1} - \bar{\mathbf{z}}_i\hat{\boldsymbol{\xi}}_{g1} - \hat{\theta}_{g1}\hat{v}_{it2})].\end{aligned}$$

As usual, the estimated APEs are derivatives or changes with respect to $(\mathbf{z}_{t1}, y_{t2})$. To get valid standard errors, we can use Problem 12.17 or the panel bootstrap – where both estimation steps are carried out with each resampling.

d. The two-step control function procedure does not require any assumptions about the relationship between u_{it1} and v_{ir2} for $t \neq r$. In other words, while adding v_{it2} as a control

function renders y_{it2} contemporaneously exogenous in the estimating equation – e_{it1} is independent of y_{it2} (and $\mathbf{z}_i$) – $\{y_{it2}\}$ is not generally strictly exogenous. An important implication is that we should not apply a method such as generalized estimating equations in the second stage.

A method that would render $\{y_{it2}\}$ strictly exogenous would be to project v_{it1} on the entire history $\{v_{ir2}, r = 1, \ldots, T\}$. There are assumptions under which the projection depends only on v_{it2} and the time average, $\bar{v}_{i2}$. So, we could write

$$v_{it1} = \theta_1 v_{it2} + \eta_1 \bar{v}_{i2} + e_{it1}$$

and assume e_{it1} is independent of $\mathbf{v}_{i2} = (v_{i12}, \ldots, v_{iT2})'$. Then e_{it1} would be uncorrelated independent of $(\mathbf{z}_i, \mathbf{y}_{i2})$ (under the other maintained assumptions). So, at each time period, $(\hat{v}_{it2}, \hat{\bar{v}}_{i2})$ can be added to the ordered probit – that is, we apply the Mundlak device to the reduced form residuals. In addition to pooled OP, one could use a GEE-like procedure in the second stage.

More flexibility would be gotten by using the more general Chamberlain formulation:

$$v_{it1} = \mathbf{v}_{i2}' \boldsymbol{\theta}_{t1} + e_{it1}$$

where $\boldsymbol{\theta}_{t1}$ is $T \times 1$ for each t. Then in each time period include $\hat{\mathbf{v}}_{i2}'$ as a set of regressors interacted with time-period dummies.

Solutions to Chapter 17 Problems

17.1. a. No. Because $\log(1) = 0$ and $\log(\cdot)$ is strictly increasing, $P[\log(1 + y) = 0] = P(y = 0) > 0$. Of course, $\log(1 + y)$ increases much more slowly than y, and so one could use $\log(1 + y)$ to reduce the influence of "unusually" large observations y_i in linear regression. Also, remembering that the type I Tobit can be obtained from a latent variable model, the transformation $\log(1 + y)$ might make the normality and homoskedasticity assumptions in the latent variable formulation more plausible.

b. We can just use ordinary least squares. OLS will be consistent for $\boldsymbol{\beta}$ (and even conditionally unbiased). Our inference should be made robust to heteroskedasticity because the restriction $r \geq -\mathbf{x}\boldsymbol{\beta}$ needs to hold, meaning r cannot be independent of $\mathbf{x}$ (unless we restrict the range of r or $\mathbf{x}$ somewhat arbitrarily).

c. Exponentiate and subtract one to get

$$y = \exp(\mathbf{x}\boldsymbol{\beta} + r) - 1 = y = \exp(\mathbf{x}\boldsymbol{\beta})\exp(r) - 1$$

Now take the expectation conditional on $\mathbf{x}$:

$$E(y|\mathbf{x}) = \exp(\mathbf{x}\boldsymbol{\beta})E[\exp(r)|\mathbf{x}] - 1.$$

If we assume r is independent of $\mathbf{x}$ then $E[\exp(r)|\mathbf{x}] = E[\exp(r)] = \eta$, and so

$$E(y|\mathbf{x}) = \eta\exp(\mathbf{x}\boldsymbol{\beta}) - 1.$$

d. Because $\eta = E[\exp(r)]$, an unbiased and consistent estimator of η would be

$$N^{-1}\sum_{i=1}^{N}\exp(r_i),$$

if we observed the random sample of errors, $\{r_i : i = 1, 2, \ldots, N\}$. Instead, we follow Duan's (1983) "smearing" approach (which is really just a method of moments approach) and replace

the errors with the OLS residuals, $\hat{r}_i$, from the regression $\log(1 + y_i)$ on $\mathbf{x}_i$. Then a consistent estimator of η is

$$\hat{\eta} = N^{-1} \sum_{i=1}^{N} \exp(\hat{r}_i),$$

which is guaranteed to be greater than one by Jensen's inequality. Note η is also greater than unity by Jensen's:

$$\eta = \mathrm{E}[\exp(r)] > \exp[\mathrm{E}(r)] = \exp(0) = 1.$$

e. The estimated conditional mean function is simply

$$\hat{\mathrm{E}}(y|\mathbf{x}) = \hat{\eta} \exp(\mathbf{x}\hat{\boldsymbol{\beta}}) - 1.$$

It is not guaranteed to be nonnegative because the estimates $\hat{\boldsymbol{\beta}}$ and $\hat{\eta}$ have not been chosen to ensure nonnegativity. It is possible that, for some vectors $\mathbf{x}$, $\hat{\eta} \exp(\mathbf{x}\hat{\boldsymbol{\beta}}) < 1$.

f. The Stata output follows. The estimated η is $\hat{\eta} = 17.18$, which is much higher than unity. None of the fitted values are negative; they range from about $.061$ to $45,202$. The largest prediction is almost 10 times above the largest observed hours in the data set, and the average of the fitted values, $3,166$, is much too high: the average of actual hours is 740.6. Therefore, for predicting *hours*, using $\log(1 + hours)$ in a linear regression is not very appealing.

```
. gen lhoursp1 = log(1 + hours)

. reg lhoursp1 nwifeinc educ exper expersq age kidslt6 kidsge6, robust

Linear regression                                      Number of obs =     753
                                                       F(  7,   745) =   73.12
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.2950
                                                       Root MSE      =  2.9367

------------------------------------------------------------------------------
             |               Robust
    lhoursp1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    nwifeinc |  -.0228321   .0098273    -2.32   0.020    -.0421247   -.0035395
```

```
        educ |   .2271644   .0507032     4.48   0.000     .1276262    .3267027
       exper |   .2968677   .0407256     7.29   0.000     .2169171    .3768182
     expersq |  -.0043383   .0013579    -3.19   0.001     -.007004   -.0016726
         age |   -.122754   .0163732    -7.50   0.000    -.1548971   -.0906109
     kidslt6 |  -1.991432   .2110337    -9.44   0.000    -2.405724   -1.577141
     kidsge6 |   .0372724   .0917873     0.41   0.685    -.1429201    .2174649
       _cons |   4.833966   1.050092     4.60   0.000     2.772473    6.895458
------------------------------------------------------------------------------

. predict xbhat
(option xb assumed; fitted values)

. predict rhat, resid

. gen exprhat = exp(rhat)

. sum exprhat

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     exprhat |       753    17.17622     69.2013   .0012194   1045.183

. gen hourshat = 17.17622*exp(xbhat) - 1

. sum hours hourshat

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       hours |       753    740.5764    871.3142          0       4950
    hourshat |       753    3166.422    5164.107    .061139   45202.41
```

g. The *R*-squared is computed in the Stata output that follows. It is about .159, which is substantially below not only the Tobit *R*-squared, .275, but also the linear regression *R*-squared, .266. For this data set, using $\log(1 + y)$ in a linear regression does not work well.

```
. corr hours hourshat
(obs=753)

             |    hours hourshat
-------------+------------------
       hours |   1.0000
    hourshat |   0.3984   1.0000


. di .3984^2
.15872256
```

h. Under the null of independence between r_i and $\mathbf{x}_i$, we should find no significant relationship between r^2 and any function of $\mathbf{x}$. Yet the *F* (that is, modified Wald) statistic for heteroskedasticity has a *p*-value of zero to more than four decimal places. Clearly, r_i is not

independent of $\mathbf{x}_i$.

```
. gen rhatsq = rhat^2

. gen xbhatsq = xbhat^2

. reg rhatsq xbhat xbhatsq
```

```
      Source |       SS       df       MS              Number of obs =     753
-------------+------------------------------           F(  2,   750) =   37.49
       Model |  5304.11081     2  2652.05541           Prob > F      =  0.0000
    Residual |   53062.278   750   70.749704           R-squared     =  0.0909
-------------+------------------------------           Adj R-squared =  0.0885
       Total |  58366.3888   752  77.6148787           Root MSE      =  8.4113

------------------------------------------------------------------------------
      rhatsq |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       xbhat |   3.840246   .4952455     7.75   0.000     2.868013    4.812478
     xbhatsq |   -.571263   .0665162    -8.59   0.000    -.7018431   -.4406829
       _cons |   4.286994   .9089565     4.72   0.000     2.502592    6.071395
------------------------------------------------------------------------------
```

17.2. a. No. The two-limit Tobit only makes sense if there is a corner at both endpoints. With $\mathrm{P}(y = 0) = 0$ the two-limit model becomes a one-limit model at unity, which means the model does not imply a zero density for $y < 0$. The estimates will be identical to the Tobit model with an upper corner at unity.

b. Over the range $(0, 1]$, $w \equiv -\log(y)$ takes values in $[0, \infty)$, with $\mathrm{P}(w = 0) = \mathrm{P}(y = 1) > 0$. Assuming that y is continuous on $(0, 1)$, w is continuous over $(0, \infty)$. So w is nonnegative, has a pile up at zero, and is continuously distributed over strictly positive values. A type I Tobit model makes logical sense.

c. It takes some work, but it is tractable. We can write $y = \exp(-w)$ but we cannot just pass the expected value through the exponential function. One way to proceed is to write $w = \max(0, \mathbf{x}\boldsymbol{\beta} + u)$ where $u|\mathbf{x} \sim \text{Normal}(0, \sigma^2)$ so $y = \exp[-\max(0, \mathbf{x}\boldsymbol{\beta} + u)]$. Then, using $\exp(0) = 1$ and splitting the integral over $u < -\mathbf{x}\boldsymbol{\beta}$ and $u \geq -\mathbf{x}\boldsymbol{\beta}$,

$$
\begin{aligned}
\mathrm{E}(y|\mathbf{x}) &= \int_{-\infty}^{\infty} \exp[-\max(0, \mathbf{x}\boldsymbol{\beta} + u)](1/\sigma)\phi(u/\sigma)du \\
&= \int_{-\infty}^{-\mathbf{x}\boldsymbol{\beta}} (1/\sigma)\phi(u/\sigma)du + \int_{-\mathbf{x}\boldsymbol{\beta}}^{\infty} \exp[-(\mathbf{x}\boldsymbol{\beta} + u)](1/\sigma)\phi(u/\sigma)du \\
&= \Phi(-\mathbf{x}\boldsymbol{\beta}/\sigma) + \exp(-\mathbf{x}\boldsymbol{\beta})[1 - \Phi(-\mathbf{x}\boldsymbol{\beta}/\sigma)] \int_{-\mathbf{x}\boldsymbol{\beta}}^{\infty} \exp(-u)(1/\sigma)\phi(u/\sigma)du \\
&= \Phi(-\mathbf{x}\boldsymbol{\beta}/\sigma) + \exp(-\mathbf{x}\boldsymbol{\beta})[1 - \Phi(-\mathbf{x}\boldsymbol{\beta}/\sigma)]\{\exp(-1)\Phi[(\mathbf{x}\boldsymbol{\beta} + 1)/\sigma]\} \\
&= \Phi(-\mathbf{x}\boldsymbol{\beta}/\sigma) + \exp(-\mathbf{x}\boldsymbol{\beta} - 1)\Phi(\mathbf{x}\boldsymbol{\beta}/\sigma)\Phi[(\mathbf{x}\boldsymbol{\beta} + 1)/\sigma]
\end{aligned}
$$

Although it is not obvious, this conditional mean function is bounded between zero and one.

17.3. a. Because $y = a_1$ if and only if $y^* \leq a_1$ we have

$$
\begin{aligned}
\mathrm{P}(y = a_1|\mathbf{x}) &= \mathrm{P}(y^* \leq a_1|\mathbf{x}) = \mathrm{P}(\mathbf{x}\boldsymbol{\beta} + u \leq a_1|\mathbf{x}) \\
&= \mathrm{P}[(u/\sigma) \leq (a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma|\mathbf{x}] \\
&= \Phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathrm{P}(y = a_2|\mathbf{x}) &= \mathrm{P}(y^* = a_2|\mathbf{x}) = \mathrm{P}(\mathbf{x}\boldsymbol{\beta} + u \geq a_2|\mathbf{x}) \\
&= \mathrm{P}[(u/\sigma) \geq (a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] = 1 - \Phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] \\
&= \Phi[-(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma].
\end{aligned}
$$

Next, for $a_1 < \mathfrak{y} < a_2$, $\mathrm{P}(y \leq \mathfrak{y}|\mathbf{x}) = \mathrm{P}(y^* \leq \mathfrak{y}|\mathbf{x}) = \phi[(\mathfrak{y} - \mathbf{x}_i\boldsymbol{\beta})/\sigma]$. Taking the derivative of this cdf with respect to y gives the pdf of y conditional on $\mathbf{x}$ for values $\mathfrak{y}$ strictly between a_1 and a_2: $(1/\sigma)\phi[(\mathfrak{y} - \mathbf{x}\boldsymbol{\beta})/\sigma]$.

b. Because $y = y^*$ when $a_1 < y^* < a_2$, $\mathrm{E}(y^*|\mathbf{x}, a_1 < y_i < a_2) = \mathrm{E}(y^*|\mathbf{x}, a_1 < y^* < a_2)$. But $y^* = \mathbf{x}\boldsymbol{\beta} + u$ and $a_1 < y^* < a_2$ if and only if $a_1 - \mathbf{x}\boldsymbol{\beta} < u < a_2 - \mathbf{x}\boldsymbol{\beta}$. Therefore, using the hint,

$$
\begin{aligned}
\mathrm{E}(y^*|\mathbf{x}, a_1 < y^* < a_2) &= \mathbf{x}\boldsymbol{\beta} + \mathrm{E}(u|\mathbf{x}, a_1 - \mathbf{x}\boldsymbol{\beta} < u < a_2 - \mathbf{x}\boldsymbol{\beta}) \\
&= \mathbf{x}\boldsymbol{\beta} + \sigma\mathrm{E}[(u/\sigma)|\mathbf{x}, (a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma < u/\sigma < (a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] \\
&= \mathbf{x}\boldsymbol{\beta} + \frac{\sigma\{\phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma] - \phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma]\}}{\{\Phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] - \Phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma]\}} \\
&= \mathrm{E}(y|\mathbf{x}, a_1 < y < a_2).
\end{aligned}
$$

Now, we can easily get $\mathrm{E}(y|\mathbf{x})$ by using the following:

$$\begin{aligned}
\mathrm{E}(y|\mathbf{x}) &= a_1 P(y = a_1|\mathbf{x}) + \mathrm{E}(y|\mathbf{x}, a_1 < y < a_2) \cdot P(a_1 < y < a_2|\mathbf{x}) + a_2 P(y_2 = a_2|\mathbf{x}) \\
&= a_1 \Phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma] \\
&\quad + \mathrm{E}(y|\mathbf{x}, a_1 < y < a_2) \cdot \{\Phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] - \Phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma]\} \\
&\quad + a_2 \Phi[(\mathbf{x}\boldsymbol{\beta} - a_2)/\sigma] \\
&= a_1 \Phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma] + (\mathbf{x}\boldsymbol{\beta}) \cdot \{\Phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] - \Phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma]\} \\
&\quad + \sigma\{\phi[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma] - \phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma]\} \\
&\quad + a_2 \Phi[(\mathbf{x}\boldsymbol{\beta} - a_2)/\sigma].
\end{aligned}$$

c. From part b it is clear that $\mathrm{E}(y^*|\mathbf{x}, a_1 < y^* < a_2) \neq \mathbf{x}\boldsymbol{\beta}$, and so it would be a fluke if OLS on the restricted sample consistently estimated $\boldsymbol{\beta}$. The linear regression of y_i on $\mathbf{x}_i$ using only those y_i such that $a_1 < y_i < a_2$ consistently estimates the linear projection of y^* on $\mathbf{x}$ in the subpopulation for which $a_1 < y^* < a_2$. Generally, there is no reason to think that this will have any simple relationship to the parameter vector $\boldsymbol{\beta}$. [In some restrictive cases, the regression on the restricted subsample could consistently estimate $\boldsymbol{\beta}$ up to a common scale coefficient.]

d. We get log-likelihood immediately from part a:

$$\begin{aligned}
\ell_i(\theta) = \; & 1[y_i = a_1] \log\{\Phi[(a_1 - \mathbf{x}_i\boldsymbol{\beta})/\sigma]\} \\
& + 1[y_i = a_2] \log\{\Phi[(\mathbf{x}_i\boldsymbol{\beta} - a_2)/\sigma]\} \\
& + 1[a_1 < y_i < a_2] \log\{(1/\sigma)\phi[(y_i - \mathbf{x}_i\boldsymbol{\beta})/\sigma]\}.
\end{aligned}$$

Note how the indicator function selects out the appropriate density for each of the three possible cases: at the left endpoint, at the right endpoint, or strictly between the endpoints.

e. After obtaining the maximum likelihood estimates $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$, just plug these into the formulas in part b. The expressions can be evaluated at interesting values of $\mathbf{x}$.

f. We can show this by brute-force differentiation of the expression in part b for $\mathrm{E}(y|\mathbf{x})$. As a shorthand, write

$$\phi_1 \equiv \phi[(a_1 - \mathbf{x}\beta)/\sigma],\ \phi_2 \equiv \phi[(a_2 - \mathbf{x}\beta)/\sigma] = \phi[(\mathbf{x}\beta - a_2)/\sigma],$$
$$\Phi_1 \equiv \Phi[(a_1 - \mathbf{x}\beta)/\sigma],\ \text{and}\ \Phi_2 \equiv \phi[(a_2 - \mathbf{x}\beta)/\sigma]$$

Then

$$\begin{aligned}\frac{\partial \mathrm{E}(y|\mathbf{x})}{\partial x_j} &= -(a_1/\sigma)\phi_1\beta_j + (a_2/\sigma)\phi_2\beta_j \\ &\quad + (\Phi_2 - \Phi_1)\beta_j + [(\mathbf{x}\boldsymbol{\beta}/\sigma)(\phi_1 - \phi_2)]\beta_j \\ &\quad + \{[(a_1 - \mathbf{x}\boldsymbol{\beta})/\sigma]\phi_1\}\beta_j - \{[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma]\phi_2\}\beta_j\end{aligned}$$

where the first two parts are the derivatives of the first and third terms, respectively, in $\mathrm{E}(y|\mathbf{x})$, and the last two lines are obtained from differentiating the second term in $\mathrm{E}(y|\mathbf{x})$. Careful inspection shows that all terms cancel except $(\Phi_2 - \Phi_1)\beta_j$, which is the expression we wanted to be left with.

The scale factor,

$$\Phi\left(\frac{a_2 - \mathbf{x}\boldsymbol{\beta}}{\sigma}\right) - \Phi\left(\frac{a_1 - \mathbf{x}\boldsymbol{\beta}}{\sigma}\right)$$

is simply the probability that a standard normal random variable falls in the interval $[(a_1 - \mathbf{x}\beta)/\sigma, (a_2 - \mathbf{x}\beta)/\sigma]$, which is necessarily between zero and one.

g. The partial effects on $\mathrm{E}(y|\mathbf{x})$ are given in f. These are estimated as

$$\Phi\left(\frac{a_2 - \mathbf{x}\hat{\boldsymbol{\beta}}}{\hat{\sigma}}\right) - \Phi\left(\frac{a_1 - \mathbf{x}\hat{\boldsymbol{\beta}}}{\hat{\sigma}}\right)$$

where the estimates are the MLEs. We could evaluate these partial effects at, say, $\bar{\mathbf{x}}$ to estimate the PEA (partial effect at the average). Or, we can estimate the scale factor for the APE of continuous x_j as

$$\hat{\rho} \equiv N^{-1}\sum_{i=1}^{N}\left[\Phi\left(\frac{a_2 - \mathbf{x}_i\hat{\boldsymbol{\beta}}}{\hat{\sigma}}\right) - \Phi\left(\frac{a_1 - \mathbf{x}_i\hat{\boldsymbol{\beta}}}{\hat{\sigma}}\right)\right].$$

Particularly for the APE, the scaled Tobit coefficients can be compared with the OLS coefficients (the $\hat{\gamma}_j$). Generally, we expect

$$\hat{\gamma}_j \approx \hat{\rho} \cdot \hat{\beta}_j,$$

where $0 < \hat{\rho} < 1$. Of course, this approximation need not be very good in a particular application often it is. It does not make sense to directly compare the magnitude of $\hat{\beta}_j$ with that of $\hat{\gamma}_j$. By the way, note that $\hat{\sigma}$ appears in the partial effects along with the $\hat{\beta}_j$.

17.4. The Stata outpus is below. The heteroskedasticity-robust standard error on *grant* is quite a bit bigger, but the robust *t* statistic is above four. (Interestingly, the heteroskedasticity-robust standard error for *union* is substantially smaller than the usual standard error.) The coefficient on *grant* implies that a firm receiving a job training grant in 1988 is estimated to provide about 27.2 more hours of job training per worker, holding firm size and union status fixed. This effect is very large considering the average hours of annual training over all 127 firms is about 16.

```
. use jtrain1

. des hrsemp grant

              storage  display     value
variable name   type   format      label      variable label
------------------------------------------------------------------------------
hrsemp          float  %9.0g                  tothrs/totrain
grant           byte   %9.0g                  = 1 if received grant

. reg hrsemp grant lemploy union if d88

      Source |       SS       df       MS              Number of obs =     127
-------------+------------------------------           F(  3,   123) =   14.58
       Model |  23232.2579     3  7744.08598           Prob > F      =  0.0000
    Residual |  65346.8909   123  531.275536           R-squared     =  0.2623
-------------+------------------------------           Adj R-squared =  0.2443
       Total |  88579.1488   126  703.009118           Root MSE      =  23.049

------------------------------------------------------------------------------
      hrsemp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       grant |   27.17647   4.769283     5.70   0.000     17.73597    36.61698
     lemploy |  -5.511867   2.012923    -2.74   0.007    -9.496324   -1.527409
```

```
       union |  -8.924901   5.392118    -1.66   0.100    -19.59827    1.748465
       _cons |   30.76978   7.345811     4.19   0.000      16.2292    45.31037
------------------------------------------------------------------------------

. reg hrsemp grant lemploy union if d88, robust

Linear regression                                      Number of obs =     127
                                                       F(  3,   123) =    7.40
                                                       Prob > F      =  0.0001
                                                       R-squared     =  0.2623
                                                       Root MSE      =  23.049

------------------------------------------------------------------------------
             |               Robust
      hrsemp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       grant |   27.17647   6.525922     4.16   0.000     14.25881    40.09414
     lemploy |  -5.511867    2.17685    -2.53   0.013    -9.820807   -1.202926
       union |  -8.924901   3.181306    -2.81   0.006     -15.2221   -2.627702
       _cons |   30.76978   8.558935     3.60   0.000      13.8279    47.71167
------------------------------------------------------------------------------
```

b. The Tobit results are below. Out of 127 firms in 1988, 38 provide no job training.

```
. count if hrsemp == 0 & d88
  38

. tobit hrsemp grant lemploy union if d88, ll(0)

Tobit regression                                  Number of obs   =        127
                                                  LR chi2(3)      =      37.46
                                                  Prob > chi2     =     0.0000
Log likelihood = -451.88026                       Pseudo R2       =     0.0398

------------------------------------------------------------------------------
      hrsemp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       grant |   36.34335   6.121823     5.94   0.000     24.22655    48.46016
     lemploy |  -4.928542   2.656817    -1.86   0.066    -10.18713     .330044
       union |  -12.63617   7.286913    -1.73   0.085    -27.05901    1.786677
       _cons |   20.32933   9.769517     2.08   0.040     .9927198    39.66594
-------------+----------------------------------------------------------------
      /sigma |   28.70726   2.229537                      24.29438    33.12014
------------------------------------------------------------------------------
  Obs. summary:         38  left-censored observations at hrsemp<=0
                        89     uncensored observations
                         0 right-censored observations
```

The language “left censored at hrsemp <= 0” is misleading for corner solution applications, but it does tell us that 38 of the 127 firms have $hrsemp = 0$. The estimate of σ is $\hat{\sigma} = 28.71$.

To get the effect of *grant* on $E(hrsemp|grant, employ, union, hrsemp > 0)$, we must compute the inverse Mills ratio with $grant = 1$ and $grant = 0$. We set

employ $= \overline{employ} = 60.87$ and *union* $= 1$. Below is the Stata session.

```
. gen xb1 = _b[_cons] + _b[grant] +_b[lemploy]*log(60.87) + _b[union]

. gen xb0 = _b[_cons] +_b[lemploy]*log(60.87) + _b[union]

. gen prob1 = normal(xb1/_b[/sigma])

. gen prob0 = normal(xb0/_b[/sigma])

. gen imr1 =  normalden(xb1/_b[/sigma])/prob1

. gen imr0 =  normalden(xb0/_b[/sigma])/prob0

. gen cm1 = xb1 + _b[/sigma]*imr1

. gen cm0 = xb0 + _b[/sigma]*imr0

. gen dcm = cm1 - cm0

. list dcm in 1

     +----------+
     |      dcm |
     |----------|
  1. | 15.09413 |
     +----------+

. gen um1 = prob1*cm1

. gen um0 = prob0*cm0

. gen dum = um1 - um0

. list dum in 1

     +----------+
     |      dum |
     |----------|
  1. | 20.81422 |
     +----------+
```

For firms already doing some job training, the grant is estimated to increase training by about 15.1 hours per employee. When we add in the effects of firms that go from no training to positive hours, the expected change is about 20.8 hours at *union* $= 1$ and the average value of *employ* in the sample. This is somewhat less than the OLS estimate we obtained earlier, 27.2.

The estimated APE is on the unconditional mean is computed below as 26.2, which is pretty close to the OLS estimate of 27.2. Bootstrapping can be used to obtain a valid standard

error.

```
. predict xb, xb
(31 missing values generated)

. sum xb if d88

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          xb |       146    9.095312    16.80602  -18.41981    48.7405

. replace xb = . if ~d88
(294 real changes made, 294 to missing)

. replace xb = . if hrsemp == . | lemploy == . | union == .
(19 real changes made, 19 to missing)

. sum xb

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          xb |       127    9.265182    17.20874  -18.41981    48.7405

. gen xb0 = xb - _b[grant]*grant
(344 missing values generated)

. gen xb1 = xb0 + _b[grant]
(344 missing values generated)

. gen prob0 = normal(xb0/_b[/sigma])
(344 missing values generated)

. gen prob1 = normal(xb1/_b[/sigma])
(344 missing values generated)

. gen imr0 =  normalden(xb0/_b[/sigma])/prob0
(344 missing values generated)

. gen imr1 =  normalden(xb1/_b[/sigma])/prob1
(344 missing values generated)

. gen cm0 = xb0 + _b[/sigma]*imr0
(344 missing values generated)

. gen cm1 = xb1 + _b[/sigma]*imr1
(344 missing values generated)

. gen um0 = prob0*cm0
(344 missing values generated)

. gen um1 = prob1*cm1
(344 missing values generated)

. gen pe = um1 - um0
(344 missing values generated)

. sum pe
```

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          pe |       127       26.23    3.272887   16.88082   30.56553
```

c. They are jointly significant at the 1.5% level, as the Stata "test" command shows.

```
. test lemploy union

 ( 1)  [model]lemploy = 0
 ( 2)  [model]union = 0

       F(  2,   124) =    4.34
            Prob > F =    0.0151
```

d. For the Tobit model, I use the square of the correlation between $y_i = hrsemp_i$ and $\hat{E}(y_i|\mathbf{x}_i)$ as an *R*-squared that can be compared with the linear model *R*-squared. After the `tobit` command in Stata, $\hat{E}(y_i|\mathbf{x}_i)$ can be gotten using the `ystar` option for predicted values. (Unfortunately, Stata's naming convention conflicts with the notation used in the text, as y^* is used to denote the unerlying latent variable, not the actual outcome.)

```
. predict hrsemph if d88 & hrsemp != ., ystar(0,.)
(344 missing values generated)

. corr hrsemp hrsemph
(obs=127)

             |   hrsemp  hrsemph
-------------+------------------
      hrsemp |   1.0000
     hrsemph |   0.5206   1.0000

. di (.5206)^2
.27102436
```

This *R*-squared is slightly above that for the linear model (.262), and so the Tobit does provide a better fit. And remember, the Tobit estimates are not chosen to maximize an *R*-squared, so the improvement in fit is effectively better.

17.5. a. The results from OLS estimation of the linear model are given below.

```
. use fringe

. reg hrbens exper age educ tenure married male white nrtheast nrthcen south
      union, robust

Linear regression                                      Number of obs =     616
```

```
                                                F( 11,   604) =   36.02
                                                Prob > F      =  0.0000
                                                R-squared     =  0.3718
                                                Root MSE      =  .53183

------------------------------------------------------------------------------
             |               Robust
      hrbens |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       exper |   .0029862   .0042485     0.70   0.482    -.0053574    .0113298
         age |  -.0022495   .0041519    -0.54   0.588    -.0104034    .0059043
        educ |    .082204   .0085122     9.66   0.000      .065487    .0989211
      tenure |   .0281931   .0037053     7.61   0.000     .0209164    .0354699
     married |   .0899016   .0499158     1.80   0.072    -.0081281    .1879312
        male |    .251898   .0496953     5.07   0.000     .1543015    .3494946
       white |    .098923   .0721337     1.37   0.171    -.0427402    .2405862
    nrtheast |  -.0834306   .0723545    -1.15   0.249    -.2255277    .0586664
     nrthcen |  -.0492621   .0626967    -0.79   0.432    -.1723922     .073868
       south |  -.0284978   .0653108    -0.44   0.663    -.1567617    .0997662
       union |   .3768401   .0535136     7.04   0.000     .2717448    .4819354
       _cons |  -.6999244   .1803555    -3.88   0.000    -1.054125   -.3457242
------------------------------------------------------------------------------
```

b. The Tobit estimates recognizing the corner at zero are

```
. tobit hrbens exper age educ tenure married male white nrtheast nrthcen south
        union, ll(0)

Tobit regression                                  Number of obs   =        616
                                                  LR chi2(11)     =     283.86
                                                  Prob > chi2     =     0.0000
Log likelihood = -519.66616                       Pseudo R2       =     0.2145

------------------------------------------------------------------------------
      hrbens |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       exper |   .0040631   .0046627     0.87   0.384    -.0050939    .0132201
         age |  -.0025859   .0044362    -0.58   0.560    -.0112981    .0061263
        educ |   .0869168   .0088168     9.86   0.000     .0696015    .1042321
      tenure |   .0287099   .0037237     7.71   0.000      .021397    .0360227
     married |   .1027574   .0538339     1.91   0.057    -.0029666    .2084814
        male |   .2556765   .0551672     4.63   0.000     .1473341     .364019
       white |   .0994408    .078604     1.27   0.206     -.054929    .2538106
    nrtheast |  -.0778461   .0775035    -1.00   0.316    -.2300547    .0743625
     nrthcen |  -.0489422   .0713965    -0.69   0.493    -.1891572    .0912729
       south |  -.0246854   .0709243    -0.35   0.728    -.1639731    .1146022
       union |   .4033519   .0522697     7.72   0.000     .3006999     .506004
       _cons |  -.8137158   .1880725    -4.33   0.000     -1.18307   -.4443616
-------------+----------------------------------------------------------------
      /sigma |   .5551027   .0165773                      .5225467    .5876588
------------------------------------------------------------------------------
  Obs. summary:         41  left-censored observations at hrbens<=0
                       575     uncensored observations
                         0 right-censored observations
```

The Tobit and OLS estimates are similar because only 41 of 616 observations, or about

6.7% of the sample, have *hrbens* = 0. As expected, the Tobit estimates are all slightly larger in magnitude; this reflects that the scale factor is always less than unity.

c. Here is what happens when $exper^2$ and $tenure^2$ are included:

```
. tobit hrbens exper age educ tenure married male white nrtheast nrthcen south
        union expersq tenuresq, ll(0)

Tobit regression                                  Number of obs   =        616
                                                  LR chi2(13)     =     315.95
                                                  Prob > chi2     =     0.0000
Log likelihood = -503.62108                       Pseudo R2       =     0.2388

------------------------------------------------------------------------------
      hrbens |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       exper |   .0306652   .0085253     3.60   0.000     .0139224     .047408
         age |  -.0040294   .0043428    -0.93   0.354    -.0125583    .0044995
        educ |   .0802587   .0086957     9.23   0.000     .0631812    .0973362
      tenure |   .0581357   .0104947     5.54   0.000      .037525    .0787463
     married |   .0714831   .0528969     1.35   0.177    -.0324014    .1753675
        male |   .2562597   .0539178     4.75   0.000     .1503703    .3621491
       white |   .0906783   .0768576     1.18   0.239    -.0602628    .2416193
    nrtheast |  -.0480194   .0760238    -0.63   0.528     -.197323    .1012841
     nrthcen |   -.033717   .0698213    -0.48   0.629    -.1708394    .1034053
       south |   -.017479   .0693418    -0.25   0.801    -.1536597    .1187017
       union |   .3874497    .051105     7.58   0.000     .2870843    .4878151
     expersq |  -.0005524   .0001487    -3.71   0.000    -.0008445   -.0002604
    tenuresq |  -.0013291   .0004098    -3.24   0.001     -.002134   -.0005242
       _cons |  -.9436572   .1853532    -5.09   0.000    -1.307673   -.5796409
-------------+----------------------------------------------------------------
      /sigma |   .5418171   .0161572                      .5100859    .5735484
------------------------------------------------------------------------------
  Obs. summary:         41  left-censored observations at hrbens<=0
                       575     uncensored observations
                         0 right-censored observations

. test expersq tenuresq

 ( 1)  [model]expersq = 0
 ( 2)  [model]tenuresq = 0

       F(  2,   603) =   16.34
            Prob > F =    0.0000
```

Both squared terms are very statistically significant as well as jointly significant. What is not clear is whether their presence would change the estimated partial effects in important ways.

d. There are nine industries, and we use *ind*1 as the base industry:

```
. tobit hrbens exper age educ tenure married male white nrtheast nrthcen south
        union expersq tenuresq ind2-ind9, ll(0)

Tobit regression                                  Number of obs   =        616
                                                  LR chi2(21)     =     388.99
                                                  Prob > chi2     =     0.0000
Log likelihood = -467.09766                       Pseudo R2       =     0.2940

------------------------------------------------------------------------------
      hrbens |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       exper |   .0267869   .0081297     3.29   0.001     .0108205    .0427534
         age |  -.0034182   .0041306    -0.83   0.408    -.0115306    .0046942
        educ |   .0789402   .0088598     8.91   0.000       .06154    .0963403
      tenure |    .053115   .0099413     5.34   0.000     .0335907    .0726393
     married |   .0547462   .0501776     1.09   0.276    -.0438005    .1532928
        male |   .2411059   .0556864     4.33   0.000     .1317401    .3504717
       white |   .1188029   .0735678     1.61   0.107    -.0256812    .2632871
    nrtheast |  -.1016799   .0721422    -1.41   0.159    -.2433643    .0400045
     nrthcen |  -.0724782   .0667174    -1.09   0.278    -.2035085    .0585521
       south |  -.0379854   .0655859    -0.58   0.563    -.1667934    .0908226
       union |   .3143174   .0506381     6.21   0.000     .2148662    .4137686
     expersq |  -.0004405   .0001417    -3.11   0.002    -.0007188   -.0001623
    tenuresq |  -.0013026   .0003863    -3.37   0.001    -.0020613    -.000544
        ind2 |  -.3731778   .3742017    -1.00   0.319    -1.108095    .3617389
        ind3 |  -.0963657    .368639    -0.26   0.794    -.8203575    .6276261
        ind4 |  -.2351539   .3716415    -0.63   0.527    -.9650425    .4947348
        ind5 |   .0209362    .373072     0.06   0.955    -.7117618    .7536342
        ind6 |  -.5083107   .3682535    -1.38   0.168    -1.231545     .214924
        ind7 |   .0033643   .3739442     0.01   0.993    -.7310468    .7377754
        ind8 |  -.6107854    .376006    -1.62   0.105    -1.349246     .127675
        ind9 |  -.3257878   .3669437    -0.89   0.375     -1.04645    .3948746
       _cons |  -.5750527   .4137824    -1.39   0.165    -1.387704    .2375989
-------------+----------------------------------------------------------------
      /sigma |   .5099298   .0151907                      .4800959    .5397637
------------------------------------------------------------------------------
  Obs. summary:         41  left-censored observations at hrbens<=0
                       575     uncensored observations
                         0 right-censored observations

. testparm ind2-ind9

 ( 1)  [model]ind2 = 0
 ( 2)  [model]ind3 = 0
 ( 3)  [model]ind4 = 0
 ( 4)  [model]ind5 = 0
 ( 5)  [model]ind6 = 0
 ( 6)  [model]ind7 = 0
 ( 7)  [model]ind8 = 0
 ( 8)  [model]ind9 = 0

       F(  8,   595) =    9.66
            Prob > F =    0.0000
```

Each industry dummy variable is individually insignificant at even the 10% level, but the joint Wald test says that they are jointly very significant. This is somewhat unusual for dummy

variables that are necessarily othogonal (so that there is not a multicollinearity problem among them). The likelihood ratio statistic is $LR = 2(503.621 - 467.098) = 73.046$, which is roughly comparable with $Q \cdot F = 8 \cdot 9.66 = 77.28$. The p-values in both cases are essentially zero.

Several estimates on the industry dummies are economically significant, with a worker in, say, industry eight earning about 61 cents less per hour in benefits than a comparable worker in industry one. [In this example, with so few observations at zero, it is roughly legitimate to use the parameter estimates as the partial effects.]

17.6. a. First, we can write $u_1 = \rho_1 v_2 + e_1$, where $\rho_1 = \text{Cov}(v_2, u_1)$, and we use the fact that $\text{Var}(v_2) = 1$. Also, $\sigma_1^2 = \rho_1^2 + \tau_1^2$ where $\tau_1^2 = \text{Var}(e_1)$. The distribution of y_1 given $(\mathbf{z}, v_2)$ can be written as $g(\mathfrak{y}_1|\mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 v_2, \sigma_1^2 - \rho_1^2)$, where $\mathfrak{y}_1$ is the generic argument. Next, we need the density of v_2 given $(\mathbf{z}, y_2)$, which is given in equations (15.55) and (15.56). To obtain the density of y_1 given $(\mathbf{z}, y_2)$, we can apply Property CD.3 in Appendix 2A. The density of $v_2|(\mathbf{z}, y_2 = 1)$ is $\phi(\mathfrak{v}_2)/\Phi(\mathbf{z}\boldsymbol{\delta}_2)$ for $\mathfrak{v}_2 > -\mathbf{z}\boldsymbol{\delta}_2$. So the density of y_1 given $(\mathbf{z}, y_2 = 1)$ is

$$\frac{1}{\Phi(\mathbf{z}\boldsymbol{\delta}_2)} \int_{-\mathbf{z}\boldsymbol{\delta}_2}^{\infty} g(\mathfrak{y}_1|\mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 \mathfrak{v}_2, \sigma_1^2 - \rho_1^2)\phi(\mathfrak{v}_2)d\mathfrak{v}_2$$

(where $\mathfrak{v}_2$ is just the dummy argument in the integration) and the density given $(\mathbf{z}, y_2 = 0)$ is

$$\frac{1}{[1 - \Phi(\mathbf{z}\boldsymbol{\delta}_2)]} \int_{-\infty}^{-\mathbf{z}\boldsymbol{\delta}_2} g(y_1|\mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 \mathfrak{v}_2, \sigma_1^2 - \rho_1^2)\phi(\mathfrak{v}_2)d\mathfrak{v}_2.$$

b. We need to combine the density obtained from part a – called it $f(\mathfrak{y}_1|y_2, \mathbf{z}; \boldsymbol{\delta}_1, \alpha_1, \rho_1, \sigma_1^2, \boldsymbol{\delta}_2)$, and let $h(\mathfrak{y}_2|\mathbf{z}; \boldsymbol{\delta}_2)$ be the probit density of y_2 given $\mathbf{z}$. Actually, it is easier to work with $\tau_1^2 = \sigma_1^2 - \rho_1^2$. Then the log-likelihood for observation i is

$$
\begin{aligned}
&\log[f(y_{i1}|y_{i2},\mathbf{z}_i;\boldsymbol{\delta}_1,\alpha_1,\rho_1,\tau_1^2,\boldsymbol{\delta}_2)] + \log[h(y_{i2}|\mathbf{z}_i;\boldsymbol{\delta}_2)] \\
&= y_{i2}\log\left(\frac{1}{\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)}\int_{-\mathbf{z}_i\boldsymbol{\delta}_2}^{\infty} g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2}+\rho_1\upsilon_2,\tau_1^2)\phi(\upsilon_2)d\upsilon_2\right) \\
&\quad + (1-y_{i2})\log\left(\frac{1}{[1-\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)]}\int_{-\infty}^{-\mathbf{z}_i\boldsymbol{\delta}_2} g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2}+\rho_1\upsilon_2,\tau_1^2)\phi(\upsilon_2)d\upsilon_2\right) \\
&\quad + y_{i2}\log[\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)] + (1-y_{i2})\log[1-\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)]
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\ell_i(\boldsymbol{\theta}) &= y_{i2}\log\left(\int_{-\mathbf{z}_i\boldsymbol{\delta}_2}^{\infty} g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2}+\rho_1\upsilon_2,\tau_1^2)\phi(\upsilon_2)d\upsilon_2\right) \\
&\quad + (1-y_{i2})\log\left(\int_{-\infty}^{-\mathbf{z}_i\boldsymbol{\delta}_2} g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2}+\rho_1\upsilon_2,\tau_1^2)\phi(\upsilon_2)d\upsilon_2\right).
\end{aligned}
$$

If $\rho_1 = 0$, the log likelihood becomes

$$
\begin{aligned}
\ell_i(\boldsymbol{\theta}) &= y_{i2}\log[g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2},\tau_1^2)\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)] \\
&\quad + (1-y_{i2})\log\{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2},\tau_1^2)[1-\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)]\} \\
&= \log[g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1+\alpha_1 y_{i2},\tau_1^2)] + y_{i2}\log[\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)] + (1-y_{i2})\log[1-\Phi(\mathbf{z}_i\boldsymbol{\delta}_2)],
\end{aligned}
$$

which is two separate log-likelihoods, one the standard Tobit for y_{i1} give $(\mathbf{z}_{i1}, y_{i2})$ and the second for probit of y_{i2} given $\mathbf{z}_i$.

c. As in the probit case (Section 15.7.3), this is another example of a forbidden regression. There is no way that $\mathrm{E}(y_1|\mathbf{z})$ has the Tobit form with $\mathbf{z}_1$ and $\Phi(\mathbf{z}\delta_2) = \mathrm{E}(y_2|\mathbf{z})$ as the explanatory variables. In fact, because $y_1 = \max(0, \mathbf{z}_1\delta_1 + \alpha_1 y_2 + u_1)$, $\mathrm{E}(y_1|\mathbf{z})$ has no simple form – although it could be computed in principle.

d. As given in the hint, it is easiest to work with the parameterization in terms of τ_1^2, as shown in part b. Passing the derivative through the integral gives

$$\frac{\partial \ell_i(\boldsymbol{\theta})}{\partial \rho_1} = y_{i2} \frac{\int_{-\mathbf{z}_i\boldsymbol{\delta}_2}^{\infty} \upsilon_2 g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2} + \rho_1 \upsilon_2, \tau_1^2)\phi(\upsilon_2)d\upsilon_2}{\int_{-\mathbf{z}_i\boldsymbol{\delta}_2}^{\infty} g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2} + \rho_1 \upsilon_2, \tau_1^2)\phi(\upsilon_2)d\upsilon_2}$$
$$+ (1 - y_{i2}) \frac{\int_{-\infty}^{-\mathbf{z}_i\boldsymbol{\delta}_2} \upsilon_2 g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2} + \rho_1 \upsilon_2, \tau_1^2)\phi(\upsilon_2)d\upsilon_2}{\int_{-\infty}^{-\mathbf{z}_i\boldsymbol{\delta}_2} g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2} + \rho_1 \upsilon_2, \tau_1^2)\phi(\upsilon_2)d\upsilon_2}.$$

where $g^{(1)}$ denotes the first derivative. When we set $\rho_1 = 0$ the first term becomes

$$y_{i2} \frac{g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2) \int_{-\mathbf{z}_i\boldsymbol{\delta}_2}^{\infty} \upsilon_2 \phi(\upsilon_2)d\upsilon_2}{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2) \int_{-\mathbf{z}_i\boldsymbol{\delta}_2}^{\infty} \phi(\upsilon_2)d\upsilon_2}$$
$$= y_{i2} \frac{g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)}{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)} \cdot \frac{\phi(\mathbf{z}_i\boldsymbol{\delta}_2)}{[1 - \Phi(-\mathbf{z}_i\boldsymbol{\delta}_2)]}$$
$$= \frac{g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)}{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)} \cdot y_{i2}\lambda(\mathbf{z}_i\boldsymbol{\delta}_2)$$

where $\lambda(a) = \phi(a)/\Phi(a)$ is the inverse Mills ratio and we use the fact that $\int_a^{\infty} \upsilon\phi(\upsilon)d\upsilon = \phi(a)$ for any $a \in \mathbb{R}$. Similarly, using $\int_{-\infty}^{a} \upsilon\phi(\upsilon)d\upsilon = -\phi(a)$, the second term is

$$\frac{g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)}{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)} \cdot [-(1 - y_{i2})\lambda(-\mathbf{z}_i\boldsymbol{\delta}_2)]$$

and so the partial derivative evaluated at $\rho_1 = 0$ is

$$\frac{g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)}{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)} [y_{i2}\lambda(\mathbf{z}_i\boldsymbol{\delta}_2) - (1 - y_{i2})\lambda(-\mathbf{z}_i\boldsymbol{\delta}_2)]$$
$$\equiv \frac{g^{(1)}(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)}{g(y_{i1}|\mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2}, \tau_1^2)} gr(y_{i2}, \mathbf{z}_i\boldsymbol{\delta}_2)$$

where $gr(y_{i2}, \mathbf{z}_i\boldsymbol{\delta}_2) \equiv y_{i2}\lambda(\mathbf{z}_i\boldsymbol{\delta}_2) - (1 - y_{i2})\lambda(-\mathbf{z}_i\boldsymbol{\delta}_2)$ is the generalized residual. The key is that this is the same partial derivative we would obtain by simply adding $gr_{i2} \equiv gr(y_{i2}, \mathbf{z}_i\boldsymbol{\delta}_2)$ as an explanatory variable and giving it a coefficient, say η_1. In other words, form the artificial model

$$y_{i1} = \max(0, \mathbf{z}_{i1}\boldsymbol{\delta}_1 + \alpha_1 y_{i2} + \eta_1 gr_{i2} + e_{i1})$$
$$e_{i1}|\mathbf{z}_{i1}, y_{i2}, gr_{i2} \sim \text{Normal}(0, \tau_1^2).$$

Of course, under the give assumptions this "model" cannot be true when $\eta_1 \neq 0$. But if we act as if it is true and compute the score for testing $\text{H}_0 : \eta_1 = 0$, we get exactly the score derived above. So we are led to a simple variable addition test. In the first stage estimate probit of y_{i2} on $\mathbf{z}_i$ to get $\hat{\boldsymbol{\delta}}_2$. Construct the generalized residuals,

$$\widehat{gr}_{i2} = y_{i2}\lambda(\mathbf{z}_i\hat{\boldsymbol{\delta}}_2) - (1 - y_{i2})\lambda(-\mathbf{z}_i\hat{\boldsymbol{\delta}}_2).$$

In the second step, estimate a Tobit model of y_{i1} on $\mathbf{z}_{i1}$, y_{i2}, $\widehat{gr}_{i2}$ and use a t test for the coefficient $\hat{\eta}_1$ on $\widehat{gr}_{i2}$. Under the null, the statistic has an asymptotic Normal$(0, 1)$ distribution, with no need to adjust the standard error for estimation of $\boldsymbol{\delta}_2$.

Incidentally, while adding the generalized residual – which acts as a kind of control function – does not generally solve the endogeneity of y_2 under the assumptions of this problem, it might be a decent approximation. It is likely to do well when ρ_1 is "close" to zero (although we then must wonder how much of a problem endogeneity is in the first case). There is some evidence that it can work well as an approximation more general, where focus would be on average partial effects. Putting in flexible functions of $\widehat{gr}_{i2}$ – such as low-order polynomials – can help even more.

If we simply assert that $\text{D}(y_1|\mathbf{z}_1, y_2, gr_2)$ follows the Tobit model given above then adding $\widehat{gr}_{i2}$ does produce consistent estimators of all parameters and average partial effects (by averaging out $\widehat{gr}_{i2}$ in the partial effect formulas for the standard Tobit). This idea is nontraditional but is in the spirit of viewing all models simply as approximations.

17.7. Let $s = 1[y > 0]$ and use Property CV.3 about conditional variances (see Appendix

2.A.2):

$$\text{Var}(y|\mathbf{x}) = \text{E}[\text{Var}(y|\mathbf{x},s)|\mathbf{x}] + \text{Var}[\text{E}(y|\mathbf{x},s)|\mathbf{x}]$$

Now because $y = s \cdot w^*$,

$$\begin{aligned}\text{E}(y|\mathbf{x},s) &= s \cdot \text{E}(w^*|\mathbf{x},s) = s \cdot \text{E}(w^*|\mathbf{x}) = s \cdot \exp(\mathbf{x}\boldsymbol{\beta}) \\ \text{Var}(y|\mathbf{x},s) &= s^2 \cdot \text{Var}(w^*|\mathbf{x},s) = s \cdot \text{Var}(w^*|\mathbf{x}) = s \cdot \eta^2[\exp(\mathbf{x}\boldsymbol{\beta})]^2\end{aligned}$$

and so

$$\begin{aligned}\text{Var}(y|\mathbf{x}) &= \text{E}[s \cdot \eta^2 \exp(2\mathbf{x}\boldsymbol{\beta})|\mathbf{x}] + \text{Var}[s \cdot \exp(\mathbf{x}\boldsymbol{\beta})|\mathbf{x}] \\ &= \text{P}(s = 1|\mathbf{x})\eta^2 \exp(2\mathbf{x}\boldsymbol{\beta}) + \text{Var}(s|\mathbf{x}) \exp(2\mathbf{x}\boldsymbol{\beta}) \\ &= \eta^2\Phi(\mathbf{x}\boldsymbol{\gamma}) \exp(2\mathbf{x}\boldsymbol{\beta}) + \Phi(\mathbf{x}\boldsymbol{\gamma})[1 - \Phi(\mathbf{x}\boldsymbol{\gamma})] \exp(2\mathbf{x}\boldsymbol{\beta})\end{aligned}$$

17.8. a. For model (1) simply use ordinary least squares. Under the conditional mean assumption, we could use a weighted least squares procedure if we suspect heteroskedasticity, as we might, and have a particular form in mind. However, we should probably not think of the linear model as a model of $\text{E}(y|\mathbf{x})$; rather, it is simply the linear projection. If we use a WLS procedure, we are effectively estimating a linear predictor in weighted variables.

For model (2) we could use nonlinear regression, or weighted nonlinear regression. The latter is attractive because of probable heteroskedasticity in $\text{Var}(y|\mathbf{x})$. We might use a variance function proportional to $\exp(\mathbf{x}\boldsymbol{\beta})$ or $[\exp(\mathbf{x}\boldsymbol{\beta})]^2$, or a quadratic in the mean function: $\text{Var}(y|\mathbf{x}) = \delta_0 + \delta_1 \exp(\mathbf{x}\boldsymbol{\beta}) + \delta_2[\exp(\mathbf{x}\boldsymbol{\beta})]^2$ which contains the previous two as a special case. We can estimate the δ_j from the OLS regression $\hat{u}_1^2$ on $1, \hat{y}_i$, and $\hat{y}_i^2$, where the hatted quantities are from a first stage NLS estimation. The fitted values are the estimated conditinal variances (and we might have to worry about whether they are all strictly positive). Other attractive options are Poisson regression – see Chapter 18 for a description of its robustness properties for estimating $\text{E}(y|\mathbf{x})$ – or regression using the Exponential quasi-log-likelihood (see Chapter

18).

Naturally, for model (3) we would use MLE.

b. We can compute an *R*-squared type measure any time we directly model $\mathrm{E}(y|\mathbf{x})$ or we have an implied model for $\mathrm{E}(y|\mathbf{x})$ (such as in the Tobit case). In each case, we obtain the fitted values, $\hat{\mathrm{E}}(y_i|\mathbf{x}_i), i = 1, \ldots, N$. Once we have fitted values we can obtain the squared correlation between y_i and $\hat{\mathrm{E}}(y_i|\mathbf{x}_i)$. These can be compared across different models and even estimation methods. Alternatively, one can use a sum-of-squared residuals form:

$$R^2 = 1 - \frac{\sum_{i=1}^{N}[y_i - \hat{\mathrm{E}}(y_i|\mathbf{x}_i)]^2}{\sum_{i=1}^{N}(y_i - \bar{y})^2}.$$

In the linear regression case with an intercept, the two ways of computing *R*-squared are identical, but the equivalance does not hold in general. In fact, the SSR version of *R*-squared can be negative in some cases. One can always compute an "adjusted" *R*-squared, too:

$$\bar{R}^2 = 1 - \frac{(N-P)^{-1}\sum_{i=1}^{N}[y_i - \hat{\mathrm{E}}(y_i|\mathbf{x}_i)]^2}{(N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{y})^2}$$

where *P* is the number of estimated parameters in the mean function.

c. This is clear from equation (17.20). If $y_i > 0$ for $i = 1, \ldots, N,$ then only the second term in the log likelihood appears. But that is just the log likelihood for the classical linear regression model where $y_i|\mathbf{x}_i \sim \text{Normal}(\mathbf{x}_i\boldsymbol{\beta}, \sigma^2)$. It is well known that the MLE of $\boldsymbol{\beta}$ in this case is the OLS estimator.

It may seem a bit odd, but if we truly believe the population follows a Tobit model – and just happen to obtain a sample where $y_i > 0$ for all *i* – then the appropriate estimate of $\mathrm{E}(y|\mathbf{x})$ is gotten from (17.14), where we plug in the usual OLS estimators for $\boldsymbol{\beta}$ and σ^2. Estimates of $\mathrm{E}(y|\mathbf{x})$ computed in this way would ensure that fitted values in the sample are all positive, even

though $\mathbf{x}_i\hat{\boldsymbol{\beta}}$ could be negative for some i.

d. If $y > 0$ in the population, a Tobit model makes no sense because $P(y = 0) > 0$ for a Tobit model. Instead, we could assume $E[\log(y)|\mathbf{x}] = \mathbf{x}\boldsymbol{\gamma}$, or, equivalently, $\log(y) = \mathbf{x}\boldsymbol{\gamma} + v$, $E(v|\mathbf{x}) = 0$. If we make the stronger assumption that v is independent of $\mathbf{x}$, then $E(y|\mathbf{x}) = \eta \cdot \exp(\mathbf{x}\boldsymbol{\gamma})$, where $\eta \equiv E[\exp(v)] > 1$. After estimating $\boldsymbol{\gamma}$ from the OLS regression $\log(y_i)$ on $\mathbf{x}_i, i = 1, \ldots, N,$ we can estimate η using Duan's (1983) estimator, as in Problem 17.1:

$$\hat{\eta} = N^{-1} \sum_{i=1}^{N} \exp(\hat{v}_i),$$

where the $\hat{v}_i$ are the OLS residuals.

17.9. a. A two-limit Tobit model, of the kind analyzed in Problem 17.3, is appropriate, with $a_1 = 0, a_2 = 10$.

b. The lower limit at zero is logically necessary considering the kind of response: the smallest percentage of one's income that can be invested in a pension plan is zero. On the other hand, the upper limit of 10 is an arbitrary corner imposed by law. One can imagine that some people at the corner $y = 10$ would choose $y > 10$ if they could. So, we can think of an underlying variable, which would be the percentage invested in the absence of any restrictions. Then, there would be no upper bound required (since we would not have to worry about 100 percent of income being invested in a pension plan).

17.10. A more general version of this problem is done in Problem 17.3, part f: set $a_1 = 0$ and let $a_2 \to \infty$.

17.11. No. OLS always consistently estimates the parameters of a linear projection provided the second moments of y and the x_j are finite and $\text{Var}(\mathbf{x})$ has full rank K – regardless

of the nature of y or **x** (discrete, continuous, some mixture). The fact that we can always consistently estimate a linear projection by OLS is why linear regression analysis is always a reasonable step for discrete outcomes (provided there is no data censoring problem of the type we discuss in Chapter 19). As discussed in Chapters 15 and 17, the linear regression coefficients often are close to estimated average partial effects from more complicated models. See Problem 17.4 part b for an example.

17.12. a. 248 out of 660, or about 37.6%, have $ecolbs_i = 0$. The positive responses range from .333 to a high of 42, but with focal points at integer values, especially one pound and two pounds. Therefore, a Tobit model cannot literally be true, but it can still lead to good estimates of the conditional mean and partial effects.

b. The linear model results are given below:

```
. use apple

. gen lecoprc = log(ecoprc)

. gen lregprc = log(regprc)

. gen lfaminc = log(faminc)

. reg ecolbs lecoprc lregprc lfaminc educ hhsize num5_17

      Source |       SS       df       MS              Number of obs =     660
-------------+------------------------------           F(  6,   653) =    4.17
       Model |  155.149478     6  25.8582463           Prob > F      =  0.0004
    Residual |  4048.98735   653  6.20059318           R-squared     =  0.0369
-------------+------------------------------           Adj R-squared =  0.0281
       Total |  4204.13682   659   6.3795703           Root MSE      =  2.4901

------------------------------------------------------------------------------
      ecolbs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lecoprc |   -2.56959   .5865181    -4.38   0.000    -3.721279   -1.417901
     lregprc |   2.204184   .5903005     3.73   0.000     1.045068    3.3633
     lfaminc |    .203861    .155967     1.31   0.192    -.1023964    .5101184
        educ |   .0251628   .0457977     0.55   0.583    -.0647657    .1150913
      hhsize |   .0015866   .0886932     0.02   0.986    -.1725717    .1757449
     num5_17 |   .1111276   .1343768     0.83   0.409    -.1527351    .3749903
       _cons |   .7307278   .7610805     0.96   0.337    -.7637326    2.225188
------------------------------------------------------------------------------
```

The price coefficients are of the expected sign: there is a negative own price effect, and a positive price effect for the substitute good, regular apples. The coefficient on log(*ecoprc*) implies that a 10% increase in *ecoprc* leads to a fall in estimated demand of about .26 lbs. At the mean value of *ecolbs*, about 1.47 lbs, this is an estimated own price elasticity of $-2.57/1.47 = -1.75$, which is very large in magnitude.

c. The test for heteroskedasticity is given below. The *F* statistic, which is asymptotically valid as a test for heteroskedasticity, gives a pretty large *p*-value, .362, so this test does not find much evidence of heteroskedasticity.

```
. predict ecolbsh
(option xb assumed; fitted values)

. gen ecolbshsq = ecolbsh^2

. predict uh, resid

. gen uhsq = uh^2

. reg uhsq  ecolbsh ecolbshsq
```

```
      Source |       SS       df       MS              Number of obs =     660
-------------+------------------------------           F(  2,   657) =    1.02
       Model |  8923.31842     2  4461.65921           Prob > F      =  0.3620
    Residual |  2880416.28   657  4384.19525           R-squared     =  0.0031
-------------+------------------------------           Adj R-squared =  0.0001
       Total |   2889339.6   659  4384.43034           Root MSE      =  66.213

------------------------------------------------------------------------------
        uhsq |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     ecolbsh |   32.61476   31.09945     1.05   0.295    -28.45153    93.68105
   ecolbshsq |    -8.9604   10.32346    -0.87   0.386    -29.23136    11.31056
       _cons |  -20.36486   21.92073    -0.93   0.353    -63.40798    22.67827
------------------------------------------------------------------------------
```

d. The fitted values were already gotten from part c. The summary statistics are

```
. sum ecolbs ecolbsh

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      ecolbs |       660     1.47399    2.525781          0         42
     ecolbsh |       660     1.47399     .485213   .2251952   2.598743

. count if ecolbs < 2.6
  541
```

```
. di 541/660
.81969697
```

The smallest fitted value is .225, and so none are negative. The largest fitted value is only about 2.6, but about 82 percent of the observations have $ecolbs_i$ below 2.6. Generally, it is difficult to find models that will track such a wide range in actual outcomes. Further, one might suspect the largest value, 42, is a mistake or an outlier. (The estimates with this one observation dropped give a similar story, but the price coefficients shrink in magnitude.)

e. The Tobit results are given below. The signs are the same as for the linear model, with the price and income variables being more statistically significant for Tobit. We know that the coefficients need to be scaled down in order to obtain the partial effects. That the Tobit coefficient on log(*ecoprc*) is about double the OLS estimate is not surprising, and we need to compute a scale factor. The scale factor for the APEs (of continuous explanatory variables) is about .547. If we multiply each Tobit coefficient by .547, we get fairly close to the OLS estimates.

