it is easy to see that the three equations (E.1) imply the following three:

$$
\begin{aligned}
\ln(e(\mathbf{p}, u)) &= \alpha_1 \ln(p_1) + c_1(p_2, p_3, u), \\
\ln(e(\mathbf{p}, u)) &= \alpha_2 \ln(p_2) + c_2(p_1, p_3, u), \\
\ln(e(\mathbf{p}, u)) &= \alpha_3 \ln(p_3) + c_3(p_1, p_2, u),
\end{aligned}
\tag{E.2}
$$

where the $c_i(\cdot)$ functions are like the constant added before to $f(x)$. But we must choose the $c_i(\cdot)$ functions so that all three of these equalities hold simultaneously. With a little thought, you will convince yourself that (E.2) then implies

$$\ln(e(\mathbf{p}, u)) = \alpha_1 \ln(p_1) + \alpha_2 \ln(p_2) + \alpha_3 \ln(p_3) + c(u),$$

where $c(u)$ is some function of u. But this means that

$$e(\mathbf{p}, u) = c(u) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

Because we must ensure that $e(\cdot)$ is strictly increasing in u, we may choose $c(u)$ to be any strictly increasing function. It does not matter which, because the implied demand behaviour will be independent of such strictly increasing transformations. For example, we may choose $c(u) = u$, so that our final solution is

$$e(\mathbf{p}, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

We leave it to you to check that this function satisfies the original system of partial differential equations and that it has all the properties required of an expenditure function. □

2.3 REVEALED PREFERENCE

So far, we have approached demand theory by assuming the consumer has preferences satisfying certain properties (complete, transitive, and strictly monotonic); then we have tried to deduce all of the observable properties of market demand that follow as a consequence (budget balancedness, symmetry, and negative semidefiniteness of the Slutsky matrix). Thus, we have begun by assuming something about things we cannot observe – preferences – to ultimately make predictions about something we can observe – consumer demand behaviour.

In his remarkable *Foundations of Economic Analysis*, Paul Samuelson (1947) suggested an alternative approach. Why not *start and finish* with observable behaviour? Samuelson showed how virtually every prediction ordinary consumer theory makes for a consumer's observable market behaviour can also (and instead) be derived from a few simple and sensible assumptions about the consumer's observable *choices* themselves, rather than about his unobservable preferences.

The basic idea is simple: if the consumer buys one bundle instead of another affordable bundle, then the first bundle is considered to be **revealed preferred** to the second. The presumption is that by actually choosing one bundle over another, the consumer conveys important information about his tastes. Instead of laying down axioms on a person's preferences as we did before, we make assumptions about the consistency of the choices that are made. We make this all a bit more formal in the following.

DEFINITION 2.1 ***Weak Axiom of Revealed Preference (WARP)***

A consumer's choice behaviour satisfies WARP if for every distinct pair of bundles x^0, x^1 *with* x^0 *chosen at prices* $\mathbf{p}^0$ *and* x^1 *chosen at prices* $\mathbf{p}^1$,

$$\mathbf{p}^0 \cdot \mathrm{x}^1 \le \mathbf{p}^0 \cdot \mathrm{x}^0 \Longrightarrow \mathbf{p}^1 \cdot \mathrm{x}^0 > \mathbf{p}^1 \cdot \mathrm{x}^1.$$

In other words, WARP holds if whenever x^0 *is revealed preferred to* x^1, x^1 *is* never *revealed preferred to* x^0.

To better understand the implications of this definition, look at Fig. 2.3. In both parts, the consumer facing $\mathbf{p}^0$ chooses x^0, and facing $\mathbf{p}^1$ chooses x^1. In Fig. 2.3(a), the consumer's choices satisfy WARP. There, x^0 is chosen when x^1 could have been, but was not, and when x^1 is chosen, the consumer could not have afforded x^0. By contrast, in Fig. 2.3(b), x^0 is again chosen when x^1 could have been, yet when x^1 is chosen, the consumer could have chosen x^0, but did not, violating WARP.

