"Ans To P Set 2, Part A"

Econ 6020: Macro Theory I

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II DGE Capital Accumulation. Part A:

Additional Problem 1.

We have, using the Cobb douglass Production function,

$$\boxed{\Delta K_{t+1} = A K_t^{\alpha} - C_t - \delta K_t \qquad (1)}$$

and

$$\boxed{\Delta C_{t+1} = -\frac{u'(C_t)}{u''(C_t)}\left[1 - \frac{1}{\beta\left[\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\right]}\right] \qquad (2)}$$

(a). Consider $\Delta\Theta > 0$ from Θ_0 to $\Theta_1 > \Theta_0$

(Since $\beta \equiv \frac{1}{1+\Theta}$, an increase in Θ is a decline in β.)

→ Note from (2) that $\Delta C_{t+1} = 0$ where

$$\beta\left[\alpha A K_S^{\alpha-1} + 1 - \delta\right] = 1 \qquad \text{or using } \beta = \frac{1}{1+\Theta}$$

$$\alpha A K_S^{\alpha-1} = \Theta + \delta \qquad \text{or}$$

↘

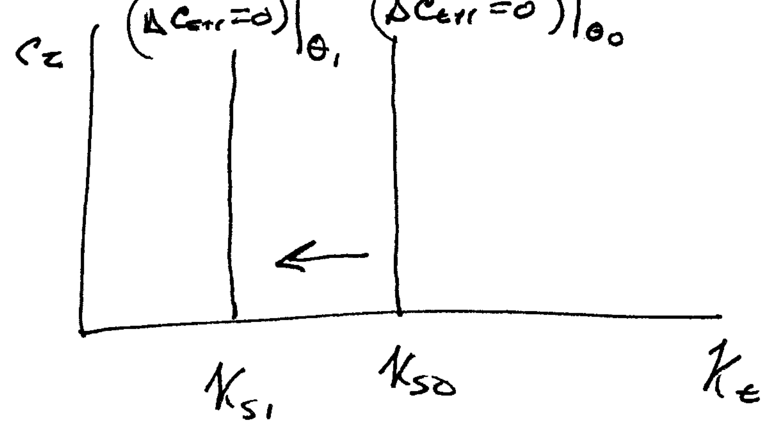
$$K_s^{\alpha-1} = \left[\frac{\theta+\delta}{\alpha A}\right] \quad \text{or}$$

$$K_s = \left[\frac{\theta+\delta}{\alpha A}\right]^{\frac{1}{\alpha-1}} \quad \text{or}$$

$$\boxed{K_s = \left[\frac{\alpha A}{\theta+\delta}\right]^{\frac{1}{1-\alpha}} \qquad (3)}$$

Based on (3) we see that $\Delta\theta > 0$ from θ_0 to $\theta_1 > \theta_0$ causes $\Delta K_s < 0$ from K_{s0} to $K_{s1} < K_{s0}$.

