谢惠民数学分析习题课讲义上册笔记整理

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2022 年 11 月 21 日

目录

第一章   引论

1.1   关于习题课教案的组织

1.1.1   书中常用记号

1. $\mathbf{N}_+$：所有正整数组成的集合.

2. $\mathbf{R}$：所有实数组成的集合 (同时也用于表示无限区间 $(-\infty, \infty)$).

3. $\mathbf{Q}$：所有有理数组成的集合.

4. $\mathbf{C}$：所有复数组成的集合.

5. $\iff$ 是等价关系的记号.$A \iff B$ 表示 A 和 B 等价. 例如，A 代表 $x > 3$，B 代表 $x - 3 > 0$，则 $x > 3 \iff x - 3 > 0$.

6. $[x]$ 是实数 x 的整数部分，即不超过 x 的最大整数. 例如，$[\sqrt{2}] = 1, [-\sqrt{2}] = -2$. 关于 $[x]$ 的基本不等式是：$[x] \leq x < [x] + 1$，或 $x - 1 < [x] \leq x$

7. 空心方块表示一个证明或解的结束.

8. $\binom{n}{k} = C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k!}$.

9. 记号 $\approx$ 表示近似值. 例如 $\sqrt{2} \approx 1.4$.

10. 复合函数 $f(g(x))$ 也写成 $(f \circ g)(x)$ 或 $f \circ g$.

11. 若 A 和 B 为两个集合，则用记号 $A - B$ 或 $A\backslash B$ 表示 A 与 B 的差集，也就是集合 $\{x | x \in A 且 x \notin B\}$.

12. 用 $O_\delta(a)$ 表示以 a 为中心，以 $\delta > 0$ 为半径的邻域. 它就是开区间 $(a - \delta, a + \delta)$(也可以用 $U_\delta(a)$ 等记号). 如不必指出半径，则可简记为 $O(a)$ (或 $U(a)$).

1.1.2   几个常用的初等不等式

几个初等不等式的证明

A.G 不等式 $a_1, a_2, \cdots, a_n$，n 个非负实数

1

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \tag{1.1}$$

$\geq$ in inequation became $= \iff a_1 = a_2 = \cdots = a_n$

Proof. method 1. induction method

$$
\begin{aligned}
&k = 1 \qquad a_1 = a_1 \\
&k = 2 \qquad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \\
&k = n \qquad \text{suppose} \quad \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \\
&k = n + 1 \\
&\qquad \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} - \frac{a_1 + a_2 + \cdots + a_n}{n} \\
&\qquad = \frac{n(a_1 + a_2 + \cdots + a_{n+1}) - (n+1)(a_1 + a_2 + \cdots + a_n)}{n(n+1)} \\
&\qquad = \frac{n a_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n(n+1)}
\end{aligned}
$$

Set $A = \frac{a_1 + a_2 + \cdots + a_n}{n}, B = \frac{n a_{n+1} - (a_1 + a_2 \cdots + a_n)}{n(n+1)}$

$$\left(\frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1}\right)^{n+1} = (A + B)^{n+1}$$

$A > 0, B \geq 0$

$$
\begin{aligned}
&(A + B)^{n+1} \geq A^{n+1} + (n+1)A^n B \\
&A^{n+1} + (n+1)A^n B = A^n(A + (n+1)B) \\
&A^n = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \geq a_1 a_2 \cdots a_n \\
&A + (n+1)B = \frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{n a_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n} = a_{n+1} \\
&\therefore (A+B)^{n+1} \geq A^n(A + (n+1)B) \geq a_1 a_2 \cdots a_n \cdot a_{n+1} \\
&\therefore \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} \geq \sqrt[n+1]{a_1 a_2 \cdots a_n a_{n+1}}
\end{aligned}
$$

使用二项式展开定理的条件

在归纳法第二步，将 $a_1, a_2, \cdots, a_{n+1}$ 重编号，使得 $n+1$ 为其中最大的数 (之一)，这使得分解式右边第二项 $(n a_{n+1} - (a_1 + a_2 + \cdots + a_n))/n(n+1)$ 一定是非负数。

method 2. Forward and Backward (Cauchy, 1897)

Forward part

$k = 2.\dfrac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$

$k = 4.\dfrac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt{(\dfrac{a_1 + a_2}{2}) \cdot (\dfrac{a_3 + a_4}{2})}.$

$\qquad \geq \sqrt{\sqrt{a_1 a_2}\sqrt{a_3 a_4}} = \sqrt[4]{a_1 a_2 a_3 a_4}.$

$k = 2^n.\text{Suppose} \quad \dfrac{a_1 + a_2 + \cdots a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdots a_{2^n}}$

$k = 2^{n+1}.$

$$\frac{a_1 + a_2 + \cdots + a_{2^n} + \cdots + a_{2^{n+1}}}{2^{n+1}} \geq \sqrt{(\frac{a_1 + a_2 + \cdots + a_2^n}{2^n}) \cdot (\frac{a_{2^n+1} + a_{2^n+2} + \cdots + a_2^{n+1}}{2^n})}$$

$$I \geq \sqrt{\sqrt[2^n]{a_1 a_2 \cdots a_{2^n}} \sqrt[2^n]{a_{2^n+1} a_{2^n+2} \cdots a_{2^{n+1}}}} = \sqrt[2^{n+1}]{a_1 a_2 \cdots a_{2^{n+1}}}$$

Backward part suppose A.G inequality is valid when $k = n$, Consider $k = n - 1.$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i = \frac{1}{n}(\frac{n}{n-1})\sum_{i=1}^{n-1} a_i$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i = \frac{1}{n}(\sum_{i=1}^{n-1} a_i + \frac{1}{n-1}\sum_{i=1}^{n-1} a_i)$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n]{(\prod_{i=1}^{n-1} a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^n \geq (\prod_{i=1}^{n-1} a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^{n-1} \geq (\prod_{i=1}^{n-1} a_i)$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i}$$

$\square$

Proposition 1.1.1. 柯西-施瓦茨不等式对 $a_1, a_2, \cdots, a_n$ 和 $b_1, b_2, \cdots, b_n$, 成立

$$|\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}|$$

Proof.

$$0 \leq \sum_{i=1}^{n}(a_i - \lambda b_i)^2 = \sum_{i=1}^{n} a_i^2 - 2\lambda \sum_{i=1}^{n} a_i b_i + \lambda^2 \sum_{i=1}^{n} b_i^2$$

由韦达定理 (视 λ 为未知数). 原方程无解或只有唯一解。

$$\Delta = b^2 - 4ac \leq 0$$

$$(-2\sum_{i=1}^{n} a_i b_i)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq 0$$

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

$\square$

练习题

Example 1.1.1. 关于 Bernoulli 不等式的推广:

(1) 证明: 当 $-2 \leq h \leq -1$ 时 Bernoulli 不等式 $(1+h)^n \geq 1 + nh$ 仍成立;

(2) 证明: 当 $h \geq 0$ 时成立不等式 $(1+h)^n \geq \frac{n(n-1)h^2}{2}$, 并推广之;

(3) 证明: 若 $a_i > -1(i = 1, 2, \ldots, n)$ 且同号, 则成立不等式

$$\prod_{i=1}^{n}(1+a_i) \geq 1 + \sum_{i=1}^{n} a_i$$

Proof. (1)

$$-2 \leq h \leq -1$$

$$-1 \leq 1+h \leq 0 \qquad\qquad -1 \leq (1+h)^n \leq 0$$

$$-2n \leq nh \leq -n \qquad\qquad 1-2n \leq 1+nh \leq 1-n$$

$$n = 0. \qquad\qquad (1+h)^0 = 1 = 1 + 0 \times h$$

$$n = 1. \qquad\qquad 1+h = 1+h$$

$$n \geq 2. \qquad\qquad 1-n \leq -2$$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1+nh \geq 1-2n$$

$$(1+h)^n \geq 1+nh$$

(2)

$$h \geq 0$$

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots \geq \frac{n(n-1)}{2}h^2$$

推广:

$$(1+h)^n \geq \binom{n}{3}h^3, \binom{n}{4}h^4, \ldots, \binom{n}{k}h^k, 0 \leq k \leq n$$

