Chapter 2

Problem: 2.E.1

Suppose $L = 3$, and consider the demand function $x(p, w)$ defined by

$$x_1(p,w) = \frac{p_2}{p_1 + p_2 + p_3}\frac{w}{p_1}$$

$$x_2(p,w) = \frac{p_3}{p_1 + p_2 + p_3}\frac{w}{p_2}$$

$$x_3(p,w) = \frac{\beta p_1}{p_1 + p_2 + p_3}\frac{w}{p_3}$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0,1)$.

Answer

To test the homogeneity, we multiply prices and wealth by α.

$$x_1(\alpha p, \alpha w) = \frac{\alpha p_2}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\cancel{\alpha} w}{\cancel{\alpha} p_1} = \frac{\cancel{\alpha} p_2}{\cancel{\alpha}(p_1 + p_2 + p_3)}\frac{w}{p_1} = \frac{p_2}{p_1 + p_2 + p_3}\frac{w}{p_1} = x_1(p,w)$$

$$x_2(p,w) = \frac{\alpha p_3}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\cancel{\alpha} w}{\cancel{\alpha} p_2} = \frac{\cancel{\alpha} p_3}{\cancel{\alpha}(p_1 + p_2 + p_3)}\frac{w}{p_2} = \frac{p_3}{p_1 + p_2 + p_3}\frac{w}{p_2} = x_2(p,w)$$

$$x_3(p,w) = \frac{\alpha \beta p_1}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\cancel{\alpha} w}{\cancel{\alpha} p_3} = \frac{\cancel{\alpha}\beta p_1}{\cancel{\alpha}(p_1 + p_2 + p_3)}\frac{w}{p_3} = \frac{\beta p_1}{p_1 + p_2 + p_3}\frac{w}{p_3} = x_3(p,w)$$

Therefore, the demand function is homogeneous of degree zero with any β.

When $\beta = 1$.

For the demand function to satisfy Walras' law, it needs to satisfy the following equation:

$$p \cdot x(p,w) = w$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ x_3(p,w) \end{bmatrix} = \frac{\cancel{p_1} p_2 w}{(p_1 + p_2 + p_3)\cancel{p_1}} + \frac{\cancel{p_2} p_3 w}{(p_1 + p_2 + p_3)\cancel{p_2}} + \frac{\beta \cancel{p_3} p_1 w}{(p_1 + p_2 + p_3)\cancel{p_3}} = \frac{p_2 w + p_3 w + p_1 w}{(p_1 + p_2 + p_3)}$$

$$= \frac{(\beta p_1 + p_2 + p_3) w}{(p_1 + p_2 + p_3)}$$

$$Only\ when\ \beta = 1, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ x_3(p,w) \end{bmatrix} = \frac{((1)p_1 + p_2 + p_3) w}{(p_1 + p_2 + p_3)} = w$$

The demand function **does not** satisfy Walras' law when $\beta \neq 1$.

Problem: 2.E.7

A consumer in a two-good economy has a demand function $x(p,w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p,w) = \frac{\alpha w}{p_1}$. Derive his demand function for the second good. Is his demand function homogeneous of degree zero?

Answer

We know Walras' law. Plug in what we have into the formula of Walras' law and derive demand function for good 2.

$$p \cdot x(p,w) = w$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \cdot \begin{bmatrix} \alpha w/p_1 \\ x_2 \end{bmatrix} = \alpha w + p_2 x_2 = w$$

$$x_2 = \frac{(1-\alpha)w}{p_2}$$

Therefore, our demand function is as follows.

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \end{bmatrix} = \begin{bmatrix} \alpha w/p_1 \\ (1-\alpha)w/p_2 \end{bmatrix}$$

Now we show that this function is homogeneous of degree zero.

$$x(\lambda p, \lambda w) = \begin{bmatrix} \lambda \alpha w/\lambda p_1 \\ (1-\alpha)\lambda w/\lambda p_2 \end{bmatrix} = \begin{bmatrix} \alpha w/p_1 \\ (1-\alpha)w/p_2 \end{bmatrix} = x(p,w)$$

Problem: 2.F.3 (a, b, c) (assume WARP holds)

You are given the following partial information about a consumer purchases. He consumes only two goods.

