Math Camp
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Homework 4 Solution

Illustrate the definitions of the following notations

(i) (X, d); (ii) $B(a;\delta), B'(a;\delta)$; (iii) $\bar{E}$; (iv) E° .

Show partial theorem 2.3 mentioned in class

(iii) The union of any collection of open subsets of X is open.
(iv) The intersection of a finite number of open subsets of X is open.

Proof. The proofs of (i) and (ii) are trivial.
(iii) Suppose $\{G_\alpha | \alpha \in I\}$ is an arbitrary collection of open sets (that means G_α is open for every $\alpha \in I$. Let us show that the set $G = \bigcup_{\alpha\in I} G_\alpha$ is open. Fix any $a \in G$. Then there exists $\alpha_0 \in I$ such that

$$a \in G_{\alpha_0}.$$

Since G_{α_0} is open, there exists an open ball $B(a;\delta)$ (or, equivalently, there exists $\delta > 0$) such that

$$B(a;\delta) \subset G_{\alpha_0}.$$

Thus, $B(a;\delta) \subset G$ because $G_{\alpha_0} \subset G$.

(iv) Suppose $G_i, i = 1, \ldots, n$ are open subsets of X. Let us show that the set $G = \cap_{i=1}^n G_i$ is also open. Fix any $a \in G$. Then $a \in G_i$ for every $i = 1, \ldots, n$. Since each G_i is open, there exists $\delta_i > 0$ such that

$$B(a;\delta_i) \subset G_i \text{for } i = 1, \ldots, n.$$

Set $\delta := \min\{\delta_i | i = 1, \ldots, n\}$. Then $\delta > 0$ and

$$B(a;\delta) \subset G.$$

Thus, G is open. □

Show The intersection of any collection of closed subsets of X is closed.

Proof. The proofs for these are simple using DeMorgan's law. Let us prove, for instance, Let $\{S_\alpha | \alpha \in I\}$ be a collection of closed sets. We are going to prove that the set

$$S = \bigcap_{\alpha\in I} S_\alpha$$

is also closed. We have that

$$S^c = [\bigcap_{\alpha\in I} S_\alpha]^c = \bigcup_{\alpha\in I} S_\alpha^c$$

is an open set because it is a union of open sets. Thus, S is closed. □

A function f is called **homogeneous of degree n** if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer and f has continuous second-order partial derivatives.

(a) Verify that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

[*Hint:* Use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.]

Solution

$$f(tx, ty) = t^2x^2 \times ty + 2tx \times t^2y^2 + 5t^3y^3 = t^3 f(x, y)$$

Thus, f is $HMG(3)$

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(tx, ty)$. Since f is homogeneous, we can write $g(t) = t^r f(x, y)$. Find $g'(t)$.

Using $g(t) = t^r f(x, y)$, it is clear that $g'(t) = rt^{r-1} f(x, y)$.

Using $g(t) = f(tx, ty)$, we get that
$g'(t) = \frac{\partial f}{\partial(tx)} \cdot \frac{d(tx)}{dt} + \frac{\partial f}{\partial(ty)} \cdot \frac{d(ty)}{dt} = x\frac{\partial f}{\partial(tx)} + y\frac{\partial f}{\partial(ty)}$.

So we have that for all t, $rt^{r-1} f(x, y) = x\frac{\partial f}{\partial(tx)} + y\frac{\partial f}{\partial(ty)}$. If we let $t = 1$, then we have that $g(1) = f(x, y)$, our original function, and $rf(x, y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$, the desired result.

Find the antiderivative $\int \frac{dx}{\sqrt{1+4x}}$

Solution

We can try to use the substitution $u = 1 + 4x$. Hence

$$du = d\,(1 + 4x) = 4dx,$$

so

$$dx = \frac{du}{4}.$$

This yields

$$\int \frac{dx}{\sqrt{1+4x}} = \int \frac{\frac{du}{4}}{\sqrt{u}} = \frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int u^{-\frac{1}{2}}du = \frac{1}{4} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{1}{4} \cdot 2u^{\frac{1}{2}} + C = \frac{u^{\frac{1}{2}}}{2} + C = \frac{\sqrt{u}}{2} + C = \frac{\sqrt{1+4x}}{2} + C.$$

