测试文件

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Chapter 1 第二章数列极限

1.1 数列极限的基本概念

1.1.1 2.1.5 练习题

Question 1 1. prove by Limit definition:
(1). $\lim_{n\to\infty}\frac{3n^2}{n^2-4}=3$.
(2). $\lim_{n\to\infty}\frac{\sin n}{n}=0$.
(3). $\lim_{n\to\infty}(1+n)^{\frac{1}{n}}=0$.
(4). $\lim_{n\to\infty}\frac{a^n}{n!}=0,(a>0)$.

Question 2 2. Suppose $a_n, n\in\mathbb{N}_+$. sequence a_n converge to a.
Prove $\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a}$.

Proof $n\to\infty a_n\to a$.
$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$.

$$|\sqrt{a_n}-\sqrt{a}|=\left|\frac{a_n-a}{\sqrt{a_n}+\sqrt{a}}\right|<\frac{\epsilon}{\sqrt{a_n}+\sqrt{a}}$$

$\therefore\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a}$. □ (check, not consider the condition$a=0$) add $a=0,\forall\epsilon\in(0,1),\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$. s.t $a_n<\epsilon^2<\epsilon$, $\sqrt{a_n}<\epsilon$.

Question 3 3. If $\lim_{n\to\infty}a_n=a$.
Prove $\lim_{n\to\infty}|a_n|=|a|$. Vice versa?

Proof $\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$.

$$\Big||a_n|-|a|\Big|\leqslant|a_n-a|<\epsilon$$

$\therefore\lim_{n\to\infty}|a_n|=|a|$
If We know $\lim_{n\to\infty}|a_n|=|a|$.
$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),\Big||a_n|-|a|\Big|<\epsilon$. We can't get $\lim_{n\to\infty}a_n=a$. For example: $a_n=\frac{1}{n}+1,a=-1$, $\lim_{n\to\infty}|a_n|=|a|$ is $\lim_{n\to\infty}|\frac{1}{n}+1|=|-1|$, but $\lim_{n\to\infty}\frac{1}{n}+1\neq-1$ □

Question 4 (1). Suppose $p(x)$ is a polynomial of x, if $\lim_{n\to\infty}a_n=a$, Prove $\lim_{n\to\infty}p(a_n)=p(a)$.
(2). Suppose $b>0$, $\lim\limits_{n\to\infty}a_n=a$. Prove $b^{a_n}=b^a$.
(3). Suppose $b>0$, $\{a_n\},a_n>0,\forall n\in\mathbb{N}$. $\lim\limits_{n\to\infty}a_n=a.a>0$. Prove $\lim\limits_{n\to\infty}\log_b a_n=\log_b a$.
(4) Suppose $b\in\mathbb{R}$, $\{a_n\},a_n>0$ when $n\in\mathbb{N}$. $\lim\limits_{n\to\infty}a_n=a$. Prove $\lim\limits_{n\to\infty}{a_n}^b=a^b$.
(5) Suppose $\lim\limits_{n\to\infty}a_n=a$. Prove $\lim\limits_{n\to\infty}\sin a_n=\sin a$.

Proof 4.(1)
$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n\geqslant N(\epsilon)$, $|a_n-a|<\epsilon$.

$p(a) = k_m a^m + k_{m-1}a^{m-1} + \cdots + k_0 a^0$.
$\therefore p(a_n) - p(a) = k_m(a_n^m - a^m) + k_{m-1}(a_n^{m-1} - a^{m-1}) + \cdots + k_0(a_n^0 - a^0)$.

$$\begin{aligned}|a_n^m - a^m| &= |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon(m-1)\cdots(a+\delta)^{m-1}\end{aligned}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a)$. □

Proof 4.(2)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.
If $b = 1$, $1^{a_n} = 1^a = 1$.
If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1)$ $\because b > 0, \epsilon \to 0$,
$\therefore b^\epsilon - 1 \to 0$. $\therefore \lim\limits_{n\to\infty} b_n^a = b^a$.
If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$.

