具体数学阅读笔记-chap1 exercise

weiyuan

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0.1 Exercises

0.1.1 Warmups

练习 1. 1 All horses are the same color; we can prove this by induction on the number of horses in a given set. Here's how: "If there's just one horse then it's the same color as itself, so the basis is trivial. For the induction step, assume that there are n horses numbered 1 to n. By the induc- tion hypothesis, horses 1 through n - 1 are the same color, and similarly horses 2 through n are the same color. But the middle horses, 2 through n - 1, can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and n must be the same color as well, by transitivity. Thus all n horses are the same color; QED." What, if anything, is wrong with this reasoning?

题目解答 1. n=1 情况下马有相同颜色

但 n=2 时该假设不一定成立

练习 2. 2

题目解答 2. 不允许在 A B 之间直接移动, 求最短的移动序列

$$\begin{array}{lllll}
k=1 & 1 & A \Rightarrow C, C \Rightarrow B & 2 & sum=2 \\
k=2 & 1 & A \Rightarrow C, C \Rightarrow B, & 2 & \\
 & 2 & A \Rightarrow C & 1 & \\
 & 1 & B \Rightarrow C, C \Rightarrow A, & 2 & \\
 & 2 & C \Rightarrow B & 1 & \\
 & 1 & A \Rightarrow C, C \Rightarrow B & 2 & sum=8 \\
k=3 & 1 & A \Rightarrow C, C \Rightarrow B, & 2 & \\
 & 2 & A \Rightarrow C & 1 & \\
 & 1 & B \Rightarrow C, C \Rightarrow A, & 2 & \\
 & 2 & C \Rightarrow B & 1 & \\
 & 1 & A \Rightarrow C, C \Rightarrow B & 2 & sum=8 \\
 & 3 & A \Rightarrow C & 1 & \\
 & 1 & B \Rightarrow C, C \Rightarrow A, & 2 & \\
 & 2 & B \Rightarrow C & 1 & \\
 & 1 & A \Rightarrow C, C \Rightarrow B, & 2 & \\
 & 2 & C \Rightarrow A & 1 & \\
 & 1 & B \Rightarrow C, C \Rightarrow A, & 2 & \\
 & 3 & C \Rightarrow B & 1 & sum=18 \\
 & 1 & A \Rightarrow C, C \Rightarrow B, & 2 & \\
 & 2 & A \Rightarrow C & 1 & \\
 & 1 & B \Rightarrow C, C \Rightarrow A, & 2 & \\
 & 2 & C \Rightarrow B & 1 & \\
 & 1 & A \Rightarrow C, C \Rightarrow B & 2 & sum=26 \\
\vdots & & & & \\
k=n & 1 & A \Rightarrow C, C \Rightarrow B & &
\end{array}$$

从前面的移动可以看出 $f(n) = 3 * f(n-1) + 2$, 设 $g(n) = f(n) + 1$, $g(1) = f(1) + 1 = 3$, $g(n) = 3g(n-1)$. $g(n) = 3^n$, $f(n) = 3^n - 1$.

练习 3. 3

题目解答 3. 是的, 以 n 个圆盘为例正确的叠放方法有 3^n 种将 ABC 视为 3 个序列, 将所有圆盘从大到小依次放置在 3 个序列中, 每个圆盘放置时有 3 种选择, 所共有 3^n 种正确的叠放方法。第二题移动 $3^n - 1$ 次, 再加上移动前所有圆盘都在 A 柱上的情况, 共有 3^n 种情况, 所以所有正确的叠放方法均会出现。

我的思考, n 个圆盘在 3 根柱子上任意放的方法有多少种?

练习 4. 4

题目解答 4. Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, under Lucas's original rules?

