

いただきます



Author: weiyuan

Date: 2022 10 24

Chapter 1

1.1

1.1.1 2.1.5

1. prove by Limit definition:

- (1). $\lim_{n \rightarrow \infty} \frac{3n^2}{n^2-4} = 3$.
- (2). $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$.
- (3). $\lim_{n \rightarrow \infty} (1+n)^{\frac{1}{n}} = 0$.
- (4). $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0, (a > 0)$.

2. Suppose $a_n, n \in \mathbb{N}_+$. sequence a_n converge to a .

Prove $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$.

Proof $n \rightarrow \infty, a_n \rightarrow a$.

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$. \square (check, not consider the condition $a = 0$) add $a = 0, \forall \epsilon \in (0, 1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$. s.t $a_n < \epsilon^2 < \epsilon, \sqrt{a_n} < \epsilon$.

3. If $\lim_{n \rightarrow \infty} a_n = a$.

Prove $\lim_{n \rightarrow \infty} |a_n| = |a|$. Vice versa?

Proof $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$||a_n| - |a|| \leq |a_n - a| < \epsilon$$

$\therefore \lim_{n \rightarrow \infty} |a_n| = |a|$

If We know $\lim_{n \rightarrow \infty} |a_n| = |a|$.

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), ||a_n| - |a|| < \epsilon$. We can't get $\lim_{n \rightarrow \infty} a_n = a$. For example: $a_n = \frac{1}{n} + 1, a = -1$, $\lim_{n \rightarrow \infty} |a_n| = |a|$ is $\lim_{n \rightarrow \infty} |\frac{1}{n} + 1| = |-1|$, but $\lim_{n \rightarrow \infty} \frac{1}{n} + 1 \neq -1$ \square

(1). Suppose $p(x)$ is a polynomial of x , if $\lim_{n \rightarrow \infty} a_n = a$, Prove $\lim_{n \rightarrow \infty} p(a_n) = p(a)$.

(2). Suppose $b > 0, \lim_{n \rightarrow \infty} a_n = a$. Prove $b^{a_n} = b^a$.

(3). Suppose $b > 0, \{a_n\}, a_n > 0, \forall n \in \mathbb{N}$. $\lim_{n \rightarrow \infty} a_n = a, a > 0$. Prove $\lim_{n \rightarrow \infty} \log_b a_n = \log_b a$.

(4) Suppose $b \in \mathbb{R}, \{a_n\}, a_n > 0$ when $n \in \mathbb{N}$. $\lim_{n \rightarrow \infty} a_n = a$. Prove $\lim_{n \rightarrow \infty} a_n^b = a^b$.

(5) Suppose $\lim_{n \rightarrow \infty} a_n = a$. Prove $\lim_{n \rightarrow \infty} \sin a_n = \sin a$.

Proof 4.(1)

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geq N(\epsilon), |a_n - a| < \epsilon$.

$p(a) = k_m a^m + k_{m-1} a^{m-1} + \dots + k_0 a^0$.

$\therefore p(a_n) - p(a) = k_m (a_n^m - a^m) + k_{m-1} (a_n^{m-1} - a^{m-1}) + \dots + k_0 (a_n^0 - a^0)$.

$$|a_n^m - a^m| = |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2} a + \dots + a^{m-1}|$$

$$< \epsilon \cdot |a_n^{m-1} + a_n^{m-2} a + \dots + a^{m-1}|$$

$$< \epsilon(m-1) \dots (a + \delta)^{m-1}$$

$\therefore \lim_{n \rightarrow \infty} p(a_n) = p(a)$. \square

Proof 4.(2)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$.

If $b = 1, 1^{a_n} = 1^a = 1$.

If $b > 1, b^{a_n} - b^a = b^a(b^{a_n-a} - 1) < b^a(b^\epsilon - 1) < b^a \cdot (b^\epsilon - 1) \because b > 0, \epsilon \rightarrow 0, \therefore b^\epsilon - 1 \rightarrow 0$.

$\therefore \lim_{n \rightarrow \infty} b_n^a = b^a$.

If $b < 1, b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$.

