

# Chapter 1

## 2020 年笔记

### 1.1 20.07.27

$$\begin{aligned} I &= \int_{\frac{\pi}{4}}^{\pi} \int_0^{2 \sin \theta} f(r \cos \theta, r \sin \theta) r dr d\theta \\ &= \left[ \int_0^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\pi - \arcsin \frac{r}{2}} + \int_{\sqrt{2}}^2 \int_{\arcsin \frac{r}{2}}^{\pi - \arcsin \frac{r}{2}} \right] f(r \cos \theta, r \sin \theta) r dr d\theta \end{aligned} \quad (1.1)$$

### 1.2 20.08.03

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2+3}\right) \cdots \left(1 - \frac{1}{1+2+\cdots+n}\right) = ? \quad (1.2)$$

$$\begin{aligned} 1 - \frac{1}{\frac{n(n+1)}{2}} &= 1 - \frac{2}{n(n+1)} \\ &= \frac{n^2 + n - 2}{n(n+1)} \\ &= \frac{(n+2)(n-1)}{n(n+1)} \end{aligned} \quad (1.3)$$

$$\begin{aligned} I &= \lim_{n \rightarrow +\infty} \frac{1 \times 4}{2 \times 3} \frac{2 \times 5}{3 \times 4} \cdots \frac{(n-2)(n+1)}{(n-1)n} \frac{(n-1)(n+2)}{n(n+1)} \\ &= \lim_{n \rightarrow +\infty} \frac{1}{3} \frac{4}{2} \frac{2}{3} \frac{5}{4} \frac{3}{5} \frac{6}{4} \cdots \frac{n+2}{n} \\ &= \lim_{n \rightarrow +\infty} \frac{1}{3} \frac{n+2}{n} \\ &= \frac{1}{3} \lim_{n \rightarrow +\infty} \frac{n+2}{n} \\ &= \frac{1}{3} \end{aligned} \quad (1.4)$$

卡特兰数  $C_n$

从 0 开始 1, 1, 2, 5, 14, 42, ...

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0$$

该公式的证明可以通过

$$\left( \left( \left( \left( \right) \right) \right) \right)$$

如图所示的括号匹配,  $C_n$  可以看成上面四组括号的合理排列形式, (合理排列意味着每一对括号都是左右对应的, 像  $) ($  这样的形式是非法的)

在  $n$  对括号的排列中, 假设最后一个括号和第  $i$  个左括号匹配。则在第  $i$  个左括号之前, 一定已经匹配上了  $(i-1)$  对左括号。如下图, 因此, 此种情况的数量为  $f(i-1) * f(n-i-1)$ 。 ( $1 \leq i \leq n$ ) 最后一个右括号可以  $1 \sim n$  个左括号匹配共  $n$  种情况。

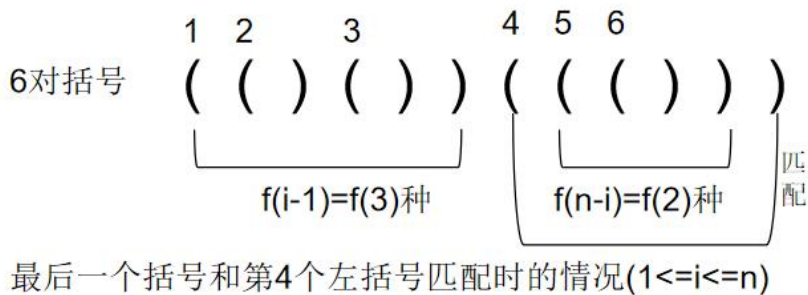


图 1.1: catalan number - proof

第  $n+1$  项

$$C(n) = \frac{C_{2n}^n}{n+1}$$

$$C(n) = C_{2n}^n - C_{2n}^{n-1} = \frac{C_{2n}^n}{n+1}$$

通项公式

$$C_1 = 1, C_n = C_{n-1} \frac{4n-2}{n+1}$$

Python 实现

```
# 打印前 n 个卡特兰数
ans, n = 1, 20
print("1:" + str(ans))
for i in range(2, n + 1):
    ans = ans * (4 * i - 2) // (i + 1)
    print(str(i) + ":" + str(ans))
```

扩展

最后留一道比较有意思的卡特兰数问题, 欢迎读者留言, 提出自己的看法。

8 个高矮不同的人需要排成两队, 每队 4 个人。其中, 每排都是从低到高排列, 且第二排的第  $i$  个人比第一排中第  $i$  个人高, 则有多少种排队方式。

