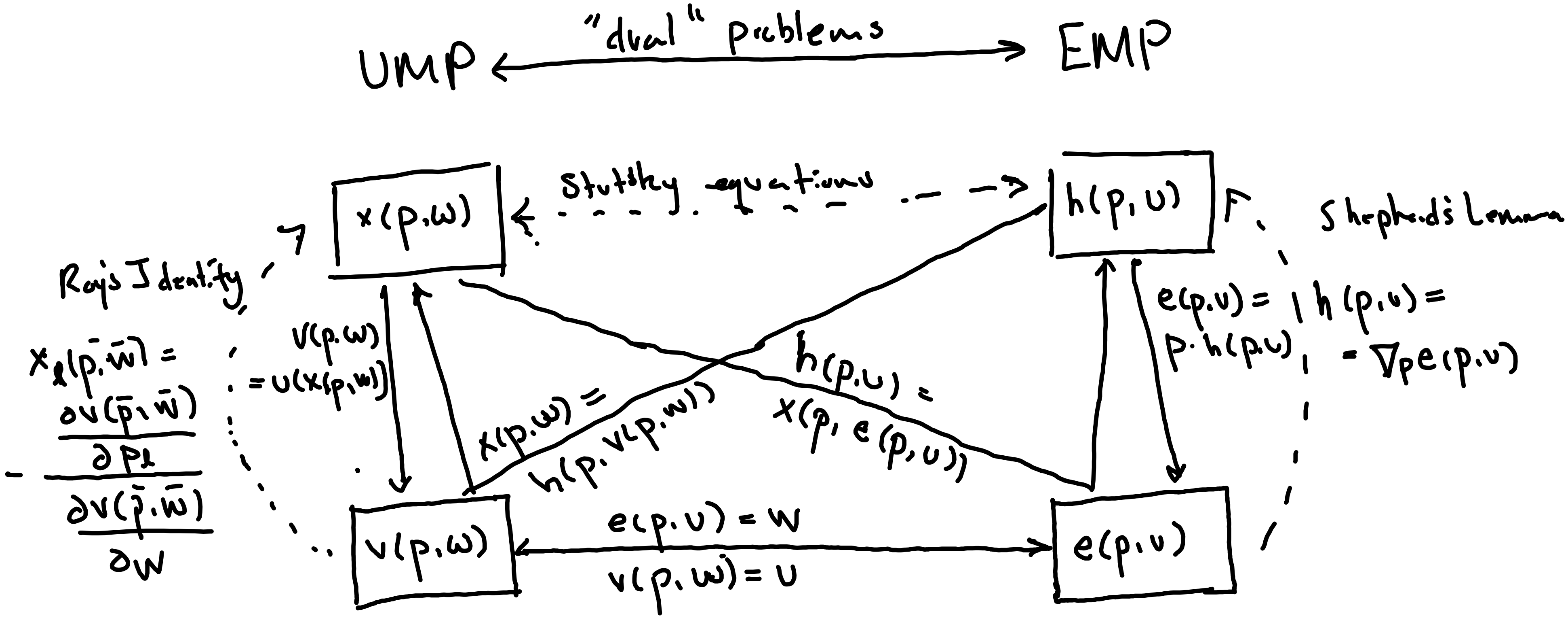
Microeconomic Theory II

Classical Demand Theory, Part 2 of 2

Duality



Expenditure Minimization Problem

The EMP computes the minimum level of wealth (expenditure) required to achieve a target utility, u. The EMP is the dual to the UMP.

$$\min\ p \cdot x \quad \text{s.t. } u(x) \geq u$$

Duality (Principle)

Prop. Suppose LNS $\succsim$ on $X = \mathbb{R}^L_+$ rep. by cont. $u(\ )$ and $p \gg 0$. Then:

i.) If x^* is optimal in the UMP when $w > 0$, then x^* is optimal in the EMP when $u = u(x^*)$. Moreover, the min expenditure in EMP is w.
$(px^* = w)$

ii) If x^* is optimal in the EMP when the required utility is $u > u(0)$, then x^* is optimal in the UMP when wealth is px^*. Moreover, max utility from UMP is u.
$(u(x^*) = u)$

