数学分析习题课讲义上册习题

Author: weiyuan

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$$\begin{aligned} I &= \int_{\frac{\pi}{4}}^{\pi}\int_{0}^{2\sin\theta} f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta \\ &= [\int_{0}^{\sqrt{2}}\int_{\frac{\pi}{4}}^{\pi-\arcsin\frac{r}{2}} + \int_{\sqrt{2}}^{2}\int_{\arcsin\frac{r}{2}}^{\pi-\arcsin\frac{r}{2}}] f(r\cos\theta, r\sin\theta) r\mathrm{d}r\mathrm{d}\theta \end{aligned} \tag{1}$$

$$\lim_{n\to+\infty}(1-\frac{1}{1+2})(1-\frac{1}{1+2})(1-\frac{1}{1+2+3})\dots(1-\frac{1}{1+2+\dots+n}) = ? \tag{2}$$

$$\begin{aligned} 1-\frac{1}{\frac{n(n+1)}{2}} &= 1-\frac{2}{n(n+1)} \\ &= \frac{n^2+n-2}{n(n+1)} \\ &= \frac{(n+2)(n-1)}{n(n+1)} \end{aligned} \tag{3}$$

$$\begin{aligned} I &= \lim_{n\to+\infty}\frac{1\times4}{2\times3}\frac{2\times5}{3\times4}\dots\frac{(n-2)(n+1)}{(n-1)n}\frac{(n-1)(n+2)}{n(n+1)} \\ &= \lim_{n\to+\infty}\frac{1}{3}\frac{4}{2}\frac{2}{4}\frac{5}{3}\frac{3}{5}\frac{6}{4}\dots\frac{n+2}{n} \\ &= \lim_{n\to+\infty}\frac{1}{3}\frac{n+2}{n} \\ &= \frac{1}{3}\lim_{n\to+\infty}\frac{n+2}{n} \\ &= \frac{1}{3} \end{aligned} \tag{4}$$

Theorem 0.1. A-G 不等式

意 n 个非负实数 $a_1, a_2, \dots, a_n$

$$\frac{a_1+a_2+\dots+a_n}{n} \geq \sqrt[n]{a_1\dots a_n} \tag{5}$$

其中等号成立 $\iff a_1 = a_2 = \dots = a_n$

Proof　数学归纳法

$n = 1$ 时结论平凡

$n = 2$　　$\frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}$

$$(a_1-a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2 \geq 0$$

$$a_1^2 + 2a_1a_2 + a_2^2 \geq 4a_1a_2$$

$$(a_1+a_2)^2 \geq 4a_1a_2$$

$$\frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}$$

$n = k$ 时，假设 $\frac{a_1+\dots+a_k}{k} \geq \sqrt[k]{a_1\dots a_k}$ 成立

$n=k+1$

$$\begin{aligned}&\frac{a_1+\cdots+a_k+a_{k+1}}{k+1}-\frac{a_1+\cdots+a_k}{k}\\=&\frac{k(a_1+\cdots+a_{k+1})-(k+1)(a_1+\cdots+a_k)}{k(k+1)}\\=&\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}\end{aligned}\tag{6}$$

we found

$$\frac{a_1+\cdots+a_k+a_{k+1}}{k+1}=\frac{a_1+\cdots+a_k}{k}+\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}$$

note

$$A:=\frac{a_1+\cdots+a_k}{k},\qquad B:=\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}$$

$$(\frac{a_1+\cdots+a_k+a_{k+1}}{k+1})^{k+1}=(A+B)^{k+1}\geq A^{k+1}+(k+1)A^kB\tag{7}$$

使用二项式展开需要对 a_i 从小到大重排，而使用 Bernoulli 不等式则只需要 $A\geq 0,(A+B)\geq 0$ 即可

$$A^{k+1}+(k+1)A^kB=A^k(A+(k+1)B)\tag{8}$$

$$\begin{aligned}A^k=(\frac{a_1+\cdots+a_k+a_{k+1}}{k+1})^{k+1}\geq a_1\ldots a_k\quad\text{assume at}(n=k)\\A+(k+1)B=\frac{a_1+\cdots+a_k}{k}+\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k}=a_{k+1}\\\because(A+B)^{k+1}\geq A^k(A+(k+1)B)\geq a_1\ldots a_ka_{k+1}\\\therefore\frac{a_1+\cdots+a_k+a_{k+1}}{k+1}\geq\sqrt[k+1]{a_1\ldots a_ka_{k+1}}\end{aligned}\tag{9}$$

使用二项式展开定理的条件：

在归纳法第二步对 $a_1\ldots a_{k+1}$ 重编号，使 a_{k+1} 为其中最大的数（之一）

这使得分解式右边第二项 $\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}$ 一定是非负数

Proof Forward and backward (Cauchy, 1897)

Forward Part:

$n=2$

$$\frac{a_1+a_2}{2}\geq\sqrt{a_1a_2}\tag{10}$$

$n=4$

$$\begin{aligned}\frac{a_1+a_2+a_3+a_4}{4}&\geq\sqrt{\frac{a_1+a_2}{2}\frac{a_3+a_4}{2}}\\&\geq\sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}}\\&\geq\sqrt[4]{a_1a_2a_3a_4}\end{aligned}\tag{11}$$

$n=2^k$ 假设不等式 $\frac{a_1+\cdots+a_{2^k}}{2^k}\geq\sqrt[2^k]{a_1\ldots a_{2^k}}$ 成立

$n=2^{k+1}$

$$\begin{aligned}\frac{a_1+\cdots+a_{2^k}+\cdots+a_{2^{k+1}}}{2^{k+1}}&\geq\sqrt{\frac{a_1+\cdots+a_{2^k}}{2^k}\frac{a_{2^k+1}+\cdots+a_{2^{k+1}}}{2^k}}\\&\geq\sqrt{\sqrt[2^k]{a_1\ldots a_{2^k}}\sqrt[2^k]{a_{2^k+1}\ldots a_{2^{k+1}}}}\\&\geq\sqrt[2^{k+1}]{a_1\ldots a_{2^{k+1}}}\end{aligned}\tag{12}$$

Backward Part: A-G 不等式对某个 $n \geq 2$ 成立，则它对 $n-1$ 也成立

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1}a_i &= \frac{1}{n}(\frac{n}{n-1})\sum_{i=1}^{n-1}a_i \\ &= \frac{1}{n}(\sum_{i=1}^{n-1}a_i + \frac{1}{n-1}\sum_{i=1}^{n-1}a_i)\end{aligned} \tag{13}$$

将 $\frac{1}{n-1}\sum_{i=1}^{n-1}a_i$ 看作 a_n

$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i \geq \sqrt[n]{(\prod_{i=1}^{n-1}a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)} \tag{14}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^n \geq \prod_{i=1}^{n-1}a_i(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i) \tag{15}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^{n-1} \geq \prod_{i=1}^{n-1}a_i \tag{16}$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1}a_i} \tag{17}$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i \geq \sqrt[n]{(\prod_{i=1}^{n-1}a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)} \tag{18}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^n \geq \prod_{i=1}^{n-1}a_i(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i) \tag{19}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^{n-1} \geq \prod_{i=1}^{n-1}a_i \tag{20}$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1}a_i} \tag{21}$$

Theorem 0.2. 柯西-施瓦茨不等式

$a_1, \ldots, a_n$ 和 $b_1, \ldots, b_n \in \mathbb{R}$，成立

$$|\sum_{i=1}^{n}a_ib_i| \leq \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2} \tag{22}$$

♡

Proof

$$\sum_{i=1}^{n}(a_i-\lambda b_i)^2 = \sum_{i=1}^{n}a_i^2 - 2\lambda\sum_{i=1}^{n}a_ib_i + \lambda^2\sum_{i=1}^{n}b_i^2 \geq 0$$

由韦达定理（视 λ 为未知数），原方程无解或只有唯一解

$$
\begin{aligned}
\Delta = b^2 - 4ac \leq 0 \\
(-2\sum_{i=1}^{n} a_i b_i)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq 0 \\
(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \\
\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}
\end{aligned}
\tag{23}
$$

Theorem 0.3. 定积分第一中值定理

函数 $f(x), g(x) \in \mathbb{C}[a,b]$. 且在 $[a,b]$ 上不变号，则存在 $\zeta \in [a,b]$，使得 $\int_a^b f(x)g(x) = f(\zeta)\int_a^b g(x)\mathrm{d}x$ ♡

Proof suppose that $g(x) \geq 0$. $f(x)$ continuous on close set,so we can get the maximum and minimum value of f. We note that m is the minimum value of $f(x), x \in [a,b]$,and M is the maximum value of $f(x)$, then we have:

$$
\begin{aligned}
mg(x) \leqslant f(x)g(x) \leqslant Mg(x) \\
m\int_a^b g(x)\mathrm{d}x \leqslant \int_a^b f(x)g(x)\mathrm{d}x \leqslant M\int_a^b g(x)\mathrm{d}x
\end{aligned}
$$

note that we don't know$\int_a^b g(x)\mathrm{d}x \neq 0$

When $\int_a^b g(x)\mathrm{d}x = 0$, then $g(x) \equiv 0$, So $\forall\zeta \in [a,b]$, the theorem works.

When $\int_a^b g(x)\mathrm{d}x = 0$, then $m \leqslant \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \leq M$

From the Intermediate Value Theorem, $f(x) \in \mathbb{C}[a,b] \quad m \leqslant f(x) \leqslant M$

$$
\begin{aligned}
\exists\zeta \in [a,b] \quad f(\zeta) = \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x} \\
\int_a^b f(x)g(x)\mathrm{d}x = f(\zeta)\int_a^b g(x)\mathrm{d}x
\end{aligned}
$$

设 $g(x)$ 在 $[a,b]$ 上连续可积，$f(x)$ 在 $[a,b]$ 上连续单调递增，且 $f'(x) \geq 0$，并对 $\forall x \in [a,b]$ 有 $f(x) \geq 0$。则存在 $\zeta \in [a,b]$，使得

$$
\int_a^b f(x)g(x)\mathrm{d}x = f(b)\int_\zeta^b g(x)\mathrm{d}x
$$

Proof set$G(x) = \int_x^b g(t)\mathrm{d}t, g(x)$在$[a,b]$ 上可积

则 $G(x), x \in [a,b]$ 存在最值，设最小值和最大值分别为 m, M

$$
G(x) = -\int_b^x g(t)\mathrm{d}t, \quad G'(x) = -g(x)
$$

$$
\begin{aligned}
\int_a^b f(x)g(x)\mathrm{d}x &= -\int_a^b f(x)\mathrm{d}G(x) \\
&= -(f(b)G(b) - f(a)G(a)) - \int_a^b G(x)f'(x)\mathrm{d}x \\
&= f(a)G(a) + \int_a^b G(x)f'(x)\mathrm{d}x
\end{aligned}
\tag{24}
$$

$$m\int_a^b f'(x)\mathrm{d}x \leqslant \int_a^b G(x)f'(x)\mathrm{d}x \leqslant M\int_a^b f'(x)\mathrm{d}x$$
$$m[f(b-f(a))] \leqslant \int_a^b G(x)f'(x)\mathrm{d}x \leqslant M[f(b-f(a))]$$

$$mf(a) \leqslant f(a)G(a) \leqslant Mf(a)$$
$$mf(b) \leqslant \int_a^b f(x)g(x)\mathrm{d}x \leqslant Mf(b)$$

From the Intermediate Value Theorem, $\exists \zeta \in [a,b] s.t. G(\zeta) = \frac{\int_a^b f(x)g(x)\mathrm{d}x}{f(b)}$
then we have

$$\int_a^b f(x)g(x)\mathrm{d}x = f(b)G(\zeta) = f(b)\int_\zeta^b g(x)\mathrm{d}x$$