是否存在 $m > 2^n - 1$

不存在。根据卢卡斯的规则, 将可能出现的移动情况分为两种:

1. 最大的圆盘不需要移动, 根据归纳法, 最多需要移动 $2^{n-1} - 1$ 次。
2. 最大的圆盘需要移动, 根据归纳法, 最多需要移动 $2^{n-1} - 1 + 1 + 2^{n-1} - 1$ 即 $2^n - 1$ 次

练习 5. 5

题目解答 5. 3 个给定集合, 共有 8 个可能子集。使用 Venn 图表示

[1] A, B, C, 三个集合的所有子集为 $\{\varnothing, A, B, C, A \cap B, A \cap C, B \cap C, A \cap B \cap C, \}$, $\{A \setminus B, A \setminus C, B \setminus A, B \setminus C, C \setminus A, C \setminus B\}, \{A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B)\}, \{A \cup B, A \cup C, B \cup C, A \cup B \cup C\}. \dots$

我认为这里所将的八个子集应当是 $\{\varnothing\}, \{A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B)\}, \{(A \cap B) \setminus C, (C \cap A) \setminus B, (B \cap C) \setminus A\}, \{A \cap B \cap C\}$. 空集和 7 个互不相交的真子集。

对于 4 个集合, Venn 图不能给出可能的 16 个子集, 因为不同的圆至多交于两点。参考答案中说的卵形 (ovals) 是什么意思?

练习 6. 6

题目解答 6. 无界区域个数 $2n$
所有区域个数 $\frac{n(n+1)}{2} + 1$
二者相减得到有界区域个数 $\frac{(n-1)(n-2)}{2}$

练习 7. 7

题目解答 7. 设 $H(n) = J(n+1) - J(n)$.
$H(2n) = 2$, 对 $n \geqslant 1$ 有

$$\begin{aligned} H(2n+1) &= J(2n+2) - J(2n+1) \\ &= (2J(n+1) - 1) - (2J(n) + 1) \\ &= 2H(n) - 2 \end{aligned}$$

但在 $n = 0$ 时, 由此推出

$$H(1) = J(2) - J(1) = 1 - 1 = 0 \neq 2$$

[1] Venn 图之后会补上

0.1.2 作业题

练习 8.

$$
\begin{aligned}
Q_0 &= \alpha \\
Q_1 &= \beta \\
Q_n &= \frac{1+Q_{n-1}}{Q_{n-2}}, \quad n > 1
\end{aligned}
$$

(hint: $Q_4 = \frac{1+\alpha}{\beta}$)

题目解答 8.

$$
\begin{array}{llllll}
Q_0 & = \alpha & & & & = \alpha \\
Q_1 & = \beta & & & & = \beta \\
Q_2 & = \dfrac{1+Q_1}{Q_0} & & & & = \dfrac{1+\beta}{\alpha} \\
Q_3 & = \dfrac{1+Q_2}{Q_1} & = \dfrac{1+\dfrac{1+\beta}{\alpha}}{\beta} & & & = \dfrac{1+\alpha+\beta}{\alpha\beta} \\
Q_4 & = \dfrac{1+Q_3}{Q_2} & = \dfrac{1+\dfrac{1+\alpha+\beta}{\alpha\beta}}{\dfrac{1+\beta}{\alpha}} & = \dfrac{\alpha\beta+1+\alpha+\beta}{\beta(1+\beta)} & & = \dfrac{1+\alpha}{\beta} \\
Q_5 & = \dfrac{1+Q_4}{Q_3} & = \dfrac{1+\dfrac{1+\alpha}{\beta}}{\dfrac{1+\alpha+\beta}{\alpha\beta}} & = \dfrac{\alpha\beta+\alpha(1+\alpha)}{1+\alpha+beta} & & = \alpha \\
Q_6 & = \dfrac{1+Q_5}{Q_4} & = \dfrac{1+\alpha}{\dfrac{1+\alpha}{\beta}} & & & = \beta
\end{array}
$$

