又一个三元不等式

单墫

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$a, b, c > 0$

$$a^2 + b + c = 11 \tag{1}$$

$$abc = 3 \tag{2}$$

证明

$$\sqrt{a+b} + \sqrt{a+c} + \sqrt{b+c} \geq 4 + \sqrt{2} \tag{3}$$

解：$a = 3, b = c = 1$，此时 (3) 中等号成立

仍用枚举法

已知 $a \leq 3$（否则 (1),(2) 不能同时成立）–> why?

1. 若 $a \leq 1$ 则

$$b + c = 11 - a^2 \geq 10 \tag{4}$$

$$(\sqrt{a+b} + \sqrt{a+c})^2 = 2a + b + c + 2\sqrt{a+b}\sqrt{a+c} > b + c \geq 10 \tag{5}$$

$$\sqrt{a+b} + \sqrt{a+c} + \sqrt{b+c} \geq \sqrt{10} + \sqrt{10} = 2\sqrt{10} > 4 + \sqrt{2} \tag{6}$$

2. 若 $1 < a \leq 3$，则

$$b + c = 11 - a^2 \geq 2, \quad bc = \frac{3}{a} \geq 1 \tag{7}$$

$$\text{let} x = a + b, y = a + c \tag{8}$$

$$x + y = 2a + b + c = 2a + 11 - a^2 \tag{9}$$

$$xy = (a+b)(a+c) = a^2 a(b+c) + bc \geq a^2 + 2a + 1 = (a+1)^2 \tag{10}$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{(\sqrt{x} + \sqrt{y})^2} \geq \sqrt{2a + 11 - a^2 + 2(a+1)} \tag{11}$$

$$\sqrt{2a + 11 - a^2 + 2(a+1)} = \sqrt{13 - a^2 + 4a} = \sqrt{17 - (a-2)^2} \tag{12}$$

$$-1 < a - 2 \leq 1 \quad (a-2)^2 \leq 1 \quad 17 - (a-2)^2 \geq 16 \therefore \sqrt{x} + \sqrt{y} \geq 4 \tag{13}$$

$$\therefore \sqrt{a+b} + \sqrt{a+c} + \sqrt{b+c} \geq 4 + \sqrt{2} \tag{14}$$