

具体数学阅读笔记-chap1 exercise

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1 Exercises

1.1 Warmups

练习 1 1 All horses are the same color; we can prove this by induction on the number of horses in a given set. Here's how: "If there's just one horse then it's the same color as itself, so the basis is trivial. For the induction step, assume that there are n horses numbered 1 to n . By the induction hypothesis, horses 1 through $n - 1$ are the same color, and similarly horses 2 through n are the same color. But the middle horses, 2 through $n - 1$, can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and n must be the same color as well, by transitivity. Thus all n horses are the same color; QED." What, if anything, is wrong with this reasoning?

题目解答 1 $n=1$ 情况下马有相同颜色

但 $n=2$ 时该假设不一定成立

练习 2 2

题目解答 2 不允许在 A B 之间直接移动, 求最短的移动序列

$$\begin{array}{llll}
 k = 1 & 1 & A \rightarrow C, C \rightarrow B & 2 \quad sum = 2 \\
 k = 2 & 1 & A \rightarrow C, C \rightarrow B, & 2 \\
 & 2 & A \rightarrow C & 1 \\
 & 1 & B \rightarrow C, C \rightarrow A, & 2 \\
 & 2 & C \rightarrow B & 1 \\
 & 1 & A \rightarrow C, C \rightarrow B & 2 \quad sum = 8 \\
 k = 3 & 1 & A \rightarrow C, C \rightarrow B, & 2 \\
 & 2 & A \rightarrow C & 1 \\
 & 1 & B \rightarrow C, C \rightarrow A, & 2 \\
 & 2 & C \rightarrow B & 1 \\
 & 1 & A \rightarrow C, C \rightarrow B & 2 \quad sum = 8 \\
 & 3 & A \rightarrow C & 1 \\
 & 1 & B \rightarrow C, C \rightarrow A, & 2 \\
 & 2 & B \rightarrow C & 1 \\
 & 1 & A \rightarrow C, C \rightarrow B, & 2 \\
 & 2 & C \rightarrow A & 1 \\
 & 1 & B \rightarrow C, C \rightarrow A, & 2 \\
 & 3 & C \rightarrow B & 1 \quad sum = 18 \\
 & 1 & A \rightarrow C, C \rightarrow B, & 2 \\
 & 2 & A \rightarrow C & 1 \\
 & 1 & B \rightarrow C, C \rightarrow A, & 2 \\
 & 2 & C \rightarrow B & 1 \\
 & 1 & A \rightarrow C, C \rightarrow B & 2 \quad sum = 26 \\
 & \vdots & & \\
 k = n & 1 & A \rightarrow C, C \rightarrow B & 2
 \end{array}$$

从前面的移动可以看出 $f(n) = 3 * f(n-1) + 2$, 设 $g(n) = f(n) + 1$, $g(1) = f(1) + 1 = 3$, $g(n) = 3g(n-1)$. $g(n) = 3^n$, $f(n) = 3^n - 1$.

练习 3 3

题目解答 3 是的, 以 n 个圆盘为例正确的叠放方法有 3^n 种将 ABC 视为 3 个序列, 将所有圆盘从大到小依次放置在 3 个序列中, 每个圆盘放置时有 3 种选择, 所共有 3^n 种正确的叠放方法。第二题移动 $3^n - 1$ 次, 再加上移动前所有圆盘都在 A 柱上的情况, 共有 3^n 种情况, 所以所有正确的叠放方法均会出现。

我的思考, n 个圆盘在 3 根柱子上任意放的方法有多少种?

练习 4 4

题目解答 4 Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, under Lucas's original rules?

是否存在 $m > 2^n - 1$

不存在。根据卢卡斯的规则, 将可能出现的移动情况分为两种:

1. 最大的圆盘不需要移动, 根据归纳法, 最多需要移动 $2^{n-1} - 1$ 次。
2. 最大的圆盘需要移动, 根据归纳法, 最多需要移动 $2^{n-1} - 1 + 1 + 2^{n-1} - 1$ 即 $2^n - 1$ 次

练习 5 5

题目解答 5 3 个给定集合, 共有 8 个可能子集。使用 Venn 图表示

¹ A, B, C , 三个集合的所有子集为 $\{\emptyset, A, B, C, A \cap B, A \cap C, B \cap C, A \cap B \cap C, \{A \setminus B, A \setminus C, B \setminus A, B \setminus C, C \setminus A, C \setminus B\}, \{A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B)\}, \{A \cup B, A \cup C, B \cup C, A \cup B \cup C\} \dots$

我认为这里所将的八个子集应当是 $\{\emptyset, \{A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B)\}, \{(A \cap B) \setminus C, (C \cap A) \setminus B, (B \cap C) \setminus A\}, \{A \cap B \cap C\}$ 。空集和 7 个互不相交的真子集。

¹Venn 图之后会补上

对于 4 个集合, Venn 图不能给出可能的 16 个子集, 因为不同的圆至多交于两点。参考答案中说的卵形 (ovals) 是什么意思?

