具体数学阅读笔记-chap2

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1 求和

1.1 求和符号

1.2 求和与递归式 Sums and recurrences

和式

$$S_n = \sum_{k=0}^{n} a_k \tag{1}$$

等价于递归式

$$\begin{cases} S_0 = a_0 \\ S_n = S_{n-1} + a_n, \quad n > 0. \end{cases} \tag{2}$$

若 $a_n = const. + k \cdot n$, 则有

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n, \quad n > 0 \end{cases} \tag{3}$$

$$
\begin{aligned}
R_1 &= R_0 + \beta + \gamma \\
R_2 &= R_0 + 2\beta + 3\gamma \\
&\ \vdots \\
R_n &= A(n)\alpha + B(n)\beta + C(n)\gamma
\end{aligned}
$$

$$
R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \tag{4}
$$

repertoire method 令 $R_n = 1$, 则 $\alpha = 1, \beta = 0, \gamma = 0$,

$$
A(n) = 1
$$

令 $R_n = n$, 则 $\alpha = 0, \beta = 1, \gamma = 0$,

$$
B(n) = n
$$

令 $R_n = n^2$, 则 $\alpha = 0, \beta = -1, \gamma = 2$,

$$
C(n) = \frac{n(n+1)}{2}
$$

例 1.1

$$
\sum_{k=0}^{n}(a+bk)
$$

解 1

$$
\begin{cases}
R_0 = a \\
R_n = R_{n-1} + a + bn
\end{cases}
$$

$$
\begin{cases}
R_0 = \alpha \\
R_n = R_{n-1} + \beta + \gamma n
\end{cases}
$$

$\alpha = \beta = a, \gamma = b$

$$
\begin{aligned}
A(n)\alpha + B(n)\beta + C(n)\gamma &= aA(n) + aB(n) + bC(n) \\
&= a + an + b\frac{n(n+1)}{2} \\
&= a(n+1) + \frac{bn(n+1)}{2}
\end{aligned}
$$

对上述递归情况进行推广

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n + \delta n^2, \quad n > 0 \end{cases} \tag{5}$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta \tag{6}$$

$\delta = 0$ 时 (5) 与 (3) 一致, 说明 $A(n), B(n), C(n)$ 不变

$R_n = n^3$

$$\begin{aligned} R_n = R_{n-1} &= n^3 - (n-1)^3 \\ &= 3n^2 - 3n + 1 \end{aligned}$$

解得 $\alpha = 0, \beta = 1, \gamma = -3, \delta = 3$

$$\begin{aligned} n^3 &= B(n) - 3C(n) + 3D(n) \\ &= n - 3\frac{n(n+1)}{2} + 3D(n) \end{aligned}$$

$$\begin{aligned} 3D(n) &= n^3 - n + 3\frac{n(n+1)}{2} \\ &= n(n+1)\left[(n-1) + \frac{3}{2}\right] \\ &= n(n+1)(n+\frac{1}{2}) \\ D(n) &= \frac{1}{3}\left((n+1)(n+\frac{1}{2})n\right) \end{aligned}$$

1.3 求和式处理

1.4 多重求和

1.4.1 Exercise 1

$$A = \begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{bmatrix} \tag{7}$$

求 $S_{\triangleleft} = \sum_{1\leqslant j\leqslant k\leqslant n} a_ja_k$ [1]

解 2 $\because a_ja_k = a_ka_j$, $\therefore$ 矩阵 A 沿主对角线对称, $S_{\triangleleft} = S_{\triangleright}$.

$$[1\leqslant j\leqslant k\leqslant n] + [1\leqslant k\leqslant j\leqslant n] = [1\leqslant j,k\leqslant n] + [1\leqslant j=k\leqslant n]$$

$$\begin{aligned} 2S_{\triangleleft} = S_{\triangleleft} + S_{\triangleright} &= S_A + S_{diag(A)} \\ &= \sum_{1\leqslant j,k\leqslant n} a_ja_k + \sum_{1\leqslant j=k\leqslant n} a_ja_k \\ &= \left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} a_k\right) + \sum_{k=1}^{n} a_k^2 \\ &= \left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2 \end{aligned}$$

$\therefore S_{\triangleleft} = \frac{1}{2}[(\sum_{k=1}^{n} a_k)^2 + \sum_{k=1}^{n} a_k^2]$

1.4.2 Exercise 2

$$S = \sum_{1\leqslant j<k\leqslant n} (a_k - a_j)(b_k - b_j) \tag{8}$$

[1]下三角形矩阵的符号是一个右上部分的直角三角形, 目前我还不会输入

解 3 交换 j,k 仍有对称性.

