HW3

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ECON 5700

Due on August 15, 2020.

1 Question 1

Derive the Taylor expansion for $f(x) = 3x^2 - 6x + 5$

Solution:

$f_x = 6x - 6$, and $f_{xx} = 6$, but $f_{xxx} = 0$, thus, the Taylor Expansion is:

$$f(x) = 3x_0^2 - 6x_0 + 5 + (6x - 6)(x - x_0) + 3(x - x_0)^2 + o(x - x_0)^n$$ [1]

2 Question 2

Derive the Maclaurin expansion for e^{kx}, k is real number.

Solution:

Let $f(x) = e^{kx}$, then $f_x = ke^{kx}$, $f_{xx} = k^2e^{kx}$, till $f^{(n)}(x) = k^ne^{kx}$.

$$\begin{aligned} f(x) &= 1 + kx + \frac{k^2}{2!}x^2 + \frac{k^3}{3!}x^3 + ... + \frac{k^n}{n!}x^n + o(x^n) \\ &= 1 + \sum_{m=1}^{n} \frac{k^m}{m!}x^m + o(x^n) \end{aligned}$$

3 Question 3

Derive the Macraulin expansion for $(1 + x)^\mu$.

Solution:

Let $f(x) = (1 + x)^\mu$, then $f_x = \mu(1 + x)^{\mu-1}$, $f_{xx} = \mu(\mu - 1)(1 + x)^{\mu-2}$.

$$\begin{aligned} f(x) &= 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!}x^3 + ... + \frac{\mu(\mu-1)...(\mu-n)}{n!}x^n + o(x^n) \\ &= \sum_{m=0}^{n} \frac{\frac{\mu!}{(\mu-m-1)!}}{m!}x^m + o(x^n) \end{aligned}$$

[*]I worked on my assignment sololy. Email: wye22@fordham.edu

[1]I am not sure whether to add Peano Remainder, so I use blue color to mark.

4 Question 4

Derive the Maclaurin expansion for $\sqrt{1+x}$.

Solution:

Let $f(x) = \sqrt{1+x}$, then $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$, and $f'''(0) = \frac{1}{8}$.

$$\begin{aligned} f(x) &= 1 + \frac{1}{2}x + (-\frac{1}{4})\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} + ... \\ &= \sum_{m=0}^{n} \binom{\frac{1}{2}}{m} x^m + o(x^n) \end{aligned}$$

5 Question 5

Find the convexity and concavity for function $f(x) = x^3 + ax + b$

Solution:

$f_x = 3x^2 + a$, $f_{xx} = 6x$, thus if $x > 0$, it' s convex function, however, it's concave when $x < 0$.

6 Question 6

Find the intervals of convexity and concavity of the function $f(x) = \frac{1}{1+x^2}$.

Solution:

$f_x = \frac{-2x}{(1+x^2)^2}$, and $f_{xx} = \frac{6x^2-2}{(1+x^2)^3}$. Let $f'' = 0 \rightarrow x_1 = -\frac{\sqrt{3}}{3}, x_2 = \frac{\sqrt{3}}{3}$. Thus, when $x < -\frac{\sqrt{3}}{3}$ or $x > \frac{\sqrt{3}}{3}$, it's convex function, otherwise, it's concave.

7 Question 7

Find the intervals of convexity and concavity of the function $f(x) = e^{\frac{1}{x}}$.

Solution:

$f_x = e^{\frac{1}{x}}(-\frac{1}{2})$, and $f_{xx} = e^{\frac{1}{x}}(\frac{2x+1}{x^4})$. Thus, if $x > -\frac{1}{2}$, it's convex function. However, when $x < -\frac{1}{2}$, it's concave function.

8 Question 8

Find the sum of the series $S = 1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + \frac{1}{4} - \frac{1}{4\sqrt{2}} + \frac{1}{8}$.

Solution:

$$S = (1 + \frac{1}{2} + \frac{1}{4})(1 - \frac{1}{\sqrt{2}}) + \frac{1}{8} = \frac{15 - 7\sqrt{2}}{8}$$

9 Question 9

Redo the examples in class on your own: (1) $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ (2) $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$ converges or diverges.

Solution:

1. Begin with ratio test.

$$\begin{aligned}
\lim \frac{\frac{3^{n+1}}{(n+1)^2}}{\frac{3^n}{n^2}} &= 3 \lim \frac{n^2}{n^2+1} \\
&= 3 \lim (1 - \frac{1}{n+1})^2 \\
&= 3 \cdot 1 \\
&= 3 > 1
\end{aligned}$$

Thus, it's divergent.

2. Begin with Ratio Test again.

$$\begin{aligned}
\lim \frac{\frac{(n+1)^3}{(\ln 2)^{n+1}}}{\frac{3^n}{(\ln 2)^n}} &= \lim \left(\frac{n+1}{n}\right)^3 \frac{1}{\ln 2} \\
&= \lim \left(1 + \frac{1}{n}\right)^3 \frac{1}{\ln 2} \\
&= \frac{1}{\ln 2} > 1
\end{aligned}$$

Thus, it's divergent