Next, let $V_1 \equiv \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ denote the eigenvector of P associated with $\lambda_2 = 1$. Thus

$$\begin{bmatrix} (p_{11}-1) & p_{21} \\ p_{12} & (p_{22}-1) \end{bmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, using $p_{21} = 1 - p_{22}$ and initial normalization $v_{11} = 1$ we have

$$(p_{11}-1) + (1-p_{22}) v_{12} = 0 \quad \text{or}$$

$$v_{12} = \frac{1-p_{11}}{1-p_{22}}. \quad \text{Thus}$$

$$V_1 = \begin{bmatrix} 1 \\ \left(\frac{1-p_{11}}{1-p_{22}}\right) \end{bmatrix}.$$

Renormalizing so that the elements of the eigenvector sum to 1

$$W_1 = \left[\frac{1}{1+\left(\frac{1-p_{11}}{1-p_{22}}\right)}\right] V_1 = \left[\frac{1}{\left(\frac{1-p_{22}+1-p_{11}}{1-p_{22}}\right)}\right] V_1$$

$$= \left(\frac{1-p_{22}}{2-p_{11}-p_{22}}\right) V_1$$

So $$W_1 = \begin{bmatrix} \dfrac{1-p_{22}}{2-p_{11}-p_{22}} \\ \left(\dfrac{1-p_{22}}{2-p_{11}-p_{22}}\right)\cdot\left(\dfrac{1-p_{11}}{1-p_{22}}\right) \end{bmatrix}$$ or

$$W_1 = \begin{bmatrix} \left(\dfrac{1-p_{22}}{2-p_{11}-p_{22}}\right) \\ \left(\dfrac{1-p_{11}}{2-p_{11}-p_{22}}\right) \end{bmatrix}$$

which verifies the second column of Hamilton's matrix T.

Note that $W_1 = \Pi$, the vector of ergodic probabilities.

To verify T^{-1} we need simply show that

$$T \cdot T^{-1} = I_2$$

$$T \cdot T^{-1} = \begin{bmatrix} \dfrac{1-p_{22}}{2-p_{11}-p_{22}} & -1 \\ \dfrac{1-p_{11}}{2-p_{11}-p_{22}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \dfrac{-(1-p_{11})}{2-p_{11}-p_{22}} & \dfrac{(1-p_{22})}{2-p_{11}-p_{22}} \end{bmatrix}$$

$$(1,1) \text{ element} = \frac{1-p_{22}}{2-p_{11}-p_{22}} + \frac{1-p_{11}}{2-p_{11}-p_{22}} = 1 \quad \checkmark$$

$$(1,2) \text{ element} = \frac{1-p_{22}}{2-p_{11}-p_{22}} - \frac{1-p_{22}}{2-p_{11}-p_{22}} = 0 \quad \checkmark$$

$$(2,1) \text{ element} = \frac{1-p_{11}}{2-p_{11}-p_{22}} + \frac{-(1-p_{11})}{2-p_{11}-p_{22}} = 0 \quad \checkmark$$

$$(2,2) \text{ element} = \frac{1-p_{11}}{2-p_{11}-p_{22}} + \frac{1-p_{22}}{2-p_{11}-p_{22}} = 1 \quad \checkmark$$

Additional Problem 3

(a) A is invertible. Thus

$$(A^{-1})A = I.$$

Transposing gives $A'(A^{-1})' = I' = I$

Thus $(A^{-1})' = (A')^{-1}$ (1) QED

(b) Begin from $P = T \Lambda T^{-1}$

Transpose to get $P' = (T^{-1})' \Lambda T'$

Post multiply by $(T')^{-1}$ to get

$$P'(T')^{-1} = (T^{-1})' \Lambda \quad \text{or, using (1),}$$

$$P'(T^{-1})' = (T^{-1})' \Lambda . \quad (2)$$

Denote the columns of $(T^{-1})'$ as follows:

$$(T^{-1})' = [\, y_{(N\times 1)}, y_{2\,(N\times 1)}, \ldots, y_{N\,(N\times 1)} \,]$$

Then eqn (2) gives

$$P'[y_1, y_2, \ldots, y_N] = [y_1, y_2, \ldots, y_N] \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_N \end{bmatrix}$$

which, multiplying out, gives

$$P'y = y$$

$$P'y_2 = y_2\lambda_2$$

$$\vdots$$

$$P'y_N = y_N\lambda_N$$

The first of these equations, $P'y = y$, confirms that the first column of $(T^{-1})'$ is the eigenvector of P' associated with its eigenvalue of 1. The first row of T^{-1} is, of course, $y'_{(1 \times N)}$.