$$S=\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)=\sum_{1\leqslant j<k\leqslant n}(a_j-a_k)(b_j-b_k)$$

$$[1\leqslant j<k\leqslant n]+[1\leqslant k<j\leqslant n]=[1\leqslant j,k\leqslant n]-[1\leqslant j=k\leqslant n]$$

$$\begin{aligned}
2S&=2\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)\\
&=\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)+\sum_{1\leqslant k<j\leqslant n}(a_k-a_j)(b_k-b_j)\\
&=\sum_{1\leqslant j,k\leqslant n}(a_k-a_j)(b_k-b_j)-\sum_{1\leqslant j=k\leqslant n}(a_k-a_j)(b_k-b_j)\\
&(a_j-a_k=0,b_j-b_k=0,[j=k]) \qquad =\sum_{1\leqslant j,k\leqslant n}(a_kb_k-a_jb_k-\\
&=\sum_{j=1}^{n}\sum_{k=1}^{n}a_kb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_jb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_kb_j+\sum_{j=1}^{n}\sum_{k=1}^{n}a_jb_j\\
&=n\sum_{k=1}^{n}a_kb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_jb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_kb_j+n\sum_{j=1}^{n}a_jb_j\\
&=2n\sum_{k=1}^{n}a_kb_k-2\sum_{j=1}^{n}a_j\sum_{k=1}^{n}b_k
\end{aligned}$$

$$S=n\sum_{k=1}^{n}a_kb_k-\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)$$

对上式结果重新排序得

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)=n\sum_{k=1}^{n}a_kb_k-\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)$$

定理 1.1 切比雪夫单调不等式 (Chebyshec's monotonic inequality)

$$
\begin{aligned}
\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) &\leqslant n\sum_{k=1}^n a_k b_k & & a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n, \text{and } b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n \\
& & & a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n, \text{and } b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \\
\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) &\geqslant n\sum_{k=1}^n a_k b_k & & a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n, \text{and } b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \\
& & & a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n, \text{and } b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n
\end{aligned}
$$

一般来说，如果 $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ 且 p 是 $\{1,\ldots,n\}$ 的一个排列。
那么不难证明：
当 $b_{p(1)} \leqslant \cdots \leqslant b_{p(n)}$ 时 $\sum_{k=1}^n a_k b_{P(k)}$ 最大.
当 $b_{p(1)} \geqslant \cdots \geqslant b_{p(n)}$ 时 $\sum_{k=1}^n a_k b_{P(k)}$ 最小.

$$\sum_{k\in K} a_k = \sum_{P(k)\in K} a_{P(k)}$$

$P(k)$ 为这些整数的任意一个排列。

$$f : J \Rightarrow K, \quad j \in J \quad f(j) \in K$$

$$\sum_{j\in J} a_{f(j)} = \sum_{k\in K} a_k \quad \#f^-(k)$$

式中 $\#f^-(k)$ 表示集合 $f^-(k) = \{j|f(j) = k\}$ 中元素的个数

$$\sum_{j\in J}[f(j) = k] = \#f^-(k)$$

$$\sum_{j\in J} a_{f(j)} = \sum_{\substack{j \in J \\ k \in K}} a_k[f(j) = k] = \sum_{k\in K} a_k \sum_{j\in J}[f(j) = k]$$

若有 $\#f^-(k) = 1$ (一一对应)[2]

$$\sum_{j\in J} a_{f(j)} = \sum_{f(j)\in K} a_{f(j)} = \sum_{k\in K} a_k$$

[2]这里还不太理解

1.4.3 Exercise 3

$$S_n = \sum_{1\leqslant j<k\leqslant n} \frac{1}{k-j}$$

首先写出前几项，尝试寻找规律：

$$\begin{aligned}
S_1 &= 0 \\
S_2 &= \frac{1}{2-1} = 1 \\
S_3 &= \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{3-2} = \frac{5}{2} \\
S_4 &= \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{3-2} + \frac{1}{4-2} + \frac{1}{4-3} = \frac{13}{3}
\end{aligned}$$

解 4 1. 先对 j 求和

$$\begin{aligned}
S_n &= \sum_{1\leqslant k\leqslant n} \sum_{1\leqslant j<k} \frac{1}{k-j} \\
&= \sum_{1\leqslant k\leqslant n} \sum_{1\leqslant (k-j)<k} \frac{1}{k-(k-j)} \quad j \Rightarrow (k-j) \\
&= \sum_{1\leqslant k\leqslant n} \sum_{0<j\leqslant k-1} \frac{1}{j} \\
&= \sum_{1\leqslant k\leqslant n} H_{k-1} \quad (H_k\text{为调和级数}) \\
&= \sum_{1\leqslant k+1\leqslant n} H_k \quad k \Rightarrow k+1 \\
&= \sum_{0\leqslant k<n} H_k
\end{aligned}$$

