Numerical Methods for the Navier-Stokes Equations

Instructor: Hong G. Im
University of Michigan
Fall 2001

Outline

What will be covered

- Summary of solution methods
  - Incompressible Navier-Stokes equations
  - Compressible Navier-Stokes equations
- High accuracy methods
  - Spatial accuracy improvement
  - Time integration methods

What will not be covered

- Non-finite difference approaches such as
  - Finite element methods (unstructured grid)
  - Spectral methods

Incompressible Navier-Stokes Equations

Incompressible Navier-Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} + \alpha \nabla^2 \mathbf{u} - \frac{\nabla p}{\rho} \qquad \mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\nabla \cdot \mathbf{u} = 0$$

The (hydrodynamic) pressure is decoupled from the rest of the solution variables. Physically, it is the pressure that drives the flow, but in practice pressure is solved such that the incompressibility condition is satisfied.

The system of ordinary differential equations (ODE's) are changed to a system of differential-algebraic equations (DAE's), where algebraic equations acts like a constraint.

Vorticity-stream function formulation

Advantages:

- Pressure does not appear explicitly (can be obtained later)
- Incompressibility is automatically satisfied (by definition of stream function)

Drawbacks:

- Limited to 2-D applications (Revised 3-D approaches are available)

Solution Methods for Incompressible N-S Equations in Primitive Formulation:

- Artificial compressibility (Chorin, 1967) – mostly steady
- Pressure correction approach – time-accurate
  - MAC (Harlow and Welch, 1965)
  - Projection method (Chorin and Temam, 1968)
  - Fractional step method (Kim and Moin, 1975)
  - SIMPLE, SIMPLER (Patankar, 1981)

Artificial Compressibility - 1

Back to a system of ODE by

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} + \alpha \nabla^2 \mathbf{u} - \frac{\nabla p}{\rho}$$

$$\frac{\partial p}{\partial t} + c^2 \nabla \cdot \mathbf{u} = 0$$

c^2 : arbitrary constant

- With properly-chosen c^2, solve until $\frac{\partial p}{\partial t} \to 0 \;\; \left(p = p(\mathbf{x})\right)$
- Originally developed for steady problems
- The term "artificial compressibility" is coined from equation of state $p = c^2 \rho$
- Possible numerical difficulties for large c^2

Artificial Compressibility - 2

The concept can be applied to a time-accurate method by using "pseudo-time stepping" at every sub-steps.

$$\beta \frac{\partial \mathbf{u}}{\partial \tau} + \frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} + \alpha \nabla^2 \mathbf{u} - \frac{\nabla p}{\rho}$$

$$\frac{\partial p}{\partial \tau} + c^2 \nabla \cdot \mathbf{u} = 0$$

At every "real" time step, take "pseudo-time stepping" using explicit time integration until $\frac{\partial \mathbf{u}}{\partial \tau} \rightarrow 0, \quad \frac{\partial p}{\partial \tau} \rightarrow 0$

Since the pseudo time scale is not physical, we can accelerate the integration however we want.

Marker-and-Cell (MAC) Method – Harlow and Welch (1965)

- Originally derived for free surface flows with staggered grid

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} + \alpha\nabla^2\mathbf{u} - \frac{\nabla p}{\rho} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0$$

Explicit integration

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = -\mathbf{u}^n\cdot\nabla_h\mathbf{u}^n + \alpha\nabla_h^2\mathbf{u}^n - \frac{\nabla_h p}{\rho}$$

Taking divergence of momentum equation,

$$\frac{\cancel{\nabla_h\cdot\mathbf{u}^{n+1}} - \cancel{\nabla_h\cdot\mathbf{u}^n}}{\Delta t} + \nabla_h\cdot(\mathbf{u}^n\cdot\nabla_h\mathbf{u}^n) = -\frac{\nabla_h^2 p}{\rho} + \alpha\nabla_h^2(\cancel{\nabla_h\cdot\mathbf{u}^n})$$

$$\nabla_h^2 p = -\rho\nabla_h\cdot(\mathbf{u}^n\cdot\nabla_h\mathbf{u}^n) \quad \text{Poisson equation}$$

