ECGA 7020
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Practice Problems

Problem 1. Take the generalized model for stochastic dynamic programming. Let $r(x,u)$ be the return function and $x' = g(x,u,z,\phi)$ be the transition function. Under this framework assume we have two shocks to the model z and ϕ.

a. Describe the transition function for the exogenous states z and ϕ.

Answer: The transition function for the two shocks would take the following form: $Q(z',\phi',z,\phi) = Prob(z_{t+1} \leq z', \phi_{t+1} \leq \phi' | z_t = z, \phi_t = \phi)$ where $Q()$ is the joint cumulative conditional distribution function.

b. Formulate the Bellman equation explicitly using the transition function for the exogenous shocks z and ϕ.

Answer: The Bellman equation takes the following form:

$$V(x,z,\phi) = \max_u \{r(x,u) + \beta \int_{z'} \int_{\phi'} V(x',z',\phi') f(z',\phi',z,\phi) d\phi' dz'\} \quad (1)$$

where $f(z',\phi',z,\phi)$ is the joint conditional probability density function.

c. Find the first order condition for the control vector u and the Benveniste-Sheinkman condition.

Answer: The first order condition is given by the following expression:

$$\frac{\partial r(x,u)}{\partial u} + \beta \int_{z'} \int_{\phi'} \frac{\partial V(x',z',\phi')}{\partial x'} \frac{\partial g(x,z,\phi,u)}{\partial u} f(z',\phi',z,\phi) d\phi' dz' = 0 \quad (2)$$

where $g(x,z,\phi,u)$ is constraint on how the endogenous state evolves. The Benveniste-Sheinkman condition is given by the following expression:

$$V'(x,z,\phi) = \frac{\partial r(x,h(x,z,\phi))}{\partial x} + \beta E\big[V'(g(x,z,\phi,h(x,z,\phi)),z',\phi') \frac{\partial g(x,z,\phi,h(x,z,\phi))}{\partial x}\big]$$

d. What conditions would one have to place on the stochastic processes z and ϕ so that usual contraction mapping theorem still applies?

Answer: In addition to the restrictions we required for the non-stochastic case, we need compactness for the set $Z = [z, \phi]$ and that the transition function $Q()$ have the Feller property (See Stokey and Lucas (1989).).

Problem 2. Start with the stochastic growth model for which $u(c_t) = ln(c_t)$, $k_{t+1} = z_t k_t^\alpha - c_t$ where we set $A = 1$ and $\delta = 1$. Furthermore assume shocks to productivity take only two values defined by:

$$z_t = \begin{cases} z^H, & \text{w.p. } p^H \\ z^L, & \text{w.p. } p^L = (1 - p^H) \end{cases}$$

where $z^H > z^L$ and $z_{t+1} \perp\!\!\!\perp z_t$.

a. Formulate the Bellman equation using the conditional expectations operator.

Answer: The Bellman equation takes the following form:

$$V(k, z) = \max_{k'} \{ln(zk^\alpha - k') + \beta E_t V(k', z')\} \tag{3}$$

b. What explicit form does the conditional expectations function take in the Bellman equation formed above?

$$V(k, z) = \max_{k'} \{ln(zk^\alpha - k') + \beta[p^H V(k', z^H) + (1 - p^H)V(k', z^L)]\} \tag{4}$$

c. Perform four iterations on the Bellman equation. What should your initial guess be for the value function? Does this initial guess matter?

Answer: For the first iteration set the value function equal to zero so that $V^0 = 0$. The initial guess does not matter for whether or not the value function converges. The initial guess only impacts the time it takes to attain convergence. For the first iteration $j = 1$ we have:

$$V^1(k, z) = \max_{k'} \{ln(zk^\alpha - k')\} \tag{5}$$

for which the optimal choice is $k' = 0$ so $V^1(k,z) = ln(zk^\alpha)$. Now for $j = 2$:

$$V^2(k,z) = \max_{k'}\{ln(zk^\alpha - k') + \beta[p^H ln(z^H k'^\alpha) + (1-p^H)ln(z^L k'^\alpha)]\} \quad (6)$$

taking the derivative with respect to k' yields the optimal $k' = \frac{\alpha\beta z k^\alpha}{1+\alpha\beta}$. This gives us the following expression for V^2:

$$V^2 = ln(zk^\alpha - \frac{\alpha\beta z k^\alpha}{1+\alpha\beta}) + \beta[p^H \alpha ln(\alpha\beta(z^H)^2 k^\alpha) + (1-p^H)\alpha ln(\alpha\beta(z^L)^2 k^\alpha) \quad (7)$$