(3) $k = 1$ 时显然成立. 使用归纳法证明. 假设 $k = n$ 时不等式 $\prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$ 成立, 证明 $k = n+1$ 时 $\prod_{i=1}^{n+1}(1+a_i) \geq 1 + \sum_{i=1}^{n+1} a_i$ 成立.

$$k = n+1 \qquad \prod_{i=1}^{n+1}(1+a_i) = \prod_{i=1}^n (1+a_i)(1+a_{n+1})$$

$$\because \prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$$

$$\prod_{i=1}^n (1+a_i)(1+a_{n+1}) \geq (1 + \sum_{i=1}^n a_i)(1+a_{n+1})$$

$$(1 + \sum_{i=1}^n a_i)(1+a_{n+1}) = 1 + \sum_{i=1}^n a_i + a_{n+1} + a_{n+1}\sum_{i=1}^n a_i$$

$$= 1 + \sum_{i=1}^{n+1} a_i + a_{n+1}\sum_{i=1}^n a_i$$

$$\geq 1 + \sum_{i=1}^{n+1} a_i$$

$\square$

Example 1.1.2. 利用 A.G. 不等式求解:

(1). $n! \leq (\frac{n+1}{2})^n$, while $n > 1$

(2). $(n!)^2 = (n \cdot 1)[(n-1) \cdot 2] \ldots (1 \cdots n)$. 证明: 当 $n > 1$ 时成立

$$n! < (\frac{n+2}{6})^n$$

(3). 比较上述两个不等式的优劣

(4). 证明: 对任意实数 r 成立:

$$(n!)^r \leq \frac{1}{n^n}(\sum_{k=1}^n k^r)^n \tag{1.2}$$

Proof. (1).

$$n > 1 \qquad n! = 1 \times 2 \times \cdots \times n < (\frac{1+2+\cdots+n}{n})^n = (\frac{(1+n)n}{2n})^n = (\frac{n+1}{2})^n$$

$\because 1 \neq 2 \neq \cdots n$, 所以不会有等号出现的情况

(2). $n > 1$

$$(n!)^2 = (n \cdot 1)[(n-1) \cdot 2] \ldots (1 \cdots n)$$

$$< (\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n})^n$$

Consider this equation

$$(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n})^n \tag{1.3}$$

$$\sum_{k=1}^{n}(n-k+1)k = (n+1)\sum_{k=1}^{n}k - \sum k^2$$
$$= (n+1)\frac{(n+1)n}{2} - \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(n+1)}{6}(3(n+1) - (2n+1))$$
$$= \frac{n(n+1)(n+2)}{6}$$

$$(n!)^2 < (\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n})^n$$
$$= (\frac{(n+1)(n+2)}{6})^n$$

$\because n+1 < n+2, \therefore n! < (\frac{n+2}{\sqrt{6}})^n$

(3). $n > 3$ 时,$\frac{n+2}{\sqrt{6}} < \frac{n+1}{2}$ (2) 的结果较好.

(4).$\forall r \in \mathbb{R}$, prove formula 1.2

$$\frac{1}{n}\sum_{k=1}^{n}k^r \geq \sqrt[n]{\prod_{k=1}^{n}k^r}$$
$$(n!)^r = \prod_{k=1}^{n}k^r \leq (\frac{1}{n}\sum_{k=1}^{n}k^r)^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$

my answer

$$\forall r \in \mathbb{R}, \qquad (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r$$
$$(n!)^r = \sum_{k=1}^{n}k^r \leq (\frac{1^r + 2^r + \cdots + n^r}{n})^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$
$$\therefore \quad (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r$$

$\square$

Example 1.1.3. $a_k > 0, k = 1, 2, \ldots, n$ 证明几何–调和平均值不等式

$$(\prod_{k=1}^{n}a_k)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}}$$

   Proof. from A.G inequality

$$\frac{\sum_{k=1}^{n}\frac{1}{a_k}}{n} \geq \sqrt[n]{\prod_{k=1}^{n}\frac{1}{a_k}} = \frac{1}{\sqrt[n]{\prod_{k=1}^{n}a_k}}$$

$$a_k > 0, \quad \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

$\square$

Example 1.1.4. $a, b, c \geq 0$. prove $\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$. 并推广到 n 个非负数的情况

Proof. 1. $\sqrt[3]{abc} = \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \leq \sqrt{\frac{ab+bc+ca}{3}}$.

2.

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}}$$

$$= \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}}$$

$$= \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}}$$

$a, b, c \geq 0$, 希望证明

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

$$\frac{ab+bc+ca}{3} \leq \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6}$$

$$\frac{ab+bc+ca}{2} \leq \frac{a^2+b^2+c^2}{6} + 2\frac{ab+bc+ca}{6} \qquad (\text{add}\frac{ab+bc+ca}{6})$$

$$\frac{ab+bc+ca}{3} \leq \frac{ab+bc+ca}{2} \leq (\frac{a+b+c}{3})^2$$

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

推广至 n 个

$$[l]n = 2 \qquad \sqrt{ab} \leq \frac{a+b}{2}$$

$$n = 3 \qquad \sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

$$n = 4 \qquad \sqrt[4]{abcd} \leq \sqrt[3]{\frac{abc+bcd+cda+dab}{4}} \leq \sqrt{\frac{a+b+c}{3}} \leq \frac{a+b+c+d}{4}$$

$$k = n \qquad \sqrt[n]{a_1 a_2 \dots a_n} \leq \sqrt{\frac{a_1 + a_2 + \dots + a_n}{n}} \leq \frac{a_1 + a_2 + \dots + a_n}{n}$$

This is

$$\sqrt[n]{\sum_{k=1}^{n} a_k} \leq \sqrt{\frac{\sum_{k=1}^{n} a_k}{k}} \leq \frac{\sum_{k=1}^{n} a_k}{k}$$

1. $\sqrt[n]{a_1 a_2 \dots a_n} = \sqrt{\sqrt[n]{a_1^2 a_2^2 \dots a_n^2}} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_n a_1}{n}}$

2. $\sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_n a_1}{n}} \leq \sqrt{\frac{a_1 + a_2 + \dots + a_n}{n}}$?

$\square$

Example 1.1.5. (1) $|\alpha + \beta| \leq |\alpha| + |\beta|$

Proof. let $\alpha = a - b, \beta = b$, the identity became $|(a - b) + b| \leq |a - b| + |b|$. This is $|a - b| \geq |a| - |b|$.

$$||a| - |b|| = \begin{cases} |a| - |b|. & a \geq b \\ |b| - |a|. & a < b \end{cases}$$

When $a \geq b$, $||a| - |b|| = |a| - |b|$. There is $|a - b| \geq |a| - |b| = ||a| - |b||$

When $a < b$, $|a - b| = |b - a| \geq |b| - |a| = ||a| - |b||$.

$\therefore$, We have$|a - b| \geq ||a| - |b||$ $\hfill\square$

(2) $\sum |a_k| \geq |\sum a_k|$

Proof. We can prove this statement by induction.

$$k = 2, \qquad |a_1| + |a_2| \geq |a_1 + a_2|$$
$$k = 3, \qquad |a_1| + |a_2| + |a_3| \geq |a_1 + a_2 + a_3|$$
$$\text{Suppose } k = n, \qquad \sum_{k=1}^{n} |a_k| \geq |\sum_{k=1}^{n} a_k|$$
$$k = n + 1, \qquad \text{prove} \sum_{k=1}^{n+1} |a_k| \geq |\sum_{k=1}^{n+1} a_k|$$

$$\sum_{k=1}^{n+1} |a_k| = \sum_{k=1}^{n} |a_k| + |a_{n+1}|$$
$$\geq |\sum_{k=1}^{n} a_k| + |a_{n+1}|$$
$$\geq |\sum_{k=1}^{n+1} a_k|$$

$$k = 2, \qquad |a_1| - |a_2| \leq |a_1 - a_2|$$
$$\text{Suppose } k = n, \qquad |a_1| - \sum_{k=2}^{n} |a_k| \leq |\sum_{k=1}^{n} a_k|$$
$$k = n + 1, \qquad \text{prove}|a_1| - \sum_{k=2}^{n+1} |a_k| \leq |\sum_{k=1}^{n+1} a_k|$$

$$\begin{aligned}
|a_1| - \sum_{k=2}^{n+1} |a_k| &= |a_1| - \sum_{k=2}^{n} |a_k| - |a_{n+1}| \\
&\leq |\sum_{k=1}^{n} a_k| - |a_{n+1}| \\
&\leq |\sum_{k=1}^{n+1} a_k|
\end{aligned}$$