因此解得

$$
\begin{array}{lllllll}
Q_i & = \{ & \alpha, & \beta, & \dfrac{1+\beta}{\alpha}, & \dfrac{1+\alpha+\beta}{\alpha\beta}, & \dfrac{1+\alpha}{\beta} & \} \\
(i\%n) & = \{ & 0, & 1, & 2, & 3, & 4, & \}
\end{array}
$$

练习 9. 反向归纳法, 从 n 到 $n-1$ 证明命题

$$
P(n):\ x_1 \dots x_n \leqslant \left(\frac{x_1+\cdots+x_n}{n}\right)^n, \quad x_i \geqslant 0, i = 1, \dots, n
$$

$n = 2$ 时为真

$$
(x_1+x_2)^2 - 4x_1x_2 = (x_1-x_2)^2 \geqslant 0
$$

a) $x_n = \frac{x_1+\cdots+x_{n-1}}{n-1}$, 证明只要 $n > 1$ 时 $P(n)$ 蕴含 $P(n-1)$.

b) 证明 $P(n)$ 和 $P(2)$ 蕴含 $P(2n)$

c) 由 a), b) 说明这就蕴含了 $P(n)$ 对所有 n 为真

题目解答 9. a) $P(n)$ 成立, $\forall n > 1$
给定 $x_n = \frac{x_1+\cdots+x_{n-1}}{n-1}$, 则有

$$\begin{aligned} x_1 \dots x_{n-1} \cdot \frac{x_1+\cdots+x_{n-1}}{n-1} &\leqslant \left(\frac{x_1+x_{n-1}+\frac{x_1+\cdots+x_{n-1}}{n-1}}{n} \right)^n \\ x_1 \dots x_{n-1} \cdot \frac{x_1+\cdots+x_{n-1}}{n-1} &\leqslant \left(\frac{x_1+\cdots+x_{n-1}}{n-1} \right)^n \\ x_1 \dots x_{n-1} &\leqslant \left(\frac{x_1+\cdots+x_{n-1}}{n-1} \right)^{n-1} \end{aligned}$$

$P(n)$ 成立

b) 由 $P(n)$ 可得

$$x_1 \dots x_n \cdot x_{n+1} \dots x_{2n} \leqslant \left(\frac{x_1+\cdots+x_n}{n} \right)^n \cdots \left(\frac{x_{n+1}+\cdots+x_{2n}}{n} \right)^n$$

记 $A = \left(\frac{x_1+\cdots+x_n}{n} \right), B = \left(\frac{x_{n+1}+\cdots+x_{2n}}{n} \right)$
由 $P(2)$ 可得

$$\begin{aligned} AB &\leqslant \left(\frac{A+B}{2} \right)^2 \\ A^n B^n = (AB)^n &\leqslant \left(\frac{A+B}{2} \right)^{2n} \\ x_1 \dots x_{2n} &\leqslant \left(\frac{x_1+\cdots+x_{2n}}{2n} \right)^{2n} \end{aligned}$$

由此推知 $P(2n)$ 成立。

c) Cauchy 向前-向后方法。

1. $P(2) \Rightarrow P(4) \Rightarrow \cdots P(2^n)$.
2. $P(n) \Rightarrow P(n-1)$.

$\therefore \forall n \geqslant 1, P(n)$ 成立

练习 10. 圆盘只能在 ABC 三根柱子上按照顺时针方向移动。记:
Q_n 为 n 个盘从 A 到 B 最少移动的次数。
R_n 为 n 个盘从 B 到 A 最少移动的次数。

题目解答 10. 先列出两种移动方式各自的迭代式:

$$Q_n = \begin{cases} 0, & n = 0 \\ 2R_{n-1} + 1, & n > 0 \end{cases} \qquad R_n = \begin{cases} 0, & n = 0 \\ Q_n + Q_{n-1} + 1, & n > 0 \end{cases}$$

这两个公式是如何得到的?

