1 2.1.5

Quetion 1. *1. prove by Limit definition:*
(1). $\lim_{n\to\infty} \frac{3n^2}{n^2-4} = 3$.
(2). $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.
(3). $\lim_{n\to\infty} (1+n)^{\frac{1}{n}} = 0$.
(4). $\lim_{n\to\infty} \frac{a^n}{n!} = 0, (a > 0)$.

Quetion 2. *2. Suppose* $a_n, n \in \mathbb{N}_+$. *sequence* a_n *converge to* a.
Prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Proof. $n \to \infty a_n \to a$.
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$|\sqrt{a_n} - \sqrt{a}| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$. □ (check, not consider the condition$a = 0$) add $a = 0, \forall \epsilon \in (0,1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$. s.t $a_n < \epsilon^2 < \epsilon$, $\sqrt{a_n} < \epsilon$. □

Quetion 3. *3. If* $\lim_{n\to\infty} a_n = a$.
Prove $\lim_{n\to\infty} |a_n| = |a|$. *Vice versa?*

Proof. $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$\Big||a_n| - |a|\Big| \leqslant |a_n - a| < \epsilon$$

$\therefore \lim_{n\to\infty} |a_n| = |a|$
If We know $\lim_{n\to\infty} |a_n| = |a|$.
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), \big||a_n| - |a|\big| < \epsilon$. We can't get $\lim_{n\to\infty} a_n = a$.
For example: $a_n = \frac{1}{n} + 1, a = -1$, $\lim_{n\to\infty} |a_n| = |a|$ is $\lim_{n\to\infty} |\frac{1}{n} + 1| = |-1|$, but $\lim_{n\to\infty} \frac{1}{n} + 1 \neq -1$ □

□

Quetion 4. *(1). Suppose* $p(x)$ *is a polynomial of* x, *if* $\lim_{n\to\infty} a_n = a$, *Prove* $\lim_{n\to\infty} p(a_n) = p(a)$.
(2). Suppose $b > 0$, $\lim\limits_{n\to\infty} a_n = a$. *Prove* $b^{a_n} = b^a$.
(3). Suppose $b > 0$, $\{a_n\}, a_n > 0, \forall n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a.a > 0$. *Prove* $\lim\limits_{n\to\infty} \log_b a_n = \log_b a$.
(4) Suppose $b \in \mathbb{R}$, $\{a_n\}, a_n > 0$ *when* $n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a$. *Prove* $\lim\limits_{n\to\infty} {a_n}^b = a^b$.

(5) Suppose $\lim_{n\to\infty} a_n = a$*. Prove* $\lim_{n\to\infty} \sin a_n = \sin a$*.*

Proof. 4.(1)
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geqslant N(\epsilon),\ |a_n - a| < \epsilon.$
$p(a) = k_m a^m + k_{m-1}a^{m-1} + \cdots + k_0 a^0.$
$\therefore p(a_n) - p(a) = k_m(a_n^m - a^m) + k_{m-1}(a_n^{m-1} - a^{m-1}) + \cdots + k_0(a_n^0 - a^0).$

$$\begin{aligned} |a_n^m - a^m| &= |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon(m-1)\cdots(a+\delta)^{m-1} \end{aligned}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a).$ □ □

Proof. 4.(2)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$
If $b = 1$, $1^{a_n} = 1^a = 1.$
If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1)$
$\because b > 0, \epsilon \to 0, \therefore b^\epsilon - 1 \to 0.$ $\therefore \lim_{n\to\infty} b_n^a = b^a.$
If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$. □

Proof. 4.(3)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$

$$\begin{aligned} \log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1) \end{aligned}$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a})$. $\because b > 0, a \neq 0,\ a_n > 0$ when $\epsilon \to 0$.
$\therefore \log_b(1 + \frac{\epsilon}{a}) \to 0.$
$\therefore \lim_{n\to\infty} \log_b a_n = \log_b a$ □

Proof. 4.(4)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$
$a_n^b = e^{b \ln a_n}$, $a_n^b - a^b = e^{b\ln a_n} - e^{b \ln a}.$

$$\begin{aligned} e^{b\ln a_n} - e^{b\ln a} &= e^{b \ln a}(e^{b\ln a_n - b\ln a} - 1) \\ &= e^{b\ln a}(e^{b \ln \frac{a_n}{a}} - 1) \end{aligned}$$

$0 < |a_n^b - a^b| < e^{b\ln a}(e^{b\ln(1+\frac{\epsilon}{a})} - 1)$
$\therefore \lim_{n\to\infty} a_n^b = a^b$ □

