Semilinear parabolic partial differential equations
Theory, approximation, and applications

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Outline

- Semilinear parabolic equation
- Finite element method for elliptic equation
- Finite element method for semilinear parabolic equation
- Application to dynamical systems
- Stochastic parabolic equation
- Computer exercises with the software Puffin

Lecture 1: Semilinear parabolic PDE

Initial-boundary value problem

$$
\begin{aligned}
u_t - \Delta u &= f(u), && x \in \Omega,\ t > 0, \\
u &= 0, && x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) &= u_0, && x \in \Omega,
\end{aligned} \tag{1}
$$

$\Omega \subset \mathbf{R}^d$, $d = 1, 2, 3$, bounded convex polygonal domain
$u = u(x,t) \in \mathbf{R}$, $u_t = \partial u / \partial t$,
$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$

$f : \mathbf{R} \to \mathbf{R}$ is twice continuously differentiable

$$
|f^{(l)}(\xi)| \le C(1 + |\xi|^{\delta+1-l}), \quad \xi \in \mathbf{R},\ l = 1, 2, \tag{2}
$$

$\delta = 2$ if $d = 3$, $\quad \delta \in [1, \infty)$ if $d = 2$.

Example: Allen-Cahn equation

$$f(\xi) = -V'(\xi)$$

$$V(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$$

$$u_t - \Delta u = -(u^3 - u)$$

Sobolev spaces

Hilbert space $H = L_2(\Omega)$, with standard norm and inner product

$$\|v\| = \Big(\int_\Omega |v|^2 \, dx \Big)^{1/2}, \quad (v, w) = \int_\Omega v \cdot w \, dx. \tag{3}$$

Sobolev spaces $H^m(\Omega)$, $m \geq 0$, norms denoted by

$$\|v\|_m = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|^2 \right)^{1/2}. \tag{4}$$

Hilbert space $V = H_0^1(\Omega)$, with norm $\|\cdot\|_1$, the functions in $H^1(\Omega)$ that vanish on $\partial\Omega$.
$V^* = H^{-1}(\Omega)$ is the dual space of V with norm

$$\|v\|_{-1} = \sup_{\chi \in V} \frac{|(v, \chi)|}{\|\chi\|_1}. \tag{5}$$

Sobolev spaces

X, Y Banach spaces
$\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X into Y
$\mathcal{L}(X) = \mathcal{L}(X,X)$
$B_X(x,R)$ denotes the closed ball in X with center x and radius R.
$B_R = B_V(0,R)$ denote the the closed ball of radius R in V:

$$B_R = \{v \in V : \|v\|_1 \leq R\}.$$

We also use the notation

$$\|v\|_{L_\infty([0,T],X)} = \sup_{t\in[0,T]} \|v(t)\|_X .$$

Abstract framework

unbounded operator $A = -\Delta$ on H
domain of definition $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$
A is a closed, densely defined, and self-adjoint positive definite operator in H with compact inverse.
nonlinear operator $f : V \to H$ defined by $f(v)(x) = f(v(x))$
The initial-boundary value problem (**??**) may then be formulated as an initial value problem in V: find $u(t) \in V$ such that

$$u' + Au = f(u),\ t > 0; \quad u(0) = u_0. \tag{6}$$

Eigenvalue problem:

$$A\varphi = \lambda\varphi$$

$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \to \infty$
orthonormal basis of eigenvectors $\{\varphi_j\}_{j=1}^{\infty}$

Abstract framework

The operator $-A$ is the infinitesimal generator of the analytic semigroup $E(t) = \exp(-tA)$ defined by

$$E(t)v = \sum_{j=1}^{\infty} e^{-t\lambda_j}(v, \varphi_j)\varphi_j, \quad v \in H, \tag{7}$$

The semigroup $E(t)$ is the solution operator of the initial value problem for the homogeneous equation,

$$u' + Au = 0,\ t > 0; \quad u(0) = u_0; \qquad u(t) = E(t)u_0$$

By Duhamel's principle: solutions of (**??**) satisfy

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds, \quad t \geq 0. \tag{8}$$

Conversely, appropriately defined solutions of the nonlinear integral equation (**??**) are solutions of the differential equation (**??**), see below. We shall mainly work with (**??**) and discretized variants of it.

Fractional powers of A

$$\|A^\alpha v\| = \Big(\sum_{j=1}^{\infty} \Big(\lambda_j^\alpha (v, \varphi_j) \Big)^2 \Big)^{1/2}, \quad \alpha \in \mathbf{R},$$
$$\mathcal{D}(A^\alpha) = \Big\{ v : \|A^\alpha v\| < \infty \Big\}, \quad \alpha \in \mathbf{R}. \tag{9}$$

elliptic regularity estimate

$$\|v\|_2 \le C\|Av\|, \quad v \in H^2(\Omega) \cap H_0^1(\Omega), \tag{10}$$

trace inequality

$$\|v\|_{L_2(\partial\Omega)} \le C\|v\|_1, \quad v \in H^1(\Omega),$$

$\mathcal{D}(A^{l/2}) = H^l(\Omega) \cap H_0^1(\Omega)$, $l = 1, 2$, with the equivalence of norms

$$c\|v\|_l \le \|A^{l/2} v\| \le C\|v\|_l, \quad v \in \mathcal{D}(A^{l/2}),\ l = 1, 2; \tag{11}$$

$\mathcal{D}(A^{-1/2}) = H^{-1}(\Omega)$

$$c\|v\|_{-1} \le \|A^{-1/2} v\| \le C\|v\|_{-1}; \tag{12}$$

Analytic semigroup

$$E(t)v = \sum_{j=1}^{\infty} e^{-t\lambda_j}(v, \varphi_j)\varphi_j, \quad v \in H,$$