This is a shift to the left of the $\Delta C_{t+1} = 0$ locus

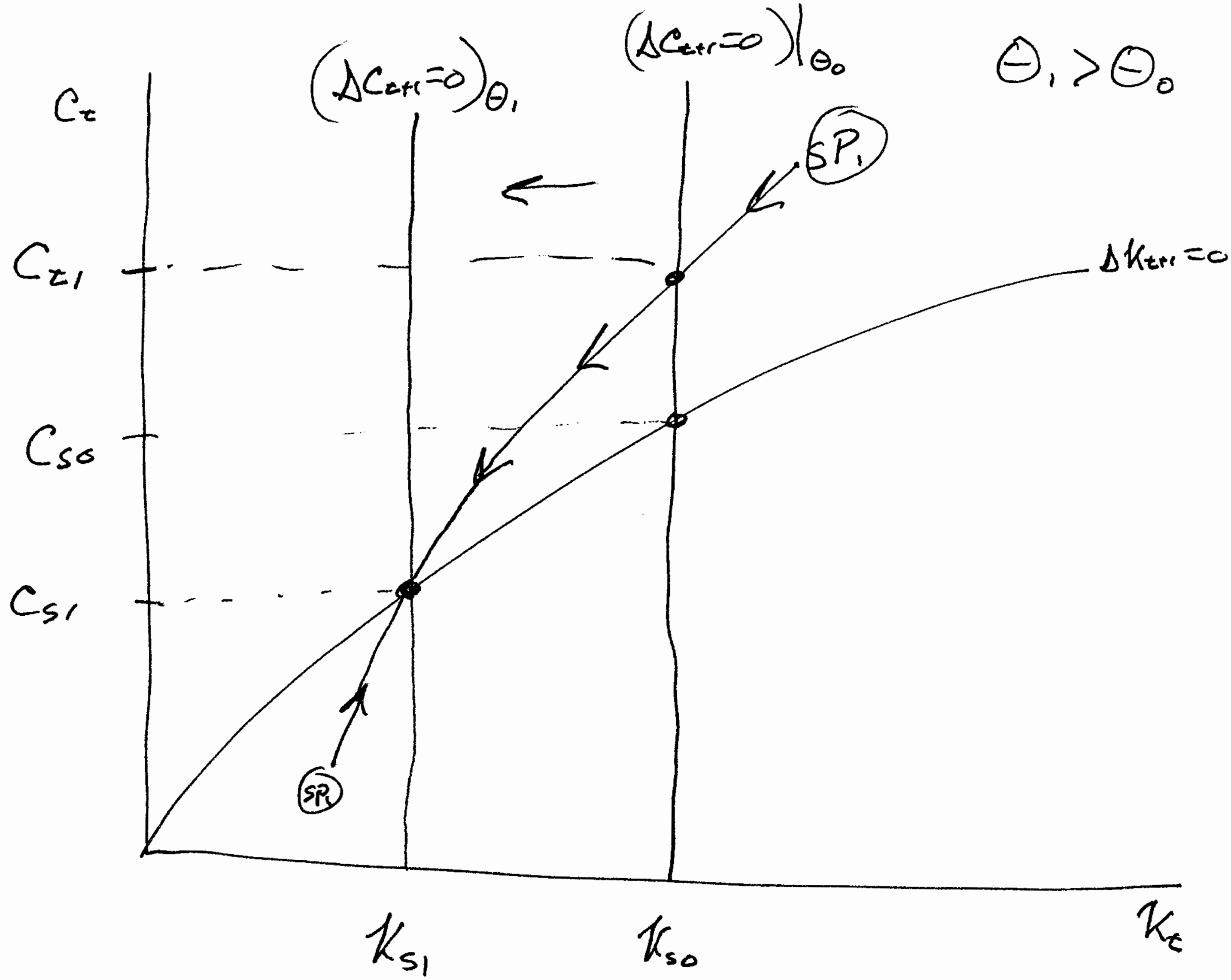


→ Next, note that $\Delta K_{t+1} = 0$ where

$$\boxed{C_t\Big|_{\Delta K_{t+1}=0} = A K_t^{\alpha} - \delta K_t \qquad (4)}$$

From (4) we see that $\Delta\theta > 0$ has no effect on the $\Delta K_{t+1} = 0$ locus.

→ Assume we start from the original steady state with $\theta = \theta_0$. Call this steady state (K_{s0}, C_{s0})



→ As shown above $\Delta\theta > 0$ to $\theta_1 > \theta_0$ causes the $\Delta C_{t+1} = 0$ curve to shift to the left. The New steady state is (K_{s1}, C_{s1}). The new saddle path is shown and labelled "SP."

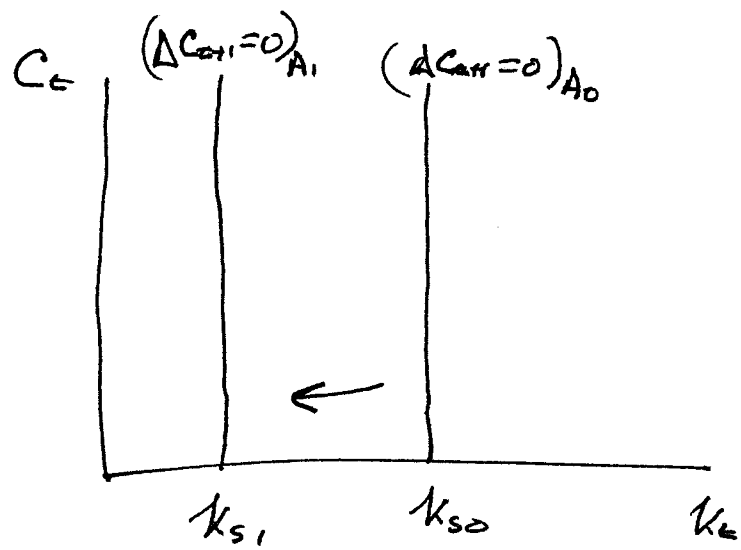
→ When θ increases to θ_1 there is an immediate increase in consumption to C_{z1}. Since the capital stock does not change output stays the same: $y_{s0} = A K_{s0}^{\alpha}$. Thus, the increase in consumption requires a decline in investment. With ~~lower~~ lower investment the capital stock depreciates. (Investment at the steady state is at the "replacement rate" - replacing depreciating capital and nothing more. The increase in consumption requires that investment fall below the ~~st~~ replacement rate and the capital stock begins to decline.)

