数学分析习题课讲义上册习题

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$$
\begin{aligned}
I &= \int_{\frac{\pi}{4}}^{\pi} \int_{0}^{2\sin\theta} f(r\cos\theta, r\sin\theta) r \mathrm{d}r \mathrm{d}\theta \\
&= [\int_{0}^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\pi - \arcsin\frac{r}{2}} + \int_{\sqrt{2}}^{2} \int_{\arcsin\frac{r}{2}}^{\pi - \arcsin\frac{r}{2}}] f(r\cos\theta, r\sin\theta) r \mathrm{d}r \mathrm{d}\theta
\end{aligned} \tag{1}
$$

$$
\lim_{n\to+\infty} (1 - \frac{1}{1+2})(1 - \frac{1}{1+2})(1 - \frac{1}{1+2+3}) \dots (1 - \frac{1}{1+2+\dots+n}) = ? \tag{2}
$$

$$
\begin{aligned}
1 - \frac{1}{\frac{n(n+1)}{2}} &= 1 - \frac{2}{n(n+1)} \\
&= \frac{n^2+n-2}{n(n+1)} \\
&= \frac{(n+2)(n-1)}{n(n+1)}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
I &= \lim_{n\to+\infty} \frac{1\times4}{2\times3} \frac{2\times5}{3\times4} \dots \frac{(n-2)(n+1)}{(n-1)n} \frac{(n-1)(n+2)}{n(n+1)} \\
&= \lim_{n\to+\infty} \frac{1}{3} \frac{4}{2} \frac{2}{4} \frac{5}{3} \frac{3}{5} \frac{6}{4} \dots \frac{n+2}{n} \\
&= \lim_{n\to+\infty} \frac{1}{3} \frac{n+2}{n} \\
&= \frac{1}{3} \lim_{n\to+\infty} \frac{n+2}{n} \\
&= \frac{1}{3}
\end{aligned} \tag{4}
$$

Theorem 1. *A-G 不等式*

任意 n 个非负实数 $a_1, a_2, \dots, a_n$

$$
\frac{a_1 + a_2 + \dots + a_n}{n} \ge \sqrt[n]{a_1 \dots a_n} \tag{5}
$$

其中等号成立 $\iff a_1 = a_2 = \dots = a_n$

Proof. 数学归纳法

$n = 1$ 时结论平凡

$n = 2$ $\quad \frac{a_1+a_2}{2} \ge \sqrt{a_1 a_2}$

$$
(a_1 - a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2 \ge 0
$$

$$
a_1^2 + 2a_1a_2 + a_2^2 \ge 4a_1a_2
$$

$$
(a_1 + a_2)^2 \ge 4a_1a_2
$$

$$
\frac{a_1 + a_2}{2} \ge \sqrt{a_1a_2}
$$

$n=k$ 时，假设 $\frac{a_1+\cdots+a_k}{k} \geq \sqrt[k]{a_1\ldots a_k}$ 成立

$n=k+1$

$$\begin{aligned}
&\frac{a_1+\cdots+a_k+a_{k+1}}{k+1}-\frac{a_1+\cdots+a_k}{k} \\
=&\frac{k(a_1+\cdots+a_{k+1})-(k+1)(a_1+\cdots+a_k)}{k(k+1)} \\
=&\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}
\end{aligned} \tag{6}$$

we found

$$\frac{a_1+\cdots+a_k+a_{k+1}}{k+1}=\frac{a_1+\cdots+a_k}{k}+\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}$$

note

$$A:=\frac{a_1+\cdots+a_k}{k}, \qquad B:=\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}$$

$$(\frac{a_1+\cdots+a_k+a_{k+1}}{k+1})^{k+1}=(A+B)^{k+1}\geq A^{k+1}+(k+1)A^kB \tag{7}$$

使用二项式展开需要对 a_i 从小到大重排，而使用 Bernoulli 不等式则只需要 $A\geq 0,(A+B)\geq 0$ 即可

$$A^{k+1}+(k+1)A^kB=A^k(A+(k+1)B) \tag{8}$$

$$\begin{aligned}
A^k=(\frac{a_1+\cdots+a_k+a_{k+1}}{k+1})^{k+1}&\geq a_1\ldots a_k \quad \text{assume at}(n=k) \\
A+(k+1)B=\frac{a_1+\cdots+a_k}{k}+\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k}&=a_{k+1} \\
\because(A+B)^{k+1}\geq A^k(A+(k+1)B)&\geq a_1\ldots a_k a_{k+1} \\
\therefore\frac{a_1+\cdots+a_k+a_{k+1}}{k+1}&\geq \sqrt[k+1]{a_1\ldots a_k a_{k+1}}
\end{aligned} \tag{9}$$

使用二项式展开定理的条件:

在归纳法第二步对 $a_1\ldots a_{k+1}$ 重编号，使 a_{k+1} 为其中最大的数（之一）

这使得分解式右边第二项 $\frac{ka_{k+1}-(a_1+\cdots+a_k)}{k(k+1)}$ 一定是非负数 □

Proof. Forward and backward (Cauchy, 1897)

Forward Part:

$n=2$

$$\frac{a_1+a_2}{2}\geq\sqrt{a_1a_2} \tag{10}$$

$n=4$

$$\begin{aligned}
\frac{a_1+a_2+a_3+a_4}{4}&\geq\sqrt{\frac{a_1+a_2}{2}\frac{a_3+a_4}{2}} \\
&\geq\sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} \\
&\geq\sqrt[4]{a_1a_2a_3a_4}
\end{aligned} \tag{11}$$

$n = 2^k$ 假设不等式 $\frac{a_1+\cdots+a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 \ldots a_{2^k}}$ 成立
$n = 2^{k+1}$

$$\begin{aligned}\frac{a_1 + \cdots + a_{2^k} + \cdots + a_{2^{k+1}}}{2^{k+1}} &\geq \sqrt{\frac{a_1 + \cdots + a_{2^k}}{2^k} \frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k}} \\ &\geq \sqrt{\sqrt[2^k]{a_1 \ldots a_{2^k}} \sqrt[2^k]{a_{2^k+1} \ldots a_{2^{k+1}}}} \\ &\geq \sqrt[2^{k+1}]{a_1 \ldots a_{2^{k+1}}}\end{aligned} \tag{12}$$

Backward Part: A-G 不等式对某个 $n \geq 2$ 成立，则它对 $n-1$ 也成立

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1} a_i &= \frac{1}{n}(\frac{n}{n-1})\sum_{i=1}^{n-1} a_i \\ &= \frac{1}{n}(\sum_{i=1}^{n-1} a_i + \frac{1}{n-1}\sum_{i=1}^{n-1} a_i)\end{aligned} \tag{13}$$

将 $\frac{1}{n-1}\sum_{i=1}^{n-1} a_i$ 看作 a_n

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n]{(\prod_{i=1}^{n-1} a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)} \tag{14}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^n \geq \prod_{i=1}^{n-1} a_i(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i) \tag{15}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^{n-1} \geq \prod_{i=1}^{n-1} a_i \tag{16}$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \tag{17}$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n]{(\prod_{i=1}^{n-1} a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)} \tag{18}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^n \geq \prod_{i=1}^{n-1} a_i(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i) \tag{19}$$

$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^{n-1} \geq \prod_{i=1}^{n-1} a_i \tag{20}$$

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \tag{21}$$

□

Theorem 2. 柯西，施瓦茨不等式
对 $a_1, \ldots, a_n$ 和 $b_1, \ldots, b_n \in \mathbb{R}$，成立

$$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \tag{22}$$

Proof.

$$\sum_{i=1}^{n}(a_i - \lambda b_i)^2 = \sum_{i=1}^{n} a_i^2 - 2\lambda \sum_{i=1}^{n} a_i b_i + \lambda^2 \sum_{i=1}^{n} b_i^2 \geq 0$$

由韦达定理（视 λ 为未知数），原方程无解或只有唯一解

$$\begin{aligned} \Delta = b^2 - 4ac \leq 0 \\ (-2\sum_{i=1}^{n} a_i b_i)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq 0 \\ (\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \\ \sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \end{aligned} \tag{23}$$

□

Theorem 3. 定积分第一中值定理
设函数 $f(x), g(x) \in \mathbb{C}[a,b]$. 且在 $[a,b]$ 上不变号，则存在 $\zeta \in [a,b]$，使得 $\int_a^b f(x)g(x) = f(\zeta)\int_a^b g(x)dx$

Proof. suppose that $g(x) \geq 0$. $f(x)$ continuous on close set,so we can get the maximum and minimum value of f. We note that m is the minimum value of $f(x), x \in [a,b]$,and M is the maximum value of $f(x)$, then we have:

$$mg(x) \leqslant f(x)g(x) \leqslant Mg(x)$$

$$m\int_a^b g(x)\mathrm{d}x \leqslant \int_a^b f(x)g(x)\mathrm{d}x \leqslant M\int_a^b g(x)\mathrm{d}x$$

note that we don't know$\int_a^b g(x)\mathrm{d}x \neq 0$
When $\int_a^b g(x)\mathrm{d}x = 0$, then $g(x) \equiv 0$, So $\forall\zeta \in [a,b]$, the theorem works.
When $\int_a^b g(x)\mathrm{d}x = 0$, then $m \leqslant \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \leq M$
From the Intermediate Value Theorem, $f(x) \in \mathbb{C}[a,b] \quad m \leqslant f(x) \leqslant M$

$$\exists\zeta\in[a,b]\quad f(\zeta)=\frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}$$
$$\int_a^b f(x)g(x)\mathrm{d}x=f(\zeta)\int_a^b g(x)\mathrm{d}x$$

