数学分析习题课讲义上册习题

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1 引论

1.1 关于习题课教案的组织

1.2 书中常用记号

1. $\mathbf{N}_+$：所有正整数组成的集合.

2. $\mathbf{R}$：所有实数组成的集合 (同时也用于表示无限区间 $(-\infty, \infty)$).

3. $\mathbf{Q}$：所有有理数组成的集合.

4. $\mathbf{C}$：所有复数组成的集合.

5. $\iff$ 是等价关系的记号.$A \iff B$ 表示 A 和 B 等价. 例如，A 代表 $x > 3$，B 代表 $x - 3 > 0$，则 $x > 3 \iff x - 3 > 0$.

6. $[x]$ 是实数 x 的整数部分,即不超过 x 的最大整数. 例如,$[\sqrt{2}] = 1, [-\sqrt{2}] = -2$. 关于 $[x]$ 的基本不等式是：$[x] \le x < [x] + 1$，或 $x - 1 < [x] \le x$

7. $\square$ 表示一个证明或解的结束.

8. $\binom{n}{k} = C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k!}$.

9. 记号 $\approx$ 表示近似值. 例如 $\sqrt{2} \approx 1.4$.

10. 复合函数 $f(g(x))$ 也写成 $(f \circ g)(x)$ 或 $f \circ g$.

11. 若 A 和 B 为两个集合，则用记号 $A - B$ 或 $A \backslash B$ 表示 A 与 B 的差集，也就是集合 $\{x | x \in A \text{且} x \notin B\}$.

12. 用 $O_\delta(a)$ 表示以 a 为中心，以 $\delta > 0$ 为半径的邻域. 它就是开区间 $(a - \delta, a + \delta)$(也可以用 $U_\delta(a)$ 等记号). 如不必指出半径，则可简记为 $O(a)$ (或 $U(a)$).

1.3 几个常用的初等不等式

1.3.1 几个初等不等式的证明

A.G 不等式 $a_1, a_2, \cdots, a_n$, n 个非负实数

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 \cdots a_n} \tag{1}$$

$\ge$ in inequation became $= \iff a_1 = a_2 = \cdots = a_n$

证明. method 1. induction method

$$
\begin{aligned}
&k=1 \qquad a_1=a_1\\
&k=2 \qquad \frac{a_1+a_2}{2}\geq\sqrt{a_1a_2}\\
&k=n \qquad \text{suppose} \quad \frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n}\\
&k=n+1\\
&\qquad \frac{a_1+a_2+\cdots+a_{n+1}}{n+1}-\frac{a_1+a_2+\cdots+a_n}{n}\\
&\qquad =\frac{n(a_1+a_2+\cdots+a_{n+1})-(n+1)(a_1+a_2+\cdots+a_n)}{n(n+1)}\\
&\qquad =\frac{na_{n+1}-(a_1+a_2+\cdots+a_n)}{n(n+1)}
\end{aligned}
$$

Set $A=\frac{a_1+a_2+\cdots+a_n}{n}$, $B=\frac{na_{n+1}-(a_1+a_2\cdots+a_n)}{n(n+1)}$

$$(\frac{a_1+a_2+\cdots+a_{n+1}}{n+1})^{n+1}=(A+B)^{n+1}$$

$A>0, B\geq 0$

$$
\begin{aligned}
&(A+B)^{n+1}\geq A^{n+1}+(n+1)A^nB\\
&A^{n+1}+(n+1)A^nB=A^n(A+(n+1)B)\\
&A^n=(\frac{a_1+a_2+\cdots+a_n}{n})^n\geq a_1a_2\cdots a_n\\
&A+(n+1)B=\frac{a_1+a_2+\cdots+a_n}{n}+\frac{na_{n+1}-(a_1+a_2+\cdots+a_n)}{n}=a_{n+1}\\
&\therefore (A+B)^{n+1}\geq A^n(A+(n+1)B)\geq a_1a_2\cdots a_n\cdot a_{n+1}\\
&\therefore \frac{a_1+a_2+\cdots+a_{n+1}}{n+1}\geq\sqrt[n+1]{a_1a_2\cdots a_na_{n+1}}
\end{aligned}
$$

使用二项式展开定理的条件

在归纳法第二步，将 $a_1,a_2,\cdots,a_{n+1}$ 重编号，使得 $n+1$ 为其中最大的数(之一)，这使得分解式右边第二项 $(na_{n+1}-(a_1+a_2+\cdots+a_n))/n(n+1)$ 一定是非负数。

method 2. Forward and Backward (Cauchy, 1897)