After some rearranging $V^2 = \alpha(1+\alpha\beta)ln(k) + \Phi(z)$ where $\Phi(z)$ is a function that does not depend k. For $j = 3$:

$$V^3 = \max_{k'}\{ln(zk^\alpha - k') + \beta[p^H \alpha(1+\alpha\beta)ln(k') + \Phi(z^H) + (1-p^H)\alpha(1+\alpha\beta)ln(k') + \Phi(z^L)]\} \quad (8)$$

After taking the derivative we solve for $k' = \frac{\alpha\beta + (\alpha\beta)^2 z k^\alpha}{1+\alpha\beta+(\alpha\beta)^2}$. Notice that is the same series derived in class. Continuing on would show that $k' = \alpha\beta z k^\alpha$.

Problem 3. Take the following model with consumption (c_t), labor $((n_t)$, and capital (k_t). The goal is to maximize the stream of discounted utility of the form:

$$W = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \quad (9)$$

where the objective is to maximize W s.t. $k_{t+1} = f(k_t, n_t) - c_t$ and $0 \leq n_t \leq 1$.

a. Formulate the Bellman equation for this problem.

Answer: The Bellman equation is as follows:

$$V(k_t) = \max_{c_t, n_t}\{u(c_t, 1-n_t) + \beta V(k_{t+1})\} \quad (10)$$

or without time subscripts:

$$V(k) = \max_{c,n}\{u(c, 1-n) + \beta V(k')\} \quad (11)$$

b. What do we hope to obtain by solving the above problem? Be specific.

Answer: The goal is the find time invariant policy functions for consumption, labor, leisure, and capital stock: $c = \phi(k)$, $n = \Psi(k)$, $l = \Omega(k)$, and $k' = \Theta(k)$ which we know to be functions of the state variable k.

c. Derive the first order conditions and envelope condition.

Answer: The first order conditions with respect to c_t and n_t are as follows:

$$u_c(c_t, 1 - n_t) - \beta V_k(k_{t+1}) = 0 \tag{12}$$

$$-u_n(c_t, 1 - n_t) + \beta f_n(k_t, n_t) V_k(k_{t+1}) = 0 \tag{13}$$

The envelope condition is given by the following expression:

$$V_k(k_t) = \beta V_k(k_{t+1}) f_k(k_t, n_t) \tag{14}$$

or

$$\frac{V_k(k_t)}{\beta f_k(k_t, n_t)} = V_k(k_{t+1}) \tag{15}$$

After plugging the above expression into Equation (12) and updating we arrive at the following:

$$V_k(k_{t+1}) = u_c(c_{t+1}, 1 - n_{t+1}) f_k(k_{t+1}, n_{t+1}) \tag{16}$$

d. Show that the ratio of the marginal utility of consumption to the marginal utility of leisure depends on the marginal product of labor.

Answer: Note that from Equation (12) we can solve for $V_k(k_{t+1}) = \frac{u_c(c_t, 1-n_t)}{\beta}$. Plugging this expression into Equation (13) we obtain:

$$-u_n(c_t, 1 - n_t) + \beta f_n(k_t, n_t) \frac{u_c(c_t, 1 - n_t)}{\beta} = 0$$

solving for the ratio of the marginal utility of leisure to the marginal utility of consumption we obtain the desired result:

$$\frac{u_n(c_t, 1 - n_t)}{u_c(c_t, 1 - n_t)} = f_n(k_t, n_t) \tag{17}$$

e. Using the envelope condition find the first order conditions absent of the value function.

Answer: Using the EC, the first-order equations become:

$$u_c(c_t, 1 - n_t) - \beta u_c(c_{t+1}, 1 - n_{t+1}) f_k(k_{t+1}, n_{t+1}) = 0 \tag{18}$$

$$-u_n(c_t, 1 - n_t) + \beta f_n(k_t, n_t) u_c(c_{t+1}, 1 - n_{t+1}) f_k(k_{t+1}, n_{t+1}) = 0 \tag{19}$$