Can left side became$||a_1| - \sum_{k=2}^{n} |a_k||$?

$$\left| |a_1| - \sum_{k=2}^{n} |a_k| \right| = |a_1| - \sum_{k=2}^{n} |a_k| \qquad |a_1| \geq \sum_{k=2}^{n} a_k \qquad (1.4)$$

$$\left| |a_1| - \sum_{k=2}^{n} |a_k| \right| = \sum_{k=2}^{n} |a_k| - |a_1| \qquad |a_1| \geq \sum_{k=2}^{n} a_k \qquad (1.5)$$

in eq1.4, the inequality is still vaild. However in eq1.5, $\sum_{k=2}^{n} |a_k| - |a_1|$ and $|a_1|$ □

(3). $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

Proof.

$$\begin{aligned}
\frac{|a+b|}{1+|a+b|} &\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \\
\frac{|a+b|}{1+|a+b|} &\leq \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
1 - \frac{|a+b|}{1+|a+b|} &\geq 1 - \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
\frac{1}{1+|a+b|} &\geq \frac{1-|a||b|}{(1+|a|)(1+|b|)} \\
1+|a|+|b|+|a||b| &\geq 1+|a+b|-|a||b|-|a||b||a+b| \\
|a|+|b|+2|a||b|+|a||b||a+b| &> 0, \text{Since} +2|a||b|+|a||b||a+b| \geq |a+b|
\end{aligned}$$

Therefore $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$ □

Example 1.1.6. (4).$|(a+b)^n - a^n| \leq (|a|+|b|)^n - |a|^n$

$$(a+b)^n - a^n = \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n$$

$$(|a|+|b|)^n - |a|^n = \binom{n}{1}|a|^{n-1}|b|^1 + \binom{n}{2}|a|^{n-2}|b|^2 + \cdots + \binom{n}{n}|a|^0|b|^n$$

$$\because |a|^j|b|^k \geq a^j b^k$$
$$\therefore \sum |a|^j|b|^k \geq |\sum a^j b^k|$$

$$|(a+b)^n - a^n| = \begin{cases} (a+b)^n - a^n, & a+b \ge a; b \ge 0 \\ a^n - (a+b)^n, & a+b < a; b < 0 \end{cases}$$

$$|(a+b)^n - a^n| \le (|a| + |b|)^n - |a|^n. \tag{1.6}$$

Proposition 1.1.2. 1.3.5(Cauchy inequality)

For $a_1, a_2, \ldots, a_n$. and $b_1, b_2, \ldots, b_n$. $a_i, b_i \in \mathbb{R}$, There is

$$\left| \sum_{i=1}^{n} a_i b_i \right| \le \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2} \tag{1.7}$$

Proof. Let's prove eq1.7

First way on book:

Use variable λ, change the inequality into nonnegative binomial.

$$0 \le \sum_{i=1}^{n} (a_i - \lambda b_i)^2 \qquad\qquad = \sum_{i=1}^{n} a_i^2 - 2\lambda \sum_{i=1}^{n} a_i b_i + \lambda^2 \sum_{i=1}^{n}$$

$$\Delta = B^2 - 4AC \qquad\qquad = (-2\sum_{i=1}^{n} a_i b_i)^2 - 4(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \le 0$$

$$(\sum_{i=1}^{n} a_i b_i)^2 \le (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$$

sqrt on both side of the inequality above, we can get

$$\left| \sum_{i=1}^{n} a_i b_i \right| \le \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$$

$\square$

6. Cauchy 不等式的不同证明

(1). 数学归纳法.

$$k = 1, \quad |ab| = \sqrt{a^2}\sqrt{b^2}$$

$$k = 1, \quad |a_1 b_1 + a_2 b_2| = \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_1^2}$$

$$\text{Suppose } k = n, \quad \left| \sum_{i=1}^{n} a_i b_i \right| = \sqrt{\sum_{i=1}^{n} a_i}\sqrt{\sum_{i=1}^{n} b_i}$$

$$k = n+1, \quad \left| \sum_{i=1}^{n+1} a_i b_i \right| = \left| \sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1} \right|$$

$$|\sum_{i=1}^{n+1} a_i b_i| = |\sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1}|$$

$$\leq |\sqrt{\sum_{i=1}^{n} a_i}\sqrt{\sum_{i=1}^{n} b_i} + a_{n+1} b_{n+1}|$$

Note that $A = \sqrt{\sum_{i=1}^{n} a_i}$, $B = \sqrt{\sum_{i=1}^{n} b_i}$

$$|\sum_{i=1}^{n+1} a_i b_i| \leq |AB + a_{n+1} b_{n+1}|$$

$$\leq \sqrt{A^2 + a_{n+1}^2}\sqrt{B^2 + b_{n+1}^2}$$

$$= \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

(2) Lagrange 恒等式

$$\sum_{i=1}^{n} a_k^2 \sum_{i=1}^{n} b_k^2 - (\sum_{i=1}^{n} |a_k b_k|) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 \tag{1.8}$$

$$(|a_k||b_i| - |a_i||b_k|)^2 = |a_k|^2|b_i|^2 - 2|a_i||a_k||b_i||b_k| + |b_k|^2|a_i|^2$$

$$= a_k^2 b_i^2 + b_k^2 a_i^2 - 2|a_i a_k b_i b_k|$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 = 2\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} b_k^2 - 2\sum_{i=1}^{n}\sum_{k=1}^{n}|a_i a_k b_i b_k|$$

$$\sum_{i=1}^{n} a_k^2 \sum_{i=1}^{n} b_k^2 - (\sum_{i=1}^{n} |a_k b_k|) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 \geq 0$$

$$\therefore (\sum_{i=1}^{n} |a_i b_i|)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

$$\because |\sum_{i=1}^{n} a_i b_i| \leq \sum_{i=1}^{n} |a_i b_i|$$

$$\therefore (|\sum_{i=1}^{n} a_i b_i|)^2 \leq (\sum_{i=1}^{n} |a_i b_i|)^2$$

$$\therefore (|\sum_{i=1}^{n} a_i b_i|)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

不等式两边开平方，得到:

$$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

(3). 用不等式 $|AB| \leq \frac{A^2+B^2}{2}$

$$|a_i b_i| \leq \frac{a_i^2 + b_i^2}{2}$$

$$|\sum_{i=1}^n a_i b_i| \leq \sum_{i=1}^n |a_i b_i| \qquad\qquad \leq \frac{\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2}{2}$$

$$\frac{\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2}{2} \geq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2} \qquad\qquad ??$$

如何用均值不等式证明 Cauchy 不等式?