□

设 $g(x)$ 在 $[a,b]$ 上连续可积，$f(x)$ 在 $[a,b]$ 上连续单调递增，且 $f'(x)\geq 0$，并对 $\forall x\in[a,b]$ 有 $f(x)\geq 0$。则存在 $\zeta\in[a,b]$，使得

$$\int_a^b f(x)g(x)\mathrm{d}x=f(b)\int_\zeta^b g(x)\mathrm{d}x$$

Proof. set$G(x)=\int_x^b g(t)\mathrm{d}t, g(x)$在$[a,b]$ 上可积

则 $G(x),x\in[a,b]$ 存在最值，设最小值和最大值分别为 m,M

$$G(x)=-\int_b^x g(t)\mathrm{d}t,\quad G'(x)=-g(x)$$

$$\begin{aligned}\int_a^b f(x)g(x)\mathrm{d}x&=-\int_a^b f(x)\mathrm{d}G(x)\\&=-(f(b)G(b)-f(a)G(a))-\int_a^b G(x)f'(x)\mathrm{d}x\\&=f(a)G(a)+\int_a^b G(x)f'(x)\mathrm{d}x\end{aligned}\tag{24}$$

$$m\int_a^b f'(x)\mathrm{d}x\leqslant\int_a^b G(x)f'(x)\mathrm{d}x\leqslant M\int_a^b f'(x)\mathrm{d}x$$
$$m[f(b-f(a))]\leqslant\int_a^b G(x)f'(x)\mathrm{d}x\leqslant M[f(b-f(a))]$$

$\because$

$$mf(a)\leqslant f(a)G(a)\leqslant Mf(a)$$
$$mf(b)\leqslant\int_a^b f(x)g(x)\mathrm{d}x\leqslant Mf(b)$$

From the Intermediate Value Theorem, $\exists\zeta\in[a,b]s.t.G(\zeta)=\frac{\int_a^b f(x)g(x)\mathrm{d}x}{f(b)}$ then we have

$$\int_a^b f(x)g(x)\mathrm{d}x=f(b)G(\zeta)=f(b)\int_\zeta^b g(x)\mathrm{d}x$$

□

1.3.2 练习题

1. 关于 Bernoulli 不等式的推广：

(1) 证明：当 $-2 \geq h \geq -1$ 时 Bernoulli 不等式 $(1+h)^n \geq 1+nh$ 仍成立；

(2) 证明：当 $h \geq 0$ 时成立不等式

$$(1+h)^n \geq \frac{n(n-1)h^2}{2} \tag{25}$$

(3) 证明：若 $a_i > -1 \quad (i = 1, 2, \ldots, n)$ 且同号，则成立不等式

solve:

(1)

$$-2 \leq h \leq -1$$

$$-1 \leq 1+h \leq 0$$

$$-1 \leq (1+h)^n \leq 0$$

$$-2n \leq nh \leq -n$$

$$1-2n \leq 1+nh \leq 1-n$$

$n = 0 \qquad (1+h)^0 = 1 = 1 + 0 * h$ 结果是平凡的

$n = 1 \qquad 1+h = 1+h$ 结果是平凡的

$n \geq 2 \qquad$ 此时 $1-n \leq -2$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1-nh \geq 1-2n$$

$$(1+h)^n \geq 1+nh$$

(2)

$$h \geq 0 \qquad (1+h)^n \geq \frac{n(n-1)h^2}{2}$$

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots \geq \frac{n(n-1)}{2}h^2$$

推广:

$$(1+h)^n \geq C_n^3 h^3, C_n^4 h^4, \ldots, C_n^k h^k, \qquad 0 \leq k \leq n$$

(3)

$$\prod_{i=1}^{n}(1+a_i) \geq 1 + \sum_{i=1}^{n} a_i$$

(a)$a_i \geq 0$, 且同号。

$$\prod_{i=1}^{n}(1+a_i) = 1 + \sum_{i=1}^{n} a_i + \sum_{i=1, i\neq j}^{n} \sum_{j=1}^{n} a_i a_j + \sum_{i=1, i\neq j,k}^{n} \sum_{j=1, j\neq k}^{n} \sum_{k=1}^{n} a_i a_j a_k + \ldots$$

$$\prod_{i=1}^{n}(1+a_i) \geq \frac{\prod_{i=1}^{n}(1+a_i)}{1+a_k} \qquad \forall k \in 1,2,\dots,n, \quad 1+a_k \geq 1$$

(b) $0 > a_i > -1$此时$1 > 1 + a_i > 0$
别人的方法：$n = 1$ 时不等式变成等式，显然成立
设 $n = k$ 时不等式也成立

$$\prod_{i=1}^{k}(1+a_i) \geq 1 + \sum_{i=1}^{k} a_i$$

则 $n = k + 1$ 时，有

$$\prod_{i=1}^{k+1}(1+a_i) = \prod_{i=1}^{k} a_i(1+a_{k+1}) \geq (1+\sum_{i=1}^{k} a_i)(1+a_{k+1})$$

$$(1+\sum_{i=1}^{k} a_i)(1+a_{k+1}) = 1 + \sum_{i=1}^{k} a_i + a_{k+1} + \sum_{i=1}^{k} a_i \cdot a_{k+1} \geq 1 + \sum_{i=1}^{k+1} a_i$$

$$\therefore \prod_{i=1}^{k+1}(1+a_i) \geq 1 + \sum_{i=1}^{k+1} a_i$$

2. 利用 A-G 不等式求解下列有关阶乘 $n!$ 的不等式
(1) 证明：当 $n > 1$ 时成立

$$n! < (\frac{n+1}{2})^n \tag{26}$$

(2) 利用 $(n!)^2 = (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n)$ 证明：当 $n > 1$ 时成立

$$n! < (\frac{n+2}{\sqrt{6}})^n \tag{27}$$

(3) 比较 (1)(2) 两个不等式的优劣，并说明原因；
(4) 证明：对任意实数 r 成立

$$(\sum_{k=1}^{n} k^r)^n \geq n^n (n!)^r \tag{28}$$

solve:
(1) when $n > 1$

$$n! = 1 \times 2 \times \cdots \times n < (\frac{1+2+\cdots+n}{n})^n$$

$$(\frac{1+2+\cdots+n}{n})^n = (\frac{n(n+1)}{2n})^n = (\frac{n+1}{2})^n$$

(2) when $n > 1$

$$(n!)^2 = (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n) < (\frac{n \cdot 1 + (n-1) \cdot 2 + \cdots + 1 \cdot n}{n})^n$$

$$n \cdot 1 + (n-1) \cdot 2 + \cdots + 1 \cdot n = \sum_{k=1}^{n} (n-k+1)k$$

$$\begin{aligned}
\sum_{k=1}^{n} (n-k+1)k &= (n+1) \sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 \\
&= (n+1) \frac{n(n+1)}{2} - \frac{n(2n+1)(n+1)}{6} \\
&= \frac{n(n+1)}{6} (3(n+1) - (2n+1)) \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned} \tag{29}$$

$$\begin{aligned}
(n!)^2 &= (n \cdot 1)((n-1) \cdot 2) \ldots (1 \cdot n) \\
&< (\frac{n \cdot 1 + (n-1) \cdot 2 + \cdots + 1 \cdot n}{n})^n \\
&= (\frac{1}{n} \frac{n(n+1)(n+2)}{6})^n \\
&= (\frac{(n+1)(n+2)}{6})^n \\
&< (\frac{n+2}{6})^{2n}
\end{aligned} \tag{30}$$

$$\therefore \qquad n! < (\frac{n+2}{\sqrt{6}})^n \tag{31}$$

(3)

$$\frac{n+1}{2} = \frac{n+2}{\sqrt{6}} \tag{32}$$

解得 $n = 1 + \sqrt{6} > 3$, $n > 3$ 时 (2) 式更精确，结果比 (1) 式更好。

(4) $\forall r \in \mathbb{R} \quad (n!)^r \leqslant \frac{1}{n^n} (\sum_{k=1}^{n} k^r)^n$ 由 A-G 不等式

$$\frac{1}{n} \sum_{k=1}^{n} k^r \geq \sqrt[n]{\prod_{k=1}^{n} k^r} \tag{33}$$

$$(n!)^r = \prod_{k=1}^{n} k^r \leqslant (\frac{1}{n} \sum_{k=1}^{n} k^r)^n = \frac{1}{n^n} (\sum_{k=1}^{n} k^r)^n \tag{34}$$

2.(4)

$$\begin{aligned}
\forall r \in \mathbb{R} \qquad (\sum_{i=1}^{n} k^r)^n \geq n^n (n!)^r \\
(n!)^r = \prod_{k=1}^{n} k^r \leqslant (\frac{1^r + 2^r + \cdots + n^r}{n})^n = \frac{1}{n^n} (\sum_{k=1}^{n} k^r)^n \quad \text{A-G inequality} \\
\therefore (\sum_{k=1}^{n} k^r)^n \geq n^n (n!)^r
\end{aligned} \tag{35}$$

3. $a_k > 0, \quad k = 1, 2, \ldots, n$ 证明几何-调和平均值不等式

$$(\prod_{k=1}^{n} a_k)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}} \tag{36}$$

Proof. from A-G inequality

$$\begin{aligned}\frac{\sum_{k=1}^{n} \frac{1}{a_k}}{n} &\geq \sqrt[n]{\prod_{k=1}^{n} \frac{1}{a_k}} \\ &= \frac{1}{\sqrt[n]{\prod_{k=1}^{n} a_k}} \\ \therefore a_k > 0, \qquad \sqrt[n]{\prod_{k=1}^{n} a_k} &\geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}\end{aligned} \tag{37}$$