Forward part

$k=2.\dfrac{a_1+a_2}{2}\geq\sqrt{a_1a_2}.$

$k=4.\dfrac{a_1+a_2+a_3+a_4}{4}\geq\sqrt{(\dfrac{a_1+a_2}{2})\cdot(\dfrac{a_3+a_4}{2})}.$

$\geq\sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}}=\sqrt[4]{a_1a_2a_3a_4}.$

$k=2^n.$Suppose $\dfrac{a_1+a_2+\cdots a_{2^n}}{2^n}\geq\sqrt[2^n]{a_1a_2\cdots a_{2^n}}$

$k=2^{n+1}.$

$\dfrac{a_1+a_2+\cdots+a_{2^n}+\cdots+a_{2^{n+1}}}{2^{n+1}}\geq\sqrt{(\dfrac{a_1+a_2+\cdots+a_2^n}{2^n})\cdot(\dfrac{a_{2^n+1}+a_{2^n+2}+\cdots+a_2^{n+1}}{2^n})}$

$I\geq\sqrt{\sqrt[2^n]{a_1a_2\cdots a_{2^n}}\sqrt[2^n]{a_{2^n+1}a_{2^n+2}\cdots a_{2^{n+1}}}}=\sqrt[2^{n+1}]{a_1a_2\cdots a_{2^{n+1}}}$

Backward part suppose A.G inequality is valid when $k=n$, Consider $k=n-1$.

$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i=\frac{1}{n}(\frac{n}{n-1})\sum_{i=1}^{n-1}a_i$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i=\frac{1}{n}(\sum_{i=1}^{n-1}a_i+\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i\geq\sqrt[n]{(\prod_{i=1}^{n-1}a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)}$$
$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^n\geq(\prod_{i=1}^{n-1}a_i)(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)$$
$$(\frac{1}{n-1}\sum_{i=1}^{n-1}a_i)^{n-1}\geq(\prod_{i=1}^{n-1}a_i)$$
$$\frac{1}{n-1}\sum_{i=1}^{n-1}a_i\geq\sqrt[n-1]{\prod_{i=1}^{n-1}a_i}$$

□

命题 1 (1.3.5)**.** 柯西-施瓦茨不等式对 $a_1,a_2,\cdots,a_n$ 和 $b_1,b_2,\cdots,b_n$，成立

$$|\sum_{i=1}^{n}a_ib_i\leq\sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}|$$

证明.
$$0 \le \sum_{i=1}^{n}(a_i-\lambda b_i)^2 = \sum_{i=1}^{n} a_i^2 - 2\lambda\sum_{i=1}^{n} a_ib_i + \lambda^2\sum_{i=1}^{n} b_i^2$$
由韦达定理 (视 λ 为未知数). 原方程无解或只有唯一解。

$$\begin{aligned}
&\Delta = b^2-4ac \le 0\\
&(-2\sum_{i=1}^{n} a_ib_i)^2 - 4\sum_{i=1}^{n} a_i^2\sum_{i=1}^{n} b_i^2 \le 0\\
&(\sum_{i=1}^{n} a_ib_i)^2 \le \sum_{i=1}^{n} a_i^2\sum_{i=1}^{n} b_i^2\\
&\sum_{i=1}^{n} a_ib_i \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}
\end{aligned}$$

□

1.3.2 练习题

例 1. 关于 *Bernoulli* 不等式的推广:
(1) 证明: 当 $-2 \le h \le -1$ 时 *Bernoulli* 不等式 $(1+h)^n \ge 1+nh$ 仍成立;
(2) 证明: 当 $h \ge 0$ 时成立不等式 $(1+h)^n \ge \frac{n(n-1)h^2}{2}$, 并推广之;
(3) 证明: 若 $a_i > -1 (i=1,2,\ldots,n)$ 且同号, 则成立不等式
$$\prod_{i=1}^{n}(1+a_i) \ge 1+\sum_{i=1}^{n} a_i$$

