Math Camp
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Homework 3 Solution

Derive the Taylor expansion for $f(x) = 3x^2 - 6x + 5$

Solution

Compute the derivatives:

$$f'(x) = 6x - 6,\ \ f''(x) = 6,\ \ f'''(x) = 0.$$

As you can see, $f^{(n)}(x) = 0$ for all $n \geq 3$. Then for $x = 1$, we get

$$f(1) = 2,\ \ f'(1) = 0,\ \ f''(1) = 6.$$

Hence, the Taylor expansion for the given function is

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!} = 2 + \frac{6(x-1)^2}{2!} = 2 + 3(x-1)^2.$$

Derive the Maclaurin expansion for $e^{kx}, k\ is\ real\ number$

Solution

Calculate the derivatives:

$$f'(x) = \left(e^{kx}\right)' = ke^{kx},\ \ f''(x) = \left(ke^{kx}\right)' = k^2e^{kx}, \ldots\ f^{(n)}(x) = k^ne^{kx}.$$

Then, at $x = 0$ we have

$$f(0) = e^0 = 1,\ \ f'(0) = ke^0 = k,\ \ f''(0) = k^2e^0 = k^2, \ldots$$
$$f^{(n)}(0) = k^ne^0 = k^n.$$

Hence, the Maclaurin expansion for the given function is

$$e^{kx} = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{k^nx^n}{n!}.$$

Derive the Maclaurin expansion for $(1+x)^{\mu}$.

Solution

Let $f(x) = (1+x)^{\mu}$, where μ is a real number and $x \neq -1$. Then we can write the derivatives as follows

$$f'(x) = \mu(1+x)^{\mu-1},$$

$$f''(x) = \mu(\mu-1)(1+x)^{\mu-2},$$

$$f'''(x) = \mu(\mu-1)(\mu-2) \cdot (1+x)^{\mu-3},$$

$$f^{(n)}(x) = \mu(\mu-1)(\mu-2)\cdots(\mu-n+1)(1+x)^{\mu-n}.$$

For $x = 0$, we obtain

$$f(0) = 1,\ \ f'(0) = \mu,\ \ f''(0) = \mu(\mu-1), \ldots$$
$$f^{(n)}(0) = \mu(\mu-1)\cdots(\mu-n+1).$$

Hence, the series expansion can be written in the form

$$(1+x)^{\mu} = 1 + \mu x + \frac{\mu(\mu-1)}{2!}x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!}x^2 + \ldots + \frac{\mu(\mu-1)\cdots(\mu-n+1)}{n!}x^n + \ldots$$

This series is called the binomial series.

Derive the Maclaurin expansion for $\sqrt{1+x}$.

Solution

Using the binomial series found in the previous example and substituting $\mu = \frac{1}{2}$, we get

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots = 1 + \frac{x}{2} - \frac{1 \cdot x^2}{2^2 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^3}{2^4 4!} + \ldots + (-1)^{n+1} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)\, x^n}{2^n n!}.$$

Keeping only the first three terms, we can write this series as

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}.$$

Find the convexity and concavity for function $f(x) = x^3 + ax + b$

Solution

Take the derivatives:

$$f'(x) = \left(x^3 + ax + b\right)' = 3x^2 + a;$$

$$f''(x) = \left(3x^2 + a\right)' = 6x.$$

We see that $f''(x) < 0$ at $x < 0$. Hence, the function is convex upward on $(-\infty, 0)$.

Find the intervals of convexity and concavity of the function $f(x) = \frac{1}{1+x^2}$

Solution

The function is defined and differentiable for all $x \in \mathbb{R}$. To determine the direction of convexity, we use the convexity test based on the second derivative. Calculate the second derivative:

$$f'(x) = \left(\frac{1}{1+x^2}\right)' = \left[\left(1+x^2\right)^{-1}\right]' = -\frac{2x}{\left(1+x^2\right)^2};$$

$$f''(x) = \left(-\frac{2x}{\left(1+x^2\right)^2}\right)' = \frac{6x^2-2}{\left(1+x^2\right)^3}.$$