$u(t) = E(t)v$ is the solution of

$$u' + Au = 0; \quad u(0) = v$$

Smoothing property ($D_t = \partial/\partial t$):

$$\|D_t^l E(t)v\| = \|A^l E(t)v\| \leq C_l t^{-l}\|v\|, \quad t > 0,\ v \in H,\ l \geq 0. \tag{13}$$

$$\begin{aligned} \|D_t^l E(t)v\|_\beta \leq C_l t^{-l-(\beta-\alpha)/2}\|v\|_\alpha, \quad t > 0,\ v \in \mathcal{D}(A^{\alpha/2}), \\ -1 \leq \alpha \leq \beta \leq 2,\ l = 0, 1. \end{aligned} \tag{14}$$

Local Lipschitz condition

$f : V \to H$ nonlinear mapping
$f' : V \to \mathcal{L}(V, H)$ Fréchet derivative
growth assumption:

$$|f^{(l)}(\xi)| \le C(1 + |\xi|^{\delta+1-l}), \quad \xi \in \mathbf{R},\ l = 1, 2, \tag{15}$$

$\delta = 2$ if $d = 3$, $\quad \delta \in [1, \infty)$ if $d = 2$.
Sobolev's inequality

$$\|v\|_{L_p} \le C\|v\|_1, \tag{16}$$

$p = 6$ if $d \le 3$, $\quad p < \infty$ if $d = 2$, $\quad p = \infty$ if $d = 1$
Hölder's inequality

$$\|v^\delta w\|_{L_r} \le \|v\|_{L_q}^\delta \|w\|_{L_p}, \quad \tfrac{\delta}{q} + \tfrac{1}{p} = \tfrac{1}{r},\ \delta > 0. \tag{17}$$

Local Lipschitz condition

$f : V \to H$ nonlinear mapping
$f' : V \to \mathcal{L}(V, H)$ Fréchet derivative

Lemma For each nonnegative number R there is a constant $C(R)$ such that, for all $u, v \in B_R \subset V$,

$$\|f'(u)\|_{\mathcal{L}(V,H)} \leq C(R) \tag{18}$$

$$\|f'(u)\|_{\mathcal{L}(H,V^*)} \leq C(R) \tag{19}$$

$$\|f(u) - f(v)\| \leq C(R)\|u - v\|_1 \tag{20}$$

$$\|f(u) - f(v)\|_{-1} \leq C(R)\|u - v\| \tag{21}$$

Proof

By (**??**) and the Hölder and Sobolev inequalities, for $z \in V = H_0^1(\Omega)$,

$$\begin{aligned}\|f'(u)z\|_{L_2} &\le C\|(1+|u|^\delta)z\|_{L_2} \le C\big(1+\|u\|_{L_q}^\delta\big)\|z\|_{L_p}\\ &\le C\big(1+\|u\|_1^\delta\big)\|z\|_1,\end{aligned}$$

where $\frac{1}{p}+\frac{\delta}{q}=\frac{1}{2}$ with $p=q=6$ if $d=3$, and with arbitrary $p\in(1,\infty)$ if $d\le 2$. This proves (**??**) and (**??**) follows.
Moreover, for any $z,\chi\in V$,

$$\begin{aligned}(f'(u)z,\chi) &\le C\big(1+\|u\|_{L_q}^\delta\big)\|z\|_{L_2}\|\chi\|_{L_p}\\ &\le C\big(1+\|u\|_1^\delta\big)\|z\|\ \|\chi\|_1,\end{aligned}$$

where $\frac{\delta}{q}+\frac{1}{2}+\frac{1}{p}=1$, i.e., with the same p and q as before. This proves (**??**) and (**??**) follows.

Local existence

Theorem. For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that (**??**) has a unique solution $u \in C([0,\tau], V)$ for any initial value $u_0 \in V$ with $\|u_0\|_1 \leq R_0$. Moreover, there is c such that $\|u\|_{L_\infty([0,\tau],V)} \leq cR_0$.

Proof

Let $u_0 \in B_{R_0}$, define

$$\mathcal{S}(u)(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds,$$

and note that (**??**) is a fixed point equation $u = \mathcal{S}(u)$. We shall choose τ and R such that we can apply Banach's fixed point theorem (the contraction mapping theorem) in the closed ball

$$\mathcal{B} = \{u \in C([0,\tau],V) : \|u\|_{L_\infty([0,\tau],V)} \leq R\}$$

in the Banach space $C([0,\tau],V)$.
We must show (i) that $\mathcal{S}$ maps $\mathcal{B}$ into itself, (ii) that $\mathcal{S}$ is a contraction on $\mathcal{B}$. In order to prove (i) we take $u \in \mathcal{B}$ and first note that the Lipschitz condition (**??**) implies that

$$\begin{aligned}
\|f(u(t))\| &\leq \|f(0)\| + \|f(u(t)) - f(0)\| \\
&\leq \|f(0)\| + C(R)\|u(t)\|_1 \\
&\leq \|f(0)\| + C(R)R, \quad 0 \leq t \leq \tau.
\end{aligned} \tag{22}$$

Proof, cont'd

Hence, using also (**??**), we get

$$\begin{aligned}\|\mathcal{S}(u)(t)\|_1 &\le \|E(t)u_0\|_1 + \int_0^t \|E(t-s)f(u(s))\|_1\,ds\\ &\le c_0\|u_0\|_1 + c_1\int_0^t (t-s)^{-1/2}\|f(u(s))\|\,ds\\ &\le c_0R_0 + 2c_1\tau^{1/2}\big(\|f(0)\| + C(R)R\big), \quad 0\le t\le\tau.\end{aligned}$$

This implies

$$\|\mathcal{S}(u)\|_{L_\infty([0,\tau],V)} \le c_0R_0 + 2c_1\tau^{1/2}\big(\|f(0)\| + C(R)R\big).$$

Choose $R = 2c_0R_0$ and $\tau = \tau(R_0)$ so small that

$$2c_1\tau^{1/2}\big(\|f(0)\| + C(R)R\big) \le \tfrac{1}{2}R. \tag{23}$$

Then $\|\mathcal{S}(u)\|_{L_\infty([0,\tau],V)} \le R$ and we conclude that $\mathcal{S}$ maps $\mathcal{B}$ into itself.

