雨滴淋湿马路问题

Q: 一维的马路长1m，雨滴直径0.01m，问平均需要多少雨滴才能把马路淋湿?

1 基本假设

假设x为马路的长度，y为单个雨滴可以覆盖的长度，马路为0到x，而雨滴中心的范围为$-y/2$ 到$x+y/2$。
我们需要求的是长度为x_0的马路被全部浇湿所用的期望雨滴数，设为$T(x_0,1)$。
其中$T(x_1,\ldots,x_n,n)$为n段干路面的期望雨滴数目，其中未湿段的长度分别为$x_1,\ldots,x_n$。
我们有边界条件$T(0,\ldots,0,n)=n\sum_{k=1}^{n}\frac{1}{k}$。比如，$T(0,0,2)=2\left(1+\frac{1}{2}\right)=3$。

2 $x_i \le y$

2.1 一段未湿马路

首先考虑简单的例子，$x\le y$，即这段马路不会被一个雨滴分割为两段，所以我们有

$$T(x,1)=\underbrace{1/p}_{\text{新淋湿路面的平均时间}}+\underbrace{2}_{\text{对称性}}\int_{-y/2}^{x-y/2}\frac{1}{x+y}\left(\underbrace{T(x-w-y/2,1)}_{\text{问题变为长度为}x-w-y/2\text{的马路}}\right)dw \tag{1}$$

其中$p=\frac{x+y}{x_0+y}$。所以

$$T(x,1)=\frac{x_0+y}{x+y}+2\int_{-y/2}^{x-y/2}\frac{1}{x+y}T(x-w-y/2,1)dw \tag{2}$$

换元得

$$T(x,1)=\frac{x_0+y}{x+y}+\frac{2}{x+y}\int_0^x T(w,1)dw \tag{3}$$

平衡方程式得

$$(x+y)T(x)=x_0+y+2\int_0^x T(w)dw$$

同时微分两侧得

$$(x+y)T'(x)+T(x)=2T(x)$$

所以

$$T(x)=C(x+y) \tag{4}$$

进而可得

$$T(x,1)=\frac{(x_0+y)(x+y)}{y^2} \tag{5}$$

现在使$x_0=x=y$，我们有

$$T(y,1)=\frac{4y^3}{y^3}=4. \tag{6}$$

2.2 两段未湿马路

考虑淋湿为两段的情况，假设$x_1,x_2\le y$, 有$p=\frac{x_1+x_2+2y}{x_0+y}$

$$\begin{aligned}T(x_1,x_2,2)=\frac{1}{p}+\frac{2(x_1+y)}{x_1+x_2+2y}&\left(\int_{-y/2}^{x_1-y/2}\frac{1}{x_1+y}T(x_1-w-y/2,x_2,2)dw+\int_{x_1-y/2}^{x_1/2}\frac{1}{x_1+y}T(x_2,1)dw\right)\\&+\frac{2(x_2+y)}{x_1+x_2+2y}\left(\int_{-y/2}^{x_2-y/2}\frac{1}{x_2+y}T(x_1,x_2-w-y/2,2)dw+\int_{x_2-y/2}^{x_2/2}\frac{1}{x_2+y}T(x_1,1)dw\right)\end{aligned}$$

所以有

$$T(x_1,x_2,2)=\frac{x_0+y}{x_1+x_2+2y}+\frac{2}{x_1+x_2+2y}\left(\int_0^{x_1}T(w,x_2,2)dw+\frac{y-x_1}{2}T(x_2,1)\right)$$
$$+\frac{2}{x_1+x_2+2y}\left(\int_0^{x_2}T(x_1,w,2)dw+\frac{y-x_2}{2}T(x_1,1)\right)$$

进而有

$$(x_1+x_2+2y)\,T(x_1,x_2,2)=x_0+y+2\left(\int_0^{x_1}T(w,x_2,2)dw+\frac{y-x_1}{2}T(x_2,1)\right)$$
$$+2\left(\int_0^{x_2}T(x_1,w,2)dw+\frac{y-x_2}{2}T(x_1,1)\right)\qquad(*)$$

