HW#2 Macroeconomics II

Mei Ye

Problem 1. Life Cycle Consumption with Quadratic Utility
Suppose that the consumer maximizes the following objective function:

$$\max \sum_{t=0}^{T} \beta^t [u(c_t)] \tag{1}$$

subject to the dynamic budget constraint (Note that this is written slightly different than we did in class; namely income is assumed to be received at the end of period t rather than at the beginning of period $t+1$):

$$A_{t+1} = (1+r)[A_t + y_t - c_t] \tag{2}$$

where

$$u(c_t) = c_t - \frac{b}{2}c_t^2 \tag{3}$$

with $b > 0$. r is a constant net interest rate, A_t is the amount of wealth available at the beginning of period t and y_t is labour income in period t.

A_t $(y_t - c_t)$ A_t

1. Do we need a transversality condition? If not, what other constraint do we need? How much wealth should the consumer have when she dies?

(1) Yes, we need a transversality condition. ⇒ which is a requirement is Dynamic programming problem.

Since, there is no bequest, ⇒ not an OLG model, the best strategy (LNS) is to spend all the money he/she has before dying (at the time of dying)

2. Assume the agent receives income A_0 in the first period. Show that the dynamic budget constraint together with the constraint you found in (1) implies the following intertemporal budget constraint:

A_0

$$\sum_{t=0}^{T} R^{-t} c_t = A_0 + \sum_{t=0}^{T} R^{-t} Y_t \tag{4}$$

where $R = (1+r)$. Note: Begin with the dynamic budget constraint (dbc) describing A_1. Then substitute this expression for A_1 into the dbc for A_2. Next substitute the expression you got for A_2 into the dbc for period 3, etc. This will allow you to derive the intertemporal budget constraint.

As in the hint, we use iteration method

$A_1 = R(A_0 + y_0 - c_0)$

$A_2 = R(A_1 + y_1 - c_1)$

$\quad = R(R(A_0 + y_0 - c_0) + y_1 - c_1)$

$A_3 = R(A_2 + y_2 - c_2)$

$\quad = R[R(R(A_0 + y_0 - c_0) + y_1 - c_1) + y_2 - c_2]$

$\vdots$

$A_{T+1} = 0 = R[\cdots\cdots(A_0 + y_0 - c_0) + y_1 - c_1) + y_2 - c_2 \cdots + y_n - c_n]$

Rearrange, we could obtain, and divided R^t

$$\sum_{t=0}^{T} R^{-t} c_t = A_0 + \sum_{t=0}^{T} R^{-t} Y_t$$

3. Set up the consumer's maximization problem. What is Bellman's Equation?

The consumer will maximize his/her life-time discounted utility given his/her intertemporal Budget constraint.

Pick A_t as state variable, c_t as control variable.

The Bellman Eqn as below w/o time as subscript:

$$V(A) = \max_{C} \{ u(c) + \beta V(A') \}$$

$$= \max_{c} [(c - \tfrac{b}{2}c^2) + \beta V[R (A + y - c)]]$$

4. Find the first order conditions (use the Envelope theorem). ⇒ If use Envelop Theorem only, how to derive Euler Eqn? I hold suspicious opinion here!

First, FOC w.r.t [c]

$$[c]: \quad u'(c) + \beta V'(A') \frac{\partial A'}{\partial c} = 0$$

$$\Rightarrow \quad 1 - bc + \beta V'(A')(-R) = 0 \Rightarrow bc = 1 - R\beta V'(A')$$

$$\Rightarrow \quad c = \frac{1 - R\beta V'(A')}{b}$$

Envelop Theorem:

$$[A]: \quad V'(A) = \beta V'(A') \cdot \frac{\partial A'}{\partial A}$$

$$= \beta V'(A') R$$

5. Derive the Euler Equation.

As $V'(A') = \dfrac{V'(A)}{\beta R}$

put this into FOC Eqn:

$$C = \frac{1 - R\beta \frac{V'(A)}{\beta R}}{b}$$

$$= \frac{1 - V'(A)}{b}$$

$$\Rightarrow \quad V'(A) = 1 - bc$$

$$\Rightarrow \quad V'(A') = 1 - bc'$$

put ~ Eqn into FOC to obtain Euler Eqn!

$$C = \frac{1 - R\beta [1 - bc']}{b} \qquad [\text{Euler}]$$

6. Derive a closed form solution for c_0.

For closed form of C_0:

As $A_{t+1} = R(A_t + y_t - C_t)$ ⇒ intemporal Budget constraint

$$C_t = \frac{1 - R\beta[1 - bc_{t+1}]}{b} \Rightarrow \text{Euler Eqn.}$$

$$bc_t = 1 - R\beta[1 - bc_{t+1}]$$

$$R\beta[1 - bc_{t+1}] = 1 - bc_t$$

$$1 - bc_{t+1} = \frac{1 - bc_t}{R\beta}$$

$$1 - \frac{1 - bc_t}{R\beta} = bc_{t+1}$$

$$\frac{R\beta - 1 + bc_t}{R\beta} = bc_{t+1}$$

$$c_{t+1} = \frac{R\beta - 1 + bc_t}{bR\beta}$$

$$c_1 = \frac{R\beta - 1 + bc_0}{bR\beta}$$

$$c_2 = \frac{R\beta - 1 + bc_1}{bR\beta} = \frac{R\beta - 1 + b\left[\frac{R\beta - 1 + bc_0}{bR\beta}\right]}{bR\beta} = \frac{R\beta - 1 + \frac{R\beta - 1 + bc_0}{R\beta}}{bR\beta}$$

$$c_3 = \frac{R\beta - 1 + bc_2}{bR\beta} = \frac{R\beta - 1 + b\left[\frac{R\beta - 1 + \frac{R\beta - 1 + bc_0}{R\beta}}{bR\beta}\right]}{bR\beta} = \frac{R\beta - 1 + \frac{R\beta - 1 + \frac{R\beta - 1 + bc_0}{R\beta}}{R\beta}}{bR\beta}$$

$$\frac{x - 1 + \frac{x - 1 + \frac{x-1}{x} + b_0}{x}}{bx}$$

$$\frac{x(x-1) + x - 1}{x}$$

1 $x - 1 + bc_0$

2 $x - 1 + \frac{x - 1 + bc_0}{x} = \frac{x(x-1) + (x-1) + bc_0}{x}$

3 $x - 1 + \frac{x - 1 + \frac{x-1}{x} + bc_0}{x} = \frac{x^2(x-1) + x(x-1) + x - 1 + bc_0}{x^2}$

$\overline{x}$

Thinking process

$\therefore$ Ded.

$$c_{T+1} = 0 = \frac{\frac{1}{(R\beta)^T}\left[\sum_{t=0}^{T} (R\beta)^t (R\beta - 1) + bc_0\right]}{bR\beta}$$

$$= \frac{\frac{1}{(R\beta)^T} \frac{1 - (R\beta)^{T-1}}{1 - R\beta} (R\beta - 1) + bc_0}{bR\beta}$$

let the numerator $= 0$

$$\frac{(R\beta)^{-T}\left(1-(R\beta)^{T-1}\right)}{1-R\beta}(R\beta-1)+bC_0=0$$

$$\Rightarrow \quad bC_0 = (R\beta)^{-T}\left(1-(R\beta)^{T-1}\right)$$

$$\Rightarrow \quad C_0 = \frac{(R\beta)^{-T}\left(1-(R\beta)^{T-1}\right)}{b}$$

$= \text{const}$ I interesting conclusion if I were right]

Problem 2. Life Cycle Utility over Consumption and Leisure Suppose that the consumer maximizes the following objective function:

$$\max \sum_{t=0}^{T} \beta^t [u(c_t) + d(l_t)] \tag{5}$$

where c_t is consumption and l is leisure. Assume that both functions, u and d are increasing and concave.

Consumption is typically measured using expenditure x_t on some group of goods and services. In reality, consumption requires expenditure plus time spent on 'home production'and households can substitute between the two. For example, it is more expensive to eat out or buy pre-prepared food, but it is less costly in terms of time. It is cheaper to buy raw foods and prepare them yourself but it is more costly in terms of time.

Assume that consumption is related to expenditure and time inputs:

$$c_t = f(\underbrace{x_t}_{\text{expenditure}}, \underbrace{s_t}_{\text{time on home product}}) \tag{6}$$

where s_t represents time spent in home production. Assume that f is increasing and concave in both of its inputs. Assume that the household can borrow or lend at the risk free rate, $R = 1 + r$. The household's dynamic budget constraint is

$$A_{t+1} = R[A_t + w_t h_t - x_t] \tag{7}$$

where h_t is hours spend working outside the home, w_t is the wage rate, which is exogenous, A_t is financial assets at the beginning of period t, and x_t is expenditure. There is no uncertainty, at time 0 households know the entire sequence of wages with certainty from 0 to T.

