On the Reduction From the Quaternion Interval System $[X] = [A][X] + [B]$ to Real Interval System *

Yifan Wei, Zhenglin Fang

School of Mathematics, China University of Mining and Technology, Xuzhou 221116, P.R. China

Abstract

In this article we tried to solve the quaternion interval linear equation $[X] = [A][X] + [B]$ by transforming it into an equivalent real interval linear equation.

Keywords: Interval Linear System; Quaternion

1 Introduction

Many practical problems led to the interval linear equation

$$[X] = [A][X] + [B], \tag{1.1}$$

where $[A]$ is an interval matrix and $[B]$ is an interval column vector (see Preliminaries below), and $[X]$ is the unknown. In general, there are different types of solutions [4], we use the *united solution* to be our solution set in this article. Existence of a solution when $[A]$ and $[B]$ consist of real intervals is completely clarified [1].

In [2], the authors introduced a way to deal with the situation where both $[A]$ and $[B]$ are made from quaternion matrix by transform it into an equivalent real interval linear equation. What we are doing in this paper is to try another way to deduce Eq.1.1 when both $[A]$ and $[B]$ are made from quaternion intervals into an equivalent real interval linear equation.

2 Preliminaries

2.1 Quaternion numbers and matrices

2.1.1 *quaternion numbers*

Quaternions are numbers of the form

$$\mathbf{q} = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k},\ (q_0, q_1, q_2, q_3) \in \mathbb{R}^4,$$

where $\boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}$ are three imaginary units with$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. It's convenient to denote a quaternion as a pair of a scalar and a 3-dimensional vector $\mathbf{q} = [q_0,\ q] = [q_0,\ (q_1, q_2, q_3)]$, with

$$\boldsymbol{i} = [0,\ (1, 0, 0)], \quad \boldsymbol{j} = [0,\ (0, 1, 0)], \quad \boldsymbol{k} = [0,\ (0, 0, 1)]$$

and

- *Addition*:
$$\mathbf{p}+\mathbf{q} := [p_0+q_0,\ p+q],$$
- *Multiplication*:
$$\mathbf{p}\cdot\mathbf{q} := [p_0q_0 - p\cdot q,\ p_0q + q_0p + p\times q],$$

which makes quaternions form a skew field. Another way to denote a quaternion is $\mathbf{q} = c_1 + c_2\boldsymbol{j}$ where $c_1 = q_0 + q_1\boldsymbol{i}$ and $c_2 = q_2 + q_3\boldsymbol{i}$ are two complex numbers.

Quaternions also have matrix representations. Let $\mathbb{H}$ be the skew field of quaternions and define the representation map $C : \mathbb{H} \to \mathbb{C}^{2\times 2}$ as

$$C(\mathbf{q}) = C(c_1 + c_2\boldsymbol{j}) := \begin{pmatrix} c_1 & c_2 \\ -\overline{c_2} & \overline{c_1} \end{pmatrix},$$

the bar over complex numbers stands for complex conjugation. One can easily check that C is a bijection, and

Theorem 2.1. *C is an isomorphism between $\mathbb{H}$ and $\mathbb{C}^{2\times 2}$.*

$$\begin{aligned} C(\boldsymbol{p}+\boldsymbol{q}) &= C(\boldsymbol{p}) + C(\boldsymbol{q}), \\ C(\boldsymbol{p}\cdot\boldsymbol{q}) &= C(\boldsymbol{p})\cdot C(\boldsymbol{q}), \end{aligned}$$

which transforms the addition and multiplication of quaternions into addition and multiplication of complex matrices.

Similarly, we can define the matrix representation of a complex number $R : \mathbb{C} \to \mathbb{R}^{2\times 2}$ as

$$R(c) = R(\mathrm{Re}(c) + \mathrm{Im}(c)\boldsymbol{i}) := \begin{pmatrix} \mathrm{Re}(c) & \mathrm{Im}(c) \\ -\mathrm{Im}(c) & \mathrm{Re}(c) \end{pmatrix},$$

which has similar properties

Theorem 2.2. *R is an isomorphism between $\mathbb{C}$ and $\mathbb{R}^{2\times 2}$*

$$\begin{aligned} R(c_1 + c_2) &= R(c_1) + R(c_2), \\ R(c_1\cdot c_2) &= R(c_1)\cdot R(c_2) \end{aligned}$$

Many other algebra properties of quaternions have been introduced by its discoverer Hamilton [3].