Projection Method – Chorin (1968), Temam (1969)

- Originally derived on a colocated grid
- Identical to MAC except for the Poisson equation

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^{t}}{\Delta t} = -\frac{1}{\rho}\nabla_h p \quad \Rightarrow \quad \mathbf{u}^{n+1} = \mathbf{u}^{t} - \frac{\Delta t}{\rho}\nabla_h p$$

$$\nabla_h \cdot \mathbf{u}^{n+1} = 0$$

$$\cancelto{0}{\nabla_h \cdot \mathbf{u}^{n+1}} = \nabla_h \cdot \mathbf{u}^{t} - \frac{\Delta t}{\rho}\nabla_h \cdot \nabla_h p$$

$$\nabla_h^2 p = \frac{\rho}{\Delta t}\nabla_h \cdot \mathbf{u}^{t}$$

MAC vs. Projection

1. Integration without pressure

$$\mathbf{u}^n = \mathbf{u}^n$$

$$\mathbf{u}^t = \mathbf{u}^n + \Delta t\left(-\mathbf{A}^n + \mathbf{D}^n\right)$$

2. Poisson equation

$$\nabla_h^2 p = -\rho \nabla_h \cdot (\mathbf{u}^n \cdot \nabla_h \mathbf{u}^n)$$

$$\nabla_h^2 p = \frac{\rho}{\Delta t} \nabla_h \cdot \mathbf{u}^t$$

3. Projection into incompressible field

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\left(-\mathbf{A}^n + \mathbf{D}^n\right) - \frac{\Delta t}{\rho} \nabla_h p$$

$$\mathbf{u}^{n+1} = \mathbf{u}^t - \frac{\Delta t}{\rho} \nabla_h p$$

SIMPLE Algorithm – Patankar (1981)

(Semi-Implicit Method for Pressure Linked Equations)

- Iterative procedure with pressure correction $p = p_0 + p'$

1. Guess the pressure field p_0
2. Solve the momentum equation (implicitly)

$$\frac{\mathbf{u}_0 - \mathbf{u}^n}{\Delta t} = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \alpha \nabla^2 \mathbf{u}_0 - \frac{\nabla p_0}{\rho}$$

3. Solve the pressure correction equation

$$\nabla^2 p' = \frac{\rho}{\Delta t} \left(\nabla \cdot \mathbf{u}_0 \right)$$

4. Correct the pressure and velocity

$$p = p_0 + p'$$

$$\mathbf{u} = \mathbf{u}_0 - \frac{\Delta t}{\rho} \nabla p'$$

5. Go to 2. Repeat the process until the solution converges.

Notes:

- Originally developed for the staggered grid system.
- The corrected velocity field satisfies the continuity equation *even if* the pressure correction is only approximate.
- Sometimes p' tends to be overestimated

$$p = p_0 + \omega p' \quad (\omega \approx 0.8) \quad \text{underrelaxation}$$

SIMPLER (SIMPLE Revised)

- Incorporating the projection method (fractional step)

1. Guess the velocity field $\mathbf{u}_0$
2. Solve momentum equation (implicitly) **without pressure**

$$\frac{\hat{\mathbf{u}} - \mathbf{u}_0}{\Delta t} = -\hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} + \alpha \nabla^2 \hat{\mathbf{u}}$$

3. Solve the pressure Poisson equation

$$\nabla^2 p^* = \frac{\rho}{\Delta t} \left(\nabla \cdot \hat{\mathbf{u}} \right)$$

Pressure Correction Method - 7

5. Solve the momentum equation with p^*

$$\frac{\mathbf{u}^* - \mathbf{u}_0}{\Delta t} = -\mathbf{u}^* \cdot \nabla \mathbf{u}^* + \alpha \nabla^2 \mathbf{u}^* - \frac{1}{\rho} \nabla p^*$$

