Consumer Choice

Definitions of Commodities

Commodities are a finite set of goods & services available for purchase, $l = 1, 2, \ldots, L$.

A commodity bundle is a vector specifying units of each of L different commodities,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{bmatrix}$$

and is a point in $\mathbb{R}^L$, the commodity space; generally, we restrict $x_i \geq 0$, or $x > 0$. Note that time (bread today & tomorrow) and in different states of nature (rain or no rain) can be viewed as elements of the commodity space.

The Consumption Set

The consumption set is a subset of commodity space $X \subset \mathbb{R}^L$ whose elements are the consumption bundles that an individual can conceivably consume given the constraints imposed by her environment. Such constraints might include time, lifespan, indivisibility, necessity, geographic, or institutional. They might also be physical.

Formally, we consider the simplest sort of constraint, the nonnegative constraint:

$$X = \mathbb{R}^L_+ = \{ x \in \mathbb{R}^L : x_l \geq 0 \ \forall \ l = 1, 2, \ldots, L \}.$$

This set is convex; we will discuss convexity later.

Competitive Budgets

We suppose that L commodities are all traded at publically quoted prices (the principle of completeness, or universality, of markets) represented by price vector

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

Note that we can accomodate $p_l \leq 0$, but for now we assume $p_l > 0$, or $p \gg 0$. We also make the price-taking assumption, that the good's price is beyond influence of the consumer.

The affordability of a bundle depends on the market prices and the consumer's wealth, w. The bundle x is affordable if $px \le w$, or

$$p_1x_1 + p_2x_2 + \dots + p_Lx_L \le w.$$

The Walrasian Budget set is $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \le w\}$, the set of all feasible consumption bundles given p, w.

The budget hyperplane, the set $\{x \in \mathbb{R}^L_+ : p \cdot x = w\}$ is the upper boundary of the budget set. (We call this the budget line for $L=2$.)

$B_{p,w}$ is a convex set because if x and x' are in $B_{p,w}$, then $x'' = \alpha x + (1-\alpha x)$ is also, for $\alpha \in [0,1]$. (This depends on the convexity of X.)

Proof: 1) x and x' are non-negative $\Rightarrow x'' \in \mathbb{R}^L_+$

2) $px \le w$ and $px' \le w$

$\Rightarrow px'' = \alpha px + (1-\alpha)px' \le w$

$\Rightarrow x'' \in B_{p,w}$

Demand Functions & Comparative Statics

The Walrasian Demand Correspondence $x(p,w)$ assigns a set of chosen consumption bundles for each p,w pair. If $x(p,w)$ is single-valued, it is a demand function.

Assumptions for $x(p,w)$

1) $x(p,w)$ is homogeneous of degree zero: $x(p,w) = x(\alpha p, \alpha w)$ for any p, w, and $\alpha > 0$. This implies $B_{p,w} = B_{\alpha p, \alpha w}$; that is, a proportional change in prices and wealth does not change the budget set. (Only relative prices matter.)

2) $x(p,w)$ satisfies Walras' law if for every $p \gg 0$ and $w > 0$, we have $px = w$ $\forall$ $x \in x(p,w)$; that is, all wealth is always spent over the course of a consumer's lifetime.

This requires (i) more is better, (ii) local non-satiation, and (iii) continuity of preferences; for now, assume these are satisfied, and that $x(p,w)$ is single-valued.

Under the assumption that $x(p,w)$ is single-valued (and sometimes also assumed continuous & differentiable), we can write it in terms of commodity-specific demand functions:

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ \vdots \\ x_L(p,w) \end{bmatrix}.$$

Comparative Statics

Wealth effects

For fixed prices $\bar{p}$, the function of wealth $x(\bar{p},w)$ is the consumer's Engel function. Its image in $\mathbb{R}^L_+$, $E_p = \{x(\bar{p},w) : w > 0\}$ is the wealth expansion path.

