Math Final
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Question #1

Question 2. Consider the following maximization problem:

Maximize $f(x,y) = xy$
subject to $x + y^2 \le 2$, $x \ge 0$, $y \ge 0$.

A.) Formulate the Lagrangian.
B.) Write out the Kuhn-Tucker conditions (marginal, complementary slackness, and non-negativity conditions).
C.) Find the maximizer for this problem.

A) For Lagrangian: we set up a Lagrangian Eqn:

$$\mathcal{L} = xy + \lambda(2 - x - y^2)$$

B) For Kuhn-Tucker Conditions:

Marginal condition	Complementary slackness	Non-negativity
$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda \le 0$	$\frac{\partial \mathcal{L}}{\partial x} \cdot x = 0 \Rightarrow x(y-\lambda) = 0$ ①	$x \ge 0$
$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda y \le 0$	$\frac{\partial \mathcal{L}}{\partial y} \cdot y = 0 \Rightarrow y(x - 2\lambda y) = 0$ ②	$y \ge 0$
$\frac{\partial \mathcal{L}}{\partial \lambda} = 2 - x - y^2 \ge 0$	$\frac{\partial \mathcal{L}}{\partial \lambda} \cdot \lambda = 0 \Rightarrow \lambda(2 - x - y^2) = 0$ ③	$\lambda \ge 0$

C) i) if $\lambda = 0$ then from ①, $xy = 0$

Thus, $x = 0$, or, $y = 0$, or both x and $y = 0$

a): if $x = 0$, $y \ne 0$

$f(x,y) = 0 \cdot y = 0$

b): if $y = 0$, $x \ne 0$

$f(x,y) = 0$

c): if $x = 0$, $y = 0$

$f(x,y) = 0$

ii) if $\lambda \neq 0$, from ③, $2-x-y^2=0$

a.) if $x=0$, $\Rightarrow y=\pm\sqrt{2}$ as $y>0 \Rightarrow y=\sqrt{2}$.

$f(x,y)= 0\cdot\sqrt{2}=0$

b.) if $y=0$, $\Rightarrow x=2$.

$f(x,y)=0$

c.) if $x\neq 0$, and $y\neq 0$

by ①, $x\neq 0 \Rightarrow y=\lambda$ ④

by ②, $y\neq 0 \Rightarrow x=2\lambda y$. ⑤

plug ④ into ⑤

$x=2y^2$. ⑥

plug ⑥ into $2-x-y^2=0$

$\Rightarrow 2-2y^2-y^2=0$

$\Rightarrow y^2=\frac{2}{3}$

$y=\frac{\sqrt{6}}{3}$ [$y=-\frac{\sqrt{6}}{3}$ is eliminated by non negativity]

$$\Rightarrow x = 2y^2 = 2 \cdot \frac{2}{3} = \frac{4}{3}$$

$$\begin{aligned} \Rightarrow f(x,y) &= xy \\ &= \frac{4}{3} \cdot \frac{\sqrt{6}}{3} \\ &= \frac{4\sqrt{6}}{9} \end{aligned}$$

Therefore, based on the analysis above,

at $\left(\frac{4}{3}, \frac{\sqrt{6}}{3}\right)$, $f(x,y)$ can obtain the maximum, which is $\frac{4\sqrt{6}}{9}$.

Question #2

Question 3. Consider the following stochastic version of the optimal growth problem. The social planner seeks to

Maximize $E_t \sum_{i=0}^{\infty} \beta^i U(c_{t+i})$, where $0 < \beta < 1$, (1)

subject to the capital accumulation constraint,

$k_{t+1+i} = (1-\delta)k_{t+i} + y_{t+i} - c_{t+i}$, where $0 \le \delta \le 1$. (2)

The production function is $y_{t+i} = F(A_{t+i}, k_{t+i})$ where $\ln A_{t+i}$ is i.i.d. $N(0, \sigma^2)$. The initial capital stock, k_t, is given. The appropriate side conditions apply but need not be discussed in your answer to this question. All variables have their usual definition.

Derive the Euler equation using the method discussed in class. (Do not use Lagrange methods.) Be careful to write out Bellman's equation and explain the value function. Also, be explicit about how the Envelope Theorem is used in the derivation of the Euler equation

Since the social planner wants to maximize total utility

$\Rightarrow \max_{\{C_{t+i}\}_{i=0}^{\infty}} E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i})$

s.t. $K_{t+i+1} = (1-\delta)K_{t+i} + y_{t+i} - C_{t+i}$

Since y_{t+i} is a function of A_t and K_t.

in this question:

The state variable is K_t, b/c from the constraint eqn,

$K_t = (1-\delta)K_{t-1} + F(A_{t-1}, K_{t-1}) - C_{t-1}$

So it's predetermined before t. Actually, A_t can also be state var, b/c A_t is exogenous, but $\ln A_t$ is iid. So, it's convenient to only consider K_t as our state variable.