2. 先对 k 求和

$$
\begin{aligned}
S_n &= \sum_{1\leqslant j\leqslant n}\sum_{j<k\leqslant n}\frac{1}{k-j}\\
&= \sum_{1\leqslant j\leqslant n}\sum_{j<(k+j)\leqslant n}\frac{1}{(k+j)-j} \quad k\Rightarrow(k+j)\\
&= \sum_{1\leqslant j\leqslant n}\sum_{0<k\leqslant n-j}\frac{1}{k}\\
&= \sum_{1\leqslant j\leqslant n}H_{n-j} \quad (H_k\text{为调和级数})\\
&= \sum_{1\leqslant n-j\leqslant n}H_k \quad j\Rightarrow n-j\\
&= \sum_{0\leqslant j<n}H_j
\end{aligned}
$$

以上两种常用的求和顺序都无法得到这个多重求和的结果，我们需要转换思路.

3. 先用 $k+j$ 替换 k (先换元，再求和)

$$
\begin{aligned}
S_n &= \sum_{1\leqslant j<(k+j)\leqslant n}\frac{1}{(k+j)-j} \quad k\Rightarrow k+j\\
&= \sum_{1\leqslant j<(k+j)\leqslant n}\frac{1}{k}\\
&= \sum_{1\leqslant k\leqslant n}\sum_{1\leqslant j\leqslant n-k}\frac{1}{k} \quad \text{首先对}j\text{求和}\\
&= \sum_{1\leqslant k\leqslant n}\frac{n-k}{k}\\
&= \sum_{1\leqslant k\leqslant n}\left(\frac{n}{k}-1\right)=nH_n-n
\end{aligned}
$$

综上可得 $\sum_{1\leqslant k\leqslant n}H_k=nH_n-n$

代数：$k+f(j)$, f 为任意函数.

用 $k-f(j)$ 替换 k ，并对 j 先求和较好。

几何：$S_n\ (n=4)$

$$\begin{array}{ccccc} & k=1 & k=2 & k=3 & k=4 \\ j=1 & & \frac{1}{1} & +\frac{1}{2} & +\frac{1}{3} \\ j=2 & & & +\frac{1}{1} & +\frac{1}{2} \\ j=3 & & & & +\frac{1}{1} \\ j=4 & & & & \end{array}$$

先对 j 求和 (按列) $H_1+H_2+H_3$ 先对 k 求和 (按行) $H_3+H_2+H_1$ $k \Rightarrow k+j$ 按对角线求和

$$\sum_{k=1}^{n} \frac{n-k}{k} = n\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} 1$$

$nH_n - n$,$n=4$

$$\begin{aligned} \frac{4}{1}+\frac{3}{2}+\frac{2}{3}+\frac{1}{4} &= \sum_{k=1}^{4} \frac{4-k}{k} \\ &= 4\sum_{k=1}^{4} \frac{1}{k} - \sum_{k=1}^{4} 1 \\ &= 4H_4 - 4 \end{aligned}$$

$$4\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-4 = \frac{4}{2}+\frac{4}{3}+\frac{4}{4}$$

$$\begin{array}{ccccc} & k-j=0 & k-j=1 & k-j=2 & k-j=3 \\ j=1 & & \frac{1}{1} & +\frac{1}{2} & +\frac{1}{3} \\ j=2 & & \frac{1}{1} & +\frac{1}{2} & \\ j=3 & & \frac{1}{1} & & \\ j=4 & & & & \end{array}$$

1.5 General methods

1.5.1 Exercise 4

求 $\Box_n = \sum_{0\leqslant k\leqslant n} k^2$, $n \geqslant 0$ 的封闭形式

$$
\begin{aligned}
\sum_{k=0}^{n} k^2 &= \sum_{k=0}^{n}[(k+1)^2 - 2k - 1] \\
&= \sum_{k=1}^{n+1} k^2 - 2\sum_{k=0}^{n} k - \sum_{k=0}^{n} 1
\end{aligned}
$$

$$
\begin{aligned}
0^2 - (n+1)^2 &= -2\sum_{k=0}^{n} k - (n+1) \\
2\sum_{k=0}^{n} k &= (n+1)^2 - (n+1) \\
\sum_{k=0}^{n} k &= \frac{(n+1)n}{2}
\end{aligned}
$$