进而有

$$T(x_1,x_2,2)=\frac{x_0+y}{y}\left(\frac{x_1+x_2+2y}{y}-\frac{1}{4}\left(1+\frac{(x_1+y)(x_2+y)}{y^2}\right)\right)=\frac{x_0+y}{4y^3}\left(15y^2-(3y-x_1)(3y-x_2)\right)\qquad(7)$$

2.3 n段末湿路段

$$\begin{aligned}&T(x_1,\ldots,x_n,n)\\=&\frac{x_0+y}{\sum_{i=1}^n x_i+ny}\\&+\sum_{i=1}^n\frac{2}{\sum_{i=1}^n x_i+ny}\left(\int_{-y/2}^{x_i-y/2}T(x_1,\ldots,x_i-w-y/2,\ldots,x_n,n)dw+\int_{x_i-y/2}^{x_i/2}T(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n,n-1)dw\right)\end{aligned}$$

所以

$$\begin{aligned}&\left(\sum_{i=1}^n x_i+ny\right)T(x_1,\ldots,x_n,n)\\=&x_0+y+2\sum_{i=1}^n\left(\int_0^{x_i}T(x_1,\ldots,w,\ldots,x_n,n)dw+\frac{y-x_i}{2}T(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n,n-1)\right)\qquad(*)\end{aligned}$$

所以

$$T(x_1,x_2,x_3,3)=\frac{x_0+y}{y}\left(\frac{2}{27}\frac{(x_1+y)(x_2+y)(x_3+y)}{y^3}+\frac{28}{27}\frac{x_1+x_2+x_3+3y}{y}-\frac{65}{108}\right.$$
$$\left.-\frac{1}{4y^2}\left((x_1+y)(x_2+y)+(x_1+y)(x_3+y)+(x_2+y)(x_3+y)\right)\right)$$

3 $x\geq y$

3.1 $y\leq x\leq 2y$

如果假设$x\geq y$，并且$y\leq x\leq 2y$

$$T(x,1)=\frac{x_0+y}{x+y}+\underbrace{2}_{\text{对称性}}\int_{-y/2}^{y/2}\frac{1}{x+y}\underbrace{T(x-w-y/2)}_{\text{问题变为长度为}x-w-y/2\text{的马路}}dw+\underbrace{2}_{\text{对称性}}\int_{y/2}^{x/2}\frac{1}{x+y}T(w-y/2,x-w-y/2,2)dw$$

且$w-y/2,x-w-y/2\leq 1$。

$$\begin{aligned}&(x+y)T(x,1)\\=&x_0+y+2\int_{-y/2}^{y/2}T(x-w-y/2,1)dw+2\int_{y/2}^{x/2}T(w-y/2,x-w-y/2,2)dw\\=&x_0+y+2\int_{x-3y/2}^{y/2}T(x-w-y/2,1)dw+2\int_{-y/2}^{x-3y/2}T(x-w-y/2,1)dw+2\int_{y/2}^{x/2}T(w-y/2,x-w-y/2,2)dw\\=&x_0+y+2\int_{x-y}^{y}T(w,1)dw+2\int_{y}^{x}T(x,1)dw+2\int_0^{(x-y)/2}T(w,x-w-y,2)dw\end{aligned}$$

假设$y \le x \le 2y$，$T(x,1)=S(x,1)$，

$$(x+y)S(x,1)=x_0+y+2\int_{x-y}^{y}T(w,1)dw+2\int_{y}^{x}S(x,1)dw+2\int_{0}^{(x-y)/2}T(w,x-w-y,2)dw \tag{8}$$

所以有

$$(x+y)S(x,1)-2\int_{y}^{x}S(w,1)dw=x_0+y+2\int_{x-y}^{y}T(w,1)dw+2\int_{0}^{(x-y)/2}T(w,x-w-y,2)dw \tag{9}$$