Proof, cont'd

To show (ii) we take $u, v \in \mathcal{B}$ and note that

$$\|f(u(t)) - f(v(t))\| \leq C(R)\|u - v\|_{L_\infty([0,\tau],V)}, \quad 0 \leq t \leq \tau.$$

Hence

$$\begin{aligned}
\|\mathcal{S}(u)(t) - \mathcal{S}(v(t))\|_1 &\leq \int_0^t \|E(t-s)(f(u(s)) - f(v(s)))\|_1 \, ds \\
&\leq c_1 \int_0^t (t-s)^{-1/2} \|f(u(s)) - f(v(s))\| \, ds \\
&\leq 2c_1 \tau^{1/2} C(R)\|u - v\|_{L_\infty([0,\tau],V)}, \quad 0 \leq t \leq \tau,
\end{aligned}$$

so that

$$\|\mathcal{S}(u) - \mathcal{S}(v)\|_{L_\infty([0,\tau],V)} \leq 2c_1 \tau^{1/2} C(R)\|u - v\|_{L_\infty([0,\tau],V)}.$$

It follows from (**??**) that $2c_1\tau^{1/2}C(R) \leq \frac{1}{2}$ and we conclude that $\mathcal{S}$ is a contraction on $\mathcal{B}$.

Hence $\mathcal{S}$ has a unique fixed point $u \in \mathcal{B}$.

Nonlinear semigroup

The initial value problem thus has a unique local solution for any initial datum $u_0 \in V$. We denote by

$$S(t, \cdot) : V \to V$$

the corresponding solution operator, so that $u(t) = S(t, u_0)$ is the solution of

$$u(t) = E(t)u_0 + \int_0^t E(t-s) f(u(s))\, ds, \quad t \geq 0. \tag{24}$$

Regularity

This will be used in our error analysis, but the theorem also shows that $u'(t) \in H$ and $Au(t) \in H$ for $t > 0$, so that any solution of the integral equation (**??**) is also a solution of the differential equation (**??**).

THEOREM Let $R \geq 0$ and $\tau > 0$ be given and let $u \in C([0,\tau], V)$ be a solution. If $\|u(t)\|_1 \leq R$ for $t \in [0,\tau]$, then

$$\|u(t)\|_2 \leq C(R,\tau)t^{-1/2}, \qquad t \in (0,\tau], \tag{25}$$

$$\|u_t(t)\|_s \leq C(R,\tau)t^{-1-(s-1)/2}, \qquad t \in (0,\tau],\ s = 0,1,2. \tag{26}$$

Proof will be provided in the notes.

Global existence

Assume we can provide a global a priori bound: there is R such that if $u \in C([0,T],V)$ is a solution then

$$\|u(t)\|_1 \leq R, \quad t \in [0,T]$$

Repeated application of the local existence theorem with $\tau = \tau(R)$ then proves existence for $t \in [0,T]$.

Example: Allen-Cahn equation

$$u_t - \Delta u = -(u^3 - u) = -V'(u)$$

$V(\xi) = \dfrac{1}{4}\xi^4 - \dfrac{1}{2}\xi^2$ bounded from below: $V(\xi) \geq -K$

$$(u_t, u_t) - (\Delta u, u_t) = -(V'(u), u_t)$$

$$(u_t, u_t) + (\nabla u, \nabla u_t) = -(V'(u), u_t)$$

$$\|u_t\|^2 + \frac{1}{2} D_t \|\nabla u\|^2 = -D_t \int_\Omega V(u)\, dx$$

$$\int_0^t \|u_t\|^2\, ds + \frac{1}{2}\|\nabla u(t)\|^2 = \frac{1}{2}\|\nabla u_0\|^2 - \int_\Omega V(u(t))\, dx + \int_\Omega V(u_0)\, dx \leq C$$

We conclude

$\|u(t)\|_1 \leq R, \quad t \in [0, \infty)$

Lecture 2: Finite element method

Elliptic equation

Let Ω be a convex polygonal domain in $\mathbf{R}^2$

$$\begin{aligned} \mathcal{A}u := -\nabla \cdot \big(a\nabla u\big) = f \quad x \in \Omega \\ u = 0 \quad x \in \partial\Omega \end{aligned}$$

$a = a(x)$ is smooth with $a(x) \geq a_0 > 0$ in $\bar{\Omega}$ and $f \in L_2$

Weak formulation: find $u \in V = H_0^1$ such that

$$a(u, v) = (f, v), \quad \forall v \in V = H_0^1$$

where

$$a(v, w) = \int_\Omega a\nabla v \cdot \nabla w \, dx \quad \text{and} \quad (f, v) = \int_\Omega f v \, dx$$

Elliptic equation

Poincaré's inequality

$$\|v\| \leq C\|\nabla v\| \quad \forall v \in V$$

implies

$$c\|v\|_1^2 \leq a_0\|\nabla v\|^2 \leq a(v,v) \leq C\|v\|_1^2 \quad \forall v \in V$$

so $a(\cdot,\cdot)$ is a scalar product and the norm $\|v\|_a = \sqrt{a(v,v)}$ is equivalent to the standard norm on V. Hence there is a unique solution in $u \in V$.

Ω is convex: $u \in H^2$ and

$$\|u\|_2 \leq C\|f\|.$$

Finite element

$\{K\}$ a set of closed triangles K, a triangulation, such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K, \quad h_K = \mathrm{diam}(K), \quad h = \max_{K \in \mathcal{T}_h} h_K.$$

Piecewise linear functions:

$$V_h = \big\{ v \in C(\bar{\Omega}) : v \text{ linear in } K \text{ for each } K,\, v = 0 \text{ on } \partial\Omega \big\}$$

$V_h \subset V = H_0^1$
$\{P_i\}_{i=1}^{M_h}$ the set of interior nodes
$v \in V_h$ is uniquely determined by its values at the P_j
pyramid functions $\{\Phi_i\}_{i=1}^{M_h} \subset V_h$, defined by

$$\Phi_i(P_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

forms a basis for V_h, $v(x) = \sum_{i=1}^{M_h} v_i \Phi_i(x)$, where $v_i = v(P_i)$.