Proof 4.(3)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1)\end{aligned}$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a})$. $\because b > 0, a \neq 0$, $a_n > 0$ when $\epsilon \to 0$. $\therefore \log_b(1 + \frac{\epsilon}{a}) \to 0$.
$\therefore \lim\limits_{n\to\infty} \log_b a_n = \log_b a$

Proof 4.(4)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.
$a_n^b = e^{b\ln a_n}$, $a_n^b - a^b = e^{b\ln a_n} - e^{b\ln a}$.

$$\begin{aligned}e^{b\ln a_n} - e^{b\ln a} &= e^{b\ln a}(e^{b\ln a_n - b\ln a} - 1) \\ &= e^{b\ln a}(e^{b\ln\frac{a_n}{a}} - 1)\end{aligned}$$

$0 < |a_n^b - a^b| < e^{b\ln a}(e^{b\ln(1+\frac{\epsilon}{a})} - 1)$
$\therefore \lim\limits_{n\to\infty} a_n^b = a^b$

Proof 4.(5)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\sin(A+B) - \sin(A-B) &= \sin A\cos B + \cos A\sin B \\ &\quad - (\sin A\cos B - \cos A\sin B) \\ &= 2\cos A\sin B \\ \sin a_n - \sin a &= 2\cos\frac{a_n + a}{2}\sin\frac{a_n - a}{2}\end{aligned}$$

$|\sin a_n - \sin a| = |2\cos\frac{a_n+a}{2}\sin\frac{a_n-a}{2}| < |2\sin\frac{a_n-a}{2}|$
$|2\sin\frac{a_n-a}{2}| < |2\frac{a_n-a}{2}| = \epsilon$
$|\sin a_n - \sin a| < \epsilon$, $\therefore \lim\limits_{n\to\infty} \sin a_n = \sin a$

Question 5 assume $a > 1$. Prove $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof $\frac{1}{n}\log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\log_a 1 = 0$.
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\} . \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon$.
$a > 1$, and $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\therefore$ when $n \to \infty$, $\sqrt[n]{n} < a^\epsilon$, take logarithm on base of a, we can get

$\frac{1}{n}\log_a n < \epsilon$
$\therefore \lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

1.2 收敛数列的基本性质

收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界
3. 收敛数列的比较定理，包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

1.2.1 思考题

Question 6 1. $\{a_n\}$ 收敛, $\{b_n\}$ 发散, $\{a_n+b_n\}$ 发散, $\{a_n\cdot b_n\}$ 可能收敛，可能发散.
2. $\{a_n\},\{b_n\}$ 都发散, $\{a_n+b_n\}$ 可能收敛，可能发散 (ex: $n+-n$, $n+-2n$), $\{a_n\cdot b_n\}$ 发散 (?).
3. $a_n \leqslant b_n \leqslant c_n, n\in\mathbb{N}_+$. 已知 $\lim\limits_{n\to\infty}(c_n-a_n)=0$. 问数列 $\{b_n\}$ 是否收敛？
4. $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$
5. $a_n \to a(n\to 0)$. $\forall n, b<a_n<c$. 是否成立 $b<a<c$?
6. $a_n \to a(n\to 0)$. and $b\leqslant a\leqslant c$, 是否存在 $N\in\mathbb{N}_+$, s.t. 当 $n>N$ 时，成立 $b\leqslant a_n\leqslant c$
7. 已知 $\lim\limits_{n\to\infty} a_n=0$, 问: 是否有 $\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0$. 反之如何?

Proof 5.4

$$\frac{n}{2n} \leqslant \frac{1}{n+1}+\cdots+\frac{1}{2n} \leqslant \frac{n}{n+1}$$

$\because \lim\limits_{n\to\infty}\frac{n}{2n}=\frac{1}{2}\lim\limits_{n\to\infty}\frac{n}{n+1}=1$, $\therefore \lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$ 收敛.

$$\begin{aligned}\frac{1}{n+1}+\cdots+\frac{1}{2n} &= \frac{1}{n}(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}}) \\ &= \int_0^1 \frac{1}{1+x}\mathrm{d}x \\ &= \ln(1+x)|_0^1 = \ln 2\end{aligned}$$

$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})=\ln 2$

Proof 5.5

不成立，应当为小于等于号。b=0, c=2, $a_n=\frac{1}{n}$, $\lim_{n\to\infty}a_n=0=c$.

Proof 5.6

不成立。$a=0, b=0, c=2$, $a_n=(-1)^n\frac{1}{n}$.
$b\leqslant a\leqslant c$, but $(-1)^{2n+1}\frac{1}{2n+1}<0=b$.

