具体数学阅读笔记-chap2

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1 求和

1.1 求和符号

1.2 求和与递归式 Sums and recurrences

和式

$$S_n = \sum_{k=0}^{n} a_k \tag{1}$$

等价于递归式

$$\begin{cases} S_0 = a_0 \\ S_n = S_{n-1} + a_n, \quad n > 0. \end{cases} \tag{2}$$

若 $a_n = const. + k \cdot n$, 则有

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n, \quad n > 0 \end{cases} \tag{3}$$

$$
\begin{aligned}
R_1 &= R_0 + \beta + \gamma \\
R_2 &= R_0 + 2\beta + 3\gamma \\
&\ \vdots \\
R_n &= A(n)\alpha + B(n)\beta + C(n)\gamma
\end{aligned}
$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \tag{4}$$

repertoire method 令 $R_n = 1$, 则 $\alpha = 1, \beta = 0, \gamma = 0$,

$$A(n) = 1$$

令 $R_n = n$, 则 $\alpha = 0, \beta = 1, \gamma = 0$,

$$B(n) = n$$

令 $R_n = n^2$, 则 $\alpha = 0, \beta = -1, \gamma = 2$,

$$C(n) = \frac{n(n+1)}{2}$$

例 1.1

$$\sum_{k=0}^{n}(a + bk)$$

解 1

$$\begin{cases} R_0 = a \\ R_n = R_{n-1} + a + bn \end{cases}$$

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n \end{cases}$$

$\alpha = \beta = a, \gamma = b$

$$
\begin{aligned}
A(n)\alpha + B(n)\beta + C(n)\gamma &= aA(n) + aB(n) + bC(n) \\
&= a + an + b\frac{n(n+1)}{2} \\
&= a(n+1) + \frac{bn(n+1)}{2}
\end{aligned}
$$

对上述递归情况进行推广

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n + \delta n^2, \quad n > 0 \end{cases} \tag{5}$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta \tag{6}$$

$\delta = 0$ 时 (5) 与 (3) 一致, 说明 $A(n), B(n), C(n)$ 不变

$R_n = n^3$

$$\begin{aligned} R_n = R_{n-1} &= n^3 - (n-1)^3 \\ &= 3n^2 - 3n + 1 \end{aligned}$$

解得 $\alpha = 0, \beta = 1, \gamma = -3, \delta = 3$

$$\begin{aligned} n^3 &= B(n) - 3C(n) + 3D(n) \\ &= n - 3\frac{n(n+1)}{2} + 3D(n) \end{aligned}$$

$$\begin{aligned} 3D(n) &= n^3 - n + 3\frac{n(n+1)}{2} \\ &= n(n+1)\left[(n-1) + \frac{3}{2}\right] \\ &= n(n+1)(n+\frac{1}{2}) \\ D(n) &= \frac{1}{3}\left((n+1)(n+\frac{1}{2})n\right) \end{aligned}$$

1.3 求和式处理

1.4 多重求和

1.4.1 Exercise 1

$$A = \begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{bmatrix} \tag{7}$$

求 $S_{\triangleleft} = \sum_{1\leqslant j\leqslant k\leqslant n} a_ja_k$ [1]

解 2 $\because a_ja_k = a_ka_j$, $\therefore$ 矩阵 A 沿主对角线对称, $S_{\triangleleft} = S_{\triangleright}$.

$$[1\leqslant j\leqslant k\leqslant n] + [1\leqslant k\leqslant j\leqslant n] = [1\leqslant j,k\leqslant n] + [1\leqslant j=k\leqslant n]$$

$$\begin{aligned} 2S_{\triangleleft} = S_{\triangleleft} + S_{\triangleright} &= S_A + S_{diag(A)} \\ &= \sum_{1\leqslant j,k\leqslant n} a_ja_k + \sum_{1\leqslant j=k\leqslant n} a_ja_k \\ &= \left(\sum_{j=1}^n a_j\right)\left(\sum_{k=1}^n a_k\right) + \sum_{k=1}^n a_k^2 \\ &= \left(\sum_{k=1}^n a_k\right)^2 + \sum_{k=1}^n a_k^2 \end{aligned}$$