Finite element method

The finite element equation: find $u_h \in V_h$ such that

$$a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h$$

Use basis:

$$u_h(x) = \sum_{i=1}^{M_h} U_i \Phi_i(x)$$

$$\sum_{j=1}^{M_h} U_j a(\Phi_j, \Phi_i) = (f, \Phi_i), \quad i = 1, \ldots, M_h,$$

Matrix form: $AU = b$, where $U = (U_i)$, $A = (a_{ij})$ is the stiffness matrix with elements $a_{ij} = a(\Phi_j, \Phi_i)$, and $b = (b_i)$ the load vector with elements $b_i = (f, \Phi_i)$.

A is symmetric and positive definite, large and sparse.

Approximation theory

Interpolation operator $I_h : C(\bar{\Omega}) \to V_h$ defined by

$$(I_h v)(x) = \sum_{i=1}^{M_h} v_i \Phi_i(x), \quad v_i = v(P_i)$$

Thus

$$(I_h v)(P_i) = v(P_i), \quad i = 1, \dots, M_h$$

Interpolation error estimates:

$$\|I_h v - v\| \le C h^2 \|v\|_2, \quad \forall v \in H^2 \cap V$$

$$\|I_h v - v\|_1 \le C h \|v\|_2, \quad \forall v \in H^2 \cap V$$

Error estimate in H^1 norm

Error $u_h - u$.

Energy norm:

$$\|v\|_a = a(v,v)^{1/2} = \Big(\int_\Omega a|\nabla v|^2 \, dx \Big)^{1/2}.$$

Standard norm:

$$\|v\|_1 = \Big(\|v\|^2 + \|\nabla v\|^2 \Big)^{1/2}$$

Theorem

$$\|u_h - u\|_a = \min_{\chi \in V_h} \|\chi - u\|_a,$$

$$\|u_h - u\|_1 \leq Ch\|u\|_2.$$

Proof

$$u \in V; \quad a(u,v) = (f,v), \quad \forall v \in V$$

$$u_h \in V_h; \quad a(u_h,\chi) = (f,\chi), \quad \forall \chi \in V_h$$

$V_h \subset V$, take $v = \chi \in V_h$ and subtract

$$a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h,$$

u_h is the orthogonal projection of u onto V_h with respect to the inner product $a(\cdot,\cdot)$

$$\|u_h - u\|_a = \min_{\chi \in V_h} \|\chi - u\|_a,$$

Equivalence of norms and interpolation error estimate:

$$\|u_h - u\|_1 \le C\|u_h - u\|_a \le C\|I_h u - u\|_a \le C\|I_h u - u\|_1 \le Ch\|u\|_2.$$

Error estimate in L_2 norm

Theorem

$$\|u_h - u\| \leq Ch^2 \|u\|_2.$$

Proof

Duality argument based on the auxiliary problem (where $e = u_h - u$)

$$\mathcal{A}\phi = e \quad \text{in } \Omega,$$
$$\phi = 0 \quad \text{on } \partial\Omega$$

Weak formulation: find $\phi \in H_0^1$ such that

$$a(w, \phi) = (w, e), \quad \forall w \in H_0^1.$$

Regularity estimate: $\|\phi\|_2 \le C\|\mathcal{A}\phi\| = C\|e\|$
Take $w = e \in H_0^1$

$$\begin{aligned}\|e\|^2 = a(e, \phi) = a(e, \phi - I_h\phi) &\le C\|e\|_1 \, \|\phi - I_h\phi\|_1 \\ &\le Ch\|e\|_1 \, \|\phi\|_2 \le Ch\|e\|_1 \, \|e\|.\end{aligned}$$

$$\|e\| \le Ch\|e\|_1 \le Ch^2\|u\|_2$$

Ritz projection

Recall $V_h \subset V$ and

$$a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h$$

u_h is the orthogonal projection of u onto V_h with respect to the inner product $a(\cdot, \cdot)$

We denote it by $R_h : V \to V_h$. It satisfies

$$a(R_h u - u, \chi) = 0, \quad \forall \chi \in V_h$$

With this notation the error estimates are:

$$\|R_h v - v\| \leq C h^2 \|v\|_2, \quad \|R_h v - v\|_1 \leq C h \|v\|_2, \quad \forall v \in H^2 \cap H_0^1$$

Also:

$$\|R_h v - v\| \leq C h \|v\|_1, \quad \forall v \in H_0^1$$

Lecture 3: Finite elements for semilinear parabolic PDE

Abstract framework

Find $u(t) \in V$ such that

$$u' + Au = f(u),\ t > 0; \quad u(0) = u_0.$$

Linear homogenous equation:

$$u' + Au = 0,\ t > 0; \quad u(0) = u_0; \qquad u(t) = E(t)u_0$$

By Duhamel's principle:

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s))\,ds, \quad t \geq 0.$$

Local existence and regularity

THEOREM For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that there is a unique solution $u \in C([0,\tau],V)$ for any initial value $u_0 \in V$ with $\|u_0\|_1 \le R_0$. Moreover, there is c such that $\|u\|_{L_\infty([0,\tau],V)} \le cR_0$.