Proof $\lim\limits_{n\to\infty}a_n=0, a_n=\frac{1}{n}.a_1a_2\ldots a_n=\frac{1}{n!}, \lim\limits_{n\to\infty}\frac{1}{n!}=0$.

$\lim\limits_{n\to\infty}a_n=0 \to \lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0 \quad \checkmark$
$\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0 \to \lim\limits_{n\to\infty}a_n=0 \quad \times$
$|a_n|<\epsilon$, $|a_{N+1}\ldots a_n|<\epsilon^{n-N}<\epsilon$, $a_n<\sqrt[n]{\epsilon}$.
for example, $a_n=\frac{n}{n+1}, a_1=\frac{1}{2}, a_2=\frac{2}{3},\ldots,a_n=\frac{n}{n+1}$.

$$a_1a_2\dots a_n = \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1} = \frac{1}{n+1}.$$

$$\lim_{n\to\infty}(a_1a_2\dots a_n)$$

$$= \lim_{n\to\infty}\frac{1}{n} = 0$$

but $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$

研究数列收敛方面的两个基本工具:

1. 夹逼定理.

2. 单调有界数列的收敛定理.

Example 1.1 2.2.2 $\lim_{n\to\infty}\frac{x_n-1}{x_n+a} = 0$,

prove $\lim_{n\to\infty} x_n = a$

Proof $\forall\epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |\frac{x_n-1}{x_n+a} - 0| < \epsilon.$

$|x_n - 1| < \epsilon|x_n + a| < 4a\cdot\epsilon.$(这个 4 是怎么取得的?)

$|x_n - a| < \epsilon|x_n + a| = \epsilon|(x_n - a) + 2a| \leqslant \epsilon(|x_n - a| + 2a).$

限制 $\epsilon < 1$, $|x_n - a| < 2\epsilon|a|/(1-\epsilon)$.

限制 $\epsilon < \frac{1}{2}$, $|x_n - a| < 2\epsilon|a|/(1-\epsilon) < 4|a|\epsilon$.

Let $\epsilon' = 4a\epsilon$, $|x_n - 1| < \epsilon'$. $\therefore \lim\lim_{n\to\infty} x_n = a$.

Example 1.2 2.2.3 $a > 0, b > 0$, 计算 $\lim\limits_{n\to\infty}(a_n + b_n)^{\frac{1}{n}}$.

Proof Suppose $a \leqslant b$.

$b = (b^b)^{\frac{1}{n}} < (a^n + b^n)^{\frac{1}{n}} \leqslant (2b^n)^{\frac{1}{n}}$.

$b < (a^n + b^n)^{\frac{1}{n}} \leqslant \sqrt[n]{2}b$, $\lim\limits_{n\to\infty} = 1$. 夹逼定理.

$\lim\limits_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}$.

两数 n 次方之和再开 n 次根号的结果由较大的值决定, a,b 中较大的值为这个数的主要部分.

Example 1.3 2.2.4 $a_n = \frac{1!+2!+\dots+n!}{n!}, n \in \mathbb{N}^+$

$\lim\limits_{n\to\infty} a_n = 1$

Example 1.4 $\lim\limits_{n\to\infty}\frac{n^3+n-7}{n+3} = +\infty$

Example 1.5 $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

调和级数 H_n 发散.

1.2.2 练习 2.2.4

Proof 1.

$\{a_n\}$ 收敛于 a, $\to$ 两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 a.

两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 a, $\to$ $\{a_n\}$ 收敛于 a.

2. 应用夹逼定理

(1). 给定 p 个正数 $a_1, a_2, \dots, a_p$. 求 $\lim\limits_{n\to\infty}\sqrt[n]{a_1^n + a_2^n + \dots a_p^n}$.

Let$a_s = \max_{1\leqslant i\leqslant p}\{a_1, a_2, \dots, a_p\}$.

Solve 1 (1).

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \dots a_p^n)^{\frac{1}{n}} \leqslant (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}}a_s$$

$n \to \infty, p^{\frac{1}{n}} \to 1$. $\lim\limits_{n\to\infty}(a_1^n + a_2^n + \dots a_p^n)^{\frac{1}{n}} = a_s$

(2). $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} x_n$

Solve 2 (2).

$$\frac{2n+1}{(n+1)} \leqslant x_n \leqslant \frac{2n+1}{\sqrt{n^2+1}}$$

$\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$, $\lim\limits_{n\to\infty} \frac{2n+1}{\sqrt{n^2+1}} = 2$. $\therefore \lim\limits_{n\to\infty} x_n = 2$

(3). $a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} a_n$

Solve 3 (3).