□

4. $a, b, c \geq 0$, proof that

$$\sqrt[3]{abc} \leqslant \sqrt{\frac{ab + bc + ca}{3}} \leq \frac{a + b + c}{3} \tag{38}$$

并推广到 n 个非负数的情况

Proof. left:

$$\begin{aligned}\sqrt[3]{abc} &= \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \\ &\leqslant \sqrt{\frac{ab + bc + ca}{3}}\end{aligned} \tag{39}$$

right:

$$\begin{aligned}\sqrt{\frac{ab + bc + ca}{3}} &\leqslant \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ &= \sqrt{\frac{2(a^2 + b^2 + c^2) + 2(ab + bc + ca)}{12}} \\ &= \sqrt{\frac{a^2 + b^2 + c^2 + ab + bc + ca}{6}}\end{aligned} \tag{40}$$

$$\because a, b, c \geq 0 \qquad \frac{ab + bc + ca}{3} \leqslant \frac{a^2 + b^2 + c^2 + ab + bc + ca}{6} \tag{41}$$

需要证明 $\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$
对该式两边平方

$$\frac{ab + bc + ca}{3} \leq \frac{(a + b + c)^2}{9} = \frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{9} \tag{42}$$

$$
\begin{aligned}
\frac{ab+bc+ca}{3} &\leqslant \frac{a^2+b^2+c^2}{6}+\frac{ab+bc+ca}{6} \\
&\leqslant \frac{a^2+b^2+c^2}{6}+\frac{ab+bc+ca}{3} \\
&= (\frac{a+b+c}{3})^2 \\
\therefore \sqrt{\frac{ab+bc+ca}{3}} &\leqslant \frac{a+b+c}{3}
\end{aligned}
\tag{43}
$$

□

Proof. 推广至 n 个

$$
\begin{aligned}
[l] n=2 \qquad & \sqrt{ab} && \leq \frac{a+b}{2} \\
n=3 \qquad & \sqrt[3]{abc} && \leq \sqrt{\frac{ab+bc+ca}{3}} && \leq \frac{a+b+c}{3} \\
n=k \qquad & \sqrt[k]{\prod_{i=1}^{k} a_i} && \leq \sqrt{\frac{\sum_{i=1}^{k} -1 a_i a_{i+1} + a_k a_1}{k}} && \leq \frac{\sum_{i=1}^{k} a_i}{k}
\end{aligned}
\tag{44}
$$

$$
1 \qquad \sqrt[k]{a_1 a_2 \dots a_k} = \sqrt{\sqrt[k]{a_1^2 a_2^2 \dots a_k^2}} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k}} \tag{45}
$$

$$
2 \qquad \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k}} \leqslant \frac{a_1 + \cdots + a_k}{k} \tag{46}
$$

$$
\begin{aligned}
\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k} &\leqslant \frac{a_1^2 + \dots a_k^2}{2k} \\
2\frac{a_1 a_2 + a_2 a_3 + \dots a_k a_1}{k} &\leqslant \frac{(a_1 + \dots a_k)^2}{2k} \\
\sqrt{\frac{a_1 \dots a_k}{k}} &\leqslant \frac{a_1 + \cdots + a_k}{\sqrt{4k}} \quad \text{wrong!}
\end{aligned}
\tag{47}
$$

□

1 引论

1.1 关于习题课教案的组织

1.2 书中常用记号

1. $\mathbf{N}_+$：所有正整数组成的集合.

2. $\mathbf{R}$：所有实数组成的集合 (同时也用于表示无限区间 $(-\infty,\infty)$).

3. $\mathbf{Q}$：所有有理数组成的集合.

4. $\mathbf{C}$：所有复数组成的集合.

5. $\iff$ 是等价关系的记号.$A \iff B$ 表示 A 和 B 等价. 例如，A 代表 $x>3$，B 代表 $x-3>0$，则 $x>3 \iff x-3>0$.

6. $[x]$ 是实数 x 的整数部分,即不超过 x 的最大整数. 例如,$[\sqrt{2}]=1,[-\sqrt{2}]=-2$. 关于 $[x]$ 的基本不等式是：$[x]\leqslant x<[x]+1$，或 $x-1<[x]\leqslant x$

7. $\square$ 表示一个证明或解的结束.

8. $\binom{n}{k}=C_n^k=\frac{n(n-1)\cdots(n-k+1)}{k!}$.

9. 记号 $\approx$ 表示近似值. 例如 $\sqrt{2}\approx 1.4$.

10. 复合函数 $f(g(x))$ 也写成 $(f\circ g)(x)$ 或 $f\circ g$.

11. 若 A 和 B 为两个集合，则用记号 $A-B$ 或 $A\backslash B$ 表示 A 与 B 的差集，也就是集合 $\{x|x\in A且x\notin B\}$.

12. 用 $O_\delta(a)$ 表示以 a 为中心，以 $\delta>0$ 为半径的邻域. 它就是开区间 $(a-\delta,a+\delta)$(也可以用 $U_\delta(a)$ 等记号). 如不必指出半径，则可简记为 $O(a)$ (或 $U(a)$).

1.3 几个常用的初等不等式

1.3.1 几个初等不等式的证明

A.G 不等式 $a_1,a_2,\cdots,a_n$, n 个非负实数

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n} \tag{48}$$

$\geq$ in inequation became $= \iff a_1=a_2=\cdots=a_n$

Proof. method 1. induction method

$$
\begin{aligned}
&k=1 \qquad a_1=a_1\\
&k=2 \qquad \frac{a_1+a_2}{2}\geq\sqrt{a_1a_2}\\
&k=n \qquad \text{suppose} \quad \frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n}\\
&k=n+1\\
&\quad \frac{a_1+a_2+\cdots+a_{n+1}}{n+1}-\frac{a_1+a_2+\cdots+a_n}{n}\\
&\quad =\frac{n(a_1+a_2+\cdots+a_{n+1})-(n+1)(a_1+a_2+\cdots+a_n)}{n(n+1)}\\
&\quad =\frac{na_{n+1}-(a_1+a_2+\cdots+a_n)}{n(n+1)}
\end{aligned}
$$

Set $A=\frac{a_1+a_2+\cdots+a_n}{n}$, $B=\frac{na_{n+1}-(a_1+a_2\cdots+a_n)}{n(n+1)}$

$$(\frac{a_1+a_2+\cdots+a_{n+1}}{n+1})^{n+1}=(A+B)^{n+1}$$

$A>0,B\geq 0$

$$
\begin{aligned}
&(A+B)^{n+1}\geq A^{n+1}+(n+1)A^nB\\
&A^{n+1}+(n+1)A^nB=A^n(A+(n+1)B)\\
&A^n=(\frac{a_1+a_2+\cdots+a_n}{n})^n\geq a_1a_2\cdots a_n\\
&A+(n+1)B=\frac{a_1+a_2+\cdots+a_n}{n}+\frac{na_{n+1}-(a_1+a_2+\cdots+a_n)}{n}=a_{n+1}\\
&\therefore (A+B)^{n+1}\geq A^n(A+(n+1)B)\geq a_1a_2\cdots a_n\cdot a_{n+1}\\
&\therefore \frac{a_1+a_2+\cdots+a_{n+1}}{n+1}\geq\sqrt[n+1]{a_1a_2\cdots a_na_{n+1}}
\end{aligned}
$$

使用二项式展开定理的条件

在归纳法第二步，将 $a_1,a_2,\cdots,a_{n+1}$ 重编号，使得 $n+1$ 为其中最大的数(之一)，这使得分解式右边第二项 $(na_{n+1}-(a_1+a_2+\cdots+a_n))/n(n+1)$ 一定是非负数。

method 2. Forward and Backward (Cauchy, 1897)

Forward part

$k = 2. \dfrac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$

$k = 4. \dfrac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt{(\dfrac{a_1 + a_2}{2}) \cdot (\dfrac{a_3 + a_4}{2})}.$

$\geq \sqrt{\sqrt{a_1 a_2}\sqrt{a_3 a_4}} = \sqrt[4]{a_1 a_2 a_3 a_4}.$

$k = 2^n$.Suppose $\dfrac{a_1 + a_2 + \cdots a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdots a_{2^n}}$

$k = 2^{n+1}$.

$$\frac{a_1 + a_2 + \cdots + a_{2^n} + \cdots + a_{2^{n+1}}}{2^{n+1}} \geq \sqrt{(\frac{a_1 + a_2 + \cdots + a_2^n}{2^n}) \cdot (\frac{a_{2^n+1} + a_{2^n+2} + \cdots + a_2^{n+1}}{2^n})}$$

$$I \geq \sqrt{\sqrt[2^n]{a_1 a_2 \cdots a_{2^n}} \sqrt[2^n]{a_{2^n+1} a_{2^n+2} \cdots a_{2^{n+1}}}} = \sqrt[2^{n+1}]{a_1 a_2 \cdots a_{2^{n+1}}}$$

Backward part suppose A.G inequality is valid when $k = n$, Consider $k = n - 1$.