	Year 1			Year 2	
	Q	P		Q	P
Good 1	100	100		120	100
Good 2	100	100		X	80

Over what range of quantities of good 2 consumed in year 2 would you conclude:

a) That his behavior is consistent with WARP?

b) That the consumer's consumption bundle in year 1 is revealed preferred to that in year 2?

c) That the consumer's consumption bundle in year 2 is revealed preferred to that in year 1?

Answer

a) The Walrasian demand function satisfies WARP if

$$p \cdot x(p', w') \leq w \quad \text{and} \quad x(p,w) \neq x(p',w') \quad \text{then} \quad p' \cdot x(p,w) > w'$$
$$\textit{And we know that } w = p \cdot x(p,w).$$

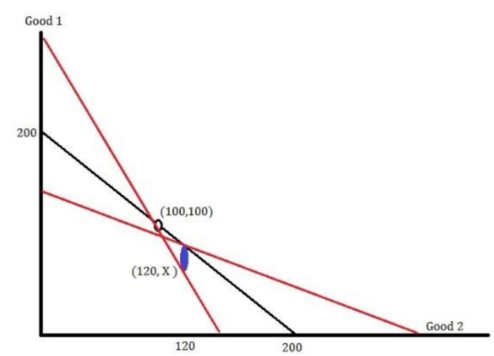
Plugging in the values to the first equation:

$$p \cdot x(p', w') \leq w$$
$$p \cdot x(p', w') \leq p \cdot x(p,w)$$
$$\begin{bmatrix}100\\100\end{bmatrix} \cdot \begin{bmatrix}120\\X\end{bmatrix} \leq \begin{bmatrix}100\\100\end{bmatrix} \cdot \begin{bmatrix}100\\100\end{bmatrix}$$
$$12000 + 100X \leq 20000$$
$$100X \leq 8000$$
$$X \leq 80$$

Doing the same with the last equation we get:

$$p' \cdot x(p,w) > w'$$
$$p' \cdot x(p,w) > p' \cdot x(p', w')$$
$$\begin{bmatrix}100\\80\end{bmatrix} \cdot \begin{bmatrix}100\\100\end{bmatrix} > \begin{bmatrix}100\\80\end{bmatrix} \cdot \begin{bmatrix}120\\X\end{bmatrix}$$
$$18000 > 12000 + 80X$$
$$6000 > 80X$$
$$75 > X$$

Also, do the problem the other way around.



Combining these, we get that his purchases are consistent with WARP if he purchases bellow or equal to 80, and bellow 75. It needs to be below 80 so that it would make the new bundle affordable to the old prices and wealth. At the same time, the new bundle must not be affordable when the prices-wealth combination change, thus the quantity of good 2 must be below 75 to conform with it. Included, there is an example graph (not to be taken seriously – this is just for intuition, it is not accurate for this problem). It shows that if the quantity of good 2 in the second year is above 80, it will not be affordable with the old prices-wealth and that if the quantity of good 2 in the second year is above 75, then the old bundle is still affordable and he wouldn't choose the new bundle (instead he would stay with the original bundle which is preferred.

$$\textit{Inconsistent (violates) WARP if } 75 > X < 80$$

$$\textit{WARP is violated when } X \in [75{,}80]$$

b) Assuming WARP, if $x(p,w) \succsim^* x(p',w')$, then it means that $p \cdot x(p',w') \leq w \ (= p \cdot x(p,w)\ )$ and $p' \cdot x(p,w) > w' \ (= p' \cdot x(p',w')\ )$. In words, it means that if the first bundle is revealed preferred to the second bundle, it means that the second bundle was affordable at the beginning prices-wealth but not as good as the first, therefore not chosen (not preferred). Also, when prices changed the second bundle was chosen only because the first bundle was not affordable (and therefore not chosen for consumption and less preferred consumption bundles was chosen).