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leqslant (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\therefore \lim\limits_{n\to\infty} a_n = 1$

(4). $a_n > 0$. $\lim\limits_{n\to\infty} a_n = a$, $a > 0$. 证明 $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$

Proof $\lim\limits_{n\to\infty} a_n = a$

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$

$$0 < a - \epsilon < a_n < a + \epsilon$$

$\therefore \sqrt[n]{a-\epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a+\epsilon}.$

$\lim\limits_{n\to\infty} \sqrt[n]{a-\epsilon} = 1$, $\lim\limits_{n\to\infty} \sqrt[n]{a+\epsilon} = 1$. $\therefore \lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1.$

3. (1). $\lim\limits_{n\to\infty} (1+x)(1+x^2)\dots(1+x^{2^n}) = \lim\limits_{n\to\infty} \prod_{i=1}^{2^n}(1+x^i)$, $|x| < 1$.

Solve 4 3.(1).

$|x| < 1, \quad 1 > x^2 > x^4 > \cdots > x^{2^n} > 0$

$x \in (0,1) \quad 1 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)^{n+1} \qquad \lim\limits_{n\to\infty} (1+x)^{n+1} = 1$

$x \in (-1,0) \quad 0 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)(1+x^2)^n \qquad \lim\limits_{n\to\infty} (1+x)(1+x^2)^n = 1$

Solve 5 3.(1). another way

$$\begin{aligned}
&\lim_{n\to\infty} (1+x)(1+x^2)\dots(1+x^n) \\
=&\lim_{n\to\infty} \frac{(1-x)(1+x)(1+x^2)\dots(1+x^n)}{1-x} \\
=&\lim_{n\to\infty} \frac{(1-x^{2^{n+1}})}{1-x} \\
=&\frac{1}{1-x}
\end{aligned}$$

Solve 6 3. (2).

$$\begin{aligned}
&\lim_{n\to\infty} (1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{n^2}) \\
=&\lim_{n\to\infty} \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\dots\cdot\frac{n-1}{n}\cdot\frac{n+1}{n} \\
=&\lim_{n\to\infty} \frac{1}{2}\frac{n+1}{n} \\
=&\lim_{n\to\infty} \frac{1}{2}
\end{aligned}$$

Solve 7 3. (3).

$$
\begin{aligned}
&\lim_{n\to\infty}\left(1-\frac{1}{1+2}\right)\left(1-\frac{1}{1+2+3}\right)\dots\left(1-\frac{1}{1+2+\cdots+n}\right)\\
=&\lim_{n\to\infty}\left(1-\frac{2}{3\times 2}\right)\left(1-\frac{2}{4\times 3}\right)\dots\left(1-\frac{2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{3\times 2-2}{3\times 2}\right)\left(\frac{4\times 3-2}{4\times 3}\right)\dots\left(\frac{(n+1)\times n-2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{4}{3\times 2}\right)\left(\frac{10}{4\times 3}\right)\dots\left(\frac{n^2+n-2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{1\times 4}{3\times 2}\right)\left(\frac{2\times 5}{4\times 3}\right)\dots\left(\frac{(n-2)\times(n+1)}{n\times(n-1)}\right)\left(\frac{(n-1)\times(n+2)}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\frac{1}{3}\times\frac{n+2}{n}\\
=&\frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}]\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{n}{n+1}\\
=&1
\end{aligned}
$$

Solve 8 3.(4).

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)}\right]\\
=&\lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}\right)+\frac{1}{2}\left(\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}\right)+\cdots+\frac{1}{2}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)\\
=&\lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{1\cdot 2}-\frac{1}{(n+1)(n+2)}\right)\\
=&\frac{1}{2}\times\frac{1}{2}\\
=&\frac{1}{4}
\end{aligned}
$$

Solve 9 3.(5).

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma)},\qquad \text{其中}\gamma\text{为正整数}\\
=&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\gamma}\left[\frac{1}{k(k+1)\dots(k+\gamma-1)}-\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\right]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\left[\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma-1)}-\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\right]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\left[\frac{1}{\gamma^{\underline{\gamma}}}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\right]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\left[\frac{1}{\gamma!}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\right]\\
=&\frac{1}{\gamma}\cdot\frac{1}{\gamma!}
\end{aligned}
$$

其中 $x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1)$, 称为下阶乘. 而 $x^{\overline{n}} = x(x+1)(x+2)\dots(x+n-1)$, 称为上阶乘.