THEOREM Let $R \ge 0$ and $\tau > 0$ be given and let $u \in C([0,\tau],V)$ be a solution. If $\|u(t)\|_1 \le R$ for $t \in [0,\tau]$, then

$$\begin{aligned} \|u(t)\|_2 &\le C(R,\tau)t^{-1/2}, && t \in (0,\tau], \\ \|u_t(t)\|_s &\le C(R,\tau)t^{-1-(s-1)/2}, && t \in (0,\tau],\ s = 0,1,2. \end{aligned}$$

Global solution on $[0,T]$ if we can prove a priori bound

$$\|u(t)\|_1 \le R, \quad t \in [0,T]$$

Finite element method

Linear elliptic equation:

$$-\nabla \cdot \big(a\nabla u\big) = f \quad x \in \Omega$$
$$u = 0 \quad x \in \partial\Omega$$

Weak formulation: find $u \in V = H_0^1$ such that

$$a(u,v) = (f,v), \quad \forall v \in V = H_0^1$$

where

$$a(v,w) = \int_\Omega a\nabla v \cdot \nabla w \, dx \quad \text{and} \quad (f,v) = \int_\Omega f v \, dx$$

The finite element equation: find $u_h \in V_h$ such that

$$a(u_h,\chi) = (f,\chi), \quad \forall \chi \in V_h$$

Error estimates

$$\|u_h - u\|_1 \leq Ch\|u\|_2$$

$$\|u_h - u\| \leq Ch^2\|u\|_2$$

Ritz projection

Recall $V_h \subset V$ and

$$a(u_h - u, \chi) = 0, \quad \forall \chi \in V_h$$

u_h is the orthogonal projection of u onto V_h with respect to the inner product $a(\cdot, \cdot)$

We denote it by $R_h : V \to V_h$. It satisfies

$$a(R_h u - u, \chi) = 0, \quad \forall \chi \in V_h$$

With this notation the error estimates are:

$$\|R_h v - v\| \leq C h^2 \|v\|_2, \quad \|R_h v - v\|_1 \leq C h \|v\|_2, \quad \forall v \in H^2 \cap H_0^1$$

Also:

$$\|R_h v - v\| \leq C h \|v\|_1, \quad \forall v \in H_0^1$$

Spatially semidiscrete approximation

$$u' + Au = f(u),\ t > 0; \quad u(0) = u_0.$$

The weak formulation: find $u(t) \in V$ such that

$$\begin{aligned} (u', v) + a(u, v) &= (f(u), v), \quad \forall v \in V,\ t > 0, \\ u(0) &= u_0, \end{aligned} \tag{27}$$

where $a(u, v) = (\nabla u, \nabla v) = (-\Delta u, v) = (Au, v)$ is the bilinear form associated with A.

Finite element spaces: $\{V_h\} \subset V$

Spatially semidiscrete finite element equation: find $u_h(t) \in V_h$ such that

$$\begin{aligned} (u_h', \chi) + a(u_h, \chi) &= (f(u_h), \chi), \quad \forall \chi \in V_h,\ t > 0, \\ u_h(0) &= u_{h,0}, \end{aligned} \tag{28}$$

where $u_{h,0} \in V_h$ is an approximation of u_0.

Abstract framework

Linear operator $A_h : V_h \to V_h$
Orthogonal projection $P_h : H \to V_h$

$$(A_h\psi, \chi) = a(\psi, \chi), \quad (P_h g, \chi) = (g, \chi) \quad \forall \psi, \chi \in V_h,\ g \in H,$$

Finite element equation becomes:

$$u_h' + A_h u_h = P_h f(u_h),\ t > 0; \quad u_h(0) = u_{h,0}.$$

A_h is self-adjoint positive definite (uniformly in h);
The Corresponding semigroup $E_h(t) = \exp(-tA_h) : V_h \to V_h$
therefore has the smoothing properties (uniformly in h):

$$\|D_t^l E_h(t)v\| = \|A_h^l E_h(t)v\| \leq C_l t^{-l}\|v\|, \quad t > 0,\ v \in V_h,\ l \geq 0.$$

Moreover, for the operator A_h we have the equivalence of norms

$$c\|v\|_1 \leq \|A_h^{1/2}v\| = \sqrt{a(v,v)} = \|A^{1/2}v\| \leq C\|v\|_1, \quad v \in V_h.$$

Abstract formulation

$\|A_h^{1/2}v\|$ controls $\|v\|_1$ and $\|v\|_1$ controls the Lipschitz constant of f.

We also have

$$\|P_h f\| \leq \|f\|, \quad f \in H,$$

and

$$\|A_h^{-1/2} P_h f\| \leq C\|f\|_{-1}, \quad f \in H, \tag{29}$$

which follows from

$$\begin{aligned}
\|A_h^{-1/2} P_h f\| &= \sup_{v_h \in V_h} \frac{|(A_h^{-1/2} P_h f, v_h)|}{\|v_h\|} = \sup_{v_h \in V_h} \frac{|(f, A_h^{-1/2} v_h)|}{\|v_h\|} \\
&= \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|A_h^{1/2} w_h\|} \leq C \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|w_h\|_1} \\
&\leq C \sup_{w \in V} \frac{|(f, w)|}{\|w\|_1} = C\|f\|_{-1}.
\end{aligned}$$

Abstract formulation

Using these inequalities we prove:

$$\|D_t^l E_h(t) P_h f\|_\beta \leq C t^{-l-(\beta-\alpha)/2} \|f\|_\alpha, \quad t > 0,\ f \in \mathcal{D}(A^{\alpha/2}),$$
$$-1 \leq \alpha \leq \beta \leq 1,\ l = 0, 1.$$

Note that the upper limit to β is 1, while it is 2 in the continuous case.

Local existence

The initial-value problem is equivalent to the integral equation

$$u_h(t) = E_h(t)u_{h,0} + \int_0^t E_h(t-s)P_h f(u_h(s))\,ds, \quad t \geq 0.$$

The proof of the previous local existence theorem carries over verbatim to the semidiscrete case. We thus have:

THEOREM For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that there is a unique solution $u_h \in C([0,\tau],V)$ for any initial value $u_{h,0} \in V_h$ with $\|u_{h,0}\|_1 \leq R_0$. Moreover, there is c such that $\|u_h\|_{L_\infty([0,\tau],V)} \leq cR_0$.

We denote by $S_h(t,\cdot)$ the corresponding (local) solution operator, so that $u_h(t) = S_h(t,u_{h,0})$ is the solution.