6. Pressure correction equation

$$\nabla^2 p' = \frac{\rho}{\Delta t} \left(\nabla \cdot \mathbf{u}^* \right)$$

7. Correct the velocity, but not the pressure

$$\mathbf{u} = \mathbf{u}^* - \frac{\Delta t}{\rho} \nabla p'$$

8. Go to 2. Repeat the process until solution is converged.

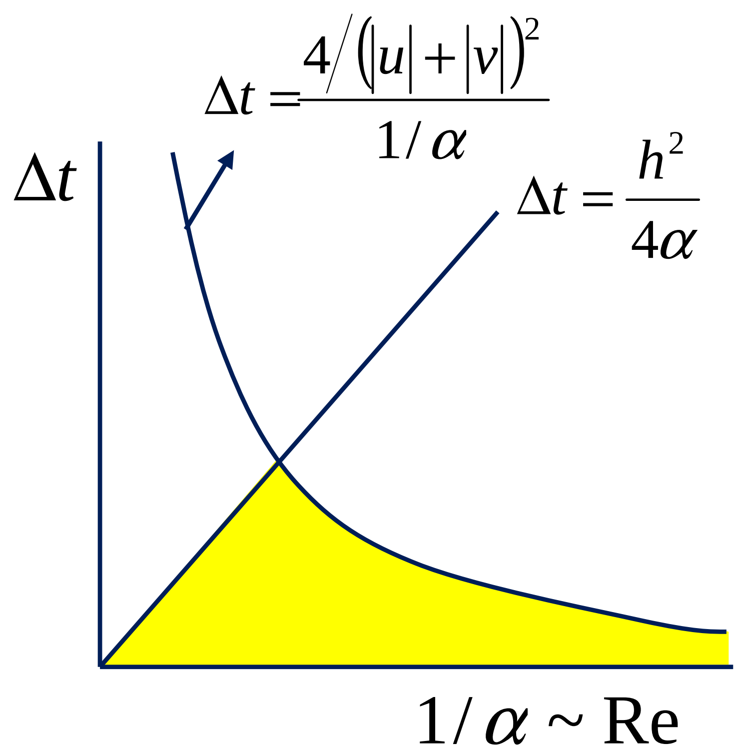
Explicit time integration in 2-D requires the stability condition:

$$\Delta t < \frac{4\alpha}{(|u| + |v|)^2} \quad \text{and} \quad \Delta t < \frac{h^2}{4\alpha}$$

High-Re flow: advection-controlled

Low-Re flow: diffusion-controlled

Use implicit schemes for appropriate terms!



$\Delta t \to 0$ at both limits!

Spatial Accuracy

- Explicit differencing - use larger stencils

$$f_j' = \frac{f_{j+1} - f_{j-1}}{2h} + O(h^2)$$

$$f_j' = \frac{f_{j-2} - 8f_{j-1} + 8f_{j+1} - f_{j+2}}{12h} + O(h^4)$$

- Tridiagonal - Padé (compact) schemes

$$f_{j-1}' + 4f_j' + f_{j+1}' = \frac{3}{h}\left(f_{j+1} - f_{j-1}\right) + O(h^4)$$