$$\begin{array}{lll}
k = 0 & Q_0 = 0 & R_0 = 0 \\
k = 1 & A \Rightarrow B & B \Rightarrow C \Rightarrow A \\
 & Q_1 = 1 & R_1 = 2 \\
k = 2 & 1 : A \Rightarrow B \Rightarrow C & 1 : B \Rightarrow C \Rightarrow A \\
 & 2 : A \Rightarrow B & 2 : B \Rightarrow C \\
 & 1 : C \Rightarrow A \Rightarrow B & 1 : A \Rightarrow B \\
 & & 2\square C \Rightarrow A \\
 & & 1 : B \Rightarrow C \Rightarrow A \\
 & Q_2 = 5 & R_2 = 7 \\
k = n & A \Rightarrow B\ Q_n & B \Rightarrow A\ R_n \\
 & (n-1)A \Rightarrow C\ R_{n-1} & (n-1)B \Rightarrow A\ R_{n-1} \\
 & n : A \Rightarrow B\ 1 & n : B \Rightarrow C\ 1 \\
 & (n-1)C \Rightarrow B\ R_{n-1} & (n-1)A \Rightarrow B\ Q_{n-1} \\
 & & n : C \Rightarrow A\ 1 \\
 & & (n-1)B \Rightarrow A\ R_{n-1} \\
 & Q_n = 2R_{n-1} + 1 & R_n = Q_{n-1} + 2R_{n-1} + 2 = Q_n + Q_{n-1} + 1
\end{array}$$

练习 11. 双重河内塔 2n 个圆盘, 第 $2k-1$ 个与第 $2k$ 个大小相同。

题目解答 11. a) 不区分相同尺寸

$$\begin{array}{ll}
n = 0 S_0 = 0 & \\
n = 1 S_1 = 2 & A \Rightarrow B, A \Rightarrow B \\
n = 2 S_2 = 6 & A \Rightarrow C, A \Rightarrow C \\
\quad A \Rightarrow B, A \Rightarrow B & \\
\quad C \Rightarrow B, C \Rightarrow B &
\end{array}$$

解得 $S_n = 2T_n = 2(2^n - 1) = 2^{n+1} - 2$

b) 在最后排列中将圆盘恢复次序需要移动几次?

$$
\begin{array}{llll}
k=0 & R_0=0 & & \\
k=1 & R_1=3 & 1.1: \ A\Rightarrow C & \\
 & & 1.2: \ A\Rightarrow B & \\
 & & 1.1: \ C\Rightarrow B & \\
k=2 & R_2=11 & 1.1: \ A\Rightarrow B & \\
 & & 1.2: \ A\Rightarrow B & \\
 & & 2.1: \ A\Rightarrow C & \\
 & & 1.2: \ B\Rightarrow C & \\
 & & 1.1: \ B\Rightarrow C & \\
 & & 2.2: \ A\Rightarrow B & \\
 & & 1.1: \ C\Rightarrow A & \\
 & & 1.2: \ C\Rightarrow A & \\
 & & 2.1: \ C\Rightarrow B & \\
 & & 1.2: \ A\Rightarrow B & \\
 & & 1.1: \ A\Rightarrow B & \\
k=n & R_n & n-1\ A\Rightarrow B & S_{n-1} \\
 & & n.1: \ A\Rightarrow C & 1 \\
 & & n-1\ B\Rightarrow C & S_{n-1} \\
 & & n.2: \ A\Rightarrow B & 1 \\
 & & n-1\ C\Rightarrow A & S_{n-1} \\
 & & n.1: \ C\Rightarrow B & 1 \\
 & & n-1\ A\Rightarrow B & S_{n-1}
\end{array}
$$

$$R_n = 4S_{n-1}+3 = 2^{n+2}-5 \quad (n \geqslant 1)$$

练习 12. 12 11 推广, m_k 个尺寸为 k 的圆盘, 不区分相同尺寸的圆盘移动一个塔最少次数 $A(m_1,\ldots,m_n)$

题目解答 12.