Find the antiderivative $\int \frac{xdx}{\sqrt{1+x^2}}$

Solution

Let $u = 1 + x^2$. Then

$$du = d\left(1+x^2\right) = 2xdx.$$

We see that

$$xdx = \frac{du}{2}.$$

Hence

$$\int \frac{xdx}{\sqrt{1+x^2}} = \int \frac{\frac{du}{2}}{\sqrt{u}} = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C = \sqrt{1+x^2} + C.$$

Find the antiderivative $\int 2^x e^x dx$

Solution

We rewrite the integral in the following way:

$$\int 2^x e^x dx = \int (2e)^x dx.$$

Denoting $2e = a$ (this is not a change of variable, since x still remains the independent variable), we get the table integral:

$$\int (2e)^x dx = \int a^x dx = \frac{a^x}{\ln a} + C = \frac{(2e)^x}{\ln(2e)} + C = \frac{2^x e^x}{\ln 2 + \ln e} + C$$
$$= \frac{2^x e^x}{\ln 2 + 1} + C.$$

Find the antiderivative $\int xe^{-x^2} dx$

Solution

Using the substitution $u = -x^2$, we have

$$du = d\left(-x^2\right) = -2xdx.$$

Note that

$$xdx = -\frac{du}{2},$$

so we can rewrite the integral in terms of the variable u and solve it:

$$\int xe^{-x^2} dx = \int e^u \left(-\frac{du}{2}\right) = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{e^{-x^2}}{2} + C.$$

Find the antiderivative $\int \frac{\ln x}{x^2} dx$

Solution

$$u = \ln x,\ \ dv = \frac{dx}{x^2}.$$

Then

$$du = \frac{dx}{x},\ \ v = \int \frac{dx}{x^2} = -\frac{1}{x}.$$

Integrating by parts, we obtain

$$\int \frac{\ln x}{x^2} dx = \ln x \cdot \left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right) \frac{dx}{x} = -\frac{\ln x}{x} + \int \frac{dx}{x^2}$$
$$= -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Find the antiderivative $\int x^2 e^x dx$

Solution

Let

$$u = x^2,\ \ dv = e^x dx.$$

Then

$$du = 2x dx,\ \ v = \int e^x dx = e^x,$$

The integral is written as

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2 \int x e^x dx.$$

We calculate the last integral by repeated integration by parts. Choosing

$$u = x,\ \ dv = e^x dx,$$

we obtain

$$du = dx,\ \ v = \int e^x dx = e^x,$$

so

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2 \left(x e^x - \int e^x dx \right)$$
$$= x^2 e^x - 2 \left(x e^x - e^x \right) + C = x^2 e^x - 2x e^x + 2e^x + C$$
$$= e^x \left(x^2 - 2x + 2 \right) + C.$$

Find the derivative of the function $g(x) = \int_0^{x^2} \sqrt{1+t^2}dt$

Solution

Since the upper limit of integration is not x, we apply the chain rule. Let $u = x^2$, then $u' = 2x$.

Consider the new function

$$h(u) = \int_0^u \sqrt{1+t^2}dt.$$

By the FTC1, we can write

$$h'(u) = \sqrt{1+u^2}.$$

As $g(x) = h(x^2)$, we have

$$g'(x) = \left[h(x^2)\right]' = h'(x^2)\cdot(x^2)' = \sqrt{1+(x^2)^2}\cdot 2x = 2x\sqrt{1+x^4}.$$

Find the derivative of the function $g(x) = \int_{\sqrt{x}}^{x}(t^2 - t)dt \ \ at\ x = 1.$

Solution

We split the integral function into two terms:

$$g(x) = \int_{\sqrt{x}}^{x} (t^2 - t)\,dt = \int_{\sqrt{x}}^{c} (t^2 - t)\,dt + \int_{c}^{x} (t^2 - t)\,dt$$

$$= \int_{c}^{x} (t^2 - t)\,dt - \int_{c}^{\sqrt{x}} (t^2 - t)\,dt,$$

where $c \in [x^2, x^3]$.