$p \cdot x(p',w') \leq w$ $p \cdot x(p',w') \leq p \cdot x(p,w)$ $\begin{bmatrix}100\\100\end{bmatrix} \cdot \begin{bmatrix}120\\X\end{bmatrix} \leq \begin{bmatrix}100\\100\end{bmatrix} \cdot \begin{bmatrix}100\\100\end{bmatrix}$ $12000 + 100X \leq 20000$ $X \leq 80$	$p' \cdot x(p,w) > w'$ $p' \cdot x(p,w) > p' \cdot x(p',w')$ $\begin{bmatrix}100\\80\end{bmatrix} \cdot \begin{bmatrix}100\\100\end{bmatrix} > \begin{bmatrix}100\\80\end{bmatrix} \cdot \begin{bmatrix}120\\X\end{bmatrix}$ $18000 > 12000 + 80X$ $X < 75$

Thus, we see that consumer's bundle $x(p,w) \succsim^* x(p',w')$ when X is below 75 and 80 at the same time, therefore below 75.

$$x(p,w) \succsim^* x(p',w') \quad when \quad X < 75$$

c) Assuming WARP, if $x(p',w') \succsim^* x(p,w)$, then this must mean that at first price-wealth (p,w) situation the second "better" bundle $x(p',w')$ was not affordable, that is way the first bundle $x(p,w)$ was chosen. And to have the second bundle $x(p',w')$ to be revealed preferred it must be chosen over the first bundle $x(p,w)$ when both are affordable. Means that at the second price-wealth situation (p',w') both were affordable.

Mathematically, this means the following two equations.

$p \cdot x(p',w') > w$ $p \cdot x(p',w') > p \cdot x(p,w)$ $\begin{bmatrix}100\\100\end{bmatrix} \cdot \begin{bmatrix}120\\X\end{bmatrix} > \begin{bmatrix}100\\100\end{bmatrix} \cdot \begin{bmatrix}100\\100\end{bmatrix}$ $12000 + 100X > 20000$ $X > 80$	$p' \cdot x(p,w) \leq w'$ $p' \cdot x(p,w) \leq p' \cdot x(p',w')$ $\begin{bmatrix}100\\80\end{bmatrix} \cdot \begin{bmatrix}100\\100\end{bmatrix} \leq \begin{bmatrix}100\\80\end{bmatrix} \cdot \begin{bmatrix}120\\X\end{bmatrix}$ $18000 \leq 12000 + 80X$ $X \geq 75$

Thus, we can conclude that bundle $x(p',w')$ is revealed preferred to $x(p,w)$ when $X > 80$.

Problem: 2.F.16 (a, b)

Consider a setting where $L = 3$ and a consumer whose consumption set is R^3. Suppose that the demand function $x(p, w)$ is

$$x_1(p,w) = \frac{p_2}{p_3}, \qquad x_2(p,w) = -\frac{p_1}{p_3}, \qquad x_3(p,w) = \frac{w}{p_3}$$

a) Show that $x(p, w)$ is homogeneous of degree zero in (p, w) and satisfies Walras' law.

b) Show that $x(p, w)$ violates WARP.

Answer

a) First, we show that this demand function is homogeneous of degree zero.

$$x_1(\alpha p, \alpha w) = \frac{\alpha p_2}{\alpha p_3} = \frac{p_2}{p_3} = x_1(p,w); \qquad x_2(\alpha p, \alpha w) = -\frac{\alpha p_1}{\alpha p_3} = -\frac{p_1}{p_3} = x_2(p,w)$$

$$x_3(\alpha p, \alpha w) = \frac{\alpha w}{\alpha p_3} = \frac{w}{p_3} = x_3(p,w)$$

Now, we show that this demand function satisfies Walras' law.

$$p \cdot x(p,w) = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} p_2/p_3 \\ -p_1/p_3 \\ w/p_3 \end{bmatrix} = \frac{p_1 p_2}{p_3} - \frac{p_1 p_2}{p_3} + \frac{p_3 w}{p_3} = w$$

b) To prove that it violates WARP by an e$p_1$xample prices and wealth.