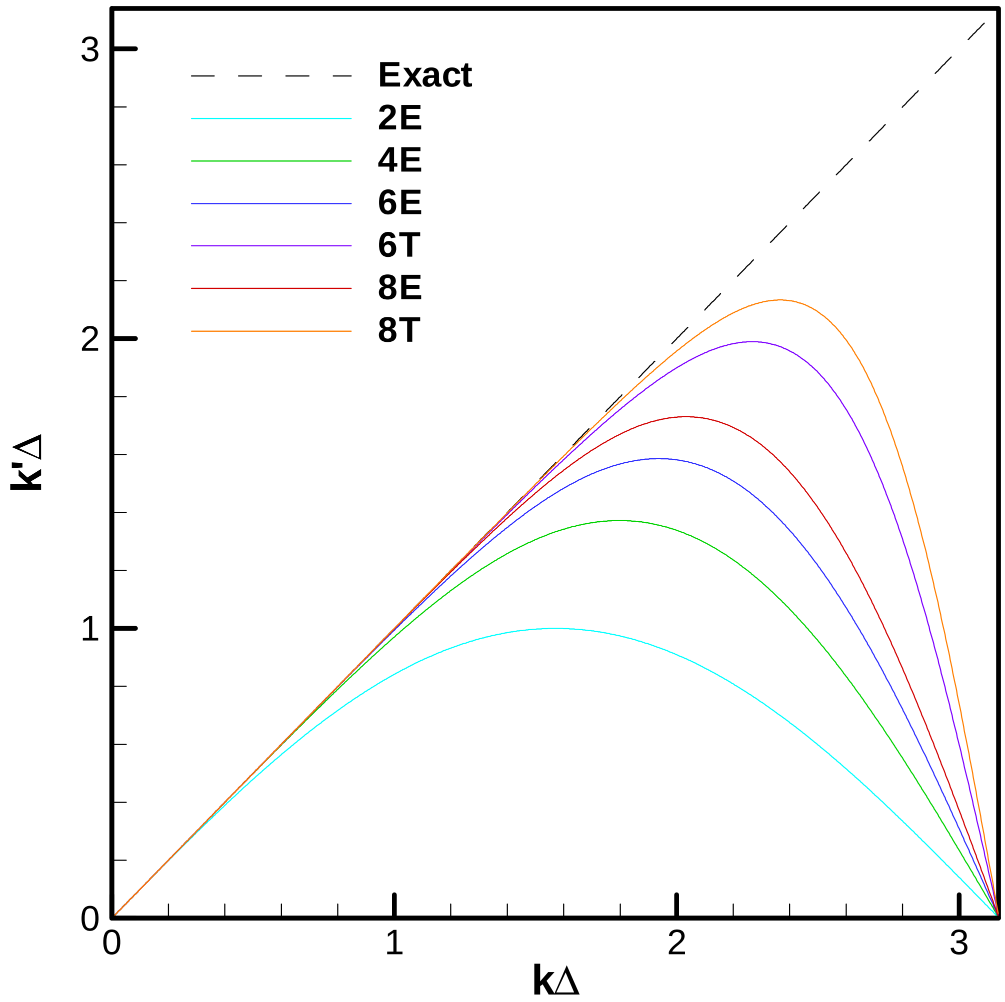
- Pentadiagonal

$$\alpha f_{j-2}' + \beta f_{j-1}' + \gamma f_j' + \delta f_{j+1}' + \varepsilon f_{j+2}' = af_{j-2} + bf_{j-1} + cf_j + df_{j+1} + ef_{j+2} + \cdots$$

WPI

Computational Fluid Dynamics I

Accuracy Improvement – Spatial 2



Ref: Kennedy, C. A. and Carpenter, M. H.,

Applied Numerical Mathematics, 14, pp. 397-433 (1994) .

Temporal Accuracy

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{A}(\mathbf{u}) + \mathbf{D}(\mathbf{u}) - \frac{1}{\rho}\nabla p$$

- Implicit – Crank-Nicolson

$$\mathbf{u}^{n+1} - \mathbf{u}^{n} = \frac{\Delta t}{2}\left[-\Big(\mathbf{A}(\mathbf{u}^{n}) + \mathbf{A}(\mathbf{u}^{n+1})\Big) + \alpha(\nabla^2\mathbf{u}^{n} + \nabla^2\mathbf{u}^{n+1})\right] - \frac{\Delta t}{\rho}\nabla p^{n+1/2}$$

Nonlinear advection term requires iteration.