Now, suppose a consumer's choice behaviour satisfies WARP. Let $\mathrm{x}(\mathbf{p}, y)$ denote the choice made by this consumer when faced with prices $\mathbf{p}$ and income y. Note well that this is *not* a demand function because we have not mentioned utility or utility maximisation – it just denotes the quantities the consumer chooses facing $\mathbf{p}$ and y. To keep this point clear in our minds, we refer to $\mathrm{x}(\mathbf{p}, y)$ as a *choice function*. In addition to WARP, we make one

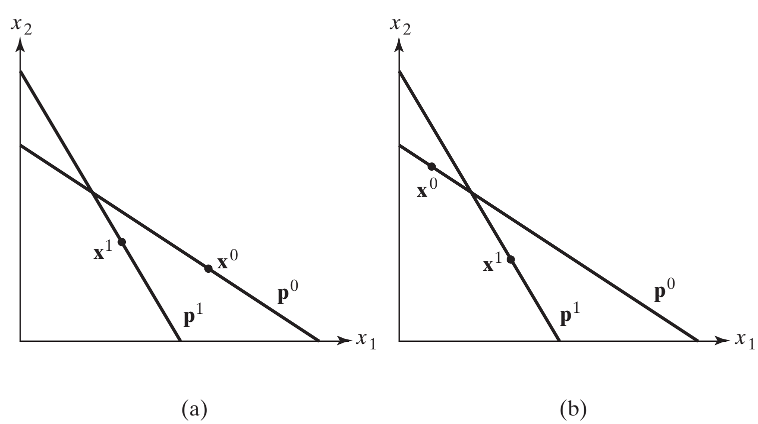


Figure 2.3. The Weak Axiom of Revealed Preference (WARP).

other assumption concerning the consumer's choice behaviour, namely, that for $\mathbf{p} \gg \mathbf{0}$, the choice $\mathrm{x}(\mathbf{p}, y)$ satisfies budget balancedness, i.e., $\mathbf{p} \cdot \mathrm{x}(\mathbf{p}, y) = y$. The implications of these two apparently mild requirements on the consumer's choice behaviour are rather remarkable.

The first consequence of WARP and budget balancedness is that the choice function $\mathrm{x}(\mathbf{p}, y)$ must be homogeneous of degree zero in $(\mathbf{p}, y)$. To see this, suppose x^0 is chosen when prices are $\mathbf{p}^0$ and income is y^0, and suppose x^1 is chosen when prices are $\mathbf{p}^1 = t\mathbf{p}^0$ and income is $y^1 = ty^0$ for $t > 0$. Because $y^1 = ty^0$, when all income is spent, we must have $\mathbf{p}^1 \cdot \mathrm{x}^1 = t\mathbf{p}^0 \cdot \mathrm{x}^0$. First, substitute $t\mathbf{p}^0$ for $\mathbf{p}^1$ in this, divide by t, and get

$$\mathbf{p}^0 \cdot \mathrm{x}^1 = \mathbf{p}^0 \cdot \mathrm{x}^0. \tag{2.3}$$

Then substitute $\mathbf{p}^1$ for $t\mathbf{p}^0$ in the same equation and get

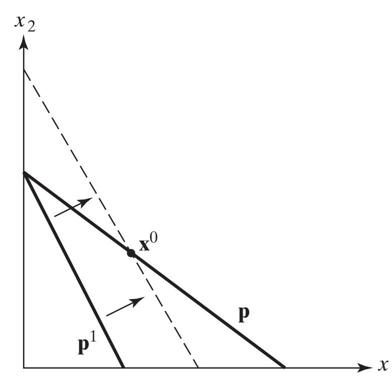
$$\mathbf{p}^1 \cdot \mathrm{x}^1 = \mathbf{p}^1 \cdot \mathrm{x}^0. \tag{2.4}$$

If x^0 and x^1 are distinct bundles for which (2.3) holds, then WARP implies that the left-hand side in (2.4) must be strictly less than the right-hand side – a contradiction. Thus, these bundles cannot be distinct, and the consumer's choice function therefore must be homogeneous of degree zero in prices and income.