Suppose we have the following prices and wealth combinations:

$$p = (1,2,1), w = 1 \quad and \quad p' = (1,1,1), w' = 2$$

In these cases, by plugging in the prices and wealth into the demand function we get the following.

$$x(p,w) = \begin{bmatrix} p_2/p_3 \\ -p_1/p_3 \\ w/p_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad and \quad x(p',w') = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

We know that WARP implies that if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$,
then the old bundle under new prices and wealth was not affordable $p' \cdot x(p, w) > w'$.

In words, WARP implies that if the later bundle was affordable under older original prices and wealth, then it means that the original bundle was not any more affordable under new prices and thus a new bundle was chosen.

Plugging values into the inequalities, we get the following.

$p \cdot x(p', w') \leq w$ $\langle 1,2,1 \rangle \cdot \langle 1,-1,2 \rangle \leq 1$ $1 - 2 + 2 \leq 1$ $1 \leq 1$	$p' \cdot x(p, w) > w'$ $\langle 1,1,1 \rangle \cdot \langle 2,-1,1 \rangle > 2$ $2 - 1 + 1 > 2$ $2 > 2$

We see that the second inequality is violated. It indicates that the old bundle was affordable under new prices but still a new bundle was chosen. In both price-wealth situations both bundles were available but the consumer switched from one bundle to another. Therefore, it violates WARP.

Problem: 2.F.17 (a, b, c)

In an L commodity world, a consumer's Walrasian demand function is

$$x_k(p,w) = \frac{w}{\sum_{l=1}^{L} p_l} \; for\; k = 1,2,3,\dots,L.$$

a) Is this demand homogeneous of degree zero in (p, w)?
b) Does it satisfy Walras' law?
c) Does it satisfy WARP?
d) Compute Slutsky substitution matrix (extra).

Answer

a) First, we show that this demand function is homogeneous of degree zero in prices and wealth.

$$x_k(\alpha p, \alpha w) = \frac{\alpha w}{\sum_{l=1}^{L} \alpha p_l} = \frac{\alpha w}{\alpha p_1 + \alpha p_2 + \cdots + \alpha p_L} = \frac{\alpha w}{\alpha (p_1 + p_2 + \cdots + p_L)} = \frac{\alpha w}{\alpha \sum_{l=1}^{L} p_l} = \frac{w}{\sum_{l=1}^{L} p_l} = x_k(p,w)$$

b) Now, we show that it satisfies Walras' law.

c) Now, we show that WARP is satisfied by this demand function.

Let's assume that the original bundle $x(p, w)$ is affordable even under new prices and wealth (p', w') and the new bundle $x(p', w')$ is also affordable under original (initial) prices and wealth $(p.w)$. Mathematically, we can state the following inequalities, then, solve them.

$$p' \cdot x(p,w) \le w' \quad and \quad p \cdot x(p',w') \le w$$

$$\begin{bmatrix} p'_1 \\ \vdots \\ p'_L \end{bmatrix} \cdot \begin{bmatrix} \left(w/\sum_{l=1}^{L} p_l \right) \\ \vdots \\ \left(w/\sum_{l=1}^{L} p_l \right) \end{bmatrix} \le w'$$

$$\frac{wp'_1 + wp'_2 + \cdots + wp'_L}{\sum_{l=1}^{L} p_l} = \frac{w(p'_1 + p'_2 + \cdots + p'_L)}{\sum_{l=1}^{L} p_l} = \frac{w \sum_{l=1}^{L} p'_l}{\sum_{l=1}^{L} p_l} \le w'$$