Proof. 4.(5)
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$

$$\begin{aligned}\sin(A+B) - \sin(A-B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B\end{aligned}$$

$$\sin a_n - \sin a = 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}$$

$|\sin a_n - \sin a| = |2 \cos \frac{a_n+a}{2} \sin \frac{a_n-a}{2}| < |2 \sin \frac{a_n-a}{2}|$
$|2 \sin \frac{a_n-a}{2}| < |2 \frac{a_n-a}{2}| = \epsilon$
$|\sin a_n - \sin a| < \epsilon, \therefore \lim_{n\to\infty} \sin a_n = \sin a$ □

Quetion 5. *assume* $a > 1$. *Prove* $\lim_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof. $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$, $\log_a 1 = 0$.
$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon.$
$a > 1$, and $\lim_{n\to\infty} \sqrt[n]{n} = 1$. $\therefore$ when $n \to \infty$, $\sqrt[n]{n} < a^{\epsilon}$, take logarithm on base of a, we can get $\frac{1}{n} \log_a n < \epsilon$
$\therefore \lim_{n\to\infty} \frac{\log_a n}{n} = 0$ □

收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界
3. 收敛数列的比较定理，包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

2 2.2.1

思考题

Quetion. *1.* $\{a_n\}$ *收敛,* $\{b_n\}$ *发散,* $\{a_n + b_n\}$ *发散,* $\{a_n \cdot b_n\}$ *可能收敛，可能发散.*
2. $\{a_n\}, \{b_n\}$ *都发散,* $\{a_n + b_n\}$ *可能收敛，可能发散 (ex:* $n + -n$, $n + -2n$*),*

$\{a_n \cdot b_n\}$ *发散 (?).*

3. $a_n \leqslant b_n \leqslant c_n$, $n \in \mathbb{N}_+$. *已知* $\lim\limits_{n\to\infty}(c_n - a_n) = 0$. *问数列* $\{b_n\}$ *是否收敛？*

4. $\lim\limits_{n\to\infty}(\frac{1}{n+1} + \cdots + \frac{1}{2n})$

5. $a_n \to a(n \to 0)$. $\forall n, b < a_n < c$. *是否成立* $b < a < c$*?*

6. $a_n \to a(n \to 0)$. *and* $b \leqslant a \leqslant c$, *是否存在* $N \in \mathbb{N}_+$, *s.t.* *当* $n > N$ *时，成立* $b \leqslant a_n \leqslant c$

7. 已知 $\lim\limits_{n\to\infty} a_n = 0$, *问: 是否有* $\lim\limits_{n\to\infty}(a_1 a_2 \ldots a_n) = 0$. *反之如何?*

Proof. 5.4

$$\frac{n}{2n} \leqslant \frac{1}{n+1} + \cdots + \frac{1}{2n} \leqslant \frac{n}{n+1}$$

$\because \lim\limits_{n\to\infty} \frac{n}{2n} = \frac{1}{2} \lim\limits_{n\to\infty} \frac{n}{n+1} = 1$, $\therefore \lim\limits_{n\to\infty}(\frac{1}{n+1} + \cdots + \frac{1}{2n})$ 收敛.

$$\begin{aligned}\frac{1}{n+1} + \cdots + \frac{1}{2n} &= \frac{1}{n}(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}) \\ &= \int_0^1 \frac{1}{1+x}\mathrm{d}x \\ &= \ln(1+x)|_0^1 = \ln 2\end{aligned}$$

$\lim\limits_{n\to\infty}(\frac{1}{n+1} + \cdots + \frac{1}{2n}) = \ln 2$ ☐

Proof. 5.5

不成立，应当为小于等于号。b=0, c=2, $a_n = \frac{1}{n}$, $\lim_{n\to\infty} a_n = 0 = c$. ☐

Proof. 5.6

不成立。$a = 0, b = 0, c = 2$, $a_n = (-1)^n \frac{1}{n}$.

$b \leqslant a \leqslant c$, but $(-1)^{2n+1}\frac{1}{2n+1} < 0 = b$. ☐

Proof. $\lim\limits_{n\to\infty} a_n = 0, a_n = \frac{1}{n}.a_1 a_2 \ldots a_n = \frac{1}{n!}, \lim\limits_{n\to\infty} \frac{1}{n!} = 0.$

$\lim\limits_{n\to\infty} a_n = 0 \to \lim\limits_{n\to\infty}(a_1 a_2 \ldots a_n) = 0 \qquad \checkmark$

$\lim\limits_{n\to\infty}(a_1 a_2 \ldots a_n) = 0 \to \lim\limits_{n\to\infty} a_n = 0 \qquad \times$