→ From (K_{S0}, C_{t1}) Capital, hence output, and consumption decline converging to their new ~~equi~~ steady state equilibrium values K_{S1}, C_{S1}, and $Y_{S1} = A K_{S1}^{\alpha}$.

b) Consider $\Delta A < 0$.

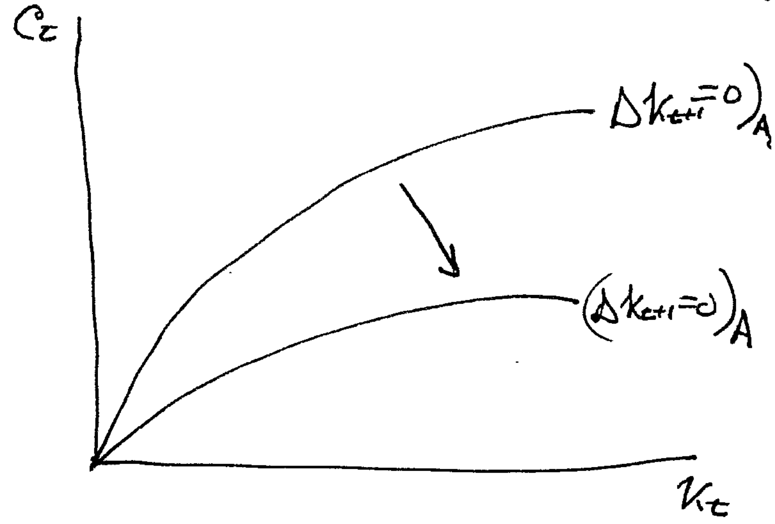
→ Note from (3) that $\Delta A < 0$ from, say, A_0 to $A_1 < A_0$, will cause a decline in K_S.

Thus $\Delta A < 0$ will shift the $\Delta C_{t+1} = 0$ locus to the left.



→ Note also from eqn (4) that a decline in A will cause the $\Delta K_{t+1} = 0$ locus to rotate down in a ~~counter~~ clockwise direction.