证明. (1)

$$\begin{aligned}
&-2 \le h \le -1 && \\
&-1 \le 1+h \le 0 && -1 \le (1+h)^n \le 0\\
&-2n \le nh \le -n && 1-2n \le 1+nh \le 1-n
\end{aligned}$$

$$\begin{aligned}
&n=0. && (1+h)^0 = 1 = 1+0\times h\\
&n=1. && 1+h = 1+h\\
&n\ge 2. && 1-n \le -2
\end{aligned}$$

$$0 \ge (1+h)^n \ge -1 \ge -2 \ge 1-n \ge 1+nh \ge 1-2n$$

$$(1+h)^n \ge 1+nh$$

(2)

$$
\begin{aligned}
&h \geq 0 \\
(1+h)^n =&1+nh+\frac{n(n-1)}{2}h^2+\cdots \geq \frac{n(n-1)}{2}h^2
\end{aligned}
$$

推广:

$$(1+h)^n \geq \binom{n}{3}h^3, \binom{n}{4}h^4, \ldots, \binom{n}{k}h^k, 0 \leq k \leq n$$

(3) $k=1$ 时显然成立. 使用归纳法证明. 假设 $k=n$ 时不等式 $\prod_{i=1}^{n}(1+a_i) \geq 1+\sum_{i=1}^{n}a_i$ 成立, 证明 $k=n+1$ 时 $\prod_{i=1}^{n+1}(1+a_i) \geq 1+\sum_{i=1}^{n+1}a_i$ 成立.

$$
\begin{aligned}
k=n+1 \qquad & \prod_{i=1}^{n+1}(1+a_i) = \prod_{i=1}^{n}(1+a_i)(1+a_{n+1}) \\
& \because \prod_{i=1}^{n}(1+a_i) \geq 1+\sum_{i=1}^{n}a_i \\
& \prod_{i=1}^{n}(1+a_i)(1+a_{n+1}) \geq (1+\sum_{i=1}^{n}a_i)(1+a_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
(1+\sum_{i=1}^{n}a_i)(1+a_{n+1}) &= 1+\sum_{i=1}^{n}a_i+a_{n+1}+a_{n+1}\sum_{i=1}^{n}a_i \\
&= 1+\sum_{i=1}^{n+1}a_i+a_{n+1}\sum_{i=1}^{n}a_i \\
&\geq 1+\sum_{i=1}^{n+1}a_i
\end{aligned}
$$

□

例 2. *利用 $A.G.$ 不等式求解:*

(1). $n! \leq (\frac{n+1}{2})^n$, while $n>1$

(2). $(n!)^2=(n\cdot 1)[(n-1)\cdot 2]\ldots(1\cdots n)$. 证明: 当 $n>1$ 时成立

$$n! < (\frac{n+2}{6})^n$$

(3). 比较上述两个不等式的优劣

(4). 证明: 对任意实数 r 成立:

$$(n!)^r \leq \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n \tag{2}$$

证明. (1).

$n > 1 \qquad n! = 1\times 2\times\cdots\times n < (\frac{1+2+\cdots+n}{n})^n = (\frac{(1+n)n}{2n})^n = (\frac{n+1}{2})^n$

$\because 1 \neq 2 \neq \cdots n$, 所以不会有等号出现的情况

(2). $n > 1$

$$\begin{aligned}(n!)^2 &= (n\cdot 1)[(n-1)\cdot 2]\ldots(1\cdots n)\\ &< (\frac{n\times 1+(n-1)\times 2+\cdots+1\times n}{n})^n\end{aligned}$$

Consider this equation

$$(\frac{n\times 1+(n-1)\times 2+\cdots+1\times n}{n})^n \tag{3}$$

$$\begin{aligned}\sum_{k=1}^{n}(n-k+1)k &= (n+1)\sum_{k=1}^{n}k - \sum_{k=1}k^2\\ &= (n+1)\frac{(n+1)n}{2} - \frac{n(n+1)(2n+1)}{6}\\ &= \frac{n(n+1)}{6}(3(n+1)-(2n+1))\\ &= \frac{n(n+1)(n+2)}{6}\end{aligned}$$

$$\begin{aligned}(n!)^2 &< (\frac{n\times 1+(n-1)\times 2+\cdots+1\times n}{n})^n\\ &= (\frac{(n+1)(n+2)}{6})^n\end{aligned}$$

$\because n+1 < n+2, \therefore n! < (\frac{n+2}{\sqrt{6}})^n$

(3). $n > 3$ 时,$\frac{n+2}{\sqrt{6}} < \frac{n+1}{2}$ (2) 的结果较好.

