Classical Demand Theory, Continued

The Expenditure Minimization Problem (EMP)

Whereas the UMP computes the maximal utility to be gained from a given wealth, w, the EMP computes the minimal level of wealth required to reach utility level v; in this way, the EMP is the dual to the UMP.

Throughout, we assume locally non-satiated $\succsim$ on $\mathbb{R}^L_+$ is represented by continuous $u(\cdot)$.

Proposition. Suppose $u(\cdot)$ is continuous representing locally non-satiated $\succsim$ on $\mathbb{R}^L_+$ & $p \gg 0$. Then we have:

i) If x^* is optimal in the UMP when $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the min expenditure level in the EMP is exactly w.

ii) If x^* is optimal in the EMP when the required utility level is $v > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the max utility level in the UMP is exactly v.

Proof. i) Suppose x^* is not optimal in the EMP w/ required $u(x^*)$. Then $\exists$ x^* such that $u(x') > u(x^*)$ and $p \cdot x' < p \cdot x^* \leq w$. By local non-satiation, we can find x'' close to x' such that $u(x'') > u(x')$ and $p \cdot x'' < w$. But this implies $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$, contradicting the optimality of x^* in the UMP. Thus, x^* must be optimal in the EMP when the required $u(x^*)$, and the min expenditure is $p \cdot x^*$. Since x^* solves the UMP w/ wealth w, by Walras' law $p \cdot x^* = w$.

ii) Since $v > u(0)$, we have $x^* \neq 0$. Hence, $p \cdot x^* > 0$. Suppose x^* is not optimal in the UMP when $w = p \cdot x$; then $\exists$ x' such that $u(x') > u(x^*)$ and $p \cdot x' \leq p \cdot x^*$. Consider bundle $x'' = \alpha x'$ where $\alpha \in (0,1)$; by continuity of $u(\cdot)$, if α is close enough to 1, we have $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$. But this contradicts optimality of x^* in the EMP. Thus, x^* must be optimal in the UMP when $w = p \cdot x^*$, and max utility is $u(x^*)$. We will show later that if x^* solves the EMP w/ required u, then $u(x^*) = u$.

From now on, we assume $u(\cdot)$ must attain values at least as large as u for some x (so $\exists$ a solution to the EMP when $p \gg 0$). (This is satisfied for any $u > u(0)$ if $u(\cdot)$ is unbounded above.)

The Expenditure Function

Given $p \gg 0$ and $u > u(0)$, the value of the EMP is denoted $e(p,u)$; this is the <u>expenditure function</u>. Its value for (p,u) is $p \cdot x^*$ where x^* is any solution to the EMP.

Proposition. Suppose continuous $u(\cdot)$ represents locally nonsatiated $\succsim$ on $X = \mathbb{R}^L_+$. Then $e(p,u)$ is

i) homogenous of degree one in p

ii) strictly increasing in u & nondecreasing in p_l for any l

iii) concave in p

iv) continuous in p & u.

Proof. See textbook, pp. 59 – 60.

The Hicksian (or Compensated) Demand Function

The set of optimal commodity vectors in the EMP is denoted $h(p,u) \subset \mathbb{R}^L_+$ & is known as the <u>Hicksian, or compensated, demand correspondence, or</u> function if single-valued.

Proposition. Suppose continuous $u(\cdot)$ represents locally nonsatiated $\succsim$ on $X = \mathbb{R}^L_+$. For any $p \gg 0$, $h(p,u)$ has the following properties:

i) homogeneity of degree zero in p: $h(\alpha p, u) = h(p,u)$ for any $p, u,$ & $\alpha > 0$.

ii) no excess utility: $\forall\, x \in h(p,u)$, $u(x) = u$.

iii) convexity / uniqueness: if $\succsim$ is convex, then $h(p,u)$ is a convex set; if $\succsim$ is strictly convex, so $u(\cdot)$ is ~~quasiconcave~~ strictly quasiconcave, then there is a unique element in $h(p,u)$.