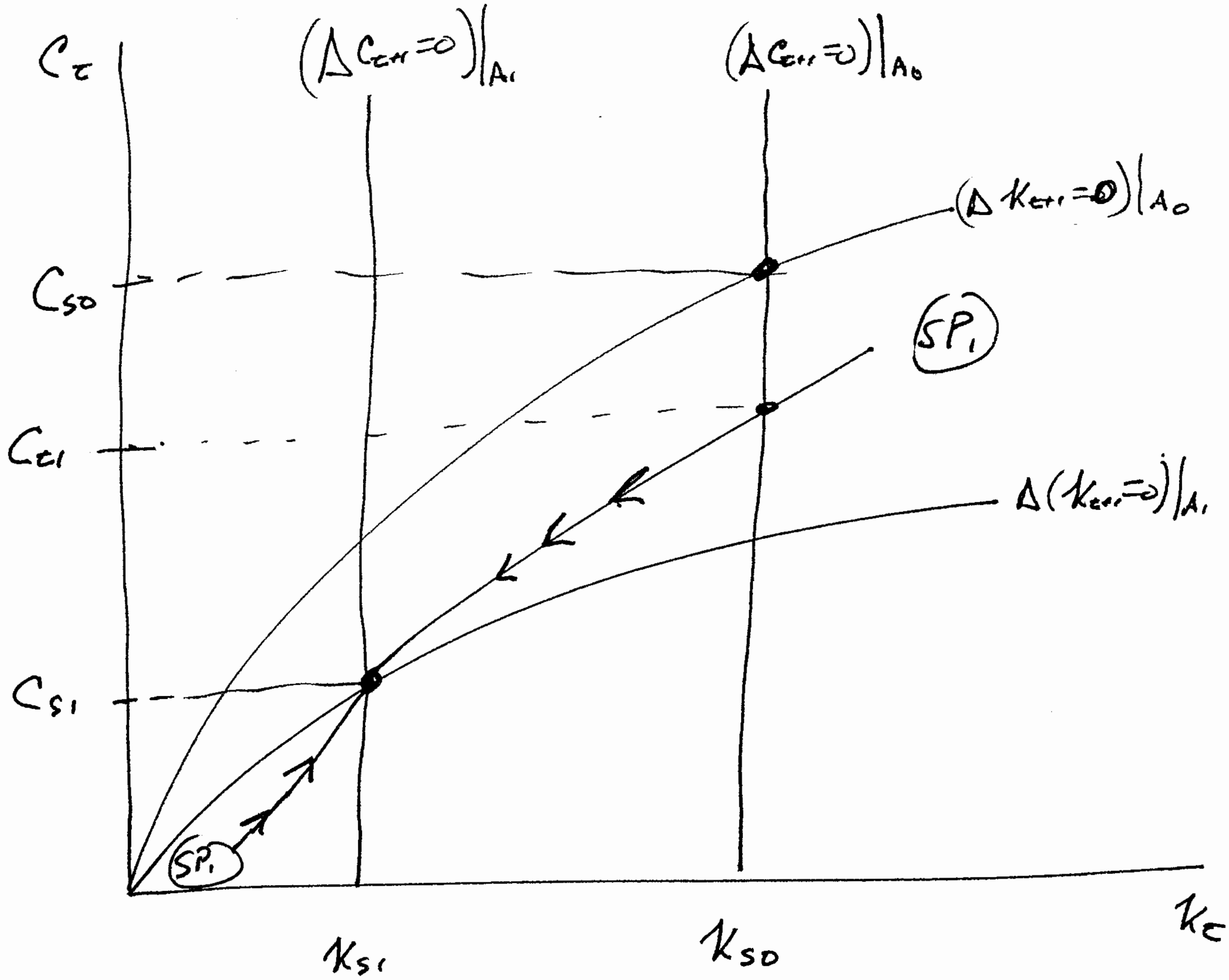
$\Delta A < 0$ will lower $C_t \Big|_{\Delta K_{t+1}=0}$ at any given K_t. The locus will still pass through the origin. Thus it rotates down.



→ putting the two shifts together (Figure Next Page) We see that the new Saddlepath will be SP_1.

→ Assume we start from C_{s0} K_{s0}. The decline in A to $A_1 < A_0$ causes optimal consumption to immediately drop to C_{t1}. Note also that $\Delta A < 0$ causes $\Delta Y_t < 0$ because $Y_t = A K_t^{\alpha}$.

→ Although consumption declines, output also declines. We can infer from the dynamics implied by the saddlepath that K_t will decline through time. This will cause further declines in Y_t. C_t will also decline through time.


C_t
$(\Delta C_{t+1}=0)|_{A_1}$
$(\Delta C_{t+1}=0)|_{A_0}$
$(\Delta k_{t+1}=0)|_{A_0}$
C_{s0}
SP_1
C_{t1}
$\Delta(k_{t+1}=0)|_{A_1}$
C_{s1}
SP_1
k_{s1}
k_{s0}
k_t

Furthermore, we can infer from declining K_t that investment at K_{S0} (after $\Delta A < 0$) is below the replacement rate.

$\rightarrow$ K_t and C_t will decline converging to the new steady state equilibrium K_{S1} C_{S1}. Note that $\Delta A < 0$ causes $\Delta K_S < 0$, $\Delta C_S < 0$ and $\Delta y_S < 0$.

(c). Consider $\Delta \delta > 0$.