(4).$\forall r \in \mathbb{R}$, prove formula 2

$$\frac{1}{n}\sum_{k=1}^{n}k^r \geq \sqrt[n]{\prod_{k=1}^{n}k^r}$$

$$(n!)^r = \prod_{k=1}^{n}k^r \leq (\frac{1}{n}\sum_{k=1}^{n}k^r)^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$

my answer

$$\forall r\in\mathbb{R}, \qquad (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r$$

$$(n!)^r = \sum_{k=1}^{n}k^r \leq (\frac{1^r+2^r+\cdots+n^r}{n})^n = \frac{1}{n^n}(\sum_{k=1}^{n}k^r)^n$$

$$\therefore \quad (\sum_{k=1}^{n}k^r)^n \geq n^n(n!)^r$$

□

例 3. $a_k > 0, k = 1, 2, \ldots, n$ 证明几何–调和平均值不等式

$$(\prod_{k=1}^{n} a_k)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

证明. from A.G inequality

$$\frac{\sum_{k=1}^{n} \frac{1}{a_k}}{n} \geq \sqrt[n]{\prod_{k=1}^{n} \frac{1}{a_k}} = \frac{1}{\sqrt[n]{\prod_{k=1}^{n} a_k}}$$

$$a_k > 0, \quad \sqrt[n]{\prod_{k=1}^{n} a_k} \geq \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}$$

□

例 4. $a, b, c \geq 0$. *prove* $\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$. 并推广到 n 个非负数的情况

证明. 1. $\sqrt[3]{abc} = \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \leq \sqrt{\frac{ab+bc+ca}{3}}$.
2.

$$\begin{aligned}\sqrt{\frac{ab+bc+ca}{3}} \leq & \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ & = \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}} \\ & = \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}}\end{aligned}$$

$a, b, c \geq 0$, 希望证明

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

$$\begin{aligned}\frac{ab+bc+ca}{3} &\leq \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6} \\ \frac{ab+bc+ca}{2} &\leq \frac{a^2+b^2+c^2}{6} + 2\frac{ab+bc+ca}{6} \qquad (\text{add} \frac{ab+bc+ca}{6}) \\ \frac{ab+bc+ca}{3} &\leq \frac{ab+bc+ca}{2} \leq (\frac{a+b+c}{3})^2 \\ \sqrt{\frac{ab+bc+ca}{3}} &\leq \frac{a+b+c}{3}\end{aligned}$$

推广至 n 个

$$\begin{aligned}[l]n = 2 \qquad & \sqrt{ab} \leq \frac{a+b}{2} \\ n = 3 \qquad & \sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3} \\ n = 4 \qquad & \sqrt[4]{abcd} \leq \sqrt[3]{\frac{abc+bcd+cda+dab}{4}} \leq \sqrt{\frac{a+b+c}{3}} \leq \frac{a+b+c+d}{4}\end{aligned}$$

$$k=n \qquad \sqrt[n]{a_1a_2\dots a_n} \le \sqrt{\frac{a_1+a_2+\cdots+a_n}{n}} \le \frac{a_1+a_2+\cdots+a_n}{n}$$

This is

$$\sqrt[n]{\sum_{k=1}^{n} a_k} \le \sqrt{\frac{\sum_{k=1}^{n} a_k}{k}} \le \frac{\sum_{k=1}^{n} a_k}{k}$$

1. $\sqrt[n]{a_1a_2\dots a_n} = \sqrt{\sqrt[n]{a_1^2a_2^2\dots a_n^2}} \le \sqrt{\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n}}$
2. $\sqrt{\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n}} \le \sqrt{\frac{a_1+a_2+\cdots+a_n}{n}}$? □

例 5. *(1)* $|\alpha+\beta| \le |\alpha|+|\beta|$

证明. let $\alpha = a-b, \beta = b$, the identity became $|(a-b)+b| \le |a-b|+|b|$. This is $|a-b| \ge |a|-|b|$.

$$||a|-|b|| = \begin{cases} |a|-|b|. & a \ge b \\ |b|-|a|. & a < b \end{cases}$$

When $a \ge b$, $||a|-|b|| = |a|-|b|$. There is $|a-b| \ge |a|-|b| = ||a|-|b||$
When $a < b$, $|a-b| = |b-a| \ge |b|-|a| = ||a|-|b||$.
$\therefore$, We have$|a-b| \ge ||a|-|b||$ □