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i = \frac{1}{n}(\frac{n}{n-1})\sum_{i=1}^{n-1} a_i$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i = \frac{1}{n}(\sum_{i=1}^{n-1} a_i + \frac{1}{n-1}\sum_{i=1}^{n-1} a_i)$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n]{(\prod_{i=1}^{n-1} a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)}$$
$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^n \geq (\prod_{i=1}^{n-1} a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)$$
$$(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i)^{n-1} \geq (\prod_{i=1}^{n-1} a_i)$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i}$$

□

Proposition 1 (1.3.5)**.** 柯西-施瓦茨不等式对 $a_1, a_2, \cdots, a_n$ 和 $b_1, b_2, \cdots, b_n$，成立

$$|\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}|$$

Proof.

$$0 \leqslant \sum_{i=1}^{n}(a_i - \lambda b_i)^2 = \sum_{i=1}^{n} a_i^2 - 2\lambda \sum_{i=1}^{n} a_i b_i + \lambda^2 \sum_{i=1}^{n} b_i^2$$

由韦达定理 (视 λ 为未知数). 原方程无解或只有唯一解。

$$\begin{aligned}
&\Delta = b^2 - 4ac \leqslant 0 \\
&(-2\sum_{i=1}^{n} a_i b_i)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leqslant 0 \\
&(\sum_{i=1}^{n} a_i b_i)^2 \leqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \\
&\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}
\end{aligned}$$

□

1.3.2 练习题

Example 1. 关于 *Bernoulli* 不等式的推广:
(1) 证明: 当 $-2 \leqslant h \leqslant -1$ 时 *Bernoulli* 不等式 $(1+h)^n \geq 1 + nh$ 仍成立;
(2) 证明: 当 $h \geq 0$ 时成立不等式 $(1+h)^n \geq \frac{n(n-1)h^2}{2}$, 并推广之;
(3) 证明: 若 $a_i > -1 (i = 1, 2, \ldots, n)$ 且同号, 则成立不等式

$$\prod_{i=1}^{n}(1 + a_i) \geq 1 + \sum_{i=1}^{n} a_i$$

Proof. (1)

$$\begin{aligned}
&-2 \leqslant h \leqslant -1 \\
&-1 \leqslant 1 + h \leqslant 0 && -1 \leqslant (1+h)^n \leqslant 0 \\
&-2n \leqslant nh \leqslant -n && 1 - 2n \leqslant 1 + nh \leqslant 1 - n
\end{aligned}$$

$$\begin{aligned}
&n = 0. && (1+h)^0 = 1 = 1 + 0 \times h \\
&n = 1. && 1 + h = 1 + h \\
&n \geq 2. && 1 - n \leqslant -2
\end{aligned}$$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1 - n \geq 1 + nh \geq 1 - 2n$$

$$(1+h)^n \geq 1 + nh$$

(2)

$$
\begin{aligned}
& h \geq 0 \\
(1+h)^n = & 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots \geq \frac{n(n-1)}{2}h^2
\end{aligned}
$$

推广:

$$(1+h)^n \geq \binom{n}{3}h^3, \binom{n}{4}h^4, \ldots, \binom{n}{k}h^k, 0 \leqslant k \leqslant n$$

(3) $k=1$ 时显然成立. 使用归纳法证明. 假设 $k=n$ 时不等式 $\prod_{i=1}^{n}(1+a_i) \geq 1+\sum_{i=1}^{n} a_i$ 成立, 证明 $k=n+1$ 时 $\prod_{i=1}^{n+1}(1+a_i) \geq 1+\sum_{i=1}^{n+1} a_i$ 成立.

$$
\begin{aligned}
k=n+1 \qquad & \prod_{i=1}^{n+1}(1+a_i) = \prod_{i=1}^{n}(1+a_i)(1+a_{n+1}) \\
& \because \prod_{i=1}^{n}(1+a_i) \geq 1+\sum_{i=1}^{n} a_i \\
& \prod_{i=1}^{n}(1+a_i)(1+a_{n+1}) \geq (1+\sum_{i=1}^{n} a_i)(1+a_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
(1+\sum_{i=1}^{n} a_i)(1+a_{n+1}) &= 1+\sum_{i=1}^{n} a_i + a_{n+1} + a_{n+1}\sum_{i=1}^{n} a_i \\
&= 1+\sum_{i=1}^{n+1} a_i + a_{n+1}\sum_{i=1}^{n} a_i \\
&\geq 1+\sum_{i=1}^{n+1} a_i
\end{aligned}
$$

□

Example 2. 利用 $A.G.$ 不等式求解:

(1). $n! \leqslant (\frac{n+1}{2})^n$, *while* $n > 1$

(2). $(n!)^2 = (n \cdot 1)[(n-1) \cdot 2] \ldots (1 \cdots n)$. 证明: 当 $n > 1$ 时成立

$$n! < (\frac{n+2}{6})^n$$

(3). 比较上述两个不等式的优劣

(4). 证明: 对任意实数 r 成立:

$$(n!)^r \leqslant \frac{1}{n^n}(\sum_{k=1}^{n} k^r)^n \tag{49}$$

Proof. (1).

$$n > 1 \qquad n! = 1 \times 2 \times \cdots \times n < (\frac{1+2+\cdots+n}{n})^n = (\frac{(1+n)n}{2n})^n = (\frac{n+1}{2})^n$$

$\because 1 \neq 2 \neq \cdots n$, 所以不会有等号出现的情况

(2). $n > 1$

$$\begin{aligned}(n!)^2 &= (n \cdot 1)[(n-1) \cdot 2] \ldots (1 \cdots n) \\ &< (\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n})^n\end{aligned}$$

Consider this equation

$$(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n})^n \tag{50}$$

$$\begin{aligned}\sum_{k=1}^{n}(n-k+1)k &= (n+1)\sum_{k=1}^{n}k - \sum_{k=1}k^2 \\ &= (n+1)\frac{(n+1)n}{2} - \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{6}(3(n+1)-(2n+1)) \\ &= \frac{n(n+1)(n+2)}{6}\end{aligned}$$

$$\begin{aligned}(n!)^2 &< (\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n})^n \\ &= (\frac{(n+1)(n+2)}{6})^n\end{aligned}$$

$\because n+1 < n+2, \therefore n! < (\frac{n+2}{\sqrt{6}})^n$

(3). $n > 3$ 时,$\frac{n+2}{\sqrt{6}} < \frac{n+1}{2}$ (2) 的结果较好.

(4). $\forall r \in \mathbb{R}$, prove formula 49

$$\frac{1}{n}\sum_{k=1}^{n}k^r \geq \sqrt[n]{\prod_{k=1}^{n}k^r}$$

$$(n!)^r = \prod_{k=1}^{n}k^r \leqslant (\frac{1}{n}\sum_{k=1}^{n}k^r)^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$

my answer

$$\begin{aligned}&\forall r \in \mathbb{R}, \qquad (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r \\ &(n!)^r = \sum_{k=1}^{n}k^r \leqslant (\frac{1^r+2^r+\cdots+n^r}{n})^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n \\ &\therefore \quad (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r\end{aligned}$$

□

Example 3. $a_k > 0, k = 1, 2, \ldots, n$ 证明几何–调和平均值不等式

$$(\prod_{k=1}^{n} a_k)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

Proof. from A.G inequality

$$\frac{\sum_{k=1}^{n} \frac{1}{a_k}}{n} \geq \sqrt[n]{\prod_{k=1}^{n} \frac{1}{a_k}} = \frac{1}{\sqrt[n]{\prod_{k=1}^{n} a_k}}$$

$$a_k > 0, \quad \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

□

Example 4. $a, b, c \geq 0$. *prove* $\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leqslant \frac{a+b+c}{3}$. 并推广到 n 个非负数的情况

Proof. 1. $\sqrt[3]{abc} = \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \leqslant \sqrt{\frac{ab+bc+ca}{3}}$.

2.

$$\begin{aligned}\sqrt{\frac{ab+bc+ca}{3}} \leqslant & \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ & = \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}} \\ & = \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}}\end{aligned}$$

$a, b, c \geq 0$, 希望证明

$$\sqrt{\frac{ab+bc+ca}{3}} \leqslant \frac{a+b+c}{3}$$

$$\begin{aligned}\frac{ab+bc+ca}{3} &\leqslant \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6} \\ \frac{ab+bc+ca}{2} &\leqslant \frac{a^2+b^2+c^2}{6} + 2\frac{ab+bc+ca}{6} \qquad (\text{add}\frac{ab+bc+ca}{6}) \\ \frac{ab+bc+ca}{3} &\leqslant \frac{ab+bc+ca}{2} \leqslant (\frac{a+b+c}{3})^2 \\ \sqrt{\frac{ab+bc+ca}{3}} &\leqslant \frac{a+b+c}{3}\end{aligned}$$

推广至 n 个

$$[l]n = 2 \qquad \sqrt{ab} \leq \frac{a+b}{2}$$

$$n = 3 \qquad \sqrt[3]{abc} \leqslant \sqrt{\frac{ab+bc+ca}{3}} \leqslant \frac{a+b+c}{3}$$

$$n = 4 \qquad \sqrt[4]{abcd} \leqslant \sqrt[3]{\frac{abc+bcd+cda+dab}{4}} \leqslant \sqrt{\frac{a+b+c}{3}} \leq \frac{a+b+c+d}{4}$$

$$k=n \qquad \sqrt[n]{a_1a_2\ldots a_n} \leqslant \sqrt{\frac{a_1+a_2+\cdots+a_n}{n}} \leq \frac{a_1+a_2+\cdots+a_n}{n}$$

This is

$$\sqrt[n]{\sum_{k=1}^{n} a_k} \leqslant \sqrt{\frac{\sum_{k=1}^{n} a_k}{k}} \leq \frac{\sum_{k=1}^{n} a_k}{k}$$