1.3.2 练习题

1. 关于 Bernoulli 不等式的推广：

(1) 证明：当 $-2 \geq h \geq -1$ 时 Bernoulli 不等式 $(1+h)^n \geq 1+nh$ 仍成立；

(2) 证明：当 $h \geq 0$ 时成立不等式

$$(1+h)^n \geq \frac{n(n-1)h^2}{2} \tag{25}$$

(3) 证明：若 $a_i > -1 \quad (i=1,2,\dots,n)$ 且同号，则成立不等式

solve:

(1)

$$-2 \leq h \leq -1$$
$$-1 \leq 1+h \leq 0$$
$$-1 \leq (1+h)^n \leq 0$$
$$-2n \leq nh \leq -n$$
$$1-2n \leq 1+nh \leq 1-n$$

$n=0 \qquad (1+h)^0 = 1 = 1+0*h$ 结果是平凡的

$n=1 \qquad 1+h = 1+h$ 结果是平凡的

$n \geq 2 \qquad$ 此时 $1-n \leq -2$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1-nh \geq 1-2n$$
$$(1+h)^n \geq 1+nh$$

(2)

$$h \geq 0 \qquad (1+h)^n \geq \frac{n(n-1)h^2}{2}$$
$$(1+h)^n = 1+nh+\frac{n(n-1)}{2}h^2+\cdots \geq \frac{n(n-1)}{2}h^2$$

推广:

$$(1+h)^n \geq C_n^3h^3, C_n^4h^4, \dots, C_n^kh^k, \qquad 0 \leq k \leq n$$

(3)

$$\prod_{i=1}^{n}(1+a_i) \geq 1+\sum_{i=1}^{n} a_i$$

(a)$a_i \geq 0$, 且同号。

$$\prod_{i=1}^{n}(1+a_i) = 1+\sum_{i=1}^{n} a_i + \sum_{i=1,i\neq j}^{n}\sum_{j=1}^{n} a_i a_j + \sum_{i=1,i\neq j,k}^{n}\sum_{j=1,j\neq k}^{n}\sum_{k=1}^{n} a_i a_j a_k + \dots$$

$$\prod_{i=1}^{n}(1+a_i) \geq \frac{\prod_{i=1}^{n}(1+a_i)}{1+a_k} \qquad \forall k \in 1,2,\dots,n, \quad 1+a_k \geq 1$$

(b) $0 > a_i > -1$此时$1 > 1+a_i > 0$

别人的方法：$n=1$ 时不等式变成等式，显然成立

设 $n=k$ 时不等式也成立

$$\prod_{i=1}^{k}(1+a_i) \geq 1+\sum_{i=1}^{k} a_i$$

则 $n=k+1$ 时，有

$$\prod_{i=1}^{k+1}(1+a_i) = \prod_{i=1}^{k} a_i(1+a_{k+1}) \geq (1+\sum_{i=1}^{k} a_i)(1+a_{k+1})$$

$$(1+\sum_{i=1}^{k} a_i)(1+a_{k+1}) = 1+\sum_{i=1}^{k} a_i + a_{k+1} + \sum_{i=1}^{k} a_i \cdot a_{k+1} \geq 1+\sum_{i=1}^{k+1} a_i$$

$$\therefore \prod_{i=1}^{k+1}(1+a_i) \geq 1+\sum_{i=1}^{k+1} a_i$$

2. 利用 A-G 不等式求解下列有关阶乘 $n!$ 的不等式

(1) 证明：当 $n>1$ 时成立

$$n! < (\frac{n+1}{2})^n \tag{26}$$

(2) 利用 $(n!)^2 = (n\cdot 1)((n-1)\cdot 2)\dots(1\cdot n)$ 证明：当 $n>1$ 时成立

$$n! < (\frac{n+2}{\sqrt{6}})^n \tag{27}$$

(3) 比较 (1)(2) 两个不等式的优劣，并说明原因；

(4) 证明：对任意实数 r 成立

$$(\sum_{k=1}^{n} k^r)^n \geq n^n (n!)^r \tag{28}$$

solve:

(1) when $n>1$

$$n! = 1\times 2\times \cdots \times n < (\frac{1+2+\cdots+n}{n})^n$$

$$(\frac{1+2+\cdots+n}{n})^n = (\frac{n(n+1)}{2n})^n = (\frac{n+1}{2})^n$$

(2) when $n>1$

$$(n!)^2 = (n\cdot 1)((n-1)\cdot 2)\dots(1\cdot n) < (\frac{n\cdot 1+(n-1)\cdot 2+\cdots+1\cdot n}{n})^n$$

$$n \cdot 1 + (n-1) \cdot 2 + \cdots + 1 \cdot n = \sum_{k=1}^{n}(n-k+1)k$$

$$\begin{aligned}\sum_{k=1}^{n}(n-k+1)k &= (n+1)\sum_{k=1}^{n}k - \sum_{k=1}^{n}k^2 \\ &= (n+1)\frac{n(n+1)}{2} - \frac{n(2n+1)(n+1)}{6} \\ &= \frac{n(n+1)}{6}(3(n+1)-(2n+1)) \\ &= \frac{n(n+1)(n+2)}{6}\end{aligned} \tag{29}$$

$$\begin{aligned}(n!)^2 &= (n\cdot 1)((n-1)\cdot 2)\ldots(1\cdot n) \\ &< (\frac{n\cdot 1 + (n-1)\cdot 2 + \cdots + 1\cdot n}{n})^n \\ &= (\frac{1}{n}\frac{n(n+1)(n+2)}{6})^n \\ &= (\frac{(n+1)(n+2)}{6})^n \\ &< (\frac{n+2}{6})^{2n}\end{aligned} \tag{30}$$

$$\therefore \qquad n! < (\frac{n+2}{\sqrt{6}})^n \tag{31}$$

(3)

$$\frac{n+1}{2} = \frac{n+2}{\sqrt{6}} \tag{32}$$

解得 $n = 1+\sqrt{6} > 3$, $n > 3$ 时 (2) 式更精确，结果比 (1) 式更好。

(4) $\forall r \in \mathbb{R} \quad (n!)^r \leqslant \frac{1}{n^n}(\sum_{k=1}^{n} k^r)^n$ 由 A-G 不等式

$$\frac{1}{n}\sum_{k=1}^{n}k^r \geq \sqrt[n]{\prod_{k=1}^{n}k^r} \tag{33}$$

$$(n!)^r = \prod_{k=1}^{n}k^r \leqslant (\frac{1}{n}\sum_{k=1}^{n}k^r)^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n \tag{34}$$

2.(4)

$$\begin{aligned}\forall r \in \mathbb{R} \qquad (\sum_{i=1}^{n}k^r)^n \geq n^n(n!)^r \\ (n!)^r = \prod_{k=1}^{n}k^r \leqslant (\frac{1^r+2^r+\cdots+n^r}{n})^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n \quad \text{A-G inequality} \\ \therefore (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r\end{aligned} \tag{35}$$

3. $a_k > 0, \quad k = 1, 2, \ldots, n$ 证明几何-调和平均值不等式

$$(\prod_{k=1}^{n}a_k)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}} \tag{36}$$

Proof from A-G inequality

$$\begin{aligned}\frac{\sum_{k=1}^{n}\frac{1}{a_k}}{n} &\geq \sqrt[n]{\prod_{k=1}^{n}\frac{1}{a_k}}\\ &= \frac{1}{\sqrt[n]{\prod_{k=1}^{n}a_k}}\\ \therefore a_k>0,\qquad \sqrt[n]{\prod_{k=1}^{n}a_k} &\geq \frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}}\end{aligned} \tag{37}$$

4. $a,b,c\geq 0$, proof that

$$\sqrt[3]{abc}\leqslant\sqrt{\frac{ab+bc+ca}{3}}\leq\frac{a+b+c}{3} \tag{38}$$

并推广到 n 个非负数的情况

Proof left:

$$\begin{aligned}\sqrt[3]{abc} &= \sqrt{\sqrt[3]{ab\cdot bc\cdot ca}}\\ &\leqslant \sqrt{\frac{ab+bc+ca}{3}}\end{aligned} \tag{39}$$

right:

$$\begin{aligned}\sqrt{\frac{ab+bc+ca}{3}} &\leqslant \sqrt{\frac{(\frac{a+b}{2})^2+(\frac{b+c}{2})^2+(\frac{c+a}{2})^2}{3}}\\ &= \sqrt{\frac{2(a^2+b^2+c^2)+2(ab+bc+ca)}{12}}\\ &= \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}}\end{aligned} \tag{40}$$

$$\because a,b,c\geq 0\qquad \frac{ab+bc+ca}{3}\leqslant\frac{a^2+b^2+c^2+ab+bc+ca}{6} \tag{41}$$

需要证明 $\sqrt{\frac{ab+bc+ca}{3}}\leq\frac{a+b+c}{3}$

对该式两边平方

$$\frac{ab+bc+ca}{3}\leq\frac{(a+b+c)^2}{9}=\frac{a^2+b^2+c^2+2ab+2bc+2ca}{9} \tag{42}$$

$$\begin{aligned}\frac{ab+bc+ca}{3} &\leqslant \frac{a^2+b^2+c^2}{6}+\frac{ab+bc+ca}{6}\\ &\leqslant \frac{a^2+b^2+c^2}{6}+\frac{ab+bc+ca}{3}\\ &= (\frac{a+b+c}{3})^2\\ \therefore\sqrt{\frac{ab+bc+ca}{3}} &\leqslant \frac{a+b+c}{3}\end{aligned} \tag{43}$$

Proof 推广至 n 个

$$\begin{array}{llll}[l]n=2 & \sqrt{ab} & \leq\frac{a+b}{2} & \\ n=3 & \sqrt[3]{abc} & \leq\sqrt{\frac{ab+bc+ca}{3}} & \leq\frac{a+b+c}{3}\\ n=k & \sqrt[k]{\prod_{i=1}^{k}a_i} & \leq\sqrt{\frac{\sum_{i=1}^{k}-1a_ia_{i+1}+a_ka_1}{k}} & \leq\frac{\sum_{i=1}^{k}a_i}{k}\end{array} \tag{44}$$

$$1 \qquad \sqrt[k]{a_1 a_2 \dots a_k} = \sqrt{\sqrt[k]{a_1^2 a_2^2 \dots a_k^2}} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k}} \tag{45}$$

$$2 \qquad \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k}} \leqslant \frac{a_1 + \cdots + a_k}{k} \tag{46}$$

$$\begin{aligned} \frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k} &\leqslant \frac{a_1^2 + \dots a_k^2}{2k} \\ 2\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k} &\leqslant \frac{(a_1 + \dots a_k)^2}{2k} \\ \sqrt{\frac{a_1 \dots a_k}{k}} &\leqslant \frac{a_1 + \cdots + a_k}{\sqrt{4k}} \quad \text{wrong!} \end{aligned} \tag{47}$$

Chapter 1 第一章

1.1 引论

1.1.1 关于习题课教案的组织

1.1.2 书中常用记号

1. $\mathbf{N}_+$：所有正整数组成的集合.
2. $\mathbf{R}$：所有实数组成的集合 (同时也用于表示无限区间 $(-\infty,\infty)$).
3. $\mathbf{Q}$：所有有理数组成的集合.
4. $\mathbf{C}$：所有复数组成的集合.
5. $\iff$ 是等价关系的记号.$A \iff B$ 表示 A 和 B 等价. 例如，A 代表 $x>3$，B 代表 $x-3>0$，则 $x>3 \iff x-3>0$.
6. $[x]$ 是实数 x 的整数部分，即不超过 x 的最大整数. 例如，$[\sqrt{2}]=1,[-\sqrt{2}]=-2$. 关于 $[x]$ 的基本不等式是：$[x]\leqslant x<[x]+1$，或 $x-1<[x]\leqslant x$
7. $\square$ 表示一个证明或解的结束.
8. $\binom{n}{k}=C_n^k=\frac{n(n-1)\cdots(n-k+1)}{k!}$.
9. 记号 $\approx$ 表示近似值. 例如 $\sqrt{2}\approx 1.4$.
10. 复合函数 $f(g(x))$ 也写成 $(f\circ g)(x)$ 或 $f\circ g$.
11. 若 A 和 B 为两个集合，则用记号 $A-B$ 或 $A\backslash B$ 表示 A 与 B 的差集，也就是集合 $\{x|x\in A 且 x\notin B\}$.
12. 用 $O_\delta(a)$ 表示以 a 为中心，以 $\delta>0$ 为半径的邻域. 它就是开区间 $(a-\delta,a+\delta)$(也可以用 $U_\delta(a)$ 等记号). 如不必指出半径，则可简记为 $O(a)$ (或 $U(a)$).