上述运算没有告诉我们 $\Box_n$ 的值，但却能推导出 $\sum_{k=0}^{n} k$ 的值。我们可以利用这种思路求解 $\Box_n$ 。

$$
\begin{aligned}
\sum_{k=0}^{n}\left[(k+1)^3 - k^3\right] &= \sum_{k=0}^{n}\left[3k^2 + 3k + 1\right] \\
(n+1)^3 - 0^3 &= 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 \\
(n+1)^3 &= 3\sum_{k=0}^{n} k^2 + 3\frac{n(n+1)}{2} + (n+1)
\end{aligned}
$$

$$
\begin{aligned}
3\sum_{k=0}^{n} k^2 &= (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1) \\
\sum_{k=0}^{n} k^2 &= \frac{1}{3}(n+1)\left((n+1)^2 - \frac{3}{2}n - 1\right) \\
\sum_{k=0}^{n} k^2 &= \frac{1}{3}(n+1)(n+\frac{1}{2})n
\end{aligned}
$$

reference book list:

1. (CRC Tables) CRC Standard Mathematical Tables
2. Handbook of Mathematical Functions
3. Sloane. Handbook of Integer Sequences

software:

Axiom MACSYMA Maple Mathematica

my: Octave maxima 熟悉标准的信息源

方法 3：建立成套方法

参考第二节的内容

方法 4：用积分替换和式 $\sum \Rightarrow \int$

$$\square_n = 1\times 1 + 1\times 4 + 1\times 9 + \cdots + 1\times n^2$$

该式近似等于 0 到 n 之间曲线 $f(x) = x^2$ 下的面积

$$
\begin{aligned}
S &= \int_0^n x^2 dx \\
&= \frac{n^3}{3}
\end{aligned}
$$

$\square_n$ 近似等于 $\frac{n^3}{3}$ 。近似的误差 $E_n = \square_n - \frac{n^3}{3}$

1. 近似误差项递归式

$$\begin{aligned}
\square_n &= \square_{n-1} + n^2 \\
E_n &= \square_n - \frac{n^3}{3} = \square_{n-1} + n^2 - \frac{n^3}{3} \\
E_{n-1} &= \square_{n-1} - \frac{(n-1)^3}{3} \\
E_n &= E_{n-1} + \frac{(n-1)^3}{3} + n^2 - \frac{n^3}{3} \\
&= E_{n-1} + \frac{-3n^2 + 3n - 1}{3} + n^2 \\
&= E_{n-1} + n - \frac{1}{3}
\end{aligned}$$

2. 对楔形误差项面积求和

$$\begin{aligned}
\square_n - \int_0^n x^2 dx &= \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 dx \right) \\
&= \sum_{k=1}^{n} \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\
&= \sum_{k=1}^{n} \left(k - \frac{1}{3} \right)
\end{aligned}$$

$$E_n = \sum_{k=1}^{n} \left(k - \frac{1}{3} \right) = \frac{n(n+1)}{2} - \frac{n}{3} = \frac{n(3n+1)}{6}$$

$$\begin{aligned}
\square_n &= \frac{n^3}{3} + E_n \\
&= \frac{n^3}{3} + \frac{n(3n+1)}{6} \\
&= \frac{n(2n^2 + 3n - 1)}{6} \\
&= \frac{n(n + \frac{1}{2})(n+1)}{3}
\end{aligned}$$

方法 5：展开和收缩

$$
\begin{aligned}
\Box_n = \sum_{1\leqslant k\leqslant n} k^2 &= \sum_{1\leqslant j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n}\sum_{j\leqslant k\leqslant n} k \\
&= \sum_{1\leqslant j\leqslant n}\left(\frac{(j+n)(n-j+1)}{2}\right) \\
&= \frac{1}{2}\sum_{1\leqslant j\leqslant n}(n(n+1)+j-j^2) \\
&= \frac{1}{2}\left[n^2(n+1)+\frac{n(n+1)}{2}-\Box_n\right] \\
&= \frac{1}{2}n(n+\frac{1}{2})(n+1)-\frac{1}{2}\Box_n
\end{aligned}
$$

$$
\begin{aligned}
\frac{3}{2}\Box_n &= \frac{1}{2}n(n+\frac{1}{2})(n+1) \\
\Box_n &= \frac{1}{3}n(n+\frac{1}{2})(n+1)
\end{aligned}
$$

方法 6：使用有限微积分

方法 7：用生成函数

1.6 有限微积分和无限微积分 Finite and infinite calculus

表 1: 有限微积分和无限微积分中的运算对比

无限微积分	有限微积分
微分算子 D	差分算子 Δ

$$
\begin{array}{ll}
Df(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} & \Delta f(x) = f(x+1) - f(x) \\
D(x^m) = mx^{m-1} & \Delta(x^3) = 3x^2 + 3x + 1
\end{array}
$$

为使差分运算在形式上与微分运算类似，引入下降阶乘幂和上升阶乘幂。

下降阶乘幂 (failing factorial power), $x^{\underline{m}}$, 读作 x 直降 m 次.