1. $\sqrt[n]{a_1a_2\ldots a_n} = \sqrt{\sqrt[n]{a_1^2a_2^2\ldots a_n^2}} \leqslant \sqrt{\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n}}$
2. $\sqrt{\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n}} \leqslant \sqrt{\frac{a_1+a_2+\cdots+a_n}{n}}$? □

Example 5. *(1)* $|\alpha+\beta| \leqslant |\alpha|+|\beta|$

Proof. let $\alpha=a-b, \beta=b$, the identity became $|(a-b)+b| \leqslant |a-b|+|b|$. This is $|a-b| \geq |a|-|b|$.

$$||a|-|b|| = \begin{cases} |a|-|b|. & a \geq b \\ |b|-|a|. & a<b \end{cases}$$

When $a \geq b$, $||a|-|b|| = |a|-|b|$. There is $|a-b| \geq |a|-|b| = ||a|-|b||$
When $a<b$, $|a-b| = |b-a| \geq |b|-|a| = ||a|-|b||$.
$\therefore$, We have$|a-b| \geq ||a|-|b||$ □

(2) $\sum |a_k| \geq |\sum a_k|$

Proof. We can prove this statement by induction.

$$\begin{aligned} k=2, &\qquad |a_1|+|a_2| \geq |a_1+a_2| \\ k=3, &\qquad |a_1|+|a_2|+|a_3| \geq |a_1+a_2+a_3| \\ \text{Suppose } k=n, &\qquad \sum_{k=1}^{n} |a_k| \geq |\sum_{k=1}^{n} a_k| \\ k=n+1, &\qquad \text{prove} \sum_{k=1}^{n+1} |a_k| \geq |\sum_{k=1}^{n+1} a_k| \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{n+1} |a_k| &= \sum_{k=1}^{n} |a_k| + |a_{n+1}| \\ &\geq |\sum_{k=1}^{n} a_k| + |a_{n+1}| \\ &\geq |\sum_{k=1}^{n+1} a_k| \end{aligned}$$

$$
\begin{aligned}
k = 2, &\qquad |a_1| - |a_2| \leqslant |a_1 - a_2| \\
\text{Suppose } k = n, &\qquad |a_1| - \sum_{k=2}^{n} |a_k| \leqslant |\sum_{k=1}^{n} a_k| \\
k = n+1, &\qquad \text{prove} |a_1| - \sum_{k=2}^{n+1} |a_k| \leqslant |\sum_{k=1}^{n+1} a_k|
\end{aligned}
$$

$$
\begin{aligned}
|a_1| - \sum_{k=2}^{n+1} |a_k| &= |a_1| - \sum_{k=2}^{n} |a_k| - |a_{n+1}| \\
&\leqslant |\sum_{k=1}^{n} a_k| - |a_{n+1}| \\
&\leqslant |\sum_{k=1}^{n+1} a_k|
\end{aligned}
$$

Can left side became$||a_1| - \sum_{k=2}^{n} |a_k||$?

$$
\left| |a_1| - \sum_{k=2}^{n} |a_k| \right| = |a_1| - \sum_{k=2}^{n} |a_k| \qquad |a_1| \geq \sum_{k=2}^{n} a_k \tag{51}
$$

$$
\left| |a_1| - \sum_{k=2}^{n} |a_k| \right| = \sum_{k=2}^{n} |a_k| - |a_1| \qquad |a_1| \geq \sum_{k=2}^{n} a_k \tag{52}
$$

in eq51, the inequality is still vaild. However in eq52, $\sum_{k=2}^{n} |a_k| - |a_1|$ and $|a_1|$ □

(3). $\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

Proof.

$$
\begin{aligned}
\frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \\
\frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a| + |b| + 2|a||b|}{(1+|a|)(1+|b|)} \\
1 - \frac{|a+b|}{1+|a+b|} &\geq 1 - \frac{|a| + |b| + 2|a||b|}{(1+|a|)(1+|b|)} \\
\frac{1}{1+|a+b|} &\geq \frac{1 - |a||b|}{(1+|a|)(1+|b|)} \\
1 + |a| + |b| + |a||b| &\geq 1 + |a+b| - |a||b| - |a||b||a+b| \\
|a| + |b| + 2|a||b| + |a||b||a+b| &> 0, \text{Since } \; + 2|a||b| + |a||b||a+b| \geq |a+b|
\end{aligned}
$$

Therefore $\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$ □

Example 6. *(4).*$|(a+b)^n - a^n| \leqslant (|a|+|b|)^n - |a|^n$

$$(a+b)^n - a^n = \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0b^n$$
$$(|a|+|b|)^n - |a|^n = \binom{n}{1}|a|^{n-1}|b|^1 + \binom{n}{2}|a|^{n-2}|b|^2 + \cdots + \binom{n}{n}|a|^0|b|^n$$

$$\because |a|^j|b|^k \geq a^jb^k$$
$$\therefore \sum |a|^j|b|^k \geq |\sum a^jb^k|$$

$$|(a+b)^n - a^n| = \begin{cases} (a+b)^n - a^n, & a+b \geq a; b \geq 0 \\ a^n - (a+b)^n, & a+b < a; b < 0 \end{cases}$$

$$|(a+b)^n - a^n| \leqslant (|a|+|b|)^n - |a|^n. \tag{53}$$

Proposition 2. *1.3.5(Cauchy inequality)*
For $a_1, a_2, \ldots, a_n$. *and* $b_1, b_2, \ldots, b_n$. $a_i, b_i \in \mathbb{R}$, *There is*

$$\Big|\sum_{i=1}^n a_ib_i\Big| \leqslant \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2} \tag{54}$$

Proof. Let's prove eq54
First way on book:
Use variable λ, change the inequality into nonnegative binomial.

$$0 \leq \sum_{i=1}^n (a_i - \lambda b_i)^2 \qquad = \sum_{i=1}^n a_i^2 - 2\lambda\sum_{i=1}^n a_ib_i + \lambda^2\sum_{i=1}^n$$
$$\Delta = B^2 - 4AC \qquad = (-2\sum_{i=1}^n a_ib_i)^2 - 4(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) \leq 0$$

$$(\sum_{i=1}^n a_ib_i)^2 \leqslant (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$$

sqrt on both side of the inequality above, we can get

$$\Big|\sum_{i=1}^n a_ib_i\Big| \leqslant \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$$

□

6. Cauchy 不等式的不同证明

(1). 数学归纳法.

$$\begin{aligned}
k=1, &\quad |ab| = \sqrt{a^2}\sqrt{b^2} \\
k=1, &\quad |a_1b_1+a_2b_2| = \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_1^2} \\
\text{Suppose } k=n, &\quad |\sum_{i=1}^{n} a_ib_i| = \sqrt{\sum_{i=1}^{n} a_i}\sqrt{\sum_{i=1}^{n} b_i} \\
k=n+1, &\quad |\sum_{i=1}^{n+1} a_ib_i| = |\sum_{i=1}^{n} a_ib_i + a_{n+1}b_{n+1}|
\end{aligned}$$

$$\begin{aligned}
|\sum_{i=1}^{n+1} a_ib_i| &= |\sum_{i=1}^{n} a_ib_i + a_{n+1}b_{n+1}| \\
&\le |\sqrt{\sum_{i=1}^{n} a_i}\sqrt{\sum_{i=1}^{n} b_i} + a_{n+1}b_{n+1}|
\end{aligned}$$

Note that $A = \sqrt{\sum_{i=1}^{n} a_i}$, $B = \sqrt{\sum_{i=1}^{n} b_i}$

$$\begin{aligned}
|\sum_{i=1}^{n+1} a_ib_i| &\le |AB + a_{n+1}b_{n+1}| \\
&\le \sqrt{A^2+a_{n+1}^2}\sqrt{B^2+b_{n+1}^2} \\
&= \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}
\end{aligned}$$

(2) Lagrange 恒等式

$$\sum_{i=1}^{n} a_k^2 \sum_{i=1}^{n} b_k^2 - (\sum_{i=1}^{n} |a_kb_k|) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 \tag{55}$$

$$\begin{aligned}
(|a_k||b_i| - |a_i||b_k|)^2 &= |a_k|^2|b_i|^2 - 2|a_i||a_k||b_i||b_k| + |b_k|^2|a_i|^2 \\
&= a_k^2b_i^2 + b_k^2a_i^2 - 2|a_ia_kb_ib_k|
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 = 2\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} b_k^2 - 2\sum_{i=1}^{n}\sum_{k=1}^{n} |a_ia_kb_ib_k| \\
\sum_{i=1}^{n} a_k^2 \sum_{i=1}^{n} b_k^2 - (\sum_{i=1}^{n} |a_kb_k|) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 \ge 0
\end{aligned}$$

$$
\begin{aligned}
&\therefore (\sum_{i=1}^{n}|a_ib_i|)^2 \leqslant \sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2 \\
&\because |\sum_{i=1}^{n}a_ib_i| \leqslant \sum_{i=1}^{n}|a_ib_i| \\
&\therefore (|\sum_{i=1}^{n}a_ib_i|)^2 \leqslant (\sum_{i=1}^{n}|a_ib_i|)^2 \\
&\therefore (|\sum_{i=1}^{n}a_ib_i|)^2 \leqslant \sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2
\end{aligned}
$$

不等式两边开平方，得到：

$$
|\sum_{i=1}^{n}a_ib_i| \leqslant \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}
$$

(3). 用不等式 $|AB| \leqslant \frac{A^2+B^2}{2}$

$$
\begin{aligned}
|a_ib_i| &\leqslant \frac{a_i^2+b_i^2}{2} \\
|\sum_{i=1}^{n}a_ib_i| &\leqslant \sum_{i=1}^{n}|a_ib_i| \qquad \leqslant \frac{\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2}{2} \\
\frac{\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2}{2} &\geq \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2} \qquad ??
\end{aligned}
$$

如何用均值不等式证明 Cauchy 不等式?