1.1.3 几个常用的初等不等式

1.1.3.1 几个初等不等式的证明

Theorem 1.1. 1. AG 不等式

个非负实数 $a_1,a_2,\cdots,a_n$

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n} \tag{1.1}$$

$\geq$ in inequation became $=\iff a_1=a_2=\cdots=a_n$

♡

Proof

method 1. induction method

$$
\begin{aligned}
&k=1 \qquad a_1=a_1\\
&k=2 \qquad \frac{a_1+a_2}{2}\geq\sqrt{a_1a_2}\\
&k=n \qquad \text{suppose} \quad \frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n}\\
&k=n+1\\
&\quad\frac{a_1+a_2+\cdots+a_{n+1}}{n+1}-\frac{a_1+a_2+\cdots+a_n}{n}\\
&\quad=\frac{n(a_1+a_2+\cdots+a_{n+1})-(n+1)(a_1+a_2+\cdots+a_n)}{n(n+1)}\\
&\quad=\frac{na_{n+1}-(a_1+a_2+\cdots+a_n)}{n(n+1)}
\end{aligned}
$$

Set $A=\frac{a_1+a_2+\cdots+a_n}{n}$, $B=\frac{na_{n+1}-(a_1+a_2\cdots+a_n)}{n(n+1)}$

$$(\frac{a_1+a_2+\cdots+a_{n+1}}{n+1})^{n+1}=(A+B)^{n+1}$$

$A>0,B\geq 0$

$$
\begin{aligned}
&(A+B)^{n+1}\geq A^{n+1}+(n+1)A^nB\\
&A^{n+1}+(n+1)A^nB=A^n(A+(n+1)B)\\
&A^n=(\frac{a_1+a_2+\cdots+a_n}{n})^n\geq a_1a_2\cdots a_n\\
&A+(n+1)B=\frac{a_1+a_2+\cdots+a_n}{n}+\frac{na_{n+1}-(a_1+a_2+\cdots+a_n)}{n}=a_{n+1}\\
&\therefore (A+B)^{n+1}\geq A^n(A+(n+1)B)\geq a_1a_2\cdots a_n\cdot a_{n+1}\\
&\therefore \frac{a_1+a_2+\cdots+a_{n+1}}{n+1}\geq\sqrt[n+1]{a_1a_2\cdots a_na_{n+1}}
\end{aligned}
$$

使用二项式展开定理的条件

在归纳法第二步，将 $a_1,a_2,\cdots,a_{n+1}$ 重编号，使得 $n+1$ 为其中最大的数 (之一)，这使得分解式右边第二项 $(na_{n+1}-(a_1+a_2+\cdots+a_n))/n(n+1)$ 一定是非负数。

method 2. Forward and Backward (Cauchy, 1897)

Forward part

$k=2.\dfrac{a_1+a_2}{2}\geq\sqrt{a_1a_2}.$

$k=4.\dfrac{a_1+a_2+a_3+a_4}{4}\geq\sqrt{(\dfrac{a_1+a_2}{2})\cdot(\dfrac{a_3+a_4}{2})}.$

$\geq\sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}}=\sqrt[4]{a_1a_2a_3a_4}.$

$k=2^n.\text{Suppose}\quad\dfrac{a_1+a_2+\cdots a_{2^n}}{2^n}\geq\sqrt[2^n]{a_1a_2\cdots a_{2^n}}$

$k=2^{n+1}.$

$$\frac{a_1+a_2+\cdots+a_{2^n}+\cdots+a_{2^{n+1}}}{2^{n+1}}\geq\sqrt{(\frac{a_1+a_2+\cdots+a_2^n}{2^n})\cdot(\frac{a_{2^n+1}+a_{2^n+2}+\cdots+a_2^{n+1}}{2^n})}$$

$$I\geq\sqrt{\sqrt[2^n]{a_1a_2\cdots a_{2^n}}\sqrt[2^n]{a_{2^n+1}a_{2^n+2}\cdots a_{2^{n+1}}}}=\sqrt[2^{n+1}]{a_1a_2\cdots a_{2^{n+1}}}$$

Backward part suppose A.G inequality is valid when $k=n$, Consider $k=n-1$.

$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i=\frac{1}{n}(\frac{n}{n-1})\sum_{i=1}^{n-1}a_i$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i=\frac{1}{n}(\sum_{i=1}^{n-1}a_i+\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i\geq\sqrt[n]{(\prod_{i=1}^{n-1}a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)}$$
$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^n\geq(\prod_{i=1}^{n-1}a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)$$
$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^{n-1}\geq(\prod_{i=1}^{n-1}a_i)$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i\geq\sqrt[n-1]{\prod_{i=1}^{n-1}a_i}$$

Proposition 1.1. 1.3.5 柯西-施瓦茨不等式

$a_1,a_2,\cdots,a_n$ 和 $b_1,b_2,\cdots,b_n$，成立

$$|\sum_{i=1}^{n}a_ib_i\leqslant\sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}|$$

Proof
$$0\leqslant\sum_{i=1}^{n}(a_i-\lambda b_i)^2=\sum_{i=1}^{n}a_i^2-2\lambda\sum_{i=1}^{n}a_ib_i+\lambda^2\sum_{i=1}^{n}b_i^2$$

由韦达定理 (视 λ 为未知数). 原方程无解或只有唯一解。

$$\Delta=b^2-4ac\leqslant 0$$
$$(-2\sum_{i=1}^{n}a_ib_i)^2-4\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2\leqslant 0$$
$$(\sum_{i=1}^{n}a_ib_i)^2\leqslant\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2$$
$$\sum_{i=1}^{n}a_ib_i\leqslant\sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}$$

1.1.3.2 练习题

Example 1.1 关于 Bernoulli 不等式的推广：
(1) 证明：当 $-2\leqslant h\leqslant -1$ 时 Bernoulli 不等式 $(1+h)^n\geq 1+nh$ 仍成立;
(2) 证明：当 $h\geq 0$ 时成立不等式 $(1+h)^n\geq\frac{n(n-1)h^2}{2}$, 并推广之;
(3) 证明：若 $a_i>-1(i=1,2,\ldots,n)$ 且同号, 则成立不等式

$$\prod_{i=1}^{n}(1+a_i)\geq 1+\sum_{i=1}^{n}a_i$$

Proof (1)

$$\begin{aligned}
&-2 \leqslant h \leqslant -1 \\
&-1 \leqslant 1+h \leqslant 0 \qquad\qquad -1 \leqslant (1+h)^n \leqslant 0 \\
&-2n \leqslant nh \leqslant -n \qquad\qquad 1-2n \leqslant 1+nh \leqslant 1-n
\end{aligned}$$

$$\begin{aligned}
&n = 0. \qquad\qquad (1+h)^0 = 1 = 1 + 0 \times h \\
&n = 1. \qquad\qquad 1+h = 1+h \\
&n \geq 2. \qquad\qquad 1-n \leqslant -2
\end{aligned}$$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1+nh \geq 1-2n$$

$$(1+h)^n \geq 1+nh$$

(2)

$$\begin{aligned}
&h \geq 0 \\
(1+h)^n =& 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots \geq \frac{n(n-1)}{2}h^2
\end{aligned}$$

推广:

$$(1+h)^n \geq \binom{n}{3}h^3, \binom{n}{4}h^4, \ldots, \binom{n}{k}h^k, 0 \leqslant k \leqslant n$$

(3) $k=1$ 时显然成立. 使用归纳法证明. 假设 $k=n$ 时不等式 $\prod_{i=1}^{n}(1+a_i) \geq 1+\sum_{i=1}^{n} a_i$ 成立, 证明 $k=n+1$ 时 $\prod_{i=1}^{n+1}(1+a_i) \geq 1+\sum_{i=1}^{n+1} a_i$ 成立.

$$\begin{aligned}
k = n+1 \qquad &\prod_{i=1}^{n+1}(1+a_i) = \prod_{i=1}^{n}(1+a_i)(1+a_{n+1}) \\
&\because \prod_{i=1}^{n}(1+a_i) \geq 1 + \sum_{i=1}^{n} a_i \\
&\prod_{i=1}^{n}(1+a_i)(1+a_{n+1}) \geq (1+\sum_{i=1}^{n} a_i)(1+a_{n+1})
\end{aligned}$$

$$\begin{aligned}
(1+\sum_{i=1}^{n} a_i)(1+a_{n+1}) &= 1 + \sum_{i=1}^{n} a_i + a_{n+1} + a_{n+1}\sum_{i=1}^{n} a_i \\
&= 1 + \sum_{i=1}^{n+1} a_i + a_{n+1}\sum_{i=1}^{n} a_i \\
&\geq 1 + \sum_{i=1}^{n+1} a_i
\end{aligned}$$

Example 1.2 利用 A.G. 不等式求解:

(1). $n! \leqslant (\frac{n+1}{2})^n$, while $n > 1$

(2). $(n!)^2 = (n \cdot 1)[(n-1) \cdot 2] \ldots (1 \cdots n)$. 证明: 当 $n > 1$ 时成立

$$n! < (\frac{n+2}{6})^n$$

(3). 比较上述两个不等式的优劣

(4). 证明: 对任意实数 r 成立:

$$(n!)^r \leqslant \frac{1}{n^n}(\sum_{k=1}^{n} k^r)^n \tag{1.2}$$

Proof (1).

$$n>1 \qquad n!=1\times 2\times\cdots\times n<(\frac{1+2+\cdots+n}{n})^n=(\frac{(1+n)n}{2n})^n=(\frac{n+1}{2})^n$$

$\because 1\neq 2\neq\cdots n$, 所以不会有等号出现的情况

(2). $n>1$

$$\begin{aligned}(n!)^2&=(n\cdot 1)[(n-1)\cdot 2]\dots(1\cdots n)\\&<(\frac{n\times 1+(n-1)\times 2+\cdots+1\times n}{n})^n\end{aligned}$$

Consider this equation

$$(\frac{n\times 1+(n-1)\times 2+\cdots+1\times n}{n})^n \tag{1.3}$$

$$\begin{aligned}\sum_{k=1}^{n}(n-k+1)k&=(n+1)\sum_{k=1}^{n}k-\sum_{k=1}k^2\\&=(n+1)\frac{(n+1)n}{2}-\frac{n(n+1)(2n+1)}{6}\\&=\frac{n(n+1)}{6}(3(n+1)-(2n+1))\\&=\frac{n(n+1)(n+2)}{6}\end{aligned}$$

$$\begin{aligned}(n!)^2&<(\frac{n\times 1+(n-1)\times 2+\cdots+1\times n}{n})^n\\&=(\frac{(n+1)(n+2)}{6})^n\end{aligned}$$

$\because n+1<n+2, \therefore n!<(\frac{n+2}{\sqrt{6}})^n$

(3). $n>3$ 时,$\frac{n+2}{\sqrt{6}}<\frac{n+1}{2}$ (2) 的结果较好.