### 1.3 20.08.07

**Theorem 1.3.1.**  $A-G$  不等式

任意  $n$  个非负实数  $a_1, a_2, \dots, a_n$

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n} \quad (1.5)$$

其中等号成立  $\iff a_1 = a_2 = \dots = a_n$

证明. 数学归纳法

$n = 1$  时结论平凡

$$n = 2 \quad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

$$(a_1 - a_2)^2 = a_1^2 - 2a_1 a_2 + a_2^2 \geq 0$$

$$a_1^2 + 2a_1 a_2 + a_2^2 \geq 4a_1 a_2$$

$$(a_1 + a_2)^2 \geq 4a_1 a_2$$

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

$n = k$  时, 假设  $\frac{a_1 + \dots + a_k}{k} \geq \sqrt[k]{a_1 \dots a_k}$  成立

$n = k + 1$

$$\begin{aligned} & \frac{a_1 + \dots + a_k + a_{k+1}}{k+1} - \frac{a_1 + \dots + a_k}{k} \\ &= \frac{k(a_1 + \dots + a_{k+1}) - (k+1)(a_1 + \dots + a_k)}{k(k+1)} \\ &= \frac{ka_{k+1} - (a_1 + \dots + a_k)}{k(k+1)} \end{aligned} \quad (1.6)$$

we found

$$\frac{a_1 + \dots + a_k + a_{k+1}}{k+1} = \frac{a_1 + \dots + a_k}{k} + \frac{ka_{k+1} - (a_1 + \dots + a_k)}{k(k+1)}$$

note

$$A := \frac{a_1 + \dots + a_k}{k}, \quad B := \frac{ka_{k+1} - (a_1 + \dots + a_k)}{k(k+1)}$$

$$\left(\frac{a_1 + \dots + a_k + a_{k+1}}{k+1}\right)^{k+1} = (A+B)^{k+1} \geq A^{k+1} + (k+1)A^k B \quad (1.7)$$

使用二项式展开需要对  $a_i$  从小到大重排, 而使用 Bernoulli 不等式则只需要  $A \geq 0, (A+B) \geq 0$  即可

$$A^{k+1} + (k+1)A^k B = A^k(A + (k+1)B) \quad (1.8)$$

$$\begin{aligned} A^k &= \left(\frac{a_1 + \dots + a_k + a_{k+1}}{k+1}\right)^{k+1} \geq a_1 \dots a_k \quad \text{assume at } (n=k) \\ A + (k+1)B &= \frac{a_1 + \dots + a_k}{k} + \frac{ka_{k+1} - (a_1 + \dots + a_k)}{k} = a_{k+1} \\ \therefore (A+B)^{k+1} &\geq A^k(A + (k+1)B) \geq a_1 \dots a_k a_{k+1} \\ \therefore \frac{a_1 + \dots + a_k + a_{k+1}}{k+1} &\geq \sqrt[k+1]{a_1 \dots a_k a_{k+1}} \end{aligned} \quad (1.9)$$

使用二项式展开定理的条件:

在归纳法第二步对  $a_1 \dots a_{k+1}$  重编号, 使  $a_{k+1}$  为其中最大的数 (之一)

这使得分解式右边第二项  $\frac{ka_{k+1} - (a_1 + \dots + a_k)}{k(k+1)}$  一定是非负数 □

证明. Forward and backward (Cauchy, 1897)

Forward Part:

$n = 2$

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \quad (1.10)$$

$n = 4$

$$\begin{aligned} \frac{a_1 + a_2 + a_3 + a_4}{4} &\geq \sqrt{\frac{a_1 + a_2}{2} \frac{a_3 + a_4}{2}} \\ &\geq \sqrt{\sqrt{a_1 a_2} \sqrt{a_3 a_4}} \\ &\geq \sqrt[4]{a_1 a_2 a_3 a_4} \end{aligned} \quad (1.11)$$

$n = 2^k$  假设不等式  $\frac{a_1 + \dots + a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 \dots a_{2^k}}$  成立