由切比雪夫不等式，有

$$
\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \leqslant (\frac{a_1+a_2+\cdots+a_n}{n})(\frac{b_1+b_2+\cdots+b_n}{n}) \tag{56}
$$

由均值不等式，有

$$
\begin{aligned}
\frac{a_1+a_2+\cdots+a_n}{n} &\leqslant \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \\
\frac{b_1+b_2+\cdots+b_n}{n} &\leqslant \sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{n}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} &\leqslant (\frac{a_1+a_2+\cdots+a_n}{n})(\frac{b_1+b_2+\cdots+b_n}{n}) \\
&\leqslant \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}\sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{n}} \\
&= \frac{1}{n}\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\sqrt{b_1^2+b_2^2+\cdots+b_n^2}
\end{aligned}
$$

This is

$$\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

Square on both side of the inequality, The calculate square root. We can get eq56:

(4). 构造复的辅助数列

$$c_k = a_k^2 - b_k^2 + 2i|a_k b_k|, \qquad k = 1, 2, \ldots, n$$

Then we use

$$\Big|\sum_{k=1}^{n} c_k\Big| \leqslant \sum_{k=1}^{n} |c_k|$$

Solve 1.

$$c_k = (|a_k| + i|b_k|)^2 = a_k^2 + b_k^2 + 2i|a_k b_k|$$
$$\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2i\sum_{k=1}^{n} |a_k b_k|$$
$$|c_k| = \sqrt{\Re^2 c_k + \Im^2 c_k} = \sqrt{(a_k^2 - b_k^2)^2 + (2a_k b_k)^2} = a_k^2 + b_k^2$$

$$\therefore \Big|\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2i\sum_{k=1}^{n} |a_k b_k|\Big| = \sqrt{\Re^2 \sum_{k=1}^{n} c_k + \Im^2 \sum_{k=1}^{n} c_k}$$
$$= \sqrt{(\sum_{k=1}^{n}(a_k^2 - b_k^2))^2 + \sum_{k=1}^{n}(2a_k b_k)^2}$$
$$= \sqrt{(\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4\sum_{k=1}^{n}(a_k b_k)^2}$$
$$\because |\sum c_k| \leqslant \sum |c_k|$$
$$\therefore (\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4\sum_{k=1}^{n}(a_k b_k)^2 \leqslant (\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2)^2$$
$$\therefore 4(\sum_{k=1}^{n} a_k b_k)^2 \leqslant 4(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)$$
$$\textit{extracting both side: } \Big|\sum_{k=1}^{n} a_k b_k\Big| \leqslant \sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2}$$

Example 7. *7. Suppose* $0 < x_i \leqslant \frac{1}{2}, i = 1, 2, \ldots, n,$ *then*

$$\frac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^n} \leqslant \frac{\prod_{i=1}^{n}(1 - x_i)}{(\sum_{i=1}^{n}(1 - x_i))^n} \tag{57}$$

Proof. Let's prove eq57 by induction method.

$$n = 2, \qquad \frac{x_1 x_2}{(x_1+x_2)^2} \leqslant \frac{(1-x_1)(1-x_2)}{((1-x_1)+(1-x_2))^2}$$

$$\frac{(x_1x_2)}{(x_1^2+2x_1x_2+x_2^2)} \leqslant \frac{1-x_1-x_2+x_1x_2}{(1-x_1)^2+2(1-x_1)(1-x_2)+(1-x_2)^2}$$
$$\frac{(x_1+x_2)^2}{(x_1x_2)} \geq \frac{((1-x_1)(1-x_2))^2}{1-x_1-x_2+x_1x_2}$$
$$\frac{x_1}{x_2}+2+\frac{x_2}{x_1} \geq \frac{1-x_1}{1-x_2}+2\frac{1-x_2}{1-x_1}$$
$$\frac{x_1}{x_2}-\frac{1-x_1}{1-x_2} \geq \frac{1-x_2}{1-x_1}-\frac{x_2}{x_1}$$
$$\frac{x_1(1-x_2)-x_2(1-x_1)}{x_2(1-x_2)} \geq \frac{x_1(1-x_2)-x_2(1-x_1)}{x_1(1-x_1)}$$
$$\frac{x_1-x_2}{x_2(1-x_2)} \geq \frac{x_1-x_2}{x_1(1-x_1)}$$

$f(x) = x - x^2, f'(x) = 1-2x > 0,$ while $x \in (0, \frac{1}{2})$
When $x_1 > x_2, 0 < x_2 < x_1 \leqslant \frac{1}{2}, x_1 - x_1^2 \geq x_2 - x_2^2, x_1 - x_2 > 0$
When $x_1 < x_2, 0 < x_1 < x_2 \leqslant \frac{1}{2}, x_1 - x_1^2 \leqslant x_2 - x_2^2, x_1 - x_2 < 0$

$$\frac{x_1-x_2}{x_2(1-x_2)} \geq \frac{x_1-x_2}{x_1(1-x_1)}$$

$$k = 2, \qquad \frac{x_1x_2}{(x_1+x_2)^2} \leqslant \frac{(1-x_1)(1-x_2)}{((1-x_1)+(1-x_2))^2}$$
$$k = 4, \qquad \frac{x_1x_2x_3x_4}{(x_1+x_2+x_3+x_4)^2} \leqslant \frac{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}{((1-x_1)+(1-x_2)+(1-x_3)+(1-x_4))^2}$$

Use Cauchy's forward and backward method, We can prove this equation

Suppose $k = n, \dfrac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} \leqslant \dfrac{\prod_{i=1}^n (1-x_i)}{(\sum_{i=1}^n (1-x_i))^2}$

$k = n-1, \qquad$ prove $\dfrac{\prod_{i=1}^{n-1} x_i}{(\sum_{i=1}^{n-1} x_i)^2} \leqslant \dfrac{\prod_{i=1}^{n-1} (1-x_i)}{(\sum_{i=1}^{n-1} (1-x_i))^2}$

We already know that

$$\frac{\sum_{i=1}^{n-1} x_i}{n-1} = \frac{1}{n}\Big(\sum_{i=1}^{n-1} x_i + \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big)$$

This trick always use in (n-1) terms tranfer to (n) terms

When the inequality holds for $n > 2$, for$k = n$, we have:

$$\frac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^n} \leqslant \frac{\prod_{i=1}^{n}(1-x_i)}{(\sum_{i=1}^{n}(1-x_i))^n}$$
$$\frac{(\sum_{i=1}^{n}(1-x_i))^n}{(\sum_{i=1}^{n} x_i)^n} \leqslant \frac{\prod_{i=1}^{n}(1-x_i)}{\prod_{i=1}^{n} x_i}$$
$$\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1-x_i))}\right)^n \geq \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n}(1-x_i)}$$

for $k = n-1$, Let $M = x_n = \frac{\sum_{i=1}^{n-1} x_i}{n-1}$. The inequality 57 left side:

$$\begin{aligned}
&\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1-x_i))}\right)^n \\
&= \left(\frac{x_1 + \cdots + x_n}{(1-x_1) + \cdots + (1-x_n)}\right)^n \\
&= \left(\frac{x_1 + \cdots + x_{n-1} + M}{(1-x_1) + \cdots + (1-x_{n-1}) + (1-M)}\right)^n \\
&= \left(\frac{x_1 + \cdots + x_{n-1} + \frac{\sum_{i=1}^{n-1} x_i}{n-1}}{(1-x_1) + \cdots + (1-x_{n-1}) + (1-\frac{\sum_{i=1}^{n-1} x_i}{n-1})}\right)^n \\
&= \left(\frac{\frac{n}{n-1}(x_1 + \cdots + x_{n-1})}{\frac{n}{n-1}((1-x_1) + \cdots + (1-x_{n-1}))}\right)^n \\
&= \left(\frac{M}{1-M}\right)^n
\end{aligned}$$

while the right side become

$$\begin{aligned}
&\frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n}(1-x_i)} \\
&= \frac{\prod_{i=1}^{n-1} x_i \cdot M}{\prod_{i=1}^{n-1}(1-x_i) \cdot (1-M)} \\
&= \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1}(1-x_i)} \frac{M}{1-M}
\end{aligned}$$

$$\left(\frac{M}{1-M}\right)^n \geq \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1}(1-x_i)} \frac{M}{1-M}$$
$$\left(\frac{M}{1-M}\right)^{n-1} \geq \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1}(1-x_i)}$$

□

1.3.1 Bernoulli inequality, Suppose that $h > -1, n \in \mathbb{N}$, Then:

$$(1+h)^n \geq 1 + nh \tag{58}$$

When $n > 1$, the inequality became equation iff $h = 0$.

Proof. When $n = 1$, $1 + h = 1 + h$
$h = 0$, $1^n = 1$
Let's consider the condition $n > 1, h \neq 0$.
i). $h > 0$, $(1+h)^n = \binom{n}{0}h^0 + \binom{n}{1}h^1 + \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n$.
$\because \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n > 0$, $\therefore (1+h)^n > 1 + nh$
ii). $-1 < h < 0, 0 < 1 + h < 1$.

$$\begin{aligned}(1+h)^n - 1 &= (1+h-1)\Big(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\Big) \\ &= h\Big(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\Big)\end{aligned}$$

$\because 1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1} < n$ when$h < 0$
$\therefore (1+h)^n > 1 + nh$

Two variable extension of the Bernoulli inequality, Suppose $h = \frac{B}{A}, A > 0, A + B > 0$, Then$1 + h > 0$ is established. □

1.3.2 Suppose $A > 0, A + B > 0, n \in \mathbb{N}$, Then the inequality is true:

$$(A+B)^n \geq A^n + nA^{n-1}B \tag{59}$$

The inequalty became equation iff $B = 0$.