根据之前的结果，我们有

$$2\int_{x-y}^{y}T(w,1)dw=2(x_0+y)\int_{x-y}^{y}\frac{w+y}{y^2}dw=\frac{x_0+y}{y^2}\left(y^2-(x-y)^2+2y(2y-x)\right)=\frac{(x_0+y)(4y^2-x^2)}{y^2} \tag{10}$$

$$\begin{aligned}2\int_{0}^{(x-y)/2}T(w,x-w-y,2)dw&=2\frac{x_0+y}{y}\int_{0}^{(x-y)/2}\left(\frac{x+y}{y}-\frac{1}{4}\left(1+\frac{(w+y)(x-w)}{y^2}\right)\right)dw\\&=\frac{x_0+y}{y}\left(\frac{x^2-y^2}{y}-\frac{x-y}{4}-\frac{1}{24y^2}(x-y)(x^2+y^2+4xy)\right)\\&=\frac{(x_0+y)(x-y)}{y}\left(\frac{x+y}{y}-\frac{1}{4}-\frac{1}{24y^2}(x^2+y^2+4xy)\right)\end{aligned}$$

所以可简化为

$$(x+y)S(x,1)-2\int_{y}^{x}S(w,1)dw=1+\frac{4y^2-x^2}{y^2}+\frac{x-y}{y}\left(\frac{x+y}{y}-\frac{1}{4}-\frac{1}{24y^2}(x^2+y^2+4xy)\right) \tag{11}$$

所以两边对x取导数得

$$\begin{aligned}(x+y)S'(x,1)-S(x,1)&=-\frac{2x}{y^2}+\frac{1}{y}\left(\frac{x+y}{y}-\frac{1}{4}-\frac{1}{24y^2}(x^2+y^2+4xy)\right)+\frac{x-y}{y}\left(\frac{1}{y}-\frac{1}{24y^2}(2x+4y)\right)\\&=\frac{1}{24y^3}\left(-48xy+24xy+24y^2-6y^2-x^2-y^2-4xy+24xy-24y^2-2x^2-2xy+4y^2\right)\\&=\frac{1}{24y^3}\left(-3y^2-3x^2-6xy\right)=-\frac{(x+y)^2}{8y^3}\ \Rightarrow\ q(x)=-\frac{1}{8y^3}(x+y)\end{aligned}$$

所以

$$u(x)=\exp\left(-\int\frac{1}{x+y}dx\right)=\exp\left(-\ln(x+y)\right)=\frac{1}{x+y}. \tag{12}$$

所以对于$y \le x \le 2y$，

$$T(x,1)=\frac{x_0+y}{y}\left(\frac{5(x+y)}{4y}-\frac{(x+y)^2}{8y^2}\right)=\frac{(x_0+y)(x+y)}{y^2}\left(\frac{5}{4}-\frac{x+y}{8y}\right). \tag{13}$$

3.2 $y \le x_1 \le 2y$，$x_2 \le y$

假设起始状态为两段，且$y \le x_1 \le 2y$，$x_2 \le y$，我们有

$$\begin{aligned}&T(x_1,x_2,2)\\&=\frac{x_0+y}{x_1+x_2+2y}+\frac{1}{x_1+x_2+2y}\left(2\int_{-y/2}^{y/2}\underbrace{T(x_1-w-y/2,x_2,2)}_{\text{问题变为2段}x_1-w-y/2\text{，}x_2\text{的马路}}dw+\int_{y/2}^{x_1-y/2}\underbrace{T(w-y/2,x_1-w-y/2,x_2,3)}_{\text{问题变为3段马路}}dw\right)\\&\quad+\frac{1}{x_1+x_2+2y}\left(2\int_{-y/2}^{x_2-y/2}T(x_1,x_2-w-y/2,2)dw+2\int_{x_2-y/2}^{x_2/2}T(x_1,1)dw\right)\end{aligned}$$