Proof. i) Suppose x^* is not optimal in the EMP w/ required $u(x^*)$. Then $\exists$ x' such that $u(x') > u(x^*)$ and $px' < px^* \leq w$. By LNS we can find x'' "close" to x' $\ni$ $u(x'') > u(x')$ and $px'' < w$. But this implies $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$, contradicts the optimality of the UMP. Thus x^* must be optimal in EMP when req. $u(x^*)$, and min expenditure is px^*. Since x^* solves the UMP w/ wealth w, by Walras' Law $p \cdot x^* = w$.

ii) Since $u > u(\emptyset)$, we have $x^* \neq \emptyset$. Hence, $px^* > \emptyset$. Suppose x^* is not optimal in the UMP when $w = px^*$; then $\exists$ x' $\ni$ $u(x') > u(x^*)$ and $p \cdot x' \leq p \cdot x^*$. Consider bundle $x'' = \alpha x'$ where $\alpha \in (\emptyset, 1)$; by continuity of $u()$, if α is close enough to 1, we have $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$. But, this contradicts optimality of x^* in the EMP. Thus x^* is optimal in the UMP when $px^* = w$ and max utility is $u(x^*)$. We can show if x^* solves the EMP w/ required u then $u(x^*) = u$.

If we assume $u(\ )$ must attain values at least as large as u for some x then $\exists$ a solution to the EMP when $p \gg \emptyset$ (i.e. for any $u > u(\emptyset)$ and $u(\cdot)$ unbounded from above)

The Hicksian (or Compensated) Demand Function $\qquad x(p,u)$

The set of optimal vectors in the EMP are denoted $h(p,u) \subset \mathbb{R}^L_+$.

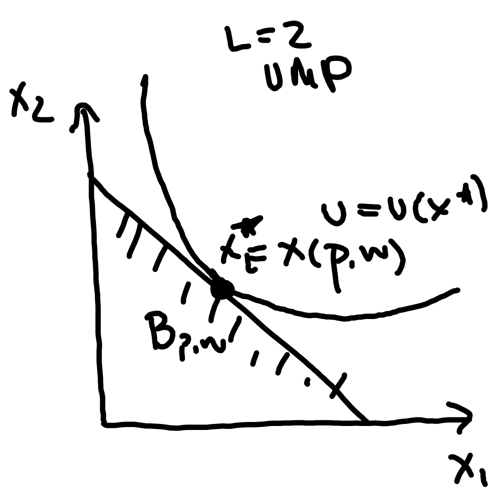
Proposition. Suppose $u(\ )$ cont. & rep. LNS $\succsim$ on $\mathbb{R}^L_+$. For any $p \gg 0$:

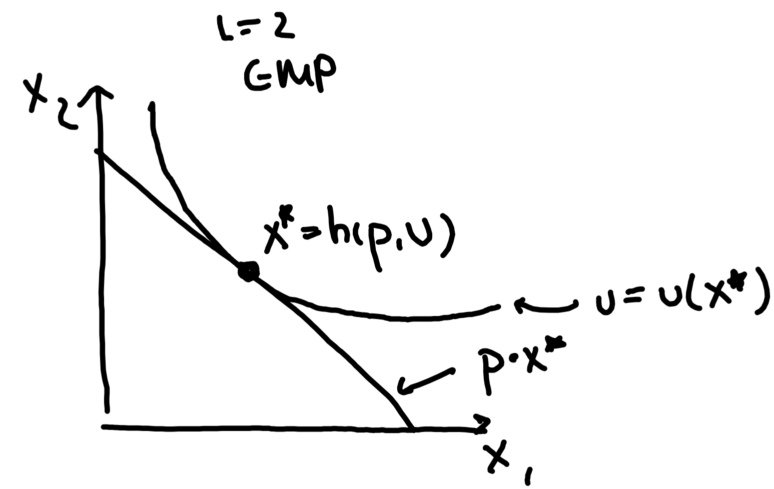
1. homogeneity of degree zero in p: $h(\alpha p, u) = h(p,u)$ for any p, u, & $\alpha > 0$.

