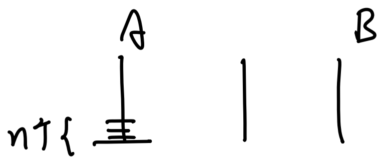
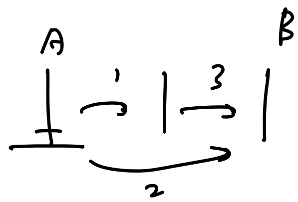
# 递归问题 (Recurrent Problems)

## 1.1 河内塔



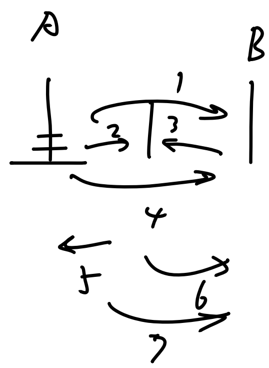

$T_0 = 0$




$T_1 = 1$

$T_2 = 3$

$T_3 = 7$




$$\left.\begin{array}{l} T_n \le 2T_{n-1}+1 \\ T_n \ge 2T_{n-1}+1 \end{array}\right\} T_n = 2T_{n-1}+1$$

$$\begin{cases} T_0 = 0 \\ T_n = 2T_{n-1}+1, \quad n>0 \end{cases} \quad (1.1)$$

递归式 recurrence

$$T_n = 2^n - 1 \quad , \quad n \geq 0 \qquad (1.2)$$

数学归纳法 mathematical induction

$\left\{\begin{array}{l} \text{1. 基础 basis} \quad n\text{取最小值}n_0\text{ 证明该命题} \\ \text{2、归纳 induction} \quad \text{假设 }n_0 \sim n-1\text{ 成立，证明 对 }n\text{ 也成立} \end{array}\right.$

$$T_{n-1} = 2^{n-1} - 1$$

$$T_n = 2T_n + 1 = 2 \cdot (2^{n-1} - 1) + 1 = 2^n - 1$$

① 研究小的情形

② 递归式 (1.1) $\quad \left\{\begin{array}{l} T_0 = 0 \\ T_n = 2T_{n-1} + 1 \quad , \quad n > 0 \end{array}\right.$

③ 递归解 (1.2) $\quad T_n = 2^n - 1 \quad , \quad n \geq 0$

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$$T_0 + 1 = 1$$

$$T_n + 1 = 2T_{n-1} + 2 = 2(T_{n-1} + 1)$$

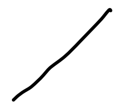
令 $U_n = T_n + 1$

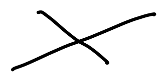
$$\left\{\begin{array}{l} U_0 = 1 \\ U_n = 2U_{n-1} \quad , \quad n > 0 \end{array}\right. \Rightarrow \begin{array}{l} U_n = 2^n \\ T_n = 2^n - 1 \end{array}$$

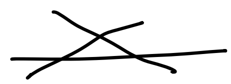
## 1.2 平面上的直线

平面上 $n$ 条直线所界定的区域最大个数 $L_n$ 是多少.

$L_0 = 1$

$L_1 = 2$ 

$L_2 = 4$ 

$L_3 = 7$ 

$\vdots$

$L_n \le L_{n-1} + n \quad , \quad n > 0$

$\underline{L_n = L_{n-1} + n}$

$= L_{n-2} + (n-1) + n$

$= \cdots$

$= L_0 + 1 + 2 + \cdots + n$

$= 1 + S_n. \qquad \underset{\text{三角形数}}{\underline{S_n}} = 1 + 2 + \cdots + n$

$S_n = \dfrac{n(n+1)}{2} \qquad L_n = \dfrac{n(n+1)}{2} + 1$

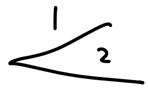
$L_n = L_{n-1} + n = \dfrac{(n-1) \cdot n}{2} + 1 + n = \dfrac{n(n+1)}{2} + 1$