且3段时$w-y/2, x_1-w-y/2\le y$。换元并平衡左右的

$$\begin{aligned}
&(x_1+x_2+2y)\,T(x_1,x_2,2)\\
&=x_0+y+2\int_{-y/2}^{x_1-3y/2}T(x_1-w-y/2,x_2,2)dw+2\int_{x_1-3y/2}^{y/2}T(x_1-w-y/2,x_2,2)dw\\
&\quad+\int_{y/2}^{x_1-y/2}T(w-y/2,x_1-w-y/2,x_2,3)dw+2\int_{-y/2}^{x_2-y/2}T(x_1,x_2-w-y/2,2)dw+(y-x_2)T(x_1,1)\\
&=x_0+y+2\int_{y}^{x_1}T(w,x_2,2)dw+2\int_{x_1-y}^{y}T(w,x_2,2)dw\\
&\quad+\int_{0}^{x_1-y}T(w,x_1-w-y,x_2,3)dw+2\int_{0}^{x_2}T(x_1,w,2)dw+(y-x_2)T(x_1,1)
\end{aligned}$$

所以我们有

$$\begin{aligned}
&(x_1+x_2+2y)\,T(x_1,x_2,2)-y-2\int_y^{x_1}T(w,x_2,2)dw-2\int_0^{x_2}T(x_1,w,2)dw\\
=&\int_0^{x_1-y}\left(\frac{2}{27}\frac{(w+y)(x_2+y)(x_1-w)}{y^3}+\frac{28}{27}\frac{x_1+x_2+2y}{y}-\frac{65}{108}-\frac{1}{4y^2}\left((x_1+y)(x_2+y)+(w+y)(x_1-w)\right)\right)dw\\
&+2\int_{x_1-y}^{y}\left(\frac{w+x_2+2y}{y}-\frac{1}{4}\left(1+\frac{(w+y)(x_2+y)}{y^2}\right)\right)dw+(y-x_2)\left(\frac{5(x_1+y)}{4y}-\frac{(x_1+y)^2}{8y^2}\right)
\end{aligned}$$

$$\begin{aligned}
&(x_1+x_2+2y)\,T(x_1,x_2,2)-y-2\int_y^{x_1}T(w,x_2,2)dw-2\int_0^{x_2}T(x_1,w,2)dw\\
=&\int_y^{x_1}\left(\frac{2}{27}\frac{w(x_2+y)(x_1+y-w)}{y^3}+\frac{28}{27}\frac{x_1+x_2+2y}{y}-\frac{65}{108}-\frac{1}{4y^2}\left((x_1+y)(x_2+y)+w(x_1+y-w)\right)\right)dw\\
&+2\int_{x_1}^{2y}\left(\frac{w+x_2+y}{y}-\frac{1}{4}\left(1+\frac{w(x_2+y)}{y^2}\right)\right)dw+(y-x_2)\left(\frac{5(x_1+y)}{4y}-\frac{(x_1+y)^2}{8y^2}\right)\\
=&\frac{1}{3}\left(\frac{1}{4y^2}-\frac{2(x_2+y)}{27y^3}\right)(x_1^3-y^3)+\frac{1}{2}\left(\frac{2(x_1+y)(x_2+y)}{27y^3}-\frac{x_1+y}{4y^2}\right)(x_1^2-y^2)\\
&+(x_1-y)\left(\frac{28}{27}\frac{x_1+x_2+2y}{y}-\frac{65}{108}-\frac{1}{4y^2}(x_1+y)(x_2+y)\right)\\
&+\left(\frac{1}{y}-\frac{x_2+y}{4y^2}\right)(4y^2-x_1^2)+2(2y-x_1)\left(\frac{x_2+y}{y}-\frac{1}{4}\right)+(y-x_2)\left(\frac{5(x_1+y)}{4y}-\frac{(x_1+y)^2}{8y^2}\right)
\end{aligned}$$