(4). $\forall r\in\mathbb{R}$, prove formula 1.2

$$\frac{1}{n}\sum_{k=1}^{n}k^r\geq\sqrt[n]{\prod_{k=1}^{n}k^r}$$

$$(n!)^r=\prod_{k=1}^{n}k^r\leqslant(\frac{1}{n}\sum_{k=1}^{n}k^r)^n=\frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$

my answer

$$\forall r\in\mathbb{R},\qquad (\sum_{k=1}^{n}k^r)^n\geq n^n(n!)^r$$

$$(n!)^r=\sum_{k=1}^{n}k^r\leqslant(\frac{1^r+2^r+\cdots+n^r}{n})^n=\frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$

$$\therefore\quad (\sum_{k=1}^{n}k^r)^n\geq n^n(n!)^r$$

Example 1.3 $a_k>0, k=1,2,\dots,n$ 证明几何–调和平均值不等式

$$(\prod_{k=1}^{n}a_k)^{\frac{1}{n}}\geq\frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}}$$

Proof from A.G inequality

$$\frac{\sum_{k=1}^{n}\frac{1}{a_k}}{n}\geq\sqrt[n]{\prod_{k=1}^{n}\frac{1}{a_k}}=\frac{1}{\sqrt[n]{\prod_{k=1}^{n}a_k}}$$

$$a_k > 0, \quad \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

Example 1.4 $a, b, c \geq 0$. prove $\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leqslant \frac{a+b+c}{3}$. 并推广到 n 个非负数的情况

Proof 1. $\sqrt[3]{abc} = \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \leqslant \sqrt{\frac{ab+bc+ca}{3}}$.

2.

$$\begin{aligned}\sqrt{\frac{ab+bc+ca}{3}} &\leqslant \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ &= \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}} \\ &= \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}}\end{aligned}$$

$a, b, c \geq 0$, 希望证明

$$\sqrt{\frac{ab+bc+ca}{3}} \leqslant \frac{a+b+c}{3}$$

$$\begin{aligned}\frac{ab+bc+ca}{3} &\leqslant \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6} \\ \frac{ab+bc+ca}{2} &\leqslant \frac{a^2+b^2+c^2}{6} + 2\frac{ab+bc+ca}{6} \qquad (\text{add} \frac{ab+bc+ca}{6}) \\ \frac{ab+bc+ca}{3} &\leqslant \frac{ab+bc+ca}{2} \leqslant (\frac{a+b+c}{3})^2 \\ \sqrt{\frac{ab+bc+ca}{3}} &\leqslant \frac{a+b+c}{3}\end{aligned}$$

推广至 n 个

$$\begin{aligned}[l] n=2 \qquad & \sqrt{ab} \leq \frac{a+b}{2} \\ n=3 \qquad & \sqrt[3]{abc} \leqslant \sqrt{\frac{ab+bc+ca}{3}} \leqslant \frac{a+b+c}{3} \\ n=4 \qquad & \sqrt[4]{abcd} \leqslant \sqrt[3]{\frac{abc+bcd+cda+dab}{4}} \leqslant \sqrt{\frac{a+b+c}{3}} \leq \frac{a+b+c+d}{4} \\ k=n \qquad & \sqrt[n]{a_1 a_2 \ldots a_n} \leqslant \sqrt{\frac{a_1+a_2+\cdots+a_n}{n}} \leq \frac{a_1+a_2+\cdots+a_n}{n}\end{aligned}$$

This is

$$\sqrt[n]{\sum_{k=1}^{n} a_k} \leqslant \sqrt{\frac{\sum_{k=1}^{n} a_k}{k}} \leq \frac{\sum_{k=1}^{n} a_k}{k}$$

1. $\sqrt[n]{a_1 a_2 \ldots a_n} = \sqrt{\sqrt[n]{a_1^2 a_2^2 \ldots a_n^2}} \leqslant \sqrt{\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n}}$

2. $\sqrt{\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n}} \leqslant \sqrt{\frac{a_1+a_2+\cdots+a_n}{n}}$?

Example 1.5 (1) $|\alpha+\beta| \leqslant |\alpha| + |\beta|$

Proof let $\alpha = a-b, \beta = b$, the identity became $|(a-b)+b| \leqslant |a-b| + |b|$. This is $|a-b| \geq |a| - |b|$.

$$||a|-|b|| = \begin{cases} |a|-|b|. & a \geq b \\ |b|-|a|. & a < b \end{cases}$$

When $a \geq b$, $||a|-|b|| = |a|-|b|$. There is $|a-b| \geq |a|-|b| = ||a|-|b||$

When $a < b$, $|a-b| = |b-a| \geq |b|-|a| = ||a|-|b||$.

$\therefore$, We have$|a-b| \geq ||a|-|b||$

(2) $\sum |a_k| \geq |\sum a_k|$

Proof We can prove this statement by induction.

$$
\begin{aligned}
k=2, &\qquad |a_1|+|a_2| \geq |a_1+a_2| \\
k=3, &\qquad |a_1|+|a_2|+|a_3| \geq |a_1+a_2+a_3| \\
\text{Suppose } k=n, &\qquad \sum_{k=1}^{n}|a_k| \geq |\sum_{k=1}^{n} a_k| \\
k=n+1, &\qquad \text{prove} \sum_{k=1}^{n+1}|a_k| \geq |\sum_{k=1}^{n+1} a_k|
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{n+1}|a_k| &= \sum_{k=1}^{n}|a_k| + |a_{n+1}| \\
&\geq |\sum_{k=1}^{n} a_k| + |a_{n+1}| \\
&\geq |\sum_{k=1}^{n+1} a_k|
\end{aligned}
$$

$$
\begin{aligned}
k=2, &\qquad |a_1|-|a_2| \leqslant |a_1-a_2| \\
\text{Suppose } k=n, &\qquad |a_1|-\sum_{k=2}^{n}|a_k| \leqslant |\sum_{k=1}^{n} a_k| \\
k=n+1, &\qquad \text{prove}|a_1|-\sum_{k=2}^{n+1}|a_k| \leqslant |\sum_{k=1}^{n+1} a_k|
\end{aligned}
$$

$$
\begin{aligned}
|a_1|-\sum_{k=2}^{n+1}|a_k| &= |a_1|-\sum_{k=2}^{n}|a_k| - |a_{n+1}| \\
&\leqslant |\sum_{k=1}^{n} a_k| - |a_{n+1}| \\
&\leqslant |\sum_{k=1}^{n+1} a_k|
\end{aligned}
$$

Can left side became$||a_1| - \sum_{k=2}^{n}|a_k||$?

$$
\left||a_1|-\sum_{k=2}^{n}|a_k|\right| = |a_1|-\sum_{k=2}^{n}|a_k| \qquad |a_1| \geq \sum_{k=2}^{n} a_k \tag{1.4}
$$

$$
\left||a_1|-\sum_{k=2}^{n}|a_k|\right| = \sum_{k=2}^{n}|a_k| - |a_1| \qquad |a_1| \geq \sum_{k=2}^{n} a_k \tag{1.5}
$$

in eq1.4, the inequality is still vaild. However in eq1.5, $\sum_{k=2}^{n}|a_k| - |a_1|$ and $|a_1|$

(3). $\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

Proof

$$
\begin{aligned}
\frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|} \\
\frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
1-\frac{|a+b|}{1+|a+b|} &\geq 1-\frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
\frac{1}{1+|a+b|} &\geq \frac{1-|a||b|}{(1+|a|)(1+|b|)} \\
1+|a|+|b|+|a||b| &\geq 1+|a+b|-|a||b|-|a||b||a+b| \\
|a|+|b|+2|a||b|+|a||b||a+b| &> 0, \text{Since } +2|a||b|+|a||b||a+b| \geq |a+b|
\end{aligned}
$$

Therefore $\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$

Example 1.6 (4).$|(a+b)^n-a^n| \leqslant (|a|+|b|)^n-|a|^n$

$$
\begin{aligned}
(a+b)^n-a^n &= \binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{n}a^0b^n \\
(|a|+|b|)^n-|a|^n &= \binom{n}{1}|a|^{n-1}|b|^1+\binom{n}{2}|a|^{n-2}|b|^2+\cdots+\binom{n}{n}|a|^0|b|^n
\end{aligned}
$$

$$
\begin{aligned}
&\because |a|^j|b|^k \geq a^jb^k \\
&\therefore \sum |a|^j|b|^k \geq |\sum a^jb^k|
\end{aligned}
$$

$$
|(a+b)^n-a^n| = \begin{cases} (a+b)^n-a^n, & a+b \geq a; b \geq 0 \\ a^n-(a+b)^n, & a+b < a; b < 0 \end{cases}
$$

$$
|(a+b)^n-a^n| \leqslant (|a|+|b|)^n-|a|^n. \tag{1.6}
$$

Proposition 1.2. 1.3.5(Cauchy inequality)

or $a_1, a_2, \ldots, a_n$. and $b_1, b_2, \ldots, b_n$. $a_i, b_i \in \mathbb{R}$, There is

$$
\left|\sum_{i=1}^n a_ib_i\right| \leqslant \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2} \tag{1.7}
$$

♠

Proof Let's prove eq1.7

First way on book:

Use variable λ, change the inequality into nonnegative binomial.

$$
\begin{aligned}
0 \leq \sum_{i=1}^n (a_i-\lambda b_i)^2 &= \sum_{i=1}^n a_i^2-2\lambda\sum_{i=1}^n a_ib_i+\lambda^2\sum_{i=1}^n \\
\Delta = B^2-4AC &= (-2\sum_{i=1}^n a_ib_i)^2-4(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) \leq 0
\end{aligned}
$$

$$
(\sum_{i=1}^n a_ib_i)^2 \leqslant (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)
$$

sqrt on both side of the inequality above, we can get

$$
\left|\sum_{i=1}^n a_ib_i\right| \leqslant \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}
$$

6. Cauchy 不等式的不同证明

(1). 数学归纳法.

$$
\begin{aligned}
k=1,\quad & |ab|=\sqrt{a^2}\sqrt{b^2}\\
k=1,\quad & |a_1b_1+a_2b_2|=\sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_1^2}\\
\text{Suppose } k=n,\quad & |\sum_{i=1}^n a_ib_i|=\sqrt{\sum_{i=1}^n a_i}\sqrt{\sum_{i=1}^n b_i}\\
k=n+1,\quad & |\sum_{i=1}^{n+1} a_ib_i|=|\sum_{i=1}^n a_ib_i+a_{n+1}b_{n+1}|
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{i=1}^{n+1} a_ib_i| &= |\sum_{i=1}^n a_ib_i+a_{n+1}b_{n+1}|\\
&\le |\sqrt{\sum_{i=1}^n a_i}\sqrt{\sum_{i=1}^n b_i}+a_{n+1}b_{n+1}|
\end{aligned}
$$

Note that $A=\sqrt{\sum_{i=1}^n a_i}$, $B=\sqrt{\sum_{i=1}^n b_i}$

$$
\begin{aligned}
|\sum_{i=1}^{n+1} a_ib_i| &\le |AB+a_{n+1}b_{n+1}|\\
&\le \sqrt{A^2+a_{n+1}^2}\sqrt{B^2+b_{n+1}^2}\\
&= \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}
\end{aligned}
$$