Proof. See textbook, pp. 61 – 62.

Hicksian Demand & The Compensated Law of Demand

Proposition. Suppose continuous $u(\cdot)$ represents locally nonsatiated $\succsim$ & $h(p,u)$ consists of a single element $\forall\, p \gg \emptyset$. Then $h(p,u)$ satisfies the compensated law of demand: for all p' and p'',

$$(p' - p'')\left[h(p'',u) - h(p',u)\right] \leq \emptyset.$$

Proof. $\forall\, p \gg \emptyset$, consumption bundle $h(p,u)$ is optimal in the EMP, and so acheives a lower expenditure at prices p than any other bundle that offers a utility of at least u. So,

$$p'' \cdot h(p'',u) \leq p'' \cdot h(p',u),$$

and

$$p' \cdot h(p'',u) \geq p' \cdot h(p',u).$$

So,

$$\underbrace{\left[p''h(p'',u) - p''h(p',u)\right]}_{\cdot\ \leq \emptyset} - \underbrace{\left[p'h(p'',u) - p'h(p',u)\right]}_{\cdot\ \leq \emptyset} = (p''-p')\left[h(p',u) - h(p'',u)\right] \leq \emptyset.$$

To review, this implies that if the price of good l changes (w/ income constant), we must have an opposite change in consumption of the good:

$$\frac{\partial h_l(p,u)}{\partial p_l} \leq \emptyset.$$

Example. Hicksian Demand & Expenditure Functions for the Cobb-Douglas Utility Function

Consider the EMP for the Cobb-Douglas Utility function, where $X = \mathbb{R}^2_+$

$$\min\ p_1 x_1 + p_2 x_2$$
$$\text{s.t. } x_1^{\alpha} x_2^{1-\alpha} = u$$

The Lagrangian is

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda\left(u - x_1^{\alpha} x_2^{1-\alpha}\right)$$

We assume that the constraint binds; our F.O.C.s are

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial x_1}: p_1 - \lambda\alpha\dfrac{u}{x_1} = 0 \implies p_1 = \lambda\alpha\dfrac{u}{x_1} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial x_2}: p_2 - \lambda(1-\alpha)\dfrac{u}{x_2} = 0 \implies p_2 = \lambda(1-\alpha)\dfrac{u}{x_2}\end{array}\right\} \implies \frac{p_1}{\alpha}x_1 = \frac{p_2}{1-\alpha}x_2 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}: u - x_1^{\alpha}x_2^{1-\alpha} = 0 \implies u = x_1^{\alpha}x_2^{1-\alpha} \quad (2)$$

Now we solve (1) for x_2:

$$x_2 = \frac{1-\alpha}{\alpha}\cdot\frac{p_1}{p_2}x_1. \quad (3)$$

Substituting (3) into (2) and solving for x_1:

$$x_1^{\alpha}\left(\frac{1-\alpha}{\alpha}\cdot\frac{p_1}{p_2}x_1\right)^{1-\alpha} = u,$$

$$\underbrace{x_1^{\alpha}x_1^{1-\alpha}}_{x_1}\left(\frac{1-\alpha}{\alpha}\cdot\frac{p_1}{p_2}\right)^{1-\alpha} = u,$$

$$x_1^{*} = \left(\frac{\alpha}{1-\alpha}\cdot\frac{p_2}{p_1}\right)^{1-\alpha}\cdot u. \quad (4)$$

Substituting (4) into (3) and simplifying,

$$x_2^{*} = \left(\frac{1-\alpha}{\alpha}\cdot\frac{p_1}{p_2}\right)^{\alpha}u. \quad (5)$$

We have $h_1(p,u) = x_1^*$ and $h_2(p,u) = x_2^*$. Calculating $e(p,u) = p\cdot h(p,u)$ yields

$$e(p,u) = \left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right]p_1^{\alpha}p_2^{1-\alpha}u$$

(If you substitute $u = e^{v(p,w)}$ as calculated previously, you get $e(p,u) = w$!)