```
. tobit ecolbs lecoprc lregprc lfaminc educ hhsize num5_17, ll(0)

Tobit regression                                  Number of obs   =        660
                                                  LR chi2(6)      =      50.79
                                                  Prob > chi2     =     0.0000
Log likelihood = -1265.7088                       Pseudo R2       =     0.0197

------------------------------------------------------------------------------
      ecolbs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lecoprc |  -5.238074   .8748606    -5.99   0.000    -6.955949     -3.5202
     lregprc |   4.261536   .8890055     4.79   0.000     2.515887    6.007185
     lfaminc |   .4149175   .2363235     1.76   0.080    -.0491269    .8789619
        educ |   .1005481    .068439     1.47   0.142    -.0338386    .2349348
      hhsize |   .0330173   .1325415     0.25   0.803    -.2272409    .2932756
     num5_17 |   .2260429   .1970926     1.15   0.252    -.1609678    .6130535
       _cons |  -1.917668   1.160126    -1.65   0.099    -4.195689    .3603525
-------------+----------------------------------------------------------------
      /sigma |   3.445719   .1268015                      3.196732    3.694706
------------------------------------------------------------------------------
  Obs. summary:        248  left-censored observations at ecolbs<=0
                       412     uncensored observations
                         0 right-censored observations
```

```
. predict xbh, xb

. gen prob = normal(xbh/_b[/sigma])

. sum prob

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        prob |       660    .5472506    .1152633   .2610003   .8191264
```

f. This question is a bit ambiguous. I will evaluate the partial effect at the mean value of *ecoprc*, *regprc*, and *faminc*, and then take the log, rather than averaging the logs. The scale factor for the APEs is given in part e: .547. The scale factor for the partial effects at the mean is .539, which is fairly close. The PAE of *lecoprc* is about −2.82, which is somewhat bigger in magnitude than the OLS estimate, −2.57.

To get the estimated elasticity, we need to estimate $\mathrm{E}(ecolbs|\mathbf{x})$ at the mean values of the covariates; we get about 1.55. So the estimated elasticity at the mean values of the covariates is about −2.82/1.55 ≈ −1.82. This is slightly larger in in magnitude than that computed for the linear model, −1.75.

```
. sum ecoprc regprc faminc educ hhsize num5_17

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      ecoprc |       660    1.081515     .295573        .59       1.59
      regprc |       660    .8827273    .2444687        .59       1.19
      faminc |       660    53.40909    35.74122          5        250
        educ |       660    14.38182    2.274014          8         20
      hhsize |       660    2.940909    1.526049          1          9
-------------+--------------------------------------------------------
     num5_17 |       660    .6212121     .994143          0          6

. di normal((_b[_cons] + _b[lecoprc]*log( 1.081515 )
     + _b[lregprc]*log(.8827273) + _b[lfaminc]*log(53.40909)
     + _b[educ]*14.38182 + _b[hhsize]*2.940909)/_b[/sigma])

.53860761

. di .53860761*_b[lecoprc]
-2.8212668

. di _b[_cons] + _b[lecoprc]*log( 1.081515 )
     + _b[lregprc]*log(.8827273) + _b[lfaminc]*log(53.40909)
     + _b[educ]* 14.38182 + _b[hhsize]*2.940909

.33398136
```

```
. di normalden((_b[_cons] + _b[lecoprc]*log( 1.081515 )
     + _b[lregprc]* log(.8827273) + _b[lfaminc]*log(53.40909)
     + _b[educ]* 14.38182 + _b[hhsize]*2.940909)/_b[/sigma])

.3970727

. di .33398136*.53860761 + _b[/sigma]*.3970727
1.5480857

. di -2.82/1.55
-1.8193548
```

g. Dropping log(*regprc*) greatly reduces the magnitude of the coefficient on log(*ecoprc*): from -5.24 to -1.82. A standard omitted variable analysis is a linear context suggests a positive correlation between *lecoprc* and *lregprc*. In fact, they are very highly positively correlated, with a correlation of about $.82$. This high correlation was built in as part of the experimental design.

```
. tobit ecolbs lecoprc lfaminc educ hhsize num5_17, ll(0)

Tobit regression                                  Number of obs   =        660
                                                  LR chi2(5)      =      27.60
                                                  Prob > chi2     =     0.0000
Log likelihood = -1277.3043                       Pseudo R2       =     0.0107

------------------------------------------------------------------------------
      ecolbs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lecoprc |  -1.822712   .5044411    -3.61   0.000    -2.813229   -.8321952
     lfaminc |   .3931692   .2395441     1.64   0.101    -.0771978    .8635362
        educ |   .1169085   .0692025     1.69   0.092    -.0189769     .252794
      hhsize |   .0222283   .1340901     0.17   0.868    -.2410699    .2855266
     num5_17 |   .2474529   .1996317     1.24   0.216    -.1445424    .6394483
       _cons |  -2.873156   1.161745    -2.47   0.014    -5.154351   -.5919615
-------------+----------------------------------------------------------------
      /sigma |   3.499092   .1291121                      3.245569    3.752616
------------------------------------------------------------------------------
  Obs. summary:        248  left-censored observations at ecolbs<=0
                       412     uncensored observations
                         0 right-censored observations

. corr lecoprc lregprc
(obs=660)

             |  lecoprc  lregprc
-------------+------------------
     lecoprc |   1.0000
     lregprc |   0.8205   1.0000
```

h. In fact, the Tobit model with prices in level form, rather than logarithms, fits a bit better

(log-likelihood = $-1,263.37$ versus $-1,265.71$).

```
. tobit ecolbs ecoprc regprc lfaminc educ hhsize num5_17, ll(0)

Tobit regression                                  Number of obs   =        660
                                                  LR chi2(6)      =      55.47
                                                  Prob > chi2     =     0.0000
Log likelihood = -1263.3702                       Pseudo R2       =     0.0215

------------------------------------------------------------------------------
      ecolbs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
      ecoprc |  -5.649516    .887358    -6.37   0.000    -7.391931   -3.907102
      regprc |   5.575299   1.063999     5.24   0.000     3.486032    7.664566
     lfaminc |   .4195658   .2354371     1.78   0.075    -.0427381    .8818696
        educ |   .1002944   .0681569     1.47   0.142    -.0335384    .2341271
      hhsize |   .0264861   .1320183     0.20   0.841    -.2327448    .2857171
     num5_17 |   .2351291   .1963111     1.20   0.231    -.1503469    .6206051
       _cons |  -1.632596   1.314633    -1.24   0.215    -4.214007    .9488146
-------------+----------------------------------------------------------------
      /sigma |   3.431504   .1262031                      3.183692    3.679316
------------------------------------------------------------------------------
  Obs. summary:        248  left-censored observations at ecolbs<=0
                       412     uncensored observations
                         0 right-censored observations
```

17.13. This extension has no practical effect on how we estimate an unobserved effects Tobit or probit model, or how we estimate a variety of unobserved effects panel data models with conditional normal heterogeneity. We simply have

$$c_i = -\left(T^{-1} \sum_{t=1}^{T} \boldsymbol{\pi}_t \right)\boldsymbol{\xi} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i \equiv \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i,$$

where $\psi \equiv -(T^{-1} \sum_{t=1}^{T} \boldsymbol{\pi}_t\boldsymbol{\xi})$. Of course, any aggregate time dummies explicitly get swept out of $\bar{\mathbf{x}}_i$ but they would usually be included in the equation.

An interesting follow-up question is: What if we standardize each $\mathbf{x}_{it}$ by its cross-sectional mean *and* variance at time t, and assume c_i is related to the mean and variance of the standardized vectors? In other words, let $\mathbf{z}_{it} \equiv (\mathbf{x}_{it} - \boldsymbol{\pi}_t)\boldsymbol{\Omega}_t^{-1/2}$, $t = 1, \ldots, T$, for each random draw i from the population, where $\boldsymbol{\Omega}_t \equiv \text{Var}(\mathbf{x}_{it})$. Then, we might assume

$$c_i|\mathbf{x}_i \sim \text{Normal}(\psi + \bar{\mathbf{z}}_i\boldsymbol{\xi}, \sigma_a^2)$$

(where, again, $\mathbf{z}_{it}$ would not contain aggregate time dummies). This is the kind of scenario that is handled by Chamberlain's more general assumption concerning the relationship between c_i and $\mathbf{x}_i$: $c_i = \psi + \sum_{t=1}^{T} \mathbf{x}_{ir}\boldsymbol{\lambda}_r + a_i$, where $\boldsymbol{\lambda}_r = \boldsymbol{\Omega}_r^{-1/2}\boldsymbol{\xi}/T$, $r = 1, 2, \ldots, T$. Alternatively, one could estimate $\boldsymbol{\pi}_t$ and $\boldsymbol{\Omega}_t$ for each t using the cross section observations $\{\mathbf{x}_{it} : i = 1, 2, \ldots, N\}$. The usual sample means and sample variance matrices, say $\hat{\boldsymbol{\pi}}_t$ and $\hat{\boldsymbol{\Omega}}_t$, are consistent and $\sqrt{N}$-asymptotically normal. Then, form $\hat{\mathbf{z}}_{it} \equiv (\mathbf{x}_{it} - \hat{\boldsymbol{\pi}}_t)\hat{\boldsymbol{\Omega}}_t^{-1/2}$, and proceed with the usual Tobit (or probit) unobserved effects analysis that includes the time averages $\bar{\hat{\mathbf{z}}}_i = T^{-1}\sum_{t=1}^{T} \hat{\mathbf{z}}_{it}$. This is a simple two-step estimation method, but accounting for the sample variation in $\hat{\boldsymbol{\pi}}_t$ and $\hat{\boldsymbol{\Omega}}_t$ analytically would be cumbersome. The panel bootstrap is an attractive alternative. Or, it may be possible to use a much larger sample to obtain $\hat{\boldsymbol{\pi}}_t$ and $\hat{\boldsymbol{\Omega}}_t$, in which case one might ignore the sampling error in the first-stage estimates.

17.14. a. Because heteroskedasticity is only in the distribution of a_i given $\mathbf{x}_i$, the density of y_{it} given $(\mathbf{x}_i, a_i)$ is the same as that implied by (17.75) and (17.76), namely, $y_{it}|\mathbf{x}_i, a_i \sim \text{Tobit}(\psi + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i, \sigma_u^2)$.

b. Let $f(y_t|\mathbf{x}_i, a_i; \boldsymbol{\eta})$ denote the Tobit density of $y_{it}|\mathbf{x}_i, a_i$ implied by part a, where $\boldsymbol{\eta}$ contains $\boldsymbol{\beta}$, ψ, $\boldsymbol{\xi}$, and σ_u^2. Then, under (17.78),

$$f(y_1, \ldots, y_T|\mathbf{x}_i, a_i; \boldsymbol{\eta}) = \prod_{t=1}^{T} f_t(y_t|\mathbf{x}_i, a_i; \boldsymbol{\eta}).$$

Therefore, to obtain $f(y_1, \ldots, y_T|\mathbf{x}_i; \boldsymbol{\theta})$, we integrate out a_i:

$$f(y_1, \ldots, y_T|\mathbf{x}_i; \theta) = \int_{-\infty}^{\infty} \prod_{t=1}^{T} f_t(y_t|\mathbf{x}_i, a_i; \eta))h(a|\mathbf{x}_i; \boldsymbol{\lambda}, \sigma_a^2)da, \tag{17.91}$$

where $h(a|\mathbf{x}_i; \boldsymbol{\lambda}, \sigma_a^2)$ denotes the normal density with mean zero and variance $\sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda})$. The

log-likelihood is obtained by plugging the y_{it} into (17.91) and taking the log.

c. The starting point is still equation (17.79), but the calculation of $\mathrm{E}[m(\psi + \mathbf{x}_t\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i, \sigma_u^2)|\mathbf{x}_i]$ is complicated by the heteroskedasticity in $\mathrm{Var}(a_i|\bar{\mathbf{x}}_i)$. Nevertheless, essentially the same argument used on page 542 shows that

$$\mathrm{E}[m(\psi + \mathbf{x}_t\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i, \sigma_u^2)|\mathbf{x}_i] = m[\psi + \mathbf{x}_t\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\xi}, \sigma_a^2 \exp(\bar{\mathbf{x}}_i\boldsymbol{\lambda}) + \sigma_u^2].$$

Given the MLEs, we can estimate the APEs from the average structural function:

$$\widehat{\mathrm{ASF}}(\mathbf{x}_t) = N^{-1}\sum_{i=1}^{N} m[\hat{\psi} + \mathbf{x}_t\hat{\boldsymbol{\beta}} + \bar{\mathbf{x}}_i\hat{\boldsymbol{\xi}}, \hat{\sigma}_a^2 \exp(\bar{\mathbf{x}}_i\hat{\boldsymbol{\lambda}}) + \hat{\sigma}_u^2];$$

we would compute changes or derivatives with respect to the elements of $\mathbf{x}_t$. Incidentally, if we drop assumption (17.78), we could use a pooled heteroskedastic Tobit procedure, and still consistently estimate the APEs.

17.15. a. The Stata output is given below. The value of the log-likelihood is $-17,599.96$.

```
. use cps91

. tobit hours nwifeinc educ exper expersq age kidlt6 kidge6, ll(0)

Tobit regression                                  Number of obs   =       5634
                                                  LR chi2(7)      =     645.55
                                                  Prob > chi2     =     0.0000
Log likelihood = -17599.958                       Pseudo R2       =     0.0180

------------------------------------------------------------------------------
       hours |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    nwifeinc |  -.2444726   .0165886   -14.74   0.000    -.2769926   -.2119525
        educ |  -6.064707   22.73817    -0.27   0.790    -50.64029    38.51087
       exper |  -8.234015   22.74967    -0.36   0.717    -52.83214    36.36411
     expersq |  -.0178206   .0041379    -4.31   0.000    -.0259325   -.0097087
         age |    8.53901   22.73703     0.38   0.707    -36.03435    53.11237
      kidlt6 |   -14.0809    1.21084   -11.63   0.000    -16.45461   -11.70719
      kidge6 |  -1.593786    1.09917    -1.45   0.147    -3.748583    .5610116
       _cons |  -56.32579   136.3411    -0.41   0.680    -323.6069    210.9553
-------------+----------------------------------------------------------------
      /sigma |   28.90194   .3998526                      28.11807     29.6858
------------------------------------------------------------------------------
  Obs. summary:       2348  left-censored observations at hours<=0
                      3286     uncensored observations
                         0 right-censored observations
```

b. The lognormal hurdle model – which has eight more parameters than the Tobit model – does fit better in this application. The log likelihood – which properly account for the fact that the linear regression for $\log(hours_i)$ is to be viewed as MLE for $D(hours|\mathbf{x}, hours > 0)$ – is about $-16,987.50$. The contribution of the probit is about $-3,538.41$ and the contribution of the lognormal distribution conditional on $hours > 0$ is $-13,449.09$. The log likelihood for the Tobit is $-17,599.96$.

```
. probit inlf nwifeinc educ exper expersq age kidlt6 kidge6

Probit regression                                 Number of obs   =       5634
                                                  LR chi2(7)      =     576.67
                                                  Prob > chi2     =     0.0000
Log likelihood = -3538.4086                       Pseudo R2       =     0.0753

------------------------------------------------------------------------------
        inlf |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    nwifeinc |  -.0091475   .0006759   -13.53   0.000    -.0104722   -.0078227
        educ |  -.0626136   .9045369    -0.07   0.945    -1.835473    1.710246
       exper |   -.157161   .9050879    -0.17   0.862    -1.931101    1.616779
     expersq |  -.0005574   .0001713    -3.25   0.001     -.000893   -.0002217
         age |   .1631286   .9044966     0.18   0.857    -1.609652    1.935909
      kidlt6 |  -.4810832    .051688    -9.31   0.000    -.5823897   -.3797767
      kidge6 |   .0409155   .0471194     0.87   0.385    -.0514367    .1332678
       _cons |  -1.489209   5.422855    -0.27   0.784    -12.11781    9.139393
------------------------------------------------------------------------------

. gen lhours = log(hours)
(2348 missing values generated)

. glm lhours nwifeinc educ exper expersq age kidlt6 kidge6

Iteration 0:   log likelihood = -1954.9002

Generalized linear models                         No. of obs      =      3286
Optimization     : ML                             Residual df     =      3278
                                                  Scale parameter =  .1928961
Deviance         =  632.3133243                   (1/df) Deviance =  .1928961
Pearson          =  632.3133243                   (1/df) Pearson  =  .1928961

Variance function: V(u) = 1                       [Gaussian]
Link function    : g(u) = u                       [Identity]

                                                  AIC             =  1.194705
Log likelihood   = -1954.900228                   BIC             = -25911.05

------------------------------------------------------------------------------
             |                 OIM
      lhours |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
```

```
-------------+----------------------------------------------------------------
    nwifeinc |  -.0018706   .0003327    -5.62   0.000    -.0025227   -.0012185
        educ |  -.2022625   .4403476    -0.46   0.646    -1.065328     .660803
       exper |  -.2074679   .4405366    -0.47   0.638    -1.070904     .655968
     expersq |  -.0001549   .0000812    -1.91   0.057     -.000314    4.33e-06
         age |   .2112264   .4403162     0.48   0.631    -.6517775     1.07423
      kidlt6 |  -.1944414   .0222299    -8.75   0.000    -.2380113   -.1508715
      kidge6 |  -.1256763   .0199962    -6.29   0.000    -.1648681   -.0864845
       _cons |   2.252439   2.640541     0.85   0.394    -2.922926    7.427805
------------------------------------------------------------------------------

. sum lhours

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      lhours |      3286    3.497927    .4467688          0   4.787492

. di 3286*r(mean)
11494.189

. di -1954.9002 - 11494.189
-13449.089

. di -3538.4086 - 13449.089
-16987.498
```

c. The ET2T model is given below. Again, to properly compare its log likelihood, we must subtract $\sum_{i=1}^{N_1} \log(hours_i)$ to obtain the final log likelihood. As must be the case, the ET2T model fits better than the lognormal hurdle model, but the improvement is very slight. In fact, the estimate of ρ is very small – about $.018$ – and not statistically different from zero. The likelihood ratio statistic gives the same result, producing p-value = $.862$. Fortunately the estimated coefficients are very similar across the two approaches, as we would hope with $\hat{\rho}$ so close to zero.

These findings are very different from what we found using the data in MROZ.RAW – see Table 17.2. There, the estimate of ρ is an implausible $-.972$. Without an exclusion restriction (in either application) it is hard to be confident of the results. But with the current data set, we are led to the lognormal hurdle model with all explanatory variables in the selection and amount equations.

```
. heckman lhours nwifeinc educ exper expersq age kidlt6 kidge6,
          select(inlf = nwifeinc educ exper expersq age kidlt6 kidge6)
```

```
Heckman selection model                         Number of obs      =      5634
(regression model with sample selection)        Censored obs       =      2348
                                                Uncensored obs     =      3286

                                                Wald chi2(7)       =     93.35
Log likelihood = -5493.294                      Prob > chi2        =    0.0000

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
lhours       |
    nwifeinc |   -.001911   .0003968    -4.82   0.000    -.0026886   -.0011333
        educ |  -.2023362   .4398277    -0.46   0.645    -1.064383    .6597103
       exper |  -.2079298   .4400233    -0.47   0.637     -1.07036    .6545001
     expersq |  -.0001578   .0000827    -1.91   0.056    -.0003198    4.21e-06
         age |   .2117355   .4398047     0.48   0.630    -.6502658    1.073737
      kidlt6 |  -.1964925   .0247921    -7.93   0.000     -.245084   -.1479009
      kidge6 |  -.1255154   .0199914    -6.28   0.000    -.1646978    -.086333
       _cons |   2.240756    2.63817     0.85   0.396    -2.929962    7.411474
-------------+----------------------------------------------------------------
inlf         |
    nwifeinc |  -.0091462    .000676   -13.53   0.000    -.0104711   -.0078214
        educ |  -.0628333   .9045172    -0.07   0.945    -1.835654    1.709988
       exper |  -.1574369   .9050687    -0.17   0.862    -1.931339    1.616465
     expersq |  -.0005568   .0001713    -3.25   0.001    -.0008925   -.0002211
         age |   .1633826   .9044773     0.18   0.857     -1.60936    1.936125
      kidlt6 |  -.4810173   .0516912    -9.31   0.000    -.5823302   -.3797043
      kidge6 |   .0410785   .0471309     0.87   0.383    -.0512964    .1334534
       _cons |  -1.491111   5.422743    -0.27   0.783    -12.11949     9.13727
-------------+----------------------------------------------------------------
     /athrho |   .0178479   .0959507     0.19   0.852    -.1702121    .2059078
    /lnsigma |  -.8239347   .0123732   -66.59   0.000    -.8481857   -.7996837
-------------+----------------------------------------------------------------
         rho |    .017846   .0959202                     -.1685871    .2030463
       sigma |   .4387021   .0054281                      .4281911    .4494711
      lambda |   .0078291   .0420881                      -.074662    .0903202
------------------------------------------------------------------------------
LR test of indep. eqns. (rho = 0):   chi2(1) =     0.03   Prob > chi2 = 0.8615
------------------------------------------------------------------------------

. di -5493.294 - 11494.189
-16987.483

. di 2*(16987.498 - 16987.483)
.03
```

d. The estimates for the amount part of the truncated normal hurdel model are given below. Because the participation equation is still the probit model we estimated earlier, we can compare the log likelihood for the truncated normal regression to that from the lognormal estimation in part b. The former is $-12,445.76$ and we already computed the latter as $-13,449.09$. Thus, in this example the TNH model fits substantially better than the LH model.

The full log likelihood for the TNH model is $-15,984.17$ and this is much larger than that for the Tobit (a special case), $-17,599.96$.

```
. truncreg hours nwifeinc educ exper expersq age kidlt6 kidge6, ll(0)
(note: 2348 obs. truncated)

Fitting full model:

Limit:   lower =          0                     Number of obs =   3286
         upper =       +inf                     Wald chi2(7)  = 132.08
Log likelihood =  -12445.76                     Prob > chi2   = 0.0000

------------------------------------------------------------------------------
       hours |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    nwifeinc |  -.0439736   .0081584    -5.39   0.000    -.0599638   -.0279835
        educ |  -9.183178   11.12374    -0.83   0.409    -30.98531    12.61896
       exper |  -9.426741   11.12822    -0.85   0.397    -31.23765    12.38417
     expersq |  -.0024584   .0019888    -1.24   0.216    -.0063564    .0014396
         age |   9.470886   11.12299     0.85   0.395    -12.32978    31.27155
      kidlt6 |  -4.779305   .5444546    -8.78   0.000    -5.846417   -3.712194
      kidge6 |  -3.370223   .4896076    -6.88   0.000    -4.329837    -2.41061
       _cons |  -21.34309   66.70579    -0.32   0.749     -152.084    109.3979
-------------+----------------------------------------------------------------
      /sigma |   10.72244   .1347352    79.58   0.000     10.45836    10.98651
------------------------------------------------------------------------------

. di -3538.4086 - 12445.76
-15984.169
```

17.16. a. Write $c_i = \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i$ and substitute to get

$$y_{it}^* = \mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i + u_{it}.$$

Now, conditional on $(\mathbf{x}_i, a_i)$, y_{it} follows a standard two-limit Tobit model. Therefore, the density is

$$\begin{aligned} f(y_t|\mathbf{x}_i, a_i; \boldsymbol{\gamma}) = {} & [\Phi((q_1 - \mathbf{x}_{it}\boldsymbol{\beta} - \psi - \bar{\mathbf{x}}_i\boldsymbol{\xi} - a_i)/\sigma_u)]^{1[y_t = q_1]} \\ & \cdot \{\sigma_u^{-1}\phi[(y_t - \mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)/\sigma_u\}^{1[q_1 < y_t < q_2]} \\ & \cdot [\Phi((-q_2 + \mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + a_i)/\sigma_u)]^{1[y_t = q_2]} \end{aligned}$$

Byt the conditional independence assumption, the joint density of $(y_{i1}, y_{i2}, \ldots, y_{iT})$ conditional on $(\mathbf{x}_i, a_i)$ is

$$\prod_{t=1}^{T} f(y_t|\mathbf{x}_i, a_i; \boldsymbol{\gamma}).$$

Now we integrate out a_i to get the joint density of $(y_{i1}, y_{i2}, \ldots, y_{iT})$ given $\mathbf{x}_i$:

$$\int_{-\infty}^{\infty} \left[\prod_{t=1}^{T} f(y_t|\mathbf{x}_i, a; \boldsymbol{\gamma}) \right] \sigma_a^{-1} \phi(a/\sigma_a) da.$$

Now the log likelihood for a random draw i is just

$$\ell_i(\boldsymbol{\theta}) \equiv \log \left\{ \int_{-\infty}^{\infty} \left[\prod_{t=1}^{T} f(y_{it}|\mathbf{x}_i, a; \boldsymbol{\gamma}) \right] \sigma_a^{-1} \phi(a/\sigma_a) da \right\}$$

where $\boldsymbol{\theta}$ is the vector of all parameters, including σ_a^2. As usual, we sum across all i to get the log likelihood for the entire cross section.

b. This is no different from any of the other CRE models that we have covered. We can easily find $\mathrm{E}(c_i)$ and $\mathrm{Var}(c_i)$ from $c_i = \psi + \bar{\mathbf{x}}_i \boldsymbol{\xi} + a_i$ because $\mathrm{E}(a_i|\mathbf{x}_i) = 0$ and $\mathrm{Var}(a_i|\mathbf{x}_i) = \sigma_a^2$. In fact,

$$\mathrm{E}(c_i) = \mathrm{E}(\psi + \bar{\mathbf{x}}_i \boldsymbol{\xi}) = \psi + \mathrm{E}(\bar{\mathbf{x}}_i) \boldsymbol{\xi}$$

and so a consistent estimator of $\mathrm{E}(c_i)$ is

$$\hat{\mu}_c = \hat{\psi} + \left(N^{-1} \sum_{i=1}^{N} \bar{\mathbf{x}}_i \right) \hat{\boldsymbol{\xi}}$$

where $\hat{\psi}$ and $\hat{\boldsymbol{\xi}}$ are the MLEs.

Next,

$$\begin{aligned} \mathrm{Var}(c_i) &= \mathrm{Var}(\psi + \bar{\mathbf{x}}_i \boldsymbol{\xi}) + \mathrm{Var}(a_i) \\ &= \boldsymbol{\xi}' \mathrm{Var}(\bar{\mathbf{x}}_i) \boldsymbol{\xi} + \sigma_a^2, \end{aligned}$$

where we use the fact that a_i and $\bar{\mathbf{x}}_i$ are uncorrelated. So a consistent estimator is

$$\hat{\sigma}_c^2 = \hat{\boldsymbol{\xi}}' \left[N^{-1} \sum_{i=1}^{N} (\bar{\mathbf{x}}_i - \bar{\mathbf{x}})'(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}) \right] \hat{\boldsymbol{\xi}} + \hat{\sigma}_a^2$$

where $\bar{\mathbf{x}} = N^{-1} \sum_{i=1}^{N} \bar{\mathbf{x}}_i$.

c. We can get the average structural function by slightly modifying equation (17.66). First, the conditional mean is

$$\begin{aligned} \mathrm{E}(y_{it}|\mathbf{x}_{it}, c_i) = {} & q_1[\Phi((q_1 - \mathbf{x}_{it}\boldsymbol{\beta} - c_i)/\sigma_u)] \\ & + [\Phi((-q_2 + \mathbf{x}_{it}\boldsymbol{\beta} + c_i)/\sigma_u) - \Phi((q_1 - \mathbf{x}_{it}\boldsymbol{\beta} - c_i)/\sigma_u)] \cdot g(q_1, q_2, \mathbf{x}_{it}\boldsymbol{\beta} + c_i, \sigma_u^2) \\ & + q_2[\Phi((-q_2 + \mathbf{x}_{it}\boldsymbol{\beta} + c_i)/\sigma_u)] \end{aligned}$$

where

$$g(q_1, q_2, z, \sigma^2) \equiv z + \frac{\{\phi[(q_2 - z)/\sigma_u] - \phi[(q_1 - z)/\sigma_u]\}}{\{\Phi[(q_2 - z)/\sigma_u] - \Phi[(q_1 - z)/\sigma_u]\}}.$$

The ASF is obtained as a function of $\mathbf{x}_t$ by averaging out c_i. But we can use iterated expectations, as usual, by first conditioning on $\bar{\mathbf{x}}_i$ and then averaging out $\bar{\mathbf{x}}_i$:

$$\mathrm{E}_{c_i}[m(\mathbf{x}_t, c_i)] = \mathrm{E}_{\bar{\mathbf{x}}_i}\{\mathrm{E}[m(\mathbf{x}_t, c_i)|\bar{\mathbf{x}}_i]\}$$

where $m(\mathbf{x}_t, c) = \mathrm{E}(y_{it}|\mathbf{x}_{it} = \mathbf{x}_t, c_i = c)$. Using the same argument from the one-limit CRE Tobit model,

$$\begin{aligned} h(\mathbf{x}_t, \bar{\mathbf{x}}_i) \equiv \mathrm{E}[m(\mathbf{x}_t, c_i)|\bar{\mathbf{x}}_i] = {} & q_1[\Phi((q_1 - \mathbf{x}_t\boldsymbol{\beta} - \psi - \bar{\mathbf{x}}_i\boldsymbol{\xi})/\sigma_v)] \\ & + [\Phi((-q_2 + \mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi})/\sigma_v)) - \Phi((q_1 - \mathbf{x}_t\boldsymbol{\beta} - \psi - \bar{\mathbf{x}}_i\boldsymbol{\xi})/\sigma_v)] \\ & \cdot g(q_1, q_2, \mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi}, \sigma_v^2) \\ & + q_2[\Phi((-q_2 + \mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi})/\sigma_v)] \end{aligned}$$

where $\sigma_v^2 = \sigma_a^2 + \sigma_u^2$. The ASF is consistently estimated as

$$\widehat{ASF}(\mathbf{x}_t) = N^{-1} \sum_{i=1}^{N} \hat{h}(\mathbf{x}_t, \bar{\mathbf{x}}_i)$$

where $\hat{h}(\cdot,\cdot)$ denotes plugging in the MLEs of all estimates. Now take derivatives and changes with respect to $\mathbf{x}_t$.

d. Without assumption (17.78), we can just use a pooled two-limit Tobit analysis to estimate $\boldsymbol{\beta}$, ψ, $\boldsymbol{\xi}$, and σ_v^2. As in the standard Tobit case, we cannot separately estimate σ_a^2 and σ_u^2. But the APEs are still identified as they depend only on σ_v^2, as shown in part c.

17.17. a. Plug in the expressions for c_{i1} and c_{i2} to get

$$\begin{aligned} y_{it1} &= \max(0, \alpha_1 y_{it2} + \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + a_{i1} + u_{it1}) \\ y_{it2} &= \mathbf{z}_{it}\boldsymbol{\pi}_2 + \psi_2 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_2 + a_{i2} + u_{it2} \end{aligned}$$

or

$$\begin{aligned} y_{it1} &= \max(0, \alpha_1 y_{it2} + \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + v_{it1}) \\ y_{it2} &= \mathbf{z}_{it}\boldsymbol{\pi}_2 + \psi_2 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_2 + v_{it2} \end{aligned}$$

where $v_{it} = a_{i1} + u_{it1}$ and $v_{it2} = a_{i2} + u_{it2}$. Given the assumptions on $\mathrm{D}(u_{it1}, u_{it2}|\mathbf{z}_i, \mathbf{a}_i)$ and $\mathrm{D}(a_{i1}, a_{i2}|\mathbf{z}_i)$, it follows that $\mathrm{D}(v_{it1}, v_{it2}|\mathbf{z}_i)$ =$\mathrm{D}(v_{it1}, v_{it2})$ is bivariate normal with mean zero. Therefore, we can write

$$\begin{aligned} v_{it1} &= \rho_1 v_{it2} + e_{it1} \\ \mathrm{D}(e_{it1}|\mathbf{z}_i, v_{it2}) &= \mathrm{E}(e_{it1}) = Normal(0, \sigma_{e_1}^2). \end{aligned}$$

It follows we can write

$$\begin{aligned} y_{it1} &= \max(0, \alpha_1 y_{it2} + \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + \rho_1 v_{it2} + e_{it1}) \\ \mathrm{D}(e_{it1}|\mathbf{z}_i, y_{i,t2}, v_{it2}) &= Normal(0, \sigma_{e_1}^2) \end{aligned}$$

and now a pooled two-step method is immediate. First, obtain the residuals $\hat{v}_{it2}$ from the pooled regression

$$y_{it2} \text{ on } \mathbf{z}_{it},\ 1,\ \bar{\mathbf{z}}_i,\ t = 1, \ldots, T; i = 1, \ldots, N.$$

Then use pooled Tobit of

$$y_{it1} \text{ on } y_{it2},\ \mathbf{z}_{it1},\ 1,\ \bar{\mathbf{z}}_i,\ \hat{v}_{it2}$$

to estimate α_1, $\boldsymbol{\delta}_1$, ψ_1, $\boldsymbol{\xi}_1$, ρ_1, and $\sigma_{e_1}^2$.

Incidentally, the statement of the problem said that $\{y_{it2}\}$ will not be strictly exogenous in the estimable equation. While that is true of the previously proposed solution, in other approaches $\{y_{it2}\}$ can be rendered strictly exogenous. Here is one possibility. Let $\mathbf{v}_{i2} = (v_{i12}, \ldots, v_{iT2})$ be the entire history on the reduced form errors. Then, given the previous assumptions, $\text{D}(v_{it1}|\mathbf{z}_i, \mathbf{v}_{i2}) = \text{D}(v_{it1}|\mathbf{v}_{i2})$. Because $v_{it1} = a_{i1} + u_{it1}$, it is reasonable to assume a Chamberlain-Mundlak representation, for example,

$$v_{it1} = \rho_1 v_{it2} + \gamma_1 \bar{v}_{i2} + e_{it1}$$

where now e_{it1} is independent of $(\mathbf{z}_i, \mathbf{v}_{i2})$ and therefore of $(\mathbf{z}_i, \mathbf{v}_{i2}, \mathbf{y}_{i2})$, where $\mathbf{y}_{i2} = (y_{i12}, \ldots, y_{iT2})$. This means that in the equation

$$y_{it1} = \max(0, \alpha_1 y_{it2} + \mathbf{z}_{it1}\boldsymbol{\delta}_1 + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + \rho_1 v_{it2} + \gamma_1 \bar{v}_{i2} + e_{it1}),$$

$\{y_{ir2}, \mathbf{z}_{ir}, v_{ir2} : r = 1, \ldots, T\}$ is strictly exogenous with respect to e_{it1}. The CF approach changes in that we add $\hat{\bar{v}}_{i2}$ as an additional explanatory variable (along with $\hat{v}_{it2}$) in using pooled Tobit. Because of strict exogeneity, approaches that attempt to exploit the serial dependence in the scores are now possible.

b. As usual, the two-step nature of the estimation needs to be accounted for by using either the delta method or the panel bootstrap. In using the delta method, the serial dependence in the scores should be accounted for. It is automatically accounted for with the panel bootstrap because the cross section units are resampled.

c. We have used this approach several times. Let $m(z, \sigma^2)$ denote the unconditional mean

function for the standard Tobit model. Then

$$\text{ASF}(y_{t2}, \mathbf{z}_{t1}) = \text{E}_{(\bar{\mathbf{z}}_i, v_{it2})}[m(\alpha_1 y_{t2} + \mathbf{z}_{t1}\boldsymbol{\delta}_1 + \psi_1 + \bar{\mathbf{z}}_i\boldsymbol{\xi}_1 + \rho_1 v_{it2}, \sigma^2_{e1})]$$

and so a consistent estimator is

$$\widehat{\text{ASF}}(y_{t2}, \mathbf{z}_{t1}) = N^{-1}\sum_{i=1}^{N} m(\hat{\alpha}_1 y_{t2} + \mathbf{z}_{t1}\hat{\boldsymbol{\delta}}_1 + \hat{\psi}_1 + \bar{\mathbf{z}}_i\hat{\boldsymbol{\xi}}_1 + \hat{\rho}_1\hat{v}_{it2}, \hat{\sigma}^2_{e1}).$$

As usual, the estimated APEs are obtained by taking derivatives or changes with respect to $(y_{t2}, \mathbf{z}_{t1})$.

17.18. a. Once we assume $\mathbf{z}$ is exogenous in the structural equation – and $\text{E}(u_1|\mathbf{z}) = 0$ ensures exogeneity – then we only need the rank condition. The assumption that $\text{E}(\mathbf{z}'\mathbf{z})$ is nonsingular is not usually restrictive. The important condition with a single endogenous explanatory variable is

$$\text{L}(y_2|\mathbf{z}) \neq \text{L}(y_2|\mathbf{z}_1),$$

so there is at least one element of $\mathbf{z}$ not in $\mathbf{z}_1$ that explains variation in y_2.

b. We can draw on the optimal instrument variables results from Section 8.6. The condition $\text{E}(u_1|\mathbf{z}) = 0$ ensures that any function of $\mathbf{z}$ is a valid instrumental variable candidate, and also implies that $\text{E}(u_1^2|\mathbf{z}) = \text{Var}(u_1|\mathbf{z})$. Because $\text{E}(u_1^2|\mathbf{z})$ is constant, from Theorem 8.5 the optimal IVs are

$$[\text{E}(y_2|\mathbf{z}), \mathbf{z}_1)].$$

If we think $\text{D}(y_2|\mathbf{z})$ follows a standard Tobit then we should obtain $\text{E}(y_2|\mathbf{z})$ from the Tobit model. Recall that if

$$\text{D}(y_2|\mathbf{z}) = \text{Tobit}(\mathbf{z}\boldsymbol{\delta}_2, \tau_2^2)$$

then

$$\mathrm{E}(y_2|\mathbf{z}) = \Phi(\mathbf{z}\boldsymbol{\delta}_2/\tau_2)\mathbf{z}\boldsymbol{\delta}_2 + \tau_2\phi(\mathbf{z}\boldsymbol{\delta}_2/\tau_2)$$

Therefore, if we run Tobit in a first stage, we get

$$\hat{m}_{i2} \equiv \hat{\mathrm{E}}(y_{i2}|\mathbf{z}_i) = \Phi(\mathbf{z}_i\hat{\boldsymbol{\delta}}_2/\hat{\tau}_2)\mathbf{z}_i\hat{\boldsymbol{\delta}}_2 + \hat{\tau}_2\phi(\mathbf{z}_i\hat{\boldsymbol{\delta}}_2/\hat{\tau}_2)$$

and then use IVs $(\hat{m}_{i2}, \mathbf{z}_{i1})$ in the equation

$$y_{i1} = \alpha_1 y_{i2} + \mathbf{z}_{i1}\boldsymbol{\delta}_1 + u_{i1}$$

by IV. This approach just identifies the parameters. We can get overidentification (if we have enough elements of $\mathbf{z}_i$) by using all of $\mathbf{z}_i$ in place of $\mathbf{z}_{i1}$.

Provided we maintain $\mathrm{E}(u_1|\mathbf{z}) = 0$, using Tobit fitted values as instruments is no less (and no more) robust than using 2SLS. As mentioned previously, *any* function of $\mathbf{z}_i$ is valid as a potential instrument. Even if the Tobit model is incorrect, we know the quasi-MLEs converge very generally. Call the plims $\boldsymbol{\delta}_2^*$ and τ_2^* and define

$$m_{i2}^* \equiv \Phi(\mathbf{z}_i\boldsymbol{\delta}_2^*/\tau_2^*)\mathbf{z}_i\boldsymbol{\delta}_2^* + \tau_2^*\phi(\mathbf{z}_i\boldsymbol{\delta}_2^*/\tau_2^*),$$

which is just a function of $\mathbf{z}_i$. Ruling out perfect collinearity in $(m_{i2}^*, \mathbf{z}_{i1})$, the rank condition is

$$\mathrm{L}(y_{i2}|m_{i2}^*, \mathbf{z}_{i1}) \neq \mathrm{L}(y_{i2}|\mathbf{z}_{i1}),$$

which simply means that m_{i2}^* should have some partial correlation with $\mathbf{z}_{i1}$, something we would expect quite generally if $\mathbf{z}_{i2}$ is partially correlated with y_{i2}.

Using $\hat{m}_{i2}$ as an instrument for y_{i2} is preferred to using it as a regressor in place of $\hat{m}_{i2}$. If we use $\hat{m}_{i2}$ as a regressor then we are effectively assuming $\mathrm{E}(y_2|\mathbf{z}) = \Phi(\mathbf{z}\boldsymbol{\delta}_2/\tau_2)\mathbf{z}\boldsymbol{\delta}_2 + \tau_2\phi(\mathbf{z}\boldsymbol{\delta}_2/\tau_2)$ (and that we have consistent estimators of the parameters in this mean). Generally, the estimates of $\boldsymbol{\delta}_1$ and α_1 would be inconsistent of the Tobit model for y_2 is misspecified. When $\hat{m}_{i2}$ as an instrument, the reduced implicit reduced

form for the IV estimation is

$$\mathrm{L}(y_2|m_{i2}^*, \mathbf{z}_{i1}) = \psi_2 m_{i2}^* + \mathbf{z}_{i1}\boldsymbol{\eta}_1$$

and we do not need $\psi_2 = 1$ and $\boldsymbol{\eta}_1 = \mathbf{0}$, as the plug-in-regressor method essentially does.

c. We can write

$$\begin{aligned} y_1 &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 v_2 + e_1 \\ \mathrm{E}(e_1|\mathbf{z}, y_2, v_2) &= 0 \end{aligned}$$

It follows that

$$\mathrm{E}(y_1|\mathbf{z}, y_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 \mathrm{E}(v_2|\mathbf{z}, y_2)$$

and we can compute $\mathrm{E}(v_2|\mathbf{z}, y_2)$ given that $\mathrm{D}(y_2|\mathbf{z}) = \text{Tobit}(\mathbf{z}\boldsymbol{\delta}_2, \tau_2^2)$. In fact, as shown in Vella (1993, *International Economic Review*),

$$\begin{aligned} \mathrm{E}(v_2|\mathbf{z}, y_2) &= 1[y_2 > 0]v_2 - 1[y_2 = 0]\tau_2\left[\frac{\phi(\mathbf{z}\boldsymbol{\delta}_2/\tau_2)}{1 - \Phi(\mathbf{z}\boldsymbol{\delta}_2/\tau_2)}\right] \\ &= 1[y_2 > 0]v_2 - 1[y_2 = 0]\tau_2\lambda(-\mathbf{z}\boldsymbol{\delta}_2/\tau_2) \end{aligned}$$

where $\lambda(\cdot) = \phi(\cdot)/\Phi(\cdot)$ is the inverse Mills ratio. (This is an example of a *generalized residual*.)

Given the Tobit MLEs, we can easily construct

$$\hat{\mathrm{E}}(v_{i2}|\mathbf{z}_i, y_{i2}) = 1[y_{i2} > 0]\hat{v}_{i2} - 1[y_{i2} = 0]\hat{\tau}_2\lambda(-\mathbf{z}_i\hat{\boldsymbol{\delta}}_2/\hat{\tau}_2)$$

in a first stage, and then in a second stage run the OLS regression

$$y_{i1} \text{ on } \mathbf{z}_{i1},\ y_{i2},\ 1[y_{i2} > 0]\hat{v}_{i2} - 1[y_{i2} = 0]\hat{\tau}_2\lambda(-\mathbf{z}_i\hat{\boldsymbol{\delta}}_2/\hat{\tau}_2)$$

to consistently estimate $\boldsymbol{\delta}_1$, α_1, and ρ_1.

Because the CF approach is based on $\mathrm{E}(y_1|\mathbf{z}, y_2)$, nothing important changes if we start with

$$y_1 = \mathbf{g}_1(\mathbf{z}_1, y_2)\boldsymbol{\beta}_1 + u_1.$$

The same reasoning as before gets us to

$$\mathrm{E}(y_1|\mathbf{z}, y_2) = \mathbf{g}_1(\mathbf{z}_1, y_2)\boldsymbol{\beta}_1 + \rho_1\{1[y_2 > 0]v_2 - 1[y_2 = 0]\tau_2\lambda(-\mathbf{z}\boldsymbol{\delta}_2/\tau_2)\}$$

and so adding the same CF as before works for consistently estimating $\boldsymbol{\beta}_1$. Of course, the interpretation of $\boldsymbol{\beta}_1$ depends on the nature of the functions in $\mathbf{g}_1(\mathbf{z}_1, y_2)$.

d. The 2SLS estimator that effectively ignores the nature of y_2 is simple and fully robust – assuming we have at least one valid instrument for y_2. Standard errors (robust to heteroskedasticity) are easy to obtain. Its primary drawback is that it may be (asymptotically) inefficient compared with the other methods. An additional shortcoming is that if we use general functions $\mathbf{g}_1(\mathbf{z}_1, y_2)$ we need to decide on instruments for any function that includes y_2. [Remember we are generally *not* allowed to plug in a fitted value to obtain $\mathbf{g}_1(\mathbf{z}_{i1}, \hat{y}_{i2})$ and then regress y_{i1} on $\mathbf{g}_1(\mathbf{z}_{i1}, \hat{y}_{i2})$.]

The method of using the Tobit fitted value as the IV for y_2 is just as robust as 2SLS estimator yet it exploits the corner solution nature of y_2. It need not be more (asymptotically) efficient than 2SLS, but it could be even if the Tobit model for y_2 is misspecified [or $\mathrm{Var}(u_1|\mathbf{z})$ is homoskedastic, or both]. That we have estimated the instruments in a first stage can be ignored in the $\sqrt{N}$-asymptotic distribution of the IV estimator. Like the 2SLS estimator, having general functions $\mathbf{g}_1(\mathbf{z}_1, y_2)$ means we would have to obtain IVs for all endogenous functions. This is almost always possible but is not always obvious.

The CF method is simple to compute but the standard errors generally have to account for the two-step estimation unless $\rho_1 = 0$. (The CF method provides a simple test of the null that y_2 is endogenous: just use a heteroskedasticity-robust t statistic for $\hat{\rho}_1$.) Another drawback to

the CF method is that it is derived assuming the Tobit model for y_2 holds. Generally, it is inconsistent if the Tobit model fails (just like using Tobit fitted values as regressors rather than instruments). An advantage of the CF method is that, as discussed in part c, it is easily applied for general functions $\mathbf{g}_1(\mathbf{z}_1, y_2)$. In such cases, the CF method is likely to be more efficient asymptotically than 2SLS or the IV method described in part b.

e. If we assume joint normality of (u_1, v_2) (and independence from $\mathbf{z}$) then MLE becomes attractive. It will give the asymptotically efficient estimators and there is no two-step estimation issue to deal with (as in the CF case). The log likelihood is a bit tricky to obtain because it depends on $\text{D}(y_1|y_2, \mathbf{z})$. We already know that $\text{D}(y_2|\mathbf{z})$ follows a Tobit. We also know $\text{D}(y_1|v_2, \mathbf{z})$ follows a classical linear regression model with mean $\mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 v_2$ and variance $\sigma_{e_1}^2$. For $y_2 > 0$, $\text{D}(y_1|y_2, \mathbf{z}) = \text{D}(y_1|v_2, \mathbf{z})$. For $y_2 = 0$, we have to integrate over $v_2 \leq -\mathbf{z}\boldsymbol{\delta}_2$, just like in Problem 17.6. When we have to two densities, we use, for each i,

$$\log[f_1(y_{i1}|y_{i2}, \mathbf{z}_i; \boldsymbol{\theta}) + \log[f_2(y_{i2}|\mathbf{z}_i; \boldsymbol{\delta}_2, \tau_2^2)$$

as the log likelihood.

17.19. a. Because of the conditional independence assumption we have

$$f(\mathfrak{y}_1, \ldots, \mathfrak{y}_G|\mathbf{x}, \mathbf{c}; \boldsymbol{\gamma}_o) = \prod_{g=1}^{G} f_g(\mathfrak{y}_g|\mathbf{x}, \mathbf{c}; \boldsymbol{\gamma}_o^g)$$

for dummy arguments $(\mathfrak{y}_1, \ldots, \mathfrak{y}_G)$.

b. To obtain the density of $(y_1, \ldots, y_G)$ given $\mathbf{x}$ we integrate out $\mathbf{c}$:

$$g(\mathfrak{y}_1, \ldots, \mathfrak{y}_G|\mathbf{x}; \boldsymbol{\gamma}_o, \boldsymbol{\delta}_o) = \int \left[\prod_{g=1}^{G} f_g(\mathfrak{y}_g|\mathbf{x}, \mathfrak{c}; \boldsymbol{\gamma}_o^g) \right] h(\mathfrak{c}|\mathbf{x}; \boldsymbol{\delta}_o) d\mathfrak{c}$$

where, in general, the integral is a multiple integral. Also, we have indicated $\mathbf{c}$ as a continuous random vector but it need not be.

c. The log likelihood for a random draw i is simply

$$\ell_i(\boldsymbol{\theta}) = \log \int \left[\prod_{g=1}^{G} f_g(y_{ig}|\mathbf{x}_i, \mathbf{c}; \boldsymbol{\gamma}^g) \right] h(\mathbf{c}|\mathbf{x}_i; \boldsymbol{\delta}) d\mathbf{c}$$

where $\boldsymbol{\theta}$ contains all parameters.

17.20. a. The Stata output is given below. The signs of the coefficients are generally what we expect: lagged hours has a positive coefficient, as does initial hours in 1980. Thus, unobserved heterogeneity that positively affects hours worked in 1980 also positively affects hours contemporaneously. The variables *nwifeinc*, *ch*0_2, and *ch*3_5 all have negative and statistically significant coefficients. The one slight puzzle is that the older children variable has a positive and just statistically significant coefficient.

```
use \mitbook1_2e\statafiles\psid80_92, clear
tsset id year
* Lagged dependent variable:
bysort id (year): gen hours_1 = L.hours
* Put initial condition in years 81-92:
by id: gen hours80 = hours[1]
* Create exogenous variables for years 81-92:
forv i=81/92 {
by id: gen nwifeinc‘i’ = nwifeinc[‘i’-80]
}
forv i=81/92 {
by id: gen ch0_2_‘i’ = ch0_2[‘i’-80]
}
forv i=81/92 {
by id: gen ch3_5_‘i’ = ch3_5[‘i’-80]
}
forv i=81/92 {
by id: gen ch6_17_‘i’ = ch6_17[‘i’-80]
}
forv i=81/92 {
by id: gen marr‘i’ = marr[‘i’-80]
}

xttobit hours hours_1 hours80 nwifeinc nwifeinc81-nwifeinc92
        ch0_2 ch0_2_81-ch0_2_92 ch3_5 ch3_5_81-ch3_5_92
        ch6_17 ch6_17_81-ch6_17_92 marr marr81-marr92 y82-y92, ll(0) re

note: marr86 omitted because of collinearity
note: marr89 omitted because of collinearity
note: marr90 omitted because of collinearity
note: marr91 omitted because of collinearity
note: marr92 omitted because of collinearity
```

```
Random-effects tobit regression                 Number of obs      =     10776
Group variable: id                              Number of groups   =       898

Random effects u_i ~Gaussian                    Obs per group: min =        12
                                                               avg =      12.
                                                               max =        12

                                                Wald chi2(73)      =   7997.27
Log likelihood  = -62882.574                    Prob > chi2        =    0.0000
```

hours	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
hours_1	.7292676	.0119746	60.90	0.000	.7057978	.7527375
hours80	.2943114	.0181831	16.19	0.000	.2586731	.3299496
nwifeinc	-1.286033	.3221528	-3.99	0.000	-1.917441	-.6546256
nwifeinc81	.6329715	1.08601	0.58	0.560	-1.49557	2.761513
nwifeinc82	.1812886	1.914677	0.09	0.925	-3.57141	3.933987
nwifeinc83	-.6567582	1.822493	-0.36	0.719	-4.228778	2.915262
nwifeinc84	-.9568491	1.344172	-0.71	0.477	-3.591379	1.677681
nwifeinc85	-1.169828	1.186202	-0.99	0.324	-3.494742	1.155085
nwifeinc86	.437133	1.142004	0.38	0.702	-1.801153	2.675419
nwifeinc87	-2.53217	1.067478	-2.37	0.018	-4.624388	-.4399525
nwifeinc88	-.8224415	.6884551	-1.19	0.232	-2.171789	.5269057
nwifeinc89	1.325135	.792212	1.67	0.094	-.2275717	2.877842
nwifeinc90	.0811052	.5898146	0.14	0.891	-1.07491	1.237121
nwifeinc91	1.550942	.861745	1.80	0.072	-.1380467	3.239932
nwifeinc92	-.6307469	.7778347	-0.81	0.417	-2.155275	.893781
ch0_2	-146.0974	21.04471	-6.94	0.000	-187.3443	-104.8506
ch0_2_81	170.7185	93.39678	1.83	0.068	-12.33577	353.7729
ch0_2_82	89.33674	97.13967	0.92	0.358	-101.0535	279.727
ch0_2_83	86.23234	100.9511	0.85	0.393	-111.6281	284.0928
ch0_2_84	-68.59569	100.3924	-0.68	0.494	-265.3612	128.1699
ch0_2_85	-11.17728	96.67042	-0.12	0.908	-200.6478	178.2933
ch0_2_86	83.53639	108.6359	0.77	0.442	-129.386	296.4588
ch0_2_87	-77.82159	110.9494	-0.70	0.483	-295.2784	139.6352
ch0_2_88	-84.03353	141.8912	-0.59	0.554	-362.1352	194.0681
ch0_2_89	48.15522	211.0668	0.23	0.820	-365.528	461.8385
ch0_2_90	17.49295	102.1897	0.17	0.864	-182.7952	217.7811
ch0_2_91	123.7578	96.82641	1.28	0.201	-66.01852	313.534
ch0_2_92	-48.17428	84.53394	-0.57	0.569	-213.8578	117.5092
ch3_5	-80.13216	17.85232	-4.49	0.000	-115.1221	-45.14226
ch3_5_81	39.44899	62.21243	0.63	0.526	-82.48514	161.3831
ch3_5_82	102.3494	72.50917	1.41	0.158	-39.766	244.4647
ch3_5_83	-38.86165	74.09378	-0.52	0.600	-184.0828	106.3595
ch3_5_84	-101.8966	94.20263	-1.08	0.279	-286.5304	82.73714
ch3_5_85	-4.967801	99.28115	-0.05	0.960	-199.5553	189.6197
ch3_5_86	-25.96859	100.6704	-0.26	0.796	-223.279	171.3418
ch3_5_87	5.59682	98.3939	0.06	0.955	-187.2517	198.4453
ch3_5_88	46.38591	93.80288	0.49	0.621	-137.4644	230.2362
ch3_5_89	-95.69263	129.6341	-0.74	0.460	-349.7709	158.3856
ch3_5_90	43.70922	129.4244	0.34	0.736	-209.9579	297.3763
ch3_5_91	147.7391	143.4973	1.03	0.303	-133.5105	428.9886
ch3_5_92	-166.5773	214.5918	-0.78	0.438	-587.1694	254.0149
ch6_17	22.18895	10.08538	2.20	0.028	2.421976	41.95593
ch6_17_81	4.64258	37.68437	0.12	0.902	-69.21744	78.5026
ch6_17_82	64.27872	55.13925	1.17	0.244	-43.79223	172.3497
ch6_17_83	-66.82245	57.25136	-1.17	0.243	-179.033	45.38815

```
   ch6_17_84 |   1.173452   56.00241     0.02   0.983    -108.5893    110.9362
   ch6_17_85 |   6.738214   54.27217     0.12   0.901    -99.63328    113.1097
   ch6_17_86 |   85.64549   57.28103     1.50   0.135    -26.62327    197.9142
   ch6_17_87 |  -65.96152    62.6244    -1.05   0.292    -188.7031    56.78006
   ch6_17_88 |    19.1112   56.21565     0.34   0.734    -91.06945    129.2918
   ch6_17_89 |    4.85883   61.37184     0.08   0.937    -115.4278    125.1454
   ch6_17_90 |   16.18911   60.09357     0.27   0.788    -101.5921    133.9703
   ch6_17_91 |  -21.25498   55.55783    -0.38   0.702    -130.1463    87.63636
   ch6_17_92 |  -7.632119   53.88032    -0.14   0.887    -113.2356    97.97137
        marr |  -199.1315    144.719    -1.38   0.169    -482.7755    84.51247
     marr81 |   127.5178    356.477     0.36   0.721    -571.1642    826.1998
     marr82 |  -13.59679    491.133    -0.03   0.978    -976.1997    949.0062
     marr83 |  -507.5586   434.9469    -1.17   0.243    -1360.039    344.9217
     marr84 |   1318.284   564.6247     2.33   0.020     211.6404    2424.928
     marr85 |  -326.1983   585.7084    -0.56   0.578    -1474.166     821.769
     marr86 |  (omitted)
     marr87 |    131.824     331.72     0.40   0.691    -518.3353    781.9832
     marr88 |  -491.7295   306.7196    -1.60   0.109    -1092.889    109.4299
     marr89 |  (omitted)
     marr90 |  (omitted)
     marr91 |  (omitted)
     marr92 |  (omitted)
        y82 |  -32.79071   25.88734    -1.27   0.205    -83.52896    17.94755
        y83 |   20.40184   25.84829     0.79   0.430    -30.25988    71.06355
        y84 |   105.7757    25.7722     4.10   0.000      55.2631    156.2883
        y85 |   26.36698   25.95325     1.02   0.310    -24.50046    77.23441
        y86 |   26.82807   25.99402     1.03   0.302    -24.11928    77.77542
        y87 |  -.1477861   26.16878    -0.01   0.995    -51.43764    51.14207
        y88 |   21.84475   26.28302     0.83   0.406    -29.66903    73.35853
        y89 |   33.76287   26.39745     1.28   0.201    -17.97518    85.50092
        y90 |   30.54594   26.52445     1.15   0.249    -21.44102     82.5329
        y91 |   29.17601   26.64107     1.10   0.273    -23.03953    81.39155
        y92 |  -27.66915   26.97277    -1.03   0.305    -80.53481    25.19651
      _cons |  -165.6397   47.85094    -3.46   0.001    -259.4258   -71.85356
-------------+----------------------------------------------------------------
    /sigma_u |   310.4876   12.44431    24.95   0.000     286.0972     334.878
    /sigma_e |   508.4561   4.327479   117.49   0.000     499.9744    516.9378
-------------+----------------------------------------------------------------
         rho |   .2716099   .0164159                      .2404141    .3046996
------------------------------------------------------------------------------

  Observation summary:      2835  left-censored observations
                            7941     uncensored observations
                               0 right-censored observations
```

b. The Stata commands below produce the scale factor for the APE of a continuous explanatory variable, evaluated at $hours_{t-1} = 0$. All other variables are averaged out, and the scale factor is for 1992. The APE for *nwifeinc* in 1992 is about $-.742$. Because *nwifeinc* is in \$1,000s, the coefficient implies that a \$10,000 increase in other sources of income decreases estimated annual hours by about 7.4. This is a small economic effect given that the average

hours in 1992 is about 1,155, and a $10,000 increase is fairly large.