(2) Lagrange 恒等式

$$
\sum_{i=1}^n a_k^2\sum_{i=1}^n b_k^2-(\sum_{i=1}^n |a_kb_k|)=\frac{1}{2}\sum_{i=1}^n\sum_{k=1}^n(|a_k||b_i|-|a_i||b_k|)^2 \tag{1.8}
$$

$$
\begin{aligned}
(|a_k||b_i|-|a_i||b_k|)^2 &= |a_k|^2|b_i|^2-2|a_i||a_k||b_i||b_k|+|b_k|^2|a_i|^2\\
&= a_k^2b_i^2+b_k^2a_i^2-2|a_ia_kb_ib_k|
\end{aligned}
$$

$$
\sum_{i=1}^n\sum_{k=1}^n(|a_k||b_i|-|a_i||b_k|)^2=2\sum_{i=1}^n a_i^2\sum_{k=1}^n b_k^2-2\sum_{i=1}^n\sum_{k=1}^n|a_ia_kb_ib_k|
$$

$$
\sum_{i=1}^n a_k^2\sum_{i=1}^n b_k^2-(\sum_{i=1}^n |a_kb_k|)=\frac{1}{2}\sum_{i=1}^n\sum_{k=1}^n(|a_k||b_i|-|a_i||b_k|)^2\ge 0
$$

$$
\begin{aligned}
&\therefore (\sum_{i=1}^n |a_ib_i|)^2\leqslant\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2\\
&\because |\sum_{i=1}^n a_ib_i|\leqslant\sum_{i=1}^n|a_ib_i|\\
&\therefore (|\sum_{i=1}^n a_ib_i|)^2\leqslant(\sum_{i=1}^n|a_ib_i|)^2\\
&\therefore (|\sum_{i=1}^n a_ib_i|)^2\leqslant\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2
\end{aligned}
$$

不等式两边开平方，得到：

$$|\sum_{i=1}^{n} a_i b_i| \leqslant \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

(3). 用不等式 $|AB| \leqslant \frac{A^2+B^2}{2}$

$$|a_i b_i| \leqslant \frac{a_i^2+b_i^2}{2}$$

$$|\sum_{i=1}^{n} a_i b_i| \leqslant \sum_{i=1}^{n} |a_i b_i| \qquad \leqslant \frac{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2}{2}$$

$$\frac{\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2}{2} \geq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \qquad ??$$

如何用均值不等式证明 Cauchy 不等式?

由切比雪夫不等式，有

$$\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \leqslant (\frac{a_1+a_2+\cdots+a_n}{n})(\frac{b_1+b_2+\cdots+b_n}{n}) \tag{1.9}$$

由均值不等式，有

$$\frac{a_1+a_2+\cdots+a_n}{n} \leqslant \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$$

$$\frac{b_1+b_2+\cdots+b_n}{n} \leqslant \sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{n}}$$

$$\begin{aligned}\therefore \frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} &\leqslant (\frac{a_1+a_2+\cdots+a_n}{n})(\frac{b_1+b_2+\cdots+b_n}{n}) \\ &\leqslant \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}\sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{n}} \\ &= \frac{1}{n}\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\sqrt{b_1^2+b_2^2+\cdots+b_n^2}\end{aligned}$$

This is

$$\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

Square on both side of the inequality, The calculate square root. We can get eq1.9:

(4). 构造复的辅助数列

$$c_k = a_k^2 - b_k^2 + 2i|a_k b_k|, \qquad k = 1, 2, \ldots, n$$

Then we use

$$\left|\sum_{k=1}^{n} c_k\right| \leqslant \sum_{k=1}^{n} |c_k|$$

$$c_k = (|a_k| + i|b_k|)^2 = a_k^2 + b_k^2 + 2i|a_k b_k|$$

$$\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2i\sum_{k=1}^{n} |a_k b_k|$$

$$|c_k| = \sqrt{\Re^2 c_k + \Im^2 c_k} = \sqrt{(a_k^2 - b_k^2)^2 + (2a_k b_k)^2} = a_k^2 + b_k^2$$

$$\therefore \left|\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2i \sum_{k=1}^{n} |a_k b_k|\right| = \sqrt{\Re^2 \sum_{k=1}^{n} c_k + \Im^2 \sum_{k=1}^{n} c_k}$$

$$= \sqrt{(\sum_{k=1}^{n}(a_k^2 - b_k^2))^2 + \sum_{k=1}^{n}(2a_k b_k)^2}$$

$$= \sqrt{(\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4\sum_{k=1}^{n}(a_k b_k)^2}$$

$$\because |\sum c_k| \leqslant \sum |c_k|$$

$$\therefore (\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4\sum_{k=1}^{n}(a_k b_k)^2 \leqslant (\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2)^2$$

$$\therefore 4(\sum_{k=1}^{n} a_k b_k)^2 \leqslant 4(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)$$

$$\text{extracting both side: } \left|\sum_{k=1}^{n} a_k b_k\right| \leqslant \sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2}$$

Example 1.7 7. Suppose $0 < x_i \leqslant \frac{1}{2}, i = 1, 2, \ldots, n$, then

$$\frac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^n} \leqslant \frac{\prod_{i=1}^{n}(1 - x_i)}{(\sum_{i=1}^{n}(1 - x_i))^n} \tag{1.10}$$

Proof Let's prove eq1.10 by induction method.

$$n = 2, \qquad \frac{x_1 x_2}{(x_1 + x_2)^2} \leqslant \frac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2}$$

$$\frac{(x_1 x_2)}{(x_1^2 + 2x_1 x_2 + x_2^2)} \leqslant \frac{1 - x_1 - x_2 + x_1 x_2}{(1 - x_1)^2 + 2(1 - x_1)(1 - x_2) + (1 - x_2)^2}$$

$$\frac{(x_1 + x_2)^2}{(x_1 x_2)} \geq \frac{((1 - x_1)(1 - x_2))^2}{1 - x_1 - x_2 + x_1 x_2}$$

$$\frac{x_1}{x_2} + 2 + \frac{x_2}{x_1} \geq \frac{1 - x_1}{1 - x_2} + 2\frac{1 - x_2}{1 - x_1}$$

$$\frac{x_1}{x_2} - \frac{1 - x_1}{1 - x_2} \geq \frac{1 - x_2}{1 - x_1} - \frac{x_2}{x_1}$$

$$\frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_2(1 - x_2)} \geq \frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_1(1 - x_1)}$$

$$\frac{x_1 - x_2}{x_2(1 - x_2)} \geq \frac{x_1 - x_2}{x_1(1 - x_1)}$$

$f(x) = x - x^2, f'(x) = 1 - 2x > 0,$ while $x \in (0, \frac{1}{2})$

When $x_1 > x_2, 0 < x_2 < x_1 \leqslant \frac{1}{2}, x_1 - x_1^2 \geq x_2 - x_2^2, x_1 - x_2 > 0$

When $x_1 < x_2, 0 < x_1 < x_2 \leqslant \frac{1}{2}, x_1 - x_1^2 \leqslant x_2 - x_2^2, x_1 - x_2 < 0$

$$\frac{x_1 - x_2}{x_2(1 - x_2)} \geq \frac{x_1 - x_2}{x_1(1 - x_1)}$$

$$k=2,\qquad \frac{x_1x_2}{(x_1+x_2)^2}\leqslant\frac{(1-x_1)(1-x_2)}{((1-x_1)+(1-x_2))^2}$$

$$k=4,\qquad \frac{x_1x_2x_3x_4}{(x_1+x_2+x_3+x_4)^2}\leqslant\frac{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}{((1-x_1)+(1-x_2)+(1-x_3)+(1-x_4))^2}$$

Use Cauchy's forward and backward method, We can prove this equation

Suppose $k=n$, $\frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2}\leqslant\frac{\prod_{i=1}^n(1-x_i)}{(\sum_{i=1}^n(1-x_i))^2}$

$k=n-1$, prove $\frac{\prod_{i=1}^{n-1} x_i}{(\sum_{i=1}^{n-1} x_i)^2}\leqslant\frac{\prod_{i=1}^{n-1}(1-x_i)}{(\sum_{i=1}^{n-1}(1-x_i))^2}$

We already know that

$$\frac{\sum_{i=1}^{n-1}x_i}{n-1}=\frac{1}{n}\Big(\sum_{i=1}^{n-1}x_i+\frac{1}{n-1}\sum_{i=1}^{n-1}x_i\Big)$$

This trick always use in (n-1) terms tranfer to (n) terms

When the inequality holds for $n>2$, for$k=n$, we have:

$$\frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^n}\leqslant\frac{\prod_{i=1}^n(1-x_i)}{(\sum_{i=1}^n(1-x_i))^n}$$
$$\frac{(\sum_{i=1}^n(1-x_i))^n}{(\sum_{i=1}^n x_i)^n}\leqslant\frac{\prod_{i=1}^n(1-x_i)}{\prod_{i=1}^n x_i}$$
$$\left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n(1-x_i))}\right)^n\geq\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n(1-x_i)}$$

for $k=n-1$, Let $M=x_n=\frac{\sum_{i=1}^{n-1}x_i}{n-1}$. The inequality 1.10 left side:

$$\begin{aligned}
&\left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n(1-x_i))}\right)^n\\
&=\left(\frac{x_1+\cdots+x_n}{(1-x_1)+\cdots+(1-x_n)}\right)^n\\
&=\left(\frac{x_1+\cdots+x_{n-1}+M}{(1-x_1)+\cdots+(1-x_{n-1})+(1-M)}\right)^n\\
&=\left(\frac{x_1+\cdots+x_{n-1}+\frac{\sum_{i=1}^{n-1}x_i}{n-1}}{(1-x_1)+\cdots+(1-x_{n-1})+(1-\frac{\sum_{i=1}^{n-1}x_i}{n-1})}\right)^n\\
&=\left(\frac{\frac{n}{n-1}(x_1+\cdots+x_{n-1})}{\frac{n}{n-1}((1-x_1)+\cdots+(1-x_{n-1}))}\right)^n\\
&=\left(\frac{M}{1-M}\right)^n
\end{aligned}$$

while the right side become

$$\begin{aligned}
&\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n(1-x_i)}\\
&=\frac{\prod_{i=1}^{n-1}x_i\cdot M}{\prod_{i=1}^{n-1}(1-x_i)\cdot(1-M)}\\
&=\frac{\prod_{i=1}^{n-1}x_i}{\prod_{i=1}^{n-1}(1-x_i)}\frac{M}{1-M}
\end{aligned}$$

$$\left(\frac{M}{1-M}\right)^n \geq \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1}(1-x_i)}\frac{M}{1-M}$$

$$\left(\frac{M}{1-M}\right)^{n-1} \geq \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1}(1-x_i)}$$

Proposition 1.3. 1.3.1 Bernoulli inequality

uppose that $h > -1, n \in \mathbb{N}$, Then:

$$(1+h)^n \geq 1 + nh \tag{1.11}$$

When $n > 1$, the inequality became equation iff $h = 0$.

Proof When $n = 1$, $1 + h = 1 + h$
$h = 0$, $1^n = 1$
Let's consider the condition $n > 1, h \neq 0$.
i). $h > 0$, $(1+h)^n = \binom{n}{0}h^0 + \binom{n}{1}h^1 + \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n$.
$\because \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n > 0$, $\therefore (1+h)^n > 1 + nh$
ii). $-1 < h < 0, 0 < 1 + h < 1$.

$$\begin{aligned}(1+h)^n - 1 &= (1+h-1)\Big(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\Big)\\ &= h\Big(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\Big)\end{aligned}$$

$\because 1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1} < n$ when$h < 0$
$\therefore (1+h)^n > 1 + nh$

Two variable extension of the Bernoulli inequality, Suppose $h = \frac{B}{A}, A > 0, A + B > 0$, Then$1 + h > 0$ is established.

Proposition 1.4. 1.3.2

uppose $A > 0, A + B > 0, n \in \mathbb{N}$, Then the inequality is true:

$$(A+B)^n \geq A^n + nA^{n-1}B \tag{1.12}$$

The inequalty became equation iff $B = 0$.