$$\frac{w}{\sum_{l=1}^{L} p_l} \le \frac{w'}{\sum_{l=1}^{L} p'_l}$$

$$p \cdot x(p',w') \le w$$

$$\begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \cdot \begin{bmatrix} \left(w' / \sum_{l=1}^{L} p'_l\right) \\ \vdots \\ \left(w' / \sum_{l=1}^{L} p'_l\right) \end{bmatrix} \leq w$$

$$\frac{w'p_1 + w'p_2 + \cdots + w'p_L}{\sum_{l=1}^{L} p'_l} = \frac{w'(p_1 + p_2 + \cdots + p_L)}{\sum_{l=1}^{L} p'_l} = \frac{w' \sum_{l=1}^{L} p_l}{\sum_{l=1}^{L} p'_l} \leq w$$

$$\frac{w'}{\sum_{l=1}^{L} p'_l} \leq \frac{w}{\sum_{l=1}^{L} p_l}$$

Now taking both of these inequalities we see that $x(p, w) = x(p', w')$. Therefore, WARP holds.

$$\frac{w}{\sum_{l=1}^{L} p_l} \leq \frac{w'}{\sum_{l=1}^{L} p'_l} \qquad and \qquad \frac{w'}{\sum_{l=1}^{L} p'_l} \leq \frac{w}{\sum_{l=1}^{L} p_l}$$

$$\frac{w}{\sum_{l=1}^{L} p_l} \leq \frac{w'}{\sum_{l=1}^{L} p'_l} \leq \frac{w}{\sum_{l=1}^{L} p_l}$$

$$\frac{w}{\sum_{l=1}^{L} p_l} = \frac{w'}{\sum_{l=1}^{L} p'_l}$$

d) Slutsky substitution matrix is computed as follows.

$$S(p, w) = D_p x(p, w) + D_w x(p, w)[x(p, w)]^T; \qquad S(p, w) = \begin{bmatrix} S_{11}(p, w) & \cdots & S_{1L}(p, w) \\ \vdots & \ddots & \vdots \\ S_{L1}(p, w) & \cdots & S_{LL}(p, w) \end{bmatrix}$$

$$D_p x(p, w) = D_p \frac{w}{\sum_{l=1}^{L} p_l} = -\frac{w}{(\sum_{l=1}^{L} p_l)^2} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} \; (all\ elements\ identically = -\frac{w}{(\sum_{l=1}^{L} p_l)^2})$$

$$D_w x(p, w) = D_w \frac{w}{\sum_{l=1}^{L} p_l} = \frac{1}{\sum_{l=1}^{L} p_l} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = x(p, w) \; (because\ p \cdot x(p, w) = w)$$

$$p \cdot x(p,w) = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \cdot \begin{bmatrix} x_1(p,w) \\ \vdots \\ x_L(p,w) \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \cdot \begin{bmatrix} \left(w/\sum_{l=1}^{L} p_l\right) \\ \vdots \\ \left(w/\sum_{l=1}^{L} p_l\right) \end{bmatrix} = \frac{wp_1 + wp_2 + wp_3 + \cdots + wp_L}{\sum_{l=1}^{L} p_l}$$

$$= \frac{w(p_1 + p_2 + p_3 + \cdots + p_L)}{\sum_{l=1}^{L} p_l} = \frac{w\sum_{l=1}^{L} p_l}{\sum_{l=1}^{L} p_l} = wS(p,w)$$

$$= -\frac{w}{(\sum_{l=1}^{L} p_l)^2} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \frac{1}{\sum_{l=1}^{L} p_l} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \frac{w}{\sum_{l=1}^{L} p_l} & \cdots & \frac{w}{\sum_{l=1}^{L} p_l} \end{bmatrix}$$

$$= \frac{w}{(\sum_{l=1}^{L} p_l)^2} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + \frac{w}{(\sum_{l=1}^{L} p_l)^2} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = 0$$

Thus, Slutsky matrix is symmetric, negative semidefinite, but not negative definite and equals zero.