练习 6 6

题目解答 6 无界区域个数 $2n$

所有区域个数 $\frac{n(n+1)}{2} + 1$

二者相减得到有界区域个数 $\frac{(n-1)(n-2)}{2}$

练习 7 7

题目解答 7 设 $H(n) = J(n+1) - J(n)$.

$H(2n) = 2$, 对 $n \geq 1$ 有

$$\begin{aligned} H(2n+1) &= J(2n+2) - J(2n+1) \\ &= (2J(n+1) - 1) - (2J(n) + 1) \\ &= 2H(n) - 2 \end{aligned}$$

但在 $n = 0$ 时, 由此推出

$$H(1) = J(2) - J(1) = 1 - 1 = 0 \neq 2$$

1.2 作业题

练习 8

$$Q_0 = \alpha$$

$$Q_1 = \beta$$

$$Q_n = \frac{1 + Q_{n-1}}{Q_{n-2}}, \quad n > 1$$

(hint: $Q_4 = \frac{1+\alpha}{\beta}$)

题目解答 8

$$\begin{aligned}
 Q_0 &= \alpha & &= \alpha \\
 Q_1 &= \beta & &= \beta \\
 Q_2 &= \frac{1+Q_1}{Q_0} & &= \frac{1+\beta}{\alpha} \\
 Q_3 &= \frac{1+Q_2}{Q_1} = \frac{1+\frac{1+\beta}{\alpha}}{\beta} & &= \frac{1+\alpha+\beta}{\alpha\beta} \\
 Q_4 &= \frac{1+Q_3}{Q_2} = \frac{1+\frac{1+\alpha+\beta}{\alpha\beta}}{\frac{1+\beta}{\alpha}} = \frac{\alpha\beta+1+\alpha+\beta}{\beta(1+\beta)} = \frac{1+\alpha}{\beta} \\
 Q_5 &= \frac{1+Q_4}{Q_3} = \frac{1+\frac{1+\alpha}{\beta}}{\frac{1+\alpha+\beta}{\alpha\beta}} = \frac{\alpha\beta+\alpha(1+\alpha)}{1+\alpha+\beta} = \alpha \\
 Q_6 &= \frac{1+Q_5}{Q_4} = \frac{1+\alpha}{\frac{1+\alpha}{\beta}} = \beta
 \end{aligned}$$

因此解得

$$\begin{aligned}
 Q_i &= \left\{ \alpha, \beta, \frac{1+\beta}{\alpha}, \frac{1+\alpha+\beta}{\alpha\beta}, \frac{1+\alpha}{\beta} \right\} \\
 (i \% n) &= \left\{ 0, 1, 2, 3, 4, \right\}
 \end{aligned}$$

练习 9 反向归纳法, 从 n 到 $n-1$ 证明命题

$$P(n): x_1 \dots x_n \leq \left(\frac{x_1 + \dots + x_n}{n} \right)^n, \quad x_i \geq 0, i = 1, \dots, n$$

$n=2$ 时为真

$$(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geq 0$$

a) $x_n = \frac{x_1 + \cdots + x_{n-1}}{n-1}$, 证明只要 $n > 1$ 时 $P(n)$ 蕴含 $P(n-1)$.

b) 证明 $P(n)$ 和 $P(2)$ 蕴含 $P(2n)$

c) 由 a), b) 说明这就蕴含了 $P(n)$ 对所有 n 为真

题目解答 9 a) $P(n)$ 成立, $\forall n > 1$

给定 $x_n = \frac{x_1 + \cdots + x_{n-1}}{n-1}$, 则有

$$\begin{aligned} x_1 \cdots x_{n-1} \cdot \frac{x_1 + \cdots + x_{n-1}}{n-1} &\leq \left(\frac{x_1 + x_{n-1} + \frac{x_1 + \cdots + x_{n-1}}{n-1}}{n} \right)^n \\ x_1 \cdots x_{n-1} \cdot \frac{x_1 + \cdots + x_{n-1}}{n-1} &\leq \left(\frac{x_1 + \cdots + x_{n-1}}{n-1} \right)^n \\ x_1 \cdots x_{n-1} &\leq \left(\frac{x_1 + \cdots + x_{n-1}}{n-1} \right)^{n-1} \end{aligned}$$