$$
x^{\underline{m}} = \underbrace{x(x-1)\ldots(x-m+1)}_{m\text{ 个因子}}, \quad (m \leqslant 0, m \in \mathbb{N})
$$

上升阶乘幂 (rising factorial power), $x^{\overline{m}}$, 读作 x 直升 m 次.

$$
x^{\overline{m}} = \underbrace{x(x+1)\ldots(x+m-1)}_{m\text{ 个因子}}, \quad (m \leqslant 0, m \in \mathbb{N})
$$

$$
\begin{aligned}
\Delta(x^{\underline{m}}) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\
&= (x+1)x\ldots(x+1-m+1) - x(x-1)\ldots(x-m+1) \\
&= (x+1-(x-m+1))x(x-1)\ldots(x-m+2) \\
&= mx(x-1)\ldots(x-m+2) \\
&= mx^{\underline{m-1}}
\end{aligned}
$$

$$
g(x) = Df(x), \iff \underbrace{\int g(x)dx}_{\text{不定积分}} = f(x) + \underbrace{C}_{\text{任意常数}} \tag{9}
$$

$$
g(x) = \Delta f(x), \iff \underbrace{\sum g(x)\delta(x)}_{\text{不定和式}} = f(x) + \underbrace{C}_{\text{满足}p(x+1)=p(x)\text{的任意函数}} \tag{10}
$$

表 2: 有限微积分和无限微积分中的运算对比

无限微积分	有限微积分
D 逆运算 $\int$ (积分算子，逆微分算子)	Δ 逆运算 $\sum$ (求和算子，逆差分算子)
微积分基本定理	
$g(x) = Df(x) \iff \int g(x)dx = f(x) + C$	$g(x) = \Delta f(x) \iff \sum g(x)\delta(x) = f(x) + C$
定积分	和式
若 $g(x) = Df(x)$ 那么	若 $g(x) = \Delta f(x)$ 那么
$\int_a^b g(x)dx = f(x)\vert_a^b = f(b) - f(a)$	$\sum_a^b g(x)\delta x = f(x)\vert_a^b = f(b) - f(a)$
$\int_b^a g(x)dx = -\int_a^b g(x)dx$	$\sum_b^a g(x)\delta x = -\sum_a^b g(x)\delta x$
$\int_a^b + \int_b^c = \int_a^c$	$\sum_a^b + \sum_b^c = \sum_a^c$
$\int_0^n x^m = \frac{x^{m+1}}{m+1}\Big\vert_0^n = \frac{n^{m+1}}{m+1}, m \neq -1$	$\sum_0^n k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big\vert_0^n = \frac{n^{\underline{m+1}}}{m+1}, m \neq -1$ $\sum_0^n k^{\overline{m}} = \frac{k^{\overline{m+1}}}{m+1}\Big\vert_0^n = \frac{n^{\overline{m+1}}}{m+1}, m \neq -1$
$(x+y)^2 = x^2 + 2xy + y^2$	$(x+y)^{\underline{2}} = x^{\underline{2}} + 2x^{\underline{2}}y^{\underline{1}} + y^{\underline{2}}$ $(x+y)^{\overline{2}} = x^{\overline{2}} + 2x^{\overline{1}}y^{\overline{1}} + y^{\overline{2}}$
$m = -1, \int_a^b x^{-1} = \ln x\Big\vert_a^b$	$m = -1, \sum_a^b k^{\underline{-1}} = H_k\Big\vert_a^b$
$\int_a^b x^m = \begin{array}{l} \frac{x^{m+1}}{m+1}\Big\vert_a^b, \ m \neq -1 \\ \ln n\Big\vert_a^b, \ m = -1 \end{array}$	$\sum_a^b k^{\underline{m}} = \begin{array}{l} \frac{k^{\underline{m+1}}}{m+1}\Big\vert_a^b, \ m \neq -1 \\ H_k\Big\vert_a^b, \ m = -1 \end{array}$ $\sum_a^b k^{\overline{m}} = \begin{array}{l} \frac{k^{\overline{m+1}}}{m+1}\Big\vert_a^b, \ m \neq -1 \\ H_{(k+1)}\Big\vert_a^b, \ m \neq -1 \end{array}$
连续性问题的解中会出现自然对数	快速排序这样的问题中会出现调和数的原因
e^x, 性质 $De^x = e^x$	$\Delta f(x) = f(x), f(x) = 2^x$ 离散指数函数

$$\sum_a^b g(x)\delta x = f(x)\big|_a^b = f(b) - f(a) \tag{11}$$

设 $g(x) = \Delta f(x) = f(x+1) - f(x)$

如果 $b = a$, 我们就有

$$\sum_a^a g(x)\delta x = f(a) - f(a) = 0 \tag{12}$$

如果 $b = a+1$, 我们就有

$$\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a) \tag{13}$$

$$\begin{aligned}\sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x &= \big[f(b+1) - f(a)\big] - \big[f(b) - f(a)\big] \\ &= f(b+1) - f(b) = g(b)\end{aligned}$$