Proof 4.(3)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$.

$$\begin{aligned} \log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b \left(\frac{a_n - a}{a} + 1 \right) < \log_b \left(\frac{\epsilon}{a} + 1 \right) \end{aligned}$$

$0 < \log_b a_n - \log_b a < \log_b \left(1 + \frac{\epsilon}{a} \right) \because b > 0, a \neq 0, a_n > 0$ when $\epsilon \rightarrow 0$. $\therefore \log_b \left(1 + \frac{\epsilon}{a} \right) \rightarrow 0$.

$\therefore \lim_{n \rightarrow \infty} \log_b a_n = \log_b a$

Proof 4.(4)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$.

$a_n^b = e^{b \ln a_n}, a_n^b - a^b = e^{b \ln a_n} - e^{b \ln a}$.

$$\begin{aligned} e^{b \ln a_n} - e^{b \ln a} &= e^{b \ln a} (e^{b \ln a_n - b \ln a} - 1) \\ &= e^{b \ln a} (e^{b \ln \frac{a_n}{a}} - 1) \end{aligned}$$

$0 < |a_n^b - a^b| < e^{b \ln a} (e^{b \ln(1 + \frac{\epsilon}{a})} - 1)$

$\therefore \lim_{n \rightarrow \infty} a_n^b = a^b$

Proof 4.(5)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$.

$$\begin{aligned} \sin(A + B) - \sin(A - B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B \end{aligned}$$

$$\sin a_n - \sin a = 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}$$

$$|\sin a_n - \sin a| = \left| 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2} \right| < \left| 2 \sin \frac{a_n - a}{2} \right|$$

$$\left| 2 \sin \frac{a_n - a}{2} \right| < \left| 2 \frac{a_n - a}{2} \right| = \epsilon$$

$$|\sin a_n - \sin a| < \epsilon, \therefore \lim_{n \rightarrow \infty} \sin a_n = \sin a$$

assume $a > 1$. Prove $\lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$

Proof $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \log_a 1 = 0$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geq N, |\sqrt[n]{n} - 1| < \epsilon$.

$a > 1$, and $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$. \therefore when $n \rightarrow \infty, \sqrt[n]{n} < a^\epsilon$, take logarithm on base of a , we can get $\frac{1}{n} \log_a n < \epsilon$

$\therefore \lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$

1.2

- 2.
- 3.
- 4.
- 5.

1.2.1

1. $\{a_n\}$, $\{b_n\}$, $\{a_n + b_n\}$, $\{a_n \cdot b_n\}$.
2. $\{a_n\}, \{b_n\}$, $\{a_n + b_n\}$ (ex: $n + -n, n + -2n$), $\{a_n \cdot b_n\}$ (?).
3. $a_n \leq b_n \leq c_n, n \in \mathbb{N}_+$. $\lim_{n \rightarrow \infty} (c_n - a_n) = 0$. $\{b_n\}$
4. $\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n})$
5. $a_n \rightarrow a (n \rightarrow 0)$. $\forall n, b < a_n < c$. $b < a < c$?
6. $a_n \rightarrow a (n \rightarrow 0)$. and $b \leq a \leq c$, $N \in \mathbb{N}_+$, s.t. $n > N$ $b \leq a_n \leq c$
7. $\lim_{n \rightarrow \infty} a_n = 0$, : $\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0$. ?

Proof 5.4

$$\frac{n}{2n} \leq \frac{1}{n+1} + \dots + \frac{1}{2n} \leq \frac{n}{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{n}{2n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1, \therefore \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n}) = 1$$

$$\begin{aligned} \frac{1}{n+1} + \dots + \frac{1}{2n} &= \frac{1}{n} \left(\frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \dots + \frac{1}{1 + \frac{n}{n}} \right) \\ &= \int_0^1 \frac{1}{1+x} dx \\ &= \ln(1+x)|_0^1 = \ln 2 \end{aligned}$$

$$\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n}) = \ln 2$$

Proof 5.5

$$b=0, c=2, a_n = \frac{1}{n}, \lim_{n \rightarrow \infty} a_n = 0 = c.$$

Proof 5.6

$$a = 0, b = 0, c = 2, a_n = (-1)^n \frac{1}{n}.$$

$$b \leq a \leq c, \text{ but } (-1)^{2n+1} \frac{1}{2n+1} < 0 = b.$$

$$\textbf{Proof} \quad \lim_{n \rightarrow \infty} a_n = 0, a_n = \frac{1}{n} \cdot a_1 a_2 \dots a_n = \frac{1}{n!}, \lim_{n \rightarrow \infty} \frac{1}{n!} = 0.$$