方程是封闭的。

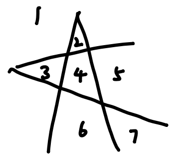
不用它本身定义 — 不产生递归式

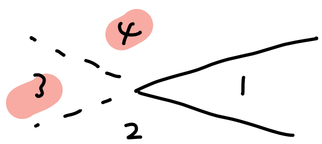
折：直线 ⇒ 折线

$Z_1 = 2$




$Z_2 = 7$




$Z_n = L_{2n} - 2n$ （锯齿点不在交点）

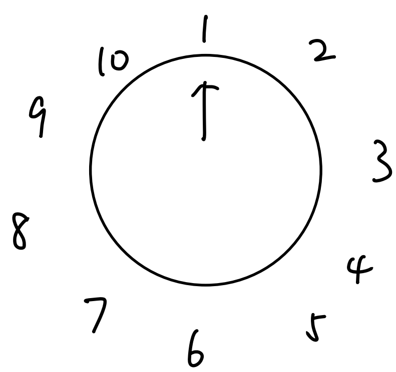
$= \frac{2n(2n+1)}{2} + 1 - 2n$

$= 2n^2 - n + 1$, $n \geq 0$

$L_n \sim \frac{1}{2}n^2$

$Z_n \sim 2n^2$

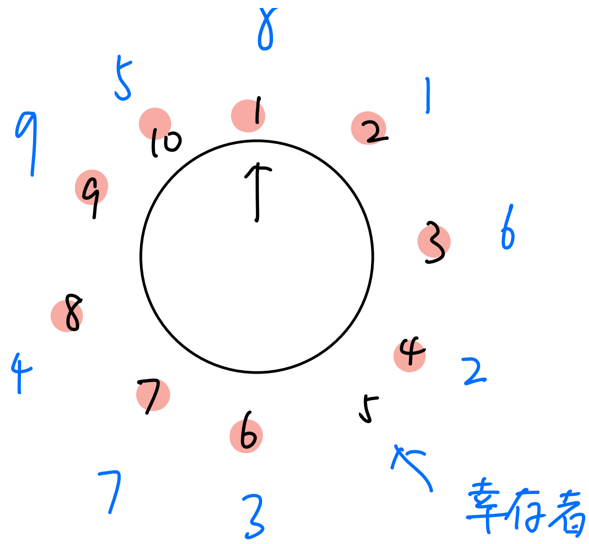
## 1.3、约瑟夫问题




$n$个人

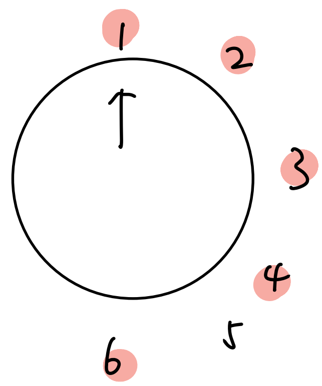