The control variable is C_t

As for the transition equation, it's

$K_{t+1} = (1-\delta)K_t + F(A_t, K_t) - C_t$

i.e., Current period's capital, tech, and consumption can determine next period's Capital.

Now, it's time to write down our Bellman Eqn:

Assume the value function in Bellman Eqn is $V(K_t)$

i.e., Max attainable value of PDV of utility given the info at current period.

So:

$$V(K_t) = \max_{C_t} E_t\left\{ u(C_t) + \beta V(K_{t+1}) \right\} \quad ①$$

$$K_{t+1} = (1-\delta)K_t + F(A_t, K_t) - C_t \quad ②$$

Thus ① can be written as:

$$V(K_t) = \max_{C_t} E_t\left\{ u(C_t) + \beta V[(1-\delta)K_t + F(A_t, K_t) - C_t] \right\} \quad ③$$

We take the first order derivative of ③ w/ respect to our control variable C_t.

$$[C_t]: E_t\left[u'(C_t) + \beta V_k(K_{t+1})(-1) \right] = 0$$

$$\Rightarrow \quad u'(C_t) = E_t\left[\beta V_k(K_{t+1}) \right] \quad ④$$

On our next step, we need to use Envelop Theorem:

Before stepping into Envelope Theorem, first elaborate about Envelop Theorem:

① If we want to evaluate $V_k(K_t)$, the value of C_t on the RHS of our Bellman Eqn, in order to maximize C_t, taken K_t as given

$$C_t^* = C^*(K_t)$$

By Envelope Theorem in ③

$$V_K(K_t) = E_t\Big[\underbrace{\frac{\partial u(C_t)}{\partial C_t}\frac{\partial C_t^*}{\partial K_t} + \beta V_K(K_{t+1})\frac{\partial K_{t+1}}{\partial C_t}\cdot\frac{\partial C_t^*}{\partial K_t}}_{\text{part A}}$$

$$\underbrace{+ \beta V_K(K_{t+1})\frac{\partial K_{t+1}}{\partial K_t}\Big|_{\text{keep } C_t \text{ is fixed}}}_{\text{Part B.}}\Big] \quad ⑤$$

Part A:

$$E_t\Big[\frac{\partial C_t^*}{\partial K_t}\underbrace{\big[u'(C_t) + \beta V_K(K_{t+1})\cdot(-1)\big]}_{\text{By ④} \ = 0}\Big]$$

So part A = 0

Part B:

can be derived as

$$E_t\Big[\beta V_K(K_{t+1})\big[(1-\delta) + F_A(A_t, K_t)\overset{\substack{=0 \\ \text{as} \\ \{A_t\} \text{ is iid.}}}{\frac{\partial A_t}{\partial K_t}}$$

$$+ F_K(A_t, K_t)\frac{\partial K_t}{\partial K_t}$$

$$+ 0\Big]$$

$$= E_t\left[\beta V_k(k_{t+1})\left(1-\delta+F_k(A_t,K_t)\right)\right] \quad (6)$$

plug (6) into (5)

$$V_k(k_t) = \beta E_t\left[V_k(k_{t+1})(1-\delta+F_k(A_t,k_t))\right] \quad (7)$$

By (4),

(7) becomes

$$V_k(k_t) = u'(c_t)\left[(1-\delta)+F_k(A_t,k_t)\right], \quad (8)$$

$$\Rightarrow V_k(k_{t+1}) = u'(c_{t+1})\left[(1-\delta)+F_k(A_{t+1},k_{t+1})\right] \quad (9)$$

plug 8, 9 into 7

$$u'(C_t)\cancel{[1-\delta+F_K(A_t,K_t)]} = \beta E_t\Big[u'(C_{t+1})[1-\delta+F_K(A_{t+1},K_{t+1})] \cdot \cancel{[1-\delta+F_K(A_t,K_t)]}\Big]$$

$$\Rightarrow u'(C_t) = \beta E_t\left[u'(C_{t+1})\left(1-\delta+F_K(A_{t+1},K_{t+1})\right)\right] \qquad (10)$$

Thus 10 is the intertemporary Euler Eqn.

Question#3 :

Question 4.

A.) Let $x,\ y \in \mathrm{R}^1$. Prove the following two inequalities:

$$|x+y| \le |x| + |y| \qquad (1)$$

$$\big||x| - |y|\big| \le |x-y| \qquad (2)$$

B.) Given the two inequalities proven in Part A, prove the following proposition:
Proposition 1: Every Cauchy Sequence in R^1 is bounded.