由切比雪夫不等式，有

$$\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \leq (\frac{a_1 + a_2 + \cdots + a_n}{n})(\frac{b_1 + b_2 + \cdots + b_n}{n}) \qquad (1.9)$$

由均值不等式，有

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \leq \sqrt{\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}}$$

$$\therefore \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \leq (\frac{a_1 + a_2 + \cdots + a_n}{n})(\frac{b_1 + b_2 + \cdots + b_n}{n})$$

$$\leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}\sqrt{\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}}$$

$$= \frac{1}{n}\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

This is

$$\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$$

Square on both side of the inequality, The calculate square root. We can get eq1.9:

(4). 构造复的辅助数列

$$c_k = a_k^2 - b_k^2 + 2|a_k b_k|, \qquad k = 1, 2, \ldots, n$$

Then we use

$$\Big|\sum_{k=1}^n c_k\Big| \leq \sum_{k=1}^n |c_k|$$

Solve 1.

$$c_k = (|a_k| + |b_k|)^2 = a_k^2 + b_k^2 + 2|a_k b_k|$$

$$\sum_{k=1}^n c_k = \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 + 2\sum_{k=1}^n |a_k b_k|$$

$$|c_k| = \sqrt{\Re^2 c_k + \Im^2 c_k} = \sqrt{(a_k^2 - b_k^2)^2 + (2a_k b_k)^2} = a_k^2 + b_k^2$$

$$\therefore \Big| \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2 \sum_{k=1}^{n} |a_k b_k| \Big| = \sqrt{\Re^2 \sum_{k=1}^{n} c_k + \Im^2 \sum_{k=1}^{n} c_k}$$

$$= \sqrt{(\sum_{k=1}^{n} (a_k^2 - b_k^2))^2 + \sum_{k=1}^{n} (2a_k b_k)^2}$$

$$= \sqrt{(\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4 \sum_{k=1}^{n} (a_k b_k)^2}$$

$$\because |\sum c_k| \le \sum |c_k|$$

$$\therefore (\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4 \sum_{k=1}^{n} (a_k b_k)^2 \le (\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2)^2$$

$$\therefore 4(\sum_{k=1}^{n} a_k b_k)^2 \le 4(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)$$

extracting both side: $\Big| \sum_{k=1}^{n} a_k b_k \Big| \le \sqrt{\sum_{k=1}^{n} a_k^2} \sqrt{\sum_{k=1}^{n} b_k^2}$

Example 1.1.7. 7. Suppose $0 < x_i \le \frac{1}{2}, i = 1, 2, \ldots, n$, then

$$\frac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^n} \le \frac{\prod_{i=1}^{n} (1 - x_i)}{(\sum_{i=1}^{n} (1 - x_i))^n} \tag{1.10}$$

Proof. Let's prove eq1.10 by induction method.

$$n = 2, \qquad \frac{x_1 x_2}{(x_1 + x_2)^2} \le \frac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2}$$

$$\frac{(x_1 x_2)}{(x_1^2 + 2x_1 x_2 + x_2^2)} \le \frac{1 - x_1 - x_2 + x_1 x_2}{(1 - x_1)^2 + 2(1 - x_1)(1 - x_2) + (1 - x_2)^2}$$

$$\frac{(x_1 + x_2)^2}{(x_1 x_2)} \ge \frac{((1 - x_1)(1 - x_2))^2}{1 - x_1 - x_2 + x_1 x_2}$$

$$\frac{x_1}{x_2} + 2 + \frac{x_2}{x_1} \ge \frac{1 - x_1}{1 - x_2} + 2 \frac{1 - x_2}{1 - x_1}$$

$$\frac{x_1}{x_2} - \frac{1 - x_1}{1 - x_2} \ge \frac{1 - x_2}{1 - x_1} - \frac{x_2}{x_1}$$

$$\frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_2(1 - x_2)} \ge \frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_1(1 - x_1)}$$

$$\frac{x_1 - x_2}{x_2(1 - x_2)} \ge \frac{x_1 - x_2}{x_1(1 - x_1)}$$

$f(x) = x - x^2, f'(x) = 1 - 2x > 0$, while $x \in (0, \frac{1}{2})$

When $x_1 > x_2, 0 < x_2 < x_1 \le \frac{1}{2}, x_1 - x_1^2 \ge x_2 - x_2^2, x_1 - x_2 > 0$

When $x_1 < x_2, 0 < x_1 < x_2 \le \frac{1}{2}, x_1 - x_1^2 \le x_2 - x_2^2, x_1 - x_2 < 0$

$$\frac{x_1 - x_2}{x_2(1 - x_2)} \ge \frac{x_1 - x_2}{x_1(1 - x_1)}$$

$k = 2, \qquad \dfrac{x_1 x_2}{(x_1 + x_2)^2} \leq \dfrac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2}$

$k = 4, \qquad \dfrac{x_1 x_2 x_3 x_4}{(x_1 + x_2 + x_3 + x_4)^2} \leq \dfrac{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)}{((1 - x_1) + (1 - x_2) + (1 - x_3) + (1 - x_4))^2}$

Use Cauchy's forward and backward method, We can prove this equation

Suppose $k = n$, $\dfrac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^2} \leq \dfrac{\prod_{i=1}^{n}(1 - x_i)}{(\sum_{i=1}^{n}(1 - x_i))^2}$

$k = n - 1, \qquad \text{prove } \dfrac{\prod_{i=1}^{n-1} x_i}{(\sum_{i=1}^{n-1} x_i)^2} \leq \dfrac{\prod_{i=1}^{n-1}(1 - x_i)}{(\sum_{i=1}^{n-1}(1 - x_i))^2}$

todo! need to complete! $\square$

Proposition 1.1.3. 1.3.1 Bernoulli inequality Suppose that $h > -1, n \in \mathbb{N}$, Then:

$$(1+h)^n \geq 1 + nh \tag{1.11}$$

When $n > 1$, the inequality became equation iff $h = 0$.

Proof. When $n = 1, 1 + h = 1 + h$

$h = 0, 1^n = 1$

Let's consider the condition $n > 1, h \neq 0$.

i). $h > 0, (1+h)^n = \binom{n}{0}h^0 + \binom{n}{1}h^1 + \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n$.

$\because \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n > 0, \therefore (1+h)^n > 1 + nh$

ii). $-1 < h < 0, 0 < 1 + h < 1$.

$$(1+h)^n - 1 = (1 + h - 1)\Big(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\Big)$$
$$= h\Big(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\Big)$$

$\because 1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1} < n$ when $h < 0$

$\therefore (1+h)^n > 1 + nh$

Two variable extension of the Bernoulli inequality, Suppose $h = \frac{B}{A}, A > 0, A + B > 0$, Then $1 + h > 0$ is established. $\qquad\square$

Proposition 1.1.4. 1.3.2 Suppose $A > 0, A + B > 0, n \in \mathbb{N}$, Then the inequality is true:

$$(A+B)^n \geq A^n + nA^{n-1}B \tag{1.12}$$

The inequalty became equation iff $B = 0$.

Proof. divide A^n on both side of the inequality 1.12. Set $h = \frac{B}{A}(A > 0)$, Then the inequality became Eq 1.11. So we can prove Eq 1.12 by prove Eq 1.11. Eq 1.11 is true when $h > -1$. $\therefore 1 + h > 0, 1 + \frac{B}{A} > 0, \because A > 0, \therefore A + B > 0$. And when $n > 1$ the equation is true iff $h = 0. \frac{B}{A} = 0, \therefore B = 0$. $\qquad\square$

Example 1.1.8. Ex 1.3.2 exercise 8

$a, c, t, g \geq 0, a + c + t + g = 1$. Prove that $a^2 + c^2 + t^2 + g^2 \geq \frac{1}{4}$.

The inequality became equatio iff $a = c = t = g = \frac{1}{4}$.

Proof. from A.G inequality,

$$\frac{a+c+t+g}{4} \geq \sqrt[4]{actg}, \quad a + c + t + g = 1 \tag{1.13}$$

$\therefore \sqrt[4]{actg} \leqslant \frac{1}{4}$

$$a + c + t + g = 1, (a + c + t + g)^2 = 1$$

$$(a + c + t + g)^2 = a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg = 1 \qquad (1.14)$$

$$a^2 + c^2 \geq 2acc^2 + t^2 \geq 2ct \qquad (1.15)$$

$$a^2 + t^2 \geq 2atc^2 + g^2 \geq 2cg \qquad (1.16)$$

$$a^2 + g^2 \geq 2agt^2 + g^2 \geq 2tg \qquad (1.17)$$

substitude $2ac, 2ag, \ldots$ in equation 1.14, we can get

$$4(a^2 + c^2 + t^2 + g^2) \geq a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg$$

Then we get the inequality 1.13. $\square$

1.2   1.4 逻辑符号与对偶法则

The law of duality: $\forall(\exists) \to \exists(\forall)$ with negative statement

Inverse proposition?

1. A have upper limit, $\exists M > 0.\forall x \in A, x \leqslant M$.

It's negative statement is 'A don't have upper limit'. $\forall M > 0, \exists x \in A, x > M$.

2. the minum item in A is b, $b \in A, \forall x \in A, x \geq b$.