隔一个删一个

n

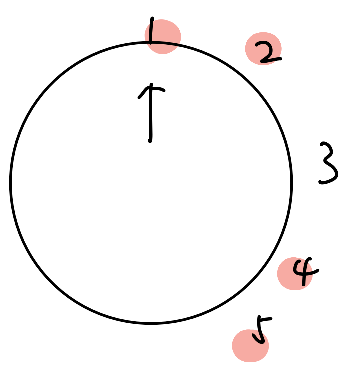



$2 \to 4 \to 6 \to 8 \to 10$

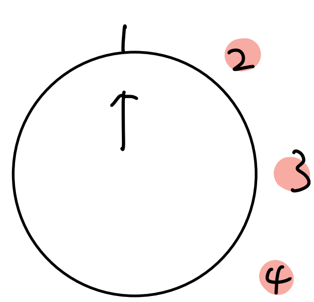
$\to 3 \to 7 \to 1 \to 9$



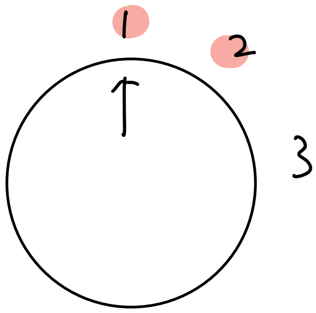
$J(6) = 5$



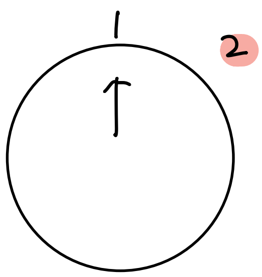
$J(5) = 3$



$J(4) = 1$




$J(3) = 3$



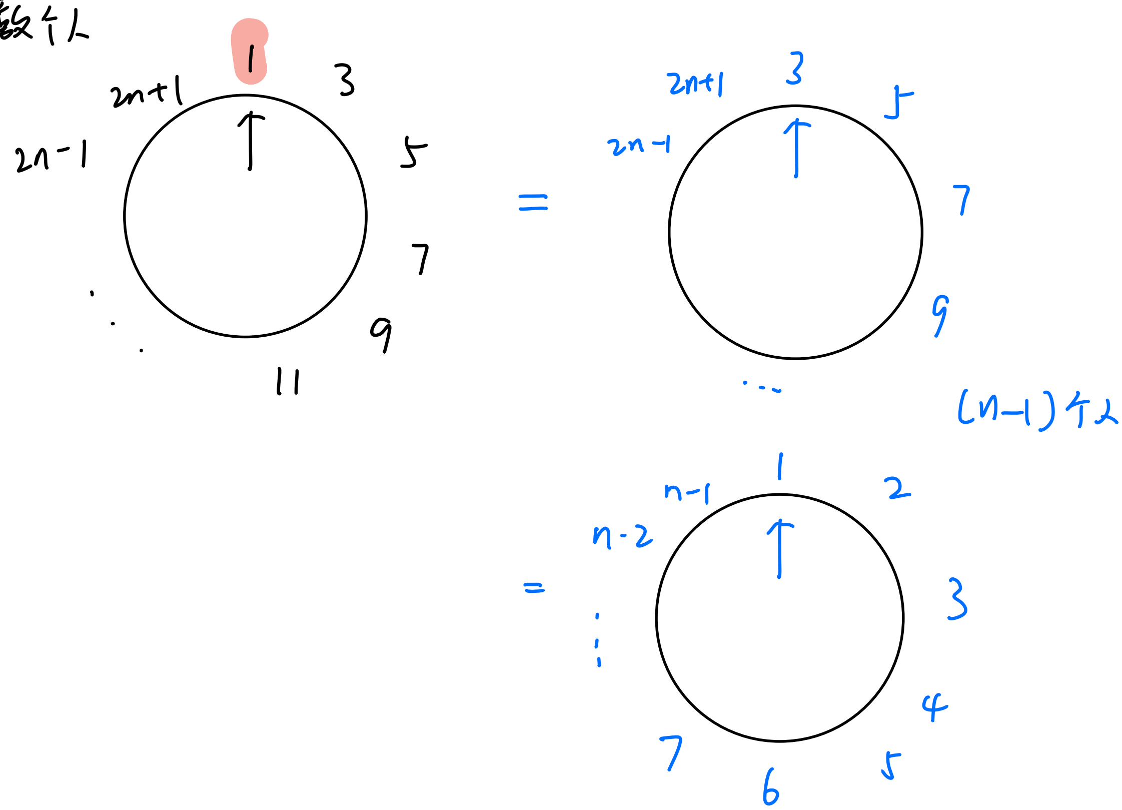

$J(2) = 1$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $J(n)$ | 1 | 1 | 3 | 1 | 3 | 5 |