```
. predict xbh, xb
(898 missing values generated)

. gen xbh_h0 = xbh - _b[hours_1]*hours_1
(898 missing values generated)

. gen scale = normal(xbh_h0/sqrt(_b[/sigma_u]^2 + _b[/sigma_e]^2))
(898 missing values generated)

. sum scale if y92

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       scale |       898    .5769774    .1692471   .0649065   .9402357

. di  .5769774*_b[nwifeinc]
-.74201219

. sum hours nwifeinc if y92

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       hours |       898    1155.318     899.656          0       3916
    nwifeinc |       898    43.57829     44.2727  -7.249999    601.504
```

c. Because *ch*0_2 is a discrete variable, we compute the difference in the conditional mean function and then average. The APE in 1992 in moving from zero to one small children is about −116.47, which means average annual hours fall by about 116.5 hours.

```
. gen xbh_c0 = xbh - _b[ch0_2]*ch0_2
(898 missing values generated)

. gen xbh_c1 = xbh_c0 + _b[ch0_2]
(898 missing values generated)

. gen mean0 = normal(xbh_c0/sqrt(_b[/sigma_u]^2 + _b[/sigma_e]^2))*xbh_c0 + sqrt
> en(xbh_c0/sqrt(_b[/sigma_u]^2 + _b[/sigma_e]^2))
(898 missing values generated)

. gen mean1 = normal(xbh_c1/sqrt(_b[/sigma_u]^2 + _b[/sigma_e]^2))*xbh_c1 + sqrt
> en(xbh_c1/sqrt(_b[/sigma_u]^2 + _b[/sigma_e]^2))
(898 missing values generated)

. gen diff = mean1 - mean0
(898 missing values generated)

. sum diff if y92

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
```

```
diff |       898   -116.4689    38.50869  -146.0974  -7.479618
```

Solutions to Chapter 18 Problems

18.1. a. This is a simple problem in univariate calculus. Write $q(\mu) \equiv \mu_o \log(\mu) - \mu$ for $\mu > 0$. Then $dq(\mu)/d\mu \equiv \mu_o/\mu - 1$, so $\mu = \mu_o$ uniquely sets the derivative to zero. The second derivative of $q(\mu)$ is $-\mu_o\mu^{-2} < 0$ for all $\mu > 0$, so the sufficient second order condition for a maximum is satisfied.

b. For the exponential case, $q(\mu) \equiv \mathrm{E}[\ell_i(\mu)] = -\mu_o/\mu - \log(\mu)$. The first order condition is $\mu_o\mu^{-2} - \mu^{-1} = 0$, which is uniquely solved by $\mu = \mu_o$. The second derivative is $-2\mu_o\mu^{-3} + \mu^{-2}$, which, when evaluated at μ_o, gives $-2\mu_o^{-2} + \mu_o^{-2} = -\mu_o^{-2} < 0$.

18.2. When $m(\mathbf{x}, \boldsymbol{\beta}) = \exp(\mathbf{x}\boldsymbol{\beta})$, we have $\mathbf{s}_i(\hat{\boldsymbol{\beta}}) = \exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})\mathbf{x}_i'\hat{u}_i/\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}}) = \mathbf{x}_i'\hat{u}_i$, where $\hat{u}_i = y_i - \exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})$. Further, the Hessian $\mathbf{H}_i(\beta)$ does not depend on y_i, and

$$\mathbf{A}_i(\hat{\boldsymbol{\beta}}) = -\mathbf{H}_i(\hat{\boldsymbol{\beta}}) = \left[\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})\right]^2\mathbf{x}_i'\mathbf{x}_i/\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}}) = \exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})\mathbf{x}_i'\mathbf{x}_i.$$

Therefore, we can write equation (18.14) as

$$\left(\sum_{i=1}^{N}\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})\mathbf{x}_i'\mathbf{x}_i\right)^{-1}\left(\sum_{i=1}^{N}\hat{u}_i^2\mathbf{x}_i'\mathbf{x}_i\right)\left(\sum_{i=1}^{N}\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})\mathbf{x}_i'\mathbf{x}_i\right)^{-1}.$$

18.3. a. The Stata output is below. Neither the price nor income variable is significant at any reasonable significance level, although the coefficient estimates are the expected sign. It does not matter whether we use the usual or robust standard errors. The two variables are jointly insignificant, too, using the usual and heteroskedasticity-robust tests (p-values = .490, .344, respectively).

```
. use smoke

. reg cigs lcigpric lincome restaurn white educ age agesq

      Source |       SS       df       MS              Number of obs =     807
-------------+------------------------------           F(  7,   799) =    6.38
       Model |  8029.43631     7  1147.06233            Prob > F      =  0.0000
```

```
    Residual |  143724.246   799  179.880158           R-squared     =  0.0529
-------------+----------------------------           Adj R-squared =  0.0446
       Total |  151753.683   806  188.280003           Root MSE      =  13.412

------------------------------------------------------------------------------
        cigs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lcigpric |  -.8509044   5.782321    -0.15   0.883    -12.20124    10.49943
     lincome |   .8690144   .7287636     1.19   0.233     -.561503    2.299532
    restaurn |  -2.865621   1.117406    -2.56   0.011    -5.059019   -.6722234
       white |  -.5592363   1.459461    -0.38   0.702    -3.424067    2.305594
        educ |  -.5017533   .1671677    -3.00   0.003     -.829893   -.1736135
         age |   .7745021   .1605158     4.83   0.000     .4594197    1.089585
       agesq |  -.0090686   .0017481    -5.19   0.000    -.0124999   -.0056373
       _cons |  -2.682435   24.22073    -0.11   0.912    -50.22621    44.86134
------------------------------------------------------------------------------

. test lcigpric lincome

 ( 1)  lcigpric = 0
 ( 2)  lincome = 0

       F(  2,   799) =    0.71
            Prob > F =    0.4899

. reg cigs lcigpric lincome restaurn white educ age agesq, robust

Linear regression                                      Number of obs =     807
                                                       F(  7,   799) =    9.38
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.0529
                                                       Root MSE      =  13.412

------------------------------------------------------------------------------
             |               Robust
        cigs |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lcigpric |  -.8509044   6.054396    -0.14   0.888     -12.7353     11.0335
     lincome |   .8690144    .597972     1.45   0.147    -.3047672    2.042796
    restaurn |  -2.865621   1.017275    -2.82   0.005    -4.862469    -.868774
       white |  -.5592363   1.378283    -0.41   0.685     -3.26472    2.146247
        educ |  -.5017533   .1624097    -3.09   0.002    -.8205533   -.1829532
         age |   .7745021   .1380317     5.61   0.000     .5035545     1.04545
       agesq |  -.0090686   .0014589    -6.22   0.000    -.0119324   -.0062048
       _cons |  -2.682435   25.90194    -0.10   0.918    -53.52632    48.16145
------------------------------------------------------------------------------

. test lcigpric lincome

 ( 1)  lcigpric = 0
 ( 2)  lincome = 0

       F(  2,   799) =    1.07
            Prob > F =    0.3441
```

b. While the price variable is still highly insignificant (p-value = .46), the income variable,

based on the usual Poisson standard errors, is very significant: $t = 5.11$. Both estimates are elasticities: the estimate price elasticity is $-.106$ and the estimated income elasticity is $.104$. Incidentally, if you drop *restaurn* – a binary indicator for restaurant smoking restrictions at the state level – then *lcigpric* becomes much more significant (using the MLE standard errors). In this data set, both *cigpric* and *restaurn* vary only at the state level, and , not surprisingly, they are significantly correlated. (States that have restaurant smoking restrictions also have higher average cigarette prices, on the order of 2.9%.)

```
. poisson cigs lcigpric lincome restaurn white educ age agesq

Poisson regression                                Number of obs   =        807
                                                  LR chi2(7)      =    1068.70
                                                  Prob > chi2     =     0.0000
Log likelihood =  -8111.519                       Pseudo R2       =     0.0618

------------------------------------------------------------------------------
        cigs |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lcigpric |  -.1059607   .1433932    -0.74   0.460    -.3870061    .1750847
     lincome |   .1037275   .0202811     5.11   0.000     .0639772    .1434779
    restaurn |  -.3636059   .0312231   -11.65   0.000    -.4248021   -.3024098
       white |  -.0552012   .0374207    -1.48   0.140    -.1285444    .0181421
        educ |  -.0594225   .0042564   -13.96   0.000    -.0677648   -.0510802
         age |   .1142571   .0049694    22.99   0.000     .1045172    .1239969
       agesq |  -.0013708    .000057   -24.07   0.000    -.0014825   -.0012592
       _cons |   .3964494   .6139626     0.65   0.518    -.8068952    1.599794
------------------------------------------------------------------------------
```

c. The GLM estimate of σ is about $\hat{\sigma} = 4.51$. This means all of the Poisson standard errors should be multiplied by this factor, as is done using the `glm` command in Stata, with the `sca(x2)` option. The *t* statistic on *lcigpric* is now very small ($-.16$), and that on *lincome* falls to 1.13 – much more in line with the linear model *t* statistic (1.19 with the usual standard errors). Clearly, using the maximum likelihood standard errors is very misleading in this example. With the GLM standard errors, the restaurant restriction variable, education, and the age variables are still significant. (There is no race effect, conditional on the other covariates.)

```
. glm cigs lcigpric lincome restaurn white educ age agesq, family(poisson)
      sca(x2)
```

```
Generalized linear models                         No. of obs      =        807
Optimization     : ML                             Residual df     =        799
                                                  Scale parameter =
Deviance         =  14752.46933                   (1/df) Deviance =  18.46367
Pearson          =  16232.70987                   (1/df) Pearson  =  20.31628

Variance function: V(u) = u                       [Poisson]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             =  20.12272
Log likelihood   = -8111.519022                   BIC             =  9404.504

------------------------------------------------------------------------------
             |                 OIM
        cigs |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lcigpric |  -.1059607   .6463244    -0.16   0.870    -1.372733    1.160812
     lincome |   .1037275   .0914144     1.13   0.257    -.0754414    .2828965
    restaurn |  -.3636059   .1407338    -2.58   0.010    -.6394391   -.0877728
       white |  -.0552011   .1686685    -0.33   0.743    -.3857854    .2753831
        educ |  -.0594225   .0191849    -3.10   0.002    -.0970243   -.0218208
         age |   .1142571   .0223989     5.10   0.000     .0703561     .158158
       agesq |  -.0013708   .0002567    -5.34   0.000     -.001874   -.0008677
       _cons |   .3964493    2.76735     0.14   0.886    -5.027457    5.820355
------------------------------------------------------------------------------
(Standard errors scaled using square root of Pearson X2-based dispersion.)

. di sqrt(20.31628)
4.5073584
```

d. The usual LR statistic is about $LR = 2 \cdot (8125.291 - 8111.519) = 27.54$, which is a very large value in a χ^2_2 distribution (p-value ≈ 0). The QLR statistic divides the usual LR statistic by $\hat{\sigma}^2 = 20.32$, so $QLR = 1.36$ (p-value $\approx .51$). As expected, the QLR statistic shows that the variables are jointly insignificant, while the LR statistic shows strong statistical significance.

```
. poisson cigs restaurn white educ age agesq

Iteration 0:   log likelihood =  -8125.618
Iteration 1:   log likelihood = -8125.2907
Iteration 2:   log likelihood = -8125.2906

Poisson regression                                Number of obs   =        807
                                                  LR chi2(5)      =    1041.16
                                                  Prob > chi2     =     0.0000
Log likelihood = -8125.2906                       Pseudo R2       =     0.0602

------------------------------------------------------------------------------
        cigs |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    restaurn |  -.3545336   .0308796   -11.48   0.000    -.4150564   -.2940107
```

```
       white |  -.0618025    .037371    -1.65   0.098    -.1350483    .0114433
        educ |  -.0532166   .0040652   -13.09   0.000    -.0611842   -.0452489
         age |   .1211174   .0048175    25.14   0.000     .1116754    .1305594
       agesq |  -.0014458   .0000553   -26.14   0.000    -.0015543   -.0013374
       _cons |   .7617484   .1095991     6.95   0.000     .5469381    .9765587
------------------------------------------------------------------------------

. di 2*(8125.291 - 8111.519)
27.544

. di 27.54/20.32
1.355315

. di chi2tail(2,1.36)
.50661699
```

e. Using the robust standard errors does not change any conclusions; in fact, most explanatory variables become slightly more significant than when we use the GLM standard errors. In this example, it is the adjustment by $\hat{\sigma} > 1$ that makes the most difference. Having fully robust standard errors has no additional effect once we account for the severe overdispersion.

```
. glm cigs lcigpric lincome restaurn white educ age agesq, family(poisson)
      robust

Generalized linear models                         No. of obs      =       807
Optimization     : ML                             Residual df     =       799
                                                  Scale parameter =
Deviance         =  14752.46933                   (1/df) Deviance =  18.46367
Pearson          =  16232.70987                   (1/df) Pearson  =  20.31628

Variance function: V(u) = u                       [Poisson]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             =  20.12272
Log pseudolikelihood = -8111.519022               BIC             =  9404.504

------------------------------------------------------------------------------
             |               Robust
        cigs |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lcigpric |  -.1059607   .6681827    -0.16   0.874    -1.415575    1.203653
     lincome |   .1037275    .083299     1.25   0.213    -.0595355    .2669906
    restaurn |  -.3636059    .140366    -2.59   0.010    -.6387182   -.0884937
       white |  -.0552011   .1632959    -0.34   0.735    -.3752553     .264853
        educ |  -.0594225   .0192058    -3.09   0.002    -.0970653   -.0217798
         age |   .1142571   .0212322     5.38   0.000     .0726427    .1558715
       agesq |  -.0013708   .0002446    -5.60   0.000    -.0018503   -.0008914
       _cons |   .3964493    2.97704     0.13   0.894    -5.438442     6.23134
------------------------------------------------------------------------------
```

f. We simply compute the turning point for the quadratic:

$\hat{\beta}_{age}/(-2\hat{\beta}_{age^2}) = .1143/[2(.00137)] \approx 41.72$, or at about 42 years of age.

g. A double-hurdle model – which separates the initial decision to smoke at all from the decision of how much to smoke – seems like a good idea. Variables such as level of education, income, and age could have very different effects on the decision to smoke versus how much to smoke. It is certainly worth investigating. One approach is to model $D(y|\mathbf{x}, y \geq 1)$ as, say, a truncated Poisson distribution, and then to model $P(y = 0|\mathbf{x})$ as a logit or probit (with parameters free to vary from the truncated Poisson distribution).

18.4. In the notation of Section 14.5.3, $r(\mathbf{w}_i, \boldsymbol{\theta}) = r(\mathbf{w}_i, \boldsymbol{\beta}) = y_i - m(\mathbf{x}_i, \boldsymbol{\beta})$, and so $R_o(\mathbf{x}_i) = -\nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \boldsymbol{\beta}_o)$. Further, $\Omega_o(\mathbf{x}_i) = \text{Var}[y_i - m(\mathbf{x}_i, \boldsymbol{\beta}_o)|\mathbf{x}_i] = \text{Var}(y_i|\mathbf{x}_i) = \sigma_o^2 m(\mathbf{x}_i, \boldsymbol{\beta}_o)$ under the GLM assumption. From equation (14.60), the asymptotic variance lower bound is

$$\text{E}\{[R_o(\mathbf{x}_i)'\boldsymbol{\Omega}_o(\mathbf{x}_i)^{-1}R_o(\mathbf{x}_i)]\}^{-1} = \sigma_o^2 \text{E}[\nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \boldsymbol{\beta}_o)'\nabla_{\boldsymbol{\beta}} m(\mathbf{x}_i, \boldsymbol{\beta}_o)/m(\mathbf{x}_i, \boldsymbol{\beta}_o)],$$

which is the same asymptotic variance for the Poisson QMLE under the GLM assumption.

18.5. a. We just use iterated expectations:

$$\begin{aligned} \text{E}(y_{it}|\mathbf{x}_i) &= \text{E}[\text{E}(y_{it}|\mathbf{x}_i, c_i)|\mathbf{x}_i] = \text{E}[c_i \exp(\mathbf{x}_{it}\boldsymbol{\beta})|\mathbf{x}_i, c_i] \\ &= \text{E}(c_i|\mathbf{x}_i)\exp(\mathbf{x}_{it}\boldsymbol{\beta}) \\ &= \exp(\alpha + \bar{\mathbf{x}}_i\boldsymbol{\gamma})\exp(\mathbf{x}_{it}\boldsymbol{\beta}) = \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\gamma}). \end{aligned}$$

b. We are explicitly testing $H_0 : \boldsymbol{\gamma} = \mathbf{0}$, but we are maintaining full independence of c_i and $\mathbf{x}_i$ under H_0. We have enough assumptions to derive $\text{Var}(\mathbf{y}_i|\mathbf{x}_i)$, the $T \times T$ conditional variance matrix of $\mathbf{y}_i$ given $\mathbf{x}_i$ under H_0. First,

$$\begin{aligned} \text{Var}(y_{it}|\mathbf{x}_i) &= \text{E}[\text{Var}(y_{it}|\mathbf{x}_i, c_i)|\mathbf{x}_i] + \text{Var}[\text{E}(y_{it}|\mathbf{x}_i, c_i)|\mathbf{x}_i] \\ &= \text{E}[c_i \exp(\mathbf{x}_{it}\boldsymbol{\beta})|\mathbf{x}_i] + \text{Var}[c_i \exp(\mathbf{x}_{it}\boldsymbol{\beta})|\mathbf{x}_i] \\ &= \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta}) + \tau^2[\exp(\mathbf{x}_{it}\boldsymbol{\beta})]^2, \end{aligned}$$

where $\tau^2 \equiv \text{Var}(c_i)$ and we have used $\text{E}(c_i|\mathbf{x}_i) = \exp(\alpha)$ under H_0. A similar, general expression holds for conditional covariances:

$$\begin{aligned}\text{Cov}(y_{it}, y_{ir}|\mathbf{x}_i) &= \text{E}[Cov(y_{it}, y_{ir}|\mathbf{x}_i, c_i)|\mathbf{x}_i] + \text{Cov}[\text{E}(y_{it}|\mathbf{x}_i, c_i), \text{E}(y_{ir}|\mathbf{x}_i, c_i)|\mathbf{x}_i] \\ &= 0 + \text{Cov}[c_i \exp(\mathbf{x}_{it}\boldsymbol{\beta}), c_i \exp(\mathbf{x}_{ir}\boldsymbol{\beta})|\mathbf{x}_i] \\ &= \tau^2 \exp(\mathbf{x}_{it}\boldsymbol{\beta}) \exp(\mathbf{x}_{ir}\boldsymbol{\beta}).\end{aligned}$$

So, under $H_0, \text{Var}(y_i|\mathbf{x}_i)$ depends on α, $\boldsymbol{\beta}$, and τ^2, all of which we can estimate. It is natural to use a score test – actually, its variable addition counterpart – to test $H_0 : \boldsymbol{\gamma} = \mathbf{0}$. First, obtain consistent estimators $\check{\alpha}$, $\check{\boldsymbol{\beta}}$ by, say, pooled Poisson QMLE. Let $\check{y}_{it} = \exp(\check{\alpha} + \mathbf{x}_{it}\check{\boldsymbol{\beta}})$ and $\check{u}_{it} = y_{it} - \check{y}_{it}$. A consistent estimator of τ^2 can be obtained from a simple pooled regression, through the origin, of

$$\check{u}_{it}^2 - \check{y}_{it} \text{ on } \exp(2\mathbf{x}_{it}\check{\boldsymbol{\beta}}),\ t = 1, \ldots, T;\ i = 1, \ldots, N.$$

Let $\tilde{\tau}^2$ be the coefficient on $\exp(2\mathbf{x}_{it}\check{\boldsymbol{\beta}})$. It is consistent for τ^2 because, under H_0,

$$\text{E}(u_{it}^2|\mathbf{x}_i) = \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta}) + \tau^2[\exp(\mathbf{x}_{it}\boldsymbol{\beta})]^2,$$

where $u_{it} \equiv y_{it} - \text{E}(y_{it}|\mathbf{x}_{it})$. We could also use the many covariance terms in estimating τ^2 because $\text{E}(u_{it}u_{ir}|\mathbf{x}_i) = \tau^2 \exp(\mathbf{x}_{it}\boldsymbol{\beta}) \exp(\mathbf{x}_{ir}\boldsymbol{\beta})$, $t \neq r$. So for all $t, r = 1, \ldots, T$, we can write

$$u_{it}u_{ir} - d_{tr} \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta}) = \tau^2 \exp(\mathbf{x}_{it}\boldsymbol{\beta}) \exp(\mathbf{x}_{ir}\boldsymbol{\beta}) + v_{itr}$$

where $\text{E}(v_{itr}|\mathbf{x}_i) = 0$ and $d_{tr} = 1[t = r]$ is a dummy variable. The pooled regression would be

$$\check{u}_{it}\check{u}_{ir} - d_{tr}\check{y}_{it} \text{ on } \exp(\mathbf{x}_{it}\check{\boldsymbol{\beta}}) \exp(\mathbf{x}_{ir}\check{\boldsymbol{\beta}})$$

Next, we construct the $T \times T$ weighting matrix for observation i, as in Section 18.7.3. The matrix $\mathbf{W}_i(\tilde{\boldsymbol{\delta}}) = \mathbf{W}(\mathbf{x}_i, \tilde{\boldsymbol{\delta}})$ has diagonal elements

$$\exp(\check{\alpha} + \mathbf{x}_{it}\check{\boldsymbol{\beta}}) + \tilde{\tau}^2 \exp(2\mathbf{x}_{it}\check{\boldsymbol{\beta}}),\ t = 1, \ldots, T$$

and off-diagonal elements

$$\tilde{\tau}^2 \exp(\mathbf{x}_{it}\tilde{\boldsymbol{\beta}}) \exp(\mathbf{x}_{ir}\tilde{\boldsymbol{\beta}}),\ t \neq r.$$

Using this weighting matrix in a MWNLS estimation problem we can simply add the time averages, $\bar{\mathbf{x}}_i$, as an additional set of explanatory variables, and test their joint signficiance. This is the VAT version of the score test.

In practice, we might want a robust form of the test that does not require $\text{Var}(\mathbf{y}_i|\mathbf{x}_i) = \mathbf{W}(\mathbf{x}_i, \boldsymbol{\delta})$ under H_0, where $\mathbf{W}(\mathbf{x}_i, \boldsymbol{\delta})$ is the matrix described above. We can just use the fully robust variance matrix reported at the bottom of page 761.

Using modern software that supports MWNLS a simpler approach is to estimate the model under the alternative and obtain a Wald test of $H_0 : \boldsymbol{\gamma} = \mathbf{0}$, where it is valid to act as if $\text{Var}(c_i|\mathbf{x}_i) = \tau^2$ because this is true under the null. This would differ from the score approach in that τ^2 would be estimated using a first stage where $\boldsymbol{\gamma}$ is also estimated. A fully robust Wald test is easy to obtain if we have any doubts about the variance-covariance structure.

Incidentally, this variance-covariance structure is different from the one used in the GEE literature for Poisson regression. With GEE and an exchangeable correlation structure, the nominal variance would be

$$\text{Var}(y_{it}|\mathbf{x}_i) = \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\gamma})$$

and the nominal covariances

$$\text{Cov}(y_{it}, y_{ir}|\mathbf{x}_i) = \rho\sqrt{\exp(\mathbf{x}_{it}\boldsymbol{\beta}) \exp(\mathbf{x}_{ir}\boldsymbol{\beta})}\,.$$

c. If we assume (18.83), (18.84) and $c_i = a_i \exp(\alpha + \bar{\mathbf{x}}_i\boldsymbol{\gamma})$ where $a_i|\mathbf{x}_i \sim \text{Gamma}(\delta, \delta)$, then testing involves estimation of a Poisson panel data model under random effects assumptions. Under these assumptions, we have

$$y_{it}|\mathbf{x}_i, a_i \sim \text{Poisson}[a_i \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\gamma})]$$
y_{it}, y_{ir} are independent conditional on $(\mathbf{x}_i, a_i)$
$a_i|\mathbf{x}_i \sim \text{Gamma}(\delta, \delta)$.

In other words, the full set of random effect Poisson assumptions holds, but where the mean function in the Poisson distribution is $a_i \exp(\alpha + \mathbf{x}_{it}\boldsymbol{\beta} + \bar{\mathbf{x}}_i\boldsymbol{\gamma})$. In practice, we just add the (nonredundant elements of) $\bar{\mathbf{x}}_i$ in each time period, along with a constant and $\mathbf{x}_{it}$, and carry out a random effects Poisson analysis. We can test $H_0 : \boldsymbol{\gamma} = 0$ using the LR, Wald, or score approaches. Any of these would be asymptotically efficient. None is robust to misspecification of the Poisson distribution or the conditional independence assumption because we have used a full distribution for $\mathbf{y}_i$ given $\mathbf{x}_i$ in the MLE analysis.

18.6. a. We know from Problem 12.6 that pooled nonlinear least squares consistently estimates $\boldsymbol{\beta}_o$ when $\boldsymbol{\beta}_o$ appears in correctly specified conditional means for each *t*. Because $\ddot{m}_{it}(\boldsymbol{\beta})$ depends on $\mathbf{x}_i$ we should show

$$\text{E}(\ddot{y}_{it}|\mathbf{x}_i) = \ddot{m}_{it}(\boldsymbol{\beta}_o), \; t = 1, \ldots, T,$$

as suggested in the hint. To this end, write

$$y_{it} = c_i + m_{it}(\mathbf{x}_i, \boldsymbol{\beta}_o) + u_{it}, \; \text{E}(u_{it}|\mathbf{x}_i, c_i) = 0, \; t = 1, \ldots, T.$$

Then subtracting off time averages gives

$$\ddot{y}_{it} = \ddot{m}_{it}(\boldsymbol{\beta}_o) + \ddot{u}_{it},$$
$$\ddot{u}_{it} \equiv u_{it} - T^{-1} \sum_{r=1}^{T} u_{ir}.$$

Because $\text{E}(\ddot{u}_{it}|\mathbf{x}_i) = 0, t = 1, \ldots, T$, consistency follows generally by Problem 12.6. We do have to make an assumption that ensures that $\boldsymbol{\beta}_o$ is identified, which restricts the way that time-constant variables can appear in $m(\mathbf{x}_{it}, \boldsymbol{\beta})$. (For example, additive time-constant variables

get swept away by the time demeaning.)

b. By the general theory of M-estimation, or by adapting either Problem 12.6 or 12.7, we can show

$$\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) = \mathbf{A}_o^{-1} N^{-1/2} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \ddot{u}_{it} + o_p(1),$$

where

$$\mathbf{A}_o = T^{-1} \sum_{r=1}^{T} \mathrm{E}\left[\nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o) \right]$$

is $P \times P$ and P is the dimension of $\boldsymbol{\beta}$. (As part of the identification assumption, we would assume that $\mathbf{A}_o$ is nonsingular.) As in the linear case, we can write, for each t,

$$\sum_{t=1}^{T} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \ddot{u}_{it} = \sum_{t=1}^{T} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' u_{it}.$$

Further, $\mathrm{Var}(\mathbf{y}_i | \mathbf{x}_i, c_i) = \sigma_o^2 \mathbf{I}_T$ is the same as $\mathrm{E}(\mathbf{u}_i \mathbf{u}_i' | \mathbf{x}_i, c_i) = \sigma_o^2 \mathbf{I}_T$, which implies

$$\mathrm{E}(u_{it}^2 | \mathbf{x}_i) = \sigma_o^2$$
$$\mathrm{E}(u_{it} u_{ir} | \mathbf{x}_i) = 0, \, t \neq r$$

Therefore, by the usual iterated expectations argument,

$$\mathrm{E}\left[u_{it}^2 \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o) \right] = \sigma_o^2 \mathrm{E}\left[\nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o) \right], \, t = 1, \ldots, T$$

and

$$\mathrm{E}\left[u_{it} u_{ir} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o) \right] = 0, \, t \neq r.$$

It follows that

$$\mathrm{Var}\left(\sum_{t=1}^{T} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' u_{it} \right) = \sigma_o^2 \left(\sum_{t=1}^{T} \mathrm{E}\left[\nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o) \right] \right).$$

Therefore, under the given assumptions,

$$\text{Avar}\left[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o)\right] = \sigma_o^2 \left(\sum_{t=1}^{T} \text{E}\left[\nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\boldsymbol{\beta}_o)\right] \right)^{-1}.$$

As in the linear case, the tricky part is in estimating σ_o^2. We can apply virtually the same argument. Let $\hat{\ddot{u}}_{it} = \ddot{y}_{it} - \ddot{m}_{it}(\hat{\boldsymbol{\beta}})$ for all i and t. Then a consistent estimator of σ_o^2 is

$$\hat{\sigma}^2 = \frac{1}{[N(T-1) - P]} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{\ddot{u}}_{it}^2,$$

where the subtraction of P is not needed but is often used as an adjustment for estimation of $\boldsymbol{\beta}_o$. Estimation of $\mathbf{A}_o$ gives

$$\hat{\mathbf{A}} = N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\hat{\boldsymbol{\beta}})' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\hat{\boldsymbol{\beta}}).$$

Then,

$$\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2 \left(\sum_{i=1}^{N} \sum_{t=1}^{T} \left[\nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\hat{\boldsymbol{\beta}})' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\hat{\boldsymbol{\beta}})\right] \right)^{-1}.$$

c. A fully robust variance matrix estimator uses $\hat{\mathbf{A}}$ and

$$\hat{\mathbf{B}} = N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{r=1}^{T} \hat{\ddot{u}}_{it} \hat{\ddot{u}}_{ir} \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\hat{\boldsymbol{\beta}})' \nabla_{\boldsymbol{\beta}} \ddot{m}_{it}(\hat{\boldsymbol{\beta}}),$$

which allows for arbitrary heteroskedasticity and serial correlation in $\{u_{it} : t = 1, \ldots, T\}$. Then $\widehat{\text{Avar}}(\hat{\boldsymbol{\beta}}) = \hat{\mathbf{A}}^{-1} \hat{\mathbf{B}} \hat{\mathbf{A}}^{-1}/N$, as usual.

Remember that the estimator of $\mathbf{A}_o$ relies on correct specification of the conditional mean; in its weakest form, $\text{E}(\ddot{y}_{it}|\mathbf{x}_i) = \ddot{m}_{it}(\boldsymbol{\beta}_o)$, which is implied by the model we started with, $\text{E}(y_{it}|\mathbf{x}_i, c_i) = c_i + m(\mathbf{x}_{it}, \boldsymbol{\beta}_o)$. If we want to allow the model to be misspecified we should use

the full Hessian, $\nabla_{\boldsymbol{\beta}}\ddot{m}_{it}(\hat{\boldsymbol{\beta}})'\nabla_{\boldsymbol{\beta}}\ddot{m}_{it}(\hat{\boldsymbol{\beta}}) - \hat{\ddot{u}}_{it}\nabla^2_{\boldsymbol{\beta}}\ddot{m}_{it}(\hat{\boldsymbol{\beta}})$ in place of $\nabla_{\boldsymbol{\beta}}\ddot{m}_{it}(\hat{\boldsymbol{\beta}})'\nabla_{\boldsymbol{\beta}}\ddot{m}_{it}(\hat{\boldsymbol{\beta}})$.

d. This is easy following the hint. For each i and given $\boldsymbol{\beta}$, $\hat{c}_i(\boldsymbol{\beta})$ is just the intercept in the simple regression of y_{it} on 1, $m_{it}(\boldsymbol{\beta}), t = 1, \ldots, T$. Therefore, $\hat{c}_i(\boldsymbol{\beta}) = \bar{y}_i - \bar{m}_i(\boldsymbol{\beta})$. Therefore, we can write problem (18.105), after concentrating out the c_i, as

$$\min_{\boldsymbol{\beta}} \sum_{i=1}^{N}\sum_{t=1}^{T}\{y_{it} - [\bar{y}_i - \bar{m}_i(\boldsymbol{\beta})] - m_i(\boldsymbol{\beta})\}^2 = \min_{\boldsymbol{\beta}} \sum_{i=1}^{N}\sum_{t=1}^{T}[\ddot{y}_{it} - \ddot{m}_{it}(\boldsymbol{\beta})]^2,$$

which is what we wanted to show. Note by treating the c_i as N parameters to estimate and using a standardi degrees-of-freedom adjustment, solving (18.105) does yield the estimate $\hat{\sigma}^2$ from part b when we use the sum of squared residuals over the degrees of freedom, $NT - N - P = N(T-1) - P$.

18.7. a. First, for each t, the density of y_{it} given $(\mathbf{x}_i = \mathbf{x}, c_i = c)$ is

$$f(y_t|\mathbf{x}, c; \boldsymbol{\beta}_o) = \exp[-c \cdot m(\mathbf{x}_t, \boldsymbol{\beta}_o)][c \cdot m(\mathbf{x}_t, \boldsymbol{\beta}_o)]^{y_t}/y_t!, \; y_t = 0, 1, 2, \ldots$$

Multiplying these together gives the joint density of $(y_{i1}, \ldots, y_{iT})$ given $(\mathbf{x}_i = \mathbf{x}, \mathbf{c}_i = c)$. Taking the log, plugging in the observed data for observation i, and dropping the factorial term gives

$$\sum_{t=1}^{T}\{-c_i m(\mathbf{x}_{it}, \boldsymbol{\beta}) + y_{it}[\log(c_i) + \log(m(\mathbf{x}_{it}, \boldsymbol{\beta}))]\}.$$

b. Taking the derivative of $\ell_i(c_i, \boldsymbol{\beta})$ with respect to c_i, setting the result to zero, and rearranging gives

$$(n_i/c_i) = \sum_{t=1}^{T} m(\mathbf{x}_{it}, \boldsymbol{\beta}).$$

Letting $c_i(\boldsymbol{\beta})$ denote the solution as a function of $\boldsymbol{\beta}$, we have $c_i(\boldsymbol{\beta}) = n_i/M_i(\boldsymbol{\beta})$, where

$M_i(\boldsymbol{\beta}) \equiv \sum_{r=1}^{T} m(\mathbf{x}_{ir}, \boldsymbol{\beta})$. The second order sufficient condition for a maximum is easily seen to hold.

c. Plugging the solution from part b into $\ell_i(c_i, \boldsymbol{\beta})$ gives

$$\begin{aligned}\ell_i[c_i(\boldsymbol{\beta}), \boldsymbol{\beta}] &= -[n_i/M_i(\boldsymbol{\beta})]M_i(\boldsymbol{\beta}) + \sum_{t=1}^{T} y_{it}\{\log[n_i/M_i(\boldsymbol{\beta})] + \log[m(\mathbf{x}_{it}, \boldsymbol{\beta})]\} \\ &= -n_i + n_i \log(n_i) + \sum_{t=1}^{T} y_{it}\{\log[m(\mathbf{x}_{it}, \boldsymbol{\beta})/M_i(\boldsymbol{\beta})]\} \\ &= \sum_{t=1}^{T} y_{it} \log[p_t(\mathbf{x}_{it}, \boldsymbol{\beta})] + (n_1 - 1)\log(n_i),\end{aligned}$$

because $p_t(\mathbf{x}_{it}, \boldsymbol{\beta}) \equiv m(\mathbf{x}_{it}, \boldsymbol{\beta})/M_i(\boldsymbol{\beta})$; see equation (18.89).

d. From part c it follows that if we maximize $\sum_{i=1}^{N} \ell_i(c_i, \boldsymbol{\beta})$ with respect to $(c_i, \ldots, c_N)$ – that is, we concentrate out these parameters – we get exactly $\sum_{i=1}^{N} \ell_i[c_i(\boldsymbol{\beta}), \boldsymbol{\beta}]$. Except for the term $\sum_{i=1}^{N}(n_i - 1)\log(n_i)$ – which does not depend on $\boldsymbol{\beta}$ – this is exactly the conditional log-likelihood for the conditional multinomial distribution obtained in Section 18.7.4. Therefore, this is another case where treating the c_i as parameters to be estimated leads us to a $\sqrt{N}$-consistent, asymptotically normal estimator of $\boldsymbol{\beta}_o$.

18.8. a. Generally, there is no simple way to recover $\mathrm{E}(y|\mathbf{x})$ from $\mathrm{E}\{\log[y/(1-y)]|\mathbf{x}\}$. In particular, if $\mathrm{E}(w|\mathbf{x}) = \mathbf{x}\boldsymbol{\alpha}$, it is *not* true that $\mathrm{E}(y|\mathbf{x}) = \exp(\mathbf{x}\boldsymbol{\alpha})/[1 + \exp(\mathbf{x}\boldsymbol{\alpha})]$. In other words, we cannot simply "undo" the log-odds transformation any more than we can undo any nonlinear transformation when trying to recover conditional means.

If we make stronger assumptions, we can recover $\mathrm{E}(y|\mathbf{x})$ from $\mathrm{E}(w|\mathbf{x})$. Suppose we write $w = \mathbf{x}\boldsymbol{\alpha} + v$ and *assume* that v is independent of $\mathbf{x}$. Assume for simplicity that v is continuous with density $g(\cdot)$. Then

$$\mathrm{E}(y|\mathbf{x} = \mathbf{x}^o) = \int_{-\infty}^{\infty} \exp(\mathbf{x}^o\boldsymbol{\alpha} + \upsilon)/[1 + \exp(\mathbf{x}^o\boldsymbol{\alpha} + \upsilon)]g(\upsilon).$$

If we parameterize $g(\cdot)$ – say, $g(\cdot;\boldsymbol{\rho})$ – and we have a consistent estimator of $\boldsymbol{\rho}$, then

$$\hat{\mathrm{E}}(y|\mathbf{x} = \mathbf{x}^o) = \int_{-\infty}^{\infty} \exp(\mathbf{x}^o\hat{\boldsymbol{\alpha}} + \upsilon)/[1 + \exp(\mathbf{x}^o\hat{\boldsymbol{\alpha}} + \upsilon)]g(\upsilon;\hat{\boldsymbol{\rho}}),$$

where $\hat{\boldsymbol{\alpha}}$ could be the OLS estimator from regressing w_i on $\mathbf{x}_i$ or the maximum likelihood estimator based on $\mathrm{D}(w|\mathbf{x})$. If $\mathrm{D}(v|\mathbf{x})$ is assumed to be normal then OLS is MLE. Even if we specify $g(\cdot;\boldsymbol{\rho})$ to be a mean-zero normal distribution, obtaining the integral is cumbersome.

There is a simpler approach that is also more robust. If we just maintain that v and $\mathbf{x}$ are independent then, by the law of large numbers, $\mathrm{E}(y|\mathbf{x} = \mathbf{x}^o)$ for a given vector $\mathbf{x}^o$ is consistently estimated by

$$N^{-1}\sum_{i=1}^{N} \exp(\mathbf{x}^o\boldsymbol{\alpha} + v_i)/[1 + \exp(\mathbf{x}^o\boldsymbol{\alpha} + v_i)],$$

where we can think of drawing random samples $\{(\mathbf{x}_i, v_i) : i = 1, 2, \ldots, N\}$. Because we cannot observe v_i, and we do not know $\boldsymbol{\alpha}$, we operationalize this formula by replacing $\boldsymbol{\alpha}$ with $\hat{\boldsymbol{\alpha}}$, including computing residuals $\hat{v}_i = w_i - \mathbf{x}_i\hat{\alpha},\ i = 1, \ldots, N$. Then

$$\hat{\mathrm{E}}(y|\mathbf{x} = \mathbf{x}^o) = N^{-1}\sum_{i=1}^{N} \exp(\mathbf{x}^o\hat{\boldsymbol{\alpha}} + \hat{v}_i)/[1 + \exp(\mathbf{x}^o\hat{\boldsymbol{\alpha}} + \hat{v}_i)].$$

This is an example of Duan's (1983) "smearing estimate." This estimator is consistent under the assumptions given – which do not require a full distribution, but do include independence between v and $\mathbf{x}$. Obtaining analytical standard errors can be done by following Probem 12.17. Bootstrapping is also valid. Unfortunately, this approach does not work if y can take on the boundary values zero or one.

The above integral for $\mathrm{E}(y|\mathbf{x} = \mathbf{x}^o)$ can be written as $\mathrm{E}(y|\mathbf{x} = \mathbf{x}^o) = r(\mathbf{x}^o\boldsymbol{\alpha})$ and so, if v and $\mathbf{x}$ are independent, then $[\partial\mathrm{E}(y|\mathbf{x})/\partial x_j]/[\partial\mathrm{E}(y|\mathbf{x})/\partial x_h] = \alpha_j/\alpha_h$: for continuous explanatory variables, the ratio of the partial effects equals the ratio of the parameters in the linear model for $w = \log[y/(1-y)]$.

b. The functional form $\mathrm{E}(y|\mathbf{x}) = \exp(\mathbf{x}\boldsymbol{\beta})/[1 + \exp(\mathbf{x}\boldsymbol{\beta})]$, and that implied by the assumptions in part a, are generally incompatible. However, as mentioned above, independence between v and $\mathbf{x}$ in $\log[y/(1-y)] = \mathbf{x}\boldsymbol{\alpha} + v$ implies that α_j/α_h is the ratio of the partial effects of continuous explanatory variables x_j and x_h. In the fractional logit model, the ratio of partial effects is β_j/β_h. Therefore, it can make sense to compare ratios of coefficients on continuous explanatory variables across the two procedures. But the magnitudes themselves are not generally comparable.

c. Because we have a full distribution of y given $\mathbf{x}$, we should use maximum likelihood, just as described in Section 17.7.

d. The functional form for $\mathrm{E}(y|\mathbf{x})$ – as a function of the parameters $\boldsymbol{\gamma}$ and σ^2 – is given in equation (17.66) where we set $a_1 = 0$, $a_2 = 1$, with the obvious change in notation:

$$\begin{aligned}\mathrm{E}(y|\mathbf{x}) &= \{\Phi[(1-\mathbf{x}\boldsymbol{\gamma})/\sigma] - \Phi[(-\mathbf{x}\boldsymbol{\gamma})/\sigma]\}\mathbf{x}\boldsymbol{\gamma} + \sigma\{\phi[((1-\mathbf{x}\boldsymbol{\gamma})/\sigma] - \phi[(-\mathbf{x}\boldsymbol{\gamma})/\sigma]\} \\ &\quad + \Phi[-(1-\mathbf{x}\boldsymbol{\gamma})/\sigma] \\ &= \{\Phi[(1-\mathbf{x}\boldsymbol{\gamma})/\sigma] - \Phi[(-\mathbf{x}\boldsymbol{\gamma})/\sigma]\}\mathbf{x}\boldsymbol{\gamma} + \sigma\{\phi[((1-\mathbf{x}\boldsymbol{\gamma})/\sigma] - \phi[(-\mathbf{x}\boldsymbol{\gamma})/\sigma]\} \\ &\quad + 1 - \Phi[(1-\mathbf{x}\boldsymbol{\gamma})/\sigma].\end{aligned}$$

This gives yet a different functional form. Nevertheless, it is easily seen from equation (17.67) that

$$[\partial\mathrm{E}(y|\mathbf{x})/\partial x_j]/[\partial\mathrm{E}(y|\mathbf{x})/\partial x_h] = \gamma_j/\gamma_h,$$

so that ratios of the coefficients on continuous variables can be compared with those from part

b.

e. Because part b only specified a conditional mean, it does not make much sense to compare the Bernoulli quasi-log-likelihood with the Tobit log-likelihood. If we are mainly interested in $\mathrm{E}(y|\mathbf{x})$ – which part b essentially maintains – it makes sense to base comparisons on goodness-of-fit for $\mathrm{E}(y|\mathbf{x})$. For each approach, we can compute a squared correlation between the y_i and the $\hat{\mathrm{E}}(y_i|\mathbf{x}_i)$, where the conditional expectations are estimated using each approach. Or, we can use a sum-of-squared residuals version (and possibly adjust for degrees-of-freedom because the Tobit model has an extra mean parameter, σ).

f. We would not expect to get similar answers for the full sample – which includes observations with $y_i = 0$ – and the subsample that excludes $y_i = 0$ (unless the fraction of excluded observations is small). Clearly, we cannot have both $\mathrm{E}(y|\mathbf{x}) = \exp(\mathbf{x\beta})/[1 + \exp(\mathbf{x\beta})]$ and $\mathrm{E}(y|\mathbf{x}, y > 0) = \exp(\mathbf{x\delta})/[1 + \exp(\mathbf{x\delta})]$. Moreover, there is no reason to expect the best fits to yield roughly the same parameter estimates.

g. Because we have assumed $\mathrm{E}(y|\mathbf{x}, y > 0) = \exp(\mathbf{x\delta})/[1 + \exp(\mathbf{x\delta})]$, we consistently estimate $\boldsymbol{\delta}$ using the sample for which $0 < y_i < 1$, provided we use the Bernoulli QMLE (or NLS or weighted NLS). There is no bias from excluding the $y_i = 0$ observations because we have specified the mean for the subpopulation with $y > 0$. (We discuss sample selection issues in Chapter 19.)

h. We would use a two-part model. Let $\hat{\boldsymbol{\delta}}$ be the Bernoulli QMLE from part g, using observations for which $y_i > 0$. To estimate $\boldsymbol{\eta}$, we run a binary response model using the binary variable $r_i = 1[y_i > 0]$. Then $P(r_i = 1|\mathbf{x}_i) = G(\mathbf{x}_i\boldsymbol{\eta})$. Probably we would use a probit or logit model. Then,

$$\hat{\mathrm{E}}(y_i|\mathbf{x}_i) = \hat{\mathrm{P}}(y_i > 0|\mathbf{x}_i) \cdot \hat{\mathrm{E}}(y_i|\mathbf{x}_i, y_i > 0)$$
$$= G(\mathbf{x}_i\hat{\boldsymbol{\eta}}) \cdot \{\exp(\mathbf{x}_i\hat{\boldsymbol{\delta}})/[1 + \exp(\mathbf{x}_i\hat{\boldsymbol{\delta}})]\}.$$

18.9. a. The Stata output follows. I first convert the dependent variable to be in $[0, 1]$, rather than $[0, 100]$; this is needed to estimate a fractional response model.

The coefficient on *ACT* means that five more points on the ACT test, other things equal, is associated with a lower attendance rate of about $.017(5) = .085$, or 8.5 percentage points. For *priGPA*, another point on the GPA (a large change) is associated with an attendance rate roughly 18.2 percentage points higher.

Twelve of the fitted values are bigger than one. This is not surprising because almost 10 percent of the students have perfect attendance rates.

```
. use attend

. sum atndrte

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     atndrte |       680    81.70956    17.04699       6.25        100

. replace atndrte = atndrte/100
(680 real changes made)

. reg atndrte ACT priGPA frosh soph

      Source |       SS       df       MS              Number of obs =     680
-------------+------------------------------           F(  4,   675) =   72.92
       Model |  5.95396289     4  1.48849072           Prob > F      =  0.0000
    Residual |  13.7777696   675  .020411511           R-squared     =  0.3017
-------------+------------------------------           Adj R-squared =  0.2976
       Total |  19.7317325   679  .029059989           Root MSE      =  .14287

------------------------------------------------------------------------------
     atndrte |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         ACT |  -.0169202    .001681   -10.07   0.000    -.0202207   -.0136196
      priGPA |   .1820163   .0112156    16.23   0.000     .1599947    .2040379
       frosh |   .0517097   .0173019     2.99   0.003     .0177377    .0856818
        soph |   .0110085    .014485     0.76   0.448    -.0174327    .0394496
       _cons |   .7087769   .0417257    16.99   0.000     .6268492    .7907046
------------------------------------------------------------------------------

. predict atndrteh_lin
(option xb assumed; fitted values)
```

```
. sum  atndrteh_lin

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
atndrteh_lin |       680    .8170956    .0936415   .4846666   1.086443

. count if atndrteh_lin > 1
   12

. count if atndrte == 1
   66
```

b. The GLM standard errors are given in the output. Note that $\hat{\sigma} \approx .0161$. In other words, the usual MLE standard errors, obtained, say, from the expected Hessian of the quasi-log-likelihood, are much too *large*. The standard errors that account for $\sigma^2 < 1$ are given by the GLM output. (If you omit the `sca(x2)` option in the `glm` command, you get the usual MLE standard errors.)

```
. glm atndrte ACT priGPA frosh soph, family(binomial) link(logit) sca(x2)
note: atndrte has noninteger values

Generalized linear models                         No. of obs      =       680
Optimization     : ML                             Residual df     =       675
                                                  Scale parameter =
Deviance         =  87.81698799                   (1/df) Deviance =  .1300992
Pearson          =  85.57283238                   (1/df) Pearson  =  .1267746

Variance function: V(u) = u*(1-u/1)               [Binomial]
Link function    : g(u) = ln(u/(1-u))             [Logit]

                                                  AIC             =  .6724981
Log likelihood   = -223.6493665                   BIC             = -4314.596

------------------------------------------------------------------------------
             |                 OIM
     atndrte |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         ACT |  -.1113802   .0113217    -9.84   0.000    -.1335703   -.0891901
      priGPA |   1.244375   .0771321    16.13   0.000     1.093199    1.395552
       frosh |   .3899318    .113436     3.44   0.001     .1676013    .6122622
        soh  |   .0928127   .0944066     0.98   0.326    -.0922209    .2778463
       _cons |   .7621699   .2859966     2.66   0.008      .201627    1.322713
------------------------------------------------------------------------------
(Standard errors scaled using square root of Pearson X2-based dispersion.)

. di (.1268)^2
.01607824
```

c. Because the coefficient on *ACT* is negative, we know that an increase in ACT score,

holding year and prior GPA fixed, actually reduces predicted attendance rate. The calculation below shows that for $priGPA - 3.0$ and $frosh = soph = 0$, when ACT increases from 25 to 30, the estimated fall in *atndrte* is about .087, or 8.7 percentage points. This is very similar to the estimate using the linear model – 8.5 percentage points – which is the same for any values of the explanatory variables.

```
. di exp(_b[_cons] + _b[ACT]*30 + _b[priGPA]*3)/(1 + exp(_b[_cons] + _b[ACT]*
      + _b[priGPA]*3)) - exp(_b[_cons] + _b[ACT]*25 + _b[priGPA]*3)
      /(1 + exp(_b[_cons] + _b[ACT]*25 + _b[priGPA]*3))
-.08671822
```

d. The *R*-squared for the linear model is about .302. For the logistic functional form, I computed the squared correlation between $atndrte_i$ and $\hat{E}(atndrte_1|\mathbf{x}_i)$. This *R*-squared is about .328, and so the logistic functional form does fit better than the linear model. And, remember that the parameters in the logistic functional form are not chosen to maximize an *R*-squared; the linear model coefficients are chosen to maximize *R*-squared given the set of explanatory variables.

```
. predict atndrteh_log
(option mu assumed; predicted mean atndrte)

. corr atndrte atndrteh_log
(obs=680)

             |  atndrte atndrt~g
-------------+------------------
     atndrte |   1.0000
atndrteh_log |   0.5725   1.0000


. di .5725^2
.32775625
```

18.10. a. The pooled Poisson estimates, with the usual pooled standard errors that assume a unit variance-mean ratio and dynamic completeness of the conditional mean, are given below. Using these nonrobust standard errors, all lags except the first are significantly different from zero.

```
. use patent

. poisson patents y77-y81 lrnd lrnd_1 lrnd_2 lrnd_3 lrnd_4

Poisson regression                                Number of obs   =       1356
                                                  LR chi2(10)     =   68767.04
                                                  Prob > chi2     =     0.0000
Log likelihood = -12194.868                       Pseudo R2       =     0.7382

------------------------------------------------------------------------------
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         y77 |  -.0732934   .0190128    -3.85   0.000    -.1105578   -.0360291
         y78 |   -.227293   .0196925   -11.54   0.000    -.2658896   -.1886965
         y79 |    -.36251   .0196912   -18.41   0.000    -.4011041   -.3239159
         y80 |  -.7066175   .0211325   -33.44   0.000    -.7480365   -.6651985
         y81 |  -2.115567   .0331249   -63.87   0.000     -2.18049   -2.050643
        lrnd |   .4406223   .0425948    10.34   0.000      .357138    .5241066
      lrnd_1 |   .0767312   .0635969     1.21   0.228    -.0479165    .2013788
      lrnd_2 |   .2452529   .0622048     3.94   0.000     .1233337    .3671721
      lrnd_3 |  -.1557527   .0630881    -2.47   0.014    -.2794031   -.0321023
      lrnd_4 |   .1619174   .0469008     3.45   0.001     .0699936    .2538412
       _cons |   1.157326   .0191835    60.33   0.000     1.119727    1.194925
------------------------------------------------------------------------------
```

b. The standard errors computed in part a can be wrong for at least two reasons. The first is that the conditional variance, $\text{Var}(y_{it}|\mathbf{x}_{it})$, may not equal the conditional mean, $\text{E}(y_{it}|\mathbf{x}_{it})$, where $\mathbf{x}_{it}$ contains the current and lagged R&D spending variables. The second is that the mean may not be dynamically complete in the sense that

$$\text{E}(y_{it}|\mathbf{x}_{it}) \neq \text{E}(y_{it}|\mathbf{x}_{it}, y_{i,t-1}, \mathbf{x}_{i,t-1}, \ldots).$$

A failure of dynamic completeness generally leads to serial correlation in the implied error terms, and cause the score of the partial quasi-log-likelihood function to be serially correlated.

A third reason the standard errors might not be valid is they use the expected Hessian form of the asymptotic variance. This form is incorrect of the conditional mean is misspecified.

c. The estimates below give $\hat{\sigma} \approx 4.14$, which shows that, even if we assume a constant variance-mean ratio and dynamic completeness of the conditional mean, we need to multiply all Poisson standard errors by just over four.

Now only the contemporaneous R&D variable is significant; none of the lags has a *t*

statistic above one.

```
. glm patents y77-y81 lrnd lrnd_1 lrnd_2 lrnd_3 lrnd_4, family(poisson)
      link(log) sca(x2)

Generalized linear models                         No. of obs      =      1356
Optimization     : ML                             Residual df     =      1345
                                                  Scale parameter =
Deviance         =  20618.28952                   (1/df) Deviance =  15.32958
Pearson          =  23082.45413                   (1/df) Pearson  =  17.16168

Variance function: V(u) = u                       [Poisson]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             =  18.00276
Log likelihood   = -12194.86797                   BIC             =  10917.75

------------------------------------------------------------------------------
             |                 OIM
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         y77 |  -.0732934   .0787636    -0.93   0.352    -.2276672    .0810803
         y78 |   -.227293   .0815794    -2.79   0.005    -.3871858   -.0674003
         y79 |    -.36251   .0815742    -4.44   0.000    -.5223925   -.2026275
         y80 |  -.7066175    .087545    -8.07   0.000    -.8782025   -.5350325
         y81 |  -2.115567   .1372256   -15.42   0.000    -2.384524   -1.846609
        lrnd |   .4406223    .176456     2.50   0.013     .0947748    .7864698
      lrnd_1 |   .0767312   .2634608     0.29   0.771    -.4396424    .5931048
      lrnd_2 |   .2452529   .2576938     0.95   0.341    -.2598177    .7503235
      lrnd_3 |  -.1557527   .2613529    -0.60   0.551    -.6679949    .3564895
      lrnd_4 |   .1619174   .1942941     0.83   0.405    -.2188921    .5427269
       _cons |   1.157326   .0794708    14.56   0.000     1.001566    1.313086
------------------------------------------------------------------------------
(Standard errors scaled using square root of Pearson X2-based dispersion.)

. di sqrt(17.16)
4.142463
```

d. The QLR statistic is just the usual LR statistic divided by $\hat{\sigma}^2 = 17.17$. The value of the unrestricted log-likelihood is $\mathcal{L}_{ur} = -12,194.87$. The value of the restricted log-likelihood (without any of the lags), using the same set of years in estimation (1976 to 1981), is $\mathcal{L}_r = -12,252.37$. Therefore,

$$QLR = 2 \cdot (12,252.37 - 12,194.87)/17.17 = 6.70.$$

With four degrees of freedom in a chi-square distribution, this leads to *p*-value = .153. The lags are jointly insignificant at the usual 5% level. The usual LR statistic is 115, which (incorrectly)

implies very strong statistical significance for the lags.

e. The Stata results are blow. With the fully robust standard errors, the contemporaneous term and the second lag are marginally significantly. The robust Wald test for the exclusion of the four lags gives *p*-value =.494. The fully robust standard errors are clearly smaller than the Poisson MLE standard errors, but they are actually smaller in some cases than the GLM standard errors from part c. The four lags are joint insignificant.

```
. glm patents y77-y81 lrnd lrnd_1 lrnd_2 lrnd_3 lrnd_4, family(poisson)
      link(log) robust cluster(cusip)

Generalized linear models                         No. of obs      =      1356
Optimization     : ML                             Residual df     =      1345
                                                  Scale parameter =
Deviance         =  20618.28952                   (1/df) Deviance =  15.32958
Pearson          =  23082.45413                   (1/df) Pearson  =  17.16168

Variance function: V(u) = u                       [Poisson]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             =  18.00276
Log pseudolikelihood = -12194.86797               BIC             =  10917.75

                                (Std. Err. adjusted for 226 clusters in cusip
------------------------------------------------------------------------------
             |               Robust
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         y77 |  -.0732934   .0317955    -2.31   0.021    -.1356115   -.0109754
         y78 |   -.227293   .0499251    -4.55   0.000    -.3251445   -.1294416
         y79 |    -.36251   .0681543    -5.32   0.000      -.49609    -.22893
         y80 |  -.7066175   .0667816   -10.58   0.000     -.837507    -.575728
         y81 |  -2.115567   .1140381   -18.55   0.000    -2.339077   -1.892056
        lrnd |   .4406223   .2409156     1.83   0.067    -.0315637    .9128083
      lrnd_1 |   .0767312   .1228435     0.62   0.532    -.1640376       .3175
      lrnd_2 |   .2452529   .1411443     1.74   0.082    -.0313848    .5218906
      lrnd_3 |  -.1557527   .2160959    -0.72   0.471     -.579293    .2677875
      lrnd_4 |   .1619174   .2679931     0.60   0.546    -.3633395    .6871743
       _cons |   1.157326   .2061445     5.61   0.000     .7532903    1.561362
------------------------------------------------------------------------------

. test lrnd_1 lrnd_2 lrnd_3 lrnd_4

 ( 1)  [patents]lrnd_1 = 0
 ( 2)  [patents]lrnd_2 = 0
 ( 3)  [patents]lrnd_3 = 0
 ( 4)  [patents]lrnd_4 = 0

           chi2(  4) =    3.40
         Prob > chi2 =    0.4937
```

f. The estimated long run elasticity is about $.441 + .077 + .245 - .156 + .162 = .769$. The `lincom` command in Stata provides a simple way to obtain a fully robust standard error. Its fully robust standard error is about .072, which gives a 95% confidence interval from about .627 to .910. As is often the case in distributed lag models, we cannot estimate the lag distribution very precisely but we can get a fairly precise estimate of the long run effect.

```
. lincom  lrnd +  lrnd_1 +  lrnd_2 +  lrnd_3 +  lrnd_4

 ( 1)  [patents]lrnd + [patents]lrnd_1 + [patents]lrnd_2 + [patents]lrnd_3
     + [patents]lrnd_4 = 0

------------------------------------------------------------------------------
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         (1) |    .768771   .0722693    10.64   0.000     .6271258    .9104162
------------------------------------------------------------------------------
```

g. The fixed effects Poisson estimates are given below. The contemporaneous spending term and second lag have much smaller effects now, while lags three and four become larger and even statistically significant – but with the third lag still having a large, negative coefficient. When we use the fully robust standard errors, only the second lag is statistically significant at conventional levels, although the third and fourth lags are close.

The estimated long-run elasticity is now only .261 and it is, at best, marginally significant with $t = 1.60$.