We will skip the mathematical introduction to duality on pp. 63-67, & instead examine directly the implications for the UMP & EMP.

Relationships b/w Demand, Indirect Utility, & Expenditure Functions

We assume throughout that continuous $u(\cdot)$ represents locally nonsatiated $\succsim$ on $X = \mathbb{R}^L_+$ and $p \gg 0$. We also assume $\succsim$ is strictly convex, so that $x(p,w)$ & $h(p,u)$ are single-valued.

Recall that $e(p,u) = p \cdot h(p,u)$.

Proposition. (Shephard's Lemma) Suppose continuous $u(\cdot)$ represents locally nonsatiated $\succsim$ on $X = \mathbb{R}^L_+$. $\forall\, p, u$, $h(p,u)$ is the derivative vector of the expenditure function w.r.t. prices:

$$h(p,u) = \nabla_p e(p,u),$$

or

$$h_l(p,u) = \frac{\partial e(p,u)}{\partial p_l} \quad \forall\, l = 1, 2, \ldots, L.$$

Proof. The text provides three proofs on pp. 68–69. We consider the F.O.C. argument here.

Assume $h(p,u) \gg 0$ is differentiable at (p,u). Via the chain rule,

$$\nabla_p e(p,u) = \nabla_p [p \cdot h(p,u)],$$
$$= h(p,u) + [p \cdot D_p h(p,u)]^T.$$

The F.O.C. for an interior solution to the EMP are $p = \lambda \nabla u(h(p,u))$; substituting yields

$$\nabla_p e(p,u) = h(p,u) + \lambda \left[\nabla u(h(p,u)) \cdot D_p h(p,u)\right]^T,$$

but since $u(h(p,u)) = u$ for all p in the EMP, $\nabla u(h(p,u)) \cdot D_p h(p,u) = 0$ — that is, since $h(p,u)$ is homogeneous of degree zero in p, we have

$$\nabla_p e(p,u) = h(p,u).$$

Proposition. Suppose continuous $u(\cdot)$ represents locally nonsatiated strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot, u)$ is continuously differentiable at (p,u) & denote its $L \times L$ derivative matrix by $D_p h(p,u)$. Then

i) $D_p h(p,u) = D_p^2 e(p,u)$,

ii) $D_p h(p,u)$ is negative semi-definite,

iii) $D_p h(p,u)$ is symmetric,

iv) $D_p h(p,u) = 0$.

Proof. See textbook. pp. 69–70.

The negative semi-definiteness of $D_p h(p,u)$ is the differential analog of the compensated law of demand, $dp \cdot dh(p,u) \le 0$. Since $dh(p,u) = D_p h(p,u) dp$, substituting gives us $dp \cdot D_p h(p,u) dp \le 0 \;\; \forall\, dp$; therefore ~~$D_p h(p,u) dp$~~ $D_p h(p,u)$ is negative semi-definite & $\partial h_l(p,u)/\partial p_l \le 0 \;\; \forall\, l$.

Because $\partial h_l(p,u)/\partial p_l \le 0$, $D_p h(p,u) p = 0$ implies ~~there~~ $\exists$ at least one good k for which $\partial h_l(p,u)/\partial p_k \ge 0$; hence, every good has at least one substitute.