Find the intervals where the derivative has a constant sign:

$$\boxed{1}\ f''(x) > 0, \quad \Rightarrow \frac{6x^2-2}{\left(1+x^2\right)^3} > 0, \quad \Rightarrow 6x^2 - 2 > 0, \quad \Rightarrow x^2 > \frac{1}{3},$$
$$\Rightarrow x \in \left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right);$$
$$\boxed{2}\ f''(x) < 0, \quad \Rightarrow \frac{6x^2-2}{\left(1+x^2\right)^3} < 0, \quad \Rightarrow 6x^2 - 2 < 0, \quad \Rightarrow x^2 < \frac{1}{3},$$
$$\Rightarrow x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

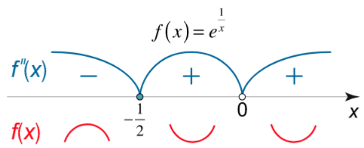
Here in solving the inequalities, we have used the fact that the denominator in the expression for the second derivative is always positive: $\left(1+x^2\right)^3 > 0$.

Thus, based on the sign of the second derivative, we establish that the given function is

- strictly convex upward for $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$;
- strictly convex downward for $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

Find the intervals of convexity and concavity of the function $f(x) = e^{\frac{1}{x}}$

Solution



First we take the derivatives:

$$f'(x) = \left(e^{\frac{1}{x}}\right)' = e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = -\frac{e^{\frac{1}{x}}}{x^2};$$

$$f''(x) = \left(-\frac{e^{\frac{1}{x}}}{x^2}\right)' = -\frac{\left(e^{\frac{1}{x}}\right)' \cdot x^2 - e^{\frac{1}{x}} \cdot (x^2)'}{x^4}$$
$$= -\frac{e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) \cdot x^2 - 2xe^{\frac{1}{x}}}{x^4} = \frac{e^{\frac{1}{x}}(1+2x)}{x^4}.$$

Solve the equation $f''(x) = 0$ and draw a sign chart for $f''(x)$.

$$f''(x) = 0, \quad \Rightarrow \frac{e^{\frac{1}{x}}(1+2x)}{x^4} = 0, \quad \Rightarrow x = -\frac{1}{2}.$$

Note that the function and its derivatives do not exist at $x = 0$, so we also indicate this point on the sign chart.

Thus, the function is convex downward on $\left(-\frac{1}{2}, 0\right)$ and $(0, +\infty)$ and convex upward on $\left(-\infty, -\frac{1}{2}\right)$.

Find the sum of the series $S = 1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + \frac{1}{4} - \frac{1}{4\sqrt{2}} + \frac{1}{8}$

Solution

This is a geometric progression with $q = -\frac{1}{\sqrt{2}}$. Since the sum of a geometric progression is given by

$$S_n = a_1 \frac{1-q^n}{1-q},$$

we have

$$S_7 = 1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + \frac{1}{4} - \frac{1}{4\sqrt{2}} + \frac{1}{8} = \frac{1-\left(-\frac{1}{\sqrt{2}}\right)^7}{1-\left(-\frac{1}{\sqrt{2}}\right)}$$
$$= \frac{1-\frac{1}{8\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\frac{8\sqrt{2}-1}{8\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}} = \frac{8\sqrt{2}-1}{8\left(\sqrt{2}+1\right)}.$$

Redo the examples in class on your own: (1) $\sum_{n=1}^{\infty}\frac{3^n}{n^2}$; (2) $\sum_{n=1}^{\infty}\frac{n^3}{(ln3)^n}$ converges or diverges

Solution

See class notes

We use the ratio test.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)^2}}{\frac{3^n}{n^2}}=\lim_{n\to\infty}\left[\frac{3^{n+1}}{3^n}\cdot\frac{n^2}{(n+1)^2}\right]=\lim_{n\to\infty}\left[3\left(\frac{n}{n+1}\right)^2\right]$$
$$=3\lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^2=3\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^2=3.$$

As it can be seen, the given series diverges.

We apply the ratio test to investigate convergence of this series:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)^3}{(\ln 3)^{n+1}}}{\frac{n^3}{(\ln 3)^n}}=\lim_{n\to\infty}\left[\frac{(\ln 3)^n}{(\ln 3)^{n+1}}\cdot\frac{(n+1)^3}{n^3}\right]$$
$$=\frac{1}{\ln 3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3=\frac{1}{\ln 3}\cdot 1=\frac{1}{\ln 3}.$$

As $\ln 3>\ln e=1$ and $\frac{1}{\ln 3}<1$, the given series converges.