由数学归纳法 $a, b \in \mathbb{N}$ 且 $b \leqslant a$ 时，$\sum_a^b g(x)\delta x$ 的确切含义是

$$\sum_a^b g(x)\delta x = \sum_{k=1}^{b-1} g(k) = \sum_{a \leqslant k < b} g(k), \quad (b \geqslant a, a, b \in \mathbb{N}) \tag{14}$$

若有 $g(x) = f(x+1) - f(x)$

$$\sum_{a \leqslant k < b} g(l) = \sum_{a \leqslant k < b} (f(k+1) - f(k)) = f(b) - f(a)$$

$$\begin{aligned}\sum_a^b g(x)\delta x = f(b) - f(a) \qquad (b < a) \\ = -(f(a) - f(b)) = -\sum_a^b g(x)\delta x\end{aligned}$$

$$\sum_a^b + \sum_b^c = \sum_a^c$$

应用：计算下降幂和式的简单方法

$$\sum_{0\leqslant k<n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\bigg|_0^n = \frac{n^{\underline{m+1}}}{m+1}, \quad \left(m,n\geqslant 0 \quad m,n\in\mathbb{N}^+\right) \tag{15}$$

$m=1$ 时，$k^{\underline{1}}=k$

$$\sum_{0\leqslant k<n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$$

$$k^2 = k(k-1)+k = k^{\underline{2}} + k^{\underline{1}}$$

$$\begin{aligned}\sum_{0\leqslant k<n} k^2 &= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\ &= \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} \\ &= \frac{1}{3}n(n-\frac{1}{2})(n-1)\end{aligned}$$

$$k^3 = k(k-1)(k-2)+3k(k-1)+k = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}$$

$$\begin{aligned}\sum_{0\leqslant k<n} k^3 &= \frac{n^{\underline{4}}}{4} + 3\frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} \\ &= \frac{n(n-1)(n-2)(n-3)}{4} + 3\frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} \\ &= \left(\frac{1}{2}n(n-1)\right)^2 = \left(\sum_{0\leqslant k<n} k\right)^2\end{aligned}$$

负指数下降幂

$$
\begin{aligned}
x^{\underline{3}} &= x(x-1)(x-2) \\
x^{\underline{2}} &= x(x-1) \\
x^{\underline{1}} &= x \\
x^{\underline{0}} &= 1 \\
x^{\underline{-1}} &= \frac{1}{x+1} \\
x^{\underline{-2}} &= \frac{1}{(x+1)(x+2)} \\
&\vdots \\
x^{\underline{-m}} &= \frac{1}{(x+1)(x+2)\dots(x+m)}
\end{aligned}
$$

$$x^{\underline{3}} \cdot \frac{1}{x-3+1} = x^{\underline{2}} \Rightarrow x^{\underline{0}} \cdot \frac{1}{x-0+1} = x^{\underline{-1}}$$

为什么选用 $x^{\underline{-1}} = \frac{1}{x+1}$ 而不是 $x^{\underline{-1}} = \frac{1}{x+1}$ 作为下降阶乘幂的拓展定义?[3]

通常幂法则 $x^{m+n} = x^m x^n$, 推广:

$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}},\ (m, n \in \mathbb{N}^+)$

$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}},\ (m, n \in \mathbb{N}^+)$

[3]当一个原有的记号被拓展包含更多种情形时，以一种使得一般性法则继续成立的方式来表述它的定义，这永远是最佳选择

my 推广至正指数下降幂

$$
\begin{aligned}
x^{\overline{3}} &= x(x+1)(x+2) \\
x^{\overline{2}} &= x(x+1) \\
x^{\overline{1}} &= x \\
x^{\overline{0}} &= 1 \\
x^{\overline{-1}} &= \frac{1}{x-1} \\
x^{\overline{-2}} &= \frac{1}{(x-1)(x-2)} \\
&\vdots \\
x^{\overline{-m}} &= \frac{1}{(x-1)(x-2)\dots(x-m)}
\end{aligned}
$$

$$
x^{\overline{3}} \cdot \frac{1}{x+3} = x^{\overline{2}} \Rightarrow x^{\overline{0}} \cdot \frac{1}{x+0-1} = x^{\overline{-1}}
$$