偶数个人

$$J(2n) = 2J(n) - 1 \qquad n \geqslant 1$$

奇数个人




$$J(2n+1) = 2J(n)+1 \qquad n \geq 1$$

递归式

$$\begin{cases} J(1) = 1 \\ J(2n) = 2J(n)-1, & n \geq 1 \\ J(2n+1) = 2J(n)+1, & n \geq 1 \end{cases}$$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $J_n$ | 1 | 1 | 3 | 1 | 3 | 5 | 7 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 1 | 3 | 5 |

$$n = 2^m + l$$

封闭形式

$$J(2^m+l) = 2l+1 \quad , \quad m \geq 0 \quad 0 \leq l < 2^m$$

proof

i) $l$ even

$$J(2^m+l) = 2J(2^{m-1}+\frac{l}{2}) - 1$$

$$= 2 \cdot (2 \cdot \frac{l}{2} + 1) - 1 = 2l+1$$

ii) $l$ odd

$$J(2^m+l) = 2J(2^{m-1}+\frac{l-1}{2}) + 1$$

$$= 2(2 \cdot \frac{l-1}{2} + 1) + 1 = 2l+1 \quad . \square$$

$$J(2n+1) - J(2n) = 2$$

推广 $n$和$J(n)$以2为基数表示

设 $n = (b_m b_{m-1} \cdots b_1 b_0)_2$

$$= b_m 2^m + b_{m-1} 2^{m-1} + \cdots + b_1 2^1 + b_0 \cdot 2^0$$

$b_m = 1$. $\quad b_i = 0$或$1$ $(0 \leq i < m, i \in N^+)$

$n = 2^m + l$

$n = (1\, b_{m-1}\, b_{m-2} \cdots b_1\, b_0)_2$

$l = (0\, b_{m-1}\, b_{m-2} \cdots b_1\, b_0)_2$

$2l = (b_{m-1}\, b_{m-2} \cdots b_1\, b_0\, 0)_2$

$2l+1 = (b_{m-1}\, b_{m-2} \cdots b_1\, b_0\, 1)_2$

$J(n) = (b_{m-1}\, b_{m-2} \cdots b_1\, b_0\, b_m)_2$

$J((b_m\, b_{m-1} \cdots b_1\, b_0)_2) = (b_{m-1}\, b_{m-2} \cdots b_1\, b_0\, b_m)_2$

n向左循环移动一位得到 J(n)！

$J((1011)_2) = (0111)_2 = (111)_2$

0移至首位会消失.

迭代足够多次数 → 所有位数字均为1.

$2^{v(n)} - 1$.　　$v(n)$ 为 n 二进制中1的个数.

ex　$n = 13$　　$(13)_{10} = (1101)_2$　　$v(13) = 3$.

$J(J(\cdots J(13)\cdots)) = 2^3 - 1 = 7$.

ex $n=23403$ $\qquad \nu(23403)=10$

$$J(J\cdots J((101101101101011)_2)\cdots))=2^{10}-1=1023$$

$\longrightarrow$ 回到第一个猜测 $J(n)=\frac{n}{2}$ $(n,\ \text{even})$

它在什么情况下成立？

已知
$n=2^m+l$ , $J(n)=2l+1$

$$J(n)=\frac{n}{2}$$

$$2l+1=(2^m+l)/2$$

$$\frac{3}{2}l=2^{m-1}-1$$

$$l=\frac{1}{3}(2^m-2)$$

$m$, odd $\qquad 2^m-2$ 是3的倍数.

$m$, even $\qquad$ 不是 .