At any (p,w), $\partial x_l(p,w)/\partial w$ is the wealth effect on the l^{th} good; if $\cdot \geq 0$, the good is normal (demand is non-decreasing in wealth), and if $\cdot < 0$, the good is inferior at (p,w). If every commodity is normal at all (p,w), we say demand is normal.

In matrix notation, wealth effects are represented as

$$D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \frac{\partial x_2(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L.$$

Price effects

When we keep wealth and all other prices constant, and vary l as a function of its own price, we have the demand curve for commodity l. The locus of points demanded in $\mathbb{R}^2$ over all possible values of p_k is the offer curve.

The derivative $\partial x_l(p,w)/\partial p_k$ is the price effect of p_k on good l.

Price effects in matrix form are:

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \frac{\partial x_1(p,w)}{\partial p_2} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \frac{\partial x_2(p,w)}{\partial p_1} & \frac{\partial x_2(p,w)}{\partial p_2} & \cdots & \frac{\partial x_2(p,w)}{\partial p_L} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} & \frac{\partial x_L(p,w)}{\partial p_2} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$$

For own-price, it is generally the case that $\frac{\partial x_l(p,w)}{\partial p_l} \leq 0$. When $\cdot >$ the good is said to be a ~~g~~ Giffen good at (p,w), and the offer curve is downward-sloping if $L=2$.

For cross-price, if $\frac{\partial x_l(p,w)}{\partial p_k} < 0$, we say the goods are complements at (p,w) and if $\cdot > 0$ we say they are substitutes; unrelated if $\cdot = 0$.

Implications of homogeneity and Walras' law for price & wealth effects

1) Because of homogeneity of degree zero, $x(\alpha p, \alpha w) - x(p,w) = 0$. If we differentiate w/ respect to α and then evaluate the derivative at $\alpha = 1$, we get:

Prop. If the Walrasian $x(p,w)$ is homogenous of degree zero, then for all p and w,

$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0 \text{ for } l = 1, \ldots, L,$$

or in matrix notation

$$D_p x(p,w)p + D_w x(p,w) w = 0.$$

We define elasticities of demand w/ respect to price and wealth (respectively) as

$$\varepsilon_{l,k}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} \cdot \frac{p_k}{x_l(p,w)} \text{ and } \varepsilon_{l,w}(p,w) = \frac{\partial x_l(p,w)}{\partial w} \cdot \frac{w}{x_l(p,w)},$$

these being the ratio of percentage change in demand to price/wealth.

Then, by substitution,

$$\sum_{k=1}^{L} \varepsilon_{l,k}(p,w) + \varepsilon_{l,w}(p,w) = 0 \text{ for } l = 1, \dots L.$$

That is, an equal percentage change in all prices and wealth leads to no change in demand.

2) By Walras' law, we know $p \cdot x(p,w) = w \; \forall p, w$. Differentiating this law

i) with respect to prices,

Prop. If the Walrasian $x(p,w)$ satisfies Walras's law, then for all p & w,

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0 \text{ for } k = 1, \dots, L,$$

or in matrix notation,

$$p \cdot D_p x(p,w) + x(p,w)^T = 0^T$$

This is Cournot aggregation: total expenditure does not change in response to a change in prices.

ii) with respect to wealth,

Prop. If the Walrasian $x(p,w)$ satisfies Walras' law, then for all p & w,

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1,$$

or in matrix notation,

$$p \cdot D_w x(p,w) = 1.$$

This is Engel aggregation: total expenditure changes by an amount equal to any wealth change.

These equations can also be rewritten in terms of elasticities.

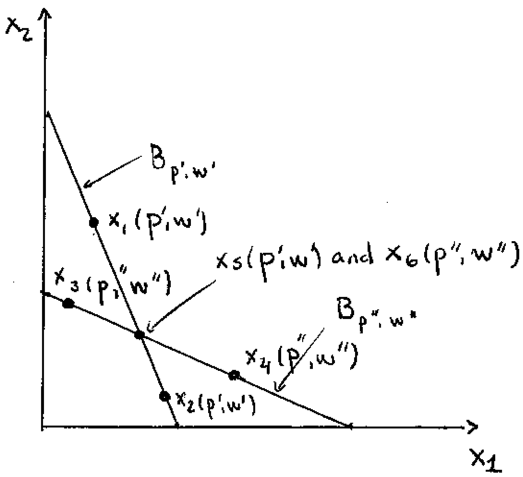
WARP and The Law of Demand

We assume $x(p,w)$ is single-valued, homogeneous of degree zero, & satisfies Walras' law.