Thus, the choice function $\mathrm{x}(\mathbf{p}, y)$ must display one of the additional properties of a demand function. In fact, as we now show, $\mathrm{x}(\mathbf{p}, y)$ must display yet another of those properties as well.

In Exercise 1.45, the notion of Slutsky-compensated demand was introduced. Let us consider the effect here of Slutsky compensation for the consumer's choice behaviour. In case you missed the exercise, the Slutsky compensation is relative to some pre-specified bundle, say x^0. The idea is to consider the choices the consumer makes as prices vary arbitrarily while his income is compensated so that he can just afford the bundle x^0. (See Fig. 2.4.) Consequently, at prices $\mathbf{p}$, his income will be $\mathbf{p} \cdot \mathrm{x}^0$. Under these circumstances, his choice behaviour will be given by $\mathrm{x}(\mathbf{p}, \mathbf{p} \cdot \mathrm{x}^0)$.

Figure 2.4. A Slutsky compensation in income.



Now fix $\mathbf{p}^0 \gg \mathbf{0}$, $y^0 > 0$, and let $\mathrm{x}^0 = \mathrm{x}(\mathbf{p}^0, y^0)$. Then if $\mathbf{p}^1$ is *any other* price vector and $\mathrm{x}^1 = \mathrm{x}(\mathbf{p}^1, \mathbf{p}^1 \cdot \mathrm{x}^0)$, WARP implies that

$$\mathbf{p}^0 \cdot \mathrm{x}^0 \leq \mathbf{p}^0 \cdot \mathrm{x}^1. \tag{2.5}$$

Indeed, if $\mathrm{x}^1 = \mathrm{x}^0$, then (2.5) holds with equality. And if $\mathrm{x}^1 \neq \mathrm{x}^0$, then because x^1 was chosen when x^0 was affordable (i.e., at prices $\mathbf{p}^1$ and income $\mathbf{p}^1 \cdot \mathrm{x}^0$), WARP implies that x^1 is not affordable whenever x^0 is chosen. Consequently, the inequality in (2.5) would be strict.

Now, note that by budget balancedness:

$$\mathbf{p}^1 \cdot \mathrm{x}^0 = \mathbf{p}^1 \cdot \mathrm{x}(\mathbf{p}^1, \mathbf{p}^1 \cdot \mathrm{x}^0). \tag{2.6}$$

Subtracting (2.5) from (2.6) then implies that for all prices $\mathbf{p}^1$,

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot \mathrm{x}^0 \geq (\mathbf{p}^1 - \mathbf{p}^0) \cdot \mathrm{x}(\mathbf{p}^1, \mathbf{p}^1 \cdot \mathrm{x}^0). \tag{2.7}$$

Because (2.7) holds for all prices $\mathbf{p}^1$, let $\mathbf{p}^1 = \mathbf{p}^0 + t\mathrm{z}$, where $t > 0$, and $\mathrm{z} \in \mathbb{R}^n$ is arbitrary. Then (2.7) becomes

$$t[\mathrm{z} \cdot \mathrm{x}^0] \geq t[\mathrm{z} \cdot \mathrm{x}(\mathbf{p}^1, \mathbf{p}^1 \cdot \mathrm{x}^0)]. \tag{2.8}$$

Dividing by $t > 0$ gives

$$\mathrm{z} \cdot \mathrm{x}^0 \geq \mathrm{z} \cdot \mathrm{x}(\mathbf{p}^0 + t\mathrm{z}, (\mathbf{p}^0 + t\mathrm{z}) \cdot \mathrm{x}^0), \tag{2.9}$$

where we have used the fact that $\mathbf{p}^1 = \mathbf{p}^0 + t\mathrm{z}$.