$J(n)=\frac{n}{2}$ 有无穷多组解

| $m$ | $\ell$ | $n=2^m+\ell$ | $J(n)=2\ell+1=\frac{n}{2}$ | $(n)_2$ |
|---|---|---|---|---|
| 1 | 0 | 2 | 1 | 10 |
| 3 | 2 | 10 | 5 | 1010 |
| 5 | 10 | 42 | 21 | 101010 |
| 7 | 42 | 170 | 85 | 10101010 |

→ 再推广 引入常数 $\alpha.\beta.\gamma.$

$$\left.\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n)+\beta, \quad n\geqslant 1 \\ f(2n+1) &= 2f(n)+\gamma, \quad n\geqslant 1 \end{aligned}\right\} (1.11)$$

| $n$ | $f(n)$ |
|---|---|
| 1 | $\alpha$ |
| 2 | $2\alpha+\beta$ |
| 3 | $2\alpha+\gamma$ |
| 4 | $4\alpha+3\beta$ |
| 5 | $4\alpha+2\beta+\gamma$ |

| | | |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 0 |
| 2 | 0 | 1 |
| 4 | 3 | 0 |
| 4 | 2 | 1 |

| | |
|---|---|
| 6 | $4\alpha + \beta + 2\gamma$ |
| 7 | $4\alpha + 3\gamma$ |
| 8 | $8\alpha + 7\beta$ |
| 9 | $8\alpha + 6\beta + \gamma$ |

| | | |
|---|---|---|
| 4 | 1 | 2 |
| 4 | 0 | 3 |
| 8 | 7 | 0 |
| 8 | 6 | 1 |

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

看起来有
$$\begin{cases} A(n) = 2^m \\ B(n) = 2^m - 1 - l \\ C(n) = l \end{cases} \quad \left| \quad \begin{aligned} & n = 2^m + l \\ & 0 \le l < 2^m \\ & (n \geq 1) \end{aligned} \right.$$

归纳证明较繁 可选特殊值组合

① $\alpha = 1$. $\beta = \gamma = 0$. 此时 $f(n) = A(n)$

式(1.11)变为

$$\begin{cases} A(1) = 1 \\ A(2n) = 2A(n), & n \geq 1 \\ A(2n+1) = 2A(n), & n \geq 1 \end{cases}$$

此时由归纳法 $A(2^m+l)=2^m$.

② $f(n)=1$.

$$\begin{cases} 1=\alpha \\ 1=2\times 1+\beta \\ 1=2\times 1+\gamma \end{cases} \Rightarrow \begin{cases} \alpha=1 \\ \beta=-1 \\ \gamma=-1 \end{cases}$$

$$\underline{A(n)-B(n)-C(n)}=f(n)=1$$

③ $f(n)=n$.

$$\begin{cases} 1=\alpha \\ 2n=2\times n+\beta \\ 2n+1=2\times n+\gamma \end{cases} \Rightarrow \begin{cases} \alpha=1 \\ \beta=0 \\ \gamma=1 \end{cases}$$

$$\underline{A(n)+C(n)}=f(n)=n$$

$$\Rightarrow \begin{cases} A(n)=2^m \quad (n=2^m+l.\ 0\le l<2^m) \\ A(n)-B(n)-C(n)=1 \\ A(n)+C(n)=n\ (=2^m+l) \end{cases}$$

$$\Rightarrow \begin{cases} A(n) = 2^m \\ B(n) = 2^m - 1 - \ell \\ C(n) = \ell \end{cases} \qquad \square$$

以上为求解递归式的成套方法

(repertoire method)

约瑟夫递归式 二进制解

$$J((b_m b_{m-1} \cdots b_1 b_0)_2) = (b_{m-1} b_{m-2} \cdots b_1 b_0 b_m)_2 \quad \text{其中} b_m = 1$$

推广约瑟夫递归式有无类似解?