Definition. $x(p,w)$ satisfies WARP if $p\cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w) \Longrightarrow$ $p'\cdot x(p,w) > w'$ for any two price-wealth situations (p,w) and (p',w').

If $p\cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$, then we know that the consumer chose $x(p,w)$ even though $x(p',w')$ was also affordable; we could say that $x(p,w)$ was "revealed prefered" to $x(p',w')$, so $x(p,w)$ must not be affordable at (p',w') if the consumer choses $x(p',w')$, or $p'x(p,w) > w'$. (Since if $x(p,w)$ were affordable, we'd have expected chosen over $x(p',w')$.)

Example. Suppose $L=2$.



a) $x_2(p',w')$ and $x_4(p'',w'')$
<u>does not</u> violate WARP

b) $x_1(p',w')$ and $x_4(p'',w'')$
<u>does not</u> violate WARP

c) $x_5(p',w')$ and $x_4(p'',w'')$
<u>does not</u> violate WARP

d) $x_3(p'',w'')$ and $x_5(p',w')$
<u>does</u> violate WARP if $x(p,w)$ is single-valued

e) $x_3(p'',w'')$ and $x_2(p',w')$
<u>does</u> violate WARP

Price Changes and WARP

A change in price alters (i) the relative cost of commodities and (ii) the consumer's real wealth. To study the implications of WARP, we must isolate changes in relative price.

Suppose prices change; to isolate the effect of <u>only</u> relative prices, we also change wealth to w' so $x(p,w)$ is just affordable at w'. Step by step:

i) start at $x(p,w)$

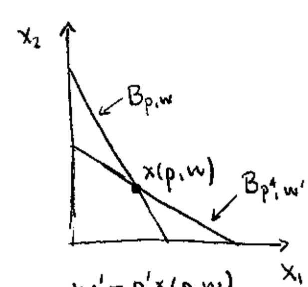
ii) new prices at p'

iii) set w' such that $p'x(p,w) = w'$

iv) the wealth adjustment is $\Delta w = \Delta p \cdot x(p,w)$ where $\Delta p = (p'-p)$.

Δw is called Slutsky <u>wealth compensation</u>

Δp is called <u>Slutsky compensated price changes</u>



$w' = p'x(p,w)$

Proposition. Suppose $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(p,w)$ satisfies WARP iff

For any compensated price change from (p,w) to $(p',w') = (p', p' \cdot x(p,w))$, we have $(p'-p)[x(p',w') - x(p,w)] \leq 0$, w/ strict inequality whenever $x(p,w) \neq x(p',w')$.

Proof. Because the proposition states iff, we must prove that the inequality is implied by WARP, and that WARP implies the inequality.

1) WARP implies the inequality

i) If $x(p,w) = x(p',w') \Rightarrow (p'-p)[x(p',w') - x(p,w)] = 0$.

ii) Suppose $x(p,w) \neq x(p',w')$, then

$$(p'-p)\left[x(p',w') - x(p,w)\right] = p'\left[x(p',w') - x(p,w)\right] - p\left[x(p',w') - x(p,w)\right]$$

We know $p'x(p',w') = w'$ by Walras' law, and
" $p'x(p,w) = w'$ by compensated price change,
so the first term $= 0$ and we need to show

$p[x(p',w') - x(p,w)] > 0$.

Because $p'x(p,w) = w'$, $x(p,w)$ is affordable under p', w'. WARP implies that $x(p',w')$ must then not be affordable at (p,w). Thus, we must have ~~$p \cdot x(p',w') > w$~~
$p \cdot x(p',w') > w$.