Linearization of Advection Terms

- For example, a 2-D equation

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{F}}{\partial y}=0$$

$$\mathbf{U}=\begin{pmatrix}\rho \\ \rho u \\ \rho v\end{pmatrix} \quad \mathbf{E}=\begin{pmatrix}\rho u \\ \rho u^2+p-\tau_{xx} \\ \rho uv-\tau_{xy}\end{pmatrix} \quad \mathbf{F}=\begin{pmatrix}\rho v \\ \rho uv-\tau_{xy} \\ \rho v^2+p-\tau_{yy}\end{pmatrix}$$

can be linearized as

$$\frac{\partial \mathbf{U}}{\partial t}+[A]\frac{\partial \mathbf{U}}{\partial x}+[B]\frac{\partial \mathbf{U}}{\partial y}=0$$

where $[A]=\frac{\partial \mathbf{E}}{\partial \mathbf{U}},\ [B]=\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$ Jacobian matrix

Fractional Step Method – Kim & Moin (1985)

- Projection method extended to higher accuracy

$$\frac{\mathbf{u}^t - \mathbf{u}^n}{\Delta t} = -\frac{1}{2}\left[3\mathbf{A}(\mathbf{u}^n) - \mathbf{A}(\mathbf{u}^{n-1})\right] + \frac{1}{2\,\mathrm{Re}}\nabla^2(\mathbf{u}^t + \mathbf{u}^n)$$

Adams-Bashforth (AB2) **Crank-Nicolson**

$$\left.\begin{aligned} &\frac{\mathbf{u}^{n+1} - \mathbf{u}^t}{\Delta t} = -\nabla\phi \\ &\nabla \cdot \mathbf{u}^{n+1} = 0 \end{aligned}\right\} \quad \nabla^2\phi = \frac{1}{\Delta t}\nabla \cdot \mathbf{u}^t$$

Note that ϕ is different from the original pressure

$$p = \phi - \frac{\Delta t}{2\,\mathrm{Re}}\nabla^2\phi$$

Treatment of implicit viscous terms

$$\frac{\mathbf{u}^t - \mathbf{u}^n}{\Delta t} = -\frac{1}{2}\left[3\mathbf{A}(\mathbf{u}^n) - \mathbf{A}(\mathbf{u}^{n-1})\right] + \frac{1}{2\,\mathrm{Re}}\nabla^2(\mathbf{u}^t + \mathbf{u}^n)$$

$$\Rightarrow \left(1 - \frac{\Delta t}{2\,\mathrm{Re}}\delta_{xx} - \frac{\Delta t}{2\,\mathrm{Re}}\delta_{yy} - \frac{\Delta t}{2\,\mathrm{Re}}\delta_{zz}\right)\left(\mathbf{u}^t - \mathbf{u}^n\right) =$$

$$-\frac{\Delta t}{2}\left[3\mathbf{A}(\mathbf{u}^n) - \mathbf{A}(\mathbf{u}^{n-1})\right] + \frac{\Delta t}{\mathrm{Re}}\left(\delta_{xx} + \delta_{yy} + \delta_{zz}\right)\mathbf{u}^n$$

Factorizing,

$$\left(1 - \frac{\Delta t}{2\,\mathrm{Re}}\delta_{xx}\right)\left(1 - \frac{\Delta t}{2\,\mathrm{Re}}\delta_{yy}\right)\left(1 - \frac{\Delta t}{2\,\mathrm{Re}}\delta_{zz}\right)\left(\mathbf{u}^t - \mathbf{u}^n\right) =$$

$$-\frac{\Delta t}{2}\left[3\mathbf{A}(\mathbf{u}^n) - \mathbf{A}(\mathbf{u}^{n-1})\right] + \frac{\Delta t}{\mathrm{Re}}\left(\delta_{xx} + \delta_{yy} + \delta_{zz}\right)\mathbf{u}^n$$

TDMA in three directions

Notes on Fractional Step Method

- Originally implemented into a staggered grid system
- Later improved with 3rd-order Runge-Kutta method

  Ref: Le & Moin, *J. Comp. Phys.*, 92:369 (1991)
- The method can be applied to a variable-density problem (e.g. subsonic combustion, two-phase flow) where Poisson equation becomes

$$\nabla^2 \phi = \frac{1}{\Delta t}\left[\nabla \cdot \left(\rho^t \mathbf{u}^t\right) + \frac{\partial \rho^t}{\partial t}\right] \qquad \rho T = 1 \text{ Eq. of State}$$

Ref: Rutland, *Ph. D. Thesis*, Stanford University (1989)

Bell, Collela and Glaz, *JCP,* 85:257 (1989)

