Problem: 6.B.2

Show that if preference relation $\succsim$ on $\mathcal{L}$ is represented by a utility function $U(\,)$ that has the expected utility form, then $\succsim$ satisfies the independence axiom.

Answer

Assume that the preference relation $\succsim$ is represented by an v.N-M expected utility function

$$U(L) = \sum_n u_n p_n \quad \text{for every } L = (p_1, \dots, p_N) \in \mathcal{L}$$

Let $L = (p_1, \dots, p_N) \in \mathcal{L}$, $L' = (p_1', \dots, p_N') \in \mathcal{L}$, $L'' = (p_1'', \dots, p_N'') \in \mathcal{L}$, and $\alpha \in (0,1)$.

Then $L \succsim L'$ if and only if

$$\sum_n u_n p_n \geq \sum_n u_n p_n'$$

This inequality is equivalent to (we add a third lottery to both sides in the same proportion)

$$\alpha \left(\sum_n u_n p_n \right) + (1-\alpha) \left(\sum_n u_n p_n'' \right) \geq \alpha \left(\sum_n u_n p_n' \right) + (1-\alpha) \left(\sum_n u_n p_n'' \right)$$

This inequality holds if and only if

$$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

Hence, $L \succsim L'$ if and only if

$$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

Thus the independence axiom holds.

Problem: 6.C.4 (a, b)

Suppose that there are N risky assets whore returns $z_n = (n = 1, \dots, N)$ per dollar invested are jointly distributed according to the distribution function $F(z_1, \dots, z_N)$. Assume also that all the returns are nonnegative with probability one. Consider an individual who has a continuous, increasing, and concave Bernoulli utility function $u(\,)$ over R_+^N. Define the utility function $U(\,)$ of this investor over R_+^N, the set of all nonnegative portfolios, by

$$U(\alpha_1, \dots, \alpha_N) = \int u(\alpha_1 z_1 + \cdots + \alpha_N z_N) dF(z_1, \dots, z_N)$$

Prove that $U(\,)$ is:

a) Increasing.
b) Concave

Answer

a) Let

$$\alpha = (\alpha_1, \dots, \alpha_N) \in R_+^N$$

$$\alpha' = (\alpha_1', \dots, \alpha_N') \in R_+^N$$

$$\alpha \geq \alpha'$$

Then we have the following:

$$\sum\nolimits_n \alpha_n z_n \geq \sum\nolimits_n \alpha_n' z_n$$

for almost every realization $(z_1, \dots, z_N)$, because all the returns are nonnegative with probability one.

Since $u(\,)$ is increasing, this implies that

$$u\left(\sum\nolimits_n \alpha_n z_n\right) \geq u\left(\sum\nolimits_n \alpha_n' z_n\right)$$

with probability one. Hence,

$$\int u\left(\sum\nolimits_n \alpha_n z_n\right) \geq \int u\left(\sum\nolimits_n \alpha_n' z_n\right)$$

That is the same as

$$U(\alpha) \geq U(\alpha')$$

Thus, U is increasing.

b) Let

$$\alpha = (\alpha_1, \dots, \alpha_N) \in R_+^N$$
$$\alpha' = (\alpha_1', \dots, \alpha_N') \in R_+^N$$
$$\lambda \in [0,1]$$

Then by concavity of $u(\,)$, we get

$$u\left(\sum_n (\lambda\alpha_n + (1-\lambda)\alpha_n')z_n\right) = u\left(\lambda\sum_n a_n z_n + (1-\lambda)\sum_n \alpha_n' z_n\right)$$
$$u\left(\lambda\sum_n a_n z_n + (1-\lambda)\sum_n \alpha_n' z_n\right) \geq \lambda u\left(\sum_n a_n z_n\right) + (1-\lambda)u\left(\sum_n \alpha_n' z_n\right)$$

which holds almost for all realization $(z_1, \dots, z_n)$.