Local a priori error estimate

Next goal: estimate the difference between the local solutions $u(t) = S(t, u_0)$ and $u_h(t) = S_h(t, u_{h,0})$

THEOREM Let $R \geq 0$ and $\tau > 0$ be given. Let $u(t)$ and $u_h(t)$ be solutions of (**??**) and (**??**) respectively, such that $u(t), u_h(t) \in B_R$ for $t \in [0, \tau]$. Then

$$\|u_h(t) - u(t)\|_1 \leq C(R,\tau) t^{-1/2} \big(\|u_{h,0} - P_h u_0\| + h \big), \qquad t \in (0, \tau],$$

$$\|u_h(t) - u(t)\| \leq C(R,\tau) \big(\|u_{h,0} - P_h u_0\| + h^2 t^{-1/2} \big), \qquad t \in (0, \tau].$$

Local: because the constant $C(R, \tau)$ grows with τ and R.
A priori: because the error is evaluated in terms of derivatives of u, which are estimated a priori in a previous regularity theorem.

Proof

Recall the Ritz projection operator $R_h : V \to V_h$ defined by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in V_h.$$

with error bounds

$$\|R_h v - v\| + h\|R_h v - v\|_1 \leq Ch^s \|v\|_s, \quad v \in V \cap H^s(\Omega), \quad s = 1, 2$$

We divide the error into two parts:

$$e(t) \equiv u_h(t) - u(t) = \big(u_h(t) - R_h u(t)\big) + \big(R_h u(t) - u(t)\big) \equiv \theta(t) + \rho(t).$$

Then, for $j = 0, 1$ and $s = 1, 2$,

$$\begin{aligned} \|\rho(t)\|_j &\leq Ch^{s-j}\|u(t)\|_s \leq C(R,\tau)h^{s-j}t^{-(s-1)/2}, && t \in (0, \tau], \\ \|\rho_t(t)\| &\leq Ch^s \|u_t(t)\|_s \leq C(R,\tau)ht^{-1-(s-1)/2}, && t \in (0, \tau]. \end{aligned}$$

Proof

It remains to estimate $\theta(t) \in V_h$. It satisfies, for $\chi \in V_h$,

$$
\begin{aligned}
(\theta_t, \chi) + a(\theta, \chi) &= (u_{h,t}, \chi) + a(u_h, \chi) - (R_h u_t, \chi) - a(R_h u, \chi) \\
&= (u_{h,t}, \chi) + a(u_h, \chi) - (R_h u_t, \chi) - a(u, \chi) \\
&= (f(u_h) - f(u), \chi) - (R_h u_t - u_t, \chi) \\
&= (f(u_h) - f(u), \chi) - (\rho_t, \chi)
\end{aligned}
$$

equivalently

$$
\theta_t + A_h \theta = P_h\big(f(u_h) - f(u) - \rho_t\big) \tag{30}
$$

Hence, by Duhamel's principle,

$$
\theta(t) = E_h(t)\theta(0) + \int_0^t E_h(t-\sigma) P_h\Big(f(u_h(\sigma)) - f(u(\sigma)) - D_\sigma \rho(\sigma)\Big)\, d\sigma
$$

Proof

Integration by parts yields

$$
\begin{aligned}
-\int_0^{t/2} E_h(t-\sigma)P_h D_\sigma \rho(\sigma)\,d\sigma = E_h(t)P_h\rho(0) - E_h(t/2)P_h\rho(t/2) \\
+ \int_0^{t/2} \big(D_\sigma E_h(t-\sigma)\big)P_h\rho(\sigma)\,d\sigma.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\theta(t) = E_h(t)P_h e(0) - E_h(t/2)P_h\rho(t/2) + \int_0^{t/2} \big(D_\sigma E_h(t-\sigma)\big)P_h\rho(\sigma)\,d\sigma \\
- \int_{t/2}^{t} E_h(t-\sigma)P_h D_\sigma\rho(\sigma)\,d\sigma + \int_0^t E_h(t-\sigma)P_h\Big(f(u_h(\sigma)) - f(u(\sigma))\Big)\,d\sigma.
\end{aligned}
$$

Proof

Using the smoothing property of $E_h(t)P_h$, the error estimates for ρ with $j=0$, $s=1$, and the Lipschitz condition $\|f(u)-f(v)\| \le C(R)\|u-v\|_1$, we obtain

$$
\begin{aligned}
\|\theta(t)\|_1 &\le Ct^{-1/2}\big(\|P_h e(0)\| + \|\rho(t/2)\|\big) + C\int_0^{t/2}(t-\sigma)^{-3/2}\|\rho(\sigma)\|\,d\sigma\\
&\qquad + C\int_{t/2}^{t}(t-\sigma)^{-1/2}\|D_\sigma\rho(\sigma)\|\,d\sigma\\
&\qquad + C\int_0^{t}(t-\sigma)^{-1/2}\|f(u_h(\sigma))-f(u(\sigma))\|\,d\sigma\\
&\le C(R,\tau)t^{-1/2}\big(\|P_h e(0)\| + h\big)\\
&\qquad + C(R,\tau)h\bigg(\int_0^{t/2}(t-\sigma)^{-3/2}\,d\sigma + \int_{t/2}^{t}(t-\sigma)^{-1/2}\sigma^{-1}\,d\sigma\bigg)\\
&\qquad + C(R)\int_0^{t}(t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma\\
&\le C(R,\tau)t^{-1/2}\big(\|P_h e(0)\| + h\big) + C(R)\int_0^{t}(t-\sigma)^{-1/2}\|e(\sigma)\|_1\,d\sigma,
\end{aligned}
$$

for $t\in(0,\tau]$.