$n = 2^{k+1}$

$$\begin{aligned} \frac{a_1 + \dots + a_{2^k} + \dots + a_{2^{k+1}}}{2^{k+1}} &\geq \sqrt{\frac{a_1 + \dots + a_{2^k}}{2^k} \frac{a_{2^k+1} + \dots + a_{2^{k+1}}}{2^k}} \\ &\geq \sqrt{\sqrt[2^k]{a_1 \dots a_{2^k}} \sqrt[2^k]{a_{2^k+1} \dots a_{2^{k+1}}}} \\ &\geq \sqrt[2^{k+1}]{a_1 \dots a_{2^{k+1}}} \end{aligned} \quad (1.12)$$

Backward Part: A-G 不等式对某个  $n \geq 2$  成立, 则它对  $n-1$  也成立

$$\begin{aligned} \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &= \frac{1}{n} \left( \frac{n}{n-1} \right) \sum_{i=1}^{n-1} a_i \\ &= \frac{1}{n} \left( \sum_{i=1}^{n-1} a_i + \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \end{aligned} \quad (1.13)$$

将  $\frac{1}{n-1} \sum_{i=1}^{n-1} a_i$  看作  $a_n$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n]{\left( \prod_{i=1}^{n-1} a_i \right) \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)} \quad (1.14)$$

$$\left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^n \geq \prod_{i=1}^{n-1} a_i \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \quad (1.15)$$

$$\left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^{n-1} \geq \prod_{i=1}^{n-1} a_i \quad (1.16)$$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \quad (1.17)$$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n]{\left( \prod_{i=1}^{n-1} a_i \right) \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)} \quad (1.18)$$

$$\left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^n \geq \prod_{i=1}^{n-1} a_i \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \quad (1.19)$$

$$\left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^{n-1} \geq \prod_{i=1}^{n-1} a_i \quad (1.20)$$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \quad (1.21)$$

□

**Theorem 1.3.2.** 柯西, 施瓦茨不等式

对  $a_1, \dots, a_n$  和  $b_1, \dots, b_n \in \mathbb{R}$ , 成立

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \quad (1.22)$$

证明.

$$\sum_{i=1}^n (a_i - \lambda b_i)^2 = \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2 \geq 0$$

由韦达定理 (视  $\lambda$  为未知数), 原方程无解或只有唯一解

$$\begin{aligned}\Delta &= b^2 - 4ac \leq 0 \\ (-2 \sum_{i=1}^n a_i b_i)^2 - 4 \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 &\leq 0 \\ (\sum_{i=1}^n a_i b_i)^2 &\leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \\ \sum_{i=1}^n a_i b_i &\leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}\end{aligned}$$

□

## 1.4 20.08.12

### 1.3.2 练习题

1. 关于 Bernoulli 不等式的推广:

- (1) 证明: 当  $-2 \geq h \geq -1$  时 Bernoulli 不等式  $(1+h)^n \geq 1+nh$  仍成立;  
 (2) 证明: 当  $h \geq 0$  时成立不等式

$$(1+h)^n \geq \frac{n(n-1)h^2}{2} \quad (1.23)$$

- (3) 证明: 若  $a_i > -1$  ( $i = 1, 2, \dots, n$ ) 且同号, 则成立不等式

solve:

(1)

$$-2 \leq h \leq -1$$

$$-1 \leq 1+h \leq 0$$

$$-1 \leq (1+h)^n \leq 0$$

$$-2n \leq nh \leq -n$$

$$1-2n \leq 1+nh \leq 1-n$$

$$n=0 \quad (1+h)^0 = 1 = 1+0 \cdot h \text{ 结果是平凡的}$$

$$n=1 \quad 1+h = 1+h \text{ 结果是平凡的}$$

$$n \geq 2 \quad \text{此时 } 1-n \leq -2$$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1-nh \geq 1-2n$$

$$(1+h)^n \geq 1+nh$$

(2)

$$h \geq 0 \quad (1+h)^n \geq \frac{n(n-1)h^2}{2}$$

$$(1+h)^n = 1+nh + \frac{n(n-1)}{2}h^2 + \dots \geq \frac{n(n-1)}{2}h^2$$

推广:

$$(1+h)^n \geq C_n^3 h^3, C_n^4 h^4, \dots, C_n^k h^k, \quad 0 \leq k \leq n$$

(3)

$$\prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$$

(a)  $a_i \geq 0$ , 且同号。

$$\prod_{i=1}^n (1 + a_i) = 1 + \sum_{i=1}^n a_i + \sum_{i=1, i \neq j}^n \sum_{j=1}^n a_i a_j + \sum_{i=1, i \neq j, k}^n \sum_{j=1, j \neq k}^n \sum_{k=1}^n a_i a_j a_k + \dots$$

$$\prod_{i=1}^n (1 + a_i) \geq \frac{\prod_{i=1}^n (1 + a_i)}{1 + a_k} \quad \forall k \in 1, 2, \dots, n, \quad 1 + a_k \geq 1$$