Question 7 2.2.4-4 $S_n = a + 3a^2 + \cdots + (2n-1)a^n$, $|a| < 1$. 求 $\{S_n\}$ 的极限.

Solve 10

$$\begin{aligned} S_n - aS_n &= a + 3a^2 + \cdots + (2n-1)a^n \\ &\quad - \quad a^2 - \cdots + (2n-3)a^n - (2n-1)a^n + 1 \\ &= a + 2a^2 + \cdots + 2aa^n - (2n-1)a^{n+1} \\ &= 2(a + a^2 + \cdots + a^n) - a - (2n-1)a^{n+1} \\ &= 2 \cdot a\frac{1-a^{n+1}}{1-a} - a - (2n-1)a^{n+1} \end{aligned}$$

$|a| < 1$, $\lim\limits_{n\to\infty} a_n = 0$
$\lim\limits_{n\to\infty}(S_n - aS_n) = (1-a)\lim \lim_{n\to\infty} S_n$

$$\begin{aligned} \lim_{n\to\infty}(S_n - aS_n) &= \lim_{n\to\infty} 2a \cdot \frac{1-a^{n+1}}{1-a} - a - (2n-1)a^{n+1} \\ &= 2a \cdot \frac{1}{1-a} - a \\ &= a\Big(\frac{2}{1-a} - a\Big) \\ &= a\frac{1+a}{1-a} \end{aligned}$$

$\therefore \lim\limits_{n\to\infty} S_n = \frac{a(a+1)}{(1-a)^2}$

Solve 11 2.2.4-5 设 $\lim\limits_{n\to\infty} x_n = A > 0$. 取 $\epsilon = \frac{A}{2}$, 则 $\exists N \in \mathbb{N}_+$. $\forall n > N$. 成立 $|x_n - A| < \frac{A}{2}$

$$A - \frac{A}{2} < x_n < A + \frac{A}{2}, \frac{A}{2} < x_n < \frac{3A}{2}$$

即 $x_n > \frac{A}{2}$.
令 $m = \min\{x_1, x_2, \dots, x_N, \frac{A}{2}\} > 0$. 则 m 为 $\{x_n\}$ 的正下界.
不一定有最小数的例子 $x_n = 1 + \frac{1}{n}$. $\lim\limits_{n\to\infty} x_n = 1$, 下界 $m = \frac{1}{2}$. 但 $\{x_n\}$ 取不到下界.

Proof 2.2.4-6 $\because \lim\limits_{n\to\infty} a_n = +\infty$. $\forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M$.
$m = \min\{a_1, a_2, \dots, a_N, M\}$, 但 M 在数列 $\{a_n\}$ 中不一定取的到!
$M = a_1 + 1, \exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$
则 $m = \min\{a_1, a_2, \dots, a_{N_1}\}$ 为数列的最小数.

Proof 2.2.4-7 构造数列
不妨设无界数列 $\{a_n\}$ 无上界.
$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M$.
取 $M_1 = 1$, 则 $\exists n_1 \in \mathbb{N}_+$ s.t. $a_{n_1} > M_1$.
取 $M_2 = \max\{a_n, 2\}$, $\exists n_2 \in \mathbb{N}_+$ s.t. $a_{n_2} > M_2$.
以此类推, 构造数列 $\{a_{n_k}\}$. s.t.$a_{n_k} > k$. 即 a_{n_k} 为无穷大量.

Proof 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散.
构造 a_n 的发散子列即可. 已知 $\tan\frac{\pi}{2} = \infty$, π 是一个无理数, 因此存在数列 $\{b_n\}$, $\lim\limits_{n\to\infty} b_n = \frac{\pi}{2}$.

Proof 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散. 参考别人的答案

由于 $\{\sin 2n\}$ 极限不存在, 又

$$\begin{aligned}\sin 2n = 2\sin n\cos n &= \frac{2\sin n\cos n}{\sin^2 n+\cos^2 n}\\ &= \frac{2\tan n}{\tan^n+1}\end{aligned}$$

若 $\{\tan n\}$ 极限存在 $\to \{\sin 2n\}$ 极限存在, 矛盾.
故 $\{\tan n\}$ 极限不存在.