C.) Prove the following Proposition.
Proposition 2: A sequence of vectors in R^M converges if all M sequences of its components converge in R^1

Proof: [Note: st in this problem represents such that]

A(1):

$$|x+y|^2 = (x+y)^2 = x^2 + 2xy + y^2$$

$$= |x|^2 + 2xy + |y|^2$$

$$\le |x|^2 + 2|x||y| + |y|^2$$

$$\le (|x| + |y|)^2$$

$$\Rightarrow |x+y| \le |x| + |y|$$

A(2):

① if $|x| \geq |y|$.

$$\Rightarrow \big||x| - |y|\big| = |x| - |y|$$

$\Rightarrow$ We need to prove:

$$|x| - |y| \le |x-y|$$

B/C

$$|x| = |x+y-y| = |x-y+y| \le |x-y| + |y|.$$

$$\Rightarrow |x| \le |x-y| + |y|$$

$$\Rightarrow |x| - |y| \le |x-y|.$$

③ if $|x| < |y|$

$\Rightarrow ||x|-|y|| = |y|-|x|$

$|y| = |y+x-x| = |y-x+x| \leq |y-x| + |x|$

$\Rightarrow |y|-|x| \leq |y-x|$

As $|y-x| = |x-y|$

$\Rightarrow |y|-|x| \leq |x-y|$

Combined w/ ① and ②,

we can get:

$$||x|-|y|| \leq |x-y|$$

□

B) Prove Every cauchy seq in $\mathbb{R}^1$ is bdd:

proof: We first define a cauchy seq $\{x_m\}_{m=1}^{\infty}$ in $\mathbb{R}^1$.

Take some sequence in x_m, denote x_i, and x_j, as x_j, x_i $\subset$ Cauchy Seq.

thus, $\exists$ an integer N, s.t $|x_i - x_j| \leq \varepsilon$ for $i, j \geq N$, and, ε is small enough, but $\varepsilon > 0$. In this seq, we can also get x_N s.t $|x_i - x_N| \leq \varepsilon$, then by (2) [Here, we take x_i as an example, but it can be any cauchy seq, iff $i \geq N$]

$$|x_i| - |x_N| \leq |x_i - x_N| \leq \varepsilon$$

$$\Rightarrow |x_i| \leq |x_N| + |x_i - x_N| = |x_N| + |x_N - x_i|$$
$$\leq |x_N| + \varepsilon$$

Because $|X_N|+\varepsilon$ is bdd for the frs $(N-1)$ terms of the sequences,

Thus, we can ~~take~~ take a value k, s.t.

$$k = \max\{ |X_1|, |X_2| \cdots, |X_{N-1}|, |X_N| \}$$

$\Rightarrow |X_i| \leq k+\varepsilon. \quad \Rightarrow$ it's bdd. $\Rightarrow$ Every Cauchy seq in $\mathbb{R}^1$ is bdd.

$\square$

C)

C.) Prove the following Proposition.
Proposition 2: A sequence of vectors in R^M converges if all M sequences of its components converge in R^1

Denote a seq $\{X_m\}_{m=1}^{\infty}$ in $\mathbb{R}^M$.

And Denote the components of $\{X_m\}$ as

$X_m = \{ X_{1m}, X_{2m}, \cdots X_{Mm} \}$

If we assume all the $\{X_{im}\}_{m=1}^{\infty}$ converges in $\mathbb{R}^1$ to limit X^*

i.e., X_{1m} converges to X_1^*, X_{2m} to X_2^* ....

$X^* = \{ X_1^*, X_2^* \cdots X_M^* \}$

For each in $\{X_{im}\}_{i=1}^{M}{}_{m=1}^{\infty}$, $\exists \; |X_{im} - X_i^*| \leq \frac{\varepsilon}{\sqrt{M}}$.

when $m \geq N_i$, N_i is large enough and ε is small enough but $\varepsilon > 0$

if we pick a number N, s.t $N = \max\{N_1, \cdots N_M\}$

Then if $m \geq N$, $\exists$

$$||X_m - X^*|| = \sqrt{(X_{1m}-X_1^*)^2 + (X_{2m}-X_2^*)^2 + \cdots + (X_{(M-1)m}-X_{M-1}^*)^2 + (X_{Mm}-X_M^*)^2}$$

$$\leq \sqrt{\frac{\varepsilon^2}{M} + \frac{\varepsilon^2}{M} + \cdots + \frac{\varepsilon^2}{M} + \frac{\varepsilon^2}{M}}$$

$$= \sqrt{\varepsilon^2} = \varepsilon.$$

$\Rightarrow ||X_m - X^*|| \leq \varepsilon.$

$\Rightarrow$ A seq of vectors in $\mathbb{R}^M$ converges.