Now for z fixed, we may choose $\bar{t} > 0$ small enough so that $\mathbf{p}^0 + t\mathrm{z} \gg \mathbf{0}$ for all $t \in [0, \bar{t}]$, because $\mathbf{p}^0 \gg \mathbf{0}$. Noting that (2.9) holds with equality when $t = 0$, (2.9) says that the function $f\colon [0, \bar{t}) \to \mathbb{R}$ defined by the right-hand side of (2.9), i.e.,

$$f(t) \equiv \mathrm{z} \cdot \mathrm{x}(\mathbf{p}^0 + t\mathrm{z}, (\mathbf{p}^0 + t\mathrm{z}) \cdot \mathrm{x}^0),$$

is maximised on $[0, \bar{t})$ at $t = 0$. Thus, we must have $f'(0) \leq 0$. But taking the derivative of $f(t)$ and evaluating at $t = 0$ gives (assuming that $\mathrm{x}(\cdot)$ is differentiable):

$$f'(0) = \sum_i \sum_j z_i \left[\frac{\partial x_i(\mathbf{p}^0, y^0)}{\partial p_j} + x_j(\mathbf{p}^0, y^0) \frac{\partial x_i(\mathbf{p}^0, y^0)}{\partial y} \right] z_j \leq 0. \tag{2.10}$$

Now, because $\mathrm{z} \in \mathbb{R}^n$ was arbitrary, (2.10) says that the matrix whose *ij*th entry is

$$\frac{\partial x_i(\mathbf{p}^0, y^0)}{\partial p_j} + x_j(\mathbf{p}^0, y^0) \frac{\partial x_i(\mathbf{p}^0, y^0)}{\partial y} \tag{2.11}$$

must be negative semidefinite. But this matrix is precisely the Slutsky matrix associated with the choice function $x(\mathbf{p}, y)$!

Thus, we have demonstrated that if a choice function satisfies WARP and budget balancedness, then it must satisfy two other properties implied by utility maximisation, namely, homogeneity of degree zero and negative semidefiniteness of the Slutsky matrix.

If we could show, in addition, that the choice function's Slutsky matrix was symmetric, then by our integrability result, that choice function would actually be a demand function because we would then be able to construct a utility function generating it.

Before pursuing this last point further, it is worthwhile to point out that if $x(\mathbf{p}, y)$ happens to be a utility-generated demand function then $x(\mathbf{p}, y)$ must satisfy WARP. To see this, suppose a utility-maximising consumer has strictly monotonic and strictly convex preferences. Then we know there will be a unique bundle demanded at every set of prices, and that bundle will always exhaust the consumer's income. (See Exercise 1.16.) So let x^0 maximise utility facing prices $\mathbf{p}^0$, let x^1 maximise utility facing $\mathbf{p}^1$, and suppose $\mathbf{p}^0 \cdot x^1 \leq \mathbf{p}^0 \cdot x^0$. Because x^1, though affordable, is not chosen, it must be because $u(x^0) > u(x^1)$. Therefore, when x^1 is chosen facing prices $\mathbf{p}^1$, it must be that x^0 is not available or that $\mathbf{p}^1 \cdot x^0 > \mathbf{p}^1 \cdot x^1$. Thus, $\mathbf{p}^0 \cdot x^1 \leq \mathbf{p}^0 \cdot x^0$ implies $\mathbf{p}^1 \cdot x^0 > \mathbf{p}^1 \cdot x^1$, so WARP is satisfied.

But again what about the other way around? What if a consumer's choice function always satisfies WARP? Must that behaviour have been generated by utility maximisation? Put another way, must there exist a utility function that would yield the observed choices as the outcome of the utility-maximising process? If the answer is yes, we say the utility function **rationalises** the observed behaviour.