$P(n)$ 成立

b) 由 $P(n)$ 可得

$$x_1 \cdots x_n \cdot x_{n+1} \cdots x_{2n} \leq \left(\frac{x_1 + \cdots + x_n}{n} \right)^n \cdots \left(\frac{x_{n+1} + \cdots + x_{2n}}{n} \right)^n$$

$$\text{记 } A = \left(\frac{x_1 + \cdots + x_n}{n} \right), B = \left(\frac{x_{n+1} + \cdots + x_{2n}}{n} \right)$$

由 $P(2)$ 可得

$$\begin{aligned} AB &\leq \left(\frac{A+B}{2}\right)^2 \\ A^n B^n &= (AB)^n \leq \left(\frac{A+B}{2}\right)^{2n} \\ x_1 \dots x_{2n} &\leq \left(\frac{x_1 + \dots + x_{2n}}{2n}\right)^{2n} \end{aligned}$$

由此推知 $P(2n)$ 成立。

c) Cauchy 向前-向后方法。

1. $P(2) \rightarrow P(4) \rightarrow \dots P(2^n)$.

2. $P(n) \rightarrow P(n-1)$.

$\therefore \forall n \geq 1, P(n)$ 成立

练习 10 圆盘只能在 ABC 三根柱子上按照顺时针方向移动。记：

Q_n 为 n 个盘从 A 到 B 最少移动的次数。

R_n 为 n 个盘从 B 到 A 最少移动的次数。

题目解答 10 先列出两种移动方式各自的迭代式：

$$Q_n = \begin{cases} 0, & n = 0 \\ 2R_{n-1} + 1, & n > 0 \end{cases} \quad R_n = \begin{cases} 0, & n = 0 \\ Q_n + Q_{n-1} + 1, & n > 0 \end{cases}$$

这两个公式是如何得到的？

$$\begin{array}{ll}
 k=0 & Q_0 = 0 & R_0 = 0 \\
 k=1 & A \rightarrow B & B \rightarrow C \rightarrow A \\
 & Q_1 = 1 & R_1 = 2 \\
 k=2 & 1 : A \rightarrow B \rightarrow C & 1 : B \rightarrow C \rightarrow A \\
 & 2 : A \rightarrow B & 2 : B \rightarrow C \\
 & 1 : C \rightarrow A \rightarrow B & 1 : A \rightarrow B \\
 & & 2C \rightarrow A \\
 & & 1 : B \rightarrow C \rightarrow A \\
 & Q_2 = 5 & R_2 = 7 \\
 k=n & A \rightarrow B \quad Q_n & B \rightarrow A \quad R_n \\
 & (n-1)A \rightarrow C \quad R_{n-1} & (n-1)B \rightarrow A \quad R_{n-1} \\
 & n : A \rightarrow B \quad 1 & n : B \rightarrow C \quad 1 \\
 & (n-1)C \rightarrow B \quad R_{n-1} & (n-1)A \rightarrow B \quad Q_{n-1} \\
 & & n : C \rightarrow A \quad 1 \\
 & & (n-1)B \rightarrow A \quad R_{n-1} \\
 & Q_n = 2R_{n-1} + 1 & R_n = Q_{n-1} + 2R_{n-1} + 2 = Q_n + Q_{n-1} + 1
 \end{array}$$