(2) $\sum |a_k| \ge |\sum a_k|$

证明. We can prove this statement by induction.

$$\begin{aligned} k=2, &\qquad |a_1|+|a_2| \ge |a_1+a_2| \\ k=3, &\qquad |a_1|+|a_2|+|a_3| \ge |a_1+a_2+a_3| \\ \text{Suppose } k=n, &\qquad \sum_{k=1}^{n} |a_k| \ge |\sum_{k=1}^{n} a_k| \\ k=n+1, &\qquad \text{prove} \sum_{k=1}^{n+1} |a_k| \ge |\sum_{k=1}^{n+1} a_k| \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{n+1} |a_k| &= \sum_{k=1}^{n} |a_k| + |a_{n+1}| \\ &\ge |\sum_{k=1}^{n} a_k| + |a_{n+1}| \\ &\ge |\sum_{k=1}^{n+1} a_k| \end{aligned}$$

$$\begin{aligned}
k=2,\qquad & |a_1|-|a_2|\le|a_1-a_2| \\
\text{Suppose } k=n,\qquad & |a_1|-\sum_{k=2}^{n}|a_k|\le|\sum_{k=1}^{n}a_k| \\
k=n+1,\qquad & \text{prove}|a_1|-\sum_{k=2}^{n+1}|a_k|\le|\sum_{k=1}^{n+1}a_k|
\end{aligned}$$

$$\begin{aligned}
|a_1|-\sum_{k=2}^{n+1}|a_k| &= |a_1|-\sum_{k=2}^{n}|a_k|-|a_{n+1}| \\
&\le|\sum_{k=1}^{n}a_k|-|a_{n+1}| \\
&\le|\sum_{k=1}^{n+1}a_k|
\end{aligned}$$

Can left side became$||a_1|-\sum_{k=2}^{n}|a_k||$?

$$\left||a_1|-\sum_{k=2}^{n}|a_k|\right|=|a_1|-\sum_{k=2}^{n}|a_k|\qquad |a_1|\ge\sum_{k=2}^{n}a_k \tag{4}$$

$$\left||a_1|-\sum_{k=2}^{n}|a_k|\right|=\sum_{k=2}^{n}|a_k|-|a_1|\qquad |a_1|\ge\sum_{k=2}^{n}a_k \tag{5}$$

in eq4, the inequality is still vaild. However in eq5, $\sum_{k=2}^{n}|a_k|-|a_1|$ and $|a_1|$ □

(3). $\frac{|a+b|}{1+|a+b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$

证明.

$$\begin{aligned}
\frac{|a+b|}{1+|a+b|} &\le \frac{|a|}{1+|a|}+\frac{|b|}{1+|b|} \\
\frac{|a+b|}{1+|a+b|} &\le \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
1-\frac{|a+b|}{1+|a+b|} &\ge 1-\frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
\frac{1}{1+|a+b|} &\ge \frac{1-|a||b|}{(1+|a|)(1+|b|)} \\
1+|a|+|b|+|a||b| &\ge 1+|a+b|-|a||b|-|a||b||a+b| \\
|a|+|b|+2|a||b|+|a||b||a+b| &> 0, \text{Since } \ +2|a||b|+|a||b||a+b|\ge|a+b|
\end{aligned}$$

Therefore $\frac{|a+b|}{1+|a+b|}\le\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$ □

例 6. *(4).*$|(a+b)^n - a^n| \le (|a|+|b|)^n - |a|^n$

$$(a+b)^n - a^n = \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0b^n$$
$$(|a|+|b|)^n - |a|^n = \binom{n}{1}|a|^{n-1}|b|^1 + \binom{n}{2}|a|^{n-2}|b|^2 + \cdots + \binom{n}{n}|a|^0|b|^n$$

$$\because |a|^j|b|^k \ge a^jb^k$$
$$\therefore \sum |a|^j|b|^k \ge |\sum a^jb^k|$$

$$|(a+b)^n - a^n| = \begin{cases} (a+b)^n - a^n, & a+b \ge a; b \ge 0 \\ a^n - (a+b)^n, & a+b < a; b < 0 \end{cases}$$

$$|(a+b)^n - a^n| \le (|a|+|b|)^n - |a|^n. \tag{6}$$

命题 2. *1.3.5(Cauchy inequality)*
For $a_1, a_2, \ldots, a_n$*. and* $b_1, b_2, \ldots, b_n$*.* $a_i, b_i \in \mathbb{R}$*, There is*

$$\Big|\sum_{i=1}^{n} a_ib_i\Big| \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \tag{7}$$