The Hicksian & Walrasian Demand Functions

Proposition. (The Slutsky Equation) Suppose continuous $u(\cdot)$ represents locally nonsatiated & strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Then $\forall (p,w)$ & $u = v(p,w)$, we have

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \quad \forall\, l,k,$$

or in matrix notation,

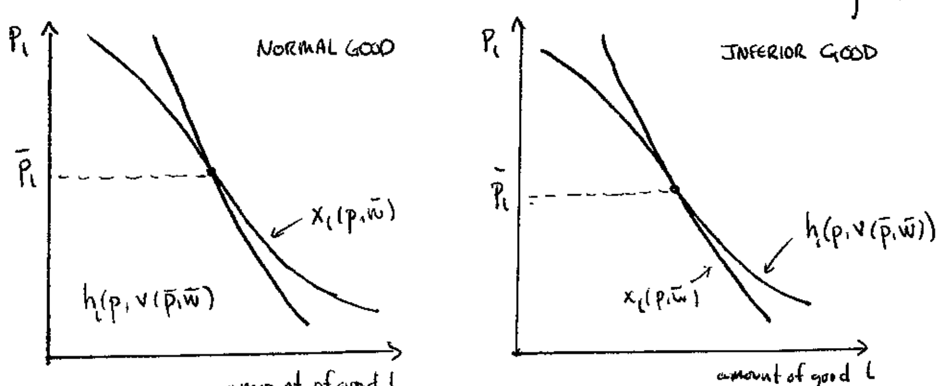
$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T.$$

Proof. Start w/ a consumer facing $(\bar{p},\bar{w})$ & attaining $\bar{u}$. By definition, for all (p,u), $h_l(p,u) = x(p, e(p,u))$. Differentiating w.r.t. p_k & evaluating at $(\bar{p},\bar{u})$, we have

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p},\bar{u}))}{\partial w} \cdot \frac{\partial e(\bar{p},\bar{u})}{\partial p_k}.$$

Since $h_k(p,u) = \frac{\partial e(p,u)}{\partial p_l} \;\forall\, k$, $\bar{w} = e(\bar{p},\bar{u})$, & $h_k(p,u) = x_k(\bar{p}, e(\bar{p},\bar{u})) = x_k(\bar{p},\bar{w})$, we have the equation above.

Below we see the Walrasian & Hicksian demand curves, holding other prices fixed.



The Slutsky equation describes the relationship b/w $x(p,\bar{w})$ & $h(p,\bar{u})$ where they cross, at $p_l = \bar{p}_l$. For a normal good, the Hicksian demand curve is steeper at this point, & for an inferior good, the Walrasian demand curve is steeper. (If a good is normal, demand falls <u>more</u> in the absence of compensation.)

The matrix of price derivatives $D_p h(p,u)$ of $h(p,u)$ is equivalent to

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

with $s_{lk}(p,w) = \partial x_l(p,w)/\partial p_k + \left[\partial x_l(p,w)/\partial w\right] x_k(p,w)$. $S(p,w)$, the Slutsky substitution matrix, is negative semi-definite, symmetric, & satisfies $S(p,w)p = 0$.

That $D_p h(p,u) = S(p,w)$ follows from additional restrictions imposed in the preference-based approach.

Proposition. (Roy's Identity) Suppose continuous $u(\cdot)$ represents locally non-satiated & strictly convex ~~$\succsim$~~ $\succsim$ on $X = \mathbb{R}^L_+$. Suppose also that $v(p,w)$ is differentiable at $(\bar{p},\bar{w}) \gg 0$. So,

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})} \nabla_p v(\bar{p},\bar{w}),$$

that is, for every $l = 1,\ldots,L$:

$$x_l(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w}.$$

Proof. The textbook has three proofs, pp. 74. We will consider the F.O.C. argument here.