As it turns out, the answer is yes – and no. If there are only two goods, then WARP implies that there *will* exist some utility function that rationalises the choices; if, however, there are more than two goods, then even if WARP holds there need *not* be such a function.

The reason for the two-good exception is related to the symmetry of the Slutsky matrix and to transitivity.

It turns out that in the two-good case, budget balancedness together with homogeneity imply that the Slutsky matrix must be symmetric. (See Exercise 2.9.) Consequently, because WARP and budget balancedness imply homogeneity as well as negative semidefiniteness, then in the case of two goods, they also imply symmetry of the Slutsky matrix. Therefore, for two goods, our integrability theorem tells us that the choice function must be utility-generated.

An apparently distinct, yet ultimately equivalent, explanation for the two-good exception is that with two goods, the pairwise ranking of bundles implied through revealed preference turns out to have no intransitive cycles. (You are, in fact, asked to show this in Exercise 2.9.) And when this is so, there will be a utility representation generating the choice function. Thus, as we mentioned earlier in the text, there is a deep connection between the symmetry of the Slutsky matrix and the transitivity of consumer preferences.

For more than two goods, WARP and budget balancedness imply neither symmetry of the Slutsky matrix nor the absence of intransitive cycles in the revealed preferred to relation. Consequently, for more than two goods, WARP and budget balancedness are not equivalent to the utility-maximisation hypothesis.

This leads naturally to the question: how must we strengthen WARP to obtain a theory of revealed preference that *is* equivalent to the theory of utility maximisation? The answer lies in the 'Strong Axiom of Revealed Preference'.

The **Strong Axiom of Revealed Preference (SARP)** is satisfied if, for every sequence of distinct bundles $x^0, x^1, \ldots, x^k$, where x^0 is revealed preferred to x^1, and x^1 is revealed preferred to $x^2, \ldots$, and x^{k-1} is revealed preferred to x^k, it is not the case that x^k is revealed preferred to x^0. SARP rules out intransitive *revealed* preferences and therefore can be used to induce a complete and transitive preference relation, $\succsim$, for which there will then exist a utility function that rationalises the observed behaviour. We omit the proof of this and instead refer the reader to Houthakker (1950) for the original argument, and to Richter (1966) for an elegant proof.

It is not difficult to show that if a consumer chooses bundles to maximise a strictly quasiconcave and strictly increasing utility function, his demand behaviour must satisfy SARP (see Exercise 2.11). Thus, a theory of demand built only on SARP, a restriction on observable choice, is essentially *equivalent* to the theory of demand built on utility maximisation. Under both SARP and the utility-maximisation hypothesis, consumer demand will be homogeneous and the Slutsky matrix will be negative semidefinite *and* symmetric.

In our analysis so far, we have focused on revealed preference axioms and consumer choice *functions*. In effect, we have been acting as though we had an infinitely large collection of price and quantity data with which to work. To many, the original allure of revealed preference theory was the promise it held of being able to begin with actual data and work from the implied utility functions to predict consumer behaviour. Because real-world data sets will never contain more than a finite number of sample points, more recent work on revealed preference has attempted to grapple directly with some of the problems that arise when the number of observations is finite.

To that end, Afriat (1967) introduced the **Generalised Axiom of Revealed Preference (GARP)**, a slightly weaker requirement than SARP, and proved an analogue of the integrability theorem (Theorem 2.6). According to **Afriat's theorem**, a *finite* set of observed price and quantity data satisfy GARP if and only if there exists a continuous, increasing, and concave utility function that rationalises the data. (Exercise 2.12 explores a weaker version of Afriat's theorem.) However, with only a finite amount of data, the consumer's preferences are not completely pinned down at bundles 'out-of-sample'. Thus, there can be many different utility functions that rationalise the (finite) data.