Proof divide A^n on both side of the inequality 1.12. Set$h = \frac{B}{A}(A > 0)$, Then the inequality became Eq 1.11. So we can prove Eq 1.12 by prove Eq 1.11. Eq 1.11 is true when $h > -1$. $\therefore 1 + h > 0, 1 + \frac{B}{A} > 0$, $\because A > 0$, $\therefore A + B > 0$. And when $n > 1$ the equation is true iff $h = 0. \frac{B}{A} = 0, \therefore B = 0$.

Example 1.8 Ex 1.3.2 exercise 8
$a, c, t, g \geq 0, a + c + t + g = 1$. Prove that $a^2 + c^2 + t^2 + g^2 \geq \frac{1}{4}$.
The inequality became equatio iff $a = c = t = g = \frac{1}{4}$.

Proof from A.G inequality,

$$\frac{a+c+t+g}{4} \geq \sqrt[4]{actg}, \quad a + c + t + g = 1 \tag{1.13}$$

$\therefore \sqrt[4]{actg} \leqslant \frac{1}{4}$

$$a + c + t + g = 1, (a+c+t+g)^2 = 1$$

$$(a+c+t+g)^2 = a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg = 1 \tag{1.14}$$

$$a^2 + c^2 \geq 2acc^2 + t^2 \geq 2ct \tag{1.15}$$

$$a^2 + t^2 \geq 2atc^2 + g^2 \geq 2cg \tag{1.16}$$

$$a^2 + g^2 \geq 2agt^2 + g^2 \geq 2tg \tag{1.17}$$

substitude $2ac, 2ag, \ldots$ in equation 1.14, we can get

$$4(a^2 + c^2 + t^2 + g^2) \geq a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg$$

Then we get the inequality 1.13.

1.2 1.4 逻辑符号与对偶法则

The law of duality: $\forall(\exists) \to \exists(\forall)$ with negative statement

Inverse proposition?

1. A have upper limit, $\exists M > 0.\forall x \in A, x \leqslant M$.

It's negative statement is 'A don't have upper limit'. $\forall M > 0, \exists x \in A, x > M$.

2. the minum item in A is b, $b \in A, \forall x \in A, x \geq b$.

It's negative statement is 'b is not the minum item in A'. $b \in A, \exists x \in A, x < b$.

3. $f \in (a, b)$ is a monotonic augmentation function, $\forall x, y \in (a, b), x < y, f(x) \leqslant f(y)$.(or $f(x) < f(y)$, depends on monotonic function's definition)

It's negative statement is '$f \in (a, b)$ isn't a monotonic augmentation function'. $\exists x, y \in (a, b), x < y, f(x) > f(y)$ (or $f(x) \geq f(y)$).

4. $f \in (a, b)$ is a monotonic function, $\forall x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) \geq 0$.

It's negative statement is '$f \in (a, b)$ isn't a monotonic function'. $\exists x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) < 0$.

(Another way $\forall x, y \in (a, b), x < y, f(x) - f(y) \geq 0$ or $f(x) - f(y) \leqslant 0$.)

5. $A \subset B$, $\forall x \in A, x \in B$.

It's negative statement is $A \subsetneq B$, $\exists x \in A, x \notin B$.

6. $A - B \neq \emptyset$, $\exists x \in A, x \in B$.

It's negative statement is $A - B = \emptyset$, $\forall x \in A, x \notin B$.

7. x_n is an infinitesimal amounts, $\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \forall n > N, |x_n| < \epsilon$.

It's negative statement is 'x_n is not an infinitesimal amounts', $\exists \epsilon > 0, \forall N \in \mathbb{N}^+, \exists n > N, |x_n| \geq \epsilon$.

8. x_n is infinitely large, $\forall M > 0, \exists N \in \mathbb{N}^+, \forall n > N, x_n > M$.

It's negative statement is 'x_n is not infinitely large', $\exists M > 0, \forall N \in \mathbb{N}^+, \exists n > N, x_n \leqslant M$.

Chapter 2 第二章数列极限

2.1 数列极限的基本概念

2.1.1 2.1.5 练习题

1. prove by Limit definition:

(1). $\lim_{n\to\infty}\frac{3n^2}{n^2-4}=3$.

(2). $\lim_{n\to\infty}\frac{\sin n}{n}=0$.

(3). $\lim_{n\to\infty}(1+n)^{\frac{1}{n}}=0$.

(4). $\lim_{n\to\infty}\frac{a^n}{n!}=0,(a>0)$.

2. Suppose $a_n, n\in\mathbb{N}_+$. sequence a_n converge to a.

Prove $\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a}$.

Proof $n\to\infty a_n\to a$.

$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$.

$$|\sqrt{a_n}-\sqrt{a}|=\left|\frac{a_n-a}{\sqrt{a_n}+\sqrt{a}}\right|<\frac{\epsilon}{\sqrt{a_n}+\sqrt{a}}$$

$\therefore\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a}$. □ (check, not consider the condition$a=0$) add $a=0,\forall\epsilon\in(0,1),\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$. s.t $a_n<\epsilon^2<\epsilon$, $\sqrt{a_n}<\epsilon$.

3. If $\lim_{n\to\infty}a_n=a$.

Prove $\lim_{n\to\infty}|a_n|=|a|$. Vice versa?

Proof $\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$.

$$\Big||a_n|-|a|\Big|\leqslant|a_n-a|<\epsilon$$

$\therefore\lim_{n\to\infty}|a_n|=|a|$

If We know $\lim_{n\to\infty}|a_n|=|a|$.

$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),\Big||a_n|-|a|\Big|<\epsilon$. We can't get $\lim_{n\to\infty}a_n=a$. For example: $a_n=\frac{1}{n}+1,a=-1,\lim_{n\to\infty}|a_n|=|a|$ is $\lim_{n\to\infty}|\frac{1}{n}+1|=|-1|$, but $\lim_{n\to\infty}\frac{1}{n}+1\neq-1$ □

(1). Suppose $p(x)$ is a polynomial of x, if $\lim_{n\to\infty}a_n=a$, Prove $\lim_{n\to\infty}p(a_n)=p(a)$.

(2). Suppose $b>0$, $\lim\limits_{n\to\infty}a_n=a$. Prove $b^{a_n}=b^a$.

(3). Suppose $b>0$, $\{a_n\},a_n>0,\forall n\in\mathbb{N}$. $\lim\limits_{n\to\infty}a_n=a.a>0$. Prove $\lim\limits_{n\to\infty}\log_b a_n=\log_b a$.

(4) Suppose $b\in\mathbb{R}$, $\{a_n\},a_n>0$ when $n\in\mathbb{N}$. $\lim\limits_{n\to\infty}a_n=a$. Prove $\lim\limits_{n\to\infty}{a_n}^b=a^b$.

(5) Suppose $\lim\limits_{n\to\infty}a_n=a$. Prove $\lim\limits_{n\to\infty}\sin a_n=\sin a$.

Proof 4.(1)

$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n\geqslant N(\epsilon),\ |a_n-a|<\epsilon$.

$p(a)=k_ma^m+k_{m-1}a^{m-1}+\cdots+k_0a^0$.

$\therefore p(a_n)-p(a)=k_m(a_n^m-a^m)+k_{m-1}(a_n^{m-1}-a^{m-1})+\cdots+k_0(a_n^0-a^0)$.

$$\begin{aligned}|a_n^m-a^m|&=|a_n-a|\cdot|a_n^{m-1}+a_n^{m-2}a+\cdots+a^{m-1}|\\&<\epsilon\cdot|a_n^{m-1}+a_n^{m-2}a+\cdots+a^{m-1}|\\&<\epsilon(m-1)\cdots(a+\delta)^{m-1}\end{aligned}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a)$. $\square$

Proof 4.(2)

$\forall\epsilon > 0, \exists N \in \mathbb{N}^{+}. \forall n \geqslant N, |a_n - a| < \epsilon$.

If $b = 1$, $1^{a_n} = 1^a = 1$.

If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1)$ $\because b > 0, \epsilon \to 0$, $\therefore b^\epsilon - 1 \to 0$. $\therefore \lim\limits_{n\to\infty} b_n^a = b^a$.

If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$.

Proof 4.(3)

$\forall\epsilon > 0, \exists N \in \mathbb{N}^{+}. \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1)\end{aligned}$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a})$. $\because b > 0, a \neq 0,\ a_n > 0$ when $\epsilon \to 0$. $\therefore \log_b(1 + \frac{\epsilon}{a}) \to 0$.

$\therefore \lim\limits_{n\to\infty} \log_b a_n = \log_b a$

Proof 4.(4)

$\forall\epsilon > 0, \exists N \in \mathbb{N}^{+}. \forall n \geqslant N, |a_n - a| < \epsilon$.

$a_n^b = e^{b\ln a_n}$, $a_n^b - a^b = e^{b\ln a_n} - e^{b\ln a}$.

$$\begin{aligned}e^{b\ln a_n} - e^{b\ln a} &= e^{b\ln a}(e^{b\ln a_n - b\ln a} - 1) \\ &= e^{b\ln a}(e^{b\ln \frac{a_n}{a}} - 1)\end{aligned}$$

$0 < |a_n^b - a^b| < e^{b\ln a}(e^{b\ln(1+\frac{\epsilon}{a})} - 1)$

$\therefore \lim\limits_{n\to\infty} a_n^b = a^b$

Proof 4.(5)

$\forall\epsilon > 0, \exists N \in \mathbb{N}^{+}. \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\sin(A+B) - \sin(A-B) &= \sin A\cos B + \cos A\sin B \\ &\quad - (\sin A\cos B - \cos A\sin B) \\ &= 2\cos A\sin B\end{aligned}$$

$$\sin a_n - \sin a = 2\cos\frac{a_n + a}{2}\sin\frac{a_n - a}{2}$$

$|\sin a_n - \sin a| = |2\cos\frac{a_n+a}{2}\sin\frac{a_n-a}{2}| < |2\sin\frac{a_n-a}{2}|$

$|2\sin\frac{a_n-a}{2}| < |2\frac{a_n-a}{2}| = \epsilon$

$|\sin a_n - \sin a| < \epsilon$, $\therefore \lim\limits_{n\to\infty} \sin a_n = \sin a$

assume $a > 1$. Prove $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof $\frac{1}{n}\log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\log_a 1 = 0$.

$\forall\epsilon > 0, \exists N \in \mathbb{N}^{+}, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon$.

$a > 1$, and $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\therefore$ when $n \to \infty$, $\sqrt[n]{n} < a^\epsilon$, take logarithm on base of a, we can get $\frac{1}{n}\log_a n < \epsilon$

$\therefore \lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

2.2 收敛数列的基本性质

收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界
3. 收敛数列的比较定理，包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

2.2.1 思考题

1. $\{a_n\}$ 收敛, $\{b_n\}$ 发散, $\{a_n+b_n\}$ 发散, $\{a_n\cdot b_n\}$ 可能收敛，可能发散.
2. $\{a_n\},\{b_n\}$ 都发散, $\{a_n+b_n\}$ 可能收敛，可能发散 (ex: $n+-n$, $n+-2n$), $\{a_n\cdot b_n\}$ 发散 (?).
3. $a_n\leqslant b_n\leqslant c_n$, $n\in\mathbb{N}_+$. 已知 $\lim\limits_{n\to\infty}(c_n-a_n)=0$. 问数列 $\{b_n\}$ 是否收敛?
4. $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$
5. $a_n\to a(n\to 0)$. $\forall n, b<a_n<c$. 是否成立 $b<a<c$?
6. $a_n\to a(n\to 0)$. and $b\leqslant a\leqslant c$, 是否存在 $N\in\mathbb{N}_+$, s.t. 当 $n>N$ 时，成立 $b\leqslant a_n\leqslant c$
7. 已知 $\lim\limits_{n\to\infty}a_n=0$, 问: 是否有 $\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0$. 反之如何?