Additional Problem 4

(a) For this Markov chain we define $\underset{(3\times1)}{\xi_{t+j}}$ as follows:

$$\xi_t = \begin{cases} (1,0,0)' & \text{if } S_t = S_1 \\ (0,1,0)' & \text{if } S_t = S_2 \\ (0,0,1)' & \text{if } S_t = S_3 \end{cases} \qquad (3)$$

(b) Given (3) we can write the stochastic process for dividends as

$$d_{t+j} = \mu\,\xi_{t+j} + \varepsilon_{t+j} \qquad (4)$$

where $\mu_{(1\times3)} = [\mu_1, \mu_2, \mu_3]$.

Note that

$$E(\varepsilon_{t+j} \mid \xi_t) = 0 \quad \text{for } j = 1, 2, 3, \dots \qquad (5)$$

Note also that

$$E(\xi_{t+j} \mid \xi_t) = P^j \xi_t \quad \text{for } j = 1, 2, 3, \dots \qquad (6)$$

Thus

$$E(d_{t+1}|\xi_t) = E\left[(\mu\xi_{t+1} + \varepsilon_{t+1}|\xi_t\right] = \mu P\xi_t \quad (7a)$$

$$E(d_{t+2}|\xi_t) = E\left[\mu\xi_{t+2} + \varepsilon_{t+2}|\xi_t\right] = \mu P^2\xi_t \quad (7b)$$

and, in general,

$$E(d_{t+j}|\xi_t) = E\left[\mu\xi_{t+j} + \varepsilon_{t+j}|\xi_t\right] = \mu P^j\xi_t \quad (7c)$$

for $j = 1, 2, 3, \ldots$

C.) Based on eqn 4, two time series variables determine d_{t+j}, ξ_{t+j} and ε_{t+j}. Since ε_{t+j} is white noise nothing in the current information set is useful in forecasting ε_{t+j} for $j = 1, 2, 3, \ldots$. Since ξ_{t+j} is a Markov chain ~~the only variable in the current information set that is useful in foreca~~

$$E(\xi_{t+j}|\xi_t, \xi_{t-1}, \xi_{t-2}, \ldots) = E(\xi_{t+j}|\xi_t).$$

Thus

$$E(d_{t+j}|\Omega_t) = E(d_{t+j}|\xi_t).$$

d.) Now

$$A_t = E_t\left[\left(\frac{1}{1+r}\right)d_{t+1} + \left(\frac{1}{1+r}\right)^2 d_{t+2} + \cdots\right]$$
$$= E\left[\left(\frac{1}{1+r}\right)d_{t+1} + \left(\frac{1}{1+r}\right)^2 d_{t+2} + \cdots \,\Big|\, \xi_t\right]$$
$$= \left(\frac{1}{1+r}\right)\left[E(d_{t+1} \mid \xi_t) + \left(\frac{1}{1+r}\right)E(d_{t+2} \mid \xi_t) + \left(\frac{1}{1+r}\right)^2 E(d_{t+3} \mid \xi_t) + \cdots\right]$$

$$= \left(\frac{1}{1+r}\right)\left[u P \xi_t + \left(\frac{1}{1+r}\right) u P^2 \xi_t + \left(\frac{1}{1+r}\right)^2 u P^3 \xi_t + \cdots\right]$$

$$\boxed{A_t = \left(\frac{1}{1+r}\right) u \left[I + \left(\frac{1}{1+r}\right)P + \left(\frac{1}{1+r}\right)^2 P^2 + \cdots\right] P \xi_t} \quad (8)$$

Note that, since the Markov chain is Ergodic, its largest eigenvalue in absolute value is 1. Thus all of the eigenvalues of $\left(\frac{1}{1+r}\right)P$ lie inside the unit circle. Thus

$$\left[I + \left(\frac{1}{1+r}\right)P + \left(\frac{1}{1+r}\right)^2 P^2 + \cdots\right] = \left[I - \left(\frac{1}{1+r}\right)P\right]^{-1}$$

and (8) becomes (cf. Hamilton p 732)

$$\boxed{A_t = \left(\frac{u}{1+r}\right)\left[I - \left(\frac{1}{1+r}\right)P\right]^{-1} P \xi_t} \quad (9)$$

Additional Problem 5

(a)

$$\begin{array}{cc} & \text{From} \rightarrow \quad \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \text{To} \downarrow \quad \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} & \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \end{array} = P$$

The ~~or~~ Markov chain is Non ergodic. It is <u>Not</u> possible to go from every state to every state: States S_1 and S_5 are absorbing states.

Since All Regular Markov Chains Are Ergodic and Since This Markov chain is NonErgodic it is Not Regular.

(b)

From $\longrightarrow$

To $\searrow$

$$\begin{array}{c|c} & \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} & \begin{bmatrix} 1 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 0 \end{bmatrix} = P \end{array}$$

Once again the Markov chain is Non Ergodic. It is Not possible to go from State S_1 to every other (to <u>ANY</u> other) state as State S_1 is an Absorbing State. The Markov chain is Not Regular because it is Non ergodic and All Regular Chains are Ergodic.