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0 \quad \checkmark$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0 \rightarrow \lim_{n \rightarrow \infty} a_n = 0 \quad \times$$

$$|a_n| < \epsilon, |a_{N+1} \dots a_n| < \epsilon^{n-N} < \epsilon, a_n < \sqrt[n]{\epsilon}.$$

$$\text{for example, } a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \dots, a_n = \frac{n}{n+1}.$$

$$a_1 a_2 \dots a_n = \frac{1}{2} \cdot \frac{2}{3} \dots \frac{n}{n+1} = \frac{1}{n+1}.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) \\ = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \end{aligned}$$

$$\text{but } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

$$1. \quad .$$

$$2. \quad .$$

Example 1.1 2.2.2 $\lim_{n \rightarrow \infty} \frac{x_n - 1}{x_n + a} = 0$,

prove $\lim_{n \rightarrow \infty} x_n = a$

Proof $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |\frac{x_n-1}{x_n+a} - 0| < \epsilon.$

$$|x_n - 1| < \epsilon |x_n + a| < 4a \cdot \epsilon. \quad (4 \quad ?)$$

$$|x_n - a| < \epsilon |x_n + a| = \epsilon |(x_n - a) + 2a| \leq \epsilon (|x_n - a| + 2a).$$

$$\epsilon < 1, |x_n - a| < 2\epsilon |a| / (1 - \epsilon).$$

$$\epsilon < \frac{1}{2}, |x_n - a| < 2\epsilon |a| / (1 - \epsilon) < 4|a|\epsilon.$$

Let $\epsilon' = 4a\epsilon$, $|x_n - 1| < \epsilon' \therefore \lim_{n \rightarrow \infty} x_n = a.$

Example 1.2 2.2.3 $a > 0, b > 0$, $\lim_{n \rightarrow \infty} (a_n + b_n)^{\frac{1}{n}}.$

Proof Suppose $a \leq b.$

$$b = (b^b)^{\frac{1}{n}} < (a^n + b^n)^{\frac{1}{n}} \leq (2b^n)^{\frac{1}{n}}.$$

$$b < (a^n + b^n)^{\frac{1}{n}} \leq \sqrt[n]{2b}, \lim_{n \rightarrow \infty} = 1. \quad .$$

$$\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

$$\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

Example 1.3 2.2.4 $a_n = \frac{1!+2!+\dots+n!}{n!}, n \in \mathbb{N}^+$

$$\lim_{n \rightarrow \infty} a_n = 1$$

Example 1.4 $\lim_{n \rightarrow \infty} \frac{n^3+n-7}{n+3} = +\infty$

Example 1.5 $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

$$H_n \quad .$$

1.2.2 2.2.4

Proof 1.

$$\{a_n\} \rightarrow a, \rightarrow \{a_{2n}\}, \{a_{2n+1}\} \rightarrow a.$$

$$\{a_{2n}\}, \{a_{2n+1}\} \rightarrow a, \rightarrow \{a_n\} \rightarrow a.$$

2.

$$(1). \quad p \quad a_1, a_2, \dots, a_p. \quad \lim_{n \rightarrow \infty} \sqrt[n]{a_1^n + a_2^n + \dots + a_p^n}.$$

$$\text{Let } a_s = \max_{1 \leq i \leq p} \{a_1, a_2, \dots, a_p\}.$$

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} \leq (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$$n \rightarrow \infty, p^{\frac{1}{n}} \rightarrow 1. \lim_{n \rightarrow \infty} (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} = a_s$$

$$(2). \quad x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}, n \in \mathbb{N}_+. \quad \lim_{n \rightarrow \infty} x_n$$

$$\frac{2n+1}{(n+1)} \leq x_n \leq \frac{2n+1}{\sqrt{n^2+1}}$$

$$\lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = 2, \lim_{n \rightarrow \infty} \frac{2n+1}{\sqrt{n^2+1}} = 2. \therefore \lim_{n \rightarrow \infty} x_n = 2$$

$$(3). \quad a_n = (1 + \frac{1}{2} + \dots + \frac{1}{n})^{\frac{1}{n}}, n \in \mathbb{N}_+. \quad \lim_{n \rightarrow \infty} a_n$$

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leq (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \therefore \lim_{n \rightarrow \infty} a_n = 1$$