Note from eqn (3) that $\Delta \delta > 0$ will reduce K_S. Note from eqn (4) that $\Delta \delta > 0$ will cause the $\Delta K_{t+1} = 0$ locus to rotate down in a clockwise direction. Thus the shift in the phase diagram is qualitatively the same as in the previous example ($\Delta A < 0$), but the economic interpretation would focus on the effect of higher depreciation of capital.

Set up the Lagrangian:

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \left\{ \beta^s u(c_{t+s}) + \lambda_{t+s} \left[(1+r) W_{t+s} - c_{t+s} - W_{t+s+1} \right] \right\}$$

F. O. C.

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u'(c_{t+s}) - \lambda_{t+s} = 0 \qquad (4)$$

$$\frac{\partial \mathcal{L}_t}{\partial W_{t+1+s}} = -\lambda_{t+s} + \lambda_{t+s+1}\left[(1+r)\right] = 0 \qquad (5)$$

From (4)

$$\lambda_{t+s} = \beta^s u'(c_{t+s}) \qquad (6)$$

Using (6) in (5) gives

$$\beta^s u'(c_{t+s}) = \beta^{s+1} u'(c_{t+s+1})(1+r) \qquad \text{or}$$

$$\boxed{u'(c_{t+s}) = \beta(1+r)u'(c_{t+s+1}) \qquad (7)}$$

Using $\beta = \frac{1}{1+\theta}$, $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, and evaluating at $s=0$

$$\boxed{C_t^{-\gamma} = \left[\frac{1+r}{1+\theta}\right] C_{t+1}^{-\gamma} \qquad (8)}$$

Eqns (7) and (8) are two versions of the intertemporal optimality condition.

To Derive the growth rate of consumption use the linearization procedure in Wickens.

A 1st order Taylor Series approx of $u'(C_{t+1})$ around C_t yields

$$u'(C_{t+1}) \cong u'(C_t) + u''(C_t)\left(C_{t+1} - C_t\right) \qquad (9)$$

Divide (9) by $u'(C_t)$ to get

$$\frac{u'(C_{t+1})}{u'(C_t)} = 1 + \frac{u''(C_t)}{u'(C_t)} \Delta C_{t+1} \qquad (10)$$

Set $s=0$ in (7) and use (10) in the Result to get

$$\Delta C_{t+1} = \frac{u'(c_t)}{u''(c_t)} \left[\frac{1}{\beta(1+r)} - 1 \right] \quad (11)$$

Divide Through by C_t to get

$$\frac{\Delta C_{t+1}}{C_t} = - \left[\frac{u'(c_t)}{u''(c_t) \cdot C_t} \right] \left[1 - \frac{1}{\beta(1+r)} \right] \quad (12)$$

Note that $u'(C_t) = C_t^{-\gamma}$

$$u''(C_t) = -\gamma \, C_t^{-(\gamma+1)}$$

so that the coefficient of relative risk aversion is

$$- \left[\frac{u''(C_t)}{u'(C_t)} \right] \cdot C_t = - \left[\frac{-\gamma C_t^{-\gamma} C_t^{-1}}{C_t^{-\gamma}} \right] \cdot C_t = \gamma \quad (13)$$

Use (13) in (12) to get

$$\frac{\Delta C_{t+1}}{C_t} = \left(\frac{1}{\gamma}\right)\left[1 - \frac{1}{\beta(1+r)}\right] \qquad (14)$$

Note that $1 - \frac{1}{\beta(1+r)} = 1 - \frac{1}{\frac{1+r}{1+\theta}} = 1 - \frac{1+\theta}{1+r}$

$$= \frac{1+r-(1+\theta)}{1+r} = \frac{r-\theta}{1+r}$$

This in (14) gives

$$\boxed{\frac{\Delta C_{t+1}}{C_t} = \left(\frac{1}{\gamma}\right)\left[\frac{r-\theta}{1+r}\right] \qquad (15)}$$

Equation (15) describes the rate of growth of consumption, $\frac{\Delta C_{t+1}}{C_t}$.

$$\frac{\Delta C_{t+1}}{C_t} > 0 \quad \text{if} \quad r > \theta$$

$$\frac{\Delta C_{t+1}}{C_t} < 0 \quad \text{if} \quad r < \theta$$

$$\frac{\Delta C_{t+1}}{C_t} = 0 \quad \text{if} \quad r = \theta$$