Walras' law implies $p \cdot x(p,w) = w$.

Thus,

$$p[x(p',w') - x(p,w)] > 0$$

and the result is yielded that the inequality is satisfied.

2) The inequality implies WARP.

This will be a proof by contradiction. Suppose that WARP doesn't hold. Then, $\exists$ some compensated price change from (p,w) to (p',w') such that $x(p,w) \neq x(p',w')$, $p\,x(p',w') = w$, and $p'x(p,w) \leq w'$. Because $x(\cdot,\cdot)$ satisfies Walras' law, this implies

$$p\cdot[x(p',w')-x(p,w)]=0 \quad \text{and} \quad p'\cdot[x(p',w')-x(p,w)]\geq 0$$

and hence,

$$(p'-p)\cdot[x(p',w')-x(p,w)]\geq 0 \text{ and } x(p,w)\neq x(p',w')$$

which contradicts the inequality holding for all compensated price changes.

The inequality can be written in shorthand as $\Delta p\cdot\Delta x\leq 0$, where $\Delta p\equiv(p'-p)$ and $\Delta x\equiv[x(p',w')-x(p,w)]$, and can be interpreted as the <u>law of demand</u>: demand and price move in opposite directions. Because this law holds for compensated price changes, we call it the <u>compensated law of demand</u>.

WARP is not sufficient to yield the law of demand for price changes that are not compensated. (Nor, in fact, are the more restrictive assumptions of preference maximization.)

Implications of the Law of Demand

Assume $x(p,w)$ is differentiable. Imagine we give the consumer compensation for a price change such that $dw=x(p,w)\,dp$ (the differential analog of $\Delta w=x(p,w)\Delta p$.

The law of demand states $dp\cdot dx\leq 0$. Using the chain rule,

$$\begin{aligned} dx &= D_p x(p,w)\,dp + D_w x(p,w)\,dw \\ &= D_p x(p,w)\,dp + D_w x(p,w)[x(p,w)\,dp] \\ &= [D_p x(p,w) + D_w x(p,w)\,x(p,w)^T]\,dp \end{aligned}$$

Now, by substitution,

$$dp\,[D_p x(p,w) + D_w x(p,w)\,x(p,w)^T]\,dp\leq 0.$$

The expression in brackets is an $L\times L$ matrix,

$$S(p,w)=\begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w)\end{bmatrix},$$

where the (l,k)th entry is $s_{lk}(p,w)=\dfrac{\partial x_l(p,w)}{\partial p_k}+\dfrac{\partial x_l(p,w)}{\partial w}x_k(p,w)$.

The matrix $S(p,w)$ is known as the substitution, or Slutsky, matrix, and its elements are known as substitution effects; these measure the differential change in consumption of commodity l (the substitution to or from other commodities) due to a differential change in the price of k when wealth is adjusted so that the consumer can just afford her original consumption bundle.

We can summarize the derivation of the Slutsky matrix in

Proposition. If $x(p,w)$ satisfies Walras' law, is homogeneous of degree zero, WARP, and is differentiable, then at any (p,w), the Slutsky matrix satisfies $v \cdot S(p,w) \cdot v \leq 0$ for any $v \in \mathbb{R}^L$; in other words, the Slutsky matrix is negative semidefinite (NSD).

Being NSD implies $s_{ll}(p,w) \leq 0$; the substitution effect w/ respect to its own price is nonpositive. This means Giffen goods must be inferior. Since

$$s_{ll}(p,w) = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} x_l(p,w) \leq 0,$$

and $\dfrac{\partial x_l(p,w)}{\partial p_l} > 0 \Rightarrow \dfrac{\partial x_l(p,w)}{\partial w} < 0.$

Note also that $S(p,w)$ is not generally symmetric, but is so for $L=2$.

Proposition. Suppose $x(p,w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p \cdot S(p,w) = 0$ and $S(p,w)p = 0$ for any (p,w).

Proof as excercise.

It follows that $S(p,w)$ is always singular (has rank less than L), and so $S(p,w)$ cannot be negative definite.