Proof 5.4

$$\frac{n}{2n}\leqslant\frac{1}{n+1}+\cdots+\frac{1}{2n}\leqslant\frac{n}{n+1}$$

$\because\lim\limits_{n\to\infty}\frac{n}{2n}=\frac{1}{2}\lim\limits_{n\to\infty}\frac{n}{n+1}=1$, $\therefore\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$ 收敛.

$$\begin{aligned}\frac{1}{n+1}+\cdots+\frac{1}{2n}&=\frac{1}{n}(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}})\\&=\int_0^1\frac{1}{1+x}\mathrm{d}x\\&=\ln(1+x)|_0^1=\ln 2\end{aligned}$$

$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})=\ln 2$

Proof 5.5

不成立，应当为小于等于号。b=0, c=2, $a_n=\frac{1}{n}$, $\lim_{n\to\infty}a_n=0=c$.

Proof 5.6

不成立。$a=0,b=0,c=2$, $a_n=(-1)^n\frac{1}{n}$.

$b\leqslant a\leqslant c$, but $(-1)^{2n+1}\frac{1}{2n+1}<0=b$.

Proof $\lim\limits_{n\to\infty}a_n=0, a_n=\frac{1}{n}.a_1a_2\ldots a_n=\frac{1}{n!}$, $\lim\limits_{n\to\infty}\frac{1}{n!}=0$.

$\lim\limits_{n\to\infty}a_n=0\to\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0\quad\checkmark$

$\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0\to\lim\limits_{n\to\infty}a_n=0\quad\times$

$|a_n|<\epsilon$, $|a_{N+1}\ldots a_n|<\epsilon^{n-N}<\epsilon$, $a_n<\sqrt[n]{\epsilon}$.

for example, $a_n=\frac{n}{n+1}, a_1=\frac{1}{2}, a_2=\frac{2}{3},\ldots,a_n=\frac{n}{n+1}$.

$$\begin{aligned}a_1a_2\ldots a_n=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}&=\frac{1}{n+1}.\\\lim_{n\to\infty}(a_1a_2\ldots a_n)&\\=\lim_{n\to\infty}\frac{1}{n}&=0\end{aligned}$$

but $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{n}{n+1}=1\neq 0$

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

Example 2.1 2.2.2 $\lim_{n\to\infty}\frac{x_n-1}{x_n+a}=0$,
prove $\lim_{n\to\infty}x_n=a$

Proof $\forall\epsilon>0,\exists N\in\mathbb{N}^+.\forall n\geqslant N,|\frac{x_n-1}{x_n+a}-0|<\epsilon$.
$|x_n-1|<\epsilon|x_n+a|<4a\cdot\epsilon$.(这个 4 是怎么取得的?)
$|x_n-a|<\epsilon|x_n+a|=\epsilon|(x_n-a)+2a|\leqslant\epsilon(|x_n-a|+2a)$.
限制 $\epsilon<1$, $|x_n-a|<2\epsilon|a|/(1-\epsilon)$.
限制 $\epsilon<\frac{1}{2}$, $|x_n-a|<2\epsilon|a|/(1-\epsilon)<4|a|\epsilon$.
Let $\epsilon'=4a\epsilon$, $|x_n-1|<\epsilon'$. $\therefore\lim\lim_{n\to\infty}x_n=a$.

Example 2.2 2.2.3 $a>0,b>0$, 计算 $\lim\limits_{n\to\infty}(a_n+b_n)^{\frac{1}{n}}$.

Proof Suppose $a\leqslant b$.
$b=(b^b)^{\frac{1}{n}}<(a^n+b^n)^{\frac{1}{n}}\leqslant(2b^n)^{\frac{1}{n}}$.
$b<(a^n+b^n)^{\frac{1}{n}}\leqslant\sqrt[n]{2}b$, $\lim\limits_{n\to\infty}=1$. 夹逼定理.
$\lim\limits_{n\to\infty}(a^n+b^n)^{\frac{1}{n}}=\max\{a,b\}$.
两数 n 次方之和再开 n 次根号的结果由较大的值决定, a,b 中较大的值为这个数的主要部分.

Example 2.3 2.2.4 $a_n=\frac{1!+2!+\cdots+n!}{n!},n\in\mathbb{N}^+$

$\lim\limits_{n\to\infty}a_n=1$

Example 2.4 $\lim\limits_{n\to\infty}\frac{n^3+n-7}{n+3}=+\infty$

Example 2.5 $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$

调和级数 H_n 发散.

2.2.2 练习 2.2.4

Proof 1.
$\{a_n\}$ 收敛于 a, $\to$ 两个子列 $\{a_{2n}\},\{a_{2n+1}\}$ 均收敛于 a.
两个子列 $\{a_{2n}\},\{a_{2n+1}\}$ 均收敛于 a, $\to$ $\{a_n\}$ 收敛于 a.

2. 应用夹逼定理
(1). 给定 p 个正数 $a_1,a_2,\ldots,a_p$. 求 $\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\ldots a_p^n}$.
Let$a_s=\max_{1\leqslant i\leqslant p}\{a_1,a_2,\ldots,a_p\}$.

$$a_s=(a_s^n)^{\frac{1}{n}}<(a_1^n+a_2^n+\ldots a_p^n)^{\frac{1}{n}}\leqslant(pa_s^n)^{\frac{1}{n}}=p^{\frac{1}{n}}a_s$$

$n\to\infty,p^{\frac{1}{n}}\to1$. $\lim\limits_{n\to\infty}(a_1^n+a_2^n+\ldots a_p^n)^{\frac{1}{n}}=a_s$

(2). $x_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$, $n\in\mathbb{N}_+$. 求 $\lim\limits_{n\to\infty}x_n$

$$\frac{2n+1}{(n+1)}\leqslant x_n\leqslant\frac{2n+1}{\sqrt{n^2+1}}$$

$\lim\limits_{n\to\infty}\frac{2n+1}{n+1}=2$, $\lim\limits_{n\to\infty}\frac{2n+1}{\sqrt{n^2+1}}=2$. $\therefore\lim\limits_{n\to\infty}x_n=2$

(3). $a_n=(1+\frac{1}{2}+\cdots+\frac{1}{n})^{\frac{1}{n}}$, $n\in\mathbb{N}_+$. 求 $\lim\limits_{n\to\infty}a_n$

$$1=(\frac{n}{n})^{\frac{1}{n}}<a_n\leqslant(1\cdot n)^{\frac{1}{n}}=\sqrt[n]{n}$$

$\lim\limits_{n\to\infty}\sqrt[n]{n}=1$, $\therefore\lim\limits_{n\to\infty}a_n=1$

(4). $a_n>0$. $\lim\limits_{n\to\infty}a_n=a$, $a>0$. 证明 $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=1$

Proof $\lim\limits_{n\to\infty}a_n=a$

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$

$$0 < a - \epsilon < a_n < a + \epsilon$$

$\therefore \sqrt[n]{a-\epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a+\epsilon}.$

$\lim\limits_{n\to\infty} \sqrt[n]{a-\epsilon} = 1, \lim\limits_{n\to\infty} \sqrt[n]{a+\epsilon} = 1. \therefore \lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1.$

3. (1). $\lim\limits_{n\to\infty} (1+x)(1+x^2)\dots(1+x^{2^n}) = \lim\limits_{n\to\infty} \prod_{i=1}^{2^n}(1+x^i), |x| < 1.$

$|x| < 1, \quad 1 > x^2 > x^4 > \cdots > x^{2^n} > 0$

$x \in (0,1) \quad 1 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)^{n+1} \qquad \lim\limits_{n\to\infty}(1+x)^{n+1} = 1$

$x \in (-1,0) \quad 0 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)(1+x^2)^n \qquad \lim\limits_{n\to\infty}(1+x)(1+x^2)^n = 1$

$$\begin{aligned}
&\lim_{n\to\infty}(1+x)(1+x^2)\dots(1+x^n)\\
=&\lim_{n\to\infty}\frac{(1-x)(1+x)(1+x^2)\dots(1+x^n)}{1-x}\\
=&\lim_{n\to\infty}\frac{(1-x^{2^{n+1}})}{1-x}\\
=&\frac{1}{1-x}
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}(1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{n^2})\\
=&\lim_{n\to\infty}\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\dots\frac{n-1}{n}\cdot\frac{n+1}{n}\\
=&\lim_{n\to\infty}\frac{1}{2}\frac{n+1}{n}\\
=&\lim_{n\to\infty}\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}\left(1-\frac{1}{1+2}\right)\left(1-\frac{1}{1+2+3}\right)\dots\left(1-\frac{1}{1+2+\dots+n}\right)\\
=&\lim_{n\to\infty}\left(1-\frac{2}{3\times 2}\right)\left(1-\frac{2}{4\times 3}\right)\dots\left(1-\frac{2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{3\times 2-2}{3\times 2}\right)\left(\frac{4\times 3-2}{4\times 3}\right)\dots\left(\frac{(n+1)\times n-2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{4}{3\times 2}\right)\left(\frac{10}{4\times 3}\right)\dots\left(\frac{n^2+n-2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{1\times 4}{3\times 2}\right)\left(\frac{2\times 5}{4\times 3}\right)\dots\left(\frac{(n-2)\times(n+1)}{n\times(n-1)}\right)\left(\frac{(n-1)\times(n+2)}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\frac{1}{3}\times\frac{n+2}{n}\\
=&\frac{1}{3}
\end{aligned}$$

$$
\begin{aligned}
&\lim_{n\to\infty}[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}]\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{n}{n+1}\\
=&1
\end{aligned}
$$

3.(4).

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)}\right]\\
=&\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}\Big)+\frac{1}{2}\Big(\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}\Big)+\cdots+\frac{1}{2}\Big(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\Big)\\
=&\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{1\cdot 2}-\frac{1}{(n+1)(n+2)}\Big)\\
=&\frac{1}{2}\times\frac{1}{2}\\
=&\frac{1}{4}
\end{aligned}
$$

3.(5).

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma)},\qquad \text{其中}\gamma\text{为正整数}\\
=&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\gamma}\Big[\frac{1}{k(k+1)\dots(k+\gamma-1)}-\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma-1)}-\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma^{\underline{\gamma}}}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma!}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]\\
=&\frac{1}{\gamma}\cdot\frac{1}{\gamma!}
\end{aligned}
$$

其中 $x^{\underline{n}}=x(x-1)(x-2)\dots(x-n+1)$, 称为下阶乘. 而 $x^{\overline{n}}=x(x+1)(x+2)\dots(x+n-1)$, 称为上阶乘.

2.2.4-4 $S_n=a+3a^2+\cdots+(2n-1)a^n$, $|a|<1$. 求 $\{S_n\}$ 的极限.

$$
\begin{aligned}
S_n-aS_n&=a+3a^2+\cdots+(2n-1)a^n\\
&\quad-\quad a^2-\cdots+(2n-3)a^n-(2n-1)a^n+1\\
&=a+2a^2+\cdots+2aa^n-(2n-1)a^{n+1}\\
&=2(a+a^2+\cdots+a^n)-a-(2n-1)a^{n+1}\\
&=2\cdot a\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}
\end{aligned}
$$

$|a|<1,\ \lim\limits_{n\to\infty}a_n=0$

$\lim\limits_{n\to\infty}(S_n - aS_n) = (1-a)\lim \lim_{n\to\infty} S_n$

$$\begin{aligned}\lim_{n\to\infty}(S_n - aS_n) &= \lim_{n\to\infty} 2a\cdot\frac{1-a^{n+1}}{1-a} - a - (2n-1)a^{n+1}\\ &= 2a\cdot\frac{1}{1-a} - a\\ &= a\Big(\frac{2}{1-a} - a\Big)\\ &= a\frac{1+a}{1-a}\end{aligned}$$

$\therefore \lim\limits_{n\to\infty} S_n = \frac{a(a+1)}{(1-a)^2}$

2.2.4-5 设 $\lim\limits_{n\to\infty} x_n = A > 0$. 取 $\epsilon = \frac{A}{2}$, 则 $\exists N \in \mathbb{N}_+$. $\forall n > N$. 成立 $|x_n - A| < \frac{A}{2}$

$$A - \frac{A}{2} < x_n < A + \frac{A}{2}, \frac{A}{2} < x_n < \frac{3A}{2}$$

即 $x_n > \frac{A}{2}$.