$$
\begin{aligned}
F(0) &= 0 \\
F(1) &= m_1 \\
F(2) &= 2F(1)+m_2 = 2m_1+m_2 \\
&\vdots \\
F(n) &= 2F(n-1)+m_n
\end{aligned}
$$

$$\begin{aligned}
A(m_1,\ldots,m_n) = F(n) &= 2F(n-1) + m_n \\
&= 2^{n-1}m_1 + 2^{n-2}m_2 + \cdots + m_n \\
&= \sum_{k=1}^{n} 2^{n-k} m_k
\end{aligned}$$

练习 13. 13

题目解答 13.

$$\begin{aligned}
k &= 1 \quad ZZ_1 = 2 + 0 = 2 \\
k &= 2 \quad ZZ_2 = 4 + 8 = 12 \\
k &= 3 \quad ZZ_3 = 6 + 25 = 31
\end{aligned}$$

对于定义了 L_n 个区域的 n 条直线, 可以用极狭窄的 Z 形线来代替。

例如, 每一对 Z 形线间有 9 个交点

$$\begin{aligned}
ZZ_n &= ZZ_{n-1} + 9n - 8, \quad (n > 0) \\
ZZ_n &= 9S_n - 8n + 1 \\
&= 9\frac{n(n+1)}{2} - 8n + 1 \\
&= \tfrac{9}{2}n^2 - \tfrac{7}{2}n + 1
\end{aligned} \tag{1}$$

练习 14. 14

题目解答 14.

$$\begin{aligned}
n &= 0 \quad P_0 = 1 \\
n &= 1 \quad P_1 = 2 \\
n &= 2 \quad P_2 = 4 \\
n &= 3 \quad P_3 = 8 \\
n &= 4 \quad P_4 = 8 + 6 = 14
\end{aligned}$$

$$P_n = P_{n-1} + L_{n-1}$$

其中

$$L_n = 1 + S_n, \quad S_n = \frac{n(n+1)}{2}$$

$\therefore P_n = P_{n-1} + 1 + \frac{n(n+1)}{2}$

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= P_0 + L_0 &&= 1 + 1 + \tfrac{0\cdot 1}{2} &&= 2 \\
P_2 &= P_1 + L_1 &&= 2 + 1 + \tfrac{1\cdot 2}{2} &&= 4 \\
P_3 &= P_2 + L_2 &&= 4 + 1 + \tfrac{2\cdot 3}{2} &&= 8 \\
P_4 &= P_3 + L_3 &&= 8 + 1 + \tfrac{3\cdot 4}{2} &&= 15 \\
P_5 &= P_4 + L_4 &&= 15 + 1 + \tfrac{4\cdot 5}{2} &&= 26
\end{aligned}$$

$$\begin{aligned}P_n &= P_{n-1} + L_{n-1}\\ &= 0 + \sum_{k=0}^{n-1}\left(1 + \frac{k(k+1)}{2}\right)\\ &= n + \frac{(n-1)n(n+1)}{6}\\ &= \frac{n(n^2+5)}{6}\end{aligned}$$

练习 15. 15 约瑟夫问题, 倒数第二个 $I(n)$

表 1: 约瑟夫问题 J(n) 与 I(n)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5
I(n)	∼	2	1	3	5	1	3	5	7	9	11	1	3	5	7	9	11	13

题目解答 15. $n > 1$ 时, $J(n), I(n)$ 有相同递归式

$$\begin{aligned}&I(2) = 2, I(1) = 1\\ &n = 2^m + 2^{m-1} + k, \quad 0 \leqslant k \leqslant 2^m + 2^{m-1}\\ &I(n) = 2k + 1\end{aligned}$$

$$n = 2^m + l, \quad I(n) = \begin{cases} J(n) + 2^{m-1}, & 0 \leqslant l < 2^{m-1}\\ J(n) - 2^m, & 2^{m-1} \leqslant l < 2^m\end{cases}$$

练习 16.