```
. xtpqml patents y77-y81 lrnd lrnd_1 lrnd_2 lrnd_3 lrnd_4, fe
note: 8 groups (48 obs) dropped because of all zero outcomes

Conditional fixed-effects Poisson regression    Number of obs      =      1308
Group variable: cusip                           Number of groups   =       218

                                                Obs per group: min =
                                                               avg =       6.
                                                               max =

                                                Wald chi2(10)      =   3002.51
Log likelihood  = -2423.7694                    Prob > chi2        =    0.0000

------------------------------------------------------------------------------
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
```

```
         y77 |  -.0210069   .0204558    -1.03   0.304    -.0610995    .0190856
         y78 |   -.108368   .0251005    -4.32   0.000     -.157564    -.059172
         y79 |  -.1721306   .0306902    -5.61   0.000    -.2322822   -.1119789
         y80 |  -.4468227    .039243   -11.39   0.000    -.5237375   -.3699079
         y81 |  -1.797958   .0547882   -32.82   0.000    -1.905341   -1.690575
        lrnd |   .0492403   .0558275     0.88   0.378    -.0601795      .15866
      lrnd_1 |   .0512096   .0666844     0.77   0.443    -.0794894    .1819086
      lrnd_2 |    .130944   .0662164     1.98   0.048     .0011622    .2607259
      lrnd_3 |  -.1909907   .0714669    -2.67   0.008    -.3310632   -.0509182
      lrnd_4 |   .2201799   .0703992     3.13   0.002     .0821999    .3581599
------------------------------------------------------------------------------
Calculating Robust Standard Errors...
------------------------------------------------------------------------------
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
patents      |
         y77 |  -.0210069    .026186    -0.80   0.422    -.0723306    .0303168
         y78 |   -.108368    .055447    -1.95   0.051    -.2170422    .0003062
         y79 |  -.1721306    .071949    -2.39   0.017     -.313148   -.0311131
         y80 |  -.4468227   .0829316    -5.39   0.000    -.6093657   -.2842797
         y81 |  -1.797958   .1380887   -13.02   0.000    -2.068607   -1.527309
        lrnd |   .0492403   .0868099     0.57   0.571     -.120904    .2193845
      lrnd_1 |   .0512096   .0600491     0.85   0.394    -.0664845    .1689038
      lrnd_2 |    .130944   .0592739     2.21   0.027     .0147694    .2471187
      lrnd_3 |  -.1909907   .1066283    -1.79   0.073    -.3999783    .0179968
      lrnd_4 |   .2201799   .1431273     1.54   0.124    -.0603446    .5007043
------------------------------------------------------------------------------
Wald chi2(10) =   366.83                          Prob > chi2 =   0.0000

. lincom  lrnd +  lrnd_1 +  lrnd_2 +  lrnd_3 +  lrnd_4

 ( 1)  [patents]lrnd + [patents]lrnd_1 + [patents]lrnd_2 + [patents]lrnd_3
     + [patents]lrnd_4 = 0

------------------------------------------------------------------------------
     patents |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         (1) |   .2605831   .1632377     1.60   0.110     -.059357    .5805231
------------------------------------------------------------------------------
```

18.11. a. For each t, the density is

$$f_t(y_t|\mathbf{x}_i, c_i) = \exp(-c_i m_{it}) m_{it}^{y_t}/y_t!, \; y_t = 0, 1, 2, \ldots$$

Under the conditional independence assumption, the joint density of $(y_{i1}, \ldots, y_{iT})$ given $(\mathbf{x}_i, c_i)$ is

$$f(y_1,\ldots,y_T|\mathbf{x}_i,c_i) = \prod_{t=1}^{T}[\exp(-c_i m_{it})(c_i m_{it})^{y_t}/y_t!]$$
$$= \left(\prod_{t=1}^{T} m_{it}^{y_t}/y_t!\right) c_i^s \exp(-c_i M_i),$$

where $M_i \equiv m_{i1} + \ldots + m_{iT}$ and $s = y_1 + \ldots + y_T$, for all nonnegative integers $\{y_t : t = 1,\ldots,T\}$.

b. To obtain the density of $(y_{i1},\ldots,y_{iT})$ given $\mathbf{x}_i$ – say $g(y_{i1},\ldots,y_{iT}|\mathbf{x}_i)$ – we integrate out with respect to the distribution of c_i (because c_i is independent of $\mathbf{x}_i$). Therefore,

$$g(y_1,\ldots,y_T|\mathbf{x}_i) = \left(\prod_{t=1}^{T} m_{it}^{y_t}/y_t!\right)\int_0^\infty c^s \exp(-c_i M_i)[\delta^\delta/\Gamma(\delta)]c^{\delta-1}\exp(-\delta c)dc.$$

Next, we follow the hint, noting that the general Gamma(α,β) density has the form $h(c) = [\beta^\alpha/\Gamma(\alpha)]c^{\alpha-1}\exp(-\beta c)$. Now

$$\int_0^\infty c^s \exp(-cM_i)[\delta^\delta/\Gamma(\delta)]c^{\delta-1}\exp(-\delta c)dc = \int_0^\infty [\delta^\delta/\Gamma(\delta)]c^{(s+\delta-1)}\exp[-(M_i+\delta)c]dc$$
$$= [\delta^\delta/\Gamma(\delta)]\left[\Gamma(s+\delta)/(M_i+\delta)^{(s+\delta)}\right]$$
$$\cdot \int_0^\infty \left[(M_i+\delta)^{(s+\delta)}/\Gamma(s+\delta)\right]c^{(s+\delta-1)}\exp[-(M_i+\delta)c]dc,$$

and the integrand is easily seen to be the Gamma$(s+\delta, M_i+\delta)$ density, and so it integrates to unity. Therefore, we have shown

$$g(y_1,\ldots,y_T|\mathbf{x}_i) = \left(\prod_{t=1}^{T} m_{it}^{y_t}/y_t!\right)[\delta^\delta/\Gamma(\delta)]\left[\Gamma(s+\delta)/(M_i+\delta)^{(s+\delta)}\right]$$

for all nonnegative integers $\{y_t : t = 1,\ldots,T\}$.

18.12. a. First, the density of y_{it} given $(\mathbf{x}_i, c_i)$ is

$$f_t(y_t|\mathbf{x}_i,c_i) = [(1/c_i)^{m_{it}}/\Gamma(m_{it})]y_t^{(m_{it}-1)}\exp[-(1/c_i)y_t],\ y_t > 0.$$

Following the hint, the density of the sum, s_i, given $(\mathbf{x}_i, c_i)$ is

$$g(s|\mathbf{x}_i, c_i) = [(1/c_i)^{M_i}/\Gamma(M_i)]s^{(M_i-1)}\exp[-(1/c_i)s],\ s > 0,$$

where $M_i = m_{i1} + \ldots + m_{iT}$. Therefore, the density of $(y_{i1}, \ldots, y_{iT})$ given $(s_i = s, \mathbf{x}_i, c_i)$ is

$$\begin{aligned}
&\left([(1/c_i)^{m_{i1}}/\Gamma(m_{i1})]y_1^{(m_{i1}-1)}\exp[-(1/c_i)y_t]\right)\cdots\left([(1/c_i)^{m_{i,T-1}}/\Gamma(m_{i,T-1})]y_{T-1}^{(m_{i,T-1}-1)}\exp[-(1/c_i)y_{T-1}]\right) \\
&\cdot([(1/c_i)^{m_{iT}}/\Gamma(m_{iT})](s - y_1 - \ldots - y_{T-1})^{m_{iT}} \cdot \exp[-(1/c_i)(s - y_1 - \ldots - y_{T-1})])/g(s|\mathbf{x}_i, c_i) \\
&= (1/c_i)^{M_i}\left(\prod_{t=1}^{T}\Gamma(m_{it})\right)^{-1}\left(\prod_{t=1}^{T}y_t^{(m_{it}-1)}\right)\exp[(-1/c_i)s]/([(1/c_i)^{M_i}/\Gamma(M_i)]s^{(M_i-1)}\exp[-(1/c_i)s]) \\
&= \Gamma(M_i)\left(\prod_{t=1}^{T}\Gamma(m_{it})\right)^{-1}\left(\prod_{t=1}^{T}y_t^{(m_{it}-1)}\right)/s^{(M_i-1)},
\end{aligned}$$

which is what we wanted to show. Note how c_i has dropped out of the density.

b. The conditional log-likelihood for observation i is

$$\begin{aligned}
\ell(\boldsymbol{\beta}; y_{i1}, \ldots, y_{iT}, \mathbf{x}_i) = &\log\{\Gamma[M_i(\boldsymbol{\beta})]\} - \sum_{t=1}^{T}\log\{\Gamma[m_{it}(\boldsymbol{\beta})]\} \\
&+ \sum_{t=1}^{T}[m_{it}(\boldsymbol{\beta}) - 1]\log(y_{it}) - [M_i(\boldsymbol{\beta}) - 1]\log(y_{i1} \cdot \cdot \cdot y_{iT}),
\end{aligned}$$

where $m_{it}(\boldsymbol{\beta}) = m_t(\mathbf{x}_i, \boldsymbol{\beta})$ and $M_i = \sum_{t=1}^{T} m_{it}(\boldsymbol{\beta})$. We can sum across all i and maximize the resulting log-likelihood with respect to $\boldsymbol{\beta}$ to obtain the fixed effects gamma estimator. The asymptotic theory is standard, provided the regression functions are smooth functions of $\boldsymbol{\beta}$ and depend on the covariates in such a way that $\boldsymbol{\beta}_o$ is identified.

18.13. a. Plug in the data, a genereric value $\boldsymbol{\beta}$, and take the natural log:

$$\ell_i(\boldsymbol{\beta}) = \mathbf{x}_i\boldsymbol{\beta} + \log(y_i)[\exp(\mathbf{x}_i\boldsymbol{\beta}) - 1].$$

Notice that this is not a member of the linear exponential family (because it is $\log(y_i)$, not y_i, that appears).

b. The gradient is

$$\nabla_{\boldsymbol{\beta}} \ell_i(\boldsymbol{\beta}) = \mathbf{x}_i + \log(y_i)\mathbf{x}_i \exp(\mathbf{x}_i\boldsymbol{\beta})$$

and taking the transpose gives

$$\begin{aligned}\mathbf{s}_i(\boldsymbol{\beta}) &= \mathbf{x}_i' + \log(y_i)\mathbf{x}_i' \exp(\mathbf{x}_i\boldsymbol{\beta}) \\ &= \mathbf{x}_i'[1 + \log(y_i)\exp(\mathbf{x}_i\boldsymbol{\beta})].\end{aligned}$$

c. Because $0 < y_i < 1$, $\log(y_i) < 0$ for all i. Therefore, we know $\mathrm{E}[\log(y_i)|\mathbf{x}_i] < 0$ for any outcome $\mathbf{x}_i$.

d. We use part c:

$$\begin{aligned}\mathrm{E}[\mathbf{s}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i] &= \mathbf{x}_i'\{1 + \mathrm{E}[\log(y_i)|\mathbf{x}_i]\exp(\mathbf{x}_i\boldsymbol{\beta}_o)\} \\ &= \mathbf{x}_i'[1 - \exp(-\mathbf{x}_i\boldsymbol{\beta}_o)\exp(\mathbf{x}_i\boldsymbol{\beta}_o)] \;= \mathbf{0}.\end{aligned}$$

e. Using $\mathbf{s}_i(\boldsymbol{\beta}) = \mathbf{x}_i'[1 + \log(y_i)\exp(\mathbf{x}_i\boldsymbol{\beta})$, the Hessian is

$$\mathbf{H}_i(\boldsymbol{\beta}) = \nabla_{\boldsymbol{\beta}}\mathbf{s}_i(\boldsymbol{\beta}) = \mathbf{x}_i'\mathbf{x}_i \log(y_i)\exp(\mathbf{x}_i\boldsymbol{\beta})$$

and so

$$\begin{aligned}\mathrm{E}[\mathbf{H}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i] &= \mathbf{x}_i'\mathbf{x}_i\mathrm{E}[\log(y_i)|\mathbf{x}_i]\exp(\mathbf{x}_i\boldsymbol{\beta}_o) \\ &= \mathbf{x}_i'\mathbf{x}_i\mathrm{E}[\log(y_i)|\mathbf{x}_i]\exp(\mathbf{x}_i\boldsymbol{\beta}_o) \\ &= -\mathbf{x}_i'\mathbf{x}_i\exp(-\mathbf{x}_i\boldsymbol{\beta}_o)\exp(\mathbf{x}_i\boldsymbol{\beta}_o) = -\mathbf{x}_i'\mathbf{x}_i,\end{aligned}$$

and so

$$-\mathrm{E}[\mathbf{H}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i] = \mathbf{x}_i'\mathbf{x}_i.$$

f. Given part e, the formula based on the expected Hessian is easiest:

$$\mathrm{Avar}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) = [\mathrm{E}(\mathbf{x}_i'\mathbf{x}_i)]^{-1}$$

and so

$$\widehat{\mathrm{Avar}}[\sqrt{N}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) = \left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_i'\mathbf{x}_i\right)^{-1} = (\mathbf{X}'\mathbf{X}/N)^{-1}.$$

g. From part d, we see that the key condition for Fisher consistency, that is, to make $E[\mathbf{s}_i(\boldsymbol{\beta}_o)|\mathbf{x}_i] = \mathbf{0}$, is that

$$E[\log(y_i)|\mathbf{x}_i] = -\exp(-\mathbf{x}_i\boldsymbol{\beta}_o).$$

In other words, the implied model for $E[\log(y_i)|\mathbf{x}_i]$ must be correct. Unfortunately, having $E(y_i|\mathbf{x}_i)$ correctly specified, that is, $E(y_i|\mathbf{x}_i) = \exp(\mathbf{x}_i\boldsymbol{\beta}_o)/[1 + \exp(\mathbf{x}_i\boldsymbol{\beta}_o)]$, generally says nothing about $E[\log(y_i)|\mathbf{x}_i]$.

h. We could use the Bernoulli QMLE to estimate the parameters in $E(y_i|\mathbf{x}_i) = \exp(\mathbf{x}_i\boldsymbol{\beta}_o)/[1 + \exp(\mathbf{x}_i\boldsymbol{\beta}_o)]$ directly, without extra assumptions about $D(y_i|\mathbf{x}_i)$.

18.14. a. The Stata output is given below with the three sets of standard errors asked for in the problem. The inference starts from the least robust and ends with the most robust.

The difference in standard errors is striking. The standard errors that effectively maintain a binomial distribution – at least its first two moments – lead to huge *t* statistics. When we allow for a scale factor – the so-called GLM variance assumption, (18.34) – the standard errors increase by at least a factor of 20. The fully robust standard errors, which allow unrestricte $\text{Var}(partic_i|employ_i, \mathbf{x}_i)$ are still larger – more than three times the standard errors produced under the GLM assumption. It seems pretty clear the binomial distribution does not hold in this application and that the actual conditional variance is not proportional to the nominal variance in the binomial distribution. It is pretty clear that we should use the fully robust standard errors, which lead to much more modest (but still quite significant) *t* statistics.

```
. glm partic mrate ltotemp age agesq sole, fam(bin employ) link(logit)

Generalized linear models                         No. of obs      =      4075
Optimization     : ML                             Residual df     =      4069
                                                  Scale parameter =
Deviance         =  2199795.239                   (1/df) Deviance =  540.6231
Pearson          =  2021563.356                   (1/df) Pearson  =  496.8207
```

```
Variance function: V(u) = u*(1-u/employ)          [Binomial]
Link function    : g(u) = ln(u/(employ-u))        [Logit]

                                                  AIC             =  544.0016
Log likelihood   = -1108397.213                   BIC             =   2165971

------------------------------------------------------------------------------
             |                 OIM
      partic |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       mrate |   .9871354   .0033855   291.58   0.000        .9805    .9937708
     ltotemp |  -.1386562    .000531  -261.10   0.000     -.139697   -.1376153
         age |   .0718575   .0001669   430.63   0.000     .0715305    .0721846
       agesq |  -.0005512   2.82e-06  -195.47   0.000    -.0005567   -.0005457
        sole |   .3419834    .003443    99.33   0.000     .3352353    .3487315
       _cons |   1.442014   .0053821   267.93   0.000     1.431465    1.452563
------------------------------------------------------------------------------

. glm partic mrate ltotemp age agesq sole, fam(bin employ) link(logit) sca(x2

Generalized linear models                         No. of obs      =      4075
Optimization     : ML                             Residual df     =      4069
                                                  Scale parameter =
Deviance         =  2199795.239                   (1/df) Deviance =  540.6231
Pearson          =  2021563.356                   (1/df) Pearson  =  496.8207

Variance function: V(u) = u*(1-u/employ)          [Binomial]
Link function    : g(u) = ln(u/(employ-u))        [Logit]

                                                  AIC             =  544.0016
Log likelihood   = -1108397.213                   BIC             =   2165971

------------------------------------------------------------------------------
             |                 OIM
      partic |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       mrate |   .9871354   .0754604    13.08   0.000     .8392358    1.135035
     ltotemp |  -.1386562   .0118368   -11.71   0.000    -.1618559   -.1154565
         age |   .0718575   .0037193    19.32   0.000     .0645678    .0791472
       agesq |  -.0005512   .0000629    -8.77   0.000    -.0006744    -.000428
        sole |   .3419834   .0767418     4.46   0.000     .1915723    .4923945
       _cons |   1.442014   .1199639    12.02   0.000     1.206889    1.677139
------------------------------------------------------------------------------
(Standard errors scaled using square root of Pearson X2-based dispersion)

. glm partic mrate ltotemp age agesq sole, fam(bin employ) link(logit) robust

Generalized linear models                         No. of obs      =      4075
Optimization     : ML                             Residual df     =      4069
                                                  Scale parameter =
Deviance         =  2199795.239                   (1/df) Deviance =  540.6231
Pearson          =  2021563.356                   (1/df) Pearson  =  496.8207

Variance function: V(u) = u*(1-u/employ)          [Binomial]
Link function    : g(u) = ln(u/(employ-u))        [Logit]

                                                  AIC             =  544.0016
Log pseudolikelihood = -1108397.213               BIC             =   2165971
```

```
------------------------------------------------------------------------------
             |               Robust
      partic |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mrate |   .9871354   .2622177     3.76   0.000     .4731982    1.501073
     ltotemp |  -.1386562   .0546138    -2.54   0.011    -.2456972   -.0316151
         age |   .0718575   .0142656     5.04   0.000     .0438974    .0998176
       agesq |  -.0005512   .0001746    -3.16   0.002    -.0008934    -.000209
        sole |   .3419834   .1145195     2.99   0.003     .1175294    .5664375
       _cons |   1.442014   .4368904     3.30   0.001     .5857248    2.298303
------------------------------------------------------------------------------
```

b. The fractional logit results for *prate* are given below – with the same kinds of standard errors in part a. In this case the usual MLE standard errors that are too large: they treat $\sigma^2 = 1$ in (18.58), which is true in the binary case but not generally. With a factional variable, $\sigma^2 < 1$. In fact, the estimate for this data set is $\hat{\sigma}^2 = .214$.

The GLM and fully robust standard errors are much closer now, with the fully robust ones typically (but not always) being slightly larger.

```
. glm prate mrate ltotemp age agesq sole, fam(bin) link(logit)
note: prate has noninteger values

Generalized linear models                         No. of obs      =      4075
Optimization     : ML                             Residual df     =      4069
                                                  Scale parameter =
Deviance         =   883.051611                   (1/df) Deviance =  .2170193
Pearson          =  871.5810654                   (1/df) Pearson  =  .2142003

Variance function: V(u) = u*(1-u/1)               [Binomial]
Link function    : g(u) = ln(u/(1-u))             [Logit]

                                                  AIC             =  .6350527
Log likelihood   = -1287.919784                   BIC             = -32941.02

------------------------------------------------------------------------------
             |                 OIM
       prate |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mrate |   1.147984   .1468736     7.82   0.000     .8601167     1.43585
     ltotemp |  -.2075898   .0290032    -7.16   0.000     -.264435   -.1507446
         age |   .0481773   .0145566     3.31   0.001     .0196469    .0767077
       agesq |  -.0004519   .0004301    -1.05   0.293    -.0012948     .000391
        sole |   .1652908     .10408     1.59   0.112    -.0387022    .3692838
       _cons |   2.355715   .2299685    10.24   0.000     1.904985    2.806445
------------------------------------------------------------------------------

. glm prate mrate ltotemp age agesq sole, fam(bin) link(logit) sca(x2)
note: prate has noninteger values

Generalized linear models                         No. of obs      =      4075
```

```
Optimization     : ML                                Residual df     =      4069
                                                     Scale parameter =
Deviance         =   883.051611                      (1/df) Deviance =  .2170193
Pearson          =  871.5810654                      (1/df) Pearson  =  .2142003

Variance function: V(u) = u*(1-u/1)                  [Binomial]
Link function    : g(u) = ln(u/(1-u))                [Logit]

                                                     AIC             =  .6350527
Log likelihood   = -1287.919784                      BIC             = -32941.02

------------------------------------------------------------------------------
             |                 OIM
       prate |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mrate |   1.147984   .0679757    16.89   0.000     1.014754    1.281213
     ltotemp |  -.2075898   .0134232   -15.47   0.000    -.2338988   -.1812808
         age |   .0481773    .006737     7.15   0.000     .0349729    .0613817
       agesq |  -.0004519    .000199    -2.27   0.023     -.000842   -.0000618
        sole |   .1652908   .0481701     3.43   0.001     .0708792    .2597024
       _cons |   2.355715   .1064335    22.13   0.000     2.147109    2.564321
------------------------------------------------------------------------------
(Standard errors scaled using square root of Pearson X2-based dispersion)

. glm prate mrate ltotemp age agesq sole, fam(bin) link(logit) robust
note: prate has noninteger values

Generalized linear models                            No. of obs      =      4075
Optimization     : ML                                Residual df     =      4069
                                                     Scale parameter =
Deviance         =   883.051611                      (1/df) Deviance =  .2170193
Pearson          =  871.5810654                      (1/df) Pearson  =  .2142003

Variance function: V(u) = u*(1-u/1)                  [Binomial]
Link function    : g(u) = ln(u/(1-u))                [Logit]

                                                     AIC             =  .6350527
Log pseudolikelihood = -1287.919784                  BIC             = -32941.02

------------------------------------------------------------------------------
             |               Robust
       prate |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       mrate |   1.147984   .0747331    15.36   0.000     1.001509    1.294458
     ltotemp |  -.2075898   .0141209   -14.70   0.000    -.2352662   -.1799134
         age |   .0481773   .0061543     7.83   0.000      .036115    .0602396
       agesq |  -.0004519   .0001764    -2.56   0.010    -.0007976   -.0001063
        sole |   .1652908   .0505915     3.27   0.001     .0661334    .2644483
       _cons |   2.355715   .1066441    22.09   0.000     2.146696    2.564734
------------------------------------------------------------------------------
```

c. It makes sense to compare the coefficients in parts a and b because both approaches could be estimating the same conditional mean function for $prate_i = partic_i/employ_i$. Generally, the binomial approach starts with

$$\mathrm{E}(y_i|\mathbf{x}_i, n_i) = n_i\Lambda(\mathbf{x}_i\boldsymbol{\beta}).$$

If we divide both sides by n_i we get

$$\frac{\mathrm{E}(y_i|\mathbf{x}_i, n_i)}{n_i} = \Lambda(\mathbf{x}_i\boldsymbol{\beta})$$

or

$$\mathrm{E}\left(\frac{y_i}{n_i}\,\middle|\,\mathbf{x}_i, n_i\right) = \Lambda(\mathbf{x}_i\boldsymbol{\beta})$$

which, of course, implies

$$\mathrm{E}\left(\frac{y_i}{n_i}\,\middle|\,\mathbf{x}_i\right) = \Lambda(\mathbf{x}_i\boldsymbol{\beta})$$

In other words, the fractional variable $w_i \equiv y_i/n_i$ follows a fractional response model with a logistic response function. So if we start with $\mathrm{E}(y_i|\mathbf{x}_i, n_i) = n_i\Lambda(\mathbf{x}_i\boldsymbol{\beta})$ then both methods consistently estimate $\boldsymbol{\beta}$.

d. The Stata output is given below. Because we want the APE on *prate*, we compute

$$\hat{\beta}_{mrate}\left(N^{-1}\sum_{i=1}^{N}\frac{\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})}{[1+\exp(\mathbf{x}_i\hat{\boldsymbol{\beta}})]^2}\right)$$

for both set of estimates. For the binomial QMLE the estimate is about .147. For the Bernoulli QMLE, the estimate is about .130. Incidentally, the linear regression estimate – coefficient on *mrate* – is about .106, so quite a bit below the other two.

```
. qui glm partic mrate ltotemp age agesq sole, fam(bin employ) link(logit)
. predict xb_bin, xb
. gen sca_bin = exp(xb_bin)/((1 + exp(xb_bin))^2)
. sum sca_bin
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     sca_bin |      4075    .1492273    .0602467   .0091082   .2499969
```

```
. di  .1492273*_b[mrate]
.14730755

. qui glm prate mrate ltotemp age agesq sole, fam(bin) link(logit)

. predict xb_ber, xb

. gen sca_ber = exp(xb_ber)/((1 + exp(xb_ber))^2)

. di sca_ber*_b[mrate]
.13000441
```

e. The Stata output is given below. The APE is about .038. If we use the linear model, we would get $.106(.25) \approx .027$, so somewhat less.

```
. qui glm prate mrate ltotemp age agesq sole, fam(bin) link(logit)

. gen xb_p50 = xb_ber - _b[mrate]*mrate + _b[mrate]*.5

. gen xb_p25 = xb_ber - _b[mrate]*mrate + _b[mrate]*.25

. gen phat_p50 = exp(xb_p50 )/(1 + exp(xb_p50 ))

. gen phat_p25 = exp(xb_p25 )/(1 + exp(xb_p25 ))

. gen diff = phat_p50 - phat_p25

. sum diff

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        diff |      4075    .0375776    .0107886   .0116919   .0689509
```

18.15. a. We can just use the usual fixed effects or first-differencing estimators. If we define $w_i \equiv \log[y_{it}/(1 - y_{it})]$ then we have

$$w_{it} = \mathbf{x}_{it}\boldsymbol{\beta} + c_i + u_{it}$$
$$\mathrm{E}(u_{it}|\mathbf{x}_i, c_i) = 0, t = 1, \ldots, T,$$

which means the key strict exogeneity assumption on $\{\mathbf{x}_{it} : t = 1, \ldots, T\}$ holds. Of course, we could use a GLS version of FE or FD, or use Chamberlain's approach.

b. Because $\log[y_{it}/(1 - y_{it})] = \mathbf{x}_{it}\boldsymbol{\beta} + v_{it}$,

$$y_{it}/(1 - y_{it}) = \exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})$$

and so

$$\frac{(1 - y_{it})}{y_{it}} = \frac{1}{\exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})}$$

or

$$\frac{1}{y_{it}} = \frac{1}{\exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})} + 1 = \frac{1 + \exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})}{\exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})}$$

which implies

$$y_{it} = \frac{\exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})}{1 + \exp(\mathbf{x}_{it}\boldsymbol{\beta} + v_{it})}.$$

The ASF is defined, for each t, as

$$\text{ASF}_t(\mathbf{x}_t) = \int_{-\infty}^{\infty} \left[\frac{\exp(\mathbf{x}_t\boldsymbol{\beta} + \upsilon)}{1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + \upsilon)} g_t(\upsilon) d\upsilon \right]$$

where $g_t(\cdot)$ is the density of v_{it}. (Of course, allowing this density to be discrete changes the integral to a sum.) We can also write

$$\text{ASF}_t(\mathbf{x}_t) = \text{E}_{v_{it}} \left[\frac{\exp(\mathbf{x}_t\boldsymbol{\beta} + v_{it})}{1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + v_{it})} \right],$$

that is, we fix the covariates at values $\mathbf{x}_t$ and average across the distribution of the unobservables, v_{it}.

c. The ASF cannot be estimated without further assumptions because we cannot estimate the expected value of $\exp(\mathbf{x}_t\boldsymbol{\beta} + v_{it})/[1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + v_{it})]$ for given $\mathbf{x}_t$ without further assumptions.

d. By the law of iterated expectations, we have

$$\text{ASF}_t(\mathbf{x}_t) = \text{E}_{\bar{\mathbf{x}}_i}\left\{\text{E}\left[\frac{\exp(\mathbf{x}_t\boldsymbol{\beta} + v_{it})}{1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + v_{it})}\,\middle|\,\bar{\mathbf{x}}_i\right]\right\}$$
$$\text{E}_{\bar{\mathbf{x}}_i}\left\{\text{E}\left[\frac{\exp(\mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + r_{it})}{1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + r_{it})}\,\middle|\,\bar{\mathbf{x}}_i\right]\right\}.$$

With r_{it} is independent of $\mathbf{x}_i$ – and we can assume $\text{E}(a_i) = 0$ due to the presence of ψ – then we can consistently estimate $\boldsymbol{\beta}$, ψ, and $\boldsymbol{\xi}$ by pooled OLS:

$$w_{it} = \text{ on } \mathbf{x}_{it}, 1, \bar{\mathbf{x}}_i, t = 1, \ldots, T; i = 1, \ldots, N.$$

(Recall this produces the FE estimator of $\boldsymbol{\beta}$.) Further, by independence,

$$\text{E}\left[\frac{\exp(\mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + r_{it})}{1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + r_{it})}\,\middle|\,\bar{\mathbf{x}}_i\right] = \int_{-\infty}^{\infty}\frac{\exp(\mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \mathfrak{r})}{1 + \exp(\mathbf{x}_t\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \mathfrak{r})}f_t(\mathfrak{r})d\mathfrak{r}$$

For fixed $\bar{\mathbf{x}}_i = \bar{\mathbf{x}}$, we can consistently estimate this expression as

$$N^{-1}\sum_{i=1}^{N}\frac{\exp(\mathbf{x}_t\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{x}}\hat{\boldsymbol{\xi}} + \hat{r}_{it})}{1 + \exp(\mathbf{x}_t\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{x}}\hat{\boldsymbol{\xi}} + \hat{r}_{it})},$$

where $\hat{r}_{it} \equiv w_{it} - \mathbf{x}_{it}\hat{\boldsymbol{\beta}} - \hat{\psi} - \bar{\mathbf{x}}_i\hat{\boldsymbol{\xi}}$ are the pooled OLS residuals. To get the ASF, we need to further average out over the distribution of $\bar{\mathbf{x}}_i$, which gives

$$\widehat{\text{ASF}}_t(\mathbf{x}_t) = N^{-1}\sum_{i=1}^{N}\frac{\exp(\mathbf{x}_t\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{x}}_i\hat{\boldsymbol{\xi}} + \hat{r}_{it})}{1 + \exp(\mathbf{x}_t\hat{\boldsymbol{\beta}} + \hat{\psi} + \bar{\mathbf{x}}_i\hat{\boldsymbol{\xi}} + \hat{r}_{it})}.$$

We use this as usual: take derivatives and changes with respect to the elements of $\mathbf{x}_t$.

18.16. (**Bonus Question**) Consider a panel data mode for $y_{it} \geq 0$ with multiplicative heterogeneity and a multiplicative idiosyncratic error:

$$y_{it} = c_i\exp(\mathbf{x}_{it}\boldsymbol{\beta})r_{it}.$$

If we assume $\{\mathbf{x}_{it} : t = 1, \ldots, T\}$ is strictly exogenous then we can estimate $\boldsymbol{\beta}$ using the fixed effects Poisson QMLE. Instead, assume we have instruments, $\{\mathbf{z}_{it} : t = 1, 2, \ldots, T\}$ that satisfy

a sequential exogeneity assumption:

$$\mathrm{E}(r_{it}|\mathbf{z}_{it},\ldots,\mathbf{z}_{i1},c_i) = \mathrm{E}(r_{it}) = 1,$$

where setting the expected value to unity is a normalization. (As usual, $\mathbf{x}_{it}$ should probably include a full set of time period dummies.)

a. Show that we can write

$$\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})} = c_i(r_{it} - r_{i,t+1}) \equiv e_{i,t+1}$$

where

$$\mathrm{E}(e_{it}|\mathbf{z}_{it},\ldots,\mathbf{z}_{i1}) = 0,\ t = 1,\ldots,T-1.$$

b. Part a implies that we can use the moment conditions

$$\mathrm{E}\left[\left.\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})}\right|\mathbf{z}_{it},\ldots,\mathbf{z}_{i1}\right] = 0,\ t = 1,\ldots,T-1$$

to estimate $\boldsymbol{\beta}$. Explain why using these moments directly can cause computational problems. (Hint: Suppose For example, if $x_{itj} > 0$ for some j and all i and t. What would happen if β_j is made larger and larger?)

c. Define the average of the population means across time as

$$\boldsymbol{\mu}_{\mathbf{x}} \equiv T^{-1}\sum_{r=1}^{T}\mathrm{E}(\mathbf{x}_{ir}).$$

Show that if you multiply the moment conditions in part b by $\exp(\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\beta})$, the resulting moment conditions are

$$\mathrm{E}\left[\left.\frac{y_{it}}{\exp[(\mathbf{x}_{it} - \boldsymbol{\mu}_{\mathbf{x}})\boldsymbol{\beta}]} - \frac{y_{i,t+1}}{\exp[(\mathbf{x}_{i,t+1} - \boldsymbol{\mu}_{\mathbf{x}})\boldsymbol{\beta}]}\right|\mathbf{z}_{it},\ldots,\mathbf{z}_{i1}\right] = 0.$$

[See Windmeijer (2002, Economics Letters).] How does this help with the computational

problem in part b?

d. What would you use in place of $\boldsymbol{\mu}_{\mathbf{x}}$ given that $\boldsymbol{\mu}_{\mathbf{x}}$ is unknown?

e. Suppose that $\{\mathbf{x}_{it} : t = 1, 2, \ldots, T\}$ is sequentially exogenenous, so that we can take $\mathbf{z}_{it} = \mathbf{x}_{it}$. Show that

$$\mathrm{E}\left[y_{it} - \frac{y_{i,t+1}}{\exp[(\mathbf{x}_{i,t+1} - \mathbf{x}_{it})\boldsymbol{\beta}]} \,\middle|\, \mathbf{x}_{it}, \ldots, \mathbf{x}_{i1} \right] = 0, \; t = 1, \ldots, T-1.$$

In other words, we can write moment conditions in terms of the first difference of the explanatory variables.

Solution

a. From $y_{it} = c_i \exp(\mathbf{x}_{it}\boldsymbol{\beta}) r_{it}$ for all $t = 1, \ldots, T$ we have

$$\begin{aligned} \frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} &= c_i r_{it} \\ \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})} &= c_i r_{i,t+1}, \end{aligned}$$

and subtracting the first equation from the second gives

$$\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})} = c_i(r_{it} - r_{i,t+1}).$$

Now

$$\begin{aligned} \mathrm{E}[c_i(r_{it} - r_{i,t+1}) | \mathbf{z}_{it}, \ldots, \mathbf{z}_{i1}, c_i)] &= c_i[\mathrm{E}(r_{it} | \mathbf{z}_{it}, \ldots, \mathbf{z}_{i1}, c_i) \\ &\quad - \mathrm{E}(r_{i,t+1} | \mathbf{z}_{it}, \ldots, \mathbf{z}_{i1}, c_i)] \\ &= c_i(1 - 1) = 0 \end{aligned}$$

b. Suppose $x_{it1} > 0$ for all i and t. Then $\beta_1 x_{it1} \to \infty$ as $\beta_j \to \infty$, which means

$$\exp(\beta_1 x_{it1} + \beta_2 x_{it2} + \ldots + \beta_K x_{itK}) \to \infty$$

for all i and t, for any values of $\beta_2, \ldots, \beta_K$. Then

$$\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})} \to 0$$

as $\beta_1 \to \infty$, and so the residual function can be made closer and closer to zero by increasing β_1 without bound.

c. Multiplying the original moment conditions by the $\exp(\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\beta})$ clearly does not does not change that they still hold:

$$\exp(\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\beta})\mathrm{E}\left[\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})} \middle| \mathbf{z}_{it}, \ldots, \mathbf{z}_{i1} \right] = 0.$$

The left hand side is simply

$$\frac{\exp(\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\beta})y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{\exp(\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\beta})y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})} = \frac{y_{it}}{\exp[(\mathbf{x}_{it} - \boldsymbol{\mu}_{\mathbf{x}})\boldsymbol{\beta}]} - \frac{y_{i,t+1}}{\exp[(\mathbf{x}_{i,t+1} - \boldsymbol{\mu}_{\mathbf{x}})\boldsymbol{\beta}]}.$$

Using these new moment conditions does not lead to the problem discussed in part b because the deviated covariates, $\mathbf{x}_{it} - \boldsymbol{\mu}_{\mathbf{x}}$, can take on both negative and positive values.

d. We would use the sample counterpart,

$$\bar{\mathbf{x}} \equiv T^{-1}\sum_{r=1}^{T}\left(N^{-1}\sum_{i=1}^{N}\mathbf{x}_{ir} \right) = (NT)^{-1}\sum_{i=1}^{N}\sum_{r=1}^{T}\mathbf{x}_{ir}.$$

In the sample, the deviated variables, $\mathbf{x}_{it} - \bar{\mathbf{x}}$, will always take on positive and negative values. Technically we should account for the estimation error in $\bar{\mathbf{x}}$ but it likely has a minor effect.

The sample moments we would like to make close to zero have the form

$$\sum_{i=1}^{N}\sum_{t=1}^{T-1}\mathbf{g}_{it}'\left[\frac{y_{it}}{\exp[(\mathbf{x}_{it} - \bar{\mathbf{x}})\boldsymbol{\beta}]} - \frac{y_{i,t+1}}{\exp[(\mathbf{x}_{i,t+1} - \bar{\mathbf{x}})\boldsymbol{\beta}]} \right]$$

where $\mathbf{g}_{it} \equiv \mathbf{g}_t(\mathbf{z}_{it}, \ldots, \mathbf{z}_{i1})$ is a function of the instruments up through time t. Or, stack the moments over time rather than sum them up to enhance efficiency. In either case, we would

use GMM with an optimal weighting matrix to set the sample moments as close to zero as possible.

e. If we can take $\mathbf{z}_{it} = \mathbf{x}_{it}$ then we know from part a that

$$\mathrm{E}[c_i(r_{it} - r_{i,t+1})|\mathbf{x}_{it}, \ldots, \mathbf{x}_{i1}, c_i)].$$

That means any function of $(\mathbf{x}_{it}, \ldots, \mathbf{x}_{i1})$ can multiply the moment conditions and we are still left with a zero conditional mean. In particular,

$$\mathrm{E}\left\{\exp(\mathbf{x}_{it}\boldsymbol{\beta})\left[\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})}\right]\middle|\mathbf{x}_{it}, \ldots, \mathbf{x}_{i1}\right\} = 0,\ t = 1, \ldots, T-1$$

and simple algebra shows

$$\exp(\mathbf{x}_{it}\boldsymbol{\beta})\left[\frac{y_{it}}{\exp(\mathbf{x}_{it}\boldsymbol{\beta})} - \frac{y_{i,t+1}}{\exp(\mathbf{x}_{i,t+1}\boldsymbol{\beta})}\right] = y_{it} - \frac{y_{i,t+1}}{\exp[(\mathbf{x}_{i,t+1} - \mathbf{x}_{it})\boldsymbol{\beta}]}.$$

Solutions to Chapter 19 Problems

19.1. If r_i is the same for any random draw i, then it is nonrandom. From equation (19.9), we can write

$$\begin{aligned} \mathrm{P}(w_i = 1|\mathbf{x}_i) &= \Phi[(\beta_1 + \beta_2 x_{i2} + \ldots + \beta_K x_{iK} - \log(r))/\sigma] \\ &= \Phi[(\beta_1 - \log(r))/\sigma + (\beta_2/\sigma)x_{i2} + \ldots + (\beta_K/\sigma)x_{iK}], \end{aligned}$$

where it is helpful to separate the intercept from the slopes. From this equation, it is clear that probit of w_i on $(1, x_{i2}, \ldots, x_{iK})$ consistently estimates $(\beta_1 - \log(r))/\sigma$, β_2/σ, ..., β_K/σ. Let $\alpha_1^* = (\beta_1 - \log(r))/\sigma$ and define $\beta_j^* = \beta_j/\sigma, j = 1, \ldots, K.$ Unfortunately, we cannot recover the original parameters because, for example, $\beta_1 = \sigma\alpha_1^* + \log(r)$, and we do not know σ. Although α_1^* is identified, and $\log(r)$ is known, we can not recover the scaled intercept $\beta_1^* \equiv \beta_1/\sigma = \alpha_1^* + \log(r)/\sigma$. Of course, we directly estimate the scaled slopes, β_j/σ, and so we can estimate the direction of the effects on $\mathrm{E}(y|\mathbf{x})$. But we cannot estimate the original intercepts or slopes. Assuming $\beta_h \neq 0$, we can estimate β_j/β_h for $j \neq h$, which means we can estimate the relative effects. Unlike in the case where the r_i vary, we cannot estimate the magnitudes of the partial effects on $\mathrm{E}(y|\mathbf{x})$.

19.2. a. It sufficies to find the density of $\log(w_i)$ conditional on $\mathbf{x}_i$; of course we arrive at the same place for the MLEs if we work with $\mathrm{D}(w_i|\mathbf{x}_i)$. Now

$$\log(w_i) = \max[\log(f), \log(y_i)]$$

and $\log(y_i) = \mathbf{x}_i\boldsymbol{\beta} + u_i$, where

$$\mathrm{D}(u_i|\mathbf{x}_i) = Normal(0, \sigma^2).$$

Let $\tilde{w}_i = \log(w_i), \tilde{f} = \log(f)$, and $\tilde{y}_i = \log(y_i)$, so that $\mathrm{D}(\tilde{y}_i|\mathbf{x}_i) = Normal(\mathbf{x}_i\boldsymbol{\beta}, \sigma^2)$. Now

$$\begin{aligned}\mathrm{P}(\tilde{w}_i = \tilde{f}|\mathbf{x}_i) &= \mathrm{P}(\tilde{y}_i \leq \tilde{f}|\mathbf{x}_i) = P\left(\frac{\mathbf{x}_i\boldsymbol{\beta} + u_i}{\sigma} \leq \frac{\tilde{f}}{\sigma}\,\middle|\,\mathbf{x}_i\right) \\ &= P\left(\frac{u_i}{\sigma} \leq \frac{\tilde{f} - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\,\middle|\,\mathbf{x}_i\right) = \Phi\left(\frac{\tilde{f} - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\end{aligned}$$

The conditional density for $\tilde{w} > \tilde{f}$ is simply the conditional density for $\tilde{y}_i$, that is,

$$\left(\frac{1}{\sigma}\right)\phi\left(\frac{\tilde{w} - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right).$$

Therefore, the density for $\tilde{w}_i$ conditional on $\mathbf{x}_i$ can be written as

$$\left[\left(\frac{1}{\sigma}\right)\phi\left(\frac{\tilde{w} - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right]^{1[\tilde{w} > \tilde{f}]}\left[\Phi\left(\frac{\tilde{f} - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right]^{1[\tilde{w} = \tilde{f}]}.$$

It follows that the log likelihood for a random draw i is

$$1[\tilde{w}_i > \tilde{f}]\log\left[\left(\frac{1}{\sigma}\right)\phi\left(\frac{\tilde{w}_i - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right] + 1[\tilde{w}_i > \tilde{f}]\log\left[\Phi\left(\frac{\tilde{f} - \mathbf{x}_i\boldsymbol{\beta}}{\sigma}\right)\right].$$

Notice that when $\tilde{f} = 0$ we get the same log likelihood as for the Type I Tobit model for corner solutions, which we covered in Chapter 17.

b. Because u is independent of $\mathbf{x}$ with a $Normal(0, \sigma^2)$ distribution,

$$\mathrm{E}[\exp(u)|\mathbf{x}] = \mathrm{E}[\exp(u)] = \exp(\sigma^2/2),$$

where the second inequality follows from the moments of a lognormal distribution. Therefore,

$$\begin{aligned}\mathrm{E}(y|\mathbf{x}) &= \exp(\mathbf{x}\boldsymbol{\beta})\mathrm{E}[\exp(u)|\mathbf{x}] = \exp(\mathbf{x}\boldsymbol{\beta})\exp(\sigma^2/2) \\ &= \exp(\mathbf{x}\boldsymbol{\beta} + \sigma^2/2).\end{aligned}$$

After using the MLE on the censored data to obtain $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$, we can use

$$\hat{\mathrm{E}}(y|\mathbf{x}) = \exp(\mathbf{x}\hat{\boldsymbol{\beta}} + \hat{\sigma}^2/2).$$

c. It is hard to see why $\mathrm{E}(w|\mathbf{x})$ would be of much interest. In most cases the floor, f, is

arbitrary, and so it is unclear why we would be interested in how the mean of the censored variable changes with the x_j. One could imagine that, if f is a minimum wage and w_i represents the observed wage for worker i, one might be interested to know how a change in a policy variable affects observed wage, on average.

19.3. a. The two-limit Tobit model from Section 17.7 could be used with limits at 0 and 10.

b. The lower bound of zero reflects the fact that pension contribution cannot be a negative percentage of income. But the upper bound of 10 percent is imposed by law, and is essentially arbitrary. If we defined a variable as the desired percentage put into the pension plan, then it could range from 0 to 100. So the upper bound of 10 can be viewed as a data censoring problem because some individuals presumably would contribute $y > 10$ if the limit were raised. But it depends on the purpose of the study: to estimate the effects within the current institutional setting or to estimate effects on pension contributions in the absense of constraints.

c. From Problem 17.3 part b, with $a_1 = 0$, we have

$$\begin{aligned} \mathrm{E}(y|\mathbf{x}) = {} & (\mathbf{x}\boldsymbol{\beta}) \cdot \{\Phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] - \Phi(-\mathbf{x}\boldsymbol{\beta}/\sigma)\} \\ & + \sigma\{\phi(\mathbf{x}\boldsymbol{\beta}/\sigma) - \phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma]\} + a_2\Phi[(\mathbf{x}\boldsymbol{\beta} - a_2)/\sigma]. \end{aligned}$$

Taking the derivative of this function with respect to a_2 gives

$$\begin{aligned} \partial\mathrm{E}(y|\mathbf{x})/\partial a_2 = {} & (\mathbf{x}\boldsymbol{\beta}/\sigma) \cdot \phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] + [(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] \cdot \phi[(a_2 - \mathbf{x}\boldsymbol{\beta})/\sigma] \\ & + \Phi[(\mathbf{x}\boldsymbol{\beta} - a_2)/\sigma] - (a_2/\sigma)\phi[(\mathbf{x}\boldsymbol{\beta} - a_2)/\sigma] \\ = {} & \Phi[(\mathbf{x}\boldsymbol{\beta} - a_2)/\sigma]. \end{aligned}$$

We can plug in $a_2 = 10$ to obtain the approximate effect of increasing the cap from 10 to 11. For a given value of $\mathbf{x}$, we would compute $\Phi[(\mathbf{x}\hat{\boldsymbol{\beta}} - 10)/\hat{\sigma}]$, where $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}$ are the MLEs. We might evaluate this expression at the sample average of $\mathbf{x}$ or at other interesting values (such as

across gender or race).

d. If $y_i < 10$ for $i = 1, \ldots, N$, $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}$ are just the usual type I Tobit estimates with lower limit at zero: there are no observations that contribute to the third piece in the log likelihood.

19.4. a. If you are interested in the effects of things like age of the building and neighborhood demographics on fire damage, given that a fire has occurred, then there is no problem. We simply need a random sample of buildings that actually caught on fire. You might want to supplement this with an analysis of the probability that buildings catch fire, given building and neighborhood characteristics. But then a two-part (hurdle) model is appropriate.

b. The issue in this case is a bit subtle because it depends on the population of interest. One possibility is, at a given point in time, to define the population of interest to be workers currently enrolled in a 401(k) plan. Then using a random sample of workers already in a 401(k) plan is appropriate. But workers currently enrolled in a plan may not represent those that may be enrolled in the future. In fact, we might think of being interested in a scenario where *all* workers are enrolled. It makes sense to think about the sensitivity of contributions to the match rate for the population of all workers. Of course, in general, using a random sample of those already enrolled leads to a sample selection problem for estimating the parameters for the larger population – much like the problem of estimating a wage offer equation (except that, in addition to not observing contributions, we would not observe a match rate for those not enrolled).

19.5. Because *IQ* and *KWW* are both indicators of *abil* we can write

$$IQ = \xi_1 abil + a_i, \; KWW = \xi_1 abil + a_i,$$

where $\xi_1, \; \xi_2 > 0$. For simplicity, I set the intercepts to zero, as this does not affect the

conclusions of the problem. The structural equation is $\log(wage) = \mathbf{z}_1\delta_1 + abil + v$. Now, given the selection mechanism described in Example 19.4 (IQ is observed if $IQ + r \geq 0$), we can assume that

$$\mathrm{E}(v|\mathbf{z}_1, abil, IQ, KWW, r) = 0, \tag{19.124}$$

which is the standard ignorability assumption with the added assumption that v is unrelated to r in the conditional mean sense. To see what else we need, write *abil* in terms of IQ and a_1 and plug into the structural equation to get

$$\log(wage) = \mathbf{z}_1\delta_1 + \xi_1^{-1}IQ + v + \xi_1^{-1}a_1.$$

Now, we want to use KWW as an instrument for IQ in this equation, and use 2SLS on the selected sample. The full set of instruments is $(\mathbf{z}_1, KWW)$. From Theorem 19.1 we need the error $u = v + \xi_1^{-1}a_1$ to satisfy $\mathrm{E}(u|\mathbf{z}_1, KWW, s) = 0$. Now, because s is a function of IQ and r, from (19.124) we have $\mathrm{E}(v|\mathbf{z}_1, KWW, s) = 0$. To ensure $\mathrm{E}(a_1|\mathbf{z}_1, KWW, s) = 0$ we can assume $\mathrm{E}(a_1|\mathbf{z}_1, KWW, r) = 0$ or, equivalently, $\mathrm{E}(a_1|\mathbf{z}_1, a_2, r) = 0$. The symmetrical assumption on a_2 is $\mathrm{E}(a_2|\mathbf{z}_1, a_1, r) = 0$. Loosely, in addition to the errors in the indicator equations being uncorrelated, they are also uncorrelated with the selection error. But for all of this to work we need to make zero conditional mean assumptions.

19.6. This is essentially given in equation (19.45), but were we allow the truncation points to depend on $\mathbf{x}_i$. Let y_i given $\mathbf{x}_i$ have density $f(y|\mathbf{x}_1, \boldsymbol{\beta}, \boldsymbol{\gamma})$, where $\boldsymbol{\beta}$ is the vector indexing $\mathrm{E}(y_i|\mathbf{x}_i)$ and $\boldsymbol{\gamma}$ is another set of parameters (often a single variance parameter). Then the density of y_i given $\mathbf{x}_i, s_i = 1$, when $s_i = 1[a_1(\mathbf{x}_i) < y_i < a_2(\mathbf{x}_i)]$, is

$$p(y|\mathbf{x}_i, s_i = 1) = \frac{f(y|\mathbf{x}_i; \boldsymbol{\beta}, \boldsymbol{\gamma})}{F(a_2(\mathbf{x}_i)|\mathbf{x}_i; \boldsymbol{\beta}, \boldsymbol{\gamma}) - F(a_1(\mathbf{x}_i)|\mathbf{x}_i; \boldsymbol{\beta}, \boldsymbol{\gamma})}, a_1(\mathbf{x}_i) < y < a_2(\mathbf{x}_i).$$

In the Hausman and Wise (1977) study, $y_i = \log(income_i)$, $a_1(\mathbf{x}_i) = -\infty$, and $a_2(\mathbf{x}_i)$ was a function of family size (which determines the official poverty level).

19.7. a. If $\mathrm{E}(u_1|v_2) = \gamma_1 v_2 + \gamma_2(v_2^2 - 1)$ then, because (u_1, v_2) is independent of $\mathbf{x}$,

$$\mathrm{E}(y_1|\mathbf{x}, v_2) = \mathbf{x}_1\boldsymbol{\beta}_1 + \mathrm{E}(u_1|v_2) = \mathbf{x}_1\boldsymbol{\beta}_1 + \gamma_1 v_2 + \gamma_2(v_2^2 - 1).$$

Now, using iterated expectations (since y_2 is a function of $(\mathbf{x}, v_2)$), we have

$$\begin{aligned}\mathrm{E}(y_1|\mathbf{x}, y_2) &= \mathbf{x}_1\boldsymbol{\beta}_1 + \gamma_1\mathrm{E}(v_2|\mathbf{x}, y_2) + \gamma_2\{\mathrm{E}(v_2^2|\mathbf{x}, y_2) - 1\} \\ &= \mathbf{x}_1\boldsymbol{\beta}_1 + \gamma_1\mathrm{E}(v_2|\mathbf{x}, y_2) + \gamma_2\{\mathrm{Var}(v_2|\mathbf{x}, y_2) + [\mathrm{E}(v_2|\mathbf{x}, y_2)]^2 - 1\}.\end{aligned}$$

We only need these expressions for $y_2 = 1$. Using $\mathrm{E}(v_2|v_2 > -\mathbf{x}\delta_2) = \lambda(\mathbf{x}\delta_2)$ and

$$\mathrm{Var}(v_2|v_2 > -\mathbf{x}\delta_2) = 1 - \lambda(\mathbf{x}\boldsymbol{\delta}_2)[\lambda(\mathbf{x}\boldsymbol{\delta}_2) + \mathbf{x}\boldsymbol{\delta}_2],$$

we have

$$\begin{aligned}\mathrm{E}(y_1|\mathbf{x}, y_2 = 1) &= \mathbf{x}_1\boldsymbol{\beta}_1 + \gamma_1\mathrm{E}(v_2|v_2 > -\mathbf{x}\boldsymbol{\delta}_2) + \gamma_2\mathrm{Var}(v_2|v_2 > -\mathbf{x}\boldsymbol{\delta}_2) \\ &= \mathbf{x}_1\boldsymbol{\beta}_1 + \gamma_1\lambda(\mathbf{x}\boldsymbol{\delta}_2) + \gamma_2\{1 - \lambda(\mathbf{x}\boldsymbol{\delta}_2)[\lambda(\mathbf{x}\boldsymbol{\delta}_2) + \mathbf{x}\boldsymbol{\delta}_2] + [\lambda(\mathbf{x}\boldsymbol{\delta}_2)]^2 - 1\} \\ &= \mathbf{x}_1\beta_1 + \gamma_1\lambda(\mathbf{x}\boldsymbol{\delta}_2) - \gamma_2\lambda(\mathbf{x}\delta_2)\mathbf{x}\delta_2.\end{aligned}$$

b. Now, we obtain $\mathbf{x}_i\hat{\boldsymbol{\delta}}_2$ and $\hat{\lambda}_{i2}$ after first-stage probit and then run the regression

$$y_{i1} \text{ on } \mathbf{x}_{i1},\ \hat{\lambda}_{i2},\ \hat{\lambda}_{i2} \cdot (\mathbf{x}_i\hat{\boldsymbol{\delta}}_2)$$

using the selected sample. We get consistent estimators of $\boldsymbol{\beta}_1, \gamma_1,$ and $-\gamma_2$.

c. A standard F test of joint significance of $\hat{\lambda}_{i2}$ and $\hat{\lambda}_{i2} \cdot (\mathbf{x}_i\hat{\boldsymbol{\delta}}_2)$ (two restrictions) in the regression from part b is a valid test, assuming homoskedasticity in the population structural model. As usual, the null is no sample selection bias.

19.8. If we replace y_2 with $\hat{y}_2$, we need to see what happens when $y_2 = \mathbf{z}\boldsymbol{\delta}_2 + v_2$ is plugged into the structural mode:

$$y_1 = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 \cdot (\mathbf{z}\boldsymbol{\delta}_2 + v_2) + u_1 \qquad (19.125)$$
$$= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 \cdot (\mathbf{z}\boldsymbol{\delta}_2) + (u_1 + \alpha_1 v_2).$$

So, the procedure is to replace $\boldsymbol{\delta}_2$ in (19.125) its $\sqrt{N}$-consistent estimator, $\hat{\boldsymbol{\delta}}_2$. The key is to note that the error term in (19.125) is $u_1 + \alpha_1 v_2$. If the selection correction is going to work whn the fitted value is plugged in for y_2, we need the expected value of $u_1 + \alpha_1 v_2$ given $(\mathbf{z}, v_3)$ to be linear in v_3 (in particular, it cannot depend on $\mathbf{z}$). Then we can write

$$\mathrm{E}(y_1|\mathbf{z}, v_3) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 \cdot (\mathbf{z}\boldsymbol{\delta}_2) + \gamma_1 v_3,$$

where $\mathrm{E}[(u_1 + \alpha_1 v_2)|v_3] = \gamma_1 v_3$ by normality. Conditioning on $y_3 = 1$ gives

$$\mathrm{E}(y_1|\mathbf{z}, y_3 = 1) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 \cdot (\mathbf{z}\boldsymbol{\delta}_2) + \gamma_1 \lambda(\mathbf{z}\boldsymbol{\delta}_3). \qquad (19.126)$$

A sufficient condition for (19.126) is that (u_1, v_2, v_3) is independent of $\mathbf{z}$ with a trivariate normal distribution. We can get by with less than this, but the nature of v_2 is restricted. If we use the IV approach – rather than plugging in fitted values – we need assume nothing about v_2; $y_2 = \mathbf{z}\boldsymbol{\delta}_2 + v_2$ is just a linear projection.

As a practical matter, if we cannot write $y_2 = \mathbf{z}\boldsymbol{\delta}_2 + v_2$, where v_2 is independent of $\mathbf{z}$ and approximately normal, then the OLS alternative will not be consistent. Thus, equations where y_2 is binary, or is some other variable that exhibits nonnormality, cannot be consistently estimated using the OLS procedure. This is why 2SLS is generally preferred.

19.9. Here is the Stata session I used to implement Procedure 19.4, although the standard errors in the second step are not adjusted to account for the first-stage Tobit estimation. Still, $\hat{v}_3$ is not statistically significant, and adding it is not really necessary in this application.