$|a_n| < \epsilon$, $|a_{N+1} \ldots a_n| < \epsilon^{n-N} < \epsilon$, $a_n < \sqrt[n]{\epsilon}$.

for example, $a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \ldots, a_n = \frac{n}{n+1}$.

$$a_1 a_2 \ldots a_n = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}.$$

$$\lim_{n\to\infty}(a_1 a_2 \ldots a_n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

but $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ ☐

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

Example 1. *2.2.2* $\lim_{n\to\infty}\frac{x_n-1}{x_n+a}=0$, *prove* $\lim_{n\to\infty}x_n=a$

Proof. $\forall\epsilon>0,\exists N\in\mathbb{N}^+.\forall n\geqslant N,|\frac{x_n-1}{x_n+a}-0|<\epsilon.$
$|x_n-1|<\epsilon|x_n+a|<4a\cdot\epsilon.$(这个 4 是怎么取得的?)
$|x_n-a|<\epsilon|x_n+a|=\epsilon|(x_n-a)+2a|\leqslant\epsilon(|x_n-a|+2a).$
限制 $\epsilon<1$, $|x_n-a|<2\epsilon|a|/(1-\epsilon)$.
限制 $\epsilon<\frac{1}{2}$, $|x_n-a|<2\epsilon|a|/(1-\epsilon)<4|a|\epsilon$.
Let $\epsilon'=4a\epsilon$, $|x_n-1|<\epsilon'$. $\therefore \lim\lim_{n\to\infty}x_n=a$. □

Example 2. *2.2.3* $a>0,b>0$ ，计算 $\lim\limits_{n\to\infty}(a_n+b_n)^{\frac{1}{n}}$.

Proof. Suppose $a\leqslant b$.
$b=(b^b)^{\frac{1}{n}}<(a^n+b^n)^{\frac{1}{n}}\leqslant(2b^n)^{\frac{1}{n}}$.
$b<(a^n+b^n)^{\frac{1}{n}}\leqslant\sqrt[n]{2}b, \lim\limits_{n\to\infty}=1$. 夹逼定理.
$\lim\limits_{n\to\infty}(a^n+b^n)^{\frac{1}{n}}=\max\{a,b\}$.
两数 n 次方之和再开 n 次根号的结果由较大的值决定, a,b 中较大的值为这个数的主要部分. □

Example 3. *2.2.4* $a_n=\frac{1!+2!+\cdots+n!}{n!}, n\in\mathbb{N}^+$

$\lim\limits_{n\to\infty}a_n=1$

Example 4. $\lim\limits_{n\to\infty}\frac{n^3+n-7}{n+3}=+\infty$

Example 5. $H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$

调和级数 H_n 发散.

2.1 练习 2.2.4

Proof. 1.
$\{a_n\}$ 收敛于 a, $\to$ 两个子列 $\{a_{2n}\},\{a_{2n+1}\}$ 均收敛于 a.
两个子列 $\{a_{2n}\},\{a_{2n+1}\}$ 均收敛于 a, $\to$ $\{a_n\}$ 收敛于 a.

□

2. 应用夹逼定理
(1). 给定 p 个正数 $a_1,a_2,\ldots,a_p$. 求 $\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\ldots a_p^n}$.
Let$a_s=\max_{1\leqslant i\leqslant p}\{a_1,a_2,\ldots,a_p\}$.

Solve. *(1).*

$$a_s=(a_s^n)^{\frac{1}{n}}<(a_1^n+a_2^n+\ldots a_p^n)^{\frac{1}{n}}\leqslant(pa_s^n)^{\frac{1}{n}}=p^{\frac{1}{n}}a_s$$

$n\to\infty, p^{\frac{1}{n}}\to 1. \lim\limits_{n\to\infty}(a_1^n+a_2^n+\ldots a_p^n)^{\frac{1}{n}}=a_s$

(2). $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} x_n$

Solve. *(2).*

$$\frac{2n+1}{(n+1)} \leqslant x_n \leqslant \frac{2n+1}{\sqrt{n^2+1}}$$

$\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$, $\lim\limits_{n\to\infty} \frac{2n+1}{\sqrt{n^2+1}} = 2$. $\therefore \lim\limits_{n\to\infty} x_n = 2$

(3). $a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} a_n$

Solve. *(3).*

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leqslant (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\therefore \lim\limits_{n\to\infty} a_n = 1$

(4). $a_n > 0$. $\lim\limits_{n\to\infty} a_n = a$, $a > 0$. 证明 $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$

Proof. $\lim\limits_{n\to\infty} a_n = a$

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon.$

$$0 < a - \epsilon < a_n < a + \epsilon$$

$\therefore \sqrt[n]{a-\epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a+\epsilon}$.