Assume that households have an endowment of one unit of time each period, which they spend on leisure, working for wages, and in home production. Thus leisure time satisfies:

$$l_t = 1 - h_t - s_t \tag{8}$$

1. What are the state variables? What are the control variables?

The state variable is A_t.

control variables are h_t, s_t, x_t. ✓

2. Write down the Bellman equation for this problem.

Set up Belman Eqn as below: [overlook time as usual]

$$V(A) = \max_{\{C, L, X\}} \{ u(C) + d(L) + \beta V(A') \}$$

$$\text{s.t.} \quad A_{t+1} = R(A_t + w_t h_t - x_t)$$

$$L_t = 1 - h_t - s_t$$

3. Derive the first order conditions (hint: there will be 3 in addition to the envelope condition)

$$V(A) = \max_{\{L, X, S\}} \{ u(f(X,S)) + d(L) + \beta V[R(A + wh - X)] \}$$

$$[L]: \quad u'(C) f_2(X,S)(-1) + d'(L) + \beta V'(A')(-RW) = 0$$

$$[X]: \quad u'(C) f_1(X,S) + \beta V'(A')(-1) = 0$$

$$[S]: \quad u'(C) f_2(X,S) + d'(L)(-1) + \beta V'(A')(-RW) = 0$$

4. Find the Euler Equation.

Using Envelope Theorem:

$[A]: \quad V'(A) = \beta V'(A') R$

$$\Rightarrow \quad V'(A') = \frac{V'(A)}{\beta R}.$$

put this into FOC in 3.

$$\begin{cases} -u'(c) f_2(x,s) + d'(l) + \cancel{\beta} \dfrac{V'(A)}{\cancel{\beta R}} (-\cancel{R} w) = 0 \\ u'(c) f_1(x,s) + \cancel{\beta} \dfrac{V'(A)}{\cancel{\beta} R} (-1) = 0 \\ u'(c) f_2(x,s) - d'(l) + \cancel{\beta} \dfrac{V'(A)}{\cancel{\beta R}} (-\cancel{R} w) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -u'(c) f_2(x,s) + d'(l) - w V'(A) = 0 \\ u'(c) f_1(x,s) - \dfrac{V'(A)}{R} = 0 \\ u'(c) f_2(x,s) - d'(l) - w V'(A) = 0 \end{cases}$$

$R\, u'(c) f_1(x,s) = V'(A).$

$\Rightarrow \quad V'(A') = R\, u'(c') f_1(x', s')$

into our envelop theorem eqn:

$$R\, u'(c') f_1(x', s') = \frac{\cancel{R} u'(c) f_1(x,s)}{\beta \cancel{R}}$$

$$\Rightarrow \quad u'(c) = \beta R u'(c') \frac{f_1(x', s')}{f_1(x, s)} \quad \text{[Euler Eqn]}$$

5. Show that $\frac{x_t}{s_t}$ is an increasing function of wages, w_t. That is, as wages rise people substitute expenditures for time in the production of consumption. Provide intuition for this result.

Proof: WTP $\frac{\partial \frac{x_t}{s_t}}{\partial w_t} > 0$

$$s_t = 1 - h_t - l_t$$

$$x_t = A_t + wh_t - \frac{1}{R} A_{t+1}$$

$$\Rightarrow \frac{x_t}{s_t} = \frac{A_t + w_t h_t - \frac{1}{R} A_{t+1}}{1 - h_t - l_t}$$

$$\Rightarrow [w_t] = \frac{h_t}{1 - h_t - l_t}$$

As the denominator > 0 $\quad h_t \geq 0$

$\therefore \frac{\partial \frac{x_t}{s_t}}{\partial w_t} \geq 0 \Rightarrow$ as wealth $\uparrow$ substitution expenditure in consumption $\uparrow$

Intuition:

As wealth $\uparrow$, people can use \$ to buy time, in which time for production is make utility $\downarrow$, instead, they can save this time and do more leisure, and spend \$ on goods that $\uparrow$ utility.