Proof. divide A^n on both side of the inequality 59. Set$h = \frac{B}{A}(A > 0)$, Then the inequality became Eq 58. So we can prove Eq 59 by prove Eq 58. Eq 58 is true when $h > -1$. $\therefore 1 + h > 0, 1 + \frac{B}{A} > 0$, $\because A > 0$, $\therefore A + B > 0$. And when $n > 1$ the equation is true iff $h = 0. \frac{B}{A} = 0, \therefore B = 0$. □

Ex 1.3.2 exercise 8
$a, c, t, g \geq 0, a + c + t + g = 1$. Prove that $a^2 + c^2 + t^2 + g^2 \geq \frac{1}{4}$.
The inequality became equatio iff $a = c = t = g = \frac{1}{4}$.

Proof. from A.G inequality,

$$\frac{a+c+t+g}{4} \geq \sqrt[4]{actg}, \quad a + c + t + g = 1 \tag{60}$$

$\therefore \sqrt[4]{actg} \leqslant \frac{1}{4}$

$$a + c + t + g = 1, (a + c + t + g)^2 = 1$$

$$(a+c+t+g)^2 = a^2+c^2+t^2+g^2+2ac+2at+2ag+2ct+2cg+2tg = 1 \quad (61)$$

$$a^2+c^2 \geq 2acc^2+t^2 \geq 2ct \quad (62)$$

$$a^2+t^2 \geq 2atc^2+g^2 \geq 2cg \quad (63)$$

$$a^2+g^2 \geq 2agt^2+g^2 \geq 2tg \quad (64)$$

substitude $2ac, 2ag, \ldots$ in equation 61, we can get

$$4(a^2+c^2+t^2+g^2) \geq a^2+c^2+t^2+g^2+2ac+2at+2ag+2ct+2cg+2tg$$

Then we get the inequality 60. □

1.4 1.4

The law of duality: $\forall(\exists) \to \exists(\forall)$ with negative statement

Inverse proposition?

1. A have upper limit, $\exists M > 0. \forall x \in A, x \leqslant M$.

It's negative statement is 'A don't have upper limit'. $\forall M > 0, \exists x \in A, x > M$.

2. the minum item in A is b, $b \in A, \forall x \in A, x \geq b$.

It's negative statement is 'b is not the minum item in A'. $b \in A, \exists x \in A, x < b$.

3. $f \in (a,b)$ is a monotonic augmentation function, $\forall x, y \in (a,b), x < y, f(x) \leqslant f(y)$.(or $f(x) < f(y)$, depends on monotonic function's definition)

It's negative statement is '$f \in (a,b)$ isn't a monotonic augmentation function'. $\exists x, y \in (a,b), x < y, f(x) > f(y)$ (or $f(x) \geq f(y)$).

4. $f \in (a,b)$ is a monotonic function, $\forall x, y, z \in (a,b), x < y < z, (f(x) - f(y))(f(y) - f(z)) \geq 0$.

It's negative statement is '$f \in (a,b)$ isn't a monotonic function'. $\exists x, y, z \in (a,b), x < y < z, (f(x) - f(y))(f(y) - f(z)) < 0$.

(Another way $\forall x, y \in (a,b), x < y, f(x) - f(y) \geq 0$ or $f(x) - f(y) \leqslant 0$.)

5. $A \subset B$, $\forall x \in A, x \in B$.

It's negative statement is $A \subsetneq B$, $\exists x \in A, x \notin B$.

6. $A - B \neq \emptyset$, $\exists x \in A, x \in B$.

It's negative statement is $A - B = \emptyset$, $\forall x \in A, x \notin B$.

7. x_n is an infinitesimal amounts, $\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \forall n > N, |x_n| < \epsilon$.

It's negative statement is 'x_n is not an infinitesimal amounts', $\exists \epsilon > 0, \forall N \in \mathbb{N}^+, \exists n > N, |x_n| \geq \epsilon$.

8. x_n is infinitely large, $\forall M > 0, \exists N \in \mathbb{N}^+, \forall n > N, x_n > M$.

It's negative statement is 'x_n is not infinitely large', $\exists M > 0, \forall N \in \mathbb{N}^+, \exists n > N, x_n \leqslant M$.

2 2.1.5

Quetion 1. *1. prove by Limit definition:*
(1). $\lim_{n\to\infty} \frac{3n^2}{n^2-4} = 3.$
(2). $\lim_{n\to\infty} \frac{\sin n}{n} = 0.$
(3). $\lim_{n\to\infty}(1+n)^{\frac{1}{n}} = 0.$
(4). $\lim_{n\to\infty} \frac{a^n}{n!} = 0, (a > 0).$

Quetion 2. *2. Suppose* $a_n, n \in \mathbb{N}_+$. *sequence* a_n *converge to* a.
Prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Proof. $n \to \infty a_n \to a$.
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$

$$|\sqrt{a_n} - \sqrt{a}| = \Big|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\Big| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$. □ (check, not consider the condition$a = 0$) add $a = 0, \forall \epsilon \in (0, 1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$. s.t $a_n < \epsilon^2 < \epsilon$, $\sqrt{a_n} < \epsilon$. □

Quetion 3. *3. If* $\lim_{n\to\infty} a_n = a$.
Prove $\lim_{n\to\infty} |a_n| = |a|$. *Vice versa?*

Proof. $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$

$$\Big||a_n| - |a|\Big| \leqslant |a_n - a| < \epsilon$$

$\therefore \lim_{n\to\infty} |a_n| = |a|$
If We know $\lim_{n\to\infty} |a_n| = |a|$.
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), \big||a_n| - |a|\big| < \epsilon$. We can't get $\lim_{n\to\infty} a_n = a$.
For example: $a_n = \frac{1}{n} + 1, a = -1$, $\lim_{n\to\infty} |a_n| = |a|$ is $\lim_{n\to\infty} |\frac{1}{n} + 1| = |-1|$, but $\lim_{n\to\infty} \frac{1}{n} + 1 \neq -1$ □

□

Quetion 4. *(1). Suppose* $p(x)$ *is a polynomial of* x*, if* $\lim_{n\to\infty} a_n = a$*, Prove* $\lim_{n\to\infty} p(a_n) = p(a)$.
(2). Suppose $b > 0$, $\lim\limits_{n\to\infty} a_n = a$. *Prove* $b^{a_n} = b^a$.
(3). Suppose $b > 0$, $\{a_n\}, a_n > 0, \forall n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a.a > 0$. *Prove* $\lim\limits_{n\to\infty} \log_b a_n = \log_b a$.
(4) Suppose $b \in \mathbb{R}$, $\{a_n\}, a_n > 0$ *when* $n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a$. *Prove* $\lim\limits_{n\to\infty} {a_n}^b = a^b$.

(5) Suppose $\lim_{n\to\infty} a_n = a$*. Prove* $\lim_{n\to\infty} \sin a_n = \sin a$*.*

Proof. 4.(1)
$\forall\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geqslant N(\epsilon),\ |a_n - a| < \epsilon.$
$p(a) = k_m a^m + k_{m-1}a^{m-1} + \cdots + k_0 a^0.$
$\therefore p(a_n) - p(a) = k_m(a_n^m - a^m) + k_{m-1}(a_n^{m-1} - a^{m-1}) + \cdots + k_0(a_n^0 - a^0).$

$$\begin{aligned} |a_n^m - a^m| &= |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon(m-1)\cdots(a+\delta)^{m-1} \end{aligned}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a).$ □ □

Proof. 4.(2)
$\forall\epsilon > 0, \exists N \in \mathbb{N}^+.\forall n \geqslant N, |a_n - a| < \epsilon.$
If $b = 1$, $1^{a_n} = 1^a = 1$.
If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1)$
$\because b > 0, \epsilon \to 0, \therefore b^\epsilon - 1 \to 0.$ $\therefore \lim_{n\to\infty} b_n^a = b^a.$
If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$. □

Proof. 4.(3)
$\forall\epsilon > 0, \exists N \in \mathbb{N}^+.\forall n \geqslant N, |a_n - a| < \epsilon.$

$$\begin{aligned} \log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1) \end{aligned}$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a})$. $\because b > 0, a \neq 0,\ a_n > 0$ when $\epsilon \to 0$.
$\therefore \log_b(1 + \frac{\epsilon}{a}) \to 0.$
$\therefore \lim_{n\to\infty} \log_b a_n = \log_b a$ □

Proof. 4.(4)
$\forall\epsilon > 0, \exists N \in \mathbb{N}^+.\forall n \geqslant N, |a_n - a| < \epsilon.$
$a_n^b = e^{b\ln a_n}$, $a_n^b - a^b = e^{b\ln a_n} - e^{b\ln a}$.

$$\begin{aligned} e^{b\ln a_n} - e^{b\ln a} &= e^{b\ln a}(e^{b\ln a_n - b\ln a} - 1) \\ &= e^{b\ln a}(e^{b\ln\frac{a_n}{a}} - 1) \end{aligned}$$

$0 < |a_n^b - a^b| < e^{b\ln a}(e^{b\ln(1+\frac{\epsilon}{a})} - 1)$
$\therefore \lim_{n\to\infty} a_n^b = a^b$ □