例如

$$
\begin{aligned}
x^{\underline{2+3}} &= x^{\underline{2}}(x-2)^{\underline{3}} = x^{\underline{3}}(x-3)^{\underline{2}} \\
x^{\underline{2-3}} &= x^{\underline{2}}(x-2)^{\underline{-3}} \\
&= x(x-1)\frac{1}{(x-2+1)(x-2+2)(x-2+3)} \\
&= x(x-1)\frac{1}{(x-1)x(x+1)} \\
&= \frac{1}{x+1}
\end{aligned}
$$

$m<0$ 时，$\Delta x^{\underline{m}} = m x^{\underline{m-1}}$ 是否仍成立?

$$\begin{aligned}\Delta x^{\underline{-2}} &= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\ &= \frac{(x+1)-(x+3)}{(x+1)(x+2)(x+3)} \\ &= -2x^{\underline{-3}}\end{aligned}$$

通常幂法则对负指数下降阶乘幂仍然成立。

离散指数函数 2^x

$$\begin{aligned}&\Delta(c^x) = c^{x+1} - c^x = (c-1)c^x \\ &c \neq 1 \frac{c^x}{c-1} \stackrel{\Delta}{\Longrightarrow} c^x \\ &\sum_{a\leqslant k<b} c^k = \sum_a^b c^x \delta x = \frac{c^x}{c-1}\Big|_a^b = \frac{c^b - c^a}{c-1},\ c \neq 1\end{aligned}$$

$$D(uv) = uDv + vDu \tag{16}$$

$$\int uDv = uv - \int vDu \tag{17}$$

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\ &= \Delta u(x)v(x+1) + u(x)\Delta v(x) \\ &= u\Delta v + Ev\Delta u\end{aligned} \tag{18}$$

其中 E 被称为移位算子.

在无限微积分中，令 $x+1 \to x$ 无限细分，避开了 E

$$\sum u\Delta v = uv - \sum Ev\Delta u \tag{19}$$

表 3: Table 55(1994), What's difference

$f=\sum g$	$\Delta f = g$
$x^{\underline{0}}=1$	0
$x^{\underline{1}}=x$	1
$x^{\underline{2}}=x(x-1)$	$2x$
$x^{\underline{m}}$	$mx^{\underline{m-1}}$
$x^{\underline{m+1}}$	$(m+1)x^{\underline{m}}$
H_x	$x^{\underline{-1}}=\frac{1}{x+1}$
2^x	2^x
c^x	$(c-1)c^x$
$\frac{c^x}{c-1}$	c^x
$cu(x)$, c is constant	$c\Delta u(x)$
$u+v$	$\Delta u+\Delta v$
uv	$u\Delta v+Ev\Delta u$, $Ev=v(x+1)$

例 1.2 $\int xe^x dx \xrightarrow{\text{离散模拟}} \sum x2^x\delta x \qquad (\sum_{k=0}^{n} k2^k)$

解 5 令 $u(x)=x, \delta v(x)=2^x$,
可得 $\delta u(x)=1, v(x)=2^x, Ev=2^{x+1}$

$$\begin{aligned}\sum x2^x\delta x &= x\cdot 2^x - \sum 2^{x+1}\cdot 1\delta x \\ &= x\cdot 2^x - 2^{x+1} + C\end{aligned}$$

$$\begin{aligned}\sum_0^n k2^k &= \sum_0^{n+1} x2x\delta x \\ &= x\cdot 2^x - 2^{x+1}\Big|_0^{n+1}\end{aligned}$$

关于第二组等式的推导，我一开始没有完全掌握，主要是对求和符号 $\sum$ 的上下标范围存在误解.

记 $\sum_{0\leqslant k\leqslant n} k2^k = S_n$

$$\begin{aligned}\sum_{0\leqslant k\leqslant n} k2^k + (n+1)2^{n+1} &= \sum_{0\leqslant k\leqslant n+1} k2^k \\ &= \sum_{1\leqslant k\leqslant n+1} (k-1)2^k + \sum_{1\leqslant k\leqslant n+1} 2^k \\ &= \sum_{0\leqslant k\leqslant n} k2^{k+1} + \sum_{1\leqslant k\leqslant n+1} 2^k \\ &= 2S_n + \sum_{1\leqslant k\leqslant n+1} 2^k\end{aligned}$$

$$\begin{aligned}S_n + (n+1)2^{n+1} &= 2S_n + \frac{2^{n+1}-2^1}{2-1} \\ S_n &= (n+1)2^{n+1} - (2^{n+1}-2) \\ &= (n-1)2^{n+1} + 2\end{aligned}$$

$$\sum_{0\leqslant k<n} H_k = nH_n - n \tag{20}$$

求解看起来更困难的和式 $\sum_{0\leqslant k<n} kH_k$

类比 $\int x\ln x dx$

$$\begin{aligned}I &= \int x\ln x dx \\ &= x^2\ln x - \int x(\ln x + x\cdot\frac{1}{x})dx \\ &= x^2\ln x - I - \int x dx \\ I &= \frac{1}{2}x^2\ln x - \frac{1}{2}x^2 + C\end{aligned}$$