```
. tobit hours exper expersq age kidslt6 kidsge6 nwifeinc motheduc fatheduc
        huseduc, ll(0)

Tobit regression                                  Number of obs   =        753
                                                  LR chi2(9)      =     261.82
                                                  Prob > chi2     =     0.0000
```

```
Log likelihood = -3823.9826                     Pseudo R2       =     0.0331

------------------------------------------------------------------------------
       hours |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       exper |   136.9463   17.27271     7.93   0.000     103.0373    170.8554
     expersq |  -1.947776   .5388933    -3.61   0.000    -3.005708   -.8898433
         age |  -54.78118   7.568762    -7.24   0.000    -69.63985    -39.9225
     kidslt6 |  -864.3263   111.6246    -7.74   0.000    -1083.463   -645.1896
     kidsge6 |  -24.68934   38.77122    -0.64   0.524    -100.8034    51.42468
    nwifeinc |  -5.312478   4.572888    -1.16   0.246    -14.28978    3.664822
    motheduc |   24.28791   16.74349     1.45   0.147    -8.582209    57.15803
    fatheduc |   6.566355   16.00859     0.41   0.682    -24.86103    37.99374
     huseduc |   3.129767   17.46452     0.18   0.858    -31.15583    37.41537
       _cons |   1548.141   437.1192     3.54   0.000     690.0075    2406.275
-------------+----------------------------------------------------------------
      /sigma |   1126.282   41.77533                      1044.271    1208.294
------------------------------------------------------------------------------
  Obs. summary:        325  left-censored observations at hours<=0
                       428     uncensored observations
                         0 right-censored observations

. predict zd3hat
(option xb assumed; fitted values)

. sum zd3hat

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      zd3hat |       753    302.7538    814.8662  -2486.756    1933.23

. gen v3hat = hours - zd3hat if hours > 0
(325 missing values generated)

. ivreg lwage exper expersq v3hat (educ = age kidslt6 kidsge6 nwifeinc
       motheduc fatheduc huseduc)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     428
-------------+------------------------------           F(  4,   423) =    9.97
       Model |  34.6676357     4  8.66690893           Prob > F      =  0.0000
    Residual |  188.659805   423  .446004268           R-squared     =  0.1552
-------------+------------------------------           Adj R-squared =  0.1472
       Total |  223.327441   427  .523015084           Root MSE      =  .66784

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |    .085618   .0213955     4.00   0.000     .0435633    .1276726
       exper |   .0378509   .0137757     2.75   0.006     .0107734    .0649283
     expersq |  -.0007453   .0004036    -1.85   0.065    -.0015386    .0000479
       v3hat |  -.0000515   .0000412    -1.25   0.211    -.0001325    .0000294
       _cons |  -.1786154   .2925231    -0.61   0.542    -.7535954    .3963645
------------------------------------------------------------------------------
Instrumented:  educ
Instruments:   exper expersq v3hat age kidslt6 kidsge6 nwifeinc motheduc
               fatheduc huseduc
------------------------------------------------------------------------------
```

If we just use 2SLS on the selected sample without including $\hat{v}_3$, and the IVs for *educ* are *motheduc*, *fatheduc*, and *huseduc*, then the estimated return to education is about 8.0%:

```
. ivreg lwage exper expersq (educ = motheduc fatheduc huseduc)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     428
-------------+------------------------------           F(  3,   424) =   11.52
       Model |  33.3927368     3  11.1309123           Prob > F      =  0.0000
    Residual |  189.934704   424  .447959208           R-squared     =  0.1495
-------------+------------------------------           Adj R-squared =  0.1435
       Total |  223.327441   427  .523015084           Root MSE      =   .6693

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
        educ |   .0803918    .021774     3.69   0.000     .0375934    .1231901
       exper |   .0430973   .0132649     3.25   0.001     .0170242    .0691704
     expersq |  -.0008628   .0003962    -2.18   0.030    -.0016415   -.0000841
       _cons |  -.1868572   .2853959    -0.65   0.513    -.7478242    .3741097
------------------------------------------------------------------------------
Instrumented:  educ
Instruments:   exper expersq motheduc fatheduc huseduc
------------------------------------------------------------------------------
```

19.10. a. Substitute the reduced forms for y_1 and y_2 into the third equation:

$$\begin{aligned} y_3 &= \max(0, \alpha_{31}(\mathbf{z}\boldsymbol{\delta}_1) + \alpha_{32}(\mathbf{z}\boldsymbol{\delta}_2) + \mathbf{z}_3\boldsymbol{\delta}_3 + v_3) \\ &\equiv \max(0, \mathbf{z}\boldsymbol{\pi}_3 + v_3), \end{aligned}$$

where $v_3 \equiv u_3 + \alpha_{31}v_1 + \alpha_{32}v_2$. Under the assumptions given, v_3 is independent of $\mathbf{z}$ and normally distributed. Thus, if we knew $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$, we could consistently estimate α_{31}, α_{32}, and $\boldsymbol{\delta}_3$ from a Tobit of y_3 on $\mathbf{z}\boldsymbol{\delta}_1, \mathbf{z}\boldsymbol{\delta}_2$, and $\mathbf{z}_3$. From the usual argument, consistent estimators are obtained by using initial consistent estimators of $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$. Estimation of $\boldsymbol{\delta}_2$ is simple: just use OLS using the entire sample. Estimation of $\boldsymbol{\delta}_1$ follows exactly as in Procedure 19.3 using the system

$$y_1 = \mathbf{z}\boldsymbol{\delta}_1 + v_1$$

$$y_3 = \max(0, \mathbf{z}\boldsymbol{\pi}_3 + v_3),$$

where y_1 is observed only when $y_3 > 0$.

Given $\hat{\boldsymbol{\delta}}_1$ and $\hat{\boldsymbol{\delta}}_2$, form $\mathbf{z}_i\hat{\boldsymbol{\delta}}_1$ and $\mathbf{z}_i\hat{\boldsymbol{\delta}}_2$ for each observation i in the sample. Then, obtain $\hat{\alpha}_{31}, \hat{\alpha}_{32},$ and $\hat{\boldsymbol{\delta}}_3$ from the Tobit

$$y_{i3} \text{ on } (\mathbf{z}_i\hat{\boldsymbol{\delta}}_1),\ (\mathbf{z}_i\hat{\boldsymbol{\delta}}_2),\ \mathbf{z}_{i3}$$

using all observations.

For identification, $(\mathbf{z}\boldsymbol{\delta}_1, \mathbf{z}\boldsymbol{\delta}_2, \mathbf{z}_3)$ can contain no exact linear dependencies. Necessary is that there must be at least two elements in $\mathbf{z}$ not also in $\mathbf{z}_3$.

Obtaining the correct asymptotic variance matrix is complicated. It is most easily done in a generalized method of moments framework. Alternatively, it is easy to use bootstrap resampling on both steps of the estimation procedure.

b. This is not very different from part a. The only difference is that $\boldsymbol{\delta}_2$ must be estimated using Procedure 19.3. Then follow the steps from part a.

c. We need to estimate the variance of u_3, σ_3^2, and then use the standard formula for the mean of a Tobit model. This gives the ASF as a function of $(y_2, y_3, \mathbf{z}_3)$ and the parameters $(\alpha_{31}, \alpha_{32}, \boldsymbol{\delta}_3, \sigma_3^2)$.

19.11. a. This follows from the usual iterated expectations argument, because $\mathbf{Z}_i$ is a function of $\mathbf{x}_i$:

$$\begin{aligned} \mathrm{E}[s_i\mathbf{Z}_i'\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)] &= \mathrm{E}\{\mathrm{E}[s_i\mathbf{Z}_i'\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)|\mathbf{x}_i, s_i]\} \\ &= \mathrm{E}\{s_i\mathbf{Z}_i'[\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)|\mathbf{x}_i, s_i]\} = \mathbf{0} \end{aligned}$$

because $\mathrm{E}[\mathbf{r}(\mathbf{w}_i, \boldsymbol{\theta}_o)|\mathbf{x}_i, s_i] = \mathbf{0}$.

b. We modify equation (14.24) from Chapter 14 to allow for selection:

$$\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left(\sum_{i=1}^{N} s_i\mathbf{Z}_i'\mathbf{r}_i(\boldsymbol{\theta})\right)'\left(N^{-1}\sum_{i=1}^{N} s_i\mathbf{Z}_i'\mathbf{Z}_i\right)^{-1}\left(\sum_{i=1}^{N} s_i\mathbf{Z}_i'\mathbf{r}_i(\boldsymbol{\theta})\right).$$

For consistency, we would have to assume that rank $\mathrm{E}(s_i\mathbf{Z}_i'\mathbf{Z}_i) = L$ – which means, that in the selected sample, the instrument matrix is not perfectly collinear – and we have to assume that $\boldsymbol{\theta}_o$ is the unique solution to $\mathrm{E}[s_i\mathbf{Z}_i'\mathbf{r}(\mathbf{w}_i,\boldsymbol{\theta}_o)] = \mathbf{0}$. For $\sqrt{N}$-asymptotic normality, we would also have to assume that rank $\mathrm{E}[s_i\mathbf{Z}_i'\nabla_{\boldsymbol{\theta}}\mathbf{r}(\mathbf{w}_i,\boldsymbol{\theta}_o)] = P$, the dimension of $\boldsymbol{\theta}$. None of the conditions can be true unless $\mathrm{P}(s_i = 1) > 0$, that is, we observe a randomly drawn observation with positive probability. But $\mathrm{P}(s_i = 1) > 0$ is not nearly sufficient, as we might not have identification in the selected population even if we have identification in the full population. (For example, we might have an instrument that varies sufficiently in the full population but not in the $s = 1$ subpopulation.)

c. Let $\breve{\boldsymbol{\theta}}$ denote the (system) nonlinear 2SLS estimator on the selected sample. For the minimum chi-square estimator, we would compute

$$\hat{\boldsymbol{\Lambda}} = N^{-1}\sum_{i=1}^{N} s_i\mathbf{Z}_i'\mathbf{r}_i(\breve{\boldsymbol{\theta}})\mathbf{r}_i(\breve{\boldsymbol{\theta}})'\mathbf{Z}_i$$

and then solve

$$\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left(\sum_{i=1}^{N} s_i\mathbf{Z}_i'\mathbf{r}_i(\boldsymbol{\theta})\right)'\hat{\boldsymbol{\Lambda}}^{-1}\left(\sum_{i=1}^{N} s_i\mathbf{Z}_i'\mathbf{r}_i(\boldsymbol{\theta})\right).$$

19.12. a. Take the expected value of (19.56) conditional on $(\mathbf{z}, y_3)$:

$$\begin{aligned}\mathrm{E}(y_1|\mathbf{z},y_3) &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1\mathrm{E}(y_2|\mathbf{z},y_3) + \mathrm{E}(u_1|\mathbf{z},y_3)\\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1\mathrm{E}(y_2|\mathbf{z},y_3)\end{aligned}$$

because $\mathrm{E}(u_1|\mathbf{z},y_3) = 0$ follows from $\mathrm{E}(u_1|\mathbf{z},v_3) = 0$.

b. Now take the expected value of (19.56) conditional on $(\mathbf{z}, v_3)$:

$$\begin{aligned} \mathrm{E}(y_1|\mathbf{z}, v_3) &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1\mathrm{E}(y_2|\mathbf{z}, v_3) + \mathrm{E}(u_1|\mathbf{z}, v_3) \\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1[\mathbf{z}\boldsymbol{\delta}_2 + \mathrm{E}(v_2|\mathbf{z}, v_3)] \\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1(\mathbf{z}\boldsymbol{\delta}_2 + \gamma_2 v_3). \end{aligned}$$

Therefore,

$$\mathrm{E}(y_1|\mathbf{z}, y_3) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1[\mathbf{z}\boldsymbol{\delta}_2 + \gamma_2\mathrm{E}(v_3|\mathbf{z}, y_3)],$$

and when $y_3 = 1$ we get the usual inverse Mill's ratio: $\mathrm{E}(v_3|\mathbf{z}, y_3 = 1) = \lambda(\mathbf{z}\boldsymbol{\delta}_3)$. So

$$\mathrm{E}(y_1|\mathbf{z}, y_3 = 1) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1[\mathbf{z}\boldsymbol{\delta}_2 + \gamma_2\lambda(\mathbf{z}\boldsymbol{\delta}_3)].$$

c. We can view it as a three-step estimation method. The first step is to obtain $\hat{\boldsymbol{\delta}}_3$ from probit of y_{i3} on $\mathbf{z}_i$, using all of the observations. Then, we can estimate $\boldsymbol{\delta}_2$ and γ_2 from standard Heckit applied to y_{i2} using the selection sample. (My initial thought was that the two steps in the Heckit method are treated as one, as it could be carried out by partial MLE.) Given $\hat{\boldsymbol{\delta}}_3$, $\hat{\boldsymbol{\delta}}_2$, and $\hat{\gamma}_2$, the final stage is the OLS regression

$$y_{i1} \text{ on } \mathbf{z}_{i1}, \mathbf{z}_i\hat{\boldsymbol{\delta}}_2 + \hat{\gamma}_2\lambda(\mathbf{z}_i\hat{\boldsymbol{\delta}}_3)$$

using the $s_{i1} = 1$ sample. Note that the final regressor, $\lambda(\mathbf{z}_i\hat{\boldsymbol{\delta}}_3)$, is simply our estimate of $\mathrm{E}(y_2|\mathbf{z}, y_3 = 1)$. Intuitively, if there is one relevant element in $\mathbf{z}_i$ not in $\mathbf{z}_{i1}$, then $\mathrm{E}(y_{i2}|\mathbf{z}_i, y_{i3} = 1)$ has sufficient variation apart from $\mathbf{z}_{i1}$ to identify $\boldsymbol{\delta}_1$ and α_1. However, I did overlook one issue when I wrote this problem: we cannot get a very good estimate of $\boldsymbol{\delta}_2$, or γ_2 for that matter, in the preliminary Heckit unless we can set an element of $\boldsymbol{\delta}_2$ equal to zero. In other words, we would really need an exclusion restriction in the reduced form of y_2 in order to get a good Heckit estimate of $\boldsymbol{\delta}_2$. Thus, this procedure seems no better – and perhaps even worse – than Procedure 19.2, even when we assume $\mathrm{E}(u_1|\mathbf{z}, v_3) = 0$.

If y_2 is always observed, then we can estimate $\boldsymbol{\delta}_2$ by a first-stage OLS regression, and we could then estimate γ_2 precisely, also, without resorting to an exclusion restriction in the reduced form of y_2.

d. Unlike Procedure 19.2, the method in part c does not work if $\mathrm{E}(u_1|\mathbf{z},y_3) \neq 0$. Therefore, there is little to recommend it.

e. If $\mathrm{E}(u_1|\mathbf{z},y_2,y_3) = 0$, we would just use OLS on the selected sample: y_{i1} on $\mathbf{z}_i$, y_{i2}.

19.13. a. There is no sample selection problem because, by definition, you have specified the distribution of y given $\mathbf{x}$ and $y > 0$. We only need to obtain a random sample from the subpopulation with $y > 0$.

b. Again, there is no sample selection bias because we have specified the conditional expectation for the population of interest. If we have a random sample from that population, NLS is generally consistent and $\sqrt{N}$-asymptotically normal.

c. We would use a standard probit model. Let $w = 1[y > 0]$. Then w given $\mathbf{x}$ follows a probit model with $\mathrm{P}(w = 1|\mathbf{x}) = \Phi(\mathbf{x}\boldsymbol{\gamma})$.

d. $\mathrm{E}(y|\mathbf{x}) = \mathrm{P}(y > 0|\mathbf{x}) \cdot \mathrm{E}(y|\mathbf{x},y > 0) = \Phi(\mathbf{x}\boldsymbol{\gamma}) \cdot \exp(\mathbf{x}\boldsymbol{\beta})$. So we would plug in the NLS estimator of $\boldsymbol{\beta}$ and the probit estimator of $\boldsymbol{\gamma}$.

e. By definition, there is no sample selection problem when you specify the conditional distribution – conditional means – for the second part. As discussed in Section 17.6.3, confusion can arise when two part models are specified with unobservables that may be correlated, as in equation (17.50):

$$y = s \cdot \exp(\mathbf{x}\boldsymbol{\beta} + u),$$
$$s = 1[\mathbf{x}\boldsymbol{\gamma} + v > 0],$$

so that $s = 0 \Leftrightarrow y = 0$. As shown in Section 17.6.3, if u and v are correlated then estimation of

$\boldsymbol{\beta}$ does use methods that are closely related to the Heckman sample selection correction. But $\boldsymbol{\beta}$ does not tell us what we need to know because both $\mathrm{E}(y|\mathbf{x})$ and $\mathrm{E}(y|\mathbf{x}, y > 0)$ are much more complicated than in the truncated normal or lognormal hurdle cases. See Section 17.6.3 for further discussion.

19.14. a. Write $u = \alpha_0(1 - s) + \alpha_1 s + e$ where, by assumption, $\mathrm{E}(e|\mathbf{z}, s) = 0$. Plugging this expression for u into (19.30) gives

$$\begin{aligned} y &= \beta_1 + \beta_2 x_2 + \ldots + \beta_K x_K + \alpha_0(1 - s) + \alpha_1 s + e \\ \mathrm{E}(e|\mathbf{z}, s) &= 0. \end{aligned}$$

Using the selected sample and applying IV corresponds to multiplying the equation through by s, and then applying 2SLS. We have

$$\begin{aligned} s \cdot y &= \beta_1 s + \beta_2(s \cdot x_2) + \ldots + \beta_K(s \cdot x_K) + \alpha_1 s + s \cdot e \\ &= (\alpha_1 + \beta_1)s + \beta_2(s \cdot x_2) + \ldots + \beta_K(s \cdot x_{\mathrm{K}}) + s \cdot e, \end{aligned}$$

where we use $s(1 - s) = 0$ and $s^2 = 1$. Because $\mathrm{E}(s \cdot e|\mathbf{z}, s) = 0$, it follows that, under the rank conditions in Theorem 19.1, 2SLS applied to the selected sample consistently estimates $(\alpha_1 + \beta_1), \beta_2, \ldots, \beta_K$.

b. This is not so much a "show" question as it is just recognizing a basic property of conditional expectations: if (u, s) is independent of $\mathbf{z}$, then $\mathrm{E}(u|\mathbf{z}, s) = \mathrm{E}(u|s)$. Because we are willing to assume something like independence between u and $\mathbf{z}$ (or, at least, a zero conditional mean), the important assumption would be independence between s and $\mathbf{z}$. But if the mean of the unobservable, u, changes with s, why would we assume that the mean of the exogenous observables, $\mathrm{E}(\mathbf{z}|s)$, does not? Even $\mathrm{E}(\mathbf{z}|s) = \mathrm{E}(\mathbf{z})$ is a strong assumption, let alone full indepdence between $\mathbf{z}$ and s.

19.15. a. We cannot use censored Tobit because that requires observing $\mathbf{x}$ when whatever

the value of y. Instead, we can use truncated Tobit: we use the distribution of y given $\mathbf{x}$ and $y > 0$. If we observed $\mathbf{x}$ always then using the truncated Normal regression model would be inefficient, but censored Tobit for $D(y|\mathbf{x})$ implies truncated Tobit for $D(y|\mathbf{x}, y > 0)$.

b. Because we have assumed y given $\mathbf{x}$ follows a standard Tobit, $E(y|\mathbf{x})$ is the parametric function

$$E(y|\mathbf{x}) = \Phi(\mathbf{x}\boldsymbol{\beta}/\sigma)\mathbf{x}\boldsymbol{\beta} + \sigma\phi(\mathbf{x}\boldsymbol{\beta}/\sigma).$$

Therefore, even though we never observe some elements of $\mathbf{x}$ when $y = 0$, we can still estimate $E(y|\mathbf{x})$ because we can estimate $\boldsymbol{\beta}$ and σ and we have an expression for $E(y|\mathbf{x})$ that (we assume) holds for all $\mathbf{x}$. To estimate $\boldsymbol{\beta}$ and σ^2 We do have to assume that $\mathbf{x}$ varies enough in the subpopulation where $y > 0$, namely, rank $E(\mathbf{x}'\mathbf{x}|y > 0) = K$. In the case where an element of $\mathbf{x}$ is a derived price, we need sufficient price variation for the population that consumes some of the good.

19.16. a. To obtain the expected value of

$$y_1 = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + u_1$$

conditional on $(\mathbf{z}, r_2, v_2)$, use the fact that y_2 is a function of $(\mathbf{z}, v_2)$, and use independence of (u_1, v_2) and $\mathbf{z}$:

$$\begin{aligned} E(y_1|\mathbf{z}, r_2, v_2) &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + E(u_1|\mathbf{z}, r_2, v_2) \\ &= \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + E(u_1|v_2). \end{aligned}$$

Now use the linearity assumption $E(u_1|v_2) = \rho_1 v_2$ to get

$$E(y_1|\mathbf{z}, r_2, v_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 v_2.$$

b. With $s_2 = 1[y_2 < w_2]$, s_2 is clearly a function of $(\mathbf{z}, r_2, v_2)$, and so s_2 is redundant in $E(y_1|\mathbf{z}, r_2, v_2, s_2)$:

$$\mathrm{E}(y_1|\mathbf{z}, r_2, v_2, s_2) = \mathrm{E}(y_1|\mathbf{z}, r_2, v_2) = \mathbf{z}_1\boldsymbol{\delta}_1 + \alpha_1 y_2 + \rho_1 v_2.$$

c. Because of part b, if we could observe v_{i2} whenever $s_{i2} = 1$ we could consistently estimate $\boldsymbol{\delta}_1$, α_1, and ρ_1 by running the regression

$$y_{i1} \text{ on } \mathbf{z}_{i1},\ y_{i2},\ v_{i2} \text{ if } s_{i2} = 1.$$

Naturally, we can replace $v_{i2} = y_{i2} - \mathbf{z}_i\boldsymbol{\delta}_2$ with $\hat{v}_{i2} \equiv y_{i2} - \mathbf{z}_i\hat{\boldsymbol{\delta}}_2$ for a consistent estimator $\hat{\boldsymbol{\delta}}_2$ of $\boldsymbol{\delta}_2$. That estimator should be from a censored normal regression using

$$w_{i2} = \min(r_{i2}, \mathbf{z}_i\boldsymbol{\delta}_2 + v_{i2})$$

and then defining

$$\hat{v}_{i2} \equiv y_{i2} - \mathbf{z}_i\hat{\boldsymbol{\delta}}_2 \text{ if } y_{i2} < r_{i2}.$$

Then run the regression

$$y_{i1} \text{ on } \mathbf{z}_{i1},\ y_{i2},\ \hat{v}_{i2} \text{ if } s_{i2} = 1.$$

We can use the delta method to obtain valid standard errors, or bootstrap both steps of the procedure. A simple test of

19.17. a. The assumption is that, conditional on $(\mathbf{x}_i, c_i)$, u_{it} is independent of the entire history of censoring values, $(r_{i1}, r_{i2}, \ldots, r_{iT})$. This a kind of strict exogeneity assumption on the censoring, which rules out the censoring values being related to current or past shocks to y. It does allow censoring to be arbitrarily correlated with heterogeneity c_i.

b. Subsititue for y_{it} to get

$$w_{it} = 1[\mathbf{x}_{it}\boldsymbol{\beta} + c_i + u_{it} > r_{it}]$$

and then substitue for c_i to get

$$
\begin{aligned}
w_{it} &= 1[\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i + a_i + u_{it} > r_{it}] \\
&= 1[(a_i + u_{it}) > r_{it} - (\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i)] \\
&= 1\left[\frac{(a_i + u_{it})}{(\sigma_a^2 + \sigma_u^2)^{1/2}} > \frac{r_{it} - (\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i)}{(\sigma_a^2 + \sigma_u^2)^{1/2}} \right].
\end{aligned}
$$

Now use the fact that $\mathrm{D}(a_i + u_{it}|\mathbf{x}_i, \mathbf{r}_i) \sim \text{Normal}(0, \sigma_a^2 + \sigma_u^2)$:

$$
\begin{aligned}
\mathrm{P}(w_{it} = 1|\mathbf{x}_i, \mathbf{r}_i) &= 1 - \Phi\left[\frac{r_{it} - (\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i)}{(\sigma_a^2 + \sigma_u^2)^{1/2}} \right] \\
&= \Phi\left[\frac{\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i - r_{it}}{(\sigma_a^2 + \sigma_u^2)^{1/2}} \right] \\
&\equiv \Phi(\mathbf{x}_{it}\boldsymbol{\beta}_{au} + \psi_{au} + \bar{\mathbf{x}}_i\boldsymbol{\xi}_{au} + \eta_{au}\bar{r}_i + \gamma_{au}r_{it})
\end{aligned}
$$

where $\boldsymbol{\beta}_{au} = \boldsymbol{\beta}/(\sigma_a^2 + \sigma_u^2)^{1/2}$, $\psi_{au} = \psi/(\sigma_a^2 + \sigma_u^2)^{1/2}$, $\boldsymbol{\xi}_{au} = \boldsymbol{\xi}/(\sigma_a^2 + \sigma_u^2)^{1/2}$, and $\gamma_{au} = -1/(\sigma_a^2 + \sigma_u^2)$.

c. From part b, we can estimate all of the scaled coefficients, including γ_{au}, by pooled probit, provided $\{\mathbf{x}_{it}\}$ and $\{r_{it}\}$ have time variation for at last some units. But

$$\boldsymbol{\beta} = -\boldsymbol{\beta}_{au}/\gamma_{au}$$

and so we just use

$$\hat{\boldsymbol{\beta}} = -\hat{\boldsymbol{\beta}}_{au}/\hat{\gamma}_{au}.$$

d. The pooled estimation from part c only allows us to estimate $\sigma_a^2 + \sigma_u^2$ and the unscaled parameters. If we add the assumption that $\{u_{it} : t = 1, 2, \ldots, T\}$ are independent then $\mathrm{Cov}(a_i + u_{it}, a_i + u_{is}) = \mathrm{Var}(a_i) = \sigma_a^2$ for all $t \neq s$. We can use a slight modification of correlated random effects probit, which takes the idiosyncratic error to have unit variance. To this end, write

$$
\begin{aligned}
w_{it} &= 1[\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i + a_i + u_{it} > r_{it}] \\
&= 1\left[\frac{(\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i - r_{it})}{\sigma_u} > \frac{(a_i + u_{it})}{\sigma_u}\right] \\
&\equiv 1\left[\frac{(\mathbf{x}_{it}\boldsymbol{\beta} + \psi + \bar{\mathbf{x}}_i\boldsymbol{\xi} + \eta\bar{r}_i - r_{it})}{\sigma_u} > g_i + e_{it}\right]
\end{aligned}
$$

where $e_{it} = u_{it}/\sigma_u$ and $g_i = a_i/\sigma_u$. This shows that if we apply the CRE probit model to w_{it} on $(\mathbf{x}_{it}, 1, \bar{\mathbf{x}}_i, \bar{r}_i, r_{it})$ we consistentlye estimate $\boldsymbol{\beta}_u = \boldsymbol{\beta}/\sigma_u$, $\psi_u = \psi/\sigma_u$, $\boldsymbol{\xi}_u = \boldsymbol{\xi}/\sigma_u$, and $\gamma_u = -1/\sigma_u$ as the coefficients and

$$\text{Var}(g_i) = \sigma_a^2/\sigma_u^2$$

as the heterogeneity variance. Thus, we can recover the original unscaled coefficients, $\sigma_u^2 = 1/\gamma_u^2$, and

$$\sigma_a^2 = \sigma_g^2/\gamma_u^2.$$

19.18. a. Conditional on $y > 0$, y follows a truncated normal distribution. So truncated normal regression would consistently estimate $\boldsymbol{\beta}$ and σ^2.

b. Because we are claiming that D($y|\mathbf{x}$) follows a type I Tobit in the population, we use the expected value derived from that assumption. Namely,

$$\text{E}(y|\mathbf{x}) = \Phi(\mathbf{x}\boldsymbol{\beta}/\sigma)\mathbf{x}\boldsymbol{\beta} + \sigma\phi(\mathbf{x}\boldsymbol{\beta}/\sigma).$$

and then we compute derivatives and changes with respect to x_j, as usual.

This differs from a hurdle model because we do not have a separate model for P($y = 0|\mathbf{x}$); we assume this is also governed by the Tobit model, so P($y = 0|\mathbf{x}$) = $1 - \Phi(\mathbf{x}\boldsymbol{\beta}/\sigma)$.

c. We could not estimate a hurdle model in this case because we have no data when $y = 0$. We have not sampled from that part of the population, and so we cannot estimate a model for P($s = 1|\mathbf{x}$) where $s = 1[y > 0]$.

19.19. a. First, if $r_i = 0$ the observation contains no information for estimating the distribution $\mathrm{D}(y_i|\mathbf{x}_i)$because then $\mathrm{P}(w_i = 0|\mathbf{x}_i) = 1$ regardless of $\mathrm{D}(y_i|\mathbf{x}_i)$. So what follows is only relevant for $r_i \in \{1, 2, 3, \ldots\}$.

For $0 \leq w < r_i$,

$$\mathrm{P}(w_i = w|\mathbf{x}_i) = \mathrm{P}(y_i = w|\mathbf{x}_i) = f_y(w|\mathbf{x}_i).$$

Next, $w_i = r_i$ if and only if $y_i \geq r_i$, and so

$$\begin{aligned}\mathrm{P}(w_i = r_i|\mathbf{x}_i) &= 1 - \mathrm{P}(y_i < r_i|\mathbf{x}_i) = 1 - \mathrm{P}(y_i \leq r_i - 1|\mathbf{x}_i) \\ &= 1 - F_y(r_i - 1|\mathbf{x}_i).\end{aligned}$$

We can write the conditional density of w_i as

$$f_w(w|\mathbf{x}_i, r_i) = [f_y(w|\mathbf{x}_i)]^{1[w<r_i]}[1 - F_y(r_i - 1|\mathbf{x}_i)]^{1[w=r_i]}, w = 0, \ldots, r_i.$$

b. In the Poisson case with an exponential mean, the conditional density of y_i is

$$f_y(y|\mathbf{x}_i; \boldsymbol{\beta}) = \frac{\exp[-\exp(\mathbf{x}_i\boldsymbol{\beta})][\exp(\mathbf{x}_i\boldsymbol{\beta})]^y}{y!}$$

and the cdf is

$$F_y(y|\mathbf{x}_i) = \exp[-\exp(\mathbf{x}_i\boldsymbol{\beta})] \sum_{h=0}^{y} \frac{[\exp(\mathbf{x}_i\boldsymbol{\beta})]^h}{h!}$$

Now just plug this into the general formula in part a.

c. Maximum likelihood estimators based on censored data are generally not robust to misspecification of the underlying population – even when that distribution is in the linear exponential family. (The log likelihood for the censored variable is not in the linear exponential family; even if it were, $\mathrm{E}(w_i|\mathbf{x}_i)$ depends on the underlying distribution.) Just like censored regression with a normal distribution is not robust for estimating the mean parameters

under nonnormality, neither is censored regression with a Poisson distribution. One way to see this is to write down the score for the general case and observe that just having $E(y_i|\mathbf{x}_i)$ correctly specified will not imply that the score has zero expectation.

d. As we know from earlier chapters, nonlinear least squares, Poisson QMLE, and other QMLEs in the LEF are robust for estimating the mean parameters. Thus, if there were no data censoring, we could use the Poisson QMLE to estimate $\boldsymbol{\beta}$. With data censoring, $E(w_i|\mathbf{x}_i)$ always depends on the underlying population distribution. Thus, in general we need to specify $D(y_i|\mathbf{x}_i)$ even if we are primarily interested in $E(y_i|\mathbf{x}_i)$.

e. Because data censoring requires us to have $D(y_i|\mathbf{x}_i)$ correctly specified, a strong case can be made for specifying flexible models for $D(y_i|\mathbf{x}_i)$ – even if we are primarily interested in $D(y_i|\mathbf{x}_i)$. For example, if we use a NegBin I or NegBin II model, these at least include the Poisson as (limiting) special cases. So, if we are pretty sure the underlying population has overdispersion, we can use one of these distributions in accounting for the right censoring. Ideally we would have a distribution that allows underdispersion, too.

Solutions to Chapter 20 Problems

20.1. a. Just use calculus and set the derivative to zero:

$$\sum_{i=1}^{N_0} p_{j_i}^{-1}(w_i - \hat{\mu}_w) = 0$$

or

$$\sum_{i=1}^{N_0} p_{j_i}^{-1} w_i = \sum_{i=1}^{N_0} p_{j_i}^{-1} \hat{\mu}_w = \left(\sum_{i=1}^{N_0} p_{j_i}^{-1} \right) \hat{\mu}_w.$$

Solving for $\hat{\mu}_w$ gives

$$\hat{\mu}_w = \left(\sum_{i=1}^{N_0} p_{j_i}^{-1} \right)^{-1} \sum_{i=1}^{N_0} p_{j_i}^{-1} w_i = \sum_{i=1}^{N_0} v_{ji} w_i$$

where

$$v_{ji} = \left(\sum_{i=h}^{N_0} p_{j_h}^{-1} \right)^{-1} p_{j_i}^{-1}.$$

b. From equation (20.7),

$$\mathrm{P}(s_i = 1|\mathbf{z}_i, w_i) = \mathrm{P}(s_i = 1|\mathbf{z}_i) = p_1 z_{i1} + \ldots + p_J z_{iJ}$$

which implies

$$\mathrm{E}(s_i|j_i, w_i) = p_{j_i}$$

because the stratum for observation i is j_i if and only if $z_{ij} = 1$. Now

$$\mathrm{E}\left[N^{-1} \sum_{i=1}^{N} (s_i/p_{j_i}) w_i \right] = N^{-1} \sum_{i=1}^{N} \mathrm{E}[(s_i/p_{j_i}) w_i]$$

and, by iterated expectations,

$$\begin{aligned}\mathrm{E}[(s_i/p_{j_i})w_i] &= \mathrm{E}\{\mathrm{E}[(s_i/p_{j_i})w_i|j_i, w_i]\} \\ &= \mathrm{E}\{[\mathrm{E}(s_i|j_i, w_i)/p_{j_i})]w_i\} \\ &= \mathrm{E}[(p_{j_i}/p_{j_i})w_i] = \mathrm{E}(w_i) = \mu_o.\end{aligned}$$

This shows $\mathrm{E}(\tilde{\mu}_w) = \mu_o$.

c. Notice that $\tilde{\mu}_w$ depends on N, the number of times we sampled the population, including when we did not record the observation. By contrast, to obtain $\hat{\mu}_w$ we need only need information on the sampling weights and data on the units actually kept. Therefore, in addition to knowing the sampling probabilities, $\tilde{\mu}_w$ requires the extra information that we know how many observations were discarded by the VP sampling scheme.

20.2. Write the log likelihood for all N observations as

$$\sum_{i=1}^{N}\sum_{h=1}^{J} z_{ih}[s_i \log(p_h) + (1 - s_i)\log(1 - p_h)].$$

For a given $j \in \{1, 2, \ldots, J\}$, take the derivative with respect to p_j, and set the result to zero:

$$\sum_{i=1}^{N} z_{ij}\left[\frac{s_i}{\hat{p}_j} - \frac{(1 - s_i)}{(1 - \hat{p}_j)}\right] \equiv 0$$

or, by obtaining a common denominator,

$$\sum_{i=1}^{N} z_{ij}\left[\frac{(1 - \hat{p}_j)s_i - \hat{p}_j(1 - s_i)}{\hat{p}_j(1 - \hat{p}_j)}\right] \equiv 0.$$

Of course, the problem only makes sense for interior solutions $0 < \hat{p}_j < 1$ so the first order condition is equivalent to

$$\sum_{i=1}^{N}[(1 - \hat{p}_j)z_{ij}s_i - \hat{p}_j z_{ij}(1 - s_i)] \equiv 0.$$

Simple algebra gives

$$\sum_{i=1}^{N} z_{ij} s_i = \sum_{i=1}^{N} \hat{p}_j z_{ij}$$

or

$$\hat{p}_j = \frac{\sum_{i=1}^{N} z_{ij} s_i}{\sum_{i=1}^{N} z_{ij}} \equiv \frac{M_j}{N_j}.$$

20.3. a. To be specific, consider the case of variable probility sampling, where the probability weights are

$$p(\mathbf{z}_i) = p_1 z_{i1} + \ldots + p_J z_{iJ} = \mathrm{P}(s_i = 1 | \mathbf{z}_i, \mathbf{w}_i)$$

where $\mathbf{w}_i = (\mathbf{x}_i, y_i)$. We can write the IPW nonliner least squares objective function as

$$\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \sum_{i=1}^{N} \frac{s_i}{p(\mathbf{z}_i)} [y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2,$$

which is for form useful for studying asymptotic properties. (For the asymptotic distribution theory, we divide the objective function by two to make the notation easier.)

b. For VP sampling, we have already assumed that each $p_j > 0$, and, because we can write

$$|s_i / p(\mathbf{z}_i)| \leq \max(p_1^{-1}, \ldots, p_J^{-1})$$

it follows that the regularity conditions sufficient for consistency of NLS on a random sample are also sufficient for NLS on a VP sample: the objective function is still continuous and the moment conditions do not need to be changed because

$$\frac{s_i}{p(\mathbf{z}_i)} [y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2 \leq \max(p_1^{-1}, \ldots, p_J^{-1}) [y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2.$$

Further, we know generally that

$$\mathrm{E}\left\{ \frac{s_i}{p(\mathbf{z}_i)} [y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2 \right\} = \mathrm{E}\{[y_i - m(\mathbf{x}_i, \boldsymbol{\theta})]^2\}$$

and so if $\boldsymbol{\theta}_o$ uniquely minimizes the right hand side, it uniquely minizes the left hand side, too.

c. The theory in Section 19.8 can be applied directly. In particular, we can use equation (19.90) because the probabilities are known, not estimated. In the formula,

$$\begin{aligned}\mathbf{A}_o &= \mathrm{E}[\mathbf{H}(\mathbf{w}_i,\boldsymbol{\theta}_o)] \;= \mathrm{E}[\mathbf{A}(\mathbf{x}_i,\boldsymbol{\theta}_o)] \\ &= \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\boldsymbol{\theta}_o)]\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}_o &= \mathrm{E}\left\{\frac{s_i}{[p(\mathbf{z}_i)]^2}\nabla_{\boldsymbol{\theta}} q(\mathbf{x}_i,\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}} q(\mathbf{x}_i,\boldsymbol{\theta}_o)\right\} \\ &= \mathrm{E}\left\{\frac{s_i u_i^2}{[p(\mathbf{z}_i)]^2}\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\boldsymbol{\theta}_o)'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\boldsymbol{\theta}_o)\right\}\end{aligned}$$

where $u_i = y_i - m(\mathbf{x}_i,\boldsymbol{\theta}_o)$. We can consistently estimate $\mathbf{A}_o$ as

$$\hat{\mathbf{A}} = N^{-1}\sum_{i=1}^{N}\left[\frac{s_i}{p(\mathbf{z}_i)}\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\hat{\boldsymbol{\theta}}_w)'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\hat{\boldsymbol{\theta}}_w)\right]$$

and $\mathbf{B}_o$ as

$$\hat{\mathbf{B}} = N^{-1}\sum_{i=1}^{N}\left[\frac{s_i\hat{u}_i^2}{[p(\mathbf{z}_i)]^2}\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\hat{\boldsymbol{\theta}}_w)'\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i,\hat{\boldsymbol{\theta}}_w)\right]$$

where $\hat{u}_i = y_i - m(\mathbf{x}_i,\hat{\boldsymbol{\theta}}_w)$ are the residuals. Then

$$\widehat{\mathrm{Avar}}(\hat{\boldsymbol{\theta}}_w) = \hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1}/N$$

(which does not actually require knowing N, as it cancels everywhere).

d. The formula does not generally simplify because $\mathrm{E}(u_i^2|\mathbf{x}_i,\mathbf{z}_i)$ might depend on $\mathbf{z}_i$ even if $\mathrm{Var}(u_i|\mathbf{x}_i) = \sigma_o^2$. [In fact, we do not even assume that $\mathrm{E}(u_i|\mathbf{x}_i,\mathbf{z}_i) = 0$ in this problem because the stratification may be endogenous.]

e. If $m(\mathbf{x},\boldsymbol{\theta})$ is misspecified we must use a more general estimator for $\mathbf{A}^*$ based on

$$\mathbf{A}^* = \mathrm{E}[\mathbf{H}(\mathbf{w}_i, \boldsymbol{\theta}^*)] = \mathrm{E}[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}^*)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \boldsymbol{\theta}^*) - u_i^* \nabla_{\boldsymbol{\theta}}^2 m(\mathbf{x}_i, \boldsymbol{\theta}^*)]$$

where $\boldsymbol{\theta}^*$ is the pseudo-true value of $\boldsymbol{\theta}$ that solves the population minimization problem and $u_i^* = y_i - m(\mathbf{x}_i, \boldsymbol{\theta}^*)$. Our estimator of $\mathbf{A}^*$ is

$$\hat{\mathbf{A}} = N^{-1} \sum_{i=1}^{N} \left\{ \frac{s_i}{p(\mathbf{z}_i)} \left[\nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \hat{\boldsymbol{\theta}}_w)' \nabla_{\boldsymbol{\theta}} m(\mathbf{x}_i, \hat{\boldsymbol{\theta}}_w) - \hat{u}_i \nabla_{\boldsymbol{\theta}}^2 m(\mathbf{x}_i, \hat{\boldsymbol{\theta}}_w) \right] \right\}.$$

The estimator of $\mathbf{B}^*$ can be the same as in part c.

20.4. First, we can write the unweighted objective function as

$$\begin{aligned} N^{-1} \sum_{j=1}^{J} \sum_{i=1}^{N_j} q(\mathbf{w}_{ij}, \boldsymbol{\theta}) &= \sum_{j=1}^{J} (N_j/N) N_j^{-1} \sum_{i=1}^{N_j} q(\mathbf{w}_{ij}, \boldsymbol{\theta}) \\ &= \sum_{j=1}^{J} H_j \left(N_j^{-1} \sum_{i=1}^{N_j} q(\mathbf{w}_{ij}, \boldsymbol{\theta}) \right), \end{aligned}$$

as suggested in the hint. Further, by the same argument as on page 860, $N_j^{-1} \sum_{i=1}^{Nj} q(\mathbf{w}_{ij}, \boldsymbol{\theta})$ converges (uniformly) to $\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{w} \in \mathcal{W}_j] = \mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x} \in \mathcal{X}_j]$, where we use the fact that the strata are determined by the conditioning variables and given by $\mathcal{X}_1$, ..., $\mathcal{X}_J$. Therefore, if $H_j \to \bar{H}_j$ as $N \to \infty$ the unweighted objective function converges uniformly to

$$\bar{H}_1 \mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x} \in \mathcal{X}_1] + \ldots + \bar{H}_J \mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x} \in \mathcal{X}_J] \tag{20.97}$$

Given that $\boldsymbol{\theta}_o$ solves (20.15) for each $\mathbf{x}$, we can also show $\boldsymbol{\theta}_o$ minimizes $\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x} \in \mathcal{X}_j]$ over Θ for each j: by iterated expectations (since the indicator $1[\mathbf{x} \in \mathcal{X}_j]$ is a function of $\mathbf{x}$),

$$\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x} \in \mathcal{X}_j] = \mathrm{E}\{\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x}] | \mathbf{x} \in \mathcal{X}_j\},$$

and if $\boldsymbol{\theta}_o$ minimizes $\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x}]$, it must also minimize $\mathrm{E}\{\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta}) | \mathbf{x}] | \mathbf{x} \in \mathcal{X}_j\}$. Therefore, $\boldsymbol{\theta}_o$ is one minimizer of (20.97) over Θ. Now we just have to show it is the unique minimizer if it uniquely minimizes $\mathrm{E}[q(\mathbf{w}, \boldsymbol{\theta})]$. Without the assumption $\bar{H}_j > 0$, $\boldsymbol{\theta}_o$ need not be the unique

minimizer of (20.97). To show uniqueness when each $\bar{H}_j$ is strictly positive, let $s_j = 1[\mathbf{x} \in \mathcal{X}_j]$. Then we can write, for any $\boldsymbol{\theta}$,

$$\mathrm{E}[q(\mathbf{w},\boldsymbol{\theta})] - \mathrm{E}[q(\mathbf{w},\boldsymbol{\theta}_o)] = \sum_{j=1}^{J} Q_j\{\mathrm{E}[q(\mathbf{w},\boldsymbol{\theta})|s_j] - \mathrm{E}[q(\mathbf{w},\boldsymbol{\theta}_o)|s_j]\},$$

where the Q_j are the population frequencies. By assumption, the left hand side is strictly positive when $\boldsymbol{\theta} \neq \boldsymbol{\theta}_o$, which means, because $Q_j > 0$ for all j, $\mathrm{E}[q(\mathbf{w},\boldsymbol{\theta})|s_j] - \mathrm{E}[q(\mathbf{w},\boldsymbol{\theta}_o)|s_j]$ must be strictly positive for at least one j; we already know that each difference is nonnegative. This, along with the fact that $\bar{H}_j > 0$, $j = 1,\ldots,J$, implies that (20.97) is uniquely minimized at $\boldsymbol{\theta}_o$.

20.5. a. The Stata output is given below. The variables with "bar" added on denote the district-level averages. Note that we can still use `xtreg` even though this is a cluster sample, not a panel data set. An alternative for obtaining the FE estimates is the `areg` command in Stata. The pooled OLS and FE estimates are identical on all explanatory variables. The pooled OLS standard errors reported below are almost certainly incorrect because the assume no within-district correlation in the unobservables.

```
. reg lavgsal bs lstaff lenroll lunch bsbar lstaffbar lenrollbar lunchbar

      Source |       SS       df       MS              Number of obs =    1848
-------------+------------------------------           F(  8,  1839) =  228.60
       Model |  49.9510474     8  6.24388093           Prob > F      =  0.0000
    Residual |  50.2303314  1839  .027313938           R-squared     =  0.4986
-------------+------------------------------           Adj R-squared =  0.4964
       Total |  100.181379  1847  .054240054           Root MSE      =  .16527

------------------------------------------------------------------------------
     lavgsal |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          bs |  -.4948449   .2199466    -2.25   0.025    -.9262162   -.0634736
      lstaff |  -.6218901   .0277027   -22.45   0.000    -.6762221   -.5675581
     lenroll |  -.0515063   .0155411    -3.31   0.001    -.0819865   -.0210262
       lunch |   .0005138   .0003452     1.49   0.137    -.0001632    .0011908
       bsbar |    .441438   .2630336     1.68   0.093     -.074438    .9573139
   lstaffbar |  -.1493942   .0370985    -4.03   0.000    -.2221538   -.0766346
  lenrollbar |   .0315714   .0184565     1.71   0.087    -.0046266    .0677694
```

```
    lunchbar |  -.0016765   .0003903    -4.30   0.000    -.0024419    -.000911
       _cons |   13.98544    .141118    99.10   0.000     13.70867    14.26221
------------------------------------------------------------------------------

. xtreg lavgsal bs lstaff lenroll lunch, fe

Fixed-effects (within) regression               Number of obs      =      1848
Group variable: distid                          Number of groups   =       537

R-sq:  within  = 0.5486                         Obs per group: min =
       between = 0.3544                                        avg =       3.
       overall = 0.4567                                        max =       162

                                                F(4,1307)          =    397.05
corr(u_i, Xb)  = 0.1433                         Prob > F           =    0.0000

------------------------------------------------------------------------------
     lavgsal |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          bs |  -.4948449    .133039    -3.72   0.000    -.7558382   -.2338515
      lstaff |  -.6218901   .0167565   -37.11   0.000    -.6547627   -.5890175
     lenroll |  -.0515063   .0094004    -5.48   0.000    -.0699478   -.0330648
       lunch |   .0005138   .0002088     2.46   0.014     .0001042    .0009234
       _cons |   13.61783   .1133406   120.15   0.000     13.39548    13.84018
-------------+----------------------------------------------------------------
     sigma_u |  .15491886
     sigma_e |  .09996638
         rho |  .70602068   (fraction of variance due to u_i)
------------------------------------------------------------------------------
F test that all u_i=0:     F(536, 1307) =     7.24           Prob > F = 0.0000
```

b. The RE estimates are given below, with and without cluster-robust standard errors. Also, the cluster-robust standard errors for FE are provided. The fully robust standard errors are bigger than the nonrobust ones, suggesting there might be additional within-district correlation even after accounting for an additive districte effect.

```
. xtreg lavgsal bs lstaff lenroll lunch  bsbar lstaffbar lenrollbar lunchbar,
        re

Random-effects GLS regression                   Number of obs      =      1848
Group variable: distid                          Number of groups   =       537

R-sq:  within  = 0.5486                         Obs per group: min =
       between = 0.4006                                        avg =       3.
       overall = 0.4831                                        max =       162

Random effects u_i ~Gaussian                    Wald chi2(8)       =   1943.89
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

------------------------------------------------------------------------------
     lavgsal |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
```

```
          bs |  -.4948449   .1334822    -3.71   0.000    -.7564652   -.2332245
      lstaff |  -.6218901   .0168123   -36.99   0.000    -.6548417   -.5889385
     lenroll |  -.0515063   .0094317    -5.46   0.000    -.0699921   -.0330205
       lunch |   .0005138   .0002095     2.45   0.014     .0001032    .0009244
       bsbar |   .2998553   .2437798     1.23   0.219    -.1779443     .777655
    lstaffbar |  -.0255493   .0418946    -0.61   0.542    -.1076613    .0565627
   lenrollbar |   .0657286   .0157977     4.16   0.000     .0347657    .0966914
     lunchbar |  -.0007259   .0004022    -1.80   0.071    -.0015143    .0000625
        _cons |   13.22003   .2136208    61.89   0.000     12.80135    13.63872
--------------+----------------------------------------------------------------
      sigma_u |  .12627558
      sigma_e |  .09996638
          rho |  .61473634   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lavgsal bs lstaff lenroll lunch  bsbar lstaffbar lenrollbar lunchbar,
        re cluster(distid)

Random-effects GLS regression                   Number of obs      =      1848
Group variable: distid                          Number of groups   =       537

R-sq:  within  = 0.5486                         Obs per group: min =
       between = 0.4006                                        avg =       3.
       overall = 0.4831                                        max =       162

Random effects u_i ~Gaussian                    Wald chi2(8)       =    556.49
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                              (Std. Err. adjusted for 537 clusters in distid
------------------------------------------------------------------------------
             |               Robust
     lavgsal |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
          bs |  -.4948449   .1939422    -2.55   0.011    -.8749646   -.1147252
      lstaff |  -.6218901   .0432281   -14.39   0.000    -.7066157   -.5371645
     lenroll |  -.0515063    .013103    -3.93   0.000    -.0771876    -.025825
       lunch |   .0005138    .000213     2.41   0.016     .0000964    .0009312
       bsbar |   .2998553   .3031961     0.99   0.323    -.2943981    .8941087
    lstaffbar |  -.0255493   .0651932    -0.39   0.695    -.1533256    .1022269
   lenrollbar |   .0657286    .020655     3.18   0.001     .0252455    .1062116
     lunchbar |  -.0007259   .0004378    -1.66   0.097    -.0015839    .0001322
        _cons |   13.22003   .2556139    51.72   0.000     12.71904    13.72103
--------------+----------------------------------------------------------------
      sigma_u |  .12627558
      sigma_e |  .09996638
          rho |  .61473634   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. xtreg lavgsal bs lstaff lenroll lunch, fe cluster(distid)

Fixed-effects (within) regression               Number of obs      =      1848
Group variable: distid                          Number of groups   =       537

R-sq:  within  = 0.5486                         Obs per group: min =
       between = 0.3544                                        avg =       3.
       overall = 0.4567                                        max =       162

                                                F(4,536)           =     57.84
corr(u_i, Xb)  = 0.1433                         Prob > F           =    0.0000
```

```
                              (Std. Err. adjusted for 537 clusters in distid
------------------------------------------------------------------------------
             |               Robust
     lavgsal |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
          bs |  -.4948449   .1937316    -2.55   0.011    -.8754112   -.1142785
      lstaff |  -.6218901   .0431812   -14.40   0.000    -.7067152   -.5370649
     lenroll |  -.0515063   .0130887    -3.94   0.000    -.0772178   -.0257948
       lunch |   .0005138   .0002127     2.42   0.016     .0000959    .0009317
       _cons |   13.61783   .2413169    56.43   0.000     13.14379    14.09187
-------------+----------------------------------------------------------------
     sigma_u |  .15491886
     sigma_e |  .09996638
         rho |  .70602068   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

c. The robust Wald test for joint significance of the four district-level averages gives a strong rejection of the null, with *p*-value = .0004. Therefore, we conclude that at least some of the variables are correlated with unobserved district effects.

```
. qui xtreg lavgsal bs lstaff lenroll lunch bsbar lstaffbar lenrollbar lunchbar
      re cluster(distid)

. test  bsbar lstaffbar lenrollbar lunchbar

 ( 1)  bsbar = 0
 ( 2)  lstaffbar = 0
 ( 3)  lenrollbar = 0
 ( 4)  lunchbar = 0

           chi2(  4) =   20.70
         Prob > chi2 =    0.0004
```

20.6. a. Only three schools in the sample have reported benefits/salary ratios of at least .5. The highest of these is about .66.

```
. count if bs >= .5
    3

. list distid bs if bs >= .5

      +-------------------+
      | distid         bs |
      |-------------------|
  68. |   9030   .6594882 |
1127. |  63160   .5747756 |
1670. |  82040   .5022581 |
      +-------------------+
```

b. The magnitude of the coefficient on *bs* falls somewhat and the cluster-robust standard error increases substantially from about .194 to .245, likely due to the reduction in variation of *bs* within the three districts listed in part a. There is still some evidence of a salary-benefits tradeoff.

```
. xtreg lavgsal bs lstaff lenroll lunch if bs < .5, fe cluster(distid)

Fixed-effects (within) regression               Number of obs      =      1845
Group variable: distid                          Number of groups   =       537

R-sq:  within  = 0.5474                         Obs per group: min =
       between = 0.3552                                        avg =       3.
       overall = 0.4567                                        max =       162

                                                F(4,536)           =     58.06
corr(u_i, Xb)  = 0.1452                         Prob > F           =    0.0000

                              (Std. Err. adjusted for 537 clusters in distid
------------------------------------------------------------------------------
             |               Robust
     lavgsal |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          bs |  -.4560107    .245449    -1.86   0.064    -.9381705    .0261492
      lstaff |  -.6226836   .0431074   -14.44   0.000    -.7073637   -.5380035
     lenroll |  -.0518125   .0131213    -3.95   0.000     -.077588    -.026037
       lunch |   .0005151   .0002157     2.39   0.017     .0000913    .0009389
       _cons |    13.6096   .2466242    55.18   0.000     13.12513    14.09407
-------------+----------------------------------------------------------------
     sigma_u |  .15486353
     sigma_e |  .10003476
         rho |  .70558835   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

c. The LAD estimates below give a point estimate that indicates a tradeoff but it is smaller in magnitude than in part a or part b. The standard error, which is not robust to cluster correlation, implies $t = -1.22$. Therefore, using LAD, there is little evidence of a tradeoff.

```
. qreg lavgsal bs lstaff lenroll lunch  bsbar lstaffbar lenrollbar lunchbar

Median regression                                       Number of obs =      1848
  Raw sum of deviations 334.3106 (about 10.482654)
  Min sum of deviations 234.7491                        Pseudo R2     =    0.2978

------------------------------------------------------------------------------
     lavgsal |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          bs |  -.3066784   .2511169    -1.22   0.222    -.7991826    .1858258
      lstaff |  -.6555687   .0313058   -20.94   0.000    -.7169673     -.59417
     lenroll |  -.0635032    .017727    -3.58   0.000    -.0982703    -.028736
```

```
       lunch |   .0005538   .0003954     1.40   0.162    -.0002217    .0013293
       bsbar |   .3679283   .3003398     1.23   0.221    -.2211145    .9569712
    lstaffbar |  -.1374073   .0421226    -3.26   0.001    -.2200204   -.0547941
   lenrollbar |   .0075581   .0210143     0.36   0.719    -.0336564    .0487726
     lunchbar |  -.0014894   .0004477    -3.33   0.001    -.0023675   -.0006113
        _cons |   14.23874   .1612496    88.30   0.000     13.92249    14.55499
------------------------------------------------------------------------------
```

20.7. a. Out of 1,683 schools, 922 have all five years of data. The fewest number of years is three. Note that the `tab` command gives includes many more observations than schools because there are multiple years per school.

```
. xtsum math4

Variable         |      Mean   Std. Dev.       Min        Max |    Observations
-----------------+--------------------------------------------+--------------
math4    overall |  63.57726   20.19047        2.9        100 |     N =    7150
         between |             16.08074      11.75      98.94 |     n =    1683
         within  |             12.37335   13.71059   122.3439 | T-bar = 4.24837

. egen tobs = sum(1), by(schid)

. count if tobs == 5 & y98
  922

. tab tobs

       tobs |      Freq.     Percent        Cum.
------------+-----------------------------------
          3 |      1,512       21.15       21.15
          4 |      1,028       14.38       35.52
          5 |      4,610       64.48      100.00
------------+-----------------------------------
      Total |      7,150      100.00
```

b. The pooled OLS estimates, with all time averages included, and the fixed effects estimates – with so-called “school fixed effects” – are given below. Variables with a “b” at the end are the within-school time averages. As expected, they are identical, including the coefficients on the year dummies.

The coefficient on *lunchb* is $-.426$, and its fully robust t statistic is -11.76. Therefore, the average poverty level over the available years has a very large effect on the math pass rate: a ten percentage point increase in the average poverty rate predicts a pass rate that is about 4.3 percentage points lower.

```
. reg math4 lavgrexp lunch lenrol y95 y96 y97 y98 lavgrexpb lunchb lenrolb
      y95b y96b y97b y98b, cluster(distid)

Linear regression                                       Number of obs =    7150
                                                        F( 14,   466) =  182.55
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.4147
                                                        Root MSE      =  15.462

                                (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lavgrexp |   6.288376    3.13387     2.01   0.045     .1301085    12.44664
       lunch |  -.0215072   .0399402    -0.54   0.590    -.0999924     .056978
      lenrol |  -2.038461   2.099636    -0.97   0.332    -6.164387    2.087466
         y95 |    11.6192   .7213934    16.11   0.000     10.20162    13.03679
         y96 |   13.05561   .9331425    13.99   0.000     11.22192     14.8893
         y97 |   10.14771   .9581113    10.59   0.000     8.264957    12.03046
         y98 |   23.41404   1.027817    22.78   0.000     21.39431    25.43377
   lavgrexpb |     2.7178    4.04162     0.67   0.502    -5.224258    10.65986
      lunchb |  -.4256461   .0361912   -11.76   0.000    -.4967642   -.3545279
     lenrolb |   .2880016    2.17219     0.13   0.895      -3.9805    4.556503
        y95b |   21.26329   15.95857     1.33   0.183    -10.09639    52.62297
        y96b |   15.69885   6.523566     2.41   0.016     2.879602     28.5181
        y97b |   20.66597   15.71006     1.32   0.189    -10.20536     51.5373
        y98b |  -8.501184   18.89568    -0.45   0.653    -45.63248    28.63011
       _cons |  -6.616139   25.07553    -0.26   0.792    -55.89125    42.65897
------------------------------------------------------------------------------

. xtreg math4 lavgrexp lunch lenrol y95 y96 y97 y98, fe cluster(distid)

Fixed-effects (within) regression               Number of obs      =      7150
Group variable: schid                           Number of groups   =      1683

R-sq:  within  = 0.3602                         Obs per group: min =
       between = 0.0292                                        avg =       4.
       overall = 0.1514                                        max =

                                                F(7,466)           =    259.90
corr(u_i, Xb)  = 0.0073                         Prob > F           =    0.0000

                                (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lavgrexp |   6.288376   3.132334     2.01   0.045     .1331271    12.44363
       lunch |  -.0215072   .0399206    -0.54   0.590    -.0999539     .0569395
      lenrol |  -2.038461   2.098607    -0.97   0.332    -6.162365    2.085443
         y95 |    11.6192   .7210398    16.11   0.000     10.20231     13.0361
         y96 |   13.05561   .9326851    14.00   0.000     11.22282     14.8884
         y97 |   10.14771   .9576417    10.60   0.000      8.26588    12.02954
         y98 |   23.41404   1.027313    22.79   0.000      21.3953    25.43278
       _cons |   11.84422   32.68429     0.36   0.717    -52.38262    76.07107
-------------+----------------------------------------------------------------
     sigma_u |   15.84958
     sigma_e |  11.325028
```

```
         rho |  .66200804   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

c. The RE estimates are given below, and they are identical to the FE estimates. The RE coefficients on the time averaegs are not identical to those for POLS. In particular, on *lunchb*, the RE coefficient is −.415, just slightly smaller in magnitude than the POLS estimate. It has a slightly smaller fully robust *t* statistic (in absolute value).

```
. xtreg math4 lavgrexp lunch lenrol y95 y96 y97 y98 lavgrexpb lunchb lenrolb
        y95b y96b y97b y98b, re cluster(distid)

Random-effects GLS regression                   Number of obs      =      7150
Group variable: schid                           Number of groups   =      1683

R-sq:  within  = 0.3602                         Obs per group: min =
       between = 0.4366                                        avg =       4.
       overall = 0.4146                                        max =

Random effects u_i ~Gaussian                    Wald chi2(14)      =   2532.10
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                              (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    lavgrexp |   6.288376    3.13387     2.01   0.045     .1461029    12.43065
       lunch |  -.0215072   .0399402    -0.54   0.590    -.0997886    .0567741
      lenrol |  -2.038461   2.099636    -0.97   0.332    -6.153671     2.07675
         y95 |    11.6192   .7213934    16.11   0.000      10.2053    13.03311
         y96 |   13.05561   .9331425    13.99   0.000     11.22668    14.88453
         y97 |   10.14771   .9581113    10.59   0.000     8.269847    12.02557
         y98 |   23.41404   1.027817    22.78   0.000     21.39956    25.42852
   lavgrexpb |   2.569862    3.99586     0.64   0.520    -5.261881     10.4016
      lunchb |  -.4153413   .0363218   -11.44   0.000    -.4865308   -.3441518
     lenrolb |   .3829623   2.157847     0.18   0.859     -3.84634    4.612264
        y95b |   18.96418   15.24131     1.24   0.213    -10.90824    48.83659
        y96b |   16.16473   6.628049     2.44   0.015     3.173993    29.15547
        y97b |   17.50964   15.42539     1.14   0.256    -12.72357    47.74285
        y98b |  -9.420143   18.25294    -0.52   0.606    -45.19524    26.35495
       _cons |  -5.159784   24.08649    -0.21   0.830    -52.36844    42.04887
-------------+----------------------------------------------------------------
     sigma_u |  10.702446
     sigma_e |  11.325028
         rho |  .47175866   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

d. When we drop the time averages of the year dummies the RE estimates are slightly different from the FE estimates. That is because we must now recognize that, with an unbalanced panel, the time averages of the year dummies are no longer constant. With a

balanced panel, the time average are $1/T$ in each case. Now, the average is either zero – if the unit does not appear in the appropriate year – or $1/T_i$ where T_i is the total number of years for unit (school) i. For example, the `list` command below shows that the school with identifier number 557 has data for the years 1994, 1997, and 1998. Therefore, $y95b$ and $y96b$ are both zero, while $y97b$ and $y98b$ are both 1/3. With an unbalanced panel, we should include the time averages of the year dummies. In effect, this is allowing certain forms of sample selection to be correlated with the unobserved school heterogeneity.

