Problem 1 (a) We have w_t - mean - zero r.v.

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t \qquad (1)$$

Define $\mu \equiv E(y_t)$ and assume that $E(y_t) = E(y_{t-1}) = E(y_{t-2})$. (We provide a basis for this assumption a bit later in the course.) Taking expectations of (1) we have

$$E(y_t) = c + \phi_1 \overset{\mu}{E y_{t-1}} + \phi_2 \overset{\mu}{E y_{t-2}} + \overset{0}{E w_t}$$

This gives

$$(1 - \phi_1 - \phi_2)\mu = c \qquad (3)$$

Substitute for c in (1) using (3) and write the result as

$$(y_t - \mu) = \phi_1 (y_{t-1} - \mu) + \phi_2 (y_{t-2} - \mu) + w_t \quad (4)$$

E.g. MA

$y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

MA(∞)

$y_t = \mu + \varepsilon_t + \sum \theta_i \varepsilon_{t-i}$

Write (4) as

$$\begin{bmatrix} (y_t - \mu) \\ (y_{t-1} - \mu) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (y_{t-1} - \mu) \\ (y_{t-2} - \mu) \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \end{bmatrix}$$

or as

$$\boxed{\xi_t = F\,\xi_{t-1} + V_t \qquad (2)}$$

where

$$\underset{(2\times 1)}{\xi_t} \equiv \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \end{bmatrix}, \quad \underset{(2\times 2)}{F} \equiv \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \underset{(2\times 1)}{V_t} \equiv \begin{bmatrix} w_t \\ 0 \end{bmatrix}$$

(b) Consider again the 2^{nd} order DE in (1). To find the Roots of this DE using C+W's method and notation we examine the homogeneous Part of the DE which in C+W is Typically written as

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0 \qquad (5)$$

We Then Solve for the Roots of the 2^{nd} order DE in (5), b_1 and b_2, by Solving the

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Characteristic equation

$$\boxed{b^2 + a_1 b + a_2 = 0 \qquad (6)}$$

Now, Re-write (1) as

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = c + w_t \qquad (1')$$

Comparing the LHS of (1′) to (5) it is clear that

$$\boxed{a_1 = -\phi_1 \quad \text{and} \quad a_2 = -\phi_2 \qquad (7)}$$

The Eigenvalues of F solve $\text{Det}(F - rI) = 0$

or $\text{Det}\begin{bmatrix} \phi_1 - r & \phi_2 \\ 1 & -r \end{bmatrix} = -r(\phi_1 - r) - \phi_2 = 0$ or

$$r^2 - \phi_1 r - \phi_2 = 0 \qquad (8)$$

using (7) This Becomes

$$\boxed{r^2 + a_1 r + a_2 = 0 \qquad (9)}$$

Comparing (9) to (6) we have that the eigenvalues of F, r_1 and r_2, are the same as the roots of the 2nd order DE, b_1 and b_2.

Problem 2: Begin from

$$1 - \phi_1 z - \phi_2 z^2 = 0 \qquad (1)$$

Multiply through by z^{-2} to get

$$z^{-2} - \phi_1 z^{-1} - \phi_2 = 0 \qquad (2)$$

Define $\boxed{\lambda \equiv z^{-1}}$ and (2) becomes

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0 \qquad (3)$$

So, if z_1 and z_2 solve (1) then $\lambda_1 = \frac{1}{z_1}$ and $\lambda_2 = \frac{1}{z_2}$ solve (3)

QED

Problem 3: We have

$$-C_t = -C_{t+1} + a_1 K_{t+1} + a_2 e_{t+1} \qquad (1a)$$

$$K_{t+1} = b_1 K_t + b_2 w_{t+1} + b_3 C_t \qquad (1b)$$

(a) write (1b) as

$$C_t = b_3^{-1}\left[K_{t+1} - b_1 K_t - b_2 w_{t+1}\right]. \qquad (4)$$

Use (4) to eliminate C_t and C_{t+1} in (1a).

$$-b_3^{-1}\left[K_{t+1} - b_1 K_t - b_2 w_{t+1}\right] =$$

$$-b_3^{-1}\left[K_{t+2} - b_1 K_{t+1} - b_2 w_{t+2}\right] + a_1 K_{t+1} + a_2 e_{t+1}$$

or, multiplying through by b_3

$$-K_{t+1} + b_1 K_t + b_2 w_{t+1} =$$

$$-K_{t+2} + b_1 K_{t+1} + b_2 w_{t+2} + b_3 a_1 K_{t+1} + b_3 a_2 e_{t+1}$$

or, collecting,

$$K_{t+2} = (1 + b_1 + a_1 b_3) K_{t+1} - b_1 K_t + \eta_{t+2} \qquad (5)$$

where

$$\eta_{t+2} = \left[b_2 w_{t+2} - b_2 w_{t+1} + a_2 b_3 e_{t+1}\right] \qquad (6)$$