练习 11 双重河内塔 $2n$ 个圆盘, 第 $2k-1$ 个与第 $2k$ 个大小相同。

题目解答 11 a) 不区分相同尺寸

$$\begin{array}{ll}
 n=0 & S_0 = 0 \\
 n=1 & S_1 = 2 & A \rightarrow B, A \rightarrow B \\
 n=2 & S_2 = 6 & A \rightarrow C, A \rightarrow C \\
 & & A \rightarrow B, A \rightarrow B \\
 & & C \rightarrow B, C \rightarrow B
 \end{array}$$

$$\text{解得 } S_n = 2T_n = 2(2^n - 1) = 2^{n+1} - 2$$

b) 在最后排列中将圆盘恢复次序需要移动几次？

$$k = 0 \quad R_0 = 0$$

$$k = 1 \quad R_1 = 3 \quad 1.1 : A \rightarrow C$$

$$1.2 : A \rightarrow B$$

$$1.1 : C \rightarrow B$$

$$k = 2 \quad R_2 = 11 \quad 1.1 : A \rightarrow B$$

$$1.2 : A \rightarrow B$$

$$2.1 : A \rightarrow C$$

$$1.2 : B \rightarrow C$$

$$1.1 : B \rightarrow C$$

$$2.2 : A \rightarrow B$$

$$1.1 : C \rightarrow A$$

$$1.2 : C \rightarrow A$$

$$2.1 : C \rightarrow B$$

$$1.2 : A \rightarrow B$$

$$1.1 : A \rightarrow B$$

$$k = n \quad R_n \quad n-1 \quad A \rightarrow B \quad S_{n-1}$$

$$n.1 : A \rightarrow C \quad 1$$

$$n-1 \quad B \rightarrow C \quad S_{n-1}$$

$$n.2 : A \rightarrow B \quad 1$$

$$n-1 \quad C \rightarrow A \quad S_{n-1}$$

$$n.1 : C \rightarrow B \quad 1$$

$$n-1 \quad A \rightarrow B \quad S_{n-1}$$

$$R_n = 4S_{n-1} + 3 = 2^{n+2} - 5 \quad (n \geq 1)$$

练习 12 12.11 推广, m_k 个尺寸为 k 的圆盘, 不区分相同尺寸的圆盘移动一个塔最少次数 $A(m_1, \dots, m_n)$

题目解答 12

$$F(0) = 0$$

$$F(1) = m_1$$

$$F(2) = 2F(1) + m_2 = 2m_1 + m_2$$

$$\vdots$$

$$F(n) = 2F(n-1) + m_n$$

$$\begin{aligned} A(m_1, \dots, m_n) &= F(n) = 2F(n-1) + m_n \\ &= 2^{n-1}m_1 + 2^{n-2}m_2 + \dots + m_n \\ &= \sum_{k=1}^n 2^{n-k}m_k \end{aligned}$$

练习 13 13

题目解答 13

$$k=1 \quad ZZ_1 = 2 + 0 = 2$$

$$k=2 \quad ZZ_2 = 4 + 8 = 12$$

$$k=3 \quad ZZ_3 = 6 + 25 = 31$$

对于定义了 L_n 个区域的 n 条直线, 可以用极狭窄的 Z 形线来代替。

例如, 每一对 Z 形线间有 9 个交点

$$\begin{aligned} ZZ_n &= ZZ_{n-1} + 9n - 8, \quad (n > 0) \\ ZZ_n &= 9S_n - 8n + 1 \\ &= 9 \frac{n(n+1)}{2} - 8n + 1 \\ &= \frac{9}{2}n^2 - \frac{7}{2}n + 1 \end{aligned} \tag{1}$$

练习 14 14

题目解答 14

$$n=0 \quad P_0=1$$

$$n=1 \quad P_1=2$$

$$n=2 \quad P_2=4$$

$$n=3 \quad P_3=8$$

$$n=4 \quad P_4=8+6=14$$

$$P_n = P_{n-1} + L_{n-1}$$

其中

$$L_n = 1 + S_n, \quad S_n = \frac{n(n+1)}{2}$$

$$\therefore P_n = P_{n-1} + 1 + \frac{n(n+1)}{2}$$

$$P_0 = 1$$

$$P_1 = P_0 + L_0 = 1 + 1 + \frac{0 \cdot 1}{2} = 2$$

$$P_2 = P_1 + L_1 = 2 + 1 + \frac{1 \cdot 2}{2} = 4$$

$$P_3 = P_2 + L_2 = 4 + 1 + \frac{2 \cdot 3}{2} = 8$$

$$P_4 = P_3 + L_3 = 8 + 1 + \frac{3 \cdot 4}{2} = 15$$

$$P_5 = P_4 + L_4 = 15 + 1 + \frac{4 \cdot 5}{2} = 26$$

$$P_n = P_{n-1} + L_{n-1}$$

$$= 0 + \sum_{k=0}^{n-1} \left(1 + \frac{k(k+1)}{2} \right)$$

$$= n + \frac{(n-1)n(n+1)}{6}$$

$$= \frac{n(n^2+5)}{6}$$

练习 15 15 约瑟夫问题, 倒数第二个 $I(n)$

表 1: 约瑟夫问题 $J(n)$ 与 $I(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5
$I(n)$	\sim	2	1	3	5	1	3	5	7	9	11	1	3	5	7	9	11	13