Boundary Conditions for Incompressible Flows

- In general, boundary condition treatment is easier than for the compressible flow formulation due to the absence of acoustics
- Typical boundary conditions:
  - Periodic: $f_N = f_1, \; f_{N+1} = f_2$ etc.
  - Inflow conditions: $f(x=0) = F(y,z,t)$
  - Outflow conditions: convective outflow condition

$$\mathbf{u}_t + U\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{at} \quad x = L \qquad \left[U = \frac{1}{A}\int u dA\right]$$

Compressible
Navier-Stokes Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0$$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho u v - \tau_{xy} \\ \rho u w - \tau_{xz} \\ (\rho E + p)u + \psi_x \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} \rho v \\ \rho u v - \tau_{xy} \\ \rho v^2 + p - \tau_{yy} \\ \rho v w - \tau_{yz} \\ (\rho E + p)v + \psi_y \end{pmatrix} \quad \mathbf{G} = \begin{pmatrix} \rho w \\ \rho u w - \tau_{xz} \\ \rho v w - \tau_{yz} \\ \rho v w^2 + p - \tau_{zz} \\ (\rho E + p)w + \psi_z \end{pmatrix}$$

where

$$\psi_x = -u\tau_{xx} - v\tau_{xy} - w\tau_{xz} + q_x$$

$$\psi_y = -u\tau_{xy} - v\tau_{yy} - w\tau_{yz} + q_y$$

$$\psi_z = -u\tau_{xz} - v\tau_{yz} - w\tau_{zz} + q_z$$

Constitutive relations

$$p = \rho R T,\ e = c_v T,\ h = c_p T$$

or

$$c_v = \frac{R}{\gamma - 1},\ p = (\gamma - 1)\rho e,\ T = \frac{(\gamma - 1)e}{R}$$

Solution methods for compressible N-S equations

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z}$$

follows the same techniques used for hyperbolic equations

For smooth solutions with viscous terms, central differencing usually works.

$\Rightarrow$ No need to worry about upwind method, flux-splitting, TVD, FCT (flux-corrected transport), etc.

In general, upwind-like methods introduces numerical dissipation, hence provides stability, but accuracy becomes a concern.

Explicit Methods

- MacCormack method
- Leap frog/DuFort-Frankel method
- Lax-Wendroff method
- Runge-Kutta method

Implicit Methods

- Beam-Warming scheme
- Runge-Kutta method

Most methods are 2nd order.
The Runge-Kutta method can be easily tailored to higher order method (both explicit and implicit).

Most of the time, an implicit integration method involves nonlinear advection terms

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z}$$

which are linearized as

$$\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n}}{\Delta t} = [A]\frac{\partial \mathbf{U}^{n+1}}{\partial x} + [B]\frac{\partial \mathbf{U}^{n+1}}{\partial y} + [C]\frac{\partial \mathbf{U}^{n+1}}{\partial z}$$

$$[A] = \left(\frac{\partial \mathbf{E}}{\partial \mathbf{U}}\right)^{n}, \quad [B] = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)^{n}, \quad [C] = \left(\frac{\partial \mathbf{G}}{\partial \mathbf{U}}\right)^{n}$$

+ ADI, factorization, etc.

Ultimately, compressible Navier-Stokes equations can be written as a system of ODE's

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z}$$

$$\Rightarrow \quad \frac{d\mathbf{U}}{dt} = F\left(t, \mathbf{U}(t)\right)$$

$$\mathbf{U}(t = t_0) = \mathbf{U}_0$$ Initial condition

Solution techniques for a system of ODE applies.

- Explicit vs. Implicit (Nonstiff vs. Stiff)
- Multi-stage vs. Multi-step

Boundary Conditions for Compressible Flows

- In general, boundary condition for the compressible flow is trickier because all the acoustic waves must be properly taken care of at the boundaries.
- Typical boundary conditions:
  - Periodic: still easy to implement
  - Both inflow and outflow conditions require treatment of characteristic waves
    (hard-wall, nonreflecting, sponge, etc).