$\lim\limits_{n\to\infty} \sqrt[n]{a-\epsilon} = 1$, $\lim\limits_{n\to\infty} \sqrt[n]{a+\epsilon} = 1$. $\therefore \lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$. □

3. (1). $\lim\limits_{n\to\infty} (1+x)(1+x^2)\dots(1+x^{2^n}) = \lim\limits_{n\to\infty} \prod_{i=1}^{2^n}(1+x^i)$, $|x| < 1$.

Solve. *3.(1).*

$|x| < 1, \quad 1 > x^2 > x^4 > \cdots > x^{2^n} > 0$

$x \in (0,1) \quad 1 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)^{n+1} \qquad \lim\limits_{n\to\infty}(1+x)^{n+1} = 1$

$x \in (-1,0) \quad 0 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)(1+x^2)^n \qquad \lim\limits_{n\to\infty}(1+x)(1+x^2)^n = 1$

Solve. *3.(1). another way*

$$\begin{aligned}
&\lim_{n\to\infty}(1+x)(1+x^2)\dots(1+x^n) \\
=&\lim_{n\to\infty}\frac{(1-x)(1+x)(1+x^2)\dots(1+x^n)}{1-x} \\
=&\lim_{n\to\infty}\frac{(1-x^{2^{n+1}})}{1-x} \\
=&\frac{1}{1-x}
\end{aligned}$$

Solve. *3. (2).*

$$\begin{aligned}
&\lim_{n\to\infty}(1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{n^2})\\
=&\lim_{n\to\infty}\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\dots\frac{n-1}{n}\cdot\frac{n+1}{n}\\
=&\lim_{n\to\infty}\frac{1}{2}\frac{n+1}{n}\\
=&\lim_{n\to\infty}\frac{1}{2}
\end{aligned}$$

Solve. *3. (3).*

$$\begin{aligned}
&\lim_{n\to\infty}\Big(1-\frac{1}{1+2}\Big)\Big(1-\frac{1}{1+2+3}\Big)\dots\Big(1-\frac{1}{1+2+\dots+n}\Big)\\
=&\lim_{n\to\infty}\Big(1-\frac{2}{3\times 2}\Big)\Big(1-\frac{2}{4\times 3}\Big)\dots\Big(1-\frac{2}{(n+1)\times n}\Big)\\
=&\lim_{n\to\infty}\Big(\frac{3\times 2-2}{3\times 2}\Big)\Big(\frac{4\times 3-2}{4\times 3}\Big)\dots\Big(\frac{(n+1)\times n-2}{(n+1)\times n}\Big)\\
=&\lim_{n\to\infty}\Big(\frac{4}{3\times 2}\Big)\Big(\frac{10}{4\times 3}\Big)\dots\Big(\frac{n^2+n-2}{(n+1)\times n}\Big)\\
=&\lim_{n\to\infty}\Big(\frac{1\times 4}{3\times 2}\Big)\Big(\frac{2\times 5}{4\times 3}\Big)\dots\Big(\frac{(n-2)\times(n+1)}{n\times(n-1)}\Big)\Big(\frac{(n-1)\times(n+2)}{(n+1)\times n}\Big)\\
=&\lim_{n\to\infty}\frac{1}{3}\times\frac{n+2}{n}\\
=&\frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n\cdot(n+1)}]\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots+\frac{1}{n}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{n}{n+1}\\
=&1
\end{aligned}$$

Solve. *3.(4).*

$$\begin{aligned}
&\lim_{n\to\infty}\Big[\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\dots+\frac{1}{n\cdot(n+1)\cdot(n+2)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}\Big)+\frac{1}{2}\Big(\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}\Big)+\dots+\frac{1}{2}\Big(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\Big)\\
=&\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{1\cdot 2}-\frac{1}{(n+1)(n+2)}\Big)\\
=&\frac{1}{2}\times\frac{1}{2}\\
=&\frac{1}{4}
\end{aligned}$$

Solve. *3.(5).*

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma)}, \qquad \text{其中}\gamma\text{为正整数}\\
=&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\gamma}\Big[\frac{1}{k(k+1)\dots(k+\gamma-1)}-\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma-1)}-\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma^{\underline{\gamma}}}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma!}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]\\
=&\frac{1}{\gamma}\cdot\frac{1}{\gamma!}
\end{aligned}
$$

其中 $x^{\underline{n}}=x(x-1)(x-2)\dots(x-n+1)$, 称为下阶乘. 而 $x^{\overline{n}}=x(x+1)(x+2)\dots(x+n-1)$, 称为上阶乘.