Proof

Since $e = \theta + \rho$ this yields

$$\|e(t)\|_1 \le C(R,\tau)t^{-1/2}\big(\|P_h e(0)\| + h\big) + C(R)\int_0^t (t-\sigma)^{-1/2}\|e(\sigma)\|_1 \, d\sigma,\ t \in (0,\tau],$$

and we use the generalized Gronwall lemma. This proves the H^1-estimate, because $P_h e(0) = u_{h,0} - P_h u_0$.

To prove the L_2 estimate we use the Lipschitz condition $\|f(u) - f(v)\|_{-1} \le C(R)\|u - v\|$ instead.

Generalized Gronwall lemma

Lemma Let the function $\varphi(t,\tau) \geq 0$ be continuous for $0 \leq \tau < t \leq T$. If

$$\varphi(t,\tau) \leq A\,(t-\tau)^{-1+\alpha} + B\int_\tau^t (t-s)^{-1+\beta}\varphi(s,\tau)\,ds, \quad 0 \leq \tau < t \leq T,$$

for some constants $A, B \geq 0$, $\alpha, \beta > 0$, then there is a constant $C = C(B,T,\alpha,\beta)$ such that

$$\varphi(t,\tau) \leq CA\,(t-\tau)^{-1+\alpha}, \quad 0 \leq \tau < t \leq T.$$

The constant in Gronwall's lemma grows exponentially with the length T of the time interval. Hence, results derived by means of this lemma are often useful only for short time intervals.

Proof

Iterating the given inequality $N-1$ times, using the identity

$$\int_\tau^t (t-s)^{-1+\alpha}(s-\tau)^{-1+\beta}\,ds = C(\alpha,\beta)\,(t-\tau)^{-1+\alpha+\beta}, \quad \alpha,\beta>0, \tag{31}$$

(Abel's integral) and estimating $(t-\tau)^\beta$ by T^β, we obtain

$$\varphi(t,\tau) \le C_1 A\,(t-\tau)^{-1+\alpha} + C_2\int_\tau^t (t-s)^{-1+N\beta}\varphi(s,\tau)\,ds, \quad 0\le\tau<t\le T,$$

where $C_1 = C_1(B,T,\alpha,\beta,N)$, $C_2 = C_2(B,\beta,N)$. We now choose the smallest N such that $-1+N\beta\ge 0$, and estimate $(t-s)^{-1+N\beta}$ by $T^{-1+N\beta}$. If $-1+\alpha\ge 0$ we obtain the desired conclusion by the standard version of Gronwall's lemma. Otherwise we set $\psi(t,\tau) = (t-\tau)^{1-\alpha}\varphi(t,\tau)$ to obtain

$$\psi(t,\tau)\le C_1 A + C_3\int_\tau^t (s-\tau)^{-1+\alpha}\psi(s,\tau)\,ds, \qquad 0\le\tau<t\le T,$$

and the standard Gronwall lemma yields $\psi(t,\tau)\le CA$ for $0\le\tau<t\le T$, which is the desired result.

Error estimate reformulated

If $S_h(t,v_h), S(t,v) \in B_R$ for $t \in [0,2\tau]$ then, for $l = 0,1$,

$$\|S_h(t,v_h) - S(t,v)\|_l \le C(R,\tau)t^{-1/2}\big(\|v_h - P_h v\| + h^{2-l}\big), \quad t \in (0,2\tau],$$

and

$$\|S_h(t,v_h) - S(t,v)\|_l \le C(R,\tau)\big(\|v_h - P_h v\| + h^{2-l}\big), \quad t \in [\tau,2\tau].$$

Time discretization

Completely discrete scheme based on the backward Euler method.

Difference quotient $\partial_t U_j = (U_j - U_{j-1})/k$

k is a time step

U_j is the approximation of $u_j = u(t_j)$ and $t_j = jk$.

The discrete solution $U_j \in V_h$ is defined by:

$$\partial_t U_j + A_h U_j = P_h f(U_j),\ t_j > 0; \quad U_0 = u_{h,0}.$$

Duhamel's principle yields

$$U_j = E_{kh}^j u_{h,0} + k \sum_{l=1}^{j} E_{kh}^{j-l-1} P_h f(U_l), \quad t_j \geq 0,$$

where $E_{kh} = (I + kA_h)^{-1}$.

Time discretization

Since A_h is self-adjoint positive definite we have (uniformly in h and k)

$$\|\partial_t^l E_{kh}^j v\| = \|A_h^l E_{kh}^j v\| \le C_l t_j^{-l} \|v\|, \quad t_j \ge t_l,\ v \in V_h,\ l \ge 0,$$

Smoothing property

$$\|\partial_t^l E_{kh}^j P_h f\|_\beta \le C t_j^{-l-(\beta-\alpha)/2} \|f\|_\alpha, \quad t_j > 0,\ f \in \mathcal{D}(A^{\alpha/2}),$$
$$-1 \le \alpha \le \beta \le 1,\ l = 0, 1.$$

Local existence and error estimates

THEOREM

For any $R_0 > 0$ there is $\tau = \tau(R_0)$ such that there is a unique solution U_j, $t_j \in [0, \tau]$, for any initial value $u_{h,0} \in V_h$ with $\|u_{h,0}\|_1 \leq R_0$. Moreover, there is c such that $\max_{t_j \in [0,\tau]} \|U_j\|_1 \leq cR_0$.

THEOREM

Let $R \geq 0$ and $\tau > 0$ be given. Let $u(t)$ and U_j be continuous and discrete solutions, such that $u(t), U_j \in B_R$ for $t, t_j \in [0, \tau]$. Then, for $k \leq k_0(R)$, we have

$$\|U_j - u(t_j)\|_1 \leq C(R,\tau)\big(\|u_{h,0} - P_h u_0\| t_j^{-1/2} + h t_j^{-1/2} + k t_j^{-1}\big), \qquad t_j \in (0,\tau],$$

$$\|U_j - u(t_j)\| \leq C(R,\tau)\big(\|u_{h,0} - P_h u_0\| + h^2 t_j^{-1/2} + k t_j^{-1/2}\big), \qquad t_j \in (0,\tau].$$

Lecture 4: Application to dynamical systems theory

Dynamical systems

Nonlinear semigroups:

$$S(t,\cdot) : V \to V$$

and

$$S_h(t,\cdot) : V_h \to V_h$$

$u(t) = S(t,v)$ is the solution of

$$u' + Au = f(u),\ t > 0; \quad u(0) = v \tag{32}$$

$u_h(t) = S_h(t,v_h)$ is the solution of

$$u_h' + A_h u_h = P_h f(u_h),\ t > 0; \quad u_h(0) = v_h \tag{33}$$

Assume that they are defined for all $t \in [0,\infty)$.