(b)  $0 > a_i > -1$  此时  $1 > 1 + a_i > 0$

别人的方法:  $n = 1$  时不等式变成等式, 显然成立

设  $n = k$  时不等式也成立

$$\prod_{i=1}^k (1 + a_i) \geq 1 + \sum_{i=1}^k a_i$$

则  $n = k + 1$  时, 有

$$\begin{aligned} \prod_{i=1}^{k+1} (1 + a_i) &= \prod_{i=1}^k a_i (1 + a_{k+1}) \geq (1 + \sum_{i=1}^k a_i) (1 + a_{k+1}) \\ (1 + \sum_{i=1}^k a_i) (1 + a_{k+1}) &= 1 + \sum_{i=1}^k a_i + a_{k+1} + \sum_{i=1}^k a_i \cdot a_{k+1} \geq 1 + \sum_{i=1}^{k+1} a_i \\ \therefore \prod_{i=1}^{k+1} (1 + a_i) &\geq 1 + \sum_{i=1}^{k+1} a_i \end{aligned}$$

2. 利用 A-G 不等式求解下列有关阶乘  $n!$  的不等式

(1) 证明: 当  $n > 1$  时成立

$$n! < \left(\frac{n+1}{2}\right)^n \quad (1.24)$$

(2) 利用  $(n!)^2 = (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n)$  证明: 当  $n > 1$  时成立

$$n! < \left(\frac{n+2}{\sqrt{6}}\right)^n \quad (1.25)$$

(3) 比较 (1)(2) 两个不等式的优劣, 并说明原因;

(4) 证明: 对任意实数  $r$  成立

$$\left(\sum_{k=1}^n k^r\right)^n \geq n^n (n!)^r \quad (1.26)$$

solve:

(1) when  $n > 1$

$$\begin{aligned} n! &= 1 \times 2 \times \dots \times n < \left(\frac{1+2+\dots+n}{n}\right)^n \\ \left(\frac{1+2+\dots+n}{n}\right)^n &= \left(\frac{n(n+1)}{2n}\right)^n = \left(\frac{n+1}{2}\right)^n \end{aligned}$$

(2) when  $n > 1$

$$\begin{aligned} (n!)^2 &= (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n) < \left(\frac{n \cdot 1 + (n-1) \cdot 2 + \dots + 1 \cdot n}{n}\right)^n \\ n \cdot 1 + (n-1) \cdot 2 + \dots + 1 \cdot n &= \sum_{k=1}^n (n-k+1)k \\ \sum_{k=1}^n (n-k+1)k &= (n+1) \sum_{k=1}^n k - \sum_{k=1}^n k^2 \\ &= (n+1) \frac{n(n+1)}{2} - \frac{n(2n+1)(n+1)}{6} \\ &= \frac{n(n+1)}{6} (3(n+1) - (2n+1)) \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

$$\begin{aligned}
(n!)^2 &= (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n) \\
&< \left( \frac{n \cdot 1 + (n-1) \cdot 2 + \dots + 1 \cdot n}{n} \right)^n \\
&= \left( \frac{1}{n} \frac{n(n+1)(n+2)}{6} \right)^n \\
&= \left( \frac{(n+1)(n+2)}{6} \right)^n \\
&< \left( \frac{n+2}{6} \right)^{2n} \\
\therefore n! &< \left( \frac{n+2}{\sqrt{6}} \right)^n
\end{aligned}$$

(3)

$$\frac{n+1}{2} = \frac{n+2}{\sqrt{6}}$$

解得  $n = 1 + \sqrt{6} > 3$ ,  $n > 3$  时 (2) 式更精确, 结果比 (1) 式更好。

(4)  $\forall r \in \mathbb{R} \quad (n!)^r \leq \frac{1}{n^n} (\sum_{k=1}^n k^r)^n$  由 A-G 不等式

$$\begin{aligned}
\frac{1}{n} \sum_{k=1}^n k^r &\geq \sqrt[n]{\prod_{k=1}^n k^r} \\
(n!)^r &= \prod_{k=1}^n k^r \leq \left( \frac{1}{n} \sum_{k=1}^n k^r \right)^n = \frac{1}{n^n} \left( \sum_{k=1}^n k^r \right)^n
\end{aligned}$$