Lag all variables in (4) 2 periods to get

$$K_t = \phi_1 K_{t-1} - b_1 K_{t-2} + \eta_t \qquad (5')$$

where

$$\phi_1 \equiv (1 + b_1 + a_1 b_3) \qquad (7)$$

and

$$\eta_t \equiv \left[b_2 w_t - b_2 w_{t-1} + a_2 b_3 e_{t-1} \right] \qquad (6')$$

Eqn (4) can be written as

$$\begin{bmatrix} K_t \\ K_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & -b_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} K_{t-1} \\ K_{t-2} \end{bmatrix} + \begin{bmatrix} \eta_t \\ 0 \end{bmatrix} \qquad (8)$$

which gives

$$\xi_t = F \xi_{t-1} + V_t \qquad (2)$$

where $\xi_t \equiv \begin{bmatrix} K_t \\ K_{t-1} \end{bmatrix}$, $F \equiv \begin{bmatrix} \phi_1 & -b_1 \\ 1 & 0 \end{bmatrix}$, and $V_t \equiv \begin{bmatrix} \eta_t \\ 0 \end{bmatrix}$

and where ϕ_1 and η_t are given by (7) and (6'), respectively.

(b) Re-write (1a, b) as

$$a_1 K_{t+1} - C_{t+1} = -C_t - a_2 e_{t+1} \qquad (1a')$$

$$K_{t+1} = b_1 K_t + b_3 C_t + b_2 w_{t+1} \qquad (1b')$$

or

$$\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ b_1 & b_3 \end{bmatrix}\begin{bmatrix} K_t \\ C_t \end{bmatrix} + \begin{bmatrix} -a_2 & 0 \\ 0 & b_2 \end{bmatrix}\begin{bmatrix} e_{t+1} \\ w_{t+1} \end{bmatrix} \qquad (9)$$

Note that $\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & a_1 \end{bmatrix}$ So, multiplying (9)

Through by $\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$ gives

$$\begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & a_1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ b_1 & b_3 \end{bmatrix}\begin{bmatrix} K_t \\ C_t \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & a_1 \end{bmatrix}\begin{bmatrix} -a_2 & 0 \\ 0 & b_2 \end{bmatrix}\begin{bmatrix} e_{t+1} \\ w_{t+1} \end{bmatrix}$$

or

$$\begin{bmatrix} K_{t+1} \\ C_{t+1} \end{bmatrix} = \begin{bmatrix} b_1 & b_3 \\ a_1 b_1 & (b_3 a_1 + 1) \end{bmatrix}\begin{bmatrix} K_t \\ C_t \end{bmatrix} + \begin{bmatrix} 0 & b_2 \\ a_2 & a_1 b_2 \end{bmatrix}\begin{bmatrix} e_{t+1} \\ w_{t+1} \end{bmatrix} \qquad (10)$$

or $X_{t+1} = A X_t + B u_{t+1}$ (3)

where

$$X_t \equiv \begin{bmatrix} K_t \\ C_t \end{bmatrix}, \quad A \equiv \begin{bmatrix} b_1 & b_3 \\ a_1 b_1 & (b_3 a_1 + 1) \end{bmatrix}, \quad u_{t+1} \equiv \begin{bmatrix} e_{t+1} \\ \omega_{t+1} \end{bmatrix}$$

and $B \equiv \begin{bmatrix} 0 & b_2 \\ a_2 & a_1 b_2 \end{bmatrix}$

(c) The Eigenvalues of F Solve

$$Det(F - r I_2) = 0 \quad \text{or}$$

$$(\phi_1 - r)(-r) + b_1 = 0 \quad \text{or, using}$$

$$r^2 - (1 + b_1 + a_1 b_3) r + b_1 = 0 \qquad (11)$$

The Eigenvalues of A solve

$$Det(A - \lambda I_2) = 0 \quad \text{or}$$

$$(b_1 - \lambda)\left[(b_3 a_1 + 1) - \lambda\right] - a_1 b_1 b_3 = 0 \quad \text{or}$$

$$\lambda^2 - \left[(b_3 a_1 + 1) + b_1\right]\lambda - a_1 b_1 b_3 + b_1(b_3 a_1 + 1) = 0 \quad \text{or}$$

$$\lambda^2 - \left[1 + b_1 + a_1 b_3\right]\lambda + b_1 = 0 \qquad (12)$$

Comparing (11) and (12) we see that the Eigenvalues of F, r_1 and r_2, are the same as the eigenvalues of A, λ_1 and λ_2.