Assume $x(p,w)$ is differentiable & $x(\bar{p},\bar{w}) \gg 0$. By the chain rule,

$$\frac{\partial v(\bar{p},\bar{w})}{\partial p_l} = \sum_{k=1}^{L} \frac{\partial u(x(\bar{p},\bar{w}))}{\partial x_k} \cdot \frac{\partial x_k(\bar{p},\bar{w})}{\partial p_l}.$$

From the UMP's F.O.C.s, we know $\frac{\partial u(x(p,w))}{\partial x_k} = p_k \lambda$. So,

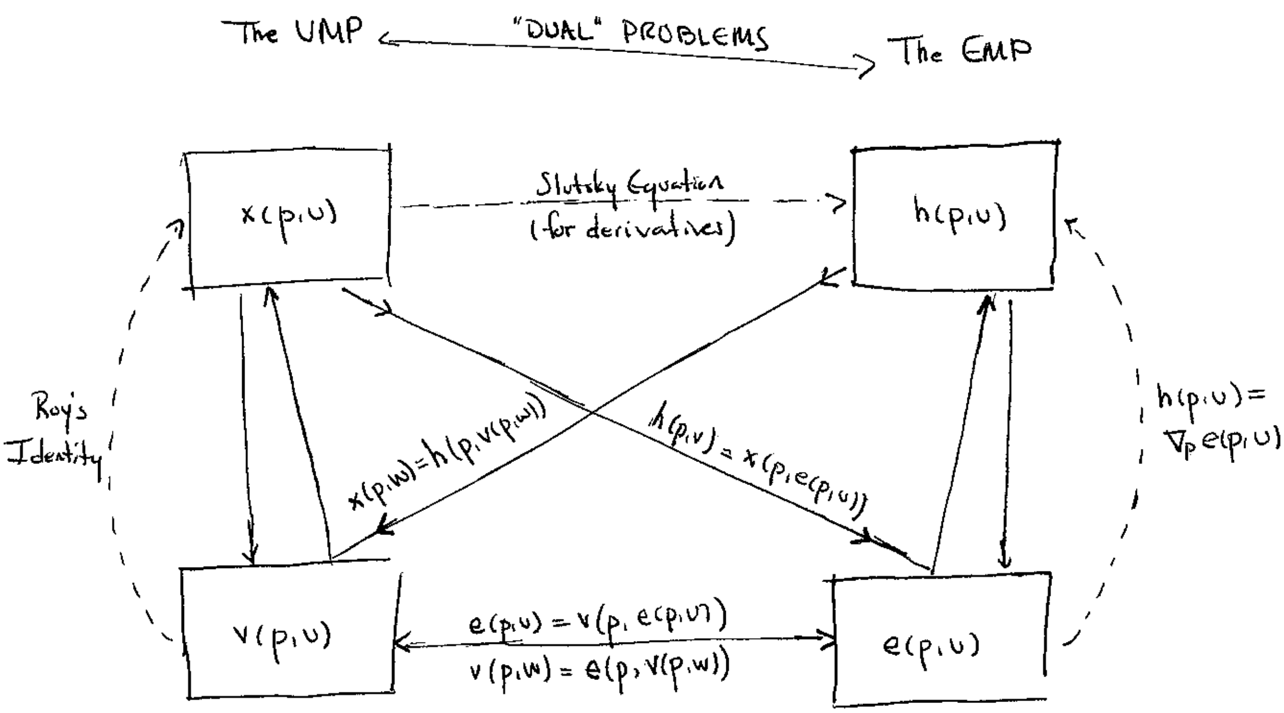
$$\frac{\partial v(\bar{p},\bar{w})}{\partial p_l} = \sum_{k=1}^{L} \lambda p_k \frac{\partial x_k(\bar{p},\bar{w})}{\partial p_l}.$$

Now, recall (Prop. 2.E.2), if Walrasian demand function $x(p,w)$ satisfies Walras' law, then $p \cdot D_p x(p,w) + x(p,w)^T = 0$, or for all p, & w. equivalently:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0 \quad \text{for } k = 1, \dots, L.$$

Thus, by substitution, & by the fact $\lambda = \partial v(\bar{p},\bar{w})/\partial w$ from the UMP (section 3.D of the text), the resulting equation is Roy's Identity, as above.

We can summarize the dual nature of the UMP & EMP as follows.



We leave Integrability & Welfare Evaluation of Economic Changes to be worked through as homework (pp. 75–92).