2. no excess utility: $\forall\, x \in h(p,u)$, $u(x) = u$.

3. convexity/uniqueness: if $\succsim$ is convex, then $h(p,u)$ is a convex set; if $\succsim$ is strictly convex, then $h(p,u)$ is single-valued.

Proof. In text, pp. 61-62.





The Expenditure Function

Given $p \gg 0$ and $u > u(0)$ the value of the EMP is denoted $e(p,u)$, the expenditure fn. It's value for (p,u) is $p \cdot x^*$ where x^* is the solution to the EMP.

Prop. Suppose $u(\,)$ cont. rep. LNS $\succsim$ on $X = \mathbb{R}^L_+$. Then $e(p,u)$:

1. homogeneous of degree one in p
2. strictly increasing in u & nondecreasing in p_ℓ for any ℓ
3. concave in p
4. conti in p & u

Proof. In text, pp. 59-60.

Hicksian Demand & the Compensated Law of Demand

Prop. Suppose cont. $u(\ )$ rep. LNS $\succsim$ & $h(p,u)$ consists of a single element $\forall\ p \gg \emptyset$. Then $h(p,u)$ satisfies the compensated law of demand (is "downward sloping"): for all p' & p''

$$\underbrace{(p'-p'')}_{\Delta p}\underbrace{[h(p'',u)-h(p',u)]}_{\Delta h} \leq \emptyset \implies \frac{\partial h_l(p,u)}{\partial p_l} \leq \emptyset \qquad (*)$$

Proof. $\forall\ p \gg \emptyset$ consumption bundle $h(p,u)$ is optimal in the EMP, and achieves a lower expenditure at p than any alternative bundle that yields utility of at least u. So, WLOG by def.

$$p''h(p'',u) \leq p''h(p',u).$$

and

$$p'h(p'',u) \geq p'\cdot h(p',u).$$

So

$$\underbrace{[p''h(p'',u)-p''h(p',u)]}_{\cdot \leq \emptyset} \underbrace{-\ [p'h(p'',u)-p'h(p',u)]}_{+\ \cdot \leq \emptyset} = (*) \leq \emptyset$$

Relationship b/w the Expenditure & Hicksian Demand Fns.

Recall $e(p,u) = p \cdot h(p,u)$.

Prop. (Shephard's Lemma). Suppose continuous $u(\,)$ represents LNS $\succsim$ on $X = \mathbb{R}^L_+$

$\forall p, u$, $h(p,u)$ is the derivative vector of the expenditure fn. w.r.t. p:

$$h(p,u) = \nabla_p e(p,u) \quad \text{or} \quad h_l(p,u) = \frac{\partial e(p,u)}{\partial p_l} \;\; \forall\, l = 1, 2, \ldots, L.$$

Proof. F.O.C. argument (second proof on pp. 68-69)

Assume $h(p,u) \gg 0$ and diff. at (p,u). Via the chain rule

$$\nabla_p e(p,u) = \nabla_p [p \cdot h(p,u)]$$

$$= h(p,u) + [p \cdot D_p h(p,u)]^T$$

The F.O.C.s for the EMP take the form $p = \lambda \nabla u(h(p,u))$; subsituting

$$= h(p,u) + \lambda [\nabla u(h(p,u)) \cdot D_p h(p,u)]^T$$

but since $u(h(p,u)) = u$ for all p in the EMP, the final term $= 0$.

This follows from $h(p,u)$ is HDZ in p.

Digging into $D_p h(p,u)$

Prop. Suppose cont. $u(\ )$ rep. LNS <u>strictly convex</u> $\succsim$ on $X = \mathbb{R}^L_+$. Suppose also $h(p,u)$ is cont. diff. at (p,u) and its $L \times L$ derivative matrix is $D_p h(p,u)$.

$$D_p h(p,u) = \begin{array}{cc} & p_k \\ l & \left[\begin{array}{ccc} \ddots & & \\ & \dfrac{\partial h_l(p,u)}{\partial p_k} & \\ & & \ddots \end{array}\right] \end{array}$$

i.) $D_p h(p,u) = D^2_p e(p,u)$

ii) $D_p h(p,u)$ is neg. semi-definite

iii) $D_p h(p,u)$ is symmetric

iv) $D_p h(p,u) p = 0$.