题目解答 15 $n > 1$ 时, $J(n), I(n)$ 有相同递归式

$$I(2) = 2, I(1) = 1$$

$$n = 2^m + 2^{m-1} + k, \quad 0 \leq k \leq 2^m + 2^{m-1}$$

$$I(n) = 2k + 1$$

$$n = 2^m + l, \quad I(n) = \begin{cases} J(n) + 2^{m-1}, & 0 \leq l < 2^{m-1} \\ J(n) - 2^m, & 2^{m-1} \leq l < 2^m \end{cases}$$

练习 16

$$\begin{cases} g(1) = \alpha \\ g(2n+j) = 3g(n) + \gamma n + \beta_j, \quad j = 0, 1, n \leq 1 \end{cases}$$

(提示, 用 $g(n) = n$)

题目解答 16 Suppose $g(n) = n$

$$g(1) = 1 = \alpha,$$

$$g(2n+j) = 2n+j = 3n + \gamma n + \beta_j.$$

$$\text{解得 } \alpha = 1, \gamma = -1, \beta_j = \begin{cases} 0, & j = 0 \\ 1, & j = 1 \end{cases}$$

(题解)

$$g(n) = a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 + d(n)\gamma$$

$n = (1b_{m-1} \dots b_1b_0)_2$ 将 n 以基数 2 展开 (写成二进制)。

$$a(n)\alpha + b(n)\beta_0 + c(n)\beta_1 = (\alpha\beta_{m-1}\beta_{m-2} \dots \beta_{b_1}\beta_{b_0})_3$$

$$g(n) = n. \quad (\alpha = 1, \beta_0 = 0, \beta_1 = 1, \gamma = -1)$$

$$a(n) + c(n) - d(n) = n$$

$$g(n) = 1. \quad (\alpha = 1, \beta_0 = -2, \beta_1 = 2, \gamma = 0)$$

$$a(n) - 2b(n) - c(n) = 1$$

$$d(n) = a(n) + c(n) - n$$

$$b(n) = \frac{1}{2}a(n) - \frac{1}{2}c(n) - \frac{1}{2}$$

$$\begin{aligned} g(n) = & a(n)\alpha + \left(\frac{1}{2}a(n) - \frac{1}{2}c(n) - \frac{1}{2}\right)\beta_0 \\ & + c(n)\beta_1 + (a(n) + c(n) - n)\gamma \end{aligned}$$

若 $\beta_i = 0 \ (i = 0, 1), \gamma = 0$

$$\begin{cases} g(1) = \alpha \\ g(2n + j) = 3g(n) \end{cases}$$

$$g(1) = \alpha, g(2) = g(3) = 3\alpha, g(4) = g(5) = g(6) = g(7) = 9\alpha, g(8) = 3g(4) = 27\alpha.$$

由此推知

$$g(n) = g(2^m + k) = 3^m, \quad a(n) = 3^m$$

继续计算 $d(n)$ 遇到困难

若 $\alpha = 0, \beta_0 = 0, \gamma = 0, g(n) = \beta_1 c(n)$

$$\begin{cases} g(1) = 0 \\ g(2n) = 3g(n) \\ g(2n + 1) = 3g(n) + 1 \end{cases}$$

表 2: m,k 变化规律

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
m	0	0	1	0	1	3	4	0	1	3	4	9	10	12	13	0	1	3	4
k	0	1		2				3								4			
			1		1	2	3		1	2	3	4	5	6	7		1	2	3

$g(2) = 0, g(3) = 1, g(4) = 0, g(5) = 1, g(6) = 3g(3) = 3, g(7) = 3g(3) + 1 = 4,$
 $g(8) = 0, g(9) = 1 \dots$

$$g(2^m + k) = g((1b_{m-1}b_{m-2} \cdots 1b_0)_2)$$

复习和重做 16 题的部分暂不录入

练习 17

$$W_{n(n+1)/2} \leq 2W_{n(n-1)/2} + T_n, \quad n > 0$$

题目解答 17 In general we have $W_m \leq 2W_{m-k} + T_k, 0 \leq k \leq M$ (This relation corresponds to transferring the top $m - k$. then using only three pegs to move the bottom $k \rightarrow T_k$, then finishing with the top $m - k \rightarrow 2 \cdot W_{m-k}$)

The stated relation turns out to be based on the unique value of k that minimizes the right-hand side of this general inequality, when $m \frac{n(n+1)}{2}$.