```
. xtreg math4 lavgrexp lunch lenrol y95 y96 y97 y98 lavgrexpb lunchb lenrolb,
       re cluster(distid)

Random-effects GLS regression                   Number of obs      =      7150
Group variable: schid                           Number of groups   =      1683

R-sq:  within  = 0.3602                         Obs per group: min =
       between = 0.4291                                        avg =       4.
       overall = 0.4105                                        max =

Random effects u_i ~Gaussian                    Wald chi2(10)      =   2073.48
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                              (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    lavgrexp |   6.222429   3.121881     1.99   0.046     .1036546     12.3412
       lunch |  -.0209812   .0402425    -0.52   0.602     -.099855    .0578926
      lenrol |   -2.06064   2.070938    -1.00   0.320    -6.119604    1.998325
         y95 |   11.78595   .7084874    16.64   0.000     10.39734    13.17456
         y96 |   13.16626     .91466    14.39   0.000     11.37356    14.95896
         y97 |   10.21612   .9441691    10.82   0.000     8.365579    12.06665
         y98 |   23.46409   1.055457    22.23   0.000     21.39544    25.53275
   lavgrexpb |   2.417603   3.887099     0.62   0.534     -5.20097    10.03618
      lunchb |  -.4088571   .0365525   -11.19   0.000    -.4804986   -.3372155
     lenrolb |   .7979708   2.109349     0.38   0.705    -3.336278     4.93222
       _cons |   2.619295   24.78096     0.11   0.916    -45.95049    51.18908
-------------+----------------------------------------------------------------
     sigma_u |  10.702446
     sigma_e |  11.325028
         rho |  .47175866   (fraction of variance due to u_i)
------------------------------------------------------------------------------

. list schid year  y95b y96b y97b y98b if schid == 557

      +-------------------------------------------------+
      | schid   year   y95b   y96b        y97b        y98b |
      |-------------------------------------------------|
```

```
 740. |   557   1994      0      0   .33333333   .33333333 |
 741. |   557   1997      0      0   .33333333   .33333333 |
 742. |   557   1998      0      0   .33333333   .33333333 |
      +-----------------------------------------------------+
```

e. The FE estimates without the year dummies are given below. The coefficient on the spending variable is more than seven times larger than when the year dummies are included. The estimate without the year dummies is very misleading. During this period in Michigan, spending was increasing and, at the same time, the definition of a passing score was changed so that more students passed the exam. Thus, without controlling for time dummies, most of the relationship between pass rates and spending is spurious.

```
. xtreg math4 lavgrexp lunch lenrol, fe cluster(distid)

Fixed-effects (within) regression               Number of obs      =      7150
Group variable: schid                           Number of groups   =      1683

R-sq:  within  = 0.1632                         Obs per group: min =
       between = 0.0001                                        avg =       4.
       overall = 0.0233                                        max =

                                                F(3,466)           =    136.54
corr(u_i, Xb)  = -0.3272                        Prob > F           =    0.0000

                              (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lavgrexp |   45.00103   2.452645    18.35   0.000     40.18141    49.82064
       lunch |   .0179948   .0377204     0.48   0.634    -.0561284     .092118
      lenrol |  -2.372125   3.403866    -0.70   0.486    -9.060952    4.316701
       _cons |  -294.8467   32.11083    -9.18   0.000    -357.9467   -231.7468
-------------+----------------------------------------------------------------
     sigma_u |  17.573721
     sigma_e |    12.9465
         rho |  .64820501   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

f. The POLS and RE estimates, without the time averages, are given below. The spending effects are larger than FE and the effect of the *lunch* variable are much larger. If we do not remove the school effect – of which a large component is demographics that do not change over time – then the poverty measure *lunch* becomes very important. From the POLS/RE

estimates with the time averages included, it is really the average poverty level over several years that has the most predictive power. Of course, the *lunch* variable does not vary across time nearly as much as it does across school. Therefore, using FE, it is difficult to separate the effect of $lunch_{it}$ from c_i.

```
. reg math4 lavgrexp lunch lenrol y95 y96 y97 y98, cluster(distid)

Linear regression                                       Number of obs =    7150
                                                        F(  7,   466) =  256.84
                                                        Prob > F      =  0.0000
                                                        R-squared     =  0.4029
                                                        Root MSE      =  15.609

                              (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lavgrexp |   8.628338   2.488897     3.47   0.001     3.737487    13.51919
       lunch |  -.4255479   .0391249   -10.88   0.000    -.5024309   -.3486648
      lenrol |  -1.294046   1.149539    -1.13   0.261    -3.552969    .9648762
         y95 |   12.09916   .8909378    13.58   0.000     10.34841    13.84992
         y96 |   13.06982   1.128072    11.59   0.000     10.85308    15.28655
         y97 |   10.29535   1.114853     9.23   0.000     8.104584    12.48611
         y98 |   23.57121    1.29055    18.26   0.000     21.03519    26.10723
       _cons |   2.758117   23.09242     0.12   0.905    -42.62005    48.13628
------------------------------------------------------------------------------

. xtreg math4 lavgrexp lunch lenrol y95 y96 y97 y98, re cluster(distid)

Random-effects GLS regression                   Number of obs      =      7150
Group variable: schid                           Number of groups   =      1683

R-sq:  within  = 0.3455                         Obs per group: min =
       between = 0.4288                                        avg =       4.
       overall = 0.4016                                        max =

Random effects u_i ~Gaussian                    Wald chi2(7)       =   1886.18
corr(u_i, X)       = 0 (assumed)                Prob > chi2        =    0.0000

                              (Std. Err. adjusted for 467 clusters in distid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    lavgrexp |   7.838068   2.157833     3.63   0.000     3.608793    12.06734
       lunch |  -.3785643   .0400361    -9.46   0.000    -.4570336   -.3000949
      lenrol |  -1.391074   .9449022    -1.47   0.141    -3.243048    .4609008
         y95 |   11.66598   .7704663    15.14   0.000      10.1559    13.17607
         y96 |   12.88762   .9420724    13.68   0.000     11.04119    14.73404
         y97 |   10.18776    .896855    11.36   0.000     8.429958    11.94557
         y98 |   23.53236   1.029968    22.85   0.000     21.51366    25.55106
```

```
       _cons |   8.166742   20.06401     0.41   0.684      -31.158    47.49148
-------------+----------------------------------------------------------------
     sigma_u |  10.702446
     sigma_e |  11.325028
         rho |  .47175866   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

g. It seems pretty clear we need to go with the FE estimate and its standard error robust to serial correlation within school and cluster correlation within district. Removing a school effect most likely gives us the least biased estimator of school spending. Clustering at the district level, rather than just at the school level, increases the standard error to 3.13 from about 2.43, and so it seems prudent to use the standard error clustered at the district level.

```
. xtreg math4 lavgrexp lunch lenrol y95 y96 y97 y98, fe cluster(schid)

Fixed-effects (within) regression               Number of obs      =      7150
Group variable: schid                           Number of groups   =      1683

R-sq:  within  = 0.3602                         Obs per group: min =
       between = 0.0292                                        avg =       4.
       overall = 0.1514                                        max =

                                                F(7,1682)          =    431.08
corr(u_i, Xb)  = 0.0073                         Prob > F           =    0.0000

                              (Std. Err. adjusted for 1683 clusters in schid
------------------------------------------------------------------------------
             |               Robust
       math4 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
    lavgrexp |   6.288376   2.431317     2.59   0.010     1.519651     11.0571
       lunch |  -.0215072   .0390732    -0.55   0.582    -.0981445      .05513
      lenrol |  -2.038461   1.789094    -1.14   0.255    -5.547545    1.470623
         y95 |    11.6192   .5358469    21.68   0.000     10.56821     12.6702
         y96 |   13.05561   .6910815    18.89   0.000     11.70014    14.41108
         y97 |   10.14771   .7326314    13.85   0.000     8.710745    11.58468
         y98 |   23.41404   .7669553    30.53   0.000     21.90975    24.91833
       _cons |   11.84422   25.16643     0.47   0.638    -37.51659    61.20503
-------------+----------------------------------------------------------------
     sigma_u |   15.84958
     sigma_e |  11.325028
         rho |  .66200804   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```

20.8. a. The information contained in $(\mathbf{x}_g, \mathbf{Z}_g, c_g)$ and $(\mathbf{x}_g, \mathbf{Z}_g, a_g)$ is the same, and so if we substitute for c_g we have

$$\begin{aligned}\mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g, c_g) &= \Phi(\alpha + \mathbf{x}_g\boldsymbol{\beta} + \mathbf{z}_{gm}\boldsymbol{\gamma} + c_g) \\ &= \Phi(\alpha + \mathbf{x}_g\boldsymbol{\beta} + \mathbf{z}_{gm}\boldsymbol{\gamma} + \eta_g + \bar{\mathbf{z}}_g\boldsymbol{\xi}_g + a_g) \\ &= \mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g, a_g).\end{aligned}$$

b. Mechanically, we can get $\mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g, a_g) = \mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g, a_g)$ from

$$\int_{-\infty}^{\infty} 1[\alpha + \mathbf{x}_g\boldsymbol{\beta} + \mathbf{z}_{gm}\boldsymbol{\gamma} + \eta_g + \bar{\mathbf{z}}_g\boldsymbol{\xi}_g + a_g + \mathrm{u} > 0]\phi(\mathrm{u})d\mathrm{u}$$

where $\phi(\cdot)$ is the standard normal distribution. If we want $\mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g)$ then we integrate out a_g with respect to the Normal$(0, \tau_g^2)$ distribution. Just as in the probit case this the same as computing

$$\mathrm{E}(1[\alpha + \mathbf{x}_g\boldsymbol{\beta} + \mathbf{z}_{gm}\boldsymbol{\gamma} + \eta_g + \bar{\mathbf{z}}_g\boldsymbol{\xi}_g + a_g + u_{gm} > 0]|\mathbf{x}_g, \mathbf{Z}_g)$$

where $(a_g + u_{gm})$ is Normal$(0, 1 + \tau_g^2)$ and independent of $(\mathbf{x}_g, \mathbf{Z}_g)$. Therefore,

$$\mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g) = \Phi\left[\frac{(\alpha + \mathbf{x}_g\boldsymbol{\beta} + \mathbf{z}_{gm}\boldsymbol{\gamma} + \eta_g + \bar{\mathbf{z}}_g\boldsymbol{\xi}_g)}{(1 + \tau_g^2)^{1/2}}\right].$$

Notice that α can just be absorbed into η_g.

c. Under the asymptotic scheme where $G \to \infty$ and the M_g are fixed, there is an upper bound, say M, with $M_g \leq M$ for all g. If we see relatively few group sizes – and lots of data per group size – we can allow the parameters to be different for each M_g, with an appropriate normalization. For example, we can have

$$\mathrm{E}(y_{gm}|\mathbf{x}_g, \mathbf{Z}_g) = \Phi\left[\frac{(\eta_{M_g} + \mathbf{x}_g\boldsymbol{\beta} + \mathbf{z}_{gm}\boldsymbol{\gamma} + \bar{\mathbf{z}}_g\boldsymbol{\xi}_{M_g})}{(1 + \tau_{M_g}^2)^{1/2}}\right].$$

where $\tau_{M_g}^2$ is set to zero for one value, such as $\tau_M^2 = 0$. We can easily estimate all of the parameters using the quasi-log likelihood associated with a "heteroskedastic probit," where we include in the heteroskedasticity function dummy variables for all but one outcome on M_g.

And, of course, we include an intercept and dummy variables in the index as well as $\bar{\mathbf{z}}_g$ and interactions with the group-size dummies.

d. If we use the Bernoulli QMLE with the mean function discussed in part c, we need to be sure that the inference is robust both to the true distribution not being Bernoulli and the within-cluster correlation.

Solutions to Chapter 21 Problems

21.1. a. We use equation (21.5). First, because we have a random sample from the treatment and control groups, $\mathrm{E}(\bar{y}_1) = \mathrm{E}(y|w = 1)$ and $\mathrm{E}(\bar{y}_0) = \mathrm{E}(y|w = 0)$. Therefore, by equation (21.5),

$$\mathrm{E}(\bar{y}_1 - \bar{y}_0) = [\mathrm{E}(y_0|w = 1) - \mathrm{E}(y_0|w = 0)] + \tau_{att}.$$

It follows that the bias term for estimating τ_{att} is given by the first term.

b. If $\mathrm{E}(y_0|w = 1) < \mathrm{E}(y_0|w = 0)$, those who participate in the program would have had lower average earnings without training than those who chose not to participate. This is a form of self-selection, and, on average, leads to an underestimate of the impact of the program.

21.2. Let $k \equiv [w - p(\mathbf{x})]y/\{p(\mathbf{x})[1 - p(\mathbf{x})]\}$. Then we know from equation (21.21) that

$$\mathrm{E}(k|\mathbf{x}) = \mu_1(\mathbf{x}) - \mu_0(\mathbf{x}) = \tau_{ate}(\mathbf{x}).$$

Define a dummy variable as $d \equiv 1[\mathbf{x} \in \mathcal{R}]$. Then, by iterated expectations and the fact that d is a function of $\mathbf{x}$,

$$\begin{aligned}\mathrm{E}(y_1 - y_0|d) &= \mathrm{E}[\mathrm{E}(y_1 - y_0|\mathbf{x}, d)|d] = \mathrm{E}[\mathrm{E}(y_1 - y_0|\mathbf{x})|d] \\ &= \mathrm{E}[\tau_{ate}(\mathbf{x})|d] = \mathrm{E}[\mathrm{E}(k|\mathbf{x})|d] = \mathrm{E}(k|d)\end{aligned}$$

It follows that $\tau_{ate,\mathcal{R}} \equiv \tau_{ate,\mathcal{R}} = \mathrm{E}(y_1 - y_0|\mathbf{x} \in \mathcal{R}) = \mathrm{E}(y_1 - y_0|d = 1) = \mathrm{E}(k|d = 1)$. Now use the simple relationship

$$\mathrm{E}(d \cdot k) = \mathrm{P}(d = 1)\mathrm{E}(k|d = 1)$$

and so

$$\mathrm{E}(k|d = 1) = \frac{\mathrm{E}(d \cdot k)}{\mathrm{P}(d = 1)} = \frac{\mathrm{E}(d \cdot k)}{\mathrm{P}(\mathbf{x} \in \mathcal{R})}.$$

If we know the propensity score, a consistent estimator of $\mathrm{E}(d \cdot k)$ would be

$$N^{-1}\sum_{i=1}^{N} 1[\mathbf{x}_i \in \mathcal{R}]k_i,$$

and a consistent estimator of $P(\mathbf{x} \in \mathcal{R})$ is just the fraction of observations with $\mathbf{x}_i \in \mathcal{R}$, call this $N_{\mathcal{R}}/N$. Combining these two estimators and using the expression for k_i gives

$$\tilde{\tau}_{ate,\mathcal{R}} = N_{\mathcal{R}}^{-1}\sum_{i=1}^{N} 1[\mathbf{x}_i \in \mathcal{R}]k_i,$$

which is simply the average of k_i over the subset of observations with $\mathbf{x}_i \in \mathcal{R}$.

21.3. a. The simple regression estimate is $\hat{\tau}_{ate} = .128$, which means that those participating in the job training program are about $.128$ *more* likely of being unemployed after completing the progam. Further, its heteroskedasticity-robust t statistic is about four. This appears to be a case of self-selection into training: those who would have a higher chance of being unemployed are also more likely to participate in job training.

```
. reg unem78 train, robust

Linear regression                                      Number of obs =    2675
                                                       F(  1,  2673) =   15.90
                                                       Prob > F      =  0.0001
                                                       R-squared     =  0.0098
                                                       Root MSE      =  .32779

------------------------------------------------------------------------------
             |               Robust
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       train |   .1283838   .0321964     3.99   0.000     .0652514    .1915162
       _cons |   .1148594   .0063922    17.97   0.000     .1023252    .1273936
------------------------------------------------------------------------------
```

b. Adding the controls listed in the problem changes the picture considerable. The estimate of τ_{ate} is now $-.199$, so participating in the job training program is estimated to reduce the unemployment probability by about $.20$. The 95% confidence interval for τ_{ate} is $[-.288, -.111]$, which clearly excludes zero.

```
. reg unem78 train age educ black hisp married re74 re75 unem75 unem74, robust

Linear regression                                      Number of obs =    2675
                                                       F( 10,  2664) =   64.36
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.3141
                                                       Root MSE      =  .27327

------------------------------------------------------------------------------
             |               Robust
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       train |  -.1993525    .045185    -4.41   0.000    -.2879538   -.1107512
         age |   .0028579   .0006397     4.47   0.000     .0016036    .0041123
        educ |   .0002969   .0020983     0.14   0.887    -.0038176    .0044114
       black |  -.0179975   .0122695    -1.47   0.143    -.0420563    .0060613
        hisp |  -.0625543   .0250947    -2.49   0.013    -.1117613   -.0133474
     married |  -.0136721   .0173229    -0.79   0.430    -.0476399    .0202957
        re74 |   .0008451    .001004     0.84   0.400    -.0011236    .0028138
        re75 |  -.0042097   .0010084    -4.17   0.000     -.006187   -.0022325
      unem75 |   .2994134   .0395227     7.58   0.000     .2219151    .3769118
      unem74 |   .2385391   .0419072     5.69   0.000     .1563652    .3207131
       _cons |   .0433446   .0358278     1.21   0.226    -.0269085    .1135978
------------------------------------------------------------------------------
```

c. After running the regressions for the untrained and trained groups separately, we obtain a fitted value (fitted probability) in each state for all 2,675 men in the sample. For each *i* we estimate the treatment effect conditional on **x** as

$$\hat{\tau}(\mathbf{x}_i) = (\hat{\alpha}_1 + \mathbf{x}_i\hat{\boldsymbol{\beta}}_1) - (\hat{\alpha}_0 + \mathbf{x}_i\hat{\boldsymbol{\beta}}_0).$$

Then

$$\hat{\tau}_{ate} = N^{-1}\sum_{i=1}^{N}\hat{\tau}(\mathbf{x}_i)$$

$$\hat{\tau}_{att} = N_1^{-1}\sum_{i=1}^{N} train_i \cdot \hat{\tau}(\mathbf{x}_i)$$

We get $\hat{\tau}_{ate} = -.203$, which is very close to the estimate when we assume $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_0$. The estimate of τ_{att} is somewhat larger in magnitude: $\hat{\tau}_{att} = -.270$.

```
. reg unem78 age educ black hisp married re74 re75 unem75 unem74 if ~train

      Source |       SS       df       MS              Number of obs =    2490
-------------+------------------------------           F(  9,  2480) =  180.15
       Model |  100.075847     9  11.1195386            Prob > F      =  0.0000
```

```
    Residual |  153.074354  2480   .06172353           R-squared     =  0.3953
-------------+------------------------------           Adj R-squared =  0.3931
       Total |  253.150201  2489  .101707594           Root MSE      =  .24844

------------------------------------------------------------------------------
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0021732   .0005579     3.90   0.000     .0010791    .0032673
        educ |  -.0014064   .0019407    -0.72   0.469     -.005212    .0023992
       black |  -.0173876   .0125968    -1.38   0.168    -.0420889    .0073136
        hisp |  -.0517355   .0285084    -1.81   0.070    -.1076382    .0041672
     married |  -.0149914   .0151672    -0.99   0.323    -.0447332    .0147503
        re74 |   .0014736   .0007966     1.85   0.064    -.0000884    .0030356
        re75 |  -.0035097   .0007814    -4.49   0.000    -.0050419   -.0019774
      unem75 |   .3435381   .0257242    13.35   0.000      .293095    .3939813
      unem74 |   .3363692   .0275345    12.22   0.000     .2823763    .3903622
       _cons |   .0500675   .0349642     1.43   0.152    -.0184946    .1186296
------------------------------------------------------------------------------

. predict unem78_0
(option xb assumed; fitted values)

. reg unem78 age educ black hisp married re74 re75 unem75 unem74 if train

      Source |       SS       df       MS              Number of obs =     185
-------------+------------------------------           F(  9,   175) =    1.68
       Model |  2.71236085     9  .301373428           Prob > F      =  0.0962
    Residual |  31.3416932   175   .17909539           R-squared     =  0.0796
-------------+------------------------------           Adj R-squared =  0.0323
       Total |  34.0540541   184  .185076381           Root MSE      =   .4232

------------------------------------------------------------------------------
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.0022981   .0046702    -0.49   0.623    -.0115153    .0069192
        educ |   -.008484   .0158595    -0.53   0.593    -.0397845    .0228166
       black |   .1374346   .1067107     1.29   0.199     -.073171    .3480401
        hisp |  -.1412636   .1655747    -0.85   0.395     -.468044    .1855168
     married |  -.0761776   .0855254    -0.89   0.374    -.2449717    .0926165
        re74 |  -.0019756   .0098056    -0.20   0.841    -.0213281     .017377
        re75 |   -.010362    .014196    -0.73   0.466    -.0383794    .0176553
      unem75 |   .1822138   .1020566     1.79   0.076    -.0192063    .3836338
      unem74 |   -.233911   .1194775    -1.96   0.052    -.4697132    .0018912
       _cons |   .3735869   .2407415     1.55   0.123    -.1015435    .8487174
------------------------------------------------------------------------------

. predict unem78_1
(option xb assumed; fitted values)

. gen te = unem78_1 - unem78_0

. sum te

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          te |      2675   -.2031515    .2448774    -1.5703   .3241221

. sum te if train
```

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          te |       185   -.2698234     .309953  -.7017545   .3241221
```

d. Using the subsample of men who were unemployed in 1974, 1975, or both gives $\hat{\tau}_{ate} = -.625$ and $\hat{\tau}_{att} = -.194$. The estimate of τ_{ate} is much larger in magnitude than on the full sample and $\hat{\tau}_{att}$ is somewhat smaller.

```
. keep if unem74 | unem75
(2240 observations deleted)

. reg unem78 age educ black hisp married re74 re75 if ~train

      Source |       SS       df       MS              Number of obs =     302
-------------+------------------------------           F(  7,   294) =   12.93
       Model |  17.2414134     7  2.46305906           Prob > F      =  0.0000
    Residual |   56.020176   294  .190544816           R-squared     =  0.2353
-------------+------------------------------           Adj R-squared =  0.2171
       Total |  73.2615894   301  .243393985           Root MSE      =  .43651

------------------------------------------------------------------------------
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0121428   .0025998     4.67   0.000     .0070262    .0172594
        educ |  -.0000954   .0090746    -0.01   0.992    -.0179548     .017764
       black |  -.0713435   .0713164    -1.00   0.318    -.2116989    .0690119
        hisp |  -.1965901   .1220144    -1.61   0.108    -.4367224    .0435422
     married |   .0610631    .075997     0.80   0.422     -.088504    .2106302
        re74 |  -.0094196   .0029031    -3.24   0.001    -.0151331   -.0037061
        re75 |  -.0190763   .0029208    -6.53   0.000    -.0248247    -.013328
       _cons |   .1819096   .1810088     1.00   0.316    -.1743278    .5381469
------------------------------------------------------------------------------

. predict unem78_0
(option xb assumed; fitted values)

. reg unem78 age educ black hisp married re74 re75 if train

      Source |       SS       df       MS              Number of obs =     133
-------------+------------------------------           F(  7,   125) =    1.90
       Model |  2.33329022     7  .333327175           Prob > F      =  0.0754
    Residual |  21.9674617   125  .175739693           R-squared     =  0.0960
-------------+------------------------------           Adj R-squared =  0.0454
       Total |  24.3007519   132  .184096605           Root MSE      =  .41921

------------------------------------------------------------------------------
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.0058054   .0049952    -1.16   0.247    -.0156914    .0040807
        educ |  -.0267626   .0175847    -1.52   0.131    -.0615649    .0080397
       black |   .1754782   .1201604     1.46   0.147    -.0623342    .4132906
        hisp |  -.1106474   .2078183    -0.53   0.595    -.5219455    .3006508
     married |  -.1606594   .1015391    -1.58   0.116    -.3616179     .040299
        re74 |  -.0150277    .066169    -0.23   0.821    -.1459844    .1159289
        re75 |  -.0269891   .0282243    -0.96   0.341    -.0828484    .0288702
```

```
       _cons |   .5632464    .253897     2.22   0.028     .0607527     1.06574
------------------------------------------------------------------------------

. predict unem78_1
(option xb assumed; fitted values)

. gen te = unem78_1 - unem78_0

. sum te

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          te |       435    -.625014    .3867973   -1.62662   .1450891

. sum te if train

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          te |       133   -.1935882    .2039181  -.7526801   .1450891
```

e. We use the entire set of data for this part. The logit model for *train* is estimated below. Of the 2,675 observations, 78 failures are completely determined. This means that the overlap assumption fails because for some values of **x** the probability of being in the training group is zero. If we are interested in the ATE then our only recourse is to redefine the population so that each unit has a nonzero chance of being in the treated group (and a nonzero chance of being in the control group, which is not a problem in this example).

```
. logit train age educ black hisp married re74 re75 unem74 unem75

Logistic regression                               Number of obs   =       2675
                                                  LR chi2(9)      =     926.52
                                                  Prob > chi2     =     0.0000
Log likelihood = -209.38931                       Pseudo R2       =     0.6887

------------------------------------------------------------------------------
       train |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.1109206   .0177106    -6.26   0.000    -.1456327   -.0762084
        educ |  -.1008807   .0561133    -1.80   0.072    -.2108608    .0090994
       black |   2.650097   .3605668     7.35   0.000     1.943399    3.356795
        hisp |   2.247747   .5908963     3.80   0.000     1.089611    3.405882
     married |  -1.560628   .2817913    -5.54   0.000    -2.112928   -1.008327
        re74 |   .0201797   .0313149     0.64   0.519    -.0411963    .0815557
        re75 |  -.2743162   .0477066    -5.75   0.000    -.3678194   -.1808129
      unem74 |   3.272456   .4887585     6.70   0.000     2.314507    4.230405
      unem75 |  -1.371405   .4545789    -3.02   0.003    -2.262363   -.4804465
       _cons |   1.794543    .979261     1.83   0.067    -.1247735    3.713859
------------------------------------------------------------------------------
Note: 78 failures and 0 successes completely determined.
```

f. The State session is below. The IPW estimate is $\hat{\tau}_{ate,psw} = -.132$. The standard error that adjusts for the first-step estimation is about .0504. If we do not take advantage of the smaller asymptotic variance due to estimating the propensity score, the standard error is .0580, which is about 15% larger. The estimate of τ_{att} is similar, about −.124.

If we assume a constant treatment effect in using regression adjustment, $\hat{\tau}_{ate,reg} = \hat{\tau}_{att,reg} = -.235$ and its standard error is .0509. Interestingly, this is very close to the standard error for $\hat{\tau}_{ate,psw}$, but the estimate is much larger in magnitude, leading to a large *t* statistic. Unfortunately, it appears separate regression are warranted, and this changes $\hat{\tau}_{ate,reg}$ to −.119 (although $\hat{\tau}_{att,reg} = -.294$). The standard error for $\hat{\tau}_{ate,reg}$ that does not even account for the randomness in the sample averages is quite large, .0911, and so $\hat{\tau}_{ate,reg}$ is barely statistically different from zero at the 10% level if we use a one-sided alternative. The IPW estimator appears to be more efficient for this application. (It could have something to do with using linear regression adjustment rather than, say, probit or logit.) The joint test of the interaction terms shows separate regressions are warranted.

```
. keep if avgre <= 15
(1513 observations deleted)

. logit train age educ black hisp married re74 re75 unem74 unem75

Logistic regression                                     Number of obs   =       1162
                                                        LR chi2(9)      =     641.37
                                                        Prob > chi2     =     0.0000
Log likelihood = -180.28028                             Pseudo R2       =     0.6401

------------------------------------------------------------------------------
       train |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.1155512   .0187215    -6.17   0.000    -.1522447   -.0788577
        educ |  -.1049275   .0591078    -1.78   0.076    -.2207766    .0109217
       black |   2.608068   .3772016     6.91   0.000     1.868767     3.34737
        hisp |   2.395905   .6292337     3.81   0.000     1.162629     3.62918
     married |  -1.631159   .3038189    -5.37   0.000    -2.226633   -1.035685
        re74 |  -.0290672     .04281    -0.68   0.497    -.1129732    .0548387
        re75 |  -.3794923   .0682029    -5.56   0.000    -.5131676    -.245817
      unem74 |   3.009282   .5221746     5.76   0.000     1.985839    4.032726
```

```
      unem75 |  -1.751808   .4995608    -3.51   0.000    -2.730929   -.7726867
       _cons |   2.695208   1.053604     2.56   0.011     .6301819    4.760234
------------------------------------------------------------------------------

. predict phat
(option pr assumed; Pr(train))

. tab train

   =1 if in |
        job |
   training |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |        982       84.51       84.51
          1 |        180       15.49      100.00
------------+-----------------------------------
      Total |      1,162      100.00

. sum train

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       train |      1162    .1549053    .3619702          0          1

. gen rhohat = r(mean)

. gen kate = ((train - phat)*unem78)/(phat*(1 - phat))

. gen katt = ((train - phat)*unem78)/(rhohat*(1 - phat))

. sum kate katt

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        kate |      1162   -.1319506    1.977683  -16.62496   56.51032
        katt |      1162   -.1243131    4.922131  -100.8678   6.455555

. * Get the correct standard error for the ATE estimate.

. gen uh = train - phat

. gen ageuh = age*uh

. gen  educuh = educ*uh

. gen  blackuh = black*uh

. gen  hispuh = hisp*uh

. gen   marrieduh = married*uh

. gen  re74uh = re74*uh

. gen re75uh = re75*uh

. gen unem74uh = unem74*uh

. gen unem75uh = unem75*uh
```

```
. reg kate uh ageuh educuh blackuh hispuh marrieduh re74uh re75uh unem74uh
      unem75uh

      Source |       SS       df       MS              Number of obs =    1162
-------------+------------------------------           F( 10,  1151) =   38.51
       Model |  1138.33705    10  113.833705            Prob > F      =  0.0000
    Residual |  3402.59957  1151  2.95621161            R-squared     =  0.2507
-------------+------------------------------           Adj R-squared =  0.2442
       Total |  4540.93661  1161  3.91122878            Root MSE      =  1.7194

------------------------------------------------------------------------------
        kate |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          uh |   3.525428   1.973887     1.79   0.074    -.3473914    7.398247
       ageuh |   .0016821   .0350983     0.05   0.962    -.0671816    .0705458
      educuh |   .2945194   .1191697     2.47   0.014     .0607052    .5283336
     blackuh |  -3.176048   .6611273    -4.80   0.000    -4.473198   -1.878898
      hispuh |  -5.475508   1.012662    -5.41   0.000    -7.462378   -3.488638
   marrieduh |   4.005544   .5475872     7.31   0.000     2.931163    5.079926
      re74uh |   .3468368    .075946     4.57   0.000     .1978287     .495845
      re75uh |  -.8364872   .1060216    -7.89   0.000    -1.044504     -.62847
     unem74uh |  -2.607257    .818097    -3.19   0.001    -4.212386   -1.002129
     unem75uh |   .2278527    .796608     0.29   0.775    -1.335114    1.790819
        _cons |  -.1319506   .0504388    -2.62   0.009    -.2309129   -.0329883
------------------------------------------------------------------------------

. di e(rmse)/sqrt(e(N))
.05043879

. di -.1320/.0504
-2.6190476

. reg kate

      Source |       SS       df       MS              Number of obs =    1162
-------------+------------------------------           F(  0,  1161) =    0.00
       Model |           0     0           .            Prob > F      =
    Residual |  4540.93661  1161  3.91122878            R-squared     =  0.0000
-------------+------------------------------           Adj R-squared =  0.0000
       Total |  4540.93661  1161  3.91122878            Root MSE      =  1.9777

------------------------------------------------------------------------------
        kate |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |  -.1319506   .0580168    -2.27   0.023      -.24578   -.0181211
------------------------------------------------------------------------------

. reg unem78 train, robust

Linear regression                                       Number of obs =    1162
                                                        F(  1,  1160) =    0.18
                                                        Prob > F      =  0.6734
                                                        R-squared     =  0.0002
                                                        Root MSE      =   .4259

------------------------------------------------------------------------------
             |               Robust
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
```

```
       train |   .0147658   .0350282     0.42   0.673    -.0539599    .0834915
       _cons |   .2352342   .0135467    17.36   0.000     .2086555    .2618129
------------------------------------------------------------------------------

. reg unem78 train age educ black hisp married re74 re75 unem74 unem75, robust

Linear regression                                      Number of obs =    1162
                                                       F( 10,  1151) =   61.27
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.3312
                                                       Root MSE      =  .34968

------------------------------------------------------------------------------
             |               Robust
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       train |  -.2349689   .0509218    -4.61   0.000    -.3348787    -.135059
         age |   .0059358   .0012367     4.80   0.000     .0035094    .0083622
        educ |   .0022623   .0042076     0.54   0.591     -.005993    .0105177
       black |  -.0202408    .022745    -0.89   0.374    -.0648671    .0243855
        hisp |   -.100478   .0399462    -2.52   0.012    -.1788536   -.0221024
     married |  -.0352163   .0272463    -1.29   0.196    -.0886743    .0182417
        re74 |  -.0010355    .002876    -0.36   0.719    -.0066783    .0046073
        re75 |  -.0177354   .0024155    -7.34   0.000    -.0224746   -.0129961
      unem74 |   .2220472    .051956     4.27   0.000     .1201081    .3239863
      unem75 |   .1439644    .048573     2.96   0.003     .0486629    .2392658
       _cons |   .1103197   .0759773     1.45   0.147    -.0387499    .2593893
------------------------------------------------------------------------------

. reg unem78 age educ black hisp married re74 re75 unem74 unem75 if ~train

      Source |       SS       df       MS              Number of obs =     982
-------------+------------------------------           F(  9,   972) =   86.39
       Model |   78.510332     9  8.72337022           Prob > F      =  0.0000
    Residual |  98.1505642   972  .100977947           R-squared     =  0.4444
-------------+------------------------------           Adj R-squared =  0.4393
       Total |  176.660896   981  .180082463           Root MSE      =  .31777

------------------------------------------------------------------------------
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0050777   .0011449     4.43   0.000     .0028309    .0073245
        educ |  -.0002579   .0039421    -0.07   0.948    -.0079938    .0074781
       black |  -.0146538   .0238818    -0.61   0.540    -.0615196    .0322119
        hisp |  -.0862098   .0524883    -1.64   0.101    -.1892132    .0167936
     married |  -.0424904   .0262258    -1.62   0.106     -.093956    .0089752
        re74 |   .0022784   .0024006     0.95   0.343    -.0024325    .0069892
        re75 |  -.0143134   .0025479    -5.62   0.000    -.0193134   -.0093133
      unem74 |   .3521536   .0435278     8.09   0.000     .2667344    .4375729
      unem75 |   .1965244   .0423339     4.64   0.000     .1134481    .2796007
       _cons |   .0770668   .0757616     1.02   0.309    -.0716084    .2257419
------------------------------------------------------------------------------

. predict unem78_0
(option xb assumed; fitted values)

. reg unem78 age educ black hisp married re74 re75 unem74 unem75 if train

      Source |       SS       df       MS              Number of obs =     180
```

```
-------------+------------------------------           F(  9,   170) =    1.57
       Model |  2.58861704     9  .287624115           Prob > F      =  0.1281
    Residual |   31.161383   170  .183302253           R-squared     =  0.0767
-------------+------------------------------           Adj R-squared =  0.0278
       Total |       33.75   179  .188547486           Root MSE      =  .42814

------------------------------------------------------------------------------
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         age |   -.002544   .0047571    -0.53   0.593    -.0119347    .0068466
        educ |  -.0086994   .0162153    -0.54   0.592    -.0407086    .0233098
       black |   .1402344    .108018     1.30   0.196     -.072995    .3534638
        hisp |  -.1480334   .1683835    -0.88   0.381    -.4804252    .1843585
     married |  -.0713415   .0879005    -0.81   0.418    -.2448585    .1021756
        re74 |   .0073134   .0145599     0.50   0.616     -.021428    .0360549
        re75 |  -.0064075   .0214837    -0.30   0.766    -.0488166    .0360016
      unem74 |  -.1885821   .1321929    -1.43   0.156    -.4495331    .0723688
      unem75 |   .1935779   .1115475     1.74   0.084    -.0266186    .4137745
       _cons |   .3229791   .2550769     1.27   0.207    -.1805469    .8265051
------------------------------------------------------------------------------

. predict unem78_1
(option xb assumed; fitted values)

. gen te = unem78_1 - unem78_0

. sum te

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          te |      1162   -.1193285    .3326819  -.9173806   .3599507

. sum te if train

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          te |       180   -.2941826    .2835388   -.728443   .2494144

. egen mage = mean(age)
. gen trainage = train*(age - mage)
. egen meduc = mean(educ)
. gen traineduc = train*(educ - meduc)
. egen mblack = mean(black)
. gen trainblack = train*(black - mblack)
. egen mhisp = mean(hisp)
. gen trainhisp = train*(hisp - mhisp)
. egen mmarried = mean(married)
. gen trainmarried = train*(married - mmarried)
. egen mre74 = mean(re74)
. gen trainre74 = train*(re74 - mre74)
. egen mre75 = mean(re75)
. gen trainre75 = train*(re75 - mre75)
. egen munem74 = mean(unem74)
. gen trainunem74 = train*(unem74 - munem74)
. egen munem75 = mean(unem75)
. gen trainunem75 = train*(unem75 - munem75)

. reg unem78 train  age educ black hisp married re74 re75 unem74 unem75
      trainage traineduc trainblack trainhisp trainmarried trainre74
```

```
      trainre75 trainunem74 trainunem75, robust

Linear regression                                      Number of obs =    1162
                                                       F( 19,  1142) =   42.62
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.3855
                                                       Root MSE      =   .3365

------------------------------------------------------------------------------
             |               Robust
      unem78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       train |  -.1193284   .0910893    -1.31   0.190    -.2980497    .0593928
         age |   .0050777   .0012214     4.16   0.000     .0026812    .0074742
        educ |  -.0002579   .0041835    -0.06   0.951     -.008466    .0079503
       black |  -.0146538   .0231254    -0.63   0.526    -.0600269    .0307192
        hisp |  -.0862098   .0424936    -2.03   0.043     -.169584   -.0028356
     married |  -.0424904   .0277233    -1.53   0.126    -.0968847    .0119039
        re74 |   .0022784   .0028885     0.79   0.430     -.003389    .0079457
        re75 |  -.0143134   .0025811    -5.55   0.000    -.0193776   -.0092491
      unem74 |   .3521536   .0566374     6.22   0.000     .2410286    .4632787
      unem75 |   .1965244   .0570442     3.45   0.001     .0846013    .3084475
    trainage |  -.0076217   .0050195    -1.52   0.129    -.0174702    .0022267
   traineduc |  -.0084415   .0152788    -0.55   0.581    -.0384192    .0215361
  trainblack |   .1548883   .0871611     1.78   0.076    -.0161256    .3259022
   trainhisp |  -.0618236   .0975772    -0.63   0.526    -.2532742    .1296271
trainmarried |  -.0288511   .0822415    -0.35   0.726    -.1902126    .1325104
   trainre74 |   .0050351   .0166047     0.30   0.762     -.027544    .0376142
   trainre75 |   .0079059   .0185161     0.43   0.669    -.0284236    .0442353
 trainunem74 |  -.5407358   .1357835    -3.98   0.000     -.807149   -.2743226
 trainunem75 |  -.0029465   .0975097    -0.03   0.976    -.1942647    .1883717
       _cons |   .0770668   .0760186     1.01   0.311    -.0720851    .2262186
------------------------------------------------------------------------------

. test  trainage traineduc trainblack trainhisp trainmarried trainre74
        trainre75 trainunem74 trainunem75

 ( 1)  trainage = 0
 ( 2)  traineduc = 0
 ( 3)  trainblack = 0
 ( 4)  trainhisp = 0
 ( 5)  trainmarried = 0
 ( 6)  trainre74 = 0
 ( 7)  trainre75 = 0
 ( 8)  trainunem74 = 0
 ( 9)  trainunem75 = 0

       F(  9,  1142) =    8.61
            Prob > F =    0.0000
```

21.4. The integral is equivalent to

$$\int_{-(\theta_0+\mathbf{x}\boldsymbol{\theta}_1+\mathbf{z}\boldsymbol{\theta}_2)}^{\infty} a\phi(a)da.$$

Because $d\phi(a)/da = -a\phi(a)$, the antiderivative of $a\phi(a)$ is simply $-\phi(a)$. Now

$$\int_{-(\theta_0+\mathbf{x}\boldsymbol{\theta}_1+\mathbf{z}\boldsymbol{\theta}_2)}^{m} a\phi(a)da = -\phi(a)\Big]_{-(\theta_0+\mathbf{x}\boldsymbol{\theta}_1+\mathbf{z}\boldsymbol{\theta}_2)}^{m} = -\phi(m) + \phi[-(\theta_0 + \mathbf{x}\theta_1 + \mathbf{z}\theta_2)]$$
$$= -\phi(m) + \phi(\theta_0 + \mathbf{x}\theta_1 + \mathbf{z}\theta_2)$$

where we use the symmetry of $\phi(\cdot)$. As $m \to \infty$, $\phi(m) \to 0$. Therefore,

$$\int_{-(\theta_0+\mathbf{x}\boldsymbol{\theta}_1+\mathbf{z}\boldsymbol{\theta}_2)}^{\infty} a\phi(a)da = \phi(\theta_0 + \mathbf{x}\theta_1 + \mathbf{z}\theta_2).$$

21.5. The Stata output to answer all parts follows.

a. The first two Stata commands are used to obtain the probit fitted values, called *PHIhat*.

b. The IV estimate of τ is -43.27 and its standard error is huge, 585.78. Clearly we can learn nothing of value from this estimate.

c. The collinearity suspected in part b is confirmed by regressing $\hat{\Phi}_i$ on the $\mathbf{x}_i$: the R-squared is .9989, which means there is almost no separate variation in $\hat{\Phi}_i$ that cannot be explained by $\mathbf{x}_i$.

d. This example illustrates why trying to achieve identification off of a nonlinearity can be fraught with problems. In cases with larger sample sizes the estimates may seem more reasonable, but we are only able to compute estimates at all because of the presumed functional form for P(w|**x**). A good general rule is that if a linear IV approach does not identify τ then we should not hope to learn anything useful by introducing nonlinearity in P(w|**x**).

```
. probit train age educ black hisp married re74 re75

Probit regression                                 Number of obs   =        445
                                                  LR chi2(7)      =       8.60
                                                  Prob > chi2     =     0.2829
Log likelihood = -297.80166                       Pseudo R2       =     0.0142

------------------------------------------------------------------------------
       train |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0066826   .0087391     0.76   0.444    -.0104458    .0238109
        educ |   .0387341   .0341574     1.13   0.257    -.0282132    .1056815
```

```
       black |  -.2216642   .2242952    -0.99   0.323    -.6612747    .2179463
        hisp |  -.5753033   .3062908    -1.88   0.060    -1.175622    .0250157
     married |   .0900855   .1703412     0.53   0.597    -.2437771    .4239482
        re74 |  -.0138226   .0155792    -0.89   0.375    -.0443572     .016712
        re75 |    .028755   .0267469     1.08   0.282    -.0236679    .0811779
       _cons |  -.5715372    .475416    -1.20   0.229    -1.503335    .3602609
------------------------------------------------------------------------------

. predict PHIhat
(option pr assumed; Pr(train))

. ivreg re78 age educ black hisp married re74 re75 (train = PHIhat)

Instrumental variables (2SLS) regression

      Source |       SS       df       MS              Number of obs =     445
-------------+------------------------------           F(  8,   436) =    0.18
       Model | -213187.422     8 -26648.4277           Prob > F      =  0.9934
    Residual |  232713.078   436  533.745593           R-squared     =
-------------+------------------------------           Adj R-squared =
       Total |  19525.6566   444  43.9767041           Root MSE      =  23.103

------------------------------------------------------------------------------
        re78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       train |  -43.26513   585.7793    -0.07   0.941    -1194.567    1108.037
         age |   .1717735   1.520127     0.11   0.910    -2.815914    3.159461
        educ |   1.067645   8.646843     0.12   0.902    -15.92703    18.06232
       black |  -6.114187   51.56931    -0.12   0.906    -107.4695    95.24116
        hisp |  -9.523185    126.287    -0.08   0.940    -257.7302    238.6838
     married |   1.432202   20.72909     0.07   0.945    -39.30917    42.17357
        re74 |  -.1443703   2.973787    -0.05   0.961     -5.98911     5.70037
        re75 |   .5327602     6.2896     0.08   0.933    -11.82894    12.89447
       _cons |   13.30468    165.517     0.08   0.936    -312.0058    338.6151
------------------------------------------------------------------------------
Instrumented:  train
Instruments:   age educ black hisp married re74 re75 PHIhat
------------------------------------------------------------------------------

. reg PHIhat age educ black hisp married re74 re75

      Source |       SS       df       MS              Number of obs =     445
-------------+------------------------------           F(  7,   437) =55245.15
       Model |  2.04859095     7  .292655851           Prob > F      =  0.0000
    Residual |  .002314965   437  5.2974e-06           R-squared     =  0.9989
-------------+------------------------------           Adj R-squared =  0.9989
       Total |  2.05090592   444  .004619157           Root MSE      =   .0023

------------------------------------------------------------------------------
      PHIhat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
         age |   .0025883   .0000158   163.53   0.000     .0025572    .0026194
        educ |   .0146708    .000062   236.45   0.000     .0145488    .0147927
       black |  -.0875955   .0004094  -213.96   0.000    -.0884002   -.0867909
        hisp |  -.2156441   .0005445  -396.02   0.000    -.2167143   -.2145739
     married |   .0351309   .0003107   113.08   0.000     .0345203    .0357415
        re74 |  -.0051274   .0000271  -189.22   0.000    -.0051807   -.0050742
        re75 |   .0108521   .0000474   228.89   0.000     .0107589    .0109453
       _cons |   .2823687   .0008635   326.99   0.000     .2806715    .2840659
```

21.6. As in Procedure 21.1, the IV estimator is consistent whether or not $G(\mathbf{x},\mathbf{z};\boldsymbol{\gamma})$ is correctly specified for $\mathrm{P}(w = 1|\mathbf{x},\mathbf{z})$. The OLS estimator from y_i on 1, $\hat{G}_i$, $\mathbf{x}_i$, $\hat{G}_i \cdot (\mathbf{x}_i - \bar{\mathbf{x}})$, $i = 1,\ldots,N$ generally requires the model for $\mathrm{P}(w = 1|\mathbf{x},\mathbf{z})$ to be correctly specified. This can be seen by writing

$$\mathrm{E}(y|\mathbf{x},\mathbf{z}) = \gamma + \tau\mathrm{E}(w|\mathbf{x},\mathbf{z}) + \mathbf{x}\boldsymbol{\beta}_0 + \mathrm{E}(w|\mathbf{x},\mathbf{z}) \cdot (\mathbf{x} - \boldsymbol{\psi})\boldsymbol{\delta},$$

which is the estimating equation underlying the OLS regression on probit fitted values and the interactions. If $\mathrm{E}(w|\mathbf{x},\mathbf{z}) = \mathrm{P}(w = 1|\mathbf{x},\mathbf{z}) \neq G(\mathbf{x},\mathbf{z};\boldsymbol{\gamma})$ for all $\boldsymbol{\gamma}$ then plugging in $\hat{G}_i$ generally produces inconsistent estimators.

Even if $G(\mathbf{x},\mathbf{z};\boldsymbol{\gamma})$ is correctly specified, the standard errors for the two-step OLS estimator are harder to obtain. One one must use the material on generated regressors in Chapter 6 or apply the bootstrap.

21.7. a. There are several options. To estimate the mean parameters, we can use Poisson regression (especially in the case where w is a count variable) or gamma regression (if w is nonnegative and continuous). Of course, we can use NLS, too (which is also a QMLE in the LEF).

As stated in the hint, if we define $r = w - \mathrm{E}(w|\mathbf{x})$ then $\mathrm{E}(r^2|\mathbf{x}) = \mathrm{Var}(w|\mathbf{x}) = \exp(\delta_0 + \mathbf{x}\boldsymbol{\delta}_1)$. Therefore, if we observed, r^2, we could use it as the dependent variable in, say, a gamma or negative binomial QMLE. In practice, we use $\hat{r}_i = w_i - \exp(\hat{\gamma}_0 + \mathbf{x}_i\hat{\boldsymbol{\gamma}}_1)$, the residuals from estimating the mean parameters.

b. By the law of large numbers,

$$N^{-1}\sum_{i=1}^{N}\left\{\frac{[w_i - \psi(\mathbf{x}_i)]y_i}{\omega(\mathbf{x}_i)}\right\} \xrightarrow{p} \beta.$$

By the usual argument, we can replace $\psi(\mathbf{x}_i)$ and $\omega(\mathbf{x}_i)$ with consistent estimators; more precisely, in a parametric context replace the unknown parameters with consistent estimators. In the case of exponential mean and variance functions,

$$\hat{\beta} = N^{-1}\sum_{i=1}^{N}\left\{\frac{[w_i - \exp(\hat{\gamma}_0 + \mathbf{x}_i\hat{\boldsymbol{\gamma}}_1)]y_i}{\exp(\hat{\delta}_0 + \mathbf{x}_i\hat{\boldsymbol{\delta}}_1)}\right\} \equiv N^{-1}\sum_{i=1}^{N}\left\{\frac{[w_i - \hat{\psi}(\mathbf{x}_i)]y_i}{\hat{\omega}(\mathbf{x}_i)}\right\}.$$

We can use Problem 12.17 to get a standard error for $\hat{\beta}$ or use the bootstrap.

c. I use Poisson regression to estimate the mean parameters and then gamm regression to estimate the variance parameters. The resulting estimate of β is about .102; the standard error is not reported. If we ignore estimation of the parameters in $\mathrm{E}(w|\mathbf{x})$ and $\mathrm{Var}(w|\mathbf{x})$ then the standard error is about .050.

There is not much reason to compute a standard error for $\hat{\beta}$ because the standard regression adjustment estimate, $\tilde{\beta}$, is very close, and provides a valid standard error. Namely, running the regression

$$re78_i \text{ on } 1, mostrn_i, age_i, educ_i, black_i, hisp_i, married_i, re74_i, re75$$

gives $\tilde{\beta} = .103$ ($se = .038$). With random assignment to the job training program it is perhaps not too surprising to see the methods give similar estimates. In fact, the simple regression estimate is .112 ($se = .038$).

```
. glm mostrn age educ black hisp married re74 re75, fam(poiss) link(log)
      robust

Generalized linear models                         No. of obs      =       445
Optimization     : ML                             Residual df     =       437
                                                  Scale parameter =
Deviance         =  6136.777616                   (1/df) Deviance =  14.04297
Pearson          =  5296.290311                   (1/df) Pearson  =  12.11966

Variance function: V(u) = u                       [Poisson]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             =  15.78862
Log pseudolikelihood = -3504.968642               BIC             =  3471.919
```

```
------------------------------------------------------------------------------
             |               Robust
      mostrn |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0037486   .0081575     0.46   0.646    -.0122398    .0197369
        educ |   .0448349   .0344975     1.30   0.194    -.0227789    .1124487
       black |  -.1809126   .1906097    -0.95   0.343    -.5545006    .1926755
        hisp |  -.4907343   .3198765    -1.53   0.125    -1.117681    .1362121
     married |   .0824876   .1620227     0.51   0.611    -.2350709    .4000462
        re74 |  -.0048997   .0140984    -0.35   0.728    -.0325322    .0227327
        re75 |   .0388417   .0197161     1.97   0.049     .0001989    .0774846
       _cons |   1.605453   .4551975     3.53   0.000     .7132821    2.497624
------------------------------------------------------------------------------

. predict mostrnh
(option mu assumed; predicted mean mostrn)

. gen rh = mostrn - mostrnh

. gen rhsq = rh^2

. glm rhsq age educ black hisp married re74 re75, fam(gamma) link(log) robust

Generalized linear models                         No. of obs      =       445
Optimization     : ML                             Residual df     =       437
                                                  Scale parameter =  .6888918
Deviance         =  251.4433046                   (1/df) Deviance =  .5753851
Pearson          =  301.0457257                   (1/df) Pearson  =  .6888918

Variance function: V(u) = u^2                     [Gamma]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             =  11.01458
Log pseudolikelihood = -2442.743803               BIC             = -2413.415

------------------------------------------------------------------------------
             |               Robust
        rhsq |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.0020813   .0054739    -0.38   0.704      -.01281    .0086475
        educ |   .0206816   .0254745     0.81   0.417    -.0292474    .0706107
       black |  -.0424931   .1083251    -0.39   0.695    -.2548063    .1698202
        hisp |  -.2107907   .2173269    -0.97   0.332    -.6367437    .2151623
     married |   .0391702   .0937328     0.42   0.676    -.1445427    .2228831
        re74 |   .0094508   .0107463     0.88   0.379    -.0116116    .0305132
        re75 |    .051212   .0190328     2.69   0.007     .0139084    .0885156
       _cons |   4.288161   .3201553    13.39   0.000     3.660668    4.915654
------------------------------------------------------------------------------

. predict omegah
(option mu assumed; predicted mean rhsq)

. sum omegah

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      omegah |       445    91.62743    28.73968    60.9556   369.0591
```

```
. gen kh = ( mostrn -  mostrnh)*re78/omegah

. sum kh

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          kh |       445    .1024405    1.047671  -3.030556   10.28323

. reg kh

      Source |       SS       df       MS              Number of obs =     445
-------------+------------------------------           F(  0,   444) =    0.00
       Model |           0     0           .           Prob > F      =
    Residual |   487.34086   444  1.09761455           R-squared     =  0.0000
-------------+------------------------------           Adj R-squared =  0.0000
       Total |   487.34086   444  1.09761455           Root MSE      =  1.0477

------------------------------------------------------------------------------
          kh |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |   .1024405   .0496644     2.06   0.040     .0048341     .200047
------------------------------------------------------------------------------

. reg re78 mostrn age educ black hisp married re74 re75, robust

Linear regression                                       Number of obs =     445
                                                        F(  8,   436) =    3.09
                                                        Prob > F      =  0.0021
                                                        R-squared     =  0.0613
                                                        Root MSE      =  6.4838

------------------------------------------------------------------------------
             |               Robust
        re78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      mostrn |    .102825   .0380686     2.70   0.007     .0280043    .1776458
         age |   .0570883   .0399249     1.43   0.153    -.0213808    .1355575
        educ |   .3980183   .1548109     2.57   0.010       .09375    .7022867
       black |  -2.150926   1.007271    -2.14   0.033    -4.130637   -.1712163
        hisp |   .1712523   1.365153     0.13   0.900    -2.511846     2.85435
     married |   -.154993   .8733899    -0.18   0.859    -1.871571    1.561585
        re74 |   .0788359   .1071444     0.74   0.462    -.1317478    .2894197
        re75 |   .0305561   .1266573     0.24   0.809    -.2183787    .2794909
       _cons |   .6004532   2.366495     0.25   0.800    -4.050703     5.25161
------------------------------------------------------------------------------

. reg re78 mostrn, robust

Linear regression                                       Number of obs =     445
                                                        F(  1,   443) =    8.66
                                                        Prob > F      =  0.0034
                                                        R-squared     =  0.0269
                                                        Root MSE      =  6.5491

------------------------------------------------------------------------------
             |               Robust
        re78 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      mostrn |   .1126397   .0382802     2.94   0.003     .0374063    .1878731
```

```
       _cons |   4.434831   .3358041    13.21   0.000     3.774864    5.094798
------------------------------------------------------------------------------
```

d. Because $E(w|\mathbf{x})$ follows a logistic regression model we can use fractional logit (that is, maximize the Bernoulli QMLE). After we have estimated the mean parameters γ_0 and $\boldsymbol{\gamma}_1$, we form the fitted values and residuals

$$\hat{w}_i = \Lambda(\hat{\gamma}_0 + \mathbf{x}_i\hat{\boldsymbol{\gamma}}_1)$$
$$\hat{r}_i = w_i - \hat{w}_i$$

and then estimate δ_0, δ_1, and δ_2 from the OLS regression

$$\hat{r}_i^2 \text{ on } 1,\ \hat{w}_i,\ \hat{w}_i^2$$

to get the variance estimates

$$\hat{\omega}_i = \hat{\delta}_0 + \hat{\delta}_1\hat{w}_i + \hat{\delta}_2\hat{w}_i^2.$$

Because the $\hat{\omega}_i$ are fitted values from a linear regression, nothing guarantees $\hat{\omega}_i > 0$ for all i, something we need for the method in part b to make sense. To avoid this problem, we might use $\text{Var}(w|\mathbf{x}) = \exp\{\delta_0 + \delta_1 E(w|\mathbf{x}) + \delta_2[E(w|\mathbf{x})]^2\}$ instead, and use the gamma QMLE with the squared residuals as the dependent variable.

e. The Stata code carries out the procedure from part d, except that, because 13 estimated variances were not positive, the exponential variance function was used instead, with a gamma QMLE. below produces the estimate $\hat{\beta} = .689$. The regression coefficient is not too different: $\tilde{\beta} = .644$ ($se = .235$)

```
. use attend
. gen ACTsq = ACT^2
. gen ACTcu = ACT^3
. gen priGPAsq = priGPA^2
. gen priGPAcu = priGPA^3
. sum atndrte
```

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     atndrte |       680    81.70956    17.04699       6.25        100

. replace atndrte = atndrte/100
(680 real changes made)

. glm atndrte  priGPA priGPAsq priGPAcu ACT ACTsq ACTcu frosh soph, fam(bin)
      link(logit) robust
note: atndrte has noninteger values

Generalized linear models                         No. of obs      =       680
Optimization     : ML                             Residual df     =       671
                                                  Scale parameter =
Deviance         =   87.0709545                   (1/df) Deviance =   .129763
Pearson          =  85.07495268                   (1/df) Pearson  =  .1267883

Variance function: V(u) = u*(1-u/1)               [Binomial]
Link function    : g(u) = ln(u/(1-u))             [Logit]

                                                  AIC             =  .6831657
Log pseudolikelihood = -223.2763498               BIC             = -4289.253

------------------------------------------------------------------------------
             |               Robust
     atndrte |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      priGPA |  -3.371154   2.195517    -1.54   0.125     -7.67429    .9319806
    priGPAsq |   1.886443   .8972586     2.10   0.036     .1278489    3.645038
    priGPAcu |  -.2454989    .118004    -2.08   0.037    -.4767825   -.0142153
         ACT |   .5538998   .6744028     0.82   0.411    -.7679054    1.875705
       ACTsq |  -.0280986   .0304868    -0.92   0.357    -.0878516    .0316544
       ACTcu |   .0003858   .0004505     0.86   0.392     -.000497    .0012687
       frosh |   .3939498   .1155299     3.41   0.001     .1675154    .6203841
        soph |   .0941678   .1006569     0.94   0.350    -.1031161    .2914517
       _cons |  -.7731446    5.13392    -0.15   0.880    -10.83544    9.289154
------------------------------------------------------------------------------

. predict atndrteh
(option mu assumed; predicted mean atndrte)

. gen rh = atndrte - atndrteh

. gen rhsq = rh^2

. gen atndrtehsq = atndrteh^2

. reg rhsq atndrteh atndrtehsq

      Source |       SS       df       MS              Number of obs =     680
-------------+------------------------------           F(  2,   677) =   37.14
       Model |  .098172929     2  .049086465           Prob > F      =  0.0000
    Residual |  .894850267   677  .001321788           R-squared     =  0.0989
-------------+------------------------------           Adj R-squared =  0.0962
       Total |  .993023196   679  .001462479           Root MSE      =  .03636

------------------------------------------------------------------------------
       rhsq |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
```

```
-------------+----------------------------------------------------------------
    atndrteh |   .1604177   .1514883     1.06   0.290    -.1370257     .457861
  atndrtehsq |  -.1854786   .0994129    -1.87   0.063    -.3806733    .0097161
       _cons |   .0137235   .0571515     0.24   0.810    -.0984919    .1259389
------------------------------------------------------------------------------

. predict omegah
(option xb assumed; fitted values)

. sum omegah

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
      omegah |       680    .0192213    .0120243  -.0049489   .0483992

. count if omegah < 0
   13

. drop omegah

. glm rhsq atndrteh atndrtehsq, fam(gamma) link(log)

Generalized linear models                         No. of obs      =       680
Optimization     : ML                             Residual df     =       677
                                                  Scale parameter =  3.781574
Deviance         =   1657.02942                   (1/df) Deviance =  2.447606
Pearson          =  2560.125871                   (1/df) Pearson  =  3.781574

Variance function: V(u) = u^2                     [Gamma]
Link function    : g(u) = ln(u)                   [Log]

                                                  AIC             = -6.298906
Log likelihood   =  2144.628097                   BIC             = -2758.427

------------------------------------------------------------------------------
             |                 OIM
        rhsq |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
    atndrteh |   21.19375   8.816922     2.40   0.016     3.912901     38.4746
  atndrtehsq |  -17.96346   5.764534    -3.12   0.002    -29.26174   -6.665185
       _cons |  -9.308977    3.33455    -2.79   0.005    -15.84458    -2.77338
------------------------------------------------------------------------------

. predict omegah
(option mu assumed; predicted mean rhsq)

. gen kh = rh*stndfnl/omegah

. sum kh

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
          kh |       680    .6890428    8.318917  -47.34537   41.90279

. reg stndfnl atndrte  priGPA priGPAsq priGPAcu ACT ACTsq ACTcu frosh soph,
      robust

Linear regression                                 Number of obs =     680
                                                  F(  9,   670) =   31.01
```

```
                                                Prob > F      =  0.0000
                                                R-squared     =  0.2356
                                                Root MSE      =   .8709

------------------------------------------------------------------------------
             |               Robust
     stndfnl |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     atndrte |   .6444118   .2345274     2.75   0.006     .1839147    1.104909
      priGPA |   1.987666   2.651676     0.75   0.454     -3.21893    7.194262
    priGPAsq |  -1.055604    1.01697    -1.04   0.300    -3.052436    .9412284
    priGPAcu |   .1842414   .1262839     1.46   0.145    -.0637183    .4322011
         ACT |   .3059699   .7236971     0.42   0.673    -1.115017    1.726957
       ACTsq |  -.0141693   .0319499    -0.44   0.658    -.0769033    .0485648
       ACTcu |   .0002633   .0004629     0.57   0.570    -.0006456    .0011722
       frosh |  -.1138172   .1035799    -1.10   0.272    -.3171976    .0895631
        soph |  -.1863224   .0870459    -2.14   0.033    -.3572381   -.0154067
       _cons |  -4.501184   5.629291    -0.80   0.424    -15.55436     6.55199
------------------------------------------------------------------------------
```

21.8. a. From (21.129) and (21.130), we are assuming

$$a = \gamma_0 + \mathbf{x}\boldsymbol{\gamma} + u$$
$$\mathrm{E}(u|\mathbf{x}, \mathbf{z}) = 0$$

and so we can write

$$y = \gamma_0 + \beta w + \mathbf{x}\boldsymbol{\gamma} + u + e$$
$$\equiv \gamma_0 + \beta w + \mathbf{x}\boldsymbol{\gamma} + r$$

where $\mathrm{E}(r|\mathbf{x}, \mathbf{z}) = \mathrm{E}(u|\mathbf{x}, \mathbf{z}) + \mathrm{E}(e|\mathbf{x}, \mathbf{z}) = 0$. Therefore, we need to add the usual rank condition: $\mathbf{z}$ must appear with nonzero coefficient vector in the linear projection of w on $(1, \mathbf{x}, \mathbf{z})$. More precisely, if

$$\mathrm{L}(w|1, \mathbf{x}, \mathbf{z}) = \xi_0 + \mathbf{x}\boldsymbol{\xi}_1 + \mathbf{z}\boldsymbol{\xi}_2$$

then $\boldsymbol{\xi}_2 \neq \mathbf{0}$.

b. If

$$w = \max(0, \pi_0 + \mathbf{x}\boldsymbol{\pi}_1 + \mathbf{z}\boldsymbol{\pi}_2 + v),$$
$$\mathrm{D}(v|\mathbf{x}, \mathbf{z}) \sim \mathrm{Normal}(0, \eta^2),$$

then

$$\mathrm{E}(w|\mathbf{x},\mathbf{z}) = \Phi(\mathbf{q}\boldsymbol{\pi}/\eta) \cdot \mathbf{q}\boldsymbol{\pi} + \eta \cdot \phi(\mathbf{q}\boldsymbol{\pi}/\eta),$$

where $\mathbf{q}\boldsymbol{\pi} \equiv \pi_0 + \mathbf{x}\boldsymbol{\pi}_1 + \mathbf{z}\boldsymbol{\pi}_2$ [see equation (17.14)]. Because $\mathrm{E}(w|\mathbf{x},\mathbf{z})$ is a function of $(\mathbf{x},\mathbf{z})$ and we have

$$\begin{aligned} y &= \gamma_0 + \beta w + \mathbf{x}\boldsymbol{\gamma} + r \\ \mathrm{E}(r|\mathbf{x},\mathbf{z}) &= 0, \end{aligned}$$

we can use $\Phi(\mathbf{q}\boldsymbol{\pi}/\eta) \cdot \mathbf{q}\boldsymbol{\pi} + \eta \cdot \phi(\mathbf{q}\boldsymbol{\pi}/\eta)$ as a valid instrument for w. (Remember, any function of $(\mathbf{x},\mathbf{z})$ is uncorrelated with r provided the second moments exist.) Because we do not know $\boldsymbol{\pi}$ or η, we replace them with estimators. In other words, use $\Phi(\mathbf{q}_i\hat{\boldsymbol{\pi}}/\hat{\eta}) \cdot \mathbf{q}_i\hat{\boldsymbol{\pi}} + \hat{\eta} \cdot \phi(\mathbf{q}_i\hat{\boldsymbol{\pi}}/\hat{\eta})$ as the IV for w_i, where $\hat{\boldsymbol{\pi}}$ and $\hat{\eta}$ are the estimates from an initial Tobit MLE.

c. By equation (14.57), the optimal IV for w is

$$\mathrm{E}(w|\mathbf{x},\mathbf{z})/\mathrm{Var}(r|\mathbf{x},\mathbf{z}).$$

But $\mathrm{E}(e|a,\mathbf{x},\mathbf{z}) = 0$ implies that e and a are uncorrelated, conditional on $(\mathbf{x},\mathbf{z})$. Therefore,

$$\begin{aligned} \mathrm{Var}(u + e|\mathbf{x},\mathbf{z}) &= \mathrm{Var}(u|\mathbf{x},\mathbf{z}) + \mathrm{Var}(e|\mathbf{x},\mathbf{z}) \\ &= \sigma_a^2 + \sigma_e^2 \equiv \sigma_r^2. \end{aligned}$$

Therefore, $\mathrm{Var}(r|\mathbf{x},\mathbf{z})$ is constant, and $\mathrm{E}(w|\mathbf{x},\mathbf{z})$ can serve as the optimal IV for w. As usual, we replace the parameters in $\mathrm{E}(w|\mathbf{x},\mathbf{z})$ with $\sqrt{N}$-consistent estimators. The results in Chapter 6 on generated instruments can be used to show that the resulting IV estimator has the same $\sqrt{N}$-asymptotic distribution as if we know $\boldsymbol{\pi}$ and η.

d. An alternative method would be to run the OLS regression

$$y_i \text{ on } 1,\ \hat{w}_i,\ \mathbf{x}_i, i = 1,\ldots,N$$

where

$$\hat{w}_i \equiv \Phi(\mathbf{q}_i\hat{\boldsymbol{\pi}}/\hat{\eta}) \cdot \mathbf{q}_i\hat{\boldsymbol{\pi}} + \hat{\eta} \cdot \phi(\mathbf{q}_i\hat{\boldsymbol{\pi}}/\hat{\eta})$$

are the estimated conditional means. While this "plug-in" approach may produce estimates similar to the IV approach, it is less preferred for the same reasons we covered for the probit case. First, using the $\hat{w}_i$ as regressors rather than instruments is less robust: using them as regressors essentially requires the Tobit model to be correctly specified for w given $(\mathbf{x}, \mathbf{z})$. Second, valid standard errors are harder to get using $\hat{w}_i$ as a regressor as opposed to an IV. Third, the plug-in procedure does not appear to be optimal within an interesting class of estimators. (By contrast, we know that the IV estimator is optimal in the class of IV estimators under the assumptions given for part c.)

e. Estimate $y_i = \eta_0 + \mathbf{x}_i\boldsymbol{\gamma} + \beta w_i + w_i \cdot (\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + error_i$ by IV, using instruments, $[1, \mathbf{x}_i, \hat{w}_i, \hat{w}_i \cdot (\mathbf{x}_i - \bar{\mathbf{x}})]$ as instruments, where $\hat{w}_i$ are the Tobit fitted values. This would be generally inefficient, as the error, *r*, is not necessarily homoskedastic. Another drawback is that this would not generate overidentifying restrictions.