But, in some cases, revealed preference does allow us to make certain 'out-of-sample' comparisons. For instance, consider Fig. 2.5. There we suppose we have observed the consumer to choose x^0 at prices $\mathbf{p}^0$ and x^1 at prices $\mathbf{p}^1$. It is easy to see that x^0 is revealed preferred to x^1. Thus, for any utility function that rationalises these data, we must have $u(x^0) > u(x^1)$, by definition. Now suppose we want to compare two bundles such as x and $\mathbf{y}$, which do not appear in our sample. Because $\mathbf{y}$ costs less than x^1 when x^1 was chosen, we may deduce that $u(x^0) > u(x^1) > u(\mathbf{y})$. Also, if more is preferred to less, the utility function must be increasing, so we have $u(x) \geq u(x^0)$. Thus, we have $u(x) \geq u(x^0) > u(x^1) > u(\mathbf{y})$ for any increasing utility function that rationalises the observed data, and so we can compare our two out-of-sample bundles directly and

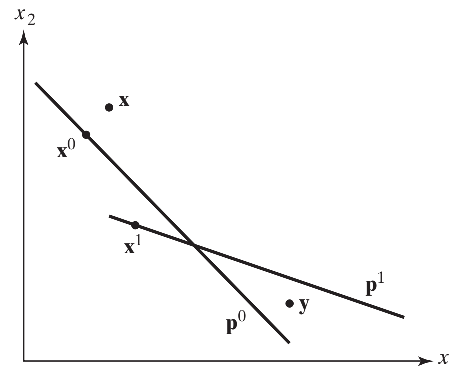


Figure 2.5. Recovering preferences that satisfy GARP.

conclude $u(\mathrm{x}) > u(\mathbf{y})$ for any increasing utility function that could possibly have generated the data we have observed.

But things do not always work out so nicely. To illustrate, say we observe the consumer to buy the single bundle $\mathrm{x}^1 = (1, 1)$ at prices $\mathbf{p}^1 = (2, 1)$. The utility function $u(\mathrm{x}) = x_1^2 x_2$ rationalises the choice we observe because the indifference curve through x^1 is tangent there to the budget constraint $2x_1 + x_2 = 3$, as you can easily verify. At the same time, the utility function $v(\mathrm{x}) = x_1(x_2 + 1)$ will *also* rationalise the choice of x^1 at $\mathbf{p}^1$ as this utility function's indifference curve through x^1 will also be tangent at x^1 to the same budget constraint. This would not be a problem if $u(\mathrm{x})$ and $v(\mathrm{x})$ were merely monotonic transforms of one another – but they are not. For when we compare the out-of-sample bundles $\mathrm{x} = (3, 1)$ and $y = (1, 7)$, in the one case, we get $u(3, 1) > u(1, 7)$, telling us the consumer prefers x to $\mathbf{y}$, and in the other, we get $v(3, 1) < v(1, 7)$, telling us he prefers $\mathbf{y}$ to x.

So for a given bundle $\mathbf{y}$, can we find all bundles x such that $u(\mathrm{x}) > u(\mathbf{y})$ for every utility function rationalises the data set? A partial solution has been provided by Varian (1982). Varian described a set of bundles such that every x in the set satisfies $u(\mathrm{x}) > u(\mathbf{y})$ for every $u(\cdot)$ that rationalises the data. Knoblauch (1992) then showed that Varian's set is a *complete* solution – that is, it contains *all* such bundles.

Unfortunately, consumption data usually contain violations of GARP. Thus, the search is now on for criteria to help decide when those violations of GARP are unimportant enough to ignore and for practical algorithms that will construct appropriate utility functions on data sets with minor violations of GARP.

2.4 UNCERTAINTY

Until now, we have assumed that decision makers act in a world of absolute certainty. The consumer knows the prices of all commodities and knows that any feasible consumption bundle can be obtained with certainty. Clearly, economic agents in the real world cannot always operate under such pleasant conditions. Many economic decisions contain some element of *un*certainty. When buying a car, for example, the consumer must consider the