(However, we cannot conclude that equality holds. Many other strategies for transferring the tower are conceivable.)

$$\text{If we set } Y_n = (W_{n(n+1)/2} - 1)/2^n$$

$$\text{we find that } Y_n \leq Y_{n-1} + 1 \text{ . hence } W_{n(n+1)/2} \leq 2^n(n - 1) + 1$$

练习 18 证明如下的一组 n 条折线定义 Z_n 个区域

$$\begin{aligned} Z_n &= L_{2n} - 2n = \frac{2n(2n+1)}{2} + 1 - 2n \\ &= 2n^2 - n + 1. \quad n \geq 0. \end{aligned} \quad (2)$$

第 j 条折线 ($1 \leq j \leq n$) 的锯齿点在 $(n^{kj}, 0)$. 并向上经过点 $(n^{2j} - n^j, 1)$ 与 $(n^{2j} - n^j - n^{-n}, 1)$

题目解答 18

练习 19 当每一个锯齿的角度为 30° 时, 有可能由 n 条折线得到 Z_n 个区域吗?

题目解答 19

练习 20 利用成套方法解递归式:

$$\begin{cases} h(1) = \alpha \\ h(2n) = 4h(n) + \gamma_0 n + \beta_0 \\ h(2n) = 4h(n) + \gamma_1 n + \beta_1 \end{cases} \quad n \geq 1$$

题目解答 20

$$h(n) = \alpha A(n) + \beta_0 B(n) + \gamma_0 C(n) + \beta_1 D(n) + \gamma_1 E(n)$$

$$1. h(1) = 1$$

$$\begin{cases} 1 = \alpha \\ 1 = 4 \times 1 + \gamma_0 n + \beta_0 \\ 1 = 4 \times 1 + \gamma_1 n + \beta_1 \end{cases}$$

$$\text{解得 } (\alpha, \beta_0, \gamma_0, \beta_1, \gamma_1) = (1, -3, 0, -3, 0)$$

$$1 = A(n) - 3B(n) - 3D(n)$$

$$2. h(n) = n$$

$$\begin{cases} 1 = \alpha \\ 2n = 4 \times n + \gamma_0 n + \beta_0 \\ 2n + 1 = 4 \times n + \gamma_1 n + \beta_1 \end{cases}$$

解得 $(\alpha, \beta_0, \gamma_0, \beta_1, \gamma_1) = (1, 0, -2, 1, -2)$

$$n = A(n) - 2C(n) + D(n) - 2E(n)$$

3. $h(n) = n^2$

$$\begin{cases} 1 & = \alpha \\ (2n)^2 & = 4 \times n^2 + \gamma_0 n + \beta_0 \\ (2n+1)^2 & = 4 \times n^2 + \gamma_1 n + \beta_1 \end{cases}$$

解得 $(\alpha, \beta_0, \gamma_0, \beta_1, \gamma_1) = (1, 0, 0, 1, 4)$

$$n^2 = A(n) + D(n) + 4E(n)$$

最终结果???

练习 21 $2n$ 个人围成圈, 前 n 个好伙计后 n 个坏家伙. 证明: 总存在一个整数 q (与 n 有关), 使得若在绕圆圈走时每隔 $q-1$ 个人处死一个, 那么所有坏家伙首先出局. (例如, $n=3$ 时取 $q=5$, $n=4$ 时取 $q=30$)

题目解答 21 We can let m be the least (or any) common multiple of $2n, 2n-1, \dots, n+1$. a non-rigorous argument suggests that "random" value of m will succeed with probability

$$\frac{n}{2n} \frac{n-1}{2n-1} \cdots \frac{1}{n+1} = \frac{1}{\binom{2n}{n}} \sim \frac{\sqrt{\pi n}}{4^n}$$

1.3 附加题

练习 22 22

题目解答 22 22

练习 23 假如约瑟夫发现自己处在 j , 但能指定 q , 隔 $q-1$ 人处死一人. 他是否能保全自己?

题目解答 23