Hence,

$$u\left(\lambda\sum_n a_n z_n + (1-\lambda)\sum_n \alpha_n' z_n\right) \geq \lambda u\left(\sum_n a_n z_n\right) + (1-\lambda)u\left(\sum_n \alpha_n' z_n\right)$$
$$\int u\left(\lambda\sum_n a_n z_n + (1-\lambda)\sum_n \alpha_n' z_n\right) dF(z_1, \dots, z_N) \geq \int \left(\lambda u\left(\sum_n a_n z_n\right) + (1-\lambda)u\left(\sum_n \alpha_n' z_n\right)\right) dF(z_1, \dots, z_N)$$
$$U(\lambda\alpha + (1-\lambda)\alpha_n') \geq \int u\left(\sum_n \alpha_n z_n\right) dF(z_1, \dots, z_N) + (1-\lambda)\int u\left(\sum_n \alpha_n' z_n\right) dF(z_1, \dots, z_N)$$
$$U(\lambda\alpha + (1-\lambda)\alpha_n') \geq \lambda U(\alpha) + (1-\lambda)U(\alpha_n')$$

Thus, $\mathbf{U}(\,)$ is concave.

Equalities used in the part b of the problem.

$$\int u\left(\sum_n (\lambda\alpha_n + (1-\lambda)\alpha_n')z_n\right) dF(z_1, \dots, z_N) = U(\lambda\alpha + (1-\lambda)\alpha_n')$$

And

$$\begin{aligned}
&\int \left(\lambda u\left(\sum_n \alpha_n z_n\right) + (1-\lambda)u\left(\sum_n \alpha_n' z_n\right)\right) dF(z_1, \dots, z_N) \\
&\qquad = \lambda\int u\left(\sum_n \alpha_n z_n\right) dF(z_1, \dots, z_N) + (1-\lambda)\int u\left(\sum_n \alpha_n' z_n\right) dF(z_1, \dots, z_N) \\
&\qquad = \lambda U(\alpha) + (1-\lambda)U(\alpha_n')
\end{aligned}$$

Problem: 6.C.15 (a, b)

Assume that in the world with uncertainty, there are two assets. The first is riskless asset that pays 1 dollar. The second pays amounts a and b with probabilities of π and $(1-\pi)$, respectively. Denote the demand for the two assets by (x_1, x_2).

Suppose that a decision maker's preferences satisfy the axioms of expected utility theory and that he is a risk averter. The decision maker's wealth is 1, and so are the prices of the assets. Therefore, the decision maker's budget constraint is given by

$$x_1 + x_2 = 1, \quad x_1, x_2 \in [0,1]$$

a) Give a simple necessary condition (involving a and b only) for the demand for the riskless asset to be strictly positive.
b) Give a simple necessary condition (involving a, b and π only) for the demand for the risky asset to be strictly positive.

Answer

Throughout the problem we assume that $a \neq b$ because otherwise there would be no uncertainty involved in the payment of the second asset.

a) If $Min\{a, b\} \geq 1$, the risky asset pays at least as high a return as the riskless asset at both states, and strictly higher return at one of them. Then all the wealth is invested to the risky asset.

   Thus, $Min\{a, b\} < 1$ is a necessary condition for the demand for the riskless asset to be strictly positive.

b) If $[\pi a + (1-\pi)b] \leq 1$, then the expected return does not exceed the payments of the riskless asset and hence the risk-averse decision maker does not demand the risky asset at all.

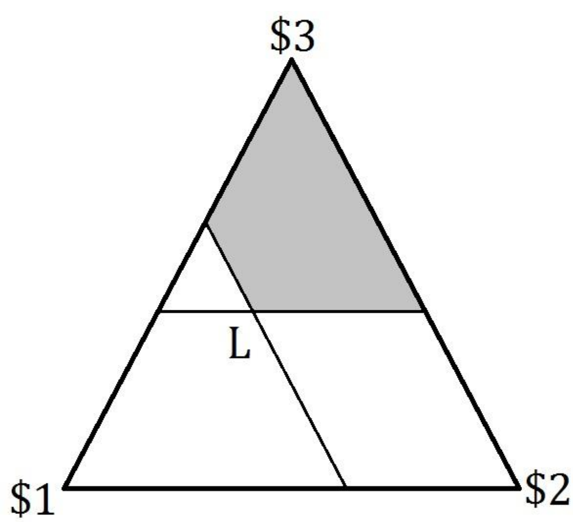
   Since, we care about the utilities of returns, not the returns itself, we have the following. $\pi u(a) + (1-\pi)u(b) > u(1)$ is a necessary condition for the demand for the risky asset to be strictly positive.