2.1.2 *quaternion matrices*

A quaternion matrix A of dimension $m\times n$ is a matrix which all entries of its m rows and n columns are quaternions, and can be uniquely expressed as $\mathbf{Q} = Q_0 + Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$ with Q_0, Q_1, Q_2, Q_3 being real matrices of dimension $m\times n$. Alternatively, a quaternion matrix can also be uniquely expressed as $\mathbf{Q} = C_1 + C_2\boldsymbol{j}$ where C_1, C_2 are two complex matrices of the same dimension of $\mathbf{Q}$.

The representation of quaternions can be naturally extended to its matrices using the convention of block matrices

$$C(\mathbf{Q}) = C(C_1 + C_2\boldsymbol{j}) := \begin{pmatrix} C_1 & C_2 \\ -\overline{C_2} & \overline{C_1} \end{pmatrix},$$

where the conjugation is taking component-wise. This extension making C a bijection from $\mathbb{H}^{m\times n}$ to $\mathbb{C}^{2m\times 2n}$.

So can the representation of complex numbers extend to real matrices

$$R(C) = R(\mathrm{Re}(C) + \mathrm{Im}(C)\boldsymbol{i}) := \begin{pmatrix} \mathrm{Re}(C) & \mathrm{Im}(C) \\ -\mathrm{Im}(C) & \mathrm{Re}(C) \end{pmatrix}.$$

2.2 Interval numbers and matrices

2.2.1 *real intervals and matrices*

A real interval $[X]$ is a set $\{x \in \mathbb{R} \mid \underline{X} \leq x \leq \overline{X}\}$. With basic arithmetic operations of intervals defined as an extension of real arithmetic

$$[X] \,\mathrm{op}\, [Y] := \{x \text{ op } y \mid x \in [X],\ y \in [Y]\},$$

where op stands for addition, subtraction, multiplication or division(when $0 \notin [Y]$).

When the upper limit and lower limit of an interval coincides we call it a **degenerated** interval, under this convention one can see that $[0] := [0, 0]$ serves like the zero element in real numbers arithmetic because $[X] + [0] = [X]$ and $[X] \cdot [0] = [0]$. Similarly $[1] := [1, 1]$ serves like 1 in real numbers multiplication.

However, here is one thing very different from real arithmetic

Theorem 2.3. *In interval number arithmetic there are no inverse elements for non-degenerate intervals. That is, for a non-degenerate interval $[X]$ there is no interval $[Y]$ such that $[X] + [Y] = [0]$ or $[X] \cdot [Y] = [1]$.*

Proof. Consider the **width** of an interval $w([X]) := \overline{X} - \underline{X}$. Select a $y \in [Y]$, it follows that $[X] + y \subseteq [X] + [Y]$ and $w([X] + [Y]) \geq w([X] + y) = w([X]) > 0$.

Similarly $w([X] \cdot [Y]) \geq w([X] \cdot y)$ for certain $y \in [Y]$. If $y \neq 0$, then we have $w([X] \cdot y) > 0$. And if $y = 0$ we have $0 \in [X] \cdot [Y] \neq [1]$. □

Addition and multiplication are commutative and associative but multiplication is not distributive over addition. For example $[0, 1] \cdot ([-1, 0] + 1) = [0, 1] \cdot [0, 1] = [0, 1]$, however $[0, 1] \cdot [-1, 0] + [0, 1] = [-1, 1] \neq [0, 1]$. Actually in general, we have

Theorem 2.4. *Interval multiplication is* **subdistributive** *over addition i.e. $[X] \cdot ([Y] + [Z]) \subseteq [X] \cdot [Y] + [X] \cdot [Z]$.*

Proof. From definition

$$\begin{aligned}
[X] \cdot ([Y] + [Z]) &= \bigcup_{x \in [X]} x([Y] + [Z]) = \bigcup_{x \in [X]} (x[Y] + x[Z]) \\
&\subseteq \bigcup_{x_1 \in [X]} x_1[Y] + \bigcup_{x_2 \in [X]} x_2[Z] = [X] \cdot [Y] + [X] \cdot [Z].
\end{aligned}$$

□

As an analog of real matrices, an interval matrix is a matrix with its all entries being interval numbers. Addition and multiplication are defined similarly. But contrary to real matrices

Corollary 2.5. *Multiplication of interval matrices is not associative.*

Thus solving interval linear equations directly can be very difficult. For more information on **interval analysis**, see [5] and [4].