It's negative statement is 'b is not the minum item in A'. $b \in A, \exists x \in A, x < b$.

3. $f \in (a, b)$ is a monotonic augmentation function, $\forall x, y \in (a, b), x < y, f(x) \leqslant f(y)$.(or $f(x) < f(y)$, depends on monotonic function's definition)

It's negative statement is '$f \in (a, b)$ isn't a monotonic augmentation function'. $\exists x, y \in (a, b), x < y, f(x) > f(y)$ (or $f(x) \geq f(y)$).

4. $f \in (a, b)$ is a monotonic function, $\forall x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) \geq 0$.

It's negative statement is '$f \in (a, b)$ isn't a monotonic function'. $\exists x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) < 0$.

(Another way $\forall x, y \in (a, b), x < y, f(x) - f(y) \geq 0$ or $f(x) - f(y) \leqslant 0$.)

5. $A \subset B, \forall x \in A, x \in B$.

It's negative statement is $A \subsetneq B, \exists x \in A, x \notin B$.

6. $A - B \neq \emptyset, \exists x \in A, x \in B$.

It's negative statement is $A - B = \emptyset, \forall x \in A, x \notin B$.

7. x_n is an infinitesimal amounts, $\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \forall n > N, |x_n| < \epsilon$.

It's negative statement is 'x_n is not an infinitesimal amounts', $\exists \epsilon > 0, \forall N \in \mathbb{N}^+, \exists n > N, |x_n| \geq \epsilon$.

8. x_n is infinitely large, $\forall M > 0, \exists N \in \mathbb{N}^+, \forall n > N, x_n > M$.

It's negative statement is 'x_n is not infinitely large', $\exists M > 0, \forall N \in \mathbb{N}^+, \exists n > N, x_n \leqslant M$.

第二章　数列极限

2.1　数列极限的基本概念

2.1.1　2.1.5 练习题

2021.5.5 1. prove by Limit definition:

(1). $\lim_{n\to\infty} \frac{3n^2}{n^2-4} = 3$.

(2). $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(3). $\lim_{n\to\infty}(1+n)^{\frac{1}{n}} = 0$.

(4). $\lim_{n\to\infty} \frac{a^n}{n!} = 0, (a > 0)$.

2. Suppose $a_n, n \in \mathbb{N}_+$. sequence a_n converge to a.
Prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Proof. $n \to \infty\, a_n \to a$.

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.　　□ (check, not consider the condition $a = 0$) add $a = 0, \forall \epsilon \in (0,1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$. s.t $a_n < \epsilon^2 < \epsilon, \sqrt{a_n} < \epsilon$.　　□

3. If $\lim_{n\to\infty} a_n = a$.
Prove $\lim_{n\to\infty} |a_n| = |a|$. Vice versa?

Proof. $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$\Big| |a_n| - |a| \Big| \leqslant |a_n - a| < \epsilon$$

$\therefore \lim_{n\to\infty} |a_n| = |a|$

If We know $\lim_{n\to\infty} |a_n| = |a|$.

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), \Big| |a_n| - |a| \Big| < \epsilon$. We can't get $\lim_{n\to\infty} a_n = a$. For example:
$a_n = \frac{1}{n} + 1, a = -1, \lim_{n\to\infty} |a_n| = |a|$ is $\lim_{n\to\infty} |\frac{1}{n} + 1| = |-1|$, but $\lim_{n\to\infty} \frac{1}{n} + 1 \neq -1$　　□
　　□

(1). Suppose $p(x)$ is a polynomial of x, if $\lim_{n\to\infty} a_n = a$, Prove $\lim_{n\to\infty} p(a_n) = p(a)$.

(2). Suppose $b > 0$, $\lim\limits_{n\to\infty} a_n = a$. Prove $b^{a_n} = b^a$.

(3). Suppose $b > 0$, $\{a_n\}, a_n > 0, \forall n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a.a > 0$. Prove $\lim\limits_{n\to\infty} \log_b a_n = \log_b a$.

(4) Suppose $b \in \mathbb{R}$, $\{a_n\}, a_n > 0$ when $n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a$. Prove $\lim\limits_{n\to\infty} a_n{}^b = a^b$.

(5) Suppose $\lim\limits_{n\to\infty} a_n = a$. Prove $\lim\limits_{n\to\infty} \sin a_n = \sin a$.

Proof. 4.(1)

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geqslant N(\epsilon), |a_n - a| < \epsilon.$

$p(a) = k_m a^m + k_{m-1} a^{m-1} + \cdots + k_0 a^0.$

$\therefore p(a_n) - p(a) = k_m(a_n^m - a^m) + k_{m-1}(a_n^{m-1} - a^{m-1}) + \cdots + k_0(a_n^0 - a^0).$

$$|a_n^m - a^m| = |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}|$$
$$< \epsilon \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}|$$
$$< \epsilon(m-1)\cdots(a+\delta)^{m-1}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a)$. $\square$ $\square$

Proof. 4.(2)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |a_n - a| < \epsilon.$

If $b = 1$, $1^{a_n} = 1^a = 1.$

If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n-a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1)$ $\because b > 0, \epsilon \to 0,$

$\therefore b^\epsilon - 1 \to 0. \therefore \lim\limits_{n\to\infty} b_n^a = b^a.$

If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$. $\square$

Proof. 4.(3)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |a_n - a| < \epsilon.$

$$\log_b a_n - \log_b a = \log_b \frac{a_n}{a}$$
$$= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1)$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a}). \because b > 0, a \neq 0, a_n > 0$ when $\epsilon \to 0. \therefore \log_b(1 + \frac{\epsilon}{a}) \to 0.$

$\therefore \lim\limits_{n\to\infty} \log_b a_n = \log_b a$ $\square$

Proof. 4.(4)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |a_n - a| < \epsilon.$

$a_n^b = e^{b\ln a_n}, a_n^b - a^b = e^{b\ln a_n} - e^{b\ln a}.$

$$e^{b\ln a_n} - e^{b\ln a} = e^{b\ln a}(e^{b\ln a_n - b\ln a} - 1)$$
$$= e^{b\ln a}(e^{b\ln \frac{a_n}{a}} - 1)$$

$$0 < |a_n^b - a^b| < e^{b\ln a}(e^{b\ln(1+\frac{\epsilon}{a})} - 1)$$
$$\therefore \lim_{n\to\infty} a_n^b = a^b \qquad \square$$

Proof. 4.(5)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$

$$\sin(A+B) - \sin(A-B) = \sin A \cos B + \cos A \sin B$$
$$- (\sin A \cos B - \cos A \sin B)$$
$$= 2\cos A \sin B$$

$$\sin a_n - \sin a = 2\cos\frac{a_n + a}{2}\sin\frac{a_n - a}{2}$$

$|\sin a_n - \sin a| = |2\cos\frac{a_n+a}{2}\sin\frac{a_n-a}{2}| < |2\sin\frac{a_n-a}{2}|$

$|2\sin\frac{a_n-a}{2}| < |2\frac{a_n-a}{2}| = \epsilon$

$|\sin a_n - \sin a| < \epsilon, \therefore \lim_{n\to\infty} \sin a_n = \sin a \qquad \square$

assume $a > 1$. Prove $\lim_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof. $\frac{1}{n}\log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim_{n\to\infty} \sqrt[n]{n} = 1, \log_a 1 = 0$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\} . \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon.$

$a > 1$, and $\lim_{n\to\infty} \sqrt[n]{n} = 1. \therefore$ when $n \to \infty, \sqrt[n]{n} < a^\epsilon$, take logarithm on base of a, we can get $\frac{1}{n}\log_a n < \epsilon$

$\therefore \lim_{n\to\infty} \frac{\log_a n}{n} = 0 \qquad \square$

2.2 收敛数列的基本性质

2021.5.6 收敛数列的性质

1. 收敛数列的极限是唯一的

2. 收敛数列一定有界

3. 收敛数列的比较定理，包括保号性定理

4. 收敛数列满足一定的四则运算规则

5. 收敛数列的每一个子列一定收敛于同一极限

2.2.1   思考题

1. $\{a_n\}$ 收敛, $\{b_n\}$ 发散, $\{a_n + b_n\}$ 发散, $\{a_n \cdot b_n\}$ 可能收敛, 可能发散.