Question 8 2.2.4-9 $S_n = \frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}, \quad n\in\mathbb{N}_+$. S_n 在 1. $p\leqslant 0$, 2. $0<p<1$ 情况下均发散

Proof 1. $p\leqslant 0$. $\lim\limits_{n\to\infty} n^{-p}=\infty$, S_n 发散.
2. $0<p<1$. $\frac{1}{n^p}>\frac{1}{n}$. $\because H_n=\sum_{k=1}^n\frac{1}{k}$(调和级数) 发散, $S_n>H_n, \therefore \{S_n\}$ 也发散.

ex2.3.5 $0<b<a$ 令 $a_0=a, b_0=b$ 递推公式

$$a_n=\frac{a_{n-1}+b_{n-1}}{2}, b_n=\sqrt{a_{n-1}b_{n-1}}, \quad n\in\mathbb{N}_+ \tag{1.1}$$

定义数列a_n,b_n. 证明这两个数列收敛于同一个极限 $AG(a,b)$.

由 AG 不等式 $a>\frac{a+b}{2}>\sqrt{ab}>b>0$, 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a,b)=\frac{\pi}{2G} \tag{1.2}$$

如果令 $a_1=\frac{a+b}{2}, b_1=\sqrt{ab}$. 则

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta+b_1^2\sin^2\theta}} \tag{1.3}$$

上面这个公式是怎么得到的:
参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}} \qquad (a>b>0) \tag{1.4}$$

这里令

$$\sin\phi=\frac{2a\sin\theta}{(a+b)+(a-b)\sin^2\theta} \tag{1.5}$$

$\theta\in[0,\frac{\pi}{2}]\to\phi\in[0,\frac{\pi}{2}]$, 取微分

$$\cos\phi\mathrm{d}\phi=2a\frac{(a+b)-(a-b)\sin^2\theta}{[(a+b)+(a-b)\sin^2\theta]^2}\cos\theta\mathrm{d}\theta \tag{1.6}$$

但是

$$\cos\phi=\frac{\sqrt{(a+b)^2-(a-b)^2\sin^2\theta}}{(a+b)+(a-b)\sin^2\theta}\cos\theta. \tag{1.7}$$

(1.6) / (1.7), 两式相除, 得到

$$\mathrm{d}\phi=2a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta}\frac{\mathrm{d}\theta}{\sqrt{(a+b)^2-(a-b)^2\sin^2\theta}} \tag{1.8}$$

另一方面

$$\sqrt{a^2\cos\phi+b^2\sin^2\phi}=a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta} \tag{1.9}$$

因而

$$\frac{\mathrm{d}\phi}{\sqrt{a^2\cos\phi+b^2\sin^2\phi}}=\frac{\mathrm{d}\theta}{\sqrt{(\frac{a+b}{2})^2\cos^2\theta+ab\sin^2\theta}}. \tag{1.10}$$

如果令 $a_1=\frac{a+b}{2}, b_1=\sqrt{ab}$, 则

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta+b_1^2\sin^2\theta}} \tag{1.11}$$

反复应用该公式, 得到

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a_n^2\cos^2\phi+b_n^2\sin^2\phi}},\qquad (n=1,2,3,\dots) \tag{1.12}$$

$$\frac{\pi}{2a_n}<G<\frac{\pi}{2b_n} \tag{1.13}$$

积分 G 可以归结到第一类全椭圆积分 $K(k)=(1+k_1)K(k_1)=\frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}=(1+k_1)\int_0^{\frac{pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{1-k_1^2\sin^2\theta}} \tag{1.14}$$

其中

$a_1=\frac{1+\sqrt{1-k^2}}{2}=\frac{1+k'}{2}, b_1=\sqrt{k'}$

$k_1=\frac{\sqrt{a_1^2-b_1^2}}{a_1}=\frac{1-k'}{1+k'}, \frac{1}{a_1}=1+k_1$

1.3 2.3 单调数列

Example 1.6 2.3.6

$$\begin{aligned}\frac{a_{n+1}}{a_n}&=\frac{\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\cdots+n!}{n!}}\\&=\frac{1}{n+1}\frac{1!+2!+\cdots+(n+1)!}{1!+2!+\cdots+n!}\\&=\frac{3+3!+\cdots+(n+1)!}{(n+1)1!+(n+1)2!+\cdots+(n+1)!}\end{aligned}$$