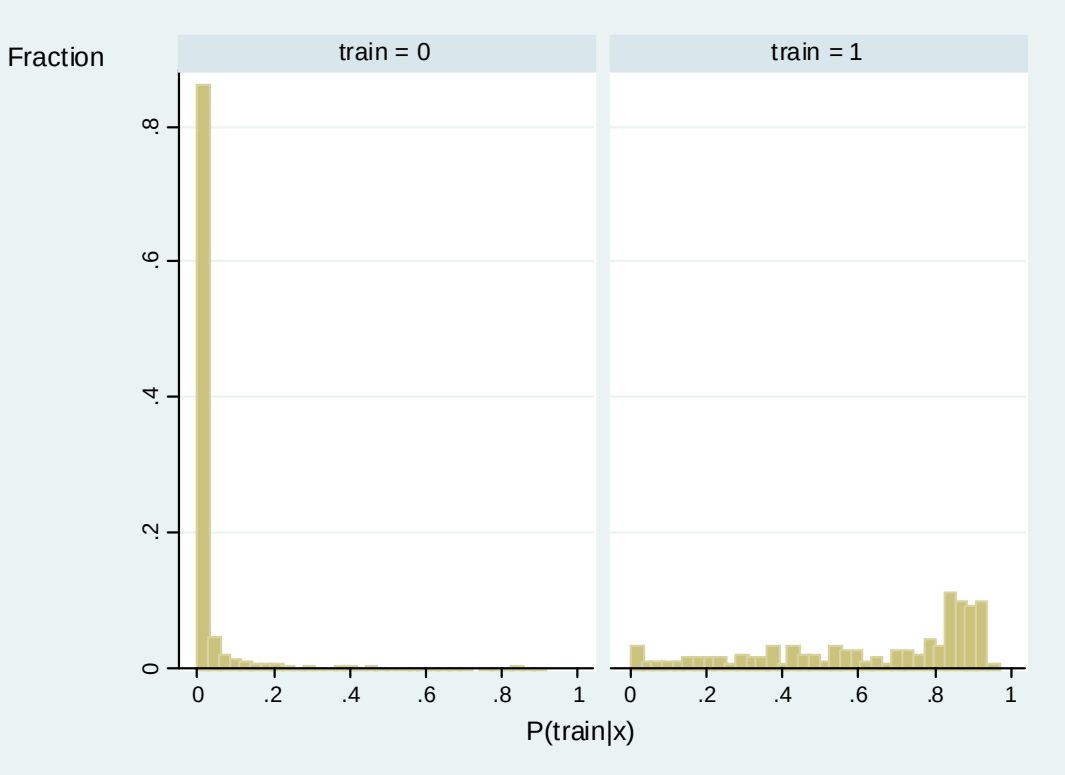
21.9. a. A histogram of the estimated propensity score for the untreated ($train = 0$) and treated ($train = 1$) cases is given below. There is a clear problem with overlap, as can be seen by studying the histogram for the control group: over 80% of units have propensity scores that are zero or practically zero. This means there are values of $\mathbf{x}$ where $p(\mathbf{x}) = 0$ or is barely distinguishable from zero.

The large differences in the histograms for the control and treatment groups spells trouble. Because $p(\mathbf{x})$ is just a particular function of $\mathbf{x}$, ideally its distribution would be similar across the control and treatment groups, and this is clearly not the case. The problems this causes is easily reasoned when thinking of matching on the propensity score. We need to find both control and treated units with similar values of $p(\mathbf{x})$, but the histograms make it clear that there are very few in the control group with $\hat{p}(\mathbf{x}_i) > .5$, whereas this is where the bulk of the

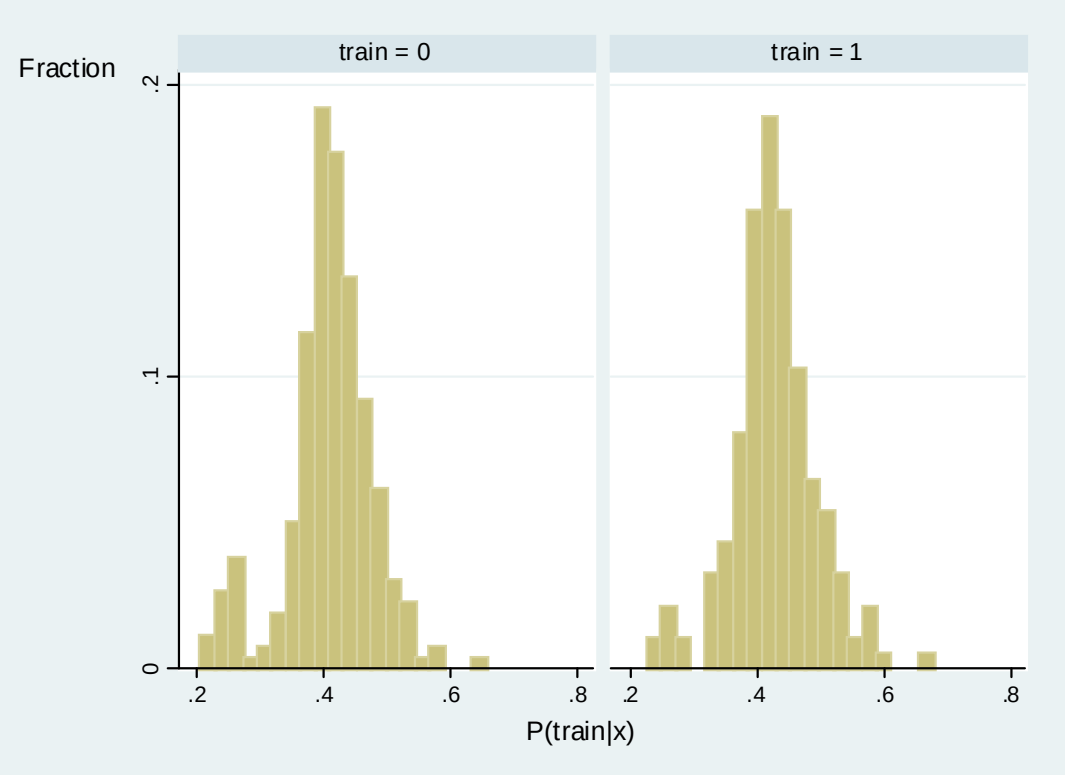
observations lie for the treated group.

For comparison, the same histograms are plotted using the experimental data in JTRAIN2.RAW. Now the histograms are virtually indistinguishable and neither has mass at zero or one.

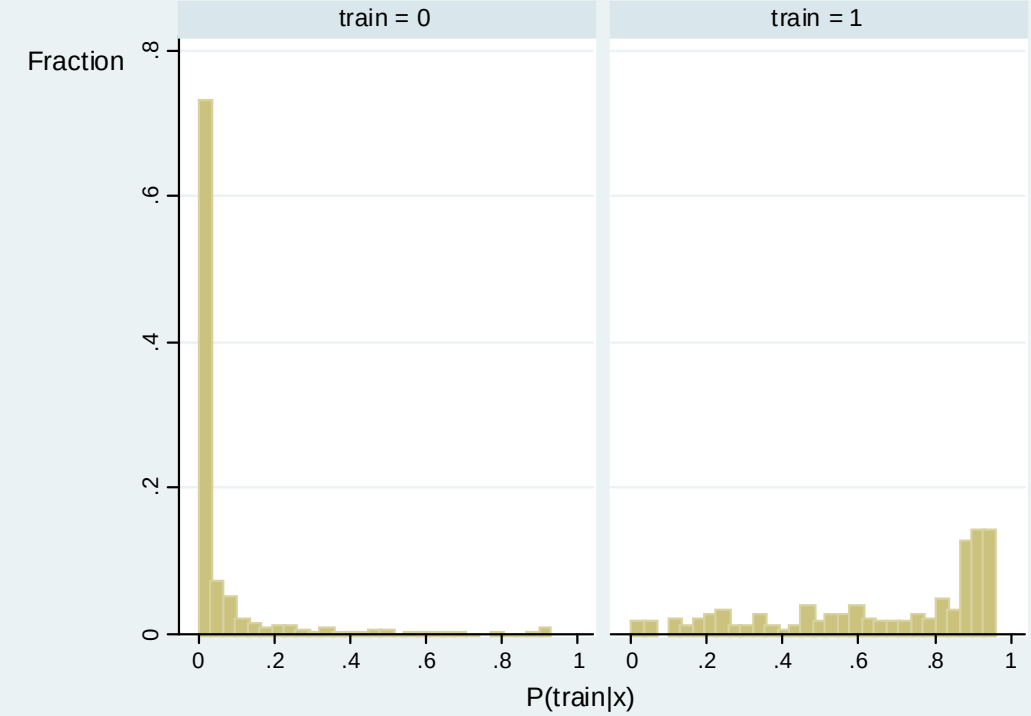
```
. use jtrain3

. logit train age educ black hisp married re74 re75

Logistic regression                                     Number of obs   =       2675
                                                        LR chi2(7)      =     872.82
                                                        Prob > chi2     =     0.0000
Log likelihood = -236.23799                             Pseudo R2       =     0.6488

------------------------------------------------------------------------------
       train |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |  -.0840291    .014761    -5.69   0.000    -.1129601    -.055098
        educ |  -.0624764   .0513973    -1.22   0.224    -.1632134    .0382605
       black |   2.242955   .3176941     7.06   0.000     1.620286    2.865624
        hisp |   2.094338   .5584561     3.75   0.000     .9997841    3.188892
     married |  -1.588358   .2602448    -6.10   0.000    -2.098428   -1.078287
        re74 |   -.117043   .0293604    -3.99   0.000    -.1745882   -.0594977
        re75 |  -.2577589   .0394991    -6.53   0.000    -.3351758   -.1803421
       _cons |   2.302714   .9112559     2.53   0.012     .5166853    4.088743
------------------------------------------------------------------------------
Note: 158 failures and 0 successes completely determined.

. predict phat
(option pr assumed; Pr(train))

. histogram phat, fraction by(train)
```


Fraction
train = 0
train = 1
0
.2
.4
.6
.8
1
P(train|x)

```
. use jtrain2

. logit train age educ black hisp married re74 re75

Logistic regression                                     Number of obs   =        445
                                                        LR chi2(7)      =       8.58
                                                        Prob > chi2     =     0.2840
Log likelihood = -297.80826                             Pseudo R2       =     0.0142

------------------------------------------------------------------------------
       train |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         age |   .0107155    .014017     0.76   0.445    -.0167572    .0381882
        educ |   .0628366   .0558026     1.13   0.260    -.0465346    .1722077
       black |  -.3553063   .3577202    -0.99   0.321    -1.056425    .3458123
        hisp |  -.9322569   .5001292    -1.86   0.062    -1.912492    .0479784
     married |   .1440193   .2734583     0.53   0.598    -.3919492    .6799878
        re74 |  -.0221324   .0252097    -0.88   0.380    -.0715425    .0272777
        re75 |   .0459029   .0429705     1.07   0.285    -.0383177    .1301235
       _cons |  -.9237055   .7693924    -1.20   0.230    -2.431687    .5842759
------------------------------------------------------------------------------

. predict phat
(option pr assumed; Pr(train))

. histogram phat, fraction by(train)
```


Fraction
train = 0
train = 1
0
.1
.2
.2
.4
.6
.8
P(train|x)

b. Using the sample restricted to $avgre \leq 15$ helps a little in that there now seem to be at least some untreated units in bins with $\hat{p}(\mathbf{x}_i) > .3$. But there are not many. The pile-up at near zero for the control group is still present. The two histograms still look very different from the experimental data in JTRAIN2.RAW.

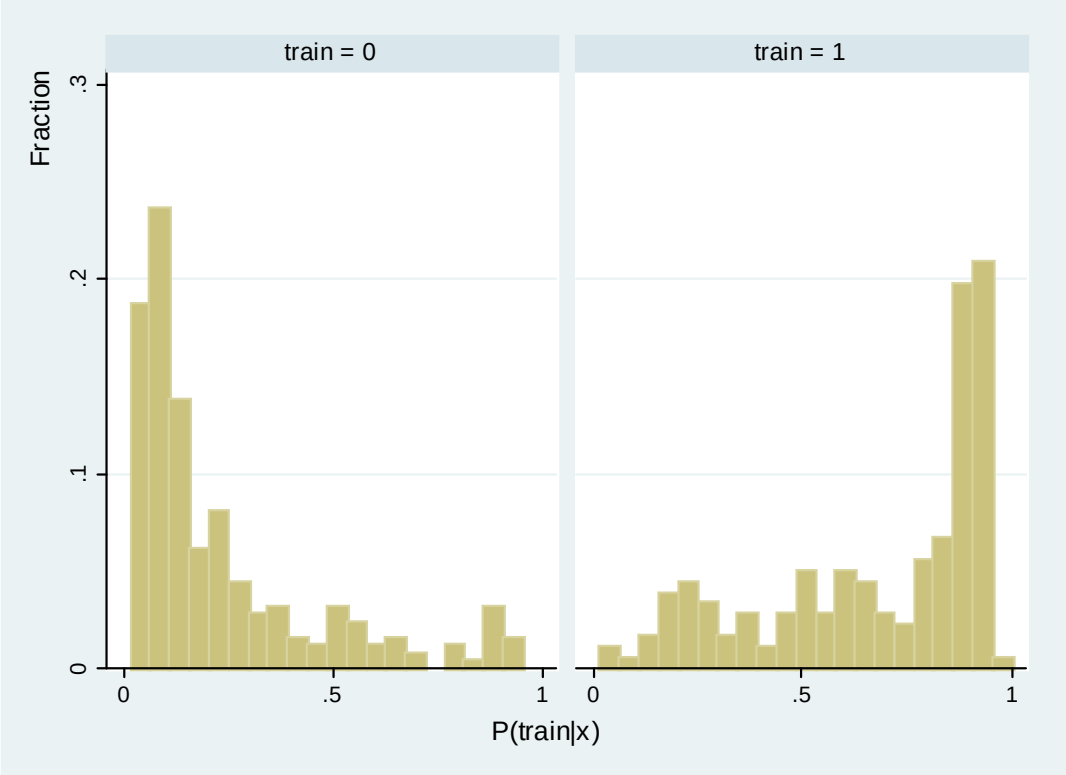

train = 0
train = 1
Fraction
0
.2
.4
.6
.8
0
.2
.4
.6
.8
1
P(train|x)

c. (Bonus Part) Suppose that using all 2,675 observations that, after estimating the logit model for *train*, we drop all data with $\hat{p}(\mathbf{x}_i) < .05$. We then reestimate the logit model using the remaining observations. How many observations are left? Obtain the resulting histograms as in part a and part b.

Solution

The Stata session is given below. Only 422 observations are left after dropping those with $\hat{p}(\mathbf{x}_i) < .05$. The histograms look much better in terms of overlap: for the most part, it appears that for $\hat{p}(\mathbf{x}_i)$ within a given bin, there a both treated and untreated observations. But the skew of the distributions is completely different (and not too suprising).

```
. use jtrain3

. qui logit train age educ black hisp married re74 re75

. predict phat
(option pr assumed; Pr(train))

. drop if phat < .05
(2253 observations deleted)

. drop phat

. qui logit train age educ black hisp married re74 re75

. predict phat
(option pr assumed; Pr(train))

. histogram phat, fraction by(train)
```


train = 0
train = 1
Fraction
0
.1
.2
.3
0
.5
1
P(train|x)

21.10. a. This is just problem in the asymptotic theory of simple regression with a binary explanatory variable. With $y_i = \mu_0 + \tau w_i + v_{i0}$ we have that w_i is independent of v_{i0}, and so there is no heteroskedasticity. It follows that (see Theorem 4.2)

$$\text{Avar}\left[\sqrt{N}(\hat{\tau} - \tau)\right] = \frac{\text{Var}(v_{i0})}{\text{Var}(w_i)} = \frac{\text{Var}(v_{i0})}{\rho(1-\rho)}$$

because $\text{P}(w_i = 1) = \rho$. This means, by definition,

$$\text{Avar}(\tilde{\tau}) = \frac{\text{Var}(v_{i0})}{N\rho(1-\rho)}.$$

b. By definition of the linear projection we can write

$$\begin{aligned} y_{i0} &= \alpha_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + u_{i0} \\ \text{E}(u_{i0}) &= 0, \ \text{E}(\mathbf{x}_i' u_{i0}) = \mathbf{0}. \end{aligned}$$

Now we just plug this into $y_i = y_{i0} + \tau w_i$:

$$y_i = \alpha_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + u_{i0} + \tau w_i = \alpha_0 + \tau w_i + \mathbf{x}_i\boldsymbol{\beta}_0 + u_{i0}.$$

The problem says to assume that w_i is independent of $(y_{i0}, \mathbf{x}_i)$ and so w_i is actually independent of $(\mathbf{x}_i, u_{i0})$ [because u_{i0} is a function of $(y_{i0}, \mathbf{x}_i)$].

c. Let $\mathbf{z}_i \equiv (w_i, \mathbf{x}_i)$ be the set of nonconstant regressors and let $\boldsymbol{\gamma} = (\tau, \boldsymbol{\beta}_0')'$. Then, as we showed in Chapter 4, if $\hat{\boldsymbol{\gamma}}$ is the OLS estimator under random sampling,

$$\sqrt{N}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) = [\text{Var}(\mathbf{z}_i)]^{-1} N^{-1/2} \sum_{i=1}^{N} (\mathbf{z}_i - \boldsymbol{\mu}_{\mathbf{z}})' u_{i0} + o_p(1).$$

Given that $\text{Cov}(\mathbf{x}_i, w_i) = \mathbf{0}$, we we restrict attention to the first element, $\sqrt{N}(\hat{\tau} - \tau)$, we get

$$\sqrt{N}(\hat{\tau} - \tau) = [\text{Var}(w_i)]^{-1} N^{-1/2} \sum_{i=1}^{N} (w_i - \rho)' u_{i0} + o_p(1).$$

Therefore, using independence between u_{i0} and w_i,

$$\text{Avar}\left[\sqrt{N}(\hat{\tau}-\tau)\right] = \frac{\text{Var}[(w_i-\rho)u_{i0}]}{[\text{Var}(w_i)]^2} = \frac{\text{Var}(u_{i0})}{\rho(1-\rho)}$$

and so

$$\text{Avar}(\hat{\tau}) = \frac{\text{Var}(u_{i0})}{N\rho(1-\rho)}.$$

d. Because $y_{i0} = \mu_0 + v_{i0}$, $\text{Var}(v_{i0}) = \text{Var}(y_{i0})$. Using the linear projection representation $y_{i0} = \alpha_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + u_{i0}$,

$$\text{Var}(y_{i0}) = \text{Var}(\mathbf{x}_i\boldsymbol{\beta}_0) + \text{Var}(u_{i0})$$

and, assuming that $\text{Var}(\mathbf{x}_i)$ has full rank, $\text{Var}(\mathbf{x}_i\boldsymbol{\beta}_0) > 0$ whenever $\boldsymbol{\beta}_0 \neq \mathbf{0}$. So

$$\text{Var}(u_{i0}) < \text{Var}(y_{i0}) = \text{Var}(v_{i0}).$$

It follows by comparing $\text{Avar}(\hat{\tau})$ and $\text{Avar}(\tilde{\tau})$ that $\text{Avar}(\hat{\tau}) < \text{Avar}(\tilde{\tau})$ whenever $\boldsymbol{\beta}_0 \neq \mathbf{0}$, that is, whenever $\mathbf{x}_i$ is correlated with y_{i0}.

e. Even though $\hat{\tau}$ is asymptotically more efficient than $\tilde{\tau}$, $\hat{\tau}$ is generally biased if $\text{E}(y_{i0}|\mathbf{x}_i) \neq \alpha_0 + \mathbf{x}_i\boldsymbol{\beta}_0$. [If $\text{E}(y_{i0}|\mathbf{x}_i) = \alpha_0 + \mathbf{x}_i\boldsymbol{\beta}_0$ then $\hat{\tau}$ would be conditionally unbiased, that is, $\text{E}(\hat{\tau}|\mathbf{W},\mathbf{X}) = \tau$.] The difference-in-means estimator $\tilde{\tau}$ is unbiased conditional on $\mathbf{W}$ because $\text{E}(y_i|\mathbf{W}) = \text{E}(y_i|w_i) = \mu_0 + \tau w_i$.

21.11. Suppose that we allow full slope, as well as intercept, heterogeneity in a linear representation of two counterfactual outcomes,

$$y_{i0} = a_{i0} + \mathbf{x}_i\mathbf{b}_{i0}$$
$$y_{i1} = a_{i1} + \mathbf{x}_i\mathbf{b}_{i1}$$

Assume that the vector $(\mathbf{x}_i, \mathbf{z}_i)$ is independent of $(a_{i0}, \mathbf{b}_{i0}, a_{i1}, \mathbf{b}_{i1})$ – which makes, as we will see, $\mathbf{z}_i$ instrumental variables candidtates in a control function or correction function setting.

a. Because $\mathbf{x}_i$ is independent of $\mathbf{b}_{ig}$, $g = 0,\ 1$,

$$\begin{aligned} \mathrm{E}(y_{ig}) &= \mathrm{E}(a_{ig}) + \mathrm{E}(\mathbf{x}_i\mathbf{b}_{ig}) = \alpha_g + \mathrm{E}(\mathbf{x}_i)\mathrm{E}(\mathbf{b}_{ig}) \\ &= \alpha_g + \boldsymbol{\psi}\boldsymbol{\beta}_g, g = 0, 1. \end{aligned}$$

b. From part a,

$$\begin{aligned} \mu_0 &= \alpha_0 + \boldsymbol{\psi}\boldsymbol{\beta}_0 \\ \mu_1 &= \alpha_1 + \boldsymbol{\psi}\boldsymbol{\beta}_1 \end{aligned}$$

and so

$$\tau = (\alpha_1 - \alpha_0) + \boldsymbol{\psi}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) = (\alpha_1 - \alpha_0) + \boldsymbol{\psi}\boldsymbol{\delta}.$$

Also,

$$y_{ig} = \alpha_g + \mathbf{x}_i\boldsymbol{\beta}_g + c_{ig} + \mathbf{x}_i\mathbf{f}_{ig}, g = 0, 1$$

and so

$$\begin{aligned} y_i &= (1 - w_i)y_{i0} + w_i y_{i1} \\ &= (1 - w_i)(\alpha_0 + \mathbf{x}_i\boldsymbol{\beta}_0 + c_{i0} + \mathbf{x}_i\mathbf{f}_{i0}) + w_i(\alpha_1 + \mathbf{x}_i\boldsymbol{\beta}_1 + c_{i1} + \mathbf{x}_i\mathbf{f}_{i1}) \\ &= \alpha_0 + (\alpha_1 - \alpha_0)w_i + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i\mathbf{x}_i(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) \\ &\quad + c_{i0} + w_i(c_{i1} - c_{i0}) + \mathbf{x}_i\mathbf{f}_{i0} + w_i\mathbf{x}_i(\mathbf{f}_{i1} - \mathbf{f}_{i0}) \\ &\equiv \alpha_0 + (\alpha_1 - \alpha_0)w_i + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i\mathbf{x}_i\boldsymbol{\delta} + c_{i0} + w_i e_i + \mathbf{x}_i\mathbf{f}_{i0} + w_i\mathbf{x}_i\mathbf{d}_i. \end{aligned}$$

Next, substitute

$$\begin{aligned} \alpha_0 &= \mu_0 - \boldsymbol{\psi}\boldsymbol{\beta}_0 \\ \alpha_1 - \alpha_0 &= \tau - \boldsymbol{\psi}\boldsymbol{\delta} \end{aligned}$$

to get

$$\begin{aligned} y_i &= \mu_0 - \boldsymbol{\psi}\boldsymbol{\beta}_0 + (\tau - \boldsymbol{\psi}\boldsymbol{\delta})w_i + \mathbf{x}_i\boldsymbol{\beta}_0 + w_i\mathbf{x}_i\boldsymbol{\delta} + c_{i0} + w_i e_i + \mathbf{x}_i\mathbf{f}_{i0} + w_i\mathbf{x}_i\mathbf{d}_i \\ &= \mu_0 + \tau w_i + (\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\delta} + c_{i0} + \mathbf{x}_i\mathbf{f}_{i0} + w_i e_i + w_i\mathbf{x}_i\mathbf{d}_i, \end{aligned}$$

which is what we wanted to show.

c. So that there is no notational conflict, write $\mathrm{E}(\mathbf{d}_i|a_i, \mathbf{x}_i, \mathbf{z}_i) = \boldsymbol{\zeta}a_i$ and keep $\boldsymbol{\theta}$ to index the

binary response model. Now take the expectation of (21.149) conditional on $(a_i, \mathbf{x}_i, \mathbf{z}_i)$, using the fact that w_i is a function of $(a_i, \mathbf{x}_i, \mathbf{z}_i)$:

$$\begin{aligned}\mathrm{E}(y_i|a_i, \mathbf{x}_i, \mathbf{z}_i) &= \mu_0 + \tau w_i + (\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\delta} \\ &\quad + \mathrm{E}(c_{i0}|a_i, \mathbf{x}_i, \mathbf{z}_i) + \mathbf{x}_i\mathrm{E}(\mathbf{f}_{i0}|a_i, \mathbf{x}_i, \mathbf{z}_i) \\ &\quad + w_i\mathrm{E}(e_i|a_i, \mathbf{x}_i, \mathbf{z}_i) + w_i\mathbf{x}_i\mathrm{E}(\mathbf{d}_i|a_i, \mathbf{x}_i, \mathbf{z}_i) \\ &= \mu_0 + \tau w_i + (\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\delta} \\ &\quad + \rho_0 a_i + a_i\mathbf{x}_i\boldsymbol{\eta}_0 + \xi w_i a_i + w_i a_i \mathbf{x}_i\boldsymbol{\zeta}\end{aligned}$$

d. Just use iterated expectations along with

$$\mathrm{E}(a_i|w_i, \mathbf{q}_i) = h(w_i, \mathbf{q}_i\boldsymbol{\theta}) \equiv w_i\lambda(\mathbf{q}_i\boldsymbol{\theta}) - (1 - w_i)\lambda(-\mathbf{q}_i\boldsymbol{\theta}).$$

So

$$\begin{aligned}\mathrm{E}(y_i|w_i, \mathbf{x}_i, \mathbf{z}_i) &= \mu_0 + \tau w_i + (\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\delta} \\ &\quad + \rho_0 h(w_i, \mathbf{q}_i\boldsymbol{\theta}) + h(w_i, \mathbf{q}_i\boldsymbol{\theta})\mathbf{x}_i\boldsymbol{\eta}_0 + \xi w_i h(w_i, \mathbf{q}_i\boldsymbol{\theta}) + w_i h(w_i, \mathbf{q}_i\boldsymbol{\theta})\mathbf{x}_i\boldsymbol{\zeta}\end{aligned}$$

e. Given $\mathrm{E}(y_i|w_i, \mathbf{x}_i, \mathbf{z}_i)$, the CF method is straightforward. In the first step, estimate probit of w_i on $\mathbf{q}_i$ to get $\hat{\boldsymbol{\theta}}$, and then compute $\hat{h}_i \equiv h(w_i, \mathbf{q}_i\hat{\boldsymbol{\theta}})$. Then run the OLS regression

$$y_i \text{ on } 1,\ w_i,\ \mathbf{x}_i - \bar{\mathbf{x}},\ w_i(\mathbf{x}_i - \bar{\mathbf{x}}),\ \hat{h}_i,\ \hat{h}_i\mathbf{x}_i,\ w_i\hat{h}_i,\ w_i\hat{h}_i\mathbf{x}_i,\ i = 1, \ldots, N.$$

We replace $\boldsymbol{\psi}$ with the sample average, $\bar{\mathbf{x}}$. The coefficient on w_i is $\hat{\tau}$.

Compared with the regression in equation (21.85), we have included the interactions $\hat{h}_i\mathbf{x}$ and $w_i\hat{h}_i\mathbf{x}_i$. These account for the random coefficients in the counterfactual equations.

f. Of course we could work through the delta method to obtain a valid asymptotic standard error for $\hat{\tau}$, but bootstrapping both steps in the procedure provides a simple alternative.

g. We just compute $\mathrm{E}(y_{ig}|\mathbf{x})$ for $g = 0, 1$:

$$\begin{aligned} \mathrm{E}(y_{ig}|\mathbf{x}_i) &= \mathrm{E}(a_{ig} + \mathbf{x}_i\mathbf{b}_{ig}|\mathbf{x}_i) \\ &= \mathrm{E}(a_{ig}|\mathbf{x}_i) + \mathbf{x}_i\mathrm{E}(\mathbf{b}_{ig}|\mathbf{x}_i) \\ &= \alpha_g + \mathbf{x}_i\boldsymbol{\beta}_g \end{aligned}$$

where the last equality holds by the independence assumption. Therefore,

$$\begin{aligned} \tau_{ate}(\mathbf{x}) &= (\alpha_1 - \alpha_0) + \mathbf{x}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) \\ &= \tau - \boldsymbol{\psi}\boldsymbol{\delta} + \mathbf{x}\boldsymbol{\delta} = \tau + (\mathbf{x} - \boldsymbol{\psi})\boldsymbol{\delta}. \end{aligned}$$

So

$$\hat{\tau}_{ate}(\mathbf{x}) = \hat{\tau} + (\mathbf{x} - \bar{\mathbf{x}})\hat{\boldsymbol{\delta}}.$$

21.12. a. The terms c_{i0} and $\mathbf{x}_i\mathbf{f}_{i0}$ have zero means conditional on $(\mathbf{x}_i, \mathbf{z}_i)$ by the independence of all heterogeneity terms – $(a_{i0}, \mathbf{b}_{i0}, a_{i1}, \mathbf{b}_{i1})$ – and $(\mathbf{x}_i, \mathbf{z}_i)$. Remember, $c_{i0} = a_{i0} - \alpha_0$ and $\mathbf{f}_{i0} = \mathbf{b}_{i0} - \boldsymbol{\beta}_0$.

b. The correction functions in this case are $\mathrm{E}(w_i e_i|\mathbf{x}_i, \mathbf{z}_i)$ and $\mathrm{E}(w_i\mathbf{x}_i\mathbf{d}_i|\mathbf{x}_i, \mathbf{z}_i) = \mathbf{x}_i\mathrm{E}(w_i\mathbf{d}_i|\mathbf{x}_i, \mathbf{z}_i)$. Now we just use the formula in equation (21.80) because $\mathrm{E}(e_i|a_i, \mathbf{x}_i, \mathbf{z}_i) = \xi a_i$ and $\mathrm{E}(\mathbf{d}_i|a_i, \mathbf{x}_i, \mathbf{z}_i) = \boldsymbol{\zeta}a_i$. Therefore,

$$\begin{aligned} \mathrm{E}(w_i e_i|\mathbf{x}_i, \mathbf{z}_i) &= \xi\phi(\mathbf{q}_i\boldsymbol{\theta}) \\ \mathrm{E}(w_i\mathbf{x}_i\mathbf{d}_i|\mathbf{x}_i, \mathbf{z}_i) &= \mathbf{x}_i\phi(\mathbf{q}_i\boldsymbol{\theta})\boldsymbol{\zeta} = \phi(\mathbf{q}_i\boldsymbol{\theta})\mathbf{x}_i\boldsymbol{\zeta}. \end{aligned}$$

c. From part b we can write

$$y_i = \mu_0 + \tau w_i + (\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \boldsymbol{\psi})\boldsymbol{\delta} + \xi\phi(\mathbf{q}_i\boldsymbol{\theta}) + \phi(\mathbf{q}_i\boldsymbol{\theta})\mathbf{x}_i\boldsymbol{\zeta} + c_{i0} + \mathbf{x}_i\mathbf{f}_{i0} + r_i$$

where

$$r_i = [w_i e_i - \mathrm{E}(w_i e_i|\mathbf{x}_i, \mathbf{z}_i)] + [w_i\mathbf{x}_i\mathbf{d}_i - \mathrm{E}(w_i\mathbf{x}_i\mathbf{d}_i|\mathbf{x}_i, \mathbf{z}_i)]$$

and so $\mathrm{E}(r_i|\mathbf{x}_i, \mathbf{z}_i) = 0$. The estimating equation, after the first-stage probit to get $\hat{\boldsymbol{\theta}}$, is

$$y_i = \mu_0 + \tau w_i + (\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\beta}_0 + w_i(\mathbf{x}_i - \bar{\mathbf{x}})\boldsymbol{\delta} + \xi\phi(\mathbf{q}_i\hat{\boldsymbol{\theta}}) + \phi(\mathbf{q}_i\hat{\boldsymbol{\theta}})\mathbf{x}_i\boldsymbol{\zeta} + error_i$$

which we can estimate using IV with instruments, say,

$$[1, \hat{\Phi}_i, (\mathbf{x}_i - \bar{\mathbf{x}}), \hat{\Phi}_i \cdot (\mathbf{x}_i - \bar{\mathbf{x}}), \hat{\phi}_i, \hat{\phi}_i \cdot \mathbf{x}_i]$$

d. Under the null hypothesis, $\xi = 0$ and $\boldsymbol{\zeta} = \mathbf{0}$. The conditions sufficient to ignore the first-stage estimator for IV estimators in Chapter 6 hold here, so we can use a standard Wald test (perhaps made robust to heteroskedasticity) to test joint significance of the $K + 1$ variables $(\hat{\phi}_i, \hat{\phi}_i \cdot \mathbf{x}_i)$. Remember, these are acting as their own instruments in the estimation.

21.13. Fractional probit or logit are natural, or some other model that keeps the fitted values in the unit interval. For the treatment rule $w_i = 1[x_i \geq c]$, let $G(\hat{\alpha}_0 + \hat{\beta}_0 x)$ be the estimated fractional response model using the data with $x_i < c$ and let $G(\hat{\alpha}_1 + \hat{\beta}_1 x)$ be the estimated model using the data with $x_i \geq c$. Probably the Bernoulli QMLE would be used. Then similar to equation (21.104),

$$\hat{\tau}_c = G(\hat{\alpha}_1 + \hat{\beta}_1 c) - G(\hat{\alpha}_0 + \hat{\beta}_0 c).$$

The delta method or bootstrapping can be used to obtain valid inference for $\tau_c = G(\alpha_1 + \beta_1 c) - G(\alpha_0 + \beta_0 c)$.

21.14. a. Just take the expected value of $y_{it}(g) = a_{itg} + \mathbf{x}_{it}\boldsymbol{\beta}_g$:

$$\mu_{gt} = \mathrm{E}(a_{itg}) + \mathrm{E}(\mathbf{x}_{it})\boldsymbol{\beta}_g = \alpha_{tg} + \boldsymbol{\psi}_t\boldsymbol{\beta}_g, g = 0, 1.$$

Therefore, for each *t*,

$$\tau_{t,ate} = (\alpha_{t1} - \alpha_{t0}) + \boldsymbol{\psi}_t(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0).$$

b. Use $\alpha_{t0} = \mu_{t0} - \boldsymbol{\psi}_t\boldsymbol{\beta}_0$ and $(\alpha_{t1} - \alpha_{t0}) = \tau_t - \boldsymbol{\psi}_t(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0)$ and plug into the equation for y_{it}:

$$
\begin{aligned}
y_{it} &= (1 - w_{it})(\alpha_{t0} + \mathbf{x}_{it}\boldsymbol{\beta}_0 + c_{it0}) + w_{it}(\alpha_{t1} + \mathbf{x}_{it}\boldsymbol{\beta}_1 + c_{it1}) \\
&= \alpha_{t0} + w_{it}(\alpha_{t1} - \alpha_{t0}) + \mathbf{x}_{it}\boldsymbol{\beta}_0 + w_{it}\mathbf{x}_{it}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) + c_{it0} + w_{it}(c_{it1} - c_{it0}) \\
&= (\mu_{t0} - \boldsymbol{\psi}_t\boldsymbol{\beta}_0) + w_{it}[\tau_t - \boldsymbol{\psi}_t(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0)] + \\
&\quad + \mathbf{x}_{it}\boldsymbol{\beta}_0 + w_{it}\mathbf{x}_{it}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_0) + c_{it0} + w_{it}(c_{it1} - c_{it0}) \\
&= \mu_{t0} + \tau_t w_{it} + (\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\beta}_0 + w_{it}(\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\delta} + c_{it0} + w_{it}e_{it}
\end{aligned}
$$

where $\boldsymbol{\delta} \equiv \boldsymbol{\beta}_1 - \boldsymbol{\beta}_0$ and $e_{it} \equiv c_{it1} - c_{it0}$.

c. Plugging in for c_{it0} and e_{it} gives

$$
\begin{aligned}
y_{it} &= \mu_{t0} + \tau_t w_{it} + (\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\beta}_0 + w_{it}(\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\delta} \\
&\quad + (\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\xi}_1 + (\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\xi}_2 + r_{it0} + w_{it}[(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\eta}_1 + (\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\eta}_2 + v_{it}] \\
&= \mu_{t0} + \tau_t w_{it} + (\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\beta}_0 + w_{it}(\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\delta} \\
&\quad + (\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\xi}_1 + (\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\xi}_2 + w_{it}(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\eta}_1 + w_{it}(\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\eta}_2 \\
&\quad + r_{it0} + w_{it}v_{it}
\end{aligned}
$$

d. As usual, we first condition on $(q_{it}, \mathbf{x}_i, \mathbf{z}_i)$:

$$
\begin{aligned}
\mathrm{E}(y_{it}|q_{it}, \mathbf{x}_i, \mathbf{z}_i) &= \mu_{t0} + \tau_t w_{it} + (\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\beta}_0 + w_{it}(\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\delta} \\
&\quad + (\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\xi}_1 + (\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\xi}_2 + w_{it}(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\eta}_1 + w_{it}(\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\eta}_2 \\
&\quad + \mathrm{E}(r_{it0}|q_{it}, \mathbf{x}_i, \mathbf{z}_i) + w_{it}\mathrm{E}(v_{it}|q_{it}, \mathbf{x}_i, \mathbf{z}_i) \\
&= \mu_{t0} + \tau_t w_{it} + (\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\beta}_0 + w_{it}(\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\delta} \\
&\quad + (\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\xi}_1 + (\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\xi}_2 + w_{it}(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\eta}_1 + w_{it}(\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\eta}_2 \\
&\quad + \alpha_0 q_{it} + \rho w_{it}q_{it}.
\end{aligned}
$$

Now condition on $(w_{it}, \mathbf{x}_i, \mathbf{z}_i)$ using iterated expectations:

$$
\begin{aligned}
\mathrm{E}(y_{it}|w_{it}, \mathbf{x}_i, \mathbf{z}_i) &= \mu_{t0} + \tau_t w_{it} + (\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\beta}_0 + w_{it}(\mathbf{x}_{it} - \boldsymbol{\psi}_t)\boldsymbol{\delta} \\
&\quad + (\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\xi}_1 + (\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\xi}_2 + w_{it}(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_{\bar{\mathbf{x}}})\boldsymbol{\eta}_1 + w_{it}(\bar{\mathbf{z}}_i - \boldsymbol{\mu}_{\bar{\mathbf{z}}})\boldsymbol{\eta}_2 \\
&\quad + \alpha_0 h(w_{it}, \mathbf{g}_{it}\boldsymbol{\theta}) + \rho w_i h(w_{it}, \mathbf{g}_{it}\boldsymbol{\theta}),
\end{aligned}
$$

where

$$
h(w_{it}, \mathbf{g}_{it}\boldsymbol{\theta}) = w_{it}\lambda(\mathbf{g}_{it}\boldsymbol{\theta}) - (1 - w_{it})\lambda(-\mathbf{g}_{it}\boldsymbol{\theta})
$$

and $\mathbf{g}_{it}\boldsymbol{\theta} = \theta_0 + \mathbf{x}_{it}\boldsymbol{\theta}_1 + \mathbf{z}_{it}\boldsymbol{\theta}_2 + \bar{\mathbf{x}}_i\boldsymbol{\theta}_3 + \bar{\mathbf{z}}_i\boldsymbol{\theta}_4$.

e. In the first step we can use pooled probit of w_{it} on 1, $\mathbf{x}_{it}$, $\mathbf{z}_{it}$, $\bar{\mathbf{x}}_i$, $\bar{\mathbf{z}}_i$ to obtain $\hat{\boldsymbol{\theta}}$ and $h(w_{it}, \mathbf{g}_{it}\hat{\boldsymbol{\theta}})$. Then we can use pooled OLS in a second step:

$$y_{it} \text{ on } 1,\ d2_t,\ ...,\ dT_t,\ w_{it},\ d2_t w_{it},\ ...,\ dT_t w_{it},\ (\mathbf{x}_{it} - \bar{\mathbf{x}}_t),\ w_{it} \cdot (\mathbf{x}_{it} - \bar{\mathbf{x}}_t),$$
$$(\bar{\mathbf{x}}_i - \bar{\mathbf{x}}),\ (\bar{\mathbf{z}}_i - \bar{\mathbf{z}}),\ w_{it} \cdot (\bar{\mathbf{x}}_i - \bar{\mathbf{x}}),\ w_{it} \cdot (\bar{\mathbf{z}}_i - \bar{\mathbf{z}}),\ \hat{h}_{it},\ w_{it} \cdot \hat{h}_{it}$$

where dr_t is a time dummy for period r, $\hat{h}_{it} \equiv h(w_{it}, \mathbf{g}_{it}\hat{\boldsymbol{\theta}})$, and overbars denote sample averages.

Note that in the conditional expectation underlying the CF method, $\mathrm{E}(y_{it}|w_{it}, \mathbf{x}_i, \mathbf{z}_i)$, $\{w_{it} : t = 1, \ldots, T\}$ is not guaranteed to be strictly exogenous. Therefore, we cannot use GLS-type methods without making extra assumptions.

21.15. a. The Stata output is given below. There are 5,735 observations, and 2,184 received a right-heart catheterization.

```
. tab rhc

      =1 if |
   received |
right heart |
catheteriza |
       tion |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |      3,551       61.92       61.92
          1 |      2,184       38.08      100.00
------------+-----------------------------------
      Total |      5,735      100.00
```

b. The Stata output, using simple regression and heteroskedasticity-robust standard errors, is given below. According to the estimate, people receiving an RHR have a .051 higher probability of dying. The estimate has a robust t statistic of 3.95, so it is statistically different from zero (and practically large – in the "wrong" direction).

```
. reg death rhc, robust

Linear regression                               Number of obs =    5735
                                                F(  1,  5733) =   15.56
```

```
                                                Prob > F      =  0.0001
                                                R-squared     =  0.0027
                                                Root MSE      =  .47673

------------------------------------------------------------------------------
             |               Robust
       death |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         rhc |   .0507212   .0128566     3.95   0.000     .0255174    .0759249
       _cons |   .6296818   .0081049    77.69   0.000     .6137931    .6455705
------------------------------------------------------------------------------
```

c. The Stata code and results are given below. We still obtain counterintuitive results: $\hat{\tau}_{ate,reg} = .078$, with a bootstrapped standard error of $.013$, and $\hat{\tau}_{att,reg} = .066$ (se $= .014$). Thus, both estimates are statistically different from zero. we can either conclude that the controls we include do not make treatment assignment ignorable or that the RHC actually increases the probability of death.

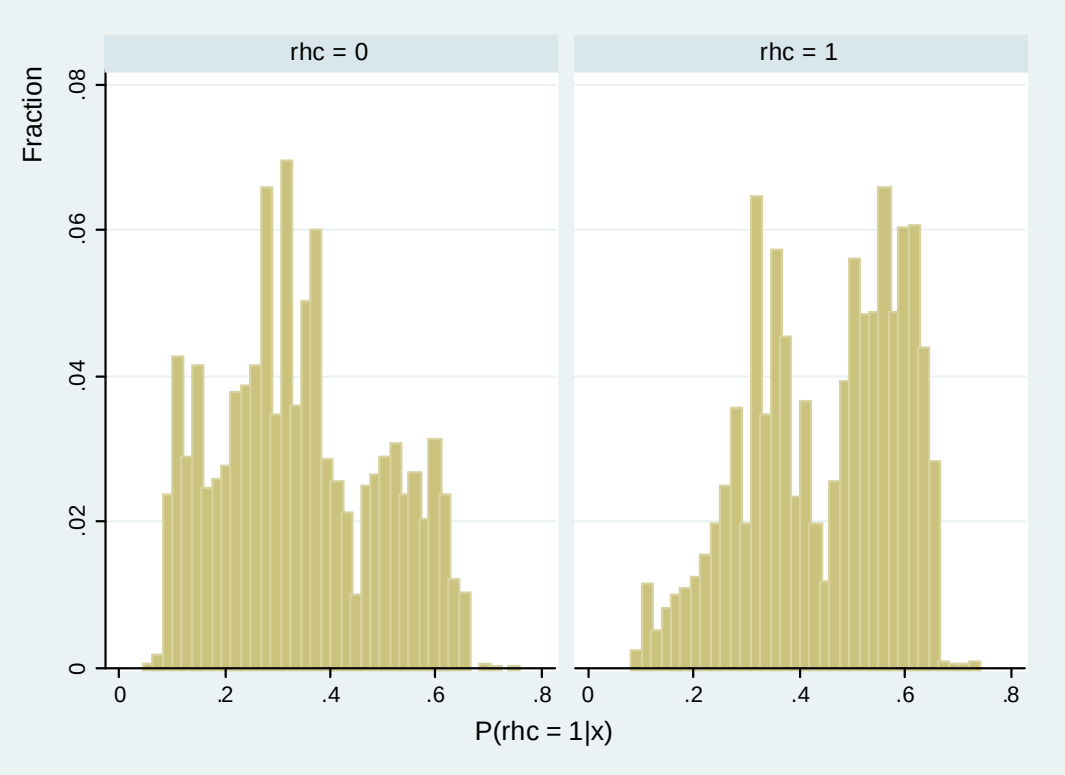
```
clear all

capture program drop ateboot

program ateboot, eclass

* Estimate logit on treatment and control groups separately
tempvar touse
gen byte ‘touse’ = 1
xi: logit death i.female i.race i.income i.cat1 i.cat2 i.ninsclas age if rhc
predict d1hat
xi: logit death i.female i.race i.income i.cat1 i.cat2 i.ninsclas age if ~rhc
predict d0hat
gen diff = d1hat - d0hat
sum diff
scalar ate = r(mean)
sum diff if rhc
scalar att = r(mean)
matrix b = (ate, att)
matrix colnames b = ate att
ereturn post b , esample(‘touse’)
ereturn display
drop d1hat d0hat diff _I*
end

use catheter

bootstrap _b[ate] _b[att], reps(1000) seed(123): ateboot

program drop ateboot

do catheter_reg

. use catheter
```

```
. bootstrap _b[ate] _b[att], reps(1000) seed(123): ateboot
(running ateboot on estimation sample)

Bootstrap replications (1000)
----+--- 1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
..................................................    50

..................................................  1000

Bootstrap results                                 Number of obs      =      5735
                                                  Replications       =      1000

      command:  ateboot
        _bs_1:  _b[ate]
        _bs_2:  _b[att]

------------------------------------------------------------------------------
             |   Observed   Bootstrap                         Normal-based
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _bs_1 |   .0776176   .0129611     5.99   0.000     .0522143    .1030208
       _bs_2 |   .0656444     .01366     4.81   0.000     .0388713    .0924175
------------------------------------------------------------------------------
.
. program drop ateboot

end of do-file
```

d. The average $\hat{p}_i$ for the treated group is about .445 and it ranges from about .085 to .737. For the control group, the numbers are .341, .044, and .738. Though the mean propensity score is somewhat higher for the treated group, the ranges are comparable. The two histograms show that for both groups the probabilities stay away from the extremes of zero and one, and for every bin representing intervals of the estimated propensity score, there are several individuals in the control and treatment groups. Overlap appears to be good.

```
. use catheter

. xi: logit rhc i.female i.race i.income i.cat1 i.cat2 i.ninsclas age
i.female          _Ifemale_0-1        (naturally coded; _Ifemale_0 omitted)
i.race            _Irace_0-2          (naturally coded; _Irace_0 omitted)
i.income          _Iincome_0-3        (naturally coded; _Iincome_0 omitted)
i.cat1            _Icat1_1-9          (naturally coded; _Icat1_1 omitted)
i.cat2            _Icat2_1-7          (naturally coded; _Icat2_1 omitted)
i.ninsclas        _Ininsclas_1-6      (naturally coded; _Ininsclas_1 omitted)

Logistic regression                               Number of obs   =      5735
                                                  LR chi2(26)     =    625.48
                                                  Prob > chi2     =    0.0000
Log likelihood = -3497.9617                       Pseudo R2       =    0.0821
```

```
------------------------------------------------------------------------------
         rhc |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
  _Ifemale_1 |   .1630495   .0587579     2.77   0.006     .0478861    .2782129
    _Irace_1 |   .0424279   .0827591     0.51   0.608    -.1197771    .2046328
    _Irace_2 |   .0393684   .1361998     0.29   0.773    -.2275784    .3063151
  _Iincome_1 |   .0443619   .0762726     0.58   0.561    -.1051296    .1938535
  _Iincome_2 |    .151793   .0892757     1.70   0.089    -.0231841    .3267701
  _Iincome_3 |   .1579471   .1140752     1.38   0.166    -.0656361    .3815303
    _Icat1_2 |    .498032    .107388     4.64   0.000     .2875553    .7085086
    _Icat1_3 |  -1.226306   .1495545    -8.20   0.000    -1.519428   -.9331849
    _Icat1_4 |  -.7173791   .1714465    -4.18   0.000    -1.053408   -.3813501
    _Icat1_5 |  -1.002513   1.085305    -0.92   0.356    -3.129671    1.124645
    _Icat1_6 |  -.6941957   .1260198    -5.51   0.000      -.94119   -.4472013
    _Icat1_7 |  -1.258815   .4833701    -2.60   0.009    -2.206203   -.3114273
    _Icat1_8 |  -.2076635   .1177652    -1.76   0.078     -.438479    .0231519
    _Icat1_9 |   1.003787   .0768436    13.06   0.000     .8531766    1.154398
    _Icat2_2 |   .9804654   1.465085     0.67   0.503    -1.891048    3.851979
    _Icat2_3 |  -.4141065   .4428411    -0.94   0.350    -1.282059    .4538461
    _Icat2_4 |  -.8864827   .8454718    -1.05   0.294    -2.543577    .7706116
    _Icat2_5 |   -.195389   .3933026    -0.50   0.619     -.966248      .57547
    _Icat2_6 |   1.034498    .369503     2.80   0.005     .3102859    1.758711
    _Icat2_7 |   .1415088   .3649828     0.39   0.698    -.5738443    .8568619
_Ininsclas_2 |   .1849583   .1216214     1.52   0.128    -.0534153    .4233318
_Ininsclas_3 |   .1082916    .152243     0.71   0.477    -.1900992    .4066824
_Ininsclas_4 |   .5216726   .1495659     3.49   0.000     .2285288    .8148164
_Ininsclas_5 |    .468176   .1122184     4.17   0.000      .248232    .6881199
_Ininsclas_6 |   .3742273   .1249122     3.00   0.003     .1294038    .6190508
         age |   .0006419    .002252     0.29   0.776    -.0037719    .0050557
       _cons |   -1.36677   .3834979    -3.56   0.000    -2.118412   -.6151284
------------------------------------------------------------------------------

. predict phat
(option pr assumed; Pr(rhc))

. sum phat if rhc

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        phat |      2184    .4449029    .1421669   .0851523   .7369323

. sum phat if ~rhc

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        phat |      3551    .3414058    .1517016   .0435625   .7379614
```


rhc = 0
rhc = 1
Fraction
0
.02
.04
.06
.08
0
.2
.4
.6
.8
P(rhc = 1|x)

e. The estimates using propensity score weighting are very similar to the regression-adjustment estimates using logit models. The PSW estimates are $\hat{\tau}_{ate.psw} = .072$ (se $= .013$) and $\hat{\tau}_{att.psw} = .063$ (se $= .014$). Unfortunately, out of the 1,000 bootstrap replications that I ran to obtain the standard errors, only 219 produced usable results because the estimated propensity score for at least some of the draws was identically zero or identically one (when the covariates perfectly classify *rhc*). But there is clearly no evidence, based on these and the regression-adjustment estimates, that RHC reduces the probability of death. It appears that RHC is being applied to patients based on variables are not included in the data set that are associated with both mortality and a doctor recommending RHC.

```
. do catheter_psw

clear all

capture program drop ateboot

program ateboot, rclass

* Estimate propensity score

xi: logit rhc i.female i.race i.income i.cat1 i.cat2 i.ninsclas age
predict phat
gen kiate = (rhc - phat)*death/(phat*(1 - phat))
sum kiate
return scalar atew = r(mean)
sum rhc
scalar rho = r(mean)
gen kiatt = (rhc - phat)*death/(1 - phat)
sum kiatt
return scalar attw = r(mean)/rho

drop phat kiate kiatt _I*

end

use catheter

bootstrap r(atew) r(attw), reps(1000) seed(123): ateboot

program drop ateboot

.
. use catheter
.
. bootstrap r(atew) r(attw), reps(1000) seed(123): ateboot
```

```
(running ateboot on estimation sample)

Bootstrap results                                     Number of obs      =      5735
                                                      Replications       =       219

      command:  ateboot
        _bs_1:  r(atew)
        _bs_2:  r(attw)

------------------------------------------------------------------------------
             |   Observed   Bootstrap                         Normal-based
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _bs_1 |    .071762   .0131705     5.45   0.000     .0459483    .0975758
       _bs_2 |   .0629458   .0140839     4.47   0.000      .035342    .0905497
------------------------------------------------------------------------------
Note: one or more parameters could not be estimated in 781 bootstrap
      replicates; standard-error estimates include only complete replications

. program drop ateboot

end of do-file
```