令 $\beta_0 = \beta$ , $\beta_1 = \gamma$

$$f(1) = \alpha$$

$$f(2n+j) = 2f(n) + \beta_j \quad , j = 0, 1 \quad , n \geq 1$$

$$\begin{aligned} f((b_m b_{m-1} \cdots b_1 b_0)_2) &= 2f((b_m b_{m-1} \cdots b_1)_2) + \beta_{b_0} \\ &= 4f((b_m b_{m-1} \cdots b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\ &\vdots \\ &= 2^m f((b_m)_2) + 2^{m-1} f((b_{m-1})_2) + \cdots + 2\beta_{b_1} + \beta_{b_0} \\ &= 2^m \alpha + 2^{m-1} \beta_{b_{m-1}} + \cdots + 2\beta_{b_1} + \beta_{b_0} \end{aligned}$$

$$f((b_m b_{m-1} \cdots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_1} \beta_{b_0})_2$$

| $n$ | $f(n)$ |
|---|---|
| 1 | $\alpha$ |
| 2 | $2\alpha+\beta$ |
| 3 | $2\alpha+\beta$ |
| 4 | $4\alpha+2\beta+\beta$ |
| 5 | $4\alpha+2\beta+\gamma$ |
| 6 | $4\alpha+2\gamma+\beta$ |
| 7 | $4\alpha+2\gamma+\gamma$ |

$n = 100 = (1100100)_2$ $\quad (\alpha, \beta, \gamma) = (1, -1, 1)$

$n = (1\ 1\ 0\ 0\ 1\ 0\ 0)_2 = 100$

$f(n) = (1\ 1\ {-1}\ {-1}\ 1\ {-1}\ {-1})_2$

$= +64+32-16-8+4-2-1 = 73$

?

$$\boxed{\begin{aligned} n &= (1\ 1\ 0\ 0\ 1\ 0\ 0)_2 = 100 \\ f(n) &= (\alpha\ \gamma\ \beta\ \beta\ \gamma\ \beta\ \beta)_2 \end{aligned}}$$

$$= (1\ 1\ {-1}\ {-1}\ 1\ {-1}\ {-1})_2$$

$$\ \ 6\ 5\ \ 4\ \ \ 3\ \ 2\ \ \ 1\ \ \ 0$$

$$= 2^6+2^5-2^4-2^3+2^2-2^1-2^0 \quad = 73$$

由于在n的二进制表示中每一块二进制数字

$(1\ 0\ 0\ 0\ \cdots\ 0)_2$ 都被变换成

$(1\,{-1}\,{-1}\,{-1}\ \cdots\ {-1})_2 = (0\ 0\ 0\ \cdots\ 0\ 1)_2$

因而推出循环移位性质

$$\boxed{\begin{aligned} f(j) &= \alpha_j & 1\le j<d \\ f(dn+j) &= c\,f(n)+\beta_j & 0\le j<d,\quad n\ge 1 \end{aligned}}$$

有变动基数的解

$$f((b_m b_{m-1}\cdots b_1 b_0)_d) = (\alpha_{b_m}\beta_{b_{m-1}}\cdots\beta_{b_1}\beta_{b_0})_c$$

基数为d　　　　基数为c

ex. $f(1)=34$

$f(2)=5$

$f(3n)=10f(n)+76, \quad n\ge 1$

$f(3n+1)=10f(n)-2, \quad n\ge 1$

$$f(3n+2) = 10f(n) + 8 \quad , \quad n \geq 1$$

计算 $f(19)$

solve: 这里有 $d=3$, $c=10$

$$19 = 2\times 3^2 + 1\times 3^0 = (201)_3$$

$$f(19) = f((201)_3) = (5 \quad 76 \quad -2)_{10}$$

$$= 5\times 10^2 + 76\times 10^1 + (-2)\times 10^0$$

$$= 500 + 760 - 2$$

$$= 1258$$

(第1章完)

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ex.

$$\begin{cases} f(1) = 34 \\ f(2) = 5 \\ f(3n) = 10f(n) + 76 \quad , \quad n \geq 1 \\ f(3n+1) = 10f(n) - 2 \quad , \quad n \geq 1 \\ f(3n+2) = 10f(n) + 8 \quad , \quad n \geq 1 \end{cases}$$

推广的约瑟夫递归式

计算 $f(19)$