可得

$$T(x_1,x_2,2)=\frac{x_0+y}{y}\left(\frac{(x_1+y)^2(x_2+y)}{27y^3}-\frac{(x_1+y)^2}{8y^2}-\frac{35(x_1+y)(x_2+y)}{108y^2}+\frac{139(x_1+y)}{108y}+\frac{x_2+y}{y}-\frac{35}{108}\right).$$

3.3 $2y\le x\le 3y$

如果假设$2y\le x\le 3y$

$$T(x,1)=\frac{x_0+y}{x+y}+2\int_{-y/2}^{y/2}\frac{1}{x+y}T(x-w-y/2,1)dw+2\int_{y/2}^{x/2}\frac{1}{x+y}T(w-y/2,x-w-y/2,2)dw$$

且$w-y/2, x-w-y/2\le 1$。所以变为

$$\begin{aligned}
(x+y)T(x,1)&=y+2\int_{-y/2}^{x-5y/2}T(x-w-y/2,1)dw+2\int_{x-5y/2}^{y/2}T(x-w-y/2,1)dw\\
&\quad+2\int_{y/2}^{x-3y/2}T(w-y/2,x-w-y/2,2)dw+2\int_{x-3y/2}^{x/2}T(w-y/2,x-w-y/2,2)dw\\
&=y+2\int_{2y}^{x}T(w,1)dw+2\int_{x-y}^{2y}T(w,1)dw+2\int_{0}^{x-2y}T(w,x-w-y,2)dw+2\int_{x-2y}^{x/2-y/2}T(w,x-w-y,2)dw
\end{aligned}$$

所以

$$
\begin{aligned}
&(x+y)S(x,1)-y-2\int_{2y}^{x}S(w,1)dw\\
&=2\int_{x-y}^{2y}\left(\frac{5(w+y)}{4y}-\frac{(w+y)^2}{8y^2}\right)dw+2\int_{x-2y}^{x/2-y/2}\left(\frac{x+y}{y}-\frac{1}{4}\left(1+\frac{(w+y)(x-w)}{y^2}\right)\right)dw\\
&\quad+2\int_{0}^{x-2y}\left(\frac{(x-w)^2(w+y)}{27y^3}-\frac{1}{8y^2}(x-w)^2-\frac{35}{108y^2}(x-w)(w+y)+\frac{139}{108y}(x-w)+\frac{w+y}{y}-\frac{35}{108}\right)dw
\end{aligned}
$$

可得

$$
T(x,1)=\frac{x_0+y}{y}\frac{x+y}{y}\left(\frac{(x+y)^2}{81y^2}-\frac{43(x+y)}{216y}+\frac{49}{36}\right) \tag{14}
$$

4 归纳分析

我们发现

$$
\frac{5}{4}-\frac{(x+y)}{8y}-1=\frac{1}{4}\left(1-\frac{x+y}{2y}\right) \tag{15}
$$

$$
\frac{49}{36}-\frac{43(x+y)}{216y}+\frac{(x+y)^2}{81y^2}-\left(\frac{5}{4}-\frac{(x+y)}{8y}\right)=\frac{1}{9}-\frac{2(x+y)}{27y}+\frac{(x+y)^2}{81y^2}=\frac{1}{9}\left(1-\frac{(x+y)}{3y}\right)^2
$$

所以我们猜想对于$x_0=x=ny$，我们有

$$
T(x,1)=(n+1)^2\sum_{k=1}^{n}\frac{1}{k^2}\left(1-\frac{n+1}{k}\right)^{k-1} \tag{16}
$$

利用计算机仿真可以得到当$x=100y$时，我们有$T(x,1)=721.1568$。所以可以验证上解为正确的。
$x=10y$时，理论结果为48.7191，100万次仿真结果为48.7132。
$x=20y$时，理论结果为110.2660，100万次仿真结果为110.2714。

5 Appendix

对于微分方程

$$
y'+p(x)y=q(x). \tag{17}
$$

我们有

$$
y=\frac{\int u(x)q(x)dx+C}{u(x)} \tag{18}
$$

其中

$$
u(x)=\exp\left(\int p(x)dx\right), \tag{19}
$$