2. $\{a_n\}, \{b_n\}$ 都发散, $\{a_n + b_n\}$ 可能收敛, 可能发散 (ex: $n + -n, n + -2n$), $\{a_n \cdot b_n\}$ 发散 (?).

3. $a_n \leqslant b_n \leqslant c_n, n \in \mathbb{N}_+$. 已知 $\lim\limits_{n \to \infty} (c_n - a_n) = 0$. 问数列 $\{b_n\}$ 是否收敛?

4. $\lim\limits_{n \to \infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right)$

5. $a_n \to a(n \to 0)$. $\forall n, b < a_n < c$. 是否成立 $b < a < c$?

6. $a_n \to a(n \to 0)$. and $b \leqslant a \leqslant c$, 是否存在 $N \in \mathbb{N}_+$, s.t. 当 $n > N$ 时, 成立 $b \leqslant a_n \leqslant c$

7. 已知 $\lim\limits_{n \to \infty} a_n = 0$, 问: 是否有 $\lim\limits_{n \to \infty} (a_1 a_2 \ldots a_n) = 0$. 反之如何?

Proof. 5.4

$$\frac{n}{2n} \leqslant \frac{1}{n+1} + \cdots + \frac{1}{2n} \leqslant \frac{n}{n+1}$$

$\because \lim\limits_{n \to \infty} \frac{n}{2n} = \frac{1}{2}$ $\lim\limits_{n \to \infty} \frac{n}{n+1} = 1, \therefore \lim\limits_{n \to \infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right)$ 收敛.

$$\frac{1}{n+1} + \cdots + \frac{1}{2n} = \frac{1}{n} \left(\frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \cdots + \frac{1}{1 + \frac{n}{n}} \right)$$
$$= \int_0^1 \frac{1}{1+x} \mathrm{d}x$$
$$= \ln(1+x)|_0^1 = \ln 2$$

$\lim\limits_{n \to \infty} \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right) = \ln 2$ $\square$

Proof. 5.5

不成立, 应当为小于等于号。b=0, c=2, $a_n = \frac{1}{n}$, $\lim_{n \to \infty} a_n = 0 = c$. $\square$

Proof. 5.6

不成立。 $a = 0, b = 0, c = 2, a_n = (-1)^n \frac{1}{n}$.

$b \leqslant a \leqslant c$, but $(-1)^{2n+1} \frac{1}{2n+1} < 0 = b$. $\square$

Proof. $\lim\limits_{n \to \infty} a_n = 0, a_n = \frac{1}{n}$. $a_1 a_2 \ldots a_n = \frac{1}{n!}$, $\lim\limits_{n \to \infty} \frac{1}{n!} = 0$.

$\lim\limits_{n \to \infty} a_n = 0 \to \lim\limits_{n \to \infty} (a_1 a_2 \ldots a_n) = 0$ $\checkmark$

$\lim\limits_{n \to \infty} (a_1 a_2 \ldots a_n) = 0 \to \lim\limits_{n \to \infty} a_n = 0$ $\times$

$|a_n| < \epsilon, |a_{N+1} \ldots a_n| < \epsilon^{n-N} < \epsilon, a_n < \sqrt[n]{\epsilon}$.

for example, $a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \ldots, a_n = \frac{n}{n+1}$.

$$a_1 a_2 \ldots a_n = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}.$$
$$\lim\limits_{n \to \infty} (a_1 a_2 \ldots a_n)$$
$$= \lim\limits_{n \to \infty} \frac{1}{n} = 0$$

but $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ □

　　研究数列收敛方面的两个基本工具:

1. 夹逼定理.

2. 单调有界数列的收敛定理.

Example 2.2.1. 2.2.2 $\lim_{n\to\infty} \frac{x_n-1}{x_n+a} = 0$,
prove $\lim_{n\to\infty} x_n = a$

　　Proof. $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |\frac{x_n-1}{x_n+a} - 0| < \epsilon.$
$|x_n - 1| < \epsilon |x_n + a| < 4a \cdot \epsilon.$(这个 4 是怎么取得的?)
$|x_n - a| < \epsilon |x_n + a| = \epsilon |(x_n - a) + 2a| \leqslant \epsilon(|x_n - a| + 2a).$
限制 $\epsilon < 1, |x_n - a| < 2\epsilon |a|/(1-\epsilon).$
限制 $\epsilon < \frac{1}{2}, |x_n - a| < 2\epsilon |a|/(1-\epsilon) < 4|a|\epsilon.$
Let $\epsilon' = 4a\epsilon, |x_n - 1| < \epsilon'. \therefore \lim \lim_{n\to\infty} x_n = a.$ □

Example 2.2.2. 2.2.3 $a > 0, b > 0$, 计算 $\lim\limits_{n\to\infty} (a_n + b_n)^{\frac{1}{n}}$.

　　Proof. Suppose $a \leqslant b$.
$b = (b^b)^{\frac{1}{n}} < (a^n + b^n)^{\frac{1}{n}} \leqslant (2b^n)^{\frac{1}{n}}.$
$b < (a^n + b^n)^{\frac{1}{n}} \leqslant \sqrt[n]{2}b, \lim\limits_{n\to\infty} = 1.$ 夹逼定理.
$\lim\limits_{n\to\infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$
两数 n 次方之和再开 n 次根号的结果由较大的值决定, a,b 中较大的值为这个数的主要部分. □

Example 2.2.3. 2.2.4 $a_n = \frac{1!+2!+\cdots+n!}{n!}, n \in \mathbb{N}^+$

　　$\lim\limits_{n\to\infty} a_n = 1$

Example 2.2.4. $\lim\limits_{n\to\infty} \frac{n^3+n-7}{n+3} = +\infty$

Example 2.2.5. $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

　　调和级数 H_n 发散.

2.2.2 练习 2.2.4

　　Proof. 1.
$\{a_n\}$ 收敛于 $a, \to$ 两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 a.
两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 $a, \to \{a_n\}$ 收敛于 a.

□

　　2. 应用夹逼定理
(1). 给定 p 个正数 $a_1, a_2, \ldots, a_p$. 求 $\lim\limits_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \ldots a_p^n}$.

Let$a_s = \max_{1 \leqslant i \leqslant p}\{a_1, a_2, \ldots, a_p\}$.

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \ldots a_p^n)^{\frac{1}{n}} \leqslant (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$n \to \infty, p^{\frac{1}{n}} \to 1. \lim\limits_{n \to \infty} (a_1^n + a_2^n + \ldots a_p^n)^{\frac{1}{n}} = a_s$

(2). $x_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}, n \in \mathbb{N}_+.$ 求 $\lim\limits_{n \to \infty} x_n$

$$\frac{2n+1}{(n+1)} \leqslant x_n \leqslant \frac{2n+1}{\sqrt{n^2+1}}$$

$\lim\limits_{n \to \infty} \dfrac{2n+1}{n+1} = 2, \lim\limits_{n \to \infty} \dfrac{2n+1}{\sqrt{n^2+1}} = 2. \therefore \lim\limits_{n \to \infty} x_n = 2$

(3). $a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}, n \in \mathbb{N}_+.$ 求 $\lim\limits_{n \to \infty} a_n$

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leqslant (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$\lim\limits_{n \to \infty} \sqrt[n]{n} = 1, \therefore \lim\limits_{n \to \infty} a_n = 1$

(4). $a_n > 0.$ $\lim\limits_{n \to \infty} a_n = a, a > 0.$ 证明 $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = 1$

Proof. $\lim\limits_{n \to \infty} a_n = a$
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |a_n - a| < \epsilon.$

$$0 < a - \epsilon < a_n < a + \epsilon$$

$\therefore \sqrt[n]{a - \epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a + \epsilon}.$
$\lim\limits_{n \to \infty} \sqrt[n]{a - \epsilon} = 1, \lim\limits_{n \to \infty} \sqrt[n]{a + \epsilon} = 1. \therefore \lim\limits_{n \to \infty} \sqrt[n]{a_n} = 1.$