Proof. In text, pp. 69-70.

The diagonals are neg. The off-diagonals, b/c $D_p h(p,u)p = 0$ and $p \gg 0$ and above, will contain at least one positive element. So, there will be at least one good $k \neq l$ for which $\partial h_l(p,u)/\partial p_k \geq 0$ — or, put more plainly, every good has at least one substitute.

$\partial x_l(p,u)/\partial p_k \geq 0$ g<u>ross</u> substitute

Relationship between Hicksian & Marshallian Demand

Prop. (the Slutsky Equations) Suppose cont. $u(\,)$ rep. LNS $\succsim$ & strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Then $\forall (p,w)$ & $u = v(p,w)$, we have

$$S_{lk}(p,w) \equiv \frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,k)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x(p,w) \quad \forall\, l,k$$

(incl. $l=k$)

or in matrix notation

$$S(p,w) \equiv D_p h(p,u) = D_p x_L(p,w) + D_w x(p,w) \cdot x(p,w)^T$$

Slutsky substitution matrix.

Proof. Start w/ a consumer facing $(\bar{p},\bar{w})$ & attaining $\bar{u}$. By def., for all (p,u), $h_l(p,u) = x(p, e(p,u))$. Diff. w.r.t. p_k and eval. at $(\bar{p},\bar{u})$:

$$\frac{\partial h_l(p,w)}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(p, e(\bar{p},\bar{u}))}{\partial w} \cdot \frac{\partial e(\bar{p},\bar{u})}{\partial p_k}$$

Since $h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}\ \forall k$, $\bar{w} = e(\bar{p},\bar{u})$, & $h_k(p,u) = x(\bar{p}, e(\bar{p},\bar{u})) = x_k(\bar{p},\bar{w})$, we have the eqn. above.

Relationship b/w Marshallian Demand and the Value Fn

Prop. (Roy's Identity) Suppose cont. $U(\ )$ rep. LNS & strictly convex $\succsim$ on $X = \mathbb{R}^L_+$.
Suppose that $v(p,w)$ is diff. at $(\bar{p},\bar{w}) \gg 0$. Then:

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})} \nabla_p v(\bar{p},\bar{w})$$

or for every $l = 1, \ldots, L$:

$$x_l(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w}.$$

Proof. Three in text, pg. 74.

Ex. 1. Suppose $L=2$ and Cobb-Douglas utility $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ for some $\alpha \in (0,1)$.

The EMP is

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad x_1^{\alpha} x_2^{1-\alpha} = u$$

The Lagrangian is

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda \left(u - x_1^{\alpha} x_2^{1-\alpha} \right)$$

Assuming an interior solution, the F.O.C.s are

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &: p_1 - \lambda \frac{\alpha u}{x_1} = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &: p_2 - \lambda \frac{(1-\alpha) u}{x_2} = 0 \end{aligned}\right\} \overset{(1.)}{\Longrightarrow} \frac{p_1}{\alpha} x_1 = \frac{p_2}{1-\alpha} x_2 \Longrightarrow \overset{(2.)}{x_2} = \frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2} x_1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} : u - x_1^{\alpha} x_2^{1-\alpha} \Longrightarrow \overset{(3.)}{u} = x_1^{\alpha} x_2^{1-\alpha} \overset{w/(2.)}{\Longrightarrow} \ldots \Longrightarrow x_1^* = \left(\frac{\alpha}{1-\alpha} \cdot \frac{p_2}{p_1} \right)^{1-\alpha} u,$$

$$x_2^* = \left(\frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2} \right)^{\alpha} u.$$

Ex. 2. From Ex. 1, find $e(p,u)$.

We have $h_1(p,u) = x_1^*$ and $h(p,u) = x_2^*$.

Then $e(p,u) = p \cdot h(p,u)$ yields (by substitution & simplification)

$$e(p,u) = \left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] p_1^{\alpha} p_2^{1-\alpha} u$$

If we substitute $u = v(p,w)$ from our exercise in the last lecture, we have

$e(p,u) = w$.