$\therefore S_{\triangleleft} = \frac{1}{2}[(\sum_{k=1}^n a_k)^2 + \sum_{k=1}^n a_k^2]$

1.4.2 Exercise 2

$$S = \sum_{1\leqslant j<k\leqslant n} (a_k - a_j)(b_k - b_j) \tag{8}$$

[1]下三角形矩阵的符号是一个右上部分的直角三角形, 目前我还不会输入

解 3 交换 j,k 仍有对称性.

$$S=\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)=\sum_{1\leqslant j<k\leqslant n}(a_j-a_k)(b_j-b_k)$$

$$[1\leqslant j<k\leqslant n]+[1\leqslant k<j\leqslant n]=[1\leqslant j,k\leqslant n]-[1\leqslant j=k\leqslant n]$$

$$\begin{aligned}
2S&=2\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)\\
&=\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)+\sum_{1\leqslant k<j\leqslant n}(a_k-a_j)(b_k-b_j)\\
&=\sum_{1\leqslant j,k\leqslant n}(a_k-a_j)(b_k-b_j)-\sum_{1\leqslant j=k\leqslant n}(a_k-a_j)(b_k-b_j)
\end{aligned}$$

$$(a_j-a_k=0,b_j-b_k=0,[j=k]) \qquad =\sum_{1\leqslant j,k\leqslant n}(a_kb_k-a_jb_k-$$

$$\begin{aligned}
&=\sum_{j=1}^{n}\sum_{k=1}^{n}a_kb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_jb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_kb_j+\sum_{j=1}^{n}\sum_{k=1}^{n}a_jb_j\\
&=n\sum_{k=1}^{n}a_kb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_jb_k-\sum_{j=1}^{n}\sum_{k=1}^{n}a_kb_j+n\sum_{j=1}^{n}a_jb_j\\
&=2n\sum_{k=1}^{n}a_kb_k-2\sum_{j=1}^{n}a_j\sum_{k=1}^{n}b_k
\end{aligned}$$

$$S=n\sum_{k=1}^{n}a_kb_k-\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)$$

对上式结果重新排序得

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)=n\sum_{k=1}^{n}a_kb_k-\sum_{1\leqslant j<k\leqslant n}(a_k-a_j)(b_k-b_j)$$

定理 1.1 切比雪夫单调不等式 (Chebyshec's monotonic inequality)

$$
\begin{array}{lll}
\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) & \leqslant n\sum_{k=1}^{n} a_k b_k & a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n, \text{and } b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n \\
 & & a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n, \text{and } b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \\
\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) & \geqslant n\sum_{k=1}^{n} a_k b_k & a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n, \text{and } b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \\
 & & a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n, \text{and } b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n
\end{array}
$$

一般来说，如果 $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ 且 p 是 $\{1,...,n\}$ 的一个排列。
那么不难证明：
当 $b_{p(1)} \leqslant \cdots \leqslant b_{p(n)}$ 时 $\sum_{k=1}^{n} a_k b_{P(k)}$ 最大.
当 $b_{p(1)} \geqslant \cdots \geqslant b_{p(n)}$ 时 $\sum_{k=1}^{n} a_k b_{P(k)}$ 最小.

$$
\sum_{k\in K} a_k = \sum_{P(k)\in K} a_{P(k)}
$$

$P(k)$ 为这些整数的任意一个排列。

$$
f : J \to K, \quad j \in J \quad f(j) \in K
$$

$$
\sum_{j\in J} a_{f(j)} = \sum_{k\in K} a_k \quad \#f^-(k)
$$

式中 $\#f^-(k)$ 表示集合 $f^-(k) = \{j|f(j) = k\}$ 中元素的个数

$$
\sum_{j\in J}[f(j) = k] = \#f^-(k)
$$

$$
\sum_{j\in J} a_{f(j)} = \sum_{\substack{j \in J \\ k \in K}} a_k[f(j) = k] = \sum_{k\in K} a_k \sum_{j\in J}[f(j) = k]
$$