证明. Let's prove eq7
First way on book:
Use variable λ, change the inequality into nonnegative binomial.

$$0 \le \sum_{i=1}^{n}(a_i - \lambda b_i)^2 \qquad = \sum_{i=1}^{n} a_i^2 - 2\lambda\sum_{i=1}^{n} a_ib_i + \lambda^2\sum_{i=1}^{n}$$
$$\Delta = B^2 - 4AC \qquad = (-2\sum_{i=1}^{n} a_ib_i)^2 - 4(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \le 0$$

$$(\sum_{i=1}^{n} a_ib_i)^2 \le (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$$

sqrt on both side of the inequality above, we can get

$$\Big|\sum_{i=1}^{n} a_ib_i\Big| \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

□

6. Cauchy 不等式的不同证明

(1). 数学归纳法.

$$
\begin{aligned}
k = 1, \quad & |ab| = \sqrt{a^2}\sqrt{b^2} \\
k = 1, \quad & |a_1 b_1 + a_2 b_2| = \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_1^2} \\
\text{Suppose } k = n, \quad & |\sum_{i=1}^{n} a_i b_i| = \sqrt{\sum_{i=1}^{n} a_i}\sqrt{\sum_{i=1}^{n} b_i} \\
k = n + 1, \quad & |\sum_{i=1}^{n+1} a_i b_i| = |\sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1}|
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{i=1}^{n+1} a_i b_i| &= |\sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1}| \\
&\leq |\sqrt{\sum_{i=1}^{n} a_i}\sqrt{\sum_{i=1}^{n} b_i} + a_{n+1} b_{n+1}|
\end{aligned}
$$

Note that $A = \sqrt{\sum_{i=1}^{n} a_i}$, $B = \sqrt{\sum_{i=1}^{n} b_i}$

$$
\begin{aligned}
|\sum_{i=1}^{n+1} a_i b_i| &\leq |AB + a_{n+1} b_{n+1}| \\
&\leq \sqrt{A^2 + a_{n+1}^2}\sqrt{B^2 + b_{n+1}^2} \\
&= \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}
\end{aligned}
$$

(2) Lagrange 恒等式

$$
\sum_{i=1}^{n} a_k^2 \sum_{i=1}^{n} b_k^2 - (\sum_{i=1}^{n} |a_k b_k|) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 \tag{8}
$$

$$
\begin{aligned}
(|a_k||b_i| - |a_i||b_k|)^2 &= |a_k|^2|b_i|^2 - 2|a_i||a_k||b_i||b_k| + |b_k|^2|a_i|^2 \\
&= a_k^2 b_i^2 + b_k^2 a_i^2 - 2|a_i a_k b_i b_k|
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 = 2\sum_{i=1}^{n} a_i^2 \sum_{k=1}^{n} b_k^2 - 2\sum_{i=1}^{n}\sum_{k=1}^{n}|a_i a_k b_i b_k| \\
\sum_{i=1}^{n} a_k^2 \sum_{i=1}^{n} b_k^2 - (\sum_{i=1}^{n} |a_k b_k|) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}(|a_k||b_i| - |a_i||b_k|)^2 \geq 0
\end{aligned}
$$

$$\therefore (\sum_{i=1}^{n}|a_ib_i|)^2 \le \sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2$$

$$\because |\sum_{i=1}^{n}a_ib_i| \le \sum_{i=1}^{n}|a_ib_i|$$

$$\therefore (|\sum_{i=1}^{n}a_ib_i|)^2 \le (\sum_{i=1}^{n}|a_ib_i|)^2$$

$$\therefore (|\sum_{i=1}^{n}a_ib_i|)^2 \le \sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2$$

不等式两边开平方，得到：

$$|\sum_{i=1}^{n}a_ib_i| \le \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}$$

(3). 用不等式 $|AB| \le \frac{A^2+B^2}{2}$

$$|a_ib_i| \le \frac{a_i^2+b_i^2}{2}$$

$$|\sum_{i=1}^{n}a_ib_i| \le \sum_{i=1}^{n}|a_ib_i| \qquad \le \frac{\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2}{2}$$

$$\frac{\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2}{2} \ge \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2} \qquad ??$$

如何用均值不等式证明 Cauchy 不等式?