Find the derivative of $g(x)$ using the FTC1 and the chain rule (for the second term):

$$\frac{d}{dx}\int_{c}^{x} (t^2 - t)\,dt = x^2 - x;$$

$$\frac{d}{dx}\int_{c}^{\sqrt{x}} (t^2 - t)\,dt = \left((\sqrt{x})^2 - \sqrt{x}\right)\cdot(\sqrt{x})' = (x - \sqrt{x})\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{\sqrt{x}}{2} - \frac{1}{2}.$$

Then

$$g'(x) = (x^2 - x) - \left(\frac{\sqrt{x}}{2} - \frac{1}{2}\right) = x^2 - x - \frac{\sqrt{x}}{2} + \frac{1}{2}.$$

At the point $x = 1$, the derivative is equal to

$$g'(1) = 1^2 - 1 - \frac{\sqrt{1}}{2} + \frac{1}{2} = 0.$$

Evaluate the integral $\int_0^2 (x^3 - x^2)dx$

Solution

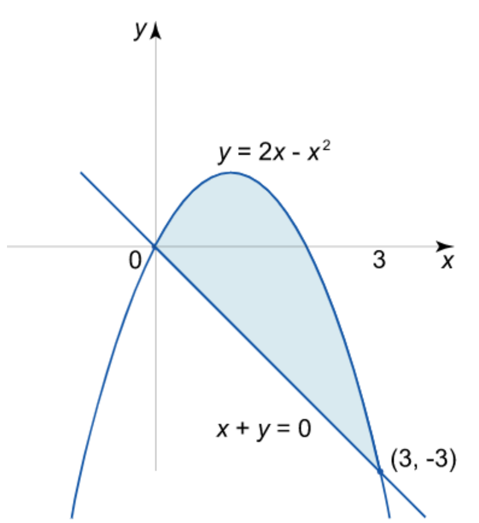
Using the Fundamental Theorem of Calculus, Part 2, we have

$$\int_0^2 (x^3 - x^2)\,dx = \left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^2 = \left(\frac{16}{4} - \frac{8}{3}\right) - 0 = \frac{4}{3}.$$

Find the area bounded by $y = 2x - x^2$ and $x + y = 0$

Solution

$$2x - x^2 = -x, \quad \Rightarrow x^2 - 3x = 0, \quad \Rightarrow x\,(x-3) = 0, \quad \Rightarrow x_1 = 0,\; x_2 = 3.$$



The upper boundary of the region is the parabola $y = 2x - x^2$, and the lower boundary is the straight line $y = -x$.

The area is given by

$$S = \int_0^3 [2x - x^2 - (-x)]\,dx = \int_0^3 (2x - x^2 + x)\,dx$$
$$= \left(x^2 - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^3 = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}.$$

Compute the integral $\int_{-2}^{2} \frac{dx}{x^3}$.

Hint: consider improper integral.

Solution

There is a discontinuity at $x = 0$, so that we must consider two improper integrals:

$$\int_{-2}^{2} \frac{dx}{x^3} = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}.$$

Using the definition of improper integral, we obtain

$$\int_{-2}^{2} \frac{dx}{x^3} = \int_{-2}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3} = \lim_{\tau \to 0+} \int_{-2}^{-\tau} \frac{dx}{x^3} + \lim_{\tau \to 0+} \int_{\tau}^{2} \frac{dx}{x^3}.$$

For the first integral,

$$\lim_{\tau \to 0+} \int_{-2}^{-\tau} \frac{dx}{x^3} = \lim_{\tau \to 0+} \left(\frac{x^{-2}}{-2} \right) \Big|_{-2}^{-\tau} = -\frac{1}{2} \lim_{\tau \to 0+} \left(\frac{1}{x^2} \right) \Big|_{-2}^{-\tau}$$

$$= -\frac{1}{2} \lim_{\tau \to 0+} \left[\frac{1}{(-\tau)^2} - \frac{1}{(-2)^2} \right] = -\frac{1}{2} \lim_{\tau \to 0+} \left(\frac{1}{\tau^2} + \frac{1}{8} \right) = \infty.$$

Since it is divergent, the initial integral also diverges.