若有 $\#f^-(k) = 1$ (一一对应)[2]

$$
\sum_{j\in J} a_{f(j)} = \sum_{f(j)\in K} a_{f(j)} = \sum_{k\in K} a_k
$$

[2]这里还不太理解

1.4.3 Exercise 3

$$S_n = \sum_{1\leqslant j<k\leqslant n} \frac{1}{k-j}$$

首先写出前几项，尝试寻找规律：

$$\begin{aligned}
S_1 &= 0 \\
S_2 &= \frac{1}{2-1} = 1 \\
S_3 &= \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{3-2} = \frac{5}{2} \\
S_4 &= \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{3-2} + \frac{1}{4-2} + \frac{1}{4-3} = \frac{13}{3}
\end{aligned}$$

解 4 1. 先对 j 求和

$$\begin{aligned}
S_n &= \sum_{1\leqslant k\leqslant n} \sum_{1\leqslant j<k} \frac{1}{k-j} \\
&= \sum_{1\leqslant k\leqslant n} \sum_{1\leqslant (k-j)<k} \frac{1}{k-(k-j)} \quad j \to (k-j) \\
&= \sum_{1\leqslant k\leqslant n} \sum_{0<j\leqslant k-1} \frac{1}{j} \\
&= \sum_{1\leqslant k\leqslant n} H_{k-1} \quad (H_k \text{为调和级数}) \\
&= \sum_{1\leqslant k+1\leqslant n} H_k \quad k \to k+1 \\
&= \sum_{0\leqslant k<n} H_k
\end{aligned}$$

2. 先对 k 求和

$$
\begin{aligned}
S_n &= \sum_{1\leqslant j\leqslant n}\sum_{j<k\leqslant n}\frac{1}{k-j} \\
&= \sum_{1\leqslant j\leqslant n}\sum_{j<(k+j)\leqslant n}\frac{1}{(k+j)-j} \quad k\to(k+j) \\
&= \sum_{1\leqslant j\leqslant n}\sum_{0<k\leqslant n-j}\frac{1}{k} \\
&= \sum_{1\leqslant j\leqslant n} H_{n-j} \quad (H_k\text{为调和级数}) \\
&= \sum_{1\leqslant n-j\leqslant n} H_k \quad j\to n-j \\
&= \sum_{0\leqslant j<n} H_j
\end{aligned}
$$

以上两种常用的求和顺序都无法得到这个多重求和的结果，我们需要转换思路.

3. 先用 $k+j$ 替换 k (先换元，再求和)

$$
\begin{aligned}
S_n &= \sum_{1\leqslant j<(k+j)\leqslant n}\frac{1}{(k+j)-j} \quad k\to k+j \\
&= \sum_{1\leqslant j<(k+j)\leqslant n}\frac{1}{k} \\
&= \sum_{1\leqslant k\leqslant n}\sum_{1\leqslant j\leqslant n-k}\frac{1}{k} \quad \text{首先对}j\text{求和} \\
&= \sum_{1\leqslant k\leqslant n}\frac{n-k}{k} \\
&= \sum_{1\leqslant k\leqslant n}\left(\frac{n}{k}-1\right) = nH_n-n
\end{aligned}
$$

综上可得 $\sum_{1\leqslant k\leqslant n} H_k = nH_n - n$

代数：$k+f(j)$, f 为任意函数.