由切比雪夫不等式，有

$$\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \le (\frac{a_1+a_2+\cdots+a_n}{n})(\frac{b_1+b_2+\cdots+b_n}{n}) \tag{9}$$

由均值不等式，有

$$\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$$

$$\frac{b_1+b_2+\cdots+b_n}{n} \le \sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{n}}$$

$$\begin{aligned}\therefore \frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} &\le (\frac{a_1+a_2+\cdots+a_n}{n})(\frac{b_1+b_2+\cdots+b_n}{n}) \\ &\le \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}\sqrt{\frac{b_1^2+b_2^2+\cdots+b_n^2}{n}} \\ &= \frac{1}{n}\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\sqrt{b_1^2+b_2^2+\cdots+b_n^2}\end{aligned}$$

This is

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

Square on both side of the inequality, The calculate square root. We can get eq9:

(4). 构造复的辅助数列

$$c_k = a_k^2 - b_k^2 + 2|a_k b_k|, \qquad k = 1, 2, \ldots, n$$

Then we use

$$\Big|\sum_{k=1}^{n} c_k\Big| \leq \sum_{k=1}^{n} |c_k|$$

解 1.

$$\begin{aligned}
&c_k = (|a_k| + |b_k|)^2 = a_k^2 + b_k^2 + 2|a_k b_k| \\
&\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sum_{k=1}^{n} |a_k b_k| \\
&|c_k| = \sqrt{\Re^2 c_k + \Im^2 c_k} = \sqrt{(a_k^2 - b_k^2)^2 + (2a_k b_k)^2} = a_k^2 + b_k^2
\end{aligned}$$

$$\therefore \Big|\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sum_{k=1}^{n} |a_k b_k|\Big| = \sqrt{\Re^2 \sum_{k=1}^{n} c_k + \Im^2 \sum_{k=1}^{n} c_k}$$

$$= \sqrt{(\sum_{k=1}^{n}(a_k^2 - b_k^2))^2 + \sum_{k=1}^{n}(2a_k b_k)^2}$$

$$= \sqrt{(\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4\sum_{k=1}^{n}(a_k b_k)^2}$$

$$\because |\sum c_k| \leq \sum |c_k|$$

$$\therefore (\sum_{k=1}^{n} a_k^2)^2 + (\sum_{k=1}^{n} a_k^2)^2 - 2(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} a_k^2) + 4\sum_{k=1}^{n}(a_k b_k)^2 \leq (\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2)^2$$

$$\therefore 4(\sum_{k=1}^{n} a_k b_k)^2 \leq 4(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)$$

extracting both side: $\Big|\sum_{k=1}^{n} a_k b_k\Big| \leq \sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2}$

例 7. *7. Suppose* $0 < x_i \leq \frac{1}{2}, i = 1, 2, \ldots, n$, *then*

$$\frac{\prod_{i=1}^{n} x_i}{(\sum_{i=1}^{n} x_i)^n} \leq \frac{\prod_{i=1}^{n}(1 - x_i)}{(\sum_{i=1}^{n}(1 - x_i))^n} \tag{10}$$

证明. Let's prove eq10 by induction method.

$$n = 2, \qquad \frac{x_1x_2}{(x_1+x_2)^2} \leq \frac{(1-x_1)(1-x_2)}{((1-x_1)+(1-x_2))^2}$$

$$\begin{aligned}
&\frac{(x_1x_2)}{(x_1^2+2x_1x_2+x_2^2)} \leq \frac{1-x_1-x_2+x_1x_2}{(1-x_1)^2+2(1-x_1)(1-x_2)+(1-x_2)^2}\\
&\frac{(x_1+x_2)^2}{(x_1x_2)} \geq \frac{((1-x_1)(1-x_2))^2}{1-x_1-x_2+x_1x_2}\\
&\frac{x_1}{x_2}+2+\frac{x_2}{x_1} \geq \frac{1-x_1}{1-x_2}+2\frac{1-x_2}{1-x_1}\\
&\frac{x_1}{x_2}-\frac{1-x_1}{1-x_2} \geq \frac{1-x_2}{1-x_1}-\frac{x_2}{x_1}\\
&\frac{x_1(1-x_2)-x_2(1-x_1)}{x_2(1-x_2)} \geq \frac{x_1(1-x_2)-x_2(1-x_1)}{x_1(1-x_1)}\\
&\frac{x_1-x_2}{x_2(1-x_2)} \geq \frac{x_1-x_2}{x_1(1-x_1)}
\end{aligned}$$