Quetion 6. *2.2.4-4* $S_n=a+3a^2+\dots+(2n-1)a^n$, $|a|<1$. 求 $\{S_n\}$ 的极限.

Solve.

$$
\begin{aligned}
S_n-aS_n&=a+3a^2+\dots+(2n-1)a^n\\
&\quad-\ a^2-\dots+(2n-3)a^n-(2n-1)a^n+1\\
&=a+2a^2+\dots+2aa^n-(2n-1)a^{n+1}\\
&=2(a+a^2+\dots+a^n)-a-(2n-1)a^{n+1}\\
&=2\cdot a\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}
\end{aligned}
$$

$|a|<1,\ \lim\limits_{n\to\infty}a_n=0$
$\lim\limits_{n\to\infty}(S_n-aS_n)=(1-a)\lim\lim_{n\to\infty}S_n$

$$
\begin{aligned}
\lim_{n\to\infty}(S_n-aS_n)&=\lim_{n\to\infty}2a\cdot\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}\\
&=2a\cdot\frac{1}{1-a}-a\\
&=a\Big(\frac{2}{1-a}-a\Big)\\
&=a\frac{1+a}{1-a}
\end{aligned}
$$

$\therefore\lim\limits_{n\to\infty}S_n=\frac{a(a+1)}{(1-a)^2}$

Solve. *2.2.4-5* 设 $\lim\limits_{n\to\infty}x_n=A>0$. 取 $\epsilon=\frac{A}{2}$, 则 $\exists N\in\mathbb{N}_+$. $\forall n>N$. 成立 $|x_n-A|<\frac{A}{2}$

$$A-\frac{A}{2}<x_n<A+\frac{A}{2},\frac{A}{2}<x_n<\frac{3A}{2}$$

即 $x_n > \frac{A}{2}$.
令 $m = \min\{x_1, x_2, \ldots, x_N, \frac{A}{2}\} > 0$. 则 m 为 $\{x_n\}$ 的正下界.
不一定有最小数的例子 $x_n = 1 + \frac{1}{n}$. $\lim\limits_{n\to\infty} x_n = 1$, 下界 $m = \frac{1}{2}$. 但 $\{x_n\}$ 取不到下界.

Proof. 2.2.4-6 $\because \lim\limits_{n\to\infty} a_n = +\infty$. $\forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M$.
$m = \min\{a_1, a_2, \ldots, a_N, M\}$, 但 M 在数列 $\{a_n\}$ 中不一定取的到!
$M = a_1 + 1$, $\exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$
则 $m = \min\{a_1, a_2, \ldots, a_{N_1}\}$ 为数列的最小数. □

Proof. 2.2.4-7 构造数列
不妨设无界数列 $\{a_n\}$ 无上界.
$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M$.
取 $M_1 = 1$, 则 $\exists n_1 \in \mathbb{N}_+$ s.t. $a_{n_1} > M_1$.
取 $M_2 = \max\{a_n, 2\}$, $\exists n_2 \in \mathbb{N}_+$ s.t. $a_{n_2} > M_2$.
以此类推，构造数列 $\{a_{n_k}\}$. s.t.$a_{n_k} > k$. 即 a_{n_k} 为无穷大量. □

Proof. 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散.
构造 a_n 的发散子列即可. 已知 $\tan\frac{\pi}{2} = \infty$, π 是一个无理数, 因此存在数列 $\{b_n\}$, $\lim\limits_{n\to\infty} b_n = \frac{\pi}{2}$. □

Proof. 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散. 参考别人的答案
由于 $\{\sin 2n\}$ 极限不存在, 又

$$
\begin{aligned}
\sin 2n = 2\sin n\cos n &= \frac{2\sin n\cos n}{\sin^2 n + \cos^2 n} \\
&= \frac{2\tan n}{\tan^n + 1}
\end{aligned}
$$

若 $\{\tan n\}$ 极限存在 $\to \{\sin 2n\}$ 极限存在, 矛盾.
故 $\{\tan n\}$ 极限不存在. □

Quetion 7. *2.2.4-9* $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$. S_n 在 *1.* $p \leqslant 0$, *2.* $0 < p < 1$ 情况下均发散

Proof. 1. $p \leqslant 0$. $\lim\limits_{n\to\infty} n^{-p} = \infty$, S_n 发散.
2. $0 < p < 1$. $\frac{1}{n^p} > \frac{1}{n}$. $\because H_n = \sum_{k=1}^{n