$$\begin{cases} g(1) &= \alpha\\ g(2n+j) &= 3g(n) + \gamma n + \beta_j, \quad j = 0, 1, n \leqslant 1\end{cases}$$

(提示, 用 $g(n) = n$)

题目解答 16. Suppose $g(n) = n$

$$g(1) = 1 = \alpha,$$
$$g(2n+j) = 2n + j = 3n + \gamma n + \beta_j.$$

解得 $\alpha = 1, \gamma = -1, \beta_j = \begin{cases} 0, & j = 0\\ 1, & j = 1\end{cases}$

(题解)

$$g(n) = a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 + d(n)\gamma$$

$n = (1b_{m-1} \ldots b_1 b_0)_2$ 将 n 以基数 2 展开 (写成二进制)。

$$a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 = (\alpha\beta_{m-1}\beta_{m-2} \ldots \beta_{b_1}\beta_{b_0})_3$$

$g(n) = n. \quad (\alpha = 1, \beta_0 = 0, \beta_1 = 1, \gamma = -1)$

$$a(n) + c(n) - d(n) = n$$

$g(n) = 1. \quad (\alpha = 1, \beta_0 = -2, \beta_1 = 2, \gamma = 0)$

$$a(n) - 2b(n) - c(n) = 1$$

$$\begin{aligned} d(n) &= a(n) + c(n) - n \\ b(n) &= \frac{1}{2}a(n) - \frac{1}{2}c(n) - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} g(n) = a(n)\alpha + (\frac{1}{2}a(n) - \frac{1}{2}c(n) - \frac{1}{2})\beta_0 \\ + c(n)\beta_1 + (a(n) + c(n) - n)\gamma \end{aligned}$$

若 $\beta_i = 0\ (i = 0, 1), \gamma = 0$

$$\begin{cases} g(1) = \alpha \\ g(2n + j) = 3g(n) \end{cases}$$

$g(1) = \alpha, g(2) = g(3) = 3\alpha, g(4) = g(5) = g(6) = g(7) = 9\alpha, g(8) = 3g(4) = 27\alpha$. 由此推知

$$g(n) = g(2^m + k) = 3^m, \quad a(n) = 3^m$$

继续计算 $d(n)$ 遇到困难

若 $\alpha = 0, \beta_0 = 0, \gamma = 0, g(n) = \beta_1 c(n)$

$$\begin{cases} g(1) = 0 \\ g(2n) = 3g(n) \\ g(2n + 1) = 3g(n) + 1 \end{cases}$$

$g(2) = 0$, $g(3) = 1$, $g(4) = 0$, $g(5) = 1$, $g(6) = 3g(3) = 3$, $g(7) = 3g(3) + 1 = 4$, $g(8) = 0$, $g(9) = 1\ldots$

表 2: m,k 变化规律

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	0	0	1	0	1	3	4	0	1	3	4	9	10	12	13	0	1	3	4
m	0	1		2				3								4			
k			1		1	2	3		1	2	3	4	5	6	7		1	2	3

$$g(2^m + k) = g((1b_{m-1}b_{m-2}\cdots_1 b_0)_2)$$

复习和重做 16 题的部分暂不录入

练习 17.

$$W_{n(n+1)/2} \leqslant 2W_{n(n-1)/2} + T_n, \quad n > 0$$

题目解答 17. In general we have $W_m \leqslant 2W_{m-k} + T_k,\ 0 \leqslant k \leqslant M$ (This relation corresponds to transferring the top $m-k$. then using only three pegs to move the bottom $k \Rightarrow T_k$, then finishing with the top $m-k \Rightarrow 2 \cdot W_{m-k}$)

The stated relation turns out to be based on the unique value of k that minimizes the right-hand side of this general inequality, when $m\frac{n(n+1)}{2}$.

(However, we cannot conclude that equality holds. Many other strategies for transferring the tower are conceivale.)