对 $\sum_{0\leqslant k<n} kH_k$，取 $u(x)=H_x\Delta$, $v(x)=x=x^{\underline{1}}$

$\Delta u(x)=x^{\underline{-1}}$, $v(x)=\frac{1}{2}x^{\underline{2}}$,$Ev(x)=v(x+1)=\frac{1}{2}(x+1)^{\underline{2}}$

$$\begin{aligned}
\sum xH_x\delta x &= \frac{1}{2}x^{\underline{2}}H_x-\sum\frac{1}{2}(x+1)^{\underline{2}}x^{\underline{-1}}\delta x\\
&= \frac{x^{\underline{2}}}{2}H_x-\sum\frac{(x+1)x}{2}\frac{1}{x+1}\delta x\\
&= \frac{x^{\underline{2}}}{2}H_x-\sum\frac{x^{\underline{1}}}{2}\delta x\\
&= \frac{x^{\underline{2}}}{2}H_x-\frac{1}{4}x^{\underline{2}}+C
\end{aligned}$$

$$\sum_{0\leqslant k<n} kH_k=\sum_{x=0}^{n-1}xH_x\delta x=\frac{(n-1)^{\underline{2}}}{2}(H_{n-1}-\frac{1}{2})$$

教材上是

$$\sum_{0\leqslant k<n} kH_k=\sum_{x=0}^{n}xH_x\delta x=\frac{n^{\underline{2}}}{2}(H_n-\frac{1}{2})$$

借助有限微积分的原理，我们很容易地记住

$$\sum_{0\leqslant k<n} k=\frac{n^2}{2}=n(n-1)/2 \tag{21}$$

myex $\sum_{0\leqslant k<n} H_k$

$$\begin{aligned}
u(x)&=H_x & \Delta v(x)&=x^{\underline{0}}=1\\
\Delta u(x)&=x^{\underline{-1}} & v(x)&=x^{\underline{1}}\\
& & Ev(x)&=v(x+1)=(x+1)^{\underline{1}}
\end{aligned}$$

$$\begin{aligned}
\sum H_x\cdot 1\delta x &= x^{\underline{1}}H_x-\sum x^{\underline{-1}}(x+1)^{\underline{1}}\delta x\\
&= x^{\underline{1}}H_x-\sum x^{\underline{0}}\delta x\\
&= x^{\underline{1}}H_x-x^{\underline{1}}+C
\end{aligned}$$

$$\sum_{0\leqslant k<n} H_k=\sum_{0}^{n}H_x\delta x=nH_n-n-(0-0)=nH_n-n$$

1.7 无限和式 Infinite sums

a_k 非负，$\sum_{k\in K} a_k$

定义 1.1 如果有 $A = const$. s.t. $\forall$ 有限子集 $F \subset K$, 均有

$$\sum_{k\in F} a_k \leqslant A$$

那么我们定义 $\sum_{k\in K} a_k$ 是最小的这样的 A (所有这样的 A 总包含一个最小元素)。若没有这样的常数 A，我们就说 $\sum_{k\in K} a_k = \infty$ 即 $\forall A \in \mathbb{R}$, $\exists$ 有限多项 a_k 组成的一个集合, 它的和超过 A

该定义与指标集 K 中可能存在的任何次序无关

特殊情形：K 为非负整数集合 $a_k \leqslant 0$ 意味着

$$\sum_{k\geqslant 0} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$$

理由：实数任意非减序列均有极限

$F \subset \mathbb{N}$, $\forall i \in F, i \leqslant n$, $\exists \sum_{k\in F} a_k \leqslant \sum_{k=0}^{n} a_k \leqslant A$. $\quad \therefore \begin{cases} A = \infty \\ A\text{为有界常数} \end{cases}$

又 $\forall A' < A$. $\exists n$. s.t $\sum_{k=0}^{n} a_k > A'$, $F = \{0, 1, \ldots, n\}$. 证明 A' 不是有界常数。

练习 1 $a_k = x^k$ 有

$$\sum_{k\geqslant 0} x^k = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \frac{1}{1-x}, & 0 \leqslant x < 1 \\ \infty, & x \geqslant 1 \end{cases}$$

练习 2

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = S - 1,\ S = 2$$

$$T = 1 + 2 + 4 + 8 + \dots$$

$$2T = 2 + 4 + 8 + \dots = T - 1,\ T = -1(\times)$$

$$T = \infty.\ (\text{另一个解})$$