$f(x) = x - x^2, f'(x) = 1 - 2x > 0,$ while $x \in (0, \frac{1}{2})$
When $x_1 > x_2, 0 < x_2 < x_1 \leq \frac{1}{2}, x_1 - x_1^2 \geq x_2 - x_2^2, x_1 - x_2 > 0$
When $x_1 < x_2, 0 < x_1 < x_2 \leq \frac{1}{2}, x_1 - x_1^2 \leq x_2 - x_2^2, x_1 - x_2 < 0$

$$\frac{x_1-x_2}{x_2(1-x_2)} \geq \frac{x_1-x_2}{x_1(1-x_1)}$$

$$k = 2, \qquad \frac{x_1x_2}{(x_1+x_2)^2} \leq \frac{(1-x_1)(1-x_2)}{((1-x_1)+(1-x_2))^2}$$

$$k = 4, \qquad \frac{x_1x_2x_3x_4}{(x_1+x_2+x_3+x_4)^2} \leq \frac{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}{((1-x_1)+(1-x_2)+(1-x_3)+(1-x_4))^2}$$

Use Cauchy's forward and backward method, We can prove this equation

Suppose $k = n, \dfrac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} \leq \dfrac{\prod_{i=1}^n (1-x_i)}{(\sum_{i=1}^n (1-x_i))^2}$

$k = n-1, \qquad$ prove $\dfrac{\prod_{i=1}^{n-1} x_i}{(\sum_{i=1}^{n-1} x_i)^2} \leq \dfrac{\prod_{i=1}^{n-1} (1-x_i)}{(\sum_{i=1}^{n-1} (1-x_i))^2}$

todo! need to complete!

We already know that

$$\frac{\sum_{i=1}^{n-1} x_i}{n-1} = \frac{1}{n}\Big(\sum_{i=1}^{n-1} x_i + \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big)$$

This trick always use in (n-1) terms tranfer to (n) terms

When the inequality holds for $n > 2$, for$k = n$, we have:

$$\frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^n} \le \frac{\prod_{i=1}^n (1-x_i)}{(\sum_{i=1}^n (1-x_i))^n}$$
$$\frac{(\sum_{i=1}^n (1-x_i))^n}{(\sum_{i=1}^n x_i)^n} \le \frac{\prod_{i=1}^n (1-x_i)}{\prod_{i=1}^n x_i}$$
$$\left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i))}\right)^n \ge \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1-x_i)}$$

for $k = n-1$, Let $M = x_n = \frac{\sum_{i=1}^{n-1} x_i}{n-1}$. The inequality 10 left side:

$$\begin{aligned}
&\left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i))}\right)^n \\
&= \left(\frac{x_1 + \cdots + x_n}{(1-x_1) + \cdots + (1-x_n)}\right)^n \\
&= \left(\frac{x_1 + \cdots + x_{n-1} + M}{(1-x_1) + \cdots + (1-x_{n-1}) + (1-M)}\right)^n \\
&= \left(\frac{x_1 + \cdots + x_{n-1} + \frac{\sum_{i=1}^{n-1} x_i}{n-1}}{(1-x_1) + \cdots + (1-x_{n-1}) + (1 - \frac{\sum_{i=1}^{n-1} x_i}{n-1})}\right)^n \\
&= \left(\frac{\frac{n}{n-1}(x_1 + \cdots + x_{n-1})}{\frac{n}{n-1}((1-x_1) + \cdots + (1-x_{n-1}))}\right)^n \\
&= \left(\frac{M}{1-M}\right)^n
\end{aligned}$$

while the right side become

$$\begin{aligned}
&\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1-x_i)} \\
&= \frac{\prod_{i=1}^{n-1} x_i \cdot M}{\prod_{i=1}^{n-1} (1-x_i) \cdot (1-M)} \\
&= \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1} (1-x_i)} \frac{M}{1-M}
\end{aligned}$$

$$\left(\frac{M}{1-M}\right)^n \ge \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1} (1-x_i)} \frac{M}{1-M}$$
$$\left(\frac{M}{1-M}\right)^{n-1} \ge \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1} (1-x_i)}$$

$\square$