Global attractor

We assume that $S(t,\cdot)$ has a global attractor $\mathcal{A}$, i.e., $\mathcal{A}$ is a compact invariant subset of V, which attracts the bounded sets of V. Thus, for any bounded set $B \subset V$ and any $\epsilon > 0$ there is $T > 0$ such that

$$S(t,B) \subset \mathcal{N}(\mathcal{A},\epsilon), \qquad t \in [T,\infty),$$

where $\mathcal{N}(\mathcal{A},\epsilon)$ denotes the ϵ-neighborhood of $\mathcal{A}$ in V. Or equivalently,

$$\delta(S(t,B),\mathcal{A}) \to 0 \qquad \text{as } t \to \infty,$$

where $\delta(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|_1$
denotes the unsymmetric semidistance between two subsets A, B of V.
Assume that $S_h(t,\cdot)$ has a global attractor $\mathcal{A}_h$ in V_h.

THEOREM $\delta(\mathcal{A}_h,\mathcal{A}) \to 0$ as $h \to 0$.

In other words: for any $\epsilon > 0$ there is $h_0 > 0$ such that $\mathcal{A}_h \subset \mathcal{N}(\mathcal{A},\epsilon)$ if $h < h_0$.

$\mathcal{A}_h$ is upper semicontinuous at $h = 0$.

Recall: Error estimate reformulated

THEOREM

If $S_h(t, v_h), S(t, v) \in B_R$ for $t \in [0, 2\tau]$ then, for $l = 0, 1$,

$$\|S_h(t, v_h) - S(t, v)\|_l \leq C(R, \tau) t^{-1/2} \big(\|v_h - P_h v\| + h^{2-l} \big), \quad t \in (0, 2\tau],$$

and

$$\|S_h(t, v_h) - S(t, v)\|_l \leq C(R, \tau) \big(\|v_h - P_h v\| + h^{2-l} \big), \quad t \in [\tau, 2\tau].$$

Exponential stability

Let $u(t) = S(t, u_0)$.

v is a perturbed solution starting at t_0,
if $v(t) = S(t - t_0, v_0)$, $t \geq t_0$, with v_0 near $u(t_0)$.

u is exponentially stable, if there are numbers $\delta, T > 0$ such that any perturbed solution $v(t) = S(t - t_0, v_0)$ with $\|v_0 - u(t_0)\|_1 < \delta$ satisfies

$$\|v(t) - u(t)\|_1 \leq \tfrac{1}{2}\|v_0 - u(t_0)\|_1, \qquad t \in [t_0 + T, \infty).$$

Under this assumption we may prove a uniform long-time error estimate.

$$\|u_h(t) - u(t)\|_1 \leq C\big(1 + t^{-1/2}\big)h, \qquad t \in [0, \infty)$$

Linearization

Let $\bar{u} \in C([0,T],V)$ be a solution with $\|\bar{u}(t)\|_1 \leq R$, $t \in [0,T]$, for some T and R.

we rewrite the differential equation

$$u' + Au + B(t)u = F(t,u),$$

where

$$B(t) = -f'(\bar{u}(t)) \in \mathcal{L}(V,H),$$
$$F(t,v) = f(v) - f'(\bar{u}(t))v.$$

Linearized homogeneous problem:

$$v' + Av + B(t)v = 0,\ t > s; \quad v(s) = \phi \tag{34}$$

$v(t) = L(t,s)\phi$ is the solution.

Expononential dichotomy

We assume that the linear evolution operator $L(t,s)$ has an exponential dichotomy in V on the interval $J=[0,T]$.

There are projections $P(t)\in\mathcal{L}(V)$, $t\in J$,
and constants $M\geq 1$, $\beta>0$
such that, for $s,t\in J$, $t\geq s$.

1. $L(t,s)P(s)=P(t)L(t,s)$.
2. The restriction $L(t,s)|_{\mathcal{R}(I-P(s))}:\mathcal{R}(I-P(s))\to\mathcal{R}(I-P(t))$ is an isomorphism. We define $L(s,t):\mathcal{R}(I-P(t))\to\mathcal{R}(I-P(s))$ to be its inverse.
3. $\|L(t,s)P(s)\|_{\mathcal{L}(X)}\leq Me^{-\beta(t-s)}$.
4. $\|L(s,t)(I-P(t))\|_{\mathcal{L}(X)}\leq Me^{-\beta(t-s)}$.

Shadowing

THEOREM
Let $\bar{u} \in C([0,T],V)$ be a solution of with

$$\max_{t\in[0,T]} \|\bar{u}(t)\|_1 \le R,$$

for some T and R.
Assume that the solution operator $L(t,s)$ of the linearized problem has an exponential dichotomy in V on the interval $[0,T]$. Then there are numbers ρ_0 and C such that, for each solution $u_h(t) = S_h(t,u_{h,0})$, $t \in [0,T]$, with

$$\max_{t\in[0,T]} \|u_h(t) - \bar{u}(t)\|_1 \le \rho_0,$$

there is a solution u such that

$$\|u_h(t) - u(t)\|_1 \le C\big(1 + t^{-1/2}\big)h, \quad t \in (0,T].$$

The numerical solution u_h is shadowed by an exact solution u in a neighborhood of $\bar{u}$.

Open problem

Numerical shadowing: prove that shadowing can be detected “a posteriori” in the numerical computation.