$\square$

$$\lim\limits_{n \to \infty} (1 + x)(1 + x^2) \dots (1 + x^{2^n}) = \lim\limits_{n \to \infty} \prod_{i=1}^{2^n} (1 + x^i), |x| < 1.$$

$|x| < 1, \quad 1 > x^2 > x^4 > \cdots > x^{2^n} > 0$

$x \in (0, 1) \quad 1 < (1 + x)(1 + x^2) \dots (1 + x^{2^n}) < (1 + x)^{n+1} \qquad \lim\limits_{n \to \infty} (1 + x)^{n+1} = 1$

$x \in (-1, 0) \quad 0 < (1 + x)(1 + x^2) \dots (1 + x^{2^n}) < (1 + x)(1 + x^2)^n \qquad \lim\limits_{n \to \infty} (1 + x)(1 + x^2)^n = 1$

$$\lim\limits_{n \to \infty} (1 + x)(1 + x^2) \dots (1 + x^n)$$
$$= \lim\limits_{n \to \infty} \frac{(1 - x)(1 + x)(1 + x^2) \dots (1 + x^n)}{1 - x}$$
$$= \lim\limits_{n \to \infty} \frac{(1 - x^{2^{n+1}})}{1 - x}$$
$$= \frac{1}{1 - x}$$

$$\lim\limits_{n \to \infty} (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2})$$
$$= \lim\limits_{n \to \infty} \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \dots \frac{n-1}{n} \cdot \frac{n+1}{n}$$
$$= \lim\limits_{n \to \infty} \frac{1}{2} \frac{n+1}{n}$$
$$= \lim\limits_{n \to \infty} \frac{1}{2}$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{1+2}\right)\left(1 - \frac{1}{1+2+3}\right)\cdots\left(1 - \frac{1}{1+2+\cdots+n}\right)$$

$$= \lim_{n \to \infty} \left(1 - \frac{2}{3 \times 2}\right)\left(1 - \frac{2}{4 \times 3}\right)\cdots\left(1 - \frac{2}{(n+1) \times n}\right)$$

$$= \lim_{n \to \infty} \left(\frac{3 \times 2 - 2}{3 \times 2}\right)\left(\frac{4 \times 3 - 2}{4 \times 3}\right)\cdots\left(\frac{(n+1) \times n - 2}{(n+1) \times n}\right)$$

$$= \lim_{n \to \infty} \left(\frac{4}{3 \times 2}\right)\left(\frac{10}{4 \times 3}\right)\cdots\left(\frac{n^2 + n - 2}{(n+1) \times n}\right)$$

$$= \lim_{n \to \infty} \left(\frac{1 \times 4}{3 \times 2}\right)\left(\frac{2 \times 5}{4 \times 3}\right)\cdots\left(\frac{(n-2) \times (n+1)}{n \times (n-1)}\right)\left(\frac{(n-1) \times (n+2)}{(n+1) \times n}\right)$$

$$= \lim_{n \to \infty} \frac{1}{3} \times \frac{n+2}{n}$$

$$= \frac{1}{3}$$

$$\lim_{n \to \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}\right]$$

$$= \lim_{n \to \infty} \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

$$= \lim_{n \to \infty} \frac{1}{1} - \frac{1}{n+1}$$

$$= \lim_{n \to \infty} \frac{n}{n+1}$$

$$= 1$$

3.(4).

$$\lim_{n \to \infty} \left[\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1) \cdot (n+2)}\right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3}\right) + \frac{1}{2}\left(\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4}\right) + \cdots + \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(\frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{4}$$

3.(5).

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma)}, \qquad 其中\gamma为正整数$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\gamma}\Big[\frac{1}{k(k+1)\dots(k+\gamma-1)}-\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]$$

$$=\lim_{n\to\infty}\frac{1}{\gamma}\Big[\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma-1)}-\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]$$

$$=\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma^{\underline{\gamma}}}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]$$

$$=\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma!}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]$$

$$=\frac{1}{\gamma}\cdot\frac{1}{\gamma!}$$

其中 $x^{\underline{n}}=x(x-1)(x-2)\dots(x-n+1)$, 称为下阶乘. 而 $x^{\overline{n}}=x(x+1)(x+2)\dots(x+n-1)$, 称为上阶乘.

2.2.4-4 $S_n=a+3a^2+\dots+(2n-1)a^n, |a|<1$. 求 $\{S_n\}$ 的极限.

$$S_n-aS_n=a+3a^2+\dots+(2n-1)a^n$$
$$\qquad -\ a^2-\dots+(2n-3)a^n-(2n-1)a^n+1$$
$$=a+2a^2+\dots+2aa^n-(2n-1)a^{n+1}$$
$$=2(a+a^2+\dots+a^n)-a-(2n-1)a^{n+1}$$
$$=2\cdot a\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}$$

$|a|<1,\ \lim\limits_{n\to\infty}a_n=0$

$\lim\limits_{n\to\infty}(S_n-aS_n)=(1-a)\lim\lim_{n\to\infty}S_n$

$$\lim_{n\to\infty}(S_n-aS_n)=\lim_{n\to\infty}2a\cdot\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}$$
$$=2a\cdot\frac{1}{1-a}-a$$
$$=a\Big(\frac{2}{1-a}-a\Big)$$
$$=a\frac{1+a}{1-a}$$

$\therefore \lim\limits_{n\to\infty}S_n=\frac{a(a+1)}{(1-a)^2}$

2.2.4-5 设 $\lim\limits_{n\to\infty}x_n=A>0$. 取 $\epsilon=\frac{A}{2}$, 则 $\exists N\in\mathbb{N}_+.\ \forall n>N.$ 成立 $|x_n-A|<\frac{A}{2}$

$$A-\frac{A}{2}<x_n<A+\frac{A}{2},\frac{A}{2}<x_n<\frac{3A}{2}$$

即 $x_n>\frac{A}{2}$.

令 $m=\min\{x_1,x_2,\dots,x_N,\frac{A}{2}\}>0$. 则 m 为 $\{x_n\}$ 的正下界.

不一定有最小数的例子 $x_n=1+\frac{1}{n}$. $\lim\limits_{n\to\infty}x_n=1$, 下界 $m=\frac{1}{2}$. 但 $\{x_n\}$ 取不到下界.

Proof. 2.2.4-6 $\because \lim\limits_{n\to\infty} a_n = +\infty$. $\forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M$.
$m = \min\{a_1, a_2, \ldots, a_N, M\}$, 但 M 在数列 $\{a_n\}$ 中不一定取的到!
$M = a_1 + 1, \exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$
则 $m = \min\{a_1, a_2, \ldots, a_{N_1}\}$ 为数列的最小数. $\square$

Proof. 2.2.4-7 构造数列
不妨设无界数列 $\{a_n\}$ 无上界.
$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M$.
取 $M_1 = 1$, 则 $\exists n_1 \in \mathbb{N}_+$ s.t. $a_{n_1} > M_1$.
取 $M_2 = \max\{a_n, 2\}, \exists n_2 \in \mathbb{N}_+$ s.t. $a_{n_2} > M_2$.
以此类推, 构造数列 $\{a_{n_k}\}$. s.t. $a_{n_k} > k$. 即 a_{n_k} 为无穷大量. $\square$

Proof. 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散.
构造 a_n 的发散子列即可. 已知 $\tan \frac{\pi}{2} = \infty, \pi$ 是一个无理数, 因此存在数列 $\{b_n\}, \lim\limits_{n\to\infty} b_n = \frac{\pi}{2}$. $\square$

Proof. 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散. 参考别人的答案
由于 $\{\sin 2n\}$ 极限不存在, 又

$$\sin 2n = 2 \sin n \cos n = \frac{2 \sin n \cos n}{\sin^2 n + \cos^2 n}$$
$$= \frac{2 \tan n}{\tan^n + 1}$$

若 $\{\tan n\}$ 极限存在 $\to \{\sin 2n\}$ 极限存在, 矛盾.
故 $\{\tan n\}$ 极限不存在. $\square$

2.2.4-9 $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$. S_n 在 1. $p \leqslant 0$, 2. $0 < p < 1$ 情况下均发散

Proof. 1. $p \leqslant 0$. $\lim\limits_{n\to\infty} n^{-p} = \infty, S_n$ 发散.
2. $0 < p < 1$. $\frac{1}{n^p} > \frac{1}{n}. \because H_n = \sum_{k=1}^{n} \frac{1}{k}$ (调和级数) 发散, $S_n > H_n, \therefore \{S_n\}$ 也发散. $\square$

2021.5.11 ex2.3.5 $0 < b < a$ 令 $a_0 = a, b_0 = b$ 递推公式

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, b_n = \sqrt{a_{n-1} b_{n-1}}, \quad n \in \mathbb{N}_+ \tag{2.1}$$

定义数列 a_n, b_n. 证明这两个数列收敛于同一个极限 $AG(a, b)$.