用 $k-f(j)$ 替换 k ，并对 j 先求和较好。

几何：$S_n\ (n=4)$

$$\begin{array}{ccccc} & k=1 & k=2 & k=3 & k=4 \\ j=1 & & \frac{1}{1} & +\frac{1}{2} & +\frac{1}{3} \\ j=2 & & & +\frac{1}{1} & +\frac{1}{2} \\ j=3 & & & & +\frac{1}{1} \\ j=4 & & & & \end{array}$$

先对 j 求和 (按列) $H_1+H_2+H_3$ 先对 k 求和 (按行) $H_3+H_2+H_1$ $k\to k+j$ 按对角线求和

$$\sum_{k=1}^{n}\frac{n-k}{k}=n\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}1$$

nH_n-n ,$n=4$

$$\begin{aligned}\frac{4}{1}+\frac{3}{2}+\frac{2}{3}+\frac{1}{4}&=\sum_{k=1}^{4}\frac{4-k}{k}\\&=4\sum_{k=1}^{4}\frac{1}{k}-\sum_{k=1}^{4}1\\&=4H_4-4\end{aligned}$$

$$4\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-4=\frac{4}{2}+\frac{4}{3}+\frac{4}{4}$$

$$\begin{array}{ccccc} & k-j=0 & k-j=1 & k-j=2 & k-j=3 \\ j=1 & & \frac{1}{1} & +\frac{1}{2} & +\frac{1}{3} \\ j=2 & & \frac{1}{1} & +\frac{1}{2} & \\ j=3 & & \frac{1}{1} & & \\ j=4 & & & & \end{array}$$

1.5 General methods

1.6 Exercise 4

求 $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$, $n \geqslant 0$ 的封闭形式

$$\begin{aligned}\sum_{k=0}^{n} k^2 &= \sum_{k=0}^{n}[(k+1)^2 - 2k - 1] \\ &= \sum_{k=1}^{n+1} k^2 - 2\sum_{k=0}^{n} k - \sum_{k=0}^{n} 1\end{aligned}$$

$$\begin{aligned}0^2 - (n+1)^2 &= -2\sum_{k=0}^{n} k - (n+1) \\ 2\sum_{k=0}^{n} k &= (n+1)^2 - (n+1) \\ \sum_{k=0}^{n} k &= \frac{(n+1)n}{2}\end{aligned}$$

上述运算没有告诉我们 $\Box_n$ 的值，但却能推导出 $\sum_{k=0}^{n} k$ 的值。我们可以利用这种思路求解 $\Box_n$ 。

$$\begin{aligned}\sum_{k=0}^{n}\left[(k+1)^3 - k^3\right] &= \sum_{k=0}^{n}\left[3k^2 + 3k + 1\right] \\ (n+1)^3 - 0^3 &= 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 \\ (n+1)^3 &= 3\sum_{k=0}^{n} k^2 + 3\frac{n(n+1)}{2} + (n+1)\end{aligned}$$

$$3\sum_{k=0}^{n} k^2 = (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)$$
$$\sum_{k=0}^{n} k^2 = \frac{1}{3}(n+1)\left((n+1)^2 - \frac{3}{2}n - 1\right)$$
$$\sum_{k=0}^{n} k^2 = \frac{1}{3}(n+1)(n+\frac{1}{2})n$$

reference book list:

1. (CRC Tables) CRC Standard Mathematical Tables
2. Handbook of Mathematical Functions
3. Sloane. Handbook of Integer Sequences

software:

Axiom MACSYMA Maple Mathematica

my: Octave maxima 熟悉标准的信息源

方法 3：建立成套方法

参考第二节的内容

方法 4：用积分替换和式 $\sum \to \int$

$$\square_n = 1\times 1 + 1\times 4 + 1\times 9 + \cdots + 1\times n^2$$

该式近似等于 0 到 n 之间曲线 $f(x) = x^2$ 下的面积

$$S = \int_0^n x^2 dx = \frac{n^3}{3}$$

$\square_n$ 近似等于 $\frac{n^3}{3}$ 。近似的误差 $E_n = \square_n - \frac{n^3}{3}$