If we set $Y_n = (W_{n(n+1)/2} - 1)/2^n$

we find that $Y_n \leqslant Y_{n-1} + 1$. hence $W_{n(n+1)/2} \leqslant 2^n(n-1) + 1$

练习 18. 证明如下的一组 n 条折线定义 Z_n 个区域

$$\begin{aligned} Z_n = L_{2n} - 2n &= \frac{2n(2n+1)}{2} + 1 - 2n \\ &= 2n^2 - n + 1. \quad n \geqslant 0. \end{aligned} \tag{2}$$

第 j 条折线 $(1 \leqslant j \leqslant n)$ 的锯齿点在 $(n^{kj}, 0)$. 并向上经过点 $(n^{2j} - n^j, 1)$ 与 $(n^{2j} - n^j - n^{-n}, 1)$

题目解答 18.

练习 19. 当每一个锯齿的角度为 30° 时, 有可能由 n 条折线得到 Z_n 个区域吗?

题目解答 19.

练习 20. 利用成套方法解递归式:

$$\begin{cases} h(1) & = \alpha \\ h(2n) & = 4h(n) \quad +\gamma_0 n \quad +\beta_0 \qquad n \geqslant 1 \\ h(2n) & = 4h(n) \quad +\gamma_1 n \quad +\beta_1 \end{cases}$$

题目解答 20.

$$h(n) = \alpha A(n) + \beta_0 B(n) + \gamma_0 C(n) + \beta_1 D(n) + \gamma_1 E(n)$$

1. $h(1) = 1$

$$\begin{cases} 1 & = \alpha \\ 1 & = 4 \times 1 \quad +\gamma_0 n \quad +\beta_0 \\ 1 & = 4 \times 1 \quad +\gamma_1 n \quad +\beta_1 \end{cases}$$

解得 $(\alpha, \beta_0, \gamma_0, \beta_1, \gamma_1) = (1, -3, 0, -3, 0)$

$$1 = A(n) - 3B(n) - 3D(n)$$

2. $h(n) = n$

$$\begin{cases} 1 & = \alpha \\ 2n & = 4 \times n \quad +\gamma_0 n \quad +\beta_0 \\ 2n+1 & = 4 \times n \quad +\gamma_1 n \quad +\beta_1 \end{cases}$$

解得 $(\alpha, \beta_0, \gamma_0, \beta_1, \gamma_1) = (1, 0, -2, 1, -2)$

$$n = A(n) - 2C(n) + D(n) - 2E(n)$$

3. $h(n) = n^2$

$$\begin{cases} 1 & = \alpha \\ (2n)^2 & = 4 \times n^2 \quad +\gamma_0 n \quad +\beta_0 \\ (2n+1)^2 & = 4 \times n^2 \quad +\gamma_1 n \quad +\beta_1 \end{cases}$$

解得 $(\alpha, \beta_0, \gamma_0, \beta_1, \gamma_1) = (1, 0, 0, 1, 4)$

$$n^2 = A(n) + D(n) + 4E(n)$$

最终结果???

练习 21. $2n$ 个人围成圈, 前 n 个好伙计后 n 个坏家伙. 证明: 总存在一个整数 q (与 n 有关), 使得若在绕圆圈走时每隔 $q-1$ 个人处死一个, 那么所有坏家伙首先出局.
(例如, $n = 3$ 时取 $q = 5$, $n = 4$ 时取 $q = 30$)

题目解答 21. We can let m be the least (or any) common multiple of $2n, 2n-1, \ldots, n+1$.
a non-rigorous argument suggests that "random" value of m will succeed with probability

$$\frac{n}{2n}\frac{n-1}{2n-1}\cdots\frac{1}{n+1} = \frac{1}{\binom{2n}{n}} \sim \frac{\sqrt{\pi n}}{4^n}$$

0.1.3 附加题

练习 22. 22

题目解答 22. 22

练习 23. 假如约瑟夫发现自己处在 j, 但能指定 q, 隔 $q-1$ 人处死一人. 他是否能保全自己?

题目解答 23.