21.16. Use the data in REGDISC.RAW to answer this question. These are simulated data of a fuzzy regression discontinuity design with forcing variable x. The discontinuity is at $x = 5$.

a. Exactly half of the observations have $x_i \geq 5$, but 58.1% (1,162 out of 2,000) of the observations are in the treatment group.

```
. sum z w

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
           z |      2000          .5     .500125          0          1
           w |      2000        .581    .4935188          0          1

. tab w

      =1 if |
    treated |      Freq.     Percent        Cum.
------------+-----------------------------------
          0 |        838       41.90       41.90
          1 |      1,162       58.10      100.00
------------+-----------------------------------
      Total |      2,000      100.00
```

b. The Stata output is given below. The graphs of the estimated probabilities (LPM and logit) are fairly similar, although the LPM estimates a larger jump in the treatment

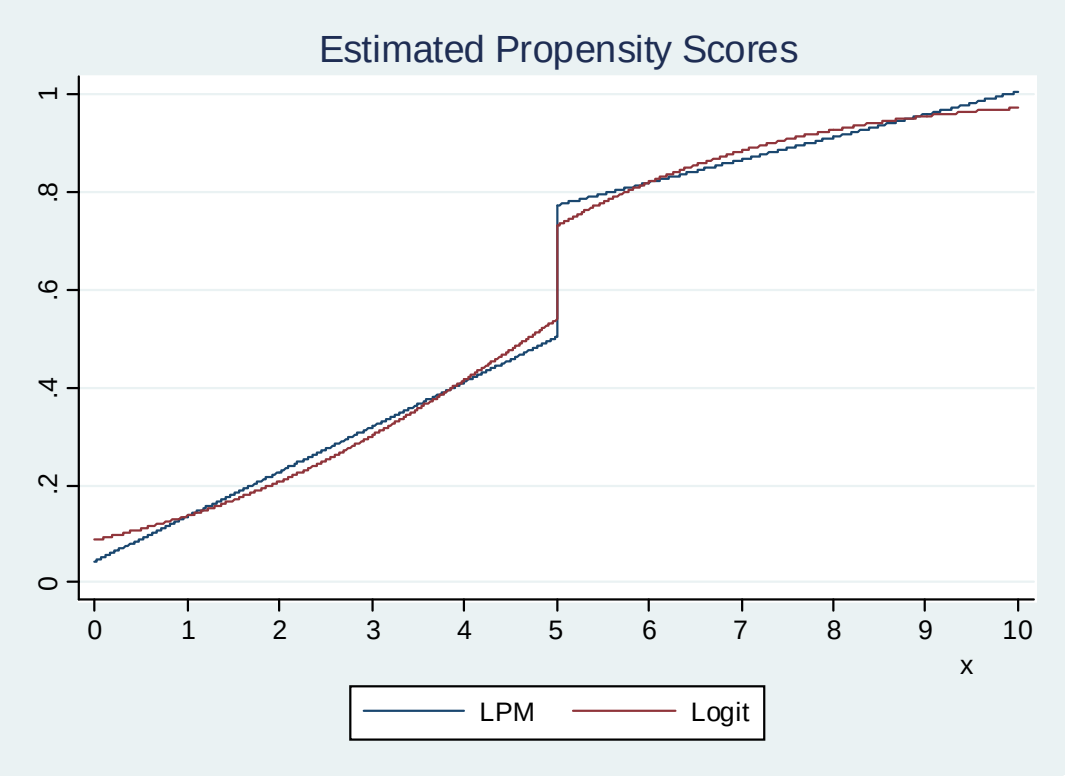
probabilities: for the LPM it is about .271 and for the logit it is about .192. In fact, the logit estimate is closer to the true jump in the propensity score at $x = 5$, which is about .186. [The treatment probability was generated from the probit model

$P(w = 1|x) = \Phi(.1 + .5 \cdot 1[x >= 5] + .3 \cdot (x - 5))$, and so the jump in the probability of treatment at $x = 5$ is $\Phi(.6) - \Phi(.1) \approx .186$.]

```
. reg w x if ~z

      Source |       SS       df       MS              Number of obs =    1000
-------------+------------------------------           F(  1,   998) =   96.61
       Model |  17.5177744     1  17.5177744           Prob > F      =  0.0000
    Residual |  180.953226   998  .181315857           R-squared     =  0.0883
-------------+------------------------------           Adj R-squared =  0.0874
       Total |     198.471   999   .19866967           Root MSE      =  .42581

------------------------------------------------------------------------------
           w |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0916522   .0093244     9.83   0.000     .0733545    .1099499
       _cons |    .043984   .0269105     1.63   0.102    -.0088237    .0967917
------------------------------------------------------------------------------

. predict wh0_lpm
(option xb assumed; fitted values)

. gen wh0_lpm_5 = _b[_cons] + _b[x]*5 in 1
(1999 missing values generated)

. reg w x if z

      Source |       SS       df       MS              Number of obs =    1000
-------------+------------------------------           F(  1,   998) =   47.59
       Model |   4.4914493     1   4.4914493           Prob > F      =  0.0000
    Residual |  94.1875507   998  .094376303           R-squared     =  0.0455
-------------+------------------------------           Adj R-squared =  0.0446
       Total |      98.679   999  .098777778           Root MSE      =  .30721

------------------------------------------------------------------------------
           w |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           x |   .0464084   .0067272     6.90   0.000     .0332073    .0596095
       _cons |   .5408787   .0513891    10.53   0.000     .4400356    .6417218
------------------------------------------------------------------------------

. predict wh1_lpm
(option xb assumed; fitted values)

. gen wh1_lpm_5 = _b[_cons] + _b[x]*5 in 1
```

```
(1999 missing values generated)

. gen jump_lpm = wh1_lpm_5 - wh0_lpm_5
(1999 missing values generated)

. gen what = what0 if ~z
(1000 missing values generated)

. replace what = what1 if z
(1000 real changes made)

. sum jump_lpm

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
    jump_lpm |         1    .2706757           .   .2706757   .2706757

. qui logit w x if ~z

. predict wh0_logit
(option pr assumed; Pr(w))

. gen wh0_logit_5 = invlogit(_b[_cons] + _b[x]*5) in 1
(1999 missing values generated)

. qui logit w x if z

. predict wh1_logit
(option pr assumed; Pr(w))

. gen wh1_logit_5 = invlogit(_b[_cons] + _b[x]*5) in 1
(1999 missing values generated)

. gen jump_logit = wh1_logit_5 - wh0_logit_5
(1999 missing values generated)

. sum jump_logit

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
  jump_logit |         1    .1922199           .   .1922199   .1922199

. gen wh_logit = wh0_logit if ~z
(1000 missing values generated)

. replace wh_logit = wh1_logit if z
(1000 real changes made)
```

```
. twoway (line wh_lpm x, sort) (line wh_logit x, sort)
```



c. We already computed the jump for the LPM estimates of the propensity score in part b. Now we need to estimate the jump in $E(y|x)$ at $x = 5$. Using the linear model, this turns out to be about $.531$. From equaton (21.107) the estimate of τ_c is the ratio of the jumps, which is about 1.96. In fact, the true effect is two, so this estimate is very close for this set of data. The data on y were generated as $y_i = 1 + 2w_i + x_i/4 + u_i$ where u_i is independent of x_i and treatment with a Normal$(0, .36)$ distribution.

```
. reg y x if ~z

      Source |       SS       df       MS              Number of obs =    1000
-------------+------------------------------           F(  1,   998) =  368.55
       Model |  409.736838     1  409.736838           Prob > F      =  0.0000
    Residual |  1109.52773   998  1.11175123           R-squared     =  0.2697
-------------+------------------------------           Adj R-squared =  0.2690
       Total |  1519.26457   999  1.52078535           Root MSE      =  1.0544

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
           x |   .4432576   .0230891    19.20   0.000     .3979488    .4885664
       _cons |   1.066599   .0666359    16.01   0.000     .9358364    1.197361
------------------------------------------------------------------------------

. gen yh0_5 = _b[_cons] + _b[x]*5 in 1
(1999 missing values generated)

. reg y x if z

      Source |       SS       df       MS              Number of obs =    1000
-------------+------------------------------           F(  1,   998) =  310.37
       Model |  230.759468     1  230.759468           Prob > F      =  0.0000
    Residual |  742.020178   998  .743507193           R-squared     =  0.2372
-------------+------------------------------           Adj R-squared =  0.2365
       Total |  972.779646   999  .973753399           Root MSE      =  .86227

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
           x |   .3326468   .0188819    17.62   0.000      .295594    .3696997
       _cons |   2.151037   .1442388    14.91   0.000      1.86799    2.434083
------------------------------------------------------------------------------

. gen yh1_5 = _b[_cons] + _b[x]*5 in 1
(1999 missing values generated)

. gen jumpy = yh1_5 - yh0_5
(1999 missing values generated)

. sum jumpy
```

```
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       jumpy |         1     .531384           .    .531384    .531384

. gen ate5 = jumpy/jump_lpm
(1999 missing values generated)

. sum ate5

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        ate5 |         1    1.963176           .   1.963176   1.963176
```

d. As is claimed in the text, the IV estimate from (21.108) is the same as the estimate from (21.107), with a very slight rounding error in the sixth digit after the decimal point. The heteroskedasticity-robust standard error is about .205. (The nonrobust standard error is about .197.)

Because the true equation for y is linear in x – with an expected jump at $x = 5$ – the IV estimator (and hence the estimate from part c) is consistent even though w_i follows a logit model rather than an LPM.

```
. ivreg y x_5 zx_5 (w = z), robust

Instrumental variables (2SLS) regression              Number of obs =    2000
                                                      F(  3,  1996) = 3588.42
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.8722
                                                      Root MSE      =   .5959

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           w |   1.963177   .2046892     9.59   0.000      1.56175    2.364604
         x_5 |    .263328   .0295197     8.92   0.000     .2054354    .3212206
        zx_5 |  -.0217891   .0214587    -1.02   0.310    -.0638729    .0202947
       _cons |    2.29689   .1352279    16.99   0.000     2.031688    2.562093
------------------------------------------------------------------------------
Instrumented:  w
Instruments:   x_5 zx_5 z
------------------------------------------------------------------------------
```

e. Using only the data with $3 < x_i < 7$ results in 800 observations, rather than 2,000. The estimate is substantially smaller than two but, more importantly, its standard error has

increased to about .327 from .205. Nevertheless, the 95% confidence interval for τ_c easily contains the true value $\tau_c = 2$.

```
. ivreg y x_5 zx_5 (w = z) if x > 3 & x < 7, robust

Instrumental variables (2SLS) regression              Number of obs =     800
                                                      F(  3,   796) =  351.50
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.7662
                                                      Root MSE      =  .61919

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           w |   1.775465   .3267695     5.43   0.000     1.134033    2.416897
         x_5 |   .3471895   .0726118     4.78   0.000     .2046563    .4897226
        zx_5 |  -.0991082   .0772654    -1.28   0.200    -.2507762    .0525599
       _cons |   2.442008   .2182725    11.19   0.000      2.01355    2.870466
------------------------------------------------------------------------------
Instrumented:  w
Instruments:   x_5 zx_5 z
------------------------------------------------------------------------------
```

Solutions to Chapter 22 Problems

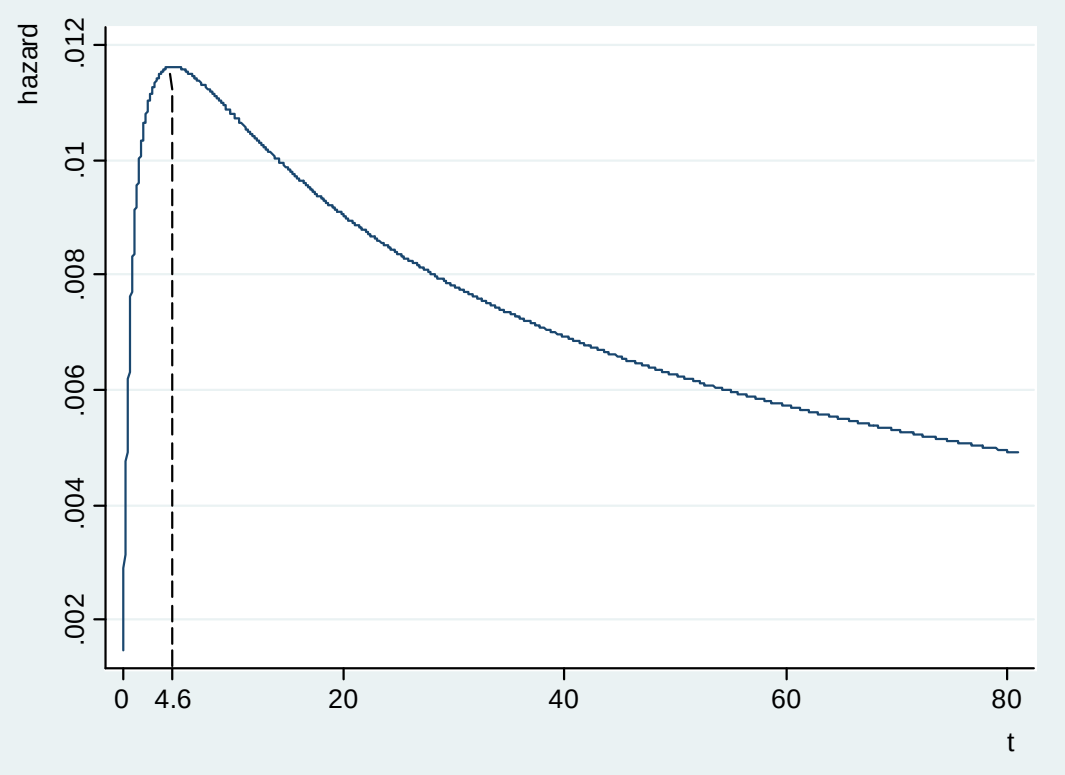
22.1. a. In Stata, there are two possibilities for estimating a lognormal duration model: the `cnreg` command (where we use the log of the duration as a response), and the `streg` command (where we specify "lognormal" as the distribution). The `streg` command is more flexible (and I use it in the next problem), but here I give the `cnreg` output. The value of the log likelihood is $-1,597.06$.

```
. use recid

. cnreg ldurat workprg priors tserved felon alcohol drugs black married educ
        age, censored(cens)

Censored-normal regression                      Number of obs   =       1445
                                                LR chi2(10)     =     166.74
                                                Prob > chi2     =     0.0000
Log likelihood =  -1597.059                     Pseudo R2       =     0.0496

------------------------------------------------------------------------------
      ldurat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     workprg |  -.0625715   .1200369    -0.52   0.602    -.2980382    .1728951
      priors |  -.1372529   .0214587    -6.40   0.000    -.1793466   -.0951592
     tserved |  -.0193305   .0029779    -6.49   0.000    -.0251721    -.013489
       felon |   .4439947   .1450865     3.06   0.002     .1593903    .7285991
     alcohol |  -.6349092   .1442166    -4.40   0.000    -.9178072   -.3520113
       drugs |  -.2981602   .1327355    -2.25   0.025    -.5585367   -.0377837
       black |  -.5427179   .1174428    -4.62   0.000    -.7730958     -.31234
     married |   .3406837   .1398431     2.44   0.015      .066365    .6150024
        educ |   .0229196   .0253974     0.90   0.367    -.0269004    .0727395
         age |   .0039103   .0006062     6.45   0.000     .0027211    .0050994
       _cons |   4.099386    .347535    11.80   0.000     3.417655    4.781117
-------------+----------------------------------------------------------------
       /sigma |    1.81047   .0623022                      1.688257    1.932683
------------------------------------------------------------------------------
  Observation summary:         0  left-censored observations
                             552     uncensored observations
                             893 right-censored observations
```

b. I graphed the hazard at the stated values of covariates using the Stata commands below. The estimated hazard initially increases, until about $t^* = 4.6$, where it reaches the value of $.0116$ (roughly). It then falls, until it hits about about $.005$ at $t = 81$. It may make sense that there are startup costs to becoming involved in crime upon release, so that the instantaneous

probability of recidivism initially increases (for about four and one-half months). After that, the hazard falls monotonically, although it does not become zero at the largest observed duration, 81 months.

```
. di = _b[_cons] + _b[felon] + _b[alcohol] + _b[drugs] + _b[priors]* 1.431834
       + _b[tserved]*19.18201 +_b[educ]* 9.702422 +_b[age]* 345.436

4.616118

. clear

. range t .1 81 5000
obs was 0, now 5000

. gen hazard = (normalden((log(t) - 4.62)/1.81)/
      (1 - normal((log(t) - 4.62)/1.81)))/(1.81*t)

egen maxhazard = max(hazard)

. list t hazard if hazard >= maxhazard

      +--------------------+
      |        t    hazard |
      |--------------------|
 277. | 4.566573   .011625 |
      +--------------------+

. twoway (line hazard t)
```


hazard
.002
.004
.006
.008
.01
.012
0
4.6
20
40
60
80
t

c. Using the only the uncensored in a linear regression analysis provides very different estimates. For example, the *alcohol* and *drugs* coefficients are much smaller in magnitude, with the latter actually changing sign and becoming very insignificant.

```
. reg ldurat workprg priors tserved felon alcohol drugs black married educ age

      Source |       SS       df       MS              Number of obs =     552
-------------+------------------------------           F( 10,   541) =    4.13
       Model |  33.7647818    10  3.37647818            Prob > F      =  0.0000
    Residual |  442.796158   541  .818477187            R-squared     =  0.0709
-------------+------------------------------           Adj R-squared =  0.0537
       Total |   476.56094   551  .864901888            Root MSE      =   .9047

------------------------------------------------------------------------------
      ldurat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     workprg |   .0923415   .0827407     1.12   0.265    -.0701909     .254874
      priors |  -.0483627   .0140418    -3.44   0.001    -.0759459   -.0207795
     tserved |  -.0067761    .001938    -3.50   0.001     -.010583   -.0029692
       felon |   .1187173    .103206     1.15   0.251    -.0840163    .3214509
     alcohol |  -.2180496   .0970583    -2.25   0.025     -.408707   -.0273923
       drugs |   .0177737   .0891098     0.20   0.842    -.1572699    .1928172
       black |  -.0008505   .0822071    -0.01   0.992    -.1623348    .1606338
     married |   .2388998   .0987305     2.42   0.016     .0449577     .432842
        educ |  -.0194548   .0189254    -1.03   0.304    -.0566312    .0177215
         age |   .0005345   .0004228     1.26   0.207     -.000296    .0013651
       _cons |   3.001025   .2438418    12.31   0.000     2.522032    3.480017
------------------------------------------------------------------------------
```

d. Treating the censored durations as if they are uncensored also gives very different estimates from the censored regression. Again, the estimated *alcohol* and *drug* effects are attenuated toward zero, although not as much as when we drop all of the censored observations. In any case, we should use censored regression analysis.

```
. reg ldurat workprg priors tserved felon alcohol drugs black married educ age

      Source |       SS       df       MS              Number of obs =    1445
-------------+------------------------------           F( 10,  1434) =   17.49
       Model |  134.350088    10  13.4350088            Prob > F      =  0.0000
    Residual |  1101.29155  1434  .767985737            R-squared     =  0.1087
-------------+------------------------------           Adj R-squared =  0.1025
       Total |  1235.64163  1444  .855707503            Root MSE      =  .87635

------------------------------------------------------------------------------
      ldurat |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval
-------------+----------------------------------------------------------------
     workprg |    .008758   .0489457     0.18   0.858    -.0872548    .1047709
```

```
      priors |  -.0590636   .0091717    -6.44   0.000     -.077055   -.0410722
     tserved |  -.0094002   .0013006    -7.23   0.000    -.0119516   -.0068488
       felon |   .1785428   .0584077     3.06   0.002     .0639691    .2931165
     alcohol |  -.2628009   .0598092    -4.39   0.000    -.3801238   -.1454779
       drugs |  -.0907441   .0549372    -1.65   0.099      -.19851    .0170217
       black |  -.1791014   .0474354    -3.78   0.000    -.2721516   -.0860511
     married |   .1344326   .0554341     2.43   0.015      .025692    .2431732
        educ |   .0053914   .0099256     0.54   0.587    -.0140789    .0248618
         age |   .0013258   .0002249     5.90   0.000     .0008847    .0017669
       _cons |   3.569168    .137962    25.87   0.000     3.298539    3.839797
------------------------------------------------------------------------------
```

22.2. a. For this question, I use the `streg` command. The `nohr` option means that the $\hat{\beta}_j$ that estimate the β_j iin equation (22.25) are reported, rather than exp($\hat{\beta}_j$). Whether or not a release was "supervised" has no discernible effect on the hazard, whereas, not surprisingly, a history of rules violation while in prison does increase the recidivism hazard.

```
. use recid.dta

. gen failed = ~cens

. stset durat, failure(failed)

     failure event:  failed != 0 & failed < .
obs. time interval:  (0, durat]
 exit on or before:  failure

------------------------------------------------------------------------------
     1445  total obs.
        0  exclusions
------------------------------------------------------------------------------
     1445  obs. remaining, representing
      552  failures in single record/single failure data
    80013  total analysis time at risk, at risk from t =         0
                              earliest observed entry t =         0
                                   last observed exit t =        81

. streg super rules workprg priors tserved felon alcohol drugs black married
        educ age, dist(weibull) nohr

         failure _d:  failed
   analysis time _t:  durat

Weibull regression -- log relative-hazard form

No. of subjects =         1445                     Number of obs   =      1445
No. of failures =          552
Time at risk    =        80013
                                                   LR chi2(12)     =    170.51
Log likelihood  =    -1630.517                     Prob > chi2     =    0.0000

------------------------------------------------------------------------------
```

```
          _t |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       super |  -.0078523   .0979703    -0.08   0.936    -.1998705     .184166
       rules |   .0386963   .0166936     2.32   0.020     .0059774    .0714151
     workprg |   .1039345   .0914158     1.14   0.256    -.0752371    .2831061
      priors |    .086349   .0136871     6.31   0.000     .0595227    .1131752
     tserved |   .0116506    .001933     6.03   0.000     .0078621    .0154392
       felon |  -.3111997   .1074569    -2.90   0.004    -.5218114   -.1005879
     alcohol |   .4510744   .1059953     4.26   0.000     .2433275    .6588214
       drugs |   .2623752   .0982732     2.67   0.008     .0697632    .4549872
       black |    .458454   .0884443     5.18   0.000     .2851063    .6318016
     married |  -.1563693     .10941    -1.43   0.153    -.3708088    .0580703
        educ |  -.0246717    .019442    -1.27   0.204    -.0627772    .0134339
         age |  -.0035167   .0005306    -6.63   0.000    -.0045567   -.0024767
       _cons |  -3.466394   .3105515   -11.16   0.000    -4.075064   -2.857724
-------------+----------------------------------------------------------------
       /ln_p |  -.2142514    .038881    -5.51   0.000    -.2904567   -.1380461
-------------+----------------------------------------------------------------
           p |   .8071455   .0313826                      .7479219    .8710585
         1/p |   1.238934   .0481709                      1.148028    1.337038
------------------------------------------------------------------------------
```

b. The lognormal estimates are given below. The estimated coefficients on *super* and *rules* are consistent with the Weibull results because a decrease in **xδ** shifts up the hazard in the lognormal case.

```
. streg super rules workprg priors tserved felon alcohol drugs black married
        educ age, dist(lognormal)

         failure _d:  failed
   analysis time _t:  durat

Lognormal regression -- accelerated failure-time form

No. of subjects =         1445                     Number of obs   =      1445
No. of failures =          552
Time at risk    =        80013
                                                   LR chi2(12)     =    172.52
Log likelihood  =   -1594.1683                     Prob > chi2     =    0.0000

------------------------------------------------------------------------------
          _t |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval
-------------+----------------------------------------------------------------
       super |   .0328411   .1280452     0.26   0.798    -.2181229    .2838052
       rules |  -.0644316   .0276338    -2.33   0.020    -.1185929   -.0102703
     workprg |  -.0883445   .1208173    -0.73   0.465     -.325142     .148453
      priors |  -.1341294   .0215358    -6.23   0.000    -.1763388   -.0919199
     tserved |  -.0156015   .0033872    -4.61   0.000    -.0222403   -.0089628
       felon |   .4345115   .1460924     2.97   0.003     .1481757    .7208473
     alcohol |  -.6415683   .1439736    -4.46   0.000    -.9237515   -.3593851
       drugs |  -.2785464   .1326321    -2.10   0.036    -.5385007   -.0185922
       black |   -.549173   .1172236    -4.68   0.000    -.7789271   -.3194189
     married |   .3308228   .1399244     2.36   0.018      .056576    .6050695
        educ |   .0234116   .0253302     0.92   0.355    -.0262347     .073058
         age |   .0036626   .0006117     5.99   0.000     .0024637    .0048614
```

```
       _cons |   4.173851   .3580214    11.66   0.000     3.472142     4.87556
-------------+----------------------------------------------------------------
     /ln_sig |   .5904906    .034399    17.17   0.000     .5230698    .6579114
-------------+----------------------------------------------------------------
       sigma |   1.804874   .0620858                      1.687199    1.930755
------------------------------------------------------------------------------
```

c. The coefficient from the lognormal model directly estimates the proportional effect of rules violations on the duration. So, one more rules violation reduces the estimated expected duration by about 6.4%. To obtain the comparable Weibull estimate, we need $-\hat{\beta}_{rules}/\hat{\alpha} = -.0387/.807 \approx -.048$, or about a 4.8% reduction for each rules violation – a bit smaller than the lognormal estimate.

22.3. a. If all durations in the sample are censored, $d_i = 0$ for all i, and so the log-likelihood is $\sum_{i=1}^{N} \log[1 - F(t_i|\mathbf{x}_i;\theta)] = \sum_{i=1}^{N} \log[1 - F(c_i|\mathbf{x}_i;\theta)]$.

b. For the Weibull case, $F(t|\mathbf{x}_i;\boldsymbol{\theta}) = 1 - \exp[-\exp(\mathbf{x}_i\boldsymbol{\beta})t^{\alpha}]$, and so the log-likelihood is $-\sum_{i=1}^{N} \exp(\mathbf{x}_i\boldsymbol{\beta})c_i^{\alpha}$.

c. Without covariates, the Weibull log-likelihood with all observations censored is $-\exp(\beta)\sum_{i=1}^{N} c_i^{\alpha}$. Because $c_i > 0$, we can choose any $\alpha > 0$ so that $\sum_{i=1}^{N} c_i^{\alpha} > 0$. But then, for any $\alpha > 0$, the log-likelihood is maximized by minimizing $\exp(\beta)$ across β. But as $\beta \to -\infty, \exp(\beta) \to 0$. Therefore, plugging any value α into the log-likelihood will lead to β getting more and more negative without bound. So no two real numbers for α and β maximize the log likelihood.

d. It is not possible to estimate duration models from flow data when all durations are right censored.

e. To have all durations censored in a large sample we would have to have $\mathrm{P}(t_i^* > c_i)$ very close to one. But if $\mathrm{P}(t_i^* < t) > 0$ for all $t > 0$ and $c_i > b > 0$,

$$\mathrm{P}(t_i^* > c_i) \leq \mathrm{P}(t_i^* > b) = 1 - \mathrm{P}(t_i^* \leq b),$$

and $\mathrm{P}(t_i^* \leq b) > 0$. So with large samples we should not expect to find that all durations have been censored.

22.4. a. The binary response d_i is equal to one if the observation is uncensored. Because t_i^* is independent of c_i conditional on $\mathbf{x}_i$,

$$\mathrm{P}(d_i = 1|\mathbf{x}_i, c_i) = P(t_i^* \leq c_i|\mathbf{x}_i) = F(c_i|\mathbf{x}_i; \boldsymbol{\theta}).$$

Therefore, the log-likelihood is

$$\sum_{i=1}^{N} \{d_i \log[F(c_i|\mathbf{x}_i; \boldsymbol{\theta})] + (1 - d_i) \log[1 - F(c_i|\mathbf{x}_i; \boldsymbol{\theta})]\},$$

which is just of the usual binary response form.

b. When the distribution is Weibull and $\mathbf{x}_i = 1$, we have (from Problem 22.3*c*), $F(c_i|\boldsymbol{\theta}) = 1 - \exp[-\exp(\beta)c_i^{\alpha}]$, and so the log-likelihood is

$$\mathcal{L}(\alpha, \beta) = \sum_{i=1}^{N} \{d_i \log[1 - \exp[-\exp(\beta)c_i^{\alpha}]] + (1 - d_i) \log(\exp[-\exp(\beta)c_i^{\alpha}])\}.$$

If $c_i = c > 0$ for all i, we have

$$\mathcal{L}(\alpha, \beta) = \sum_{i=1}^{N} \{d_i \log[1 - \exp[-\exp(\beta)c^{\alpha}]] + (1 - d_i) \log(\exp[-\exp(\beta)c^{\alpha}])\}.$$

If we define $\rho \equiv \exp[-\exp(\beta)c^{\alpha}]$ then $0 < \rho < 1$, and the log-likelihood can be written as

$$\sum_{i=1}^{N} [d_i \log(\rho) + (1 - d_i) \log(1 - \rho)].$$

In other words, the log-likelihood is the same for all combinations of α and β that give the same value of ρ. While we can consistently estimate ρ – the fraction of uncensored observations is the maximum likelihood estimator – we cannot recover estimates of α and β. In

other words, α and β are not identified.

c. In the log-normal case, the log-likelihood is based on

$P[\log(t_i^*) \leq \log(c_i)|\mathbf{x}_i] = \phi[(1/\sigma)\log(c_i) - (1/\sigma)\mathbf{x}_i\boldsymbol{\beta}]$, because

$\log(t_i^*)|(\mathbf{x}_i, c_i) \sim Normal(\mathbf{x}_i\boldsymbol{\beta}, \sigma^2)$. The log-likelihood is

$$\begin{aligned}\mathcal{L}(\boldsymbol{\beta}, \sigma^2) &= \sum_{i=1}^{N} \ell_i(\boldsymbol{\beta}, \sigma^2) \\ &= \sum_{i=1}^{N} \{d_i \log(\phi[(1/\sigma)\log(c_i) - (1/\sigma)\mathbf{x}_i\boldsymbol{\beta}]) + (1 - d_i)\log(1 - \phi[(1/\sigma)\log(c_i) - (1/\sigma)\mathbf{x}_i\boldsymbol{\beta}])\}.\end{aligned}$$

Even though $x_{i1} \equiv 1$, $1/\sigma$ is identified as the coefficient on $\log(c_i)$, provided c_i varies. Then, of course, we can identify $\boldsymbol{\beta}$ because $\boldsymbol{\beta}/\sigma$ is generally identified from the probit log-likelihood. [We would need to assume the usual condition that rank $E(\mathbf{x}_i'\mathbf{x}_i) = K$.] If $c_i = c$ for all i then the intercept effectively becomes $(1/\sigma)\log(c) - \beta_1/\sigma$; along with β_j/σ, $j = 2, \ldots, K$, these are the only parameters we can identify. We cannot separately identify $\boldsymbol{\beta}$ or σ. This is the same situation we faced in Section 19.2.1.

22.5. a. We have

$$\begin{aligned}P(t_i^* \leq t|\mathbf{x}_i, a_i, c_i, s_i = 1) &= P(t_i^* \leq t|\mathbf{x}_i, t_i^* > b - a_i) \\ &= P(t_i^* \leq t, t_i^* > b - a_i|\mathbf{x}_i)/P(t_i^* > b - a_i|\mathbf{x}_i) = P(b - a_i < t_i^* \leq t|\mathbf{x}_i)/P(t_i^* > b - a_i|\mathbf{x}_i) \\ &= [F(t|\mathbf{x}_i) - F(b - a_i|\mathbf{x}_i)/[1 - F(b - a_i|\mathbf{x}_i)]\end{aligned}$$

where we use the fact that $t < b - a_i$.

b. The derivative of the cdf in part a, with respect to t, is simply $f(t|\mathbf{x}_i)/[1 - F(b - a_i|\mathbf{x}_i)]$.

c. Because $s_i = 1[t_i^* > b - a_i]$ and $t_i = c_i$ if and only if $t_i^* \geq c_i$, we have

$$\begin{aligned}P(t_i = c_i|\mathbf{x}_i, a_i, c_i, s_i = 1) &= P(t_i^* \geq c_i|\mathbf{x}_i, t_i^* > b - a_i) \\ &= P(t_i^* \geq c_i|\mathbf{x}_i)/P(t_i^* \geq b - a_i|\mathbf{x}_i) \\ &= [1 - F(c_i|\mathbf{x}_i)]/[1 - F(b - a_i|\mathbf{x}_i)]\end{aligned}$$

where the third equality follows because $c_i > b - a_i$.

d. Parts b and c show that the density of the censored variable, t_i, conditional on $(\mathbf{x}_i, a_i, c_i, s_i = 1)$, can be written as

$$\frac{[f(t|\mathbf{x}_i)]^{1[t<c_i]}[1 - F(c_i|\mathbf{x}_i)]^{1[t=c_i]}}{[1 - F(b - a_i|\mathbf{x}_i)]}.$$

Showing the dependence on the parameters, plugging in t_i, noting that $d_i = 1[t_i < c_i]$, and taking the log gives

$$d_i \log[f(t|\mathbf{x}_i;\boldsymbol{\theta})] + (1 - d_i)\log[1 - F(c_i|\mathbf{x}_i;\boldsymbol{\theta})] - \log[1 - F(b - a_i|\mathbf{x}_i;\boldsymbol{\theta})].$$

Summing across all N observations gives equation (22.30).

22.6. In what follows, we initially suppress dependence on the parameters.

a. Because $s_i = 1[t_i^* \geq b - a_i]$,

$$\begin{aligned} \mathrm{P}(a_i \leq a, s_i = 1|\mathbf{x}_i) &= \mathrm{P}(a_i \leq a, t_i^* \geq b - a_i|\mathbf{x}_i) \\ &= \int_0^a \int_{b-\mathfrak{u}}^{\infty} q(\mathfrak{u}, \mathfrak{v}|\mathbf{x}_i) d\mathfrak{v} d\mathfrak{u}, \end{aligned}$$

where $q(\cdot, \cdot|\mathbf{x}_i)$ denotes the joint density of (a_i, t_i^*) given $\mathbf{x}_i$. By conditional independence, $q(a, t|\mathbf{x}_i) = k(a|\mathbf{x}_i)f(t|\mathbf{x}_i)$, and so the double integral is

$$\int_0^a \left(\int_{b-\mathfrak{u}}^{\infty} f(\mathfrak{v}|\mathbf{x}_i) d\mathfrak{v} \right) k(\mathfrak{u}|\mathbf{x}_i) d\mathfrak{u} = \int_0^a [1 - F(b - \mathfrak{u}|\mathbf{x}_i)] k(\mathfrak{u}|\mathbf{x}_i) d\mathfrak{u}$$

because $\int_{b-\mathfrak{u}}^{\infty} f(\mathfrak{v}|\mathbf{x}_i) d\mathfrak{v} = [1 - F(b - \mathfrak{u}|\mathbf{x}_i)]$.

b. From the hint, we first compute $\mathrm{E}(s_i = 1|a_i, \mathbf{x}_i) = P(t_i^* \geq b - a_i|\mathbf{x}_i) = 1 - F(b - a_i|\mathbf{x}_i)$. Next, we compute the expected value of this with respect to the distribution $\mathrm{D}(a_i|\mathbf{x}_i)$, which is simply equation (22.32).

c. The conditional cdf is obtained by dividing the answer from part a by the answer from part b. The density is just the derivative of the resulting expression with respect to a; by the

fundamental theorem of calculus, the derivative is (22.31).

d. When $b = 1$ and $k(a|\mathbf{x}_i) = 1$, all $0 < a < 1$, the numerator of (22.31) is just $1 - F(1 - a|\mathbf{x}_i)$. The denominator is simply $\int_0^1 [1 - F(1 - \text{u}|\mathbf{x}_i)]d\text{u}$.

e. In the Weibull case, $1 - F(1 - a|\mathbf{x}_i) = \exp[-\exp(\mathbf{x}_i\beta)(1 - a)^\alpha]$ and the denominator is $\int_0^1 \exp[-\exp(\mathbf{x}_i\boldsymbol{\beta})(1 - \text{u})^\alpha]d\text{u}$. This integral cannot be solved in closed form unless $\alpha = 1$.

22.7. a. For notational simplicity, the parameters in the densities are suppressed. Then, by equation (22.22) and $\text{D}(a_i|c_i, \mathbf{x}_i) = \text{D}(a_i|\mathbf{x}_i)$, the density of (a_i, t_i^*) given $(c_i, \mathbf{x}_i)$ does not depend on c_i and is given by $k(a|\mathbf{x}_i)f(t|\mathbf{x}_i)$ for $0 < a < b$ and $0 < t < \infty$. This is also the conditional density of (a_i, t_i) given $(c_i, \mathbf{x}_i)$ for $t < c_i$, that is, for values of t corresponding to being uncensored. For $t = c_i$, the density is $k(a|\mathbf{x}_i)[1 - F(c_i|\mathbf{x}_i)]$ by the usual right censoring argument. Now, the probability of observing the random draw $(a_i, c_i, \mathbf{x}_i, t_i)$, conditional on $\mathbf{x}_i$, is $\text{P}(t_i^* \geq b - a_i, \mathbf{x}_i)$, which is exactly (22.32). From the standard result for densities for truncated distributions, the density of (a_i, t_i) given $(c_i, d_i, \mathbf{x}_i)$ and $s_i = 1$ is

$$k(a|\mathbf{x}_i)[f(t|\mathbf{x}_i)]^{d_i}[1 - F(c_i|\mathbf{x}_i)]^{(1-d_i)}/P(s_i = 1|\mathbf{x}_i),$$

for all combinations (a, t) such as that $s_i = 1$. Putting in the observed data, inserting the parameters, and taking the log gives (22.56).

b. We have the usual trade-off between robustness and efficiency. Using the log likelihood (20.56) results in more efficient estimators provided we have the two densities correctly specified; (20.30) requires us to only specify $f(\cdot|\mathbf{x}_i)$.

22.8. a. Again I suppress the parameters in the densities. Let $z_i \equiv t_i^* - b$, so that, by assumption, a_i and z_i are independent conditional on $\mathbf{x}_i$. The conditional density of z_i is simply $g(z|\mathbf{x}_i) \equiv f(z + b|\mathbf{x}_i)$, $z > -b$. By the usual convolution formula for the density of a sum of

independent random variables,

$$h(r|\mathbf{x}_i) = \int_0^b k(\mathrm{u}|\mathbf{x}_i)g(r - \mathrm{u}|\mathbf{x}_i)d\mathrm{u} = \int_0^b k(\mathrm{u}|\mathbf{x}_i)f(r + b - \mathrm{u}|\mathbf{x}_i)d\mathrm{u},$$

where $f(r + b - \mathrm{u}|\mathbf{x}_i) = 0$ if $r + b - \mathrm{u} \leq 0$. When $r > 0, r + b - \mathrm{u} > 0$ for all $0 \leq \mathrm{u} \leq b$, and so we need not modify the formula.

b. As usual for right censoring, $\mathrm{P}(r_i = q|\mathbf{x}_i) = \mathrm{P}(r_i^* > q|\mathbf{x}_i) = 1 - H(q|\mathbf{x}_i)$.

c. The argument is now essentially the same as the first stock sampling argument treated in Section 22.3.3, except that we cannot condition on a_i. Instead, for $0 < r < q$, $\mathrm{P}(r_i \leq r|\mathbf{x}_i, s_i = 1) = [H(r|\mathbf{x}_i) - H(0|\mathbf{x}_i)]/\mathrm{P}(s_i = 1|\mathbf{x}_i)$ – and so the density for $0 < r < q$ is $h(r|\mathbf{x}_i)/P(s_i = 1|\mathbf{x}_i)$ – and $\mathrm{P}(r_i = q|\mathbf{x}_i, s_i = 1) = [1 - H(q|\mathbf{x}_i)]/P(s_i = 1|\mathbf{x}_i)$, where $\mathrm{P}(s_i = 1|\mathbf{x}_i)$ is given by (22.32).

d. With $b = 1$ and a uniform distribution for a_i, the log-likelihood reduces to

$$d_i \log[h(r_i|\mathbf{x}_i; \boldsymbol{\theta}, \boldsymbol{\eta})] + (1 - d_i) \log[1 - H(r_i|\mathbf{x}_i; \boldsymbol{\theta}, \boldsymbol{\eta})] - \log\left\{\int_0^1 [1 - F(1 - \mathrm{u}|\mathbf{x}_i; \boldsymbol{\theta})]d\mathrm{u}\right\}.$$

22.9. a. Let ω be the value for type B people. Then we must have

$$\rho\eta + (1 - \rho)\omega = 1$$

or $\omega = (1 - \rho\eta)/(1 - \rho)$.

b. The cdf conditional on $(\mathbf{x}, v)$ is $F(t|\mathbf{x}, v; \alpha, \boldsymbol{\beta}) = 1 - \exp[-v \exp(\mathbf{x}\boldsymbol{\beta})t^\alpha]$. Therefore, the cdf conditional on $\mathbf{x}$ is obtained by averaging out v:

$$\begin{aligned} G(t|\mathbf{x}; \alpha, \boldsymbol{\beta}, \eta, \rho) = {} & \rho\{1 - \exp[-\eta \exp(\mathbf{x}\boldsymbol{\beta})t^\alpha]\} \\ & + (1 - \rho)\{1 - \exp[-((1 - \rho\eta)/(1 - \rho)) \exp(\mathbf{x}\boldsymbol{\beta})t^\alpha]\}. \end{aligned}$$

c. The density function is just the derivative with respect to t:

$$g(t|\mathbf{x};\alpha,\boldsymbol{\beta},\eta,\rho) = \rho\eta\exp(\mathbf{x}\boldsymbol{\beta})t^{\alpha-1}\exp[-\eta\exp(\mathbf{x}\boldsymbol{\beta})t^{\alpha}] + (1-\rho\eta)\exp(\mathbf{x}\boldsymbol{\beta})\alpha t^{\alpha-1}\exp[-((1-\rho\eta)/(1-\rho))\exp(\mathbf{x}\boldsymbol{\beta})t^{\alpha}]\}$$

If none of the durations are censore, the log-likelihood for each observation i is obtained by taking the log of this density in plugging in $(\mathbf{x}_i, t_i)$. If we have right censored data with censoring values c_i, the log likelihood takes on the usual form:

$$d_i \log[g(t_i|\mathbf{x}_i;\alpha,\boldsymbol{\beta},\eta,\rho)] + (1-d_i)\log[G(t_i|\mathbf{x}_i;\alpha,\boldsymbol{\beta},\eta,\rho)]$$

where d_i is the dummy variable equal to one of the observation is not censored. The log likelihood for the entire sample is a very smooth function of all parameters. As a computational device, it might be better to replace η with, say, $\exp(\xi)/[1+\exp(\xi)]$ for $-\infty < \xi < \infty$ and similarly for ρ.

22.10. a. If $\mathrm{P}(T > a_{m-1}) = 0$ then $\mathrm{P}(T > a_m) = 0$ because $a_m > a_{m-1}$ in which case the equality is trivial. So assume that $\mathrm{P}(T > a_{m-1}) > 0$. Then, by definition of conditional probability,

$$\begin{aligned}\mathrm{P}(T > a_m | T > a_{m-1}) &= \mathrm{P}(T > a_m, T > a_{m-1})/\mathrm{P}(T > a_{m-1}) \\ &= \mathrm{P}(T > a_m)/\mathrm{P}(T > a_{m-1})\end{aligned}$$

since the events $\{T > a_m, T > a_{m-1}\}$ and $\{T > a_m\}$ are identical when $a_m > a_{m-1}$. Rearranging the equality gives the result.

b. We can use induction to obtain an algebraically simple proof. First, equation (22.48) holds trivially when $m = 1$: $\mathrm{P}(T > a_1) = \mathrm{P}(T > a_1 | T > 0)$ because $\mathrm{P}(T > 0) = 1$. Now assume that (22.48) holds for any $m \geq 1$. We show it holds for $m + 1$. By part a,

$$\begin{aligned}\mathrm{P}(T > a_{m+1}) &= \mathrm{P}(T > a_{m+1} | T > a_m)\mathrm{P}(T > a_m) \\ &= \mathrm{P}(T > a_{m+1} | T > a_m) \prod_{r=1}^{m} \mathrm{P}(T > a_r | T > a_{r-1})\end{aligned}$$

because $\mathrm{P}(T > a_m) = \prod_{r=1}^{m} \mathrm{P}(T > a_r | T > a_{r-1})$ by the induction hypothesis. It follows that

$$\mathrm{P}(T > a_{m+1}) = \prod_{r=1}^{m+1} \mathrm{P}(T > a_r | T > a_{r-1})$$

and this completes the proof.

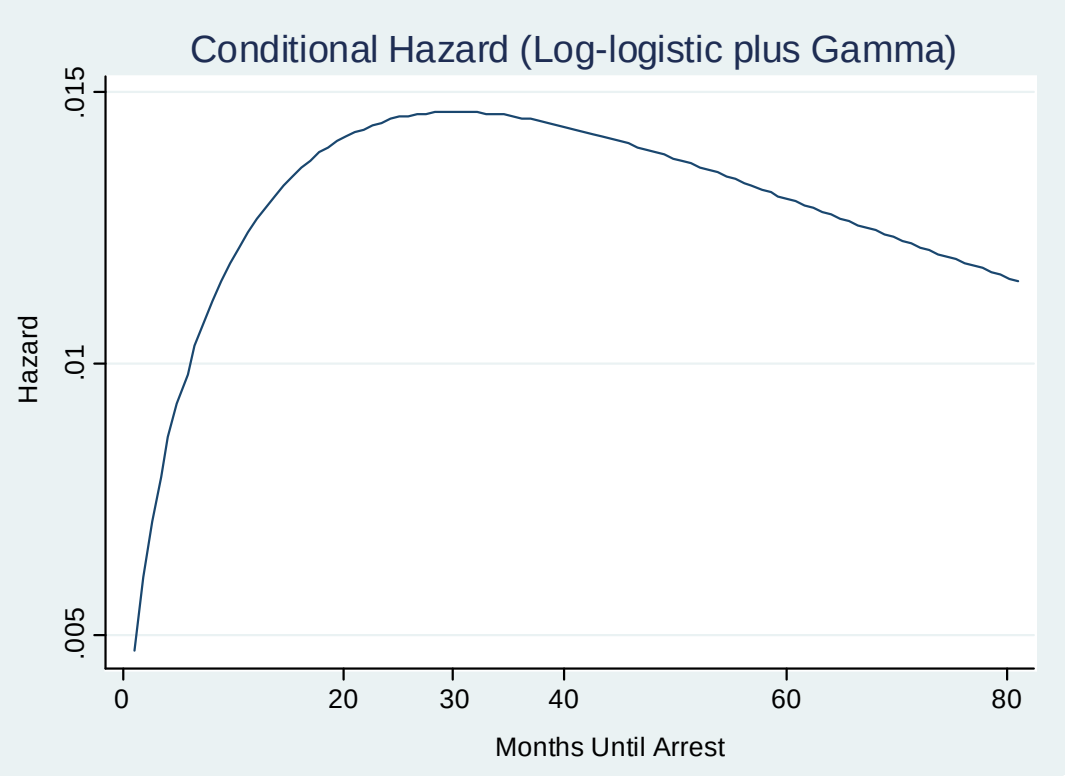
22.11. a. The estimates from the log-logistic model with gamma-distributed hazard are given below. For comparison purposes, the Stata output used to produce Table 22.2 follows. The log likelihood for the log-logistic model is $-1,587.92$ and that for the Weibull model (both with gamma heterogeneity) is $-1,584.92$. The Weibull model fits somewhat better. (A Vuong model selection statistic could be computed to see if the fit is statistically better.)

```
. streg  workprg priors tserved felon alcohol drugs black married educ age,
         d(loglogistic) fr(gamma)

         failure _d:  failed
   analysis time _t:  durat

Loglogistic regression -- accelerated failure-time form
                       Gamma frailty

No. of subjects =         1445                     Number of obs   =      1445
No. of failures =          552
Time at risk    =        80013
                                                   LR chi2(10)     =    132.67
Log likelihood  =   -1587.9185                     Prob > chi2     =    0.0000

------------------------------------------------------------------------------
          _t |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     workprg |   .0098501     .11888     0.08   0.934    -.2231504    .2428505
      priors |  -.1488187   .0231592    -6.43   0.000      -.19421   -.1034275
     tserved |  -.0190707   .0036147    -5.28   0.000    -.0261553   -.0119861
       felon |   .4219072   .1502377     2.81   0.005     .1274467    .7163677
     alcohol |  -.6597875    .147537    -4.47   0.000    -.9489547   -.3706203
       drugs |  -.2156168    .131043    -1.65   0.100    -.4724563    .0412227
       black |  -.4355534   .1209008    -3.60   0.000    -.6725147   -.1985921
     married |   .4345344   .1353415     3.21   0.001       .16927    .6997988
        educ |   .0243336   .0265813     0.92   0.360    -.0277648    .0764321
         age |   .0032531    .000633     5.14   0.000     .0020124    .0044937
       _cons |    3.34599   .3570848     9.37   0.000     2.646117    4.045864
-------------+----------------------------------------------------------------
     /ln_gam |  -.3648587   .0716481    -5.09   0.000    -.5052864   -.2244309
     /ln_the |   .8176437   .1998151     4.09   0.000     .4260133    1.209274
-------------+----------------------------------------------------------------
       gamma |   .6942948   .0497449                      .6033327    .7989708
```

```
       theta |   2.265156   .4526125                      1.531141    3.351051
------------------------------------------------------------------------------
Likelihood-ratio test of theta=0: chibar2(01) =    46.05 Prob>=chibar2 = 0.000

. streg  workprg priors tserved felon alcohol drugs black married educ age,
         d(weibull) fr(gamma) nohr

         failure _d:  nocens
   analysis time _t:  durat


Weibull regression -- log relative-hazard form
                      Gamma frailty

No. of subjects =         1445                     Number of obs   =      1445
No. of failures =          552
Time at risk    =        80013
                                                   LR chi2(10)     =    143.82
Log likelihood  =   -1584.9172                     Prob > chi2     =    0.0000

------------------------------------------------------------------------------
          _t |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     workprg |   .0073827   .2038775     0.04   0.971    -.3922099    .4069753
      priors |   .2431142   .0421543     5.77   0.000     .1604933    .3257352
     tserved |   .0349363   .0070177     4.98   0.000     .0211818    .0486908
       felon |  -.7909533   .2666084    -2.97   0.003    -1.313496   -.2684104
     alcohol |   1.173558   .2805222     4.18   0.000     .6237451    1.723372
       drugs |   .2847665   .2233072     1.28   0.202    -.1529074    .7224405
       black |   .7715762   .2038289     3.79   0.000      .372079    1.171073
     married |  -.8057042   .2578214    -3.13   0.002    -1.311025   -.3003834
        educ |  -.0271193    .044901    -0.60   0.546    -.1151237     .060885
         age |  -.0052162   .0009974    -5.23   0.000    -.0071711   -.0032613
       _cons |  -5.393658    .720245    -7.49   0.000    -6.805312   -3.982004
-------------+----------------------------------------------------------------
       /ln_p |   .5352553   .0951206     5.63   0.000     .3488225    .7216882
     /ln_the |   1.790243   .1788498    10.01   0.000     1.439703    2.140782
-------------+----------------------------------------------------------------
           p |   1.707884   .1624549                      1.417398    2.057904
         1/p |   .5855198    .055695                      .4859312    .7055184
       theta |   5.990906   1.071472                      4.219445    8.506084
------------------------------------------------------------------------------
Likelihood-ratio test of theta=0: chibar2(01) =    96.23 Prob>=chibar2 = 0.000
```

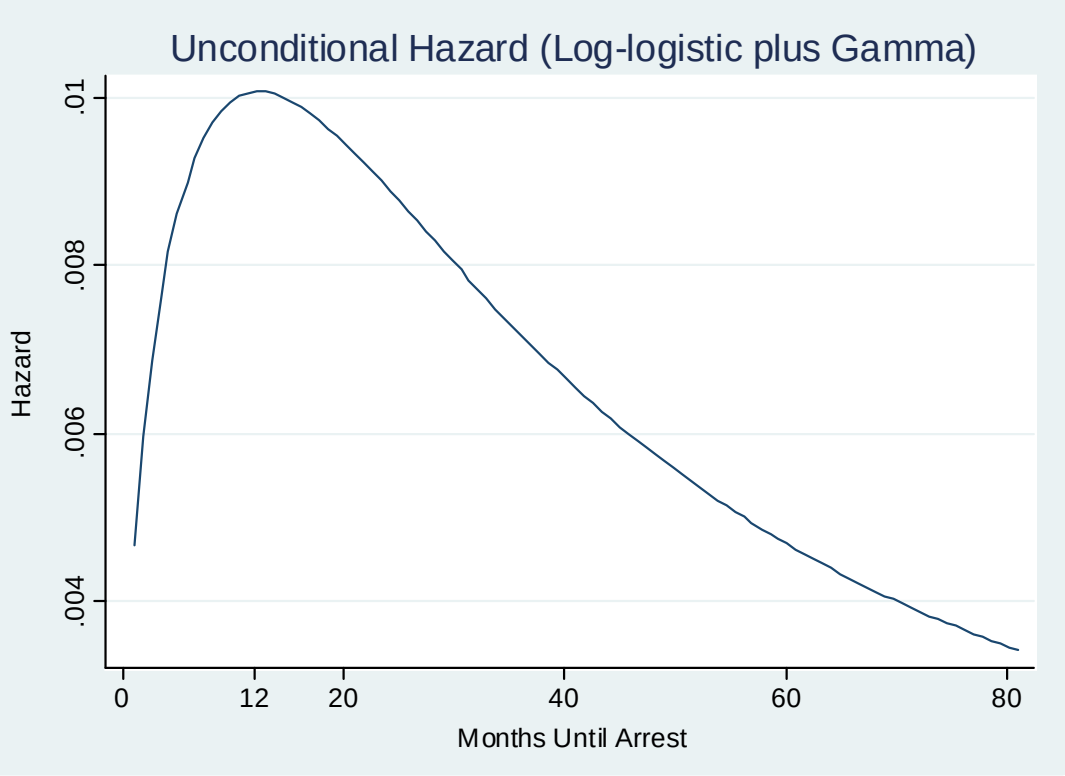
b. The conditional hazard is plotted below. Its shape is very different from the Weibull case, which is always upward sloping. (See Figure 22.3.) The hazard for the log-logistic model with gamma heterogeneity has its maximum value near 30 weeks, and it falls off gradually after that.

```
. stcurve, haz alpha1
```

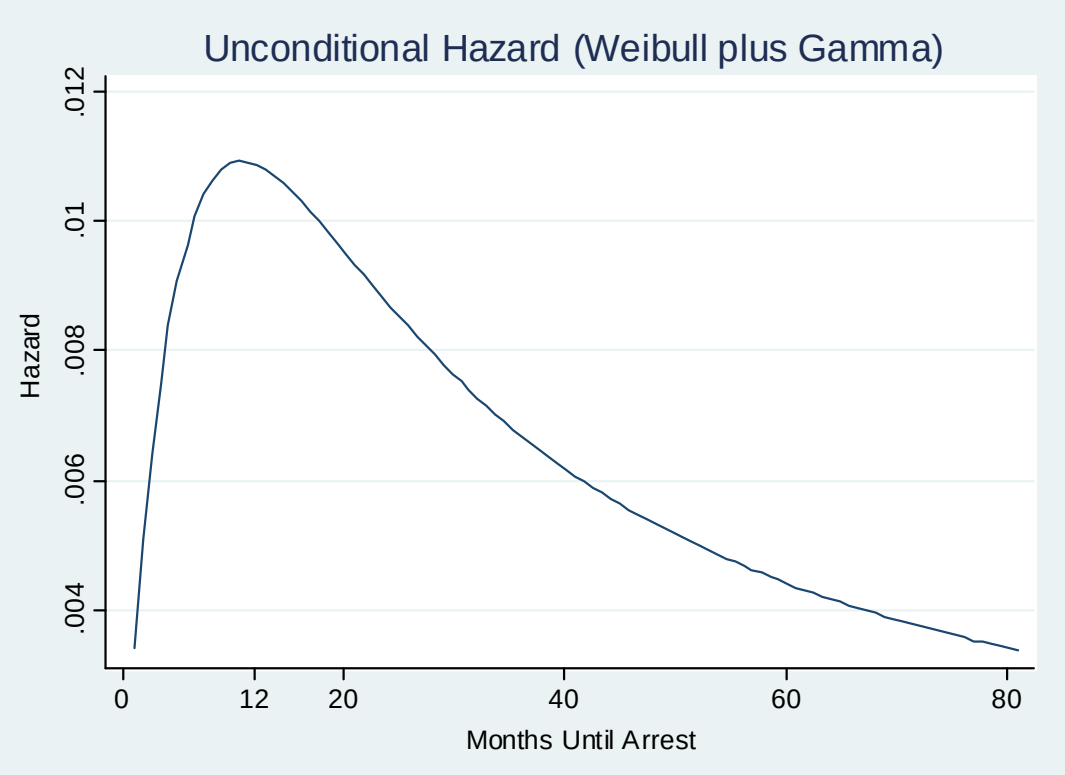


c. The unconditional hazard is plotted below. While it also has a hump shape, the maximum value of the unconditional hazard is around 12 – which is well below that for the conditional hazard.

```
. stcurve, haz
(option unconditional assumed)
```



There is a final point worth making about this example that builds on the discussion in Section 22.3.4. Below is the graph in Figure 22.4, reproduced for convenience. It is the unconditional hazard for the Weibull model with gamma heterogeneity. While it differs somewhat from the unconditional hazard for the log-logistic model with gamma heterogeneity, it is practically very similar. Both hazards have sharp increases until about 12 months, and then fall off to zero more gradually. In other words, when we study features of the distribution $D(t_i^*|\mathbf{x}_i)$ – which is what we can generally hope to identify – we get pretty similar findings. Yet the hazards based on $D(t_i^*|\mathbf{x}_i, v_i)$ are very different. Recall that the hazard for $D(t_i^*|\mathbf{x}_i, v_i)$ in the Weibull case is of the proportional hazard form while that for the log-logistic is not (which is apparent when studying the plots of the conditional hazards). Given that the models fit the data roughly equally well, and that they give similar shapes for the hazard of $D(t_i^*|\mathbf{x}_i)$, it seems that trying to decide whether the conditional hazard has a hump shape, as in part b, or is strictly increasing, as in Figure 22.3, seems pretty hopeless.


Unconditional Hazard (Weibull plus Gamma)
Hazard
.004
.006
.008
.01
.012
0
12
20
40
60
80
Months Until Arrest

Incidentally, if we use a gamma distribution for $\mathrm{D}(t_i^*|\mathbf{x}_i, v_i)$ with gamma heterogeneity, we obtain an unconditional hazard very similar to the Weibull and log-logistic cases. The conditional hazard is similar to the log-logistic case, and the log likelihood value is $-1,585.09$.