Solve 12 $n>2$ 时, 分母每一项大于等于分子对应项.. $n>2$ 后a_n 单调减少. 由于 0 是下界, 因此 a_n 单调有界, 数列收敛.

$$\begin{aligned}a_{n+1}&=\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}\\&=\frac{1!+2!+\cdots+n!}{n!}\frac{1}{n+1}+1\\&=1+\frac{a_n}{n+1}\end{aligned}$$

设 $n\to\infty$ 时, $a_n\to a$

$$a=1+\left(\frac{1}{n+1}\to 0\right)=1+0,\qquad \therefore a=1$$

$\therefore \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1!+2!+\cdots+n!}{n!}=1$

1.3.1 2.3.2 练习题

Question 9 证明, 若x_n 单调, 则$|x_n|$ 至少从某项开始后单调, 又问: 反之如何?

Proof 分类讨论, 不妨设 $x_1\geqslant 0$

1. x_n 单调递增, $|x_n|$ 从第一项开始单调.

2. x_n 单调递减, 且 $|x_n| \geqslant 0$. $|x_n|$ 从第一项开始单调.
3. x_n 单调递减, 且 $\exists N$ s.t. $x_n < 0$(第一个负数项). 则$|x_n|$ 从第 N 项 (x_N) 开始单调.
反之该结论不成立.
反例: $x_n = \frac{(-1)^n}{n}$, $|x_n|$ 单调递减. 但 $x_{2k} = \frac{1}{2k} > 0 > x_{2k-1} = \frac{-1}{2k-1}$

Question 10 设a_n 单调增加, b_n 单调减少, 且有 $\lim\limits_{n\to\infty}(a_n - b_n) = 0$.
证明: 数列a_n 和b_n 都收敛, 且极限相等.

Proof $\lim\limits_{n\to\infty}(a_n - b_n) = 0, \forall\epsilon > 0, \exists N \in \mathbb{N}_+, \text{s.t.} \forall n > N, |a_n - b_n - 0| < \epsilon$.
$b_n - \epsilon < a_n < b_n + \epsilon$, 同时有 $a_n - \epsilon < b_n < a_n + \epsilon$.
b_n 单调减少, $\therefore \exists N_2, \forall m < N_2$, $b_m > b_n + \epsilon$.
使用反证法证明 b_m 是a_n 的上界.
假设 b_m 不是a_n 的上界, 则存在 $a_n > b_m > b_n + \epsilon$, 这与 $|a_n - b_n| < \epsilon$ 矛盾.
$\therefore$ b_m 是a_n 的上界, 根据单调有界收敛准则, a_n 收敛. 同理可证b_n 收敛. $\lim\limits_{n\to\infty}(a_n - b_n) = 0$.
$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

Question 11 按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.
2. 单调减少有下界的数列的极限不大于数列中的任何一项.

Question 12 设 $x_n = \frac{2}{3} \cdot \frac{3}{5} \cdots \frac{n+1}{2n+1}$, $n \in \mathbb{N}_+$, 求数列x_n 的极限.

Solve 13

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)+1}{2(n+1)+1} = \frac{n+2}{2n+3} < 1. \qquad (n > 0) \tag{1.15}$$

x_n 单调递减. $\because x_n > 0$, $\therefore x_n$ 有下界, x_n 收敛.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$, 由夹逼定理, $\lim\limits_{n\to\infty} x_n = 0$

Question 13 6. 在例题 2.2.6 的基础上证明: 当 $p > 1$ 时, 数列S_n 收敛. 其中

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$$

(S_n 就是 p 级数, 当 $p = 1$ 时为调和级数.)

Proof S_n 单调递增, 记 $\frac{1}{2^{p-1}} = r$, 则 $0 < r < 1$.

$$\begin{aligned}
\frac{1}{2^p} + \frac{1}{3^p} &< \frac{1}{2^p} + \frac{1}{2^p} && = \frac{1}{2^{p-1}} = r \\
\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} &< \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} && = \frac{1}{4^{p-1}} = r^2 \\
\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} &< \frac{1}{(2^k)^p} + \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^k)^p} && = \frac{1}{(2^k)^{p-1}} = r^k
\end{aligned}$$