令 $m = \min\{x_1, x_2, \dots, x_N, \frac{A}{2}\} > 0$. 则 m 为 $\{x_n\}$ 的正下界.

不一定有最小数的例子 $x_n = 1 + \frac{1}{n}$. $\lim\limits_{n\to\infty} x_n = 1$, 下界 $m = \frac{1}{2}$. 但 $\{x_n\}$ 取不到下界.

Proof 2.2.4-6 $\because \lim\limits_{n\to\infty} a_n = +\infty$. $\forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M$.

$m = \min\{a_1, a_2, \dots, a_N, M\}$, 但 M 在数列 $\{a_n\}$ 中不一定取的到!

$M = a_1 + 1, \exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$

则 $m = \min\{a_1, a_2, \dots, a_{N_1}\}$ 为数列的最小数.

Proof 2.2.4-7 构造数列

不妨设无界数列 $\{a_n\}$ 无上界.

$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M$.

取 $M_1 = 1$, 则 $\exists n_1 \in \mathbb{N}_+$ s.t. $a_{n_1} > M_1$.

取 $M_2 = \max\{a_n, 2\}$, $\exists n_2 \in \mathbb{N}_+$ s.t. $a_{n_2} > M_2$.

以此类推，构造数列 $\{a_{n_k}\}$. s.t.$a_{n_k} > k$. 即 a_{n_k} 为无穷大量.

Proof 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散.

构造 a_n 的发散子列即可. 已知 $\tan\frac{\pi}{2} = \infty$, π 是一个无理数, 因此存在数列 $\{b_n\}$, $\lim\limits_{n\to\infty} b_n = \frac{\pi}{2}$.

Proof 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散. 参考别人的答案

由于 $\{\sin 2n\}$ 极限不存在, 又

$$\begin{aligned}\sin 2n = 2\sin n\cos n &= \frac{2\sin n\cos n}{\sin^2 n + \cos^2 n}\\ &= \frac{2\tan n}{\tan^n + 1}\end{aligned}$$

若 $\{\tan n\}$ 极限存在 $\to \{\sin 2n\}$ 极限存在, 矛盾.

故 $\{\tan n\}$ 极限不存在.

2.2.4-9 $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$. S_n 在 1. $p \leqslant 0$, 2. $0 < p < 1$ 情况下均发散

Proof 1. $p \leqslant 0$. $\lim\limits_{n\to\infty} n^{-p} = \infty$, S_n 发散.

2. $0 < p < 1$. $\frac{1}{n^p} > \frac{1}{n}$. $\because H_n = \sum_{k=1}^{n}\frac{1}{k}$(调和级数) 发散, $S_n > H_n, \therefore \{S_n\}$ 也发散.

ex2.3.5 $0 < b < a$ 令 $a_0 = a, b_0 = b$ 递推公式

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, b_n = \sqrt{a_{n-1}b_{n-1}}, \quad n \in \mathbb{N}_+ \tag{2.1}$$

定义数列a_n,b_n. 证明这两个数列收敛于同一个极限 $AG(a, b)$.

由 AG 不等式 $a > \frac{a+b}{2} > \sqrt{ab} > b > 0$, 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a, b) = \frac{\pi}{2G} \tag{2.2}$$

如果令 $a_1=\frac{a+b}{2}, b_1=\sqrt{ab}$. 则

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta+b_1^2\sin^2\theta}} \tag{2.3}$$

上面这个公式是怎么得到的:
参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}\qquad (a>b>0) \tag{2.4}$$

这里令

$$\sin\phi=\frac{2a\sin\theta}{(a+b)+(a-b)\sin^2\theta} \tag{2.5}$$

$\theta\in[0,\frac{\pi}{2}]\to\phi\in[0,\frac{\pi}{2}]$, 取微分

$$\cos\phi\mathrm{d}\phi=2a\frac{(a+b)-(a-b)\sin^2\theta}{[(a+b)+(a-b)\sin^2\theta]^2}\cos\theta\mathrm{d}\theta \tag{2.6}$$

但是

$$\cos\phi=\frac{\sqrt{(a+b)^2-(a-b)^2\sin^2\theta}}{(a+b)+(a-b)\sin^2\theta}\cos\theta. \tag{2.7}$$

(2.6) / (2.7), 两式相除, 得到

$$\mathrm{d}\phi=2a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta}\frac{\mathrm{d}\theta}{\sqrt{(a+b)^2-(a-b)^2\sin^2\theta}} \tag{2.8}$$

另一方面

$$\sqrt{a^2\cos\phi+b^2\sin^2\phi}=a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta} \tag{2.9}$$

因而

$$\frac{\mathrm{d}\phi}{\sqrt{a^2\cos\phi+b^2\sin^2\phi}}=\frac{\mathrm{d}\theta}{\sqrt{(\frac{a+b}{2})^2\cos^2\theta+ab\sin^2\theta}}. \tag{2.10}$$

如果令 $a_1=\frac{a+b}{2}, b_1=\sqrt{ab}$, 则

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta+b_1^2\sin^2\theta}} \tag{2.11}$$

反复应用该公式, 得到

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a_n^2\cos^2\phi+b_n^2\sin^2\phi}},\qquad (n=1,2,3,\dots) \tag{2.12}$$

$$\frac{\pi}{2a_n}<G<\frac{\pi}{2b_n} \tag{2.13}$$

积分 G 可以归结到第一类全椭圆积分 $K(k)=(1+k_1)K(k_1)=\frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}=(1+k_1)\int_0^{\frac{pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{1-k_1^2\sin^2\theta}} \tag{2.14}$$

其中
$a_1=\frac{1+\sqrt{1-k^2}}{2}=\frac{1+k'}{2}, b_1=\sqrt{k'}$
$k_1=\frac{\sqrt{a_1^2-b_1^2}}{a_1}=\frac{1-k'}{1+k'}, \frac{1}{a_1}=1+k_1$

2.3 2.3 单调数列

Example 2.6 2.3.6

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\cdots+n!}{n!}} \\ &= \frac{1}{n+1}\frac{1!+2!+\cdots+(n+1)!}{1!+2!+\cdots+n!} \\ &= \frac{3+3!+\cdots+(n+1)!}{(n+1)1!+(n+1)2!+\cdots+(n+1)!}\end{aligned}$$

$n>2$ 时, 分母每一项大于等于分子对应项.. $n>2$ 后a_n 单调减少. 由于 0 是下界, 因此 a_n 单调有界, 数列收敛.

$$\begin{aligned}a_{n+1} &= \frac{1!+2!+\cdots+(n+1)!}{(n+1)!} \\ &= \frac{1!+2!+\cdots+n!}{n!}\frac{1}{n+1}+1 \\ &= 1+\frac{a_n}{n+1}\end{aligned}$$

设 $n\to\infty$ 时, $a_n\to a$

$$a=1+\left(\frac{1}{n+1}\to 0\right)=1+0, \qquad \therefore a=1$$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1!+2!+\cdots+n!}{n!} = 1$

2.3.1 2.3.2 练习题

证明, 若x_n 单调, 则$|x_n|$ 至少从某项开始后单调, 又问: 反之如何?

Proof 分类讨论, 不妨设 $x_1 \geqslant 0$

1. x_n 单调递增, $|x_n|$ 从第一项开始单调.
2. x_n 单调递减, 且 $|x_n|\geqslant 0$. $|x_n|$ 从第一项开始单调.
3. x_n 单调递减, 且 $\exists N$ s.t. $x_n<0$(第一个负数项). 则$|x_n|$ 从第 N 项 (x_N) 开始单调.

反之该结论不成立.

反例: $x_n=\frac{(-1)^n}{n}$, $|x_n|$ 单调递减. 但 $x_{2k}=\frac{1}{2k}>0>x_{2k-1}=\frac{-1}{2k-1}$

设a_n 单调增加, b_n 单调减少, 且有 $\lim\limits_{n\to\infty}(a_n-b_n)=0$.

证明: 数列a_n 和b_n 都收敛, 且极限相等.

Proof $\lim\limits_{n\to\infty}(a_n-b_n)=0, \forall\epsilon>0, \exists N\in\mathbb{N}_+, \text{s.t.} \forall n>N, |a_n-b_n-0|<\epsilon$.

$b_n-\epsilon<a_n<b_n+\epsilon$, 同时有 $a_n-\epsilon<b_n<a_n+\epsilon$.

b_n 单调减少, $\therefore \exists N_2, \forall m<N_2, b_m>b_n+\epsilon$.

使用反证法证明 b_m 是a_n 的上界.

假设 b_m 不是a_n 的上界，则存在 $a_n>b_m>b_n+\epsilon$, 这与 $|a_n-b_n|<\epsilon$ 矛盾.

$\therefore$ b_m 是a_n 的上界, 根据单调有界收敛准则, a_n 收敛. 同理可证b_n 收敛. $\lim\limits_{n\to\infty}(a_n-b_n)=0$. $\therefore \lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty} b_n$.

按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.
2. 单调减少有下界的数列的极限不大于数列中的任何一项.

设 $x_n=\frac{2}{3}\cdot\frac{3}{5}\cdots\frac{n+1}{2n+1}$, $n\in\mathbb{N}_+$, 求数列x_n 的极限.

$$\frac{x_{n+1}}{x_n}=\frac{(n+1)+1}{2(n+1)+1}=\frac{n+2}{2n+3}<1. \qquad (n>0) \tag{2.15}$$

x_n 单调递减. $\because x_n > 0$, $\therefore x_n$ 有下界, x_n 收敛.

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$, 由夹逼定理, $\lim\limits_{n\to\infty} x_n = 0$

6. 在例题 2.2.6 的基础上证明: 当 $p > 1$ 时，数列S_n 收敛. 其中

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$$

(S_n 就是 p 级数, 当 $p = 1$ 时为调和级数.)

Proof S_n 单调递增, 记 $\frac{1}{2^{p-1}} = r$, 则 $0 < r < 1$.

$$\begin{aligned}
\frac{1}{2^p} + \frac{1}{3^p} &< \frac{1}{2^p} + \frac{1}{2^p} &&= \frac{1}{2^{p-1}} = r \\
\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} &< \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} &&= \frac{1}{4^{p-1}} = r^2 \\
\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} &< \frac{1}{(2^k)^p} + \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^k)^p} &&= \frac{1}{(2^k)^{p-1}} = r^k
\end{aligned}$$

由此可知

$$S_n \leqslant S_{2^n-1} < 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} < \frac{1}{1-r}$$

S_n 单调递增有上界，由单调有界收敛准则知S_n 收敛。

7. 设 $0 < x_0 < \frac{\pi}{2}$, $x_n = \sin x_{n-1}$. $n \in \mathbb{N}_+$.

证明x_n 收敛，并求其极限。

Proof $x_0 \in (0, \frac{\pi}{2})$, $\sin x$,

$0 < x_1 = \sin x_0 < x_0 < \frac{\pi}{2}$.

$0 < x_2 = \sin x_1 < x_1 < \frac{\pi}{2}$.

$0 < \cdots < x_n < x_{n-1} < \cdots < x_2 < x_1 < \frac{\pi}{2}$.

x_n 单调递减有下界，x_n 收敛。

$a = \sin a, \quad a \in [0, \frac{\pi}{2}]$

解得 $a = 0$，$\therefore \lim\limits_{n\to\infty} x_n = 0$.