2.2.2 *Complex intervals, quaternion intervals and their matrices*

A complex interval is a set $[X]_{\mathbb{C}}$ of the form

$$[X]_{\mathbb{C}} = [A] + [B]\boldsymbol{i} := \{a + b\boldsymbol{i} \mid a \in [A], b \in [B]\},$$

where $[A]$ is the real part of $[X]_{\mathbb{C}}$ denoted by $\mathrm{Re}([X]_{\mathbb{C}})$ and $[B]$ is the imaginary part of $[X]_{\mathbb{C}}$ denoted by $\mathrm{Im}([X]_{\mathbb{C}})$.

Addition and multiplication are simple substitutions of real intervals for real numbers, e.g.

$$\begin{aligned}[X]_{\mathbb{C}} \cdot [Y]_{\mathbb{C}} :=& \mathrm{Re}([X]_{\mathbb{C}}) \cdot \mathrm{Re}([Y]_{\mathbb{C}}) - \mathrm{Im}([X]_{\mathbb{C}}) \cdot \mathrm{Im}([Y]_{\mathbb{C}}) \\ &+ (\mathrm{Re}([X]_{\mathbb{C}}) \cdot \mathrm{Im}([Y]_{\mathbb{C}}) + \mathrm{Im}([X]_{\mathbb{C}}) \cdot \mathrm{Re}([Y]_{\mathbb{C}}))\boldsymbol{i}.\end{aligned}$$

Complex matrices are defined through the same analogy. So are quaternion intervals and matrices.

3 Our Solution

For a quaternion interval matrix $[A]_{\mathbb{H}} = [A_1]_{\mathbb{C}} + [A_2]_{\mathbb{C}}\boldsymbol{j}$ where $[A_1]_{\mathbb{C}}$ and $[A_2]_{\mathbb{C}}$ are two complex interval matrices, define

$$\Phi([A]_{\mathbb{H}}) = \begin{pmatrix} [A_1]_{\mathbb{C}} \\ -\overline{[A_2]_{\mathbb{C}}} \end{pmatrix}$$

to be the first column of the complex interval matrix representation of $[A]_{\mathbb{C}}$.

Then we have

Theorem 3.1. *Φ is a bijection and*

$$\begin{aligned}\Phi([A]_{\mathbb{H}} + [B]_{\mathbb{H}}) &= \Phi([A]_{\mathbb{H}}) + \Phi([B]_{\mathbb{H}}), \\ \Phi([A]_{\mathbb{H}}[B]_{\mathbb{H}}) &= C([A]_{\mathbb{H}})\Phi([B]_{\mathbb{H}}).\end{aligned}$$

Therefore the quaternion interval linear equation 1.1 can be transformed by taking Φ on both sides of the equation

$$\begin{aligned}\Phi(X) &= \Phi([A]_{\mathbb{H}}X + [B]_{\mathbb{H}}) \\ &= \Phi([A]_{\mathbb{H}}X) + \Phi([B]_{\mathbb{H}}) \\ &= C([A]_{\mathbb{H}})\Phi(X) + \Phi([B]_{\mathbb{H}}),\end{aligned}$$

Take one step further, define

$$\phi([A]_{\mathbb{C}}) := \phi([A_1] + [A_2]\boldsymbol{i}) := \begin{pmatrix} [A_1] \\ -[A_2] \end{pmatrix}$$

to be the first column of the real interval matrix representation of the complex interval matrix $[A]_{\mathbb{C}}$.

ϕ shares similar properties with Φ,

$$\begin{aligned}\phi([A]_{\mathbb{C}} + [B]_{\mathbb{C}}) &= \phi([A]_{\mathbb{C}}) + \phi([B]_{\mathbb{C}}), \\ \phi([A]_{\mathbb{C}}[B]_{\mathbb{C}}) &= \phi(([A_1][B_1] - [A_2][B_2]) + ([A_1][B_2] + [A_2][B_1])\boldsymbol{i}), \\ &= R([A]_{\mathbb{C}})\phi([B]_{\mathbb{C}})\end{aligned}$$

leading to the desired equivalent real interval linear equation,

$$X = R \circ C([A]_{\mathbb{H}})X + \phi \circ \Phi([B]_{\mathbb{H}}).$$

References

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