由 AG 不等式 $a > \frac{a+b}{2} > \sqrt{ab} > b > 0$, 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a, b) = \frac{\pi}{2G} \tag{2.2}$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$. 则

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \tag{2.3}$$

上面这个公式是怎么得到的:
参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} \qquad (a > b > 0) \tag{2.4}$$

这里令

$$\sin \phi = \frac{2a \sin \theta}{(a+b) + (a-b) \sin^2 \theta} \tag{2.5}$$

$\theta \in [0, \frac{\pi}{2}] \rightarrow \phi \in [0, \frac{\pi}{2}]$, 取微分

$$\cos \phi \mathrm{d}\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{[(a+b) + (a-b) \sin^2 \theta]^2} \cos \theta \mathrm{d}\theta \tag{2.6}$$

但是

$$\cos \phi = \frac{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}}{(a+b) + (a-b) \sin^2 \theta} \cos \theta. \tag{2.7}$$

(2.6) / (2.7), 两式相除, 得到

$$\mathrm{d}\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \frac{\mathrm{d}\theta}{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}} \tag{2.8}$$

另一方面

$$\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \tag{2.9}$$

因而

$$\frac{\mathrm{d}\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \frac{\mathrm{d}\theta}{\sqrt{(\frac{a+b}{2})^2 \cos^2 \theta + ab \sin^2 \theta}}. \tag{2.10}$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$, 则

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \tag{2.11}$$

反复应用该公式, 得到

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}}, \qquad (n = 1, 2, 3, \dots) \tag{2.12}$$

$$\frac{\pi}{2a_n} < G < \frac{\pi}{2b_n} \tag{2.13}$$

积分 G 可以归结到第一类全椭圆积分 $K(k) = (1+k_1)K(k_1) = \frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = (1 + k_1) \int_0^{\frac{pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - k_1^2 \sin^2 \theta}} \tag{2.14}$$

其中
$a_1 = \frac{1 + \sqrt{1 - k^2}}{2} = \frac{1 + k'}{2}, b_1 = \sqrt{k'}$
$k_1 = \frac{\sqrt{a_1^2 - b_1^2}}{a_1} = \frac{1 - k'}{1 + k'}, \frac{1}{a_1} = 1 + k_1$

2.3 2.3 单调数列

2021.05.12

Example 2.3.1. 2.3.6

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\cdots+n!}{n!}}$$

$$= \frac{1}{n+1} \frac{1!+2!+\cdots+(n+1)!}{1!+2!+\cdots+n!}$$

$$= \frac{3+3!+\cdots+(n+1)!}{(n+1)1!+(n+1)2!+\cdots+(n+1)!}$$

$n > 2$ 时, 分母每一项大于等于分子对应项.. $n > 2$ 后 a_n 单调减少. 由于 0 是下界, 因此 a_n 单调有界, 数列收敛.

$$a_{n+1} = \frac{1!+2!+\cdots+(n+1)!}{(n+1)!}$$

$$= \frac{1!+2!+\cdots+n!}{n!} \frac{1}{n+1} + 1$$

$$= 1 + \frac{a_n}{n+1}$$

设 $n \to \infty$ 时, $a_n \to a$

$$a = 1 + \left(\frac{1}{n+1} \to 0\right) = 1+0, \qquad \therefore a = 1$$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1!+2!+\cdots+n!}{n!} = 1$

2.3.1 2.3.2 练习题

证明, 若 x_n 单调, 则 $|x_n|$ 至少从某项开始后单调, 又问: 反之如何?

Proof. 分类讨论, 不妨设 $x_1 \geqslant 0$
1. x_n 单调递增, $|x_n|$ 从第一项开始单调.
2. x_n 单调递减, 且 $|x_n| \geqslant 0$. $|x_n|$ 从第一项开始单调.
3. x_n 单调递减, 且 $\exists N$ s.t. $x_n < 0$(第一个负数项). 则 $|x_n|$ 从第 N 项 (x_N) 开始单调.
反之该结论不成立.
反例: $x_n = \frac{(-1)^n}{n}$, $|x_n|$ 单调递减. 但 $x_{2k} = \frac{1}{2k} > 0 > x_{2k-1} = \frac{-1}{2k-1}$ $\qquad \square$

设 a_n 单调增加, b_n 单调减少, 且有 $\lim\limits_{n\to\infty}(a_n - b_n) = 0$.
证明: 数列 a_n 和 b_n 都收敛, 且极限相等.

Proof. $\lim\limits_{n\to\infty}(a_n - b_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N}_+, \text{s.t.} \forall n > N, |a_n - b_n - 0| < \epsilon.$
$b_n - \epsilon < a_n < b_n + \epsilon$, 同时有 $a_n - \epsilon < b_n < a_n + \epsilon$.
b_n 单调减少, $\therefore \exists N_2, \forall m < N_2, b_m > b_n + \epsilon.$

使用反证法证明 b_m 是a_n 的上界.

假设 b_m 不是a_n 的上界, 则存在 $a_n > b_m > b_n + \epsilon$, 这与 $|a_n - b_n| < \epsilon$ 矛盾.

$\therefore b_m$ 是a_n 的上界, 根据单调有界收敛准则, a_n 收敛. 同理可证b_n 收敛. $\lim\limits_{n\to\infty}(a_n - b_n) = 0$.

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$. □

按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.

2. 单调减少有下界的数列的极限不大于数列中的任何一项.

设 $x_n = \frac{2}{3} \cdot \frac{3}{5} \cdots \frac{n+1}{2n+1}, n \in \mathbb{N}_+$, 求数列$x_n$ 的极限.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)+1}{2(n+1)+1} = \frac{n+2}{2n+3} < 1. \qquad (n > 0) \tag{2.15}$$

x_n 单调递减. $\because x_n > 0, \therefore x_n$ 有下界, x_n 收敛.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$, 由夹逼定理, $\lim\limits_{n\to\infty} x_n = 0$

6. 在例题 2.2.6 的基础上证明: 当 $p > 1$ 时, 数列S_n 收敛. 其中

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$$

(S_n 就是 p 级数, 当 $p = 1$ 时为调和级数.)

Proof. S_n 单调递增, 记 $\frac{1}{2^{p-1}} = r$, 则 $0 < r < 1$.

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}} = r$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{4^{p-1}} = r^2$$

$$\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} < \frac{1}{(2^k)^p} + \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^k)^p} = \frac{1}{(2^k)^{p-1}} = r^k$$

□

由此可知

$$S_n \leqslant S_{2^n-1} < 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} < \frac{1}{1-r}$$

S_n 单调递增有上界, 由单调有界收敛准则知S_n 收敛。

7. 设 $0 < x_0 < \frac{\pi}{2}, x_n = \sin x_{n-1}. n \in \mathbb{N}_+$.

证明x_n 收敛, 并求其极限。

Proof. $x_0 \in (0, \frac{\pi}{2})$, $\sin x$,

$0 < x_1 = \sin x_0 < x_0 < \frac{\pi}{2}$.

$0 < x_2 = \sin x_1 < x_1 < \frac{\pi}{2}$.

$0 < \cdots < x_n < x_{n-1} < \cdots < x_2 < x_1 < \frac{\pi}{2}$.

x_n 单调递减有下界，x_n 收敛。

$a = \sin a, \quad a \in [0, \frac{\pi}{2}]$

解得 $a = 0$，$\therefore \lim\limits_{n \to \infty} x_n = 0$. $\hfill\square$