Problem 6.D.1

The purpose of this exercise is to prove Proposition 4.D.1 in a two-dimensional probability simplex. Suppose that there are three monetary outcomes: 1 dollar, 2 dollars, and 3 dollars. Consider the probability simplex of Figure 6.B.1(b).

a) For a given lottery L over these outcomes, determine the region of the probability simplex in which lie the lotteries whose distributions first-order stochastically dominate the distribution of L.
b) Given a lottery L, determine the region of the probability simplex in which lie the lotteries L' such that $F(x) \leq G(x)$ for every x, where $F(\,)$ is the distribution of L' and $G(\,)$ is the distribution of L. [Notice that we get the same region as in **(a)**].

Answer



6.D.2 Prove that if $F(\cdot)$ first-order stochastically dominates $G(\cdot)$, then the mean of x under $F(\cdot)$, $\int x dF(x)$, exceeds that under $G(\cdot)$, $\int x dG(x)$. Also provide an example where $\int x dF(x) > \int x dG(x)$, but $F(\cdot)$ does not first-order stochastically dominate $G(\cdot)$.

6.D.2 [First printing errata: The phrase "the mean of x under $F(\cdot)$, $\int x dF(x)$, exceeds that under $G(\cdot)$, $\int x dG(x)$" should be "the mean of x under $G(\cdot)$, $\int x dG(x)$, cannot exceed that under $F(\cdot)$, $\int x dF(x)$". That is, the equality of the two means should be allowed.] For the first assertion, simply put $u(x) = x$ and apply Definition 6.D.1. As for the second, let $p \in (0,1/2)$ and consider the following two distributions:

$$F(z) = \begin{cases} 0 & \text{if } z < 0, \\ p & \text{if } 0 \le z < 2, \\ 1 & \text{if } 2 \le z, \end{cases}$$

$$G(z) = \begin{cases} 0 & \text{if } z < 1, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Then $F(1/2) = p > 0 = G(1/2)$ and $\int x dF(x) = 2(1-p) > 1 = \int x dG(x)$. Hence $F(\cdot)$ does not first-order stochastically dominate $G(\cdot)$, but the mean of $F(\cdot)$ is larger than that of $G(\cdot)$.

Definitions: First Order Stochastic Dominance (FOSD):

$F(\cdot)$ FOSD $G(\cdot)$ if for every nondecreasing $u: \mathbb{R} \to \mathbb{R}$:

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

or $F(x) \le G(x)$ for every x.

Second Order Stochastic Dominance (SOSD):

For $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ SOSD $G(\cdot)$ (is less risky than) if for every nondecreasing concave function $u: \mathbb{R}_+ \to \mathbb{R}$:

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

Prove: if $F(\cdot)$ FOSD $G(\cdot)$ then mean of x under $F(\cdot)$ exceeds that under $G(\cdot)$:

$$\int x \, dF(x) \geq \int x \, dG(x)$$

Give an example where $\int x \, dF(x) \geq \int x \, dG(x)$ but $F(\cdot)$ does not FOSD $G(\cdot)$.

Answer: From FOSD, if $F(\cdot)$ FOSD $G(\cdot)$:

$$\int u(x) \, dF(x) \geq \int u(x) \, dG(x)$$

for every nondecreasing $u(x)$.

Let $u(x) = x$ (so that u remains nondecreasing).

Then plugging into the above inequality we obtain:

$$\int x \, dF(x) \geq \int x \, dG(x)$$

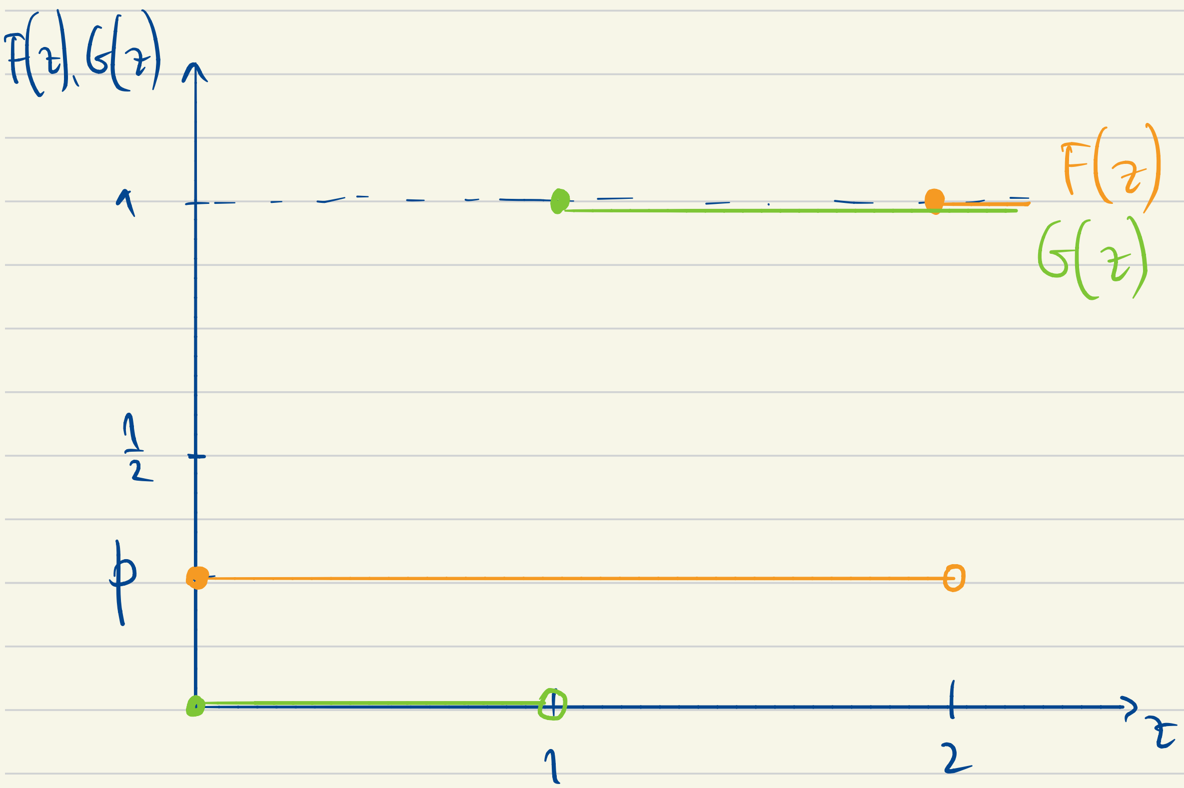
and so the mean of x under $F(\cdot)$ exceed (or is equal) to that under $G(\cdot)$.

Now, for an example where $\int x \, dF(x) > \int x \, dG(x)$ but $F(\cdot)$ does not FOSD $G(\cdot)$.

Let $p \in \left(0, \frac{1}{2}\right)$ and consider the (cumulative) distributions:

$$F(z) = \begin{cases} 0 & \text{if} \quad z < 0 \\ p & \text{if} \quad 0 \leq z < 2 \\ 1 & \text{if} \quad z \geq 2 \end{cases}$$

$$G(z) = \begin{cases} 0 & \text{if} \quad z < 1 \\ 1 & \text{if} \quad z \geq 1 \end{cases}$$



Graphically, $F(z)$ FOSD $G(z)$ if $F(z)$ lies below $G(z)$ (or at the same level, but never above) for all possible values of z. (higher prob of bigger payoff)

Note that this does not happen for $z \in [0, 1)$.

- $\int x \, dF(x) = 2(1-p)$

- $\int x \, dG(x) = 1$

Thus, $\int x \, dF(x) > \int x \, dG(x)$ for $0 < p < \frac{1}{2}$.

But, for example, $z = \frac{1}{2}$:

- $F\left(\frac{1}{2}\right) = p$

- $G\left(\frac{1}{2}\right) = 0$

So $F\left(\frac{1}{2}\right) > G\left(\frac{1}{2}\right)$ (will also hold for $0 \leq z < 1$)

Thus $F(\cdot)$ does not FOSD $G(\cdot)$.

Hence the mean of $F(\cdot)$ is larger than that of $G(\cdot)$ but $F(\cdot)$ does not FOSD $G(\cdot)$.

In conclusion: if $F(\cdot)$ FOSD $G(\cdot)$ $\Rightarrow \int x\, dF(x) > \int x\, dG(x)$

but $\int x\, dF(x) > \int x\, dG(x) \not\Rightarrow$ $F(\cdot)$ FOSD $G(\cdot)$.