Proof. 4.(5)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |a_n - a| < \epsilon.$

$$\begin{aligned}\sin(A+B) - \sin(A-B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2\cos A \sin B\end{aligned}$$

$$\sin a_n - \sin a = 2\cos\frac{a_n + a}{2}\sin\frac{a_n - a}{2}$$

$|\sin a_n - \sin a| = |2\cos\frac{a_n+a}{2}\sin\frac{a_n-a}{2}| < |2\sin\frac{a_n-a}{2}|$
$|2\sin\frac{a_n-a}{2}| < |2\frac{a_n-a}{2}| = \epsilon$
$|\sin a_n - \sin a| < \epsilon, \therefore \lim\limits_{n\to\infty} \sin a_n = \sin a$ □

Quetion 5. *assume* $a > 1$. *Prove* $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof. $\frac{1}{n}\log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\log_a 1 = 0$.
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon.$
$a > 1$, and $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\therefore$ when $n \to \infty$, $\sqrt[n]{n} < a^\epsilon$, take logarithm on base of a, we can get $\frac{1}{n}\log_a n < \epsilon$
$\therefore \lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$ □

收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界
3. 收敛数列的比较定理，包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

3 2.2.1

思考题

Quetion. *1.* $\{a_n\}$ *收敛，*$\{b_n\}$ *发散，*$\{a_n + b_n\}$ *发散，*$\{a_n \cdot b_n\}$ *可能收敛，可能发散.*
2. $\{a_n\}, \{b_n\}$ *都发散，*$\{a_n + b_n\}$ *可能收敛，可能发散 (ex:* $n + -n$, $n + -2n$*),*

$\{a_n \cdot b_n\}$ *发散 (?).*

3. $a_n \leqslant b_n \leqslant c_n$, $n \in \mathbb{N}_+$. *已知* $\lim\limits_{n\to\infty}(c_n - a_n) = 0$. *问数列* $\{b_n\}$ *是否收敛？*

4. $\lim\limits_{n\to\infty}(\frac{1}{n+1} + \cdots + \frac{1}{2n})$

5. $a_n \to a(n \to 0)$. $\forall n, b < a_n < c$. *是否成立* $b < a < c$*?*

6. $a_n \to a(n \to 0)$. *and* $b \leqslant a \leqslant c$, *是否存在* $N \in \mathbb{N}_+$, *s.t. 当* $n > N$ *时，成立* $b \leqslant a_n \leqslant c$

7. 已知 $\lim\limits_{n\to\infty} a_n = 0$, *问: 是否有* $\lim\limits_{n\to\infty}(a_1 a_2 \dots a_n) = 0$. *反之如何?*

Proof. 5.4

$$\frac{n}{2n} \leqslant \frac{1}{n+1} + \cdots + \frac{1}{2n} \leqslant \frac{n}{n+1}$$

$\because \lim\limits_{n\to\infty} \frac{n}{2n} = \frac{1}{2} \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$, $\therefore \lim\limits_{n\to\infty}(\frac{1}{n+1} + \cdots + \frac{1}{2n})$ 收敛.

$$\begin{aligned}\frac{1}{n+1} + \cdots + \frac{1}{2n} &= \frac{1}{n}(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}) \\ &= \int_0^1 \frac{1}{1+x}\mathrm{d}x \\ &= \ln(1+x)|_0^1 = \ln 2\end{aligned}$$

$\lim\limits_{n\to\infty}(\frac{1}{n+1} + \cdots + \frac{1}{2n}) = \ln 2$ □

Proof. 5.5

不成立，应当为小于等于号。b=0, c=2, $a_n = \frac{1}{n}$, $\lim_{n\to\infty} a_n = 0 = c$. □

Proof. 5.6

不成立。$a = 0, b = 0, c = 2$, $a_n = (-1)^n \frac{1}{n}$.

$b \leqslant a \leqslant c$, but $(-1)^{2n+1}\frac{1}{2n+1} < 0 = b$. □

Proof. $\lim\limits_{n\to\infty} a_n = 0, a_n = \frac{1}{n}.a_1 a_2 \dots a_n = \frac{1}{n!}, \lim\limits_{n\to\infty} \frac{1}{n!} = 0$.

$\lim\limits_{n\to\infty} a_n = 0 \to \lim\limits_{n\to\infty}(a_1 a_2 \dots a_n) = 0 \qquad \checkmark$

$\lim\limits_{n\to\infty}(a_1 a_2 \dots a_n) = 0 \to \lim\limits_{n\to\infty} a_n = 0 \qquad \times$

$|a_n| < \epsilon$, $|a_{N+1} \dots a_n| < \epsilon^{n-N} < \epsilon$, $a_n < \sqrt[n]{\epsilon}$.

for example, $a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \dots, a_n = \frac{n}{n+1}$.

$$a_1 a_2 \dots a_n = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}.$$

$$\lim_{n\to\infty}(a_1 a_2 \dots a_n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

but $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ □

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

Example 8. *2.2.2* $\lim_{n\to\infty}\frac{x_n-1}{x_n+a}=0$, *prove* $\lim_{n\to\infty}x_n=a$

Proof. $\forall\epsilon>0,\exists N\in\mathbb{N}^+.\forall n\geqslant N,|\frac{x_n-1}{x_n+a}-0|<\epsilon$.
$|x_n-1|<\epsilon|x_n+a|<4a\cdot\epsilon$.(这个 4 是怎么取得的?)
$|x_n-a|<\epsilon|x_n+a|=\epsilon|(x_n-a)+2a|\leqslant\epsilon(|x_n-a|+2a)$.
限制 $\epsilon<1$, $|x_n-a|<2\epsilon|a|/(1-\epsilon)$.
限制 $\epsilon<\frac{1}{2}$, $|x_n-a|<2\epsilon|a|/(1-\epsilon)<4|a|\epsilon$.
Let $\epsilon'=4a\epsilon$, $|x_n-1|<\epsilon'$. $\therefore\lim\lim_{n\to\infty}x_n=a$. □

Example 9. *2.2.3* $a>0,b>0$ ，计算 $\lim\limits_{n\to\infty}(a_n+b_n)^{\frac{1}{n}}$.

Proof. Suppose $a\leqslant b$.
$b=(b^b)^{\frac{1}{n}}<(a^n+b^n)^{\frac{1}{n}}\leqslant(2b^n)^{\frac{1}{n}}$.
$b<(a^n+b^n)^{\frac{1}{n}}\leqslant\sqrt[n]{2}b$, $\lim\limits_{n\to\infty}=1$. 夹逼定理.
$\lim\limits_{n\to\infty}(a^n+b^n)^{\frac{1}{n}}=\max\{a,b\}$.
两数 n 次方之和再开 n 次根号的结果由较大的值决定, a,b 中较大的值为这个数的主要部分. □

Example 10. *2.2.4* $a_n=\frac{1!+2!+\cdots+n!}{n!},n\in\mathbb{N}^+$

$\lim\limits_{n\to\infty}a_n=1$

Example 11. $\lim\limits_{n\to\infty}\frac{n^3+n-7}{n+3}=+\infty$

Example 12. $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$

调和级数 H_n 发散.

3.1 练习 2.2.4

Proof. 1.
$\{a_n\}$ 收敛于 a, $\to$ 两个子列 $\{a_{2n}\},\{a_{2n+1}\}$ 均收敛于 a.
两个子列 $\{a_{2n}\},\{a_{2n+1}\}$ 均收敛于 a, $\to\{a_n\}$ 收敛于 a.

□

2. 应用夹逼定理
(1). 给定 p 个正数 $a_1,a_2,\ldots,a_p$. 求 $\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\ldots a_p^n}$.
Let$a_s=\max_{1\leqslant i\leqslant p}\{a_1,a_2,\ldots,a_p\}$.

Solve. *(1).*

$$a_s=(a_s^n)^{\frac{1}{n}}<(a_1^n+a_2^n+\ldots a_p^n)^{\frac{1}{n}}\leqslant(pa_s^n)^{\frac{1}{n}}=p^{\frac{1}{n}}a_s$$

$n\to\infty,p^{\frac{1}{n}}\to1.\lim\limits_{n\to\infty}(a_1^n+a_2^n+\ldots a_p^n)^{\frac{1}{n}}=a_s$

(2). $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} x_n$

Solve. *(2).*
$$\frac{2n+1}{(n+1)} \leqslant x_n \leqslant \frac{2n+1}{\sqrt{n^2+1}}$$
$\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$, $\lim\limits_{n\to\infty} \frac{2n+1}{\sqrt{n^2+1}} = 2$. $\therefore \lim\limits_{n\to\infty} x_n = 2$

(3). $a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} a_n$

Solve. *(3).*
$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leqslant (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$
$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\therefore \lim\limits_{n\to\infty} a_n = 1$

(4). $a_n > 0$. $\lim\limits_{n\to\infty} a_n = a$, $a > 0$. 证明 $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$

Proof. $\lim\limits_{n\to\infty} a_n = a$
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$
$$0 < a - \epsilon < a_n < a + \epsilon$$
$\therefore \sqrt[n]{a-\epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a+\epsilon}$.
$\lim\limits_{n\to\infty} \sqrt[n]{a-\epsilon} = 1$, $\lim\limits_{n\to\infty} \sqrt[n]{a+\epsilon} = 1$. $\therefore \lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$. □

3. (1). $\lim\limits_{n\to\infty} (1+x)(1+x^2)\dots(1+x^n) = \lim\limits_{n\to\infty} \prod_{i=1}^{n}(1+x^i)$, $|x| < 1$.