## 1.5 20.08.13

2.(4)

$$\begin{aligned}
\forall r \in \mathbb{R} \quad \left( \sum_{i=1}^n k^r \right)^n &\geq n^n (n!)^r \\
(n!)^r &= \prod_{k=1}^n k^r \leq \left( \frac{1^r + 2^r + \dots + n^r}{n} \right)^n = \frac{1}{n^n} \left( \sum_{k=1}^n k^r \right)^n \quad \text{A-G inequality} \\
\therefore \left( \sum_{k=1}^n k^r \right)^n &\geq n^n (n!)^r
\end{aligned} \tag{1.27}$$

3.  $a_k > 0, \quad k = 1, 2, \dots, n$  证明几何-调和平均值不等式

$$\left( \prod_{k=1}^n a_k \right)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}} \tag{1.28}$$

证明. from A-G inequality

$$\begin{aligned}
\frac{\sum_{k=1}^n \frac{1}{a_k}}{n} &\geq \sqrt[n]{\prod_{k=1}^n \frac{1}{a_k}} \\
&= \frac{1}{\sqrt[n]{\prod_{k=1}^n a_k}} \\
\therefore a_k > 0, \quad \sqrt[n]{\prod_{k=1}^n a_k} &\geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}
\end{aligned} \tag{1.29}$$

□

4.  $a, b, c \geq 0$ , proof that

$$\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3} \tag{1.30}$$

并推广到  $n$  个非负数的情况

证明. left:

$$\begin{aligned}\sqrt[3]{abc} &= \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \\ &\leq \sqrt{\frac{ab + bc + ca}{3}}\end{aligned}\quad (1.31)$$

right:

$$\begin{aligned}\sqrt{\frac{ab + bc + ca}{3}} &\leq \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ &= \sqrt{\frac{2(a^2 + b^2 + c^2) + 2(ab + bc + ca)}{12}} \\ &= \sqrt{\frac{a^2 + b^2 + c^2 + ab + bc + ca}{6}}\end{aligned}\quad (1.32)$$

$$\because a, b, c \geq 0 \quad \frac{ab + bc + ca}{3} \leq \frac{a^2 + b^2 + c^2 + ab + bc + ca}{6}$$

需要证明  $\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$

对该式两边平方

$$\frac{ab + bc + ca}{3} \leq \frac{(a + b + c)^2}{9} = \frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{9} \quad (1.33)$$

$$\begin{aligned}\frac{ab + bc + ca}{3} &\leq \frac{a^2 + b^2 + c^2}{6} + \frac{ab + bc + ca}{6} \\ &\leq \frac{a^2 + b^2 + c^2}{6} + \frac{ab + bc + ca}{3} \\ &= \left(\frac{a + b + c}{3}\right)^2\end{aligned}\quad (1.34)$$

$$\therefore \sqrt{\frac{ab + bc + ca}{3}} \leq \frac{a + b + c}{3}$$

□

证明. 推广至 n 个

$$\begin{aligned}n = 2 \quad \sqrt{ab} &\leq \frac{a + b}{2} \\ n = 3 \quad \sqrt[3]{abc} &\leq \sqrt{\frac{ab + bc + ca}{3}} \leq \frac{a + b + c}{3} \\ n = k \quad \sqrt[k]{\prod_{i=1}^k a_i} &\leq \sqrt{\frac{\sum_{i=1}^k -1 a_i a_{i+1} + a_k a_1}{k}} \leq \frac{\sum_{i=1}^k a_i}{k}\end{aligned}\quad (1.35)$$

$$1 \quad \sqrt[k]{a_1 a_2 \dots a_k} = \sqrt{\sqrt[k]{a_1^2 a_2^2 \dots a_k^2}} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k}} \quad (1.36)$$

$$2 \quad \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k}} \leq \frac{a_1 + \dots + a_k}{k} \quad (1.37)$$

$$\begin{aligned}\frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k} &\leq \frac{a_1^2 + \dots + a_k^2}{2k} \\ 2 \frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k} &\leq \frac{(a_1 + \dots + a_k)^2}{2k} \\ \sqrt{\frac{a_1 \dots a_k}{k}} &\leq \frac{a_1 + \dots + a_k}{\sqrt{4k}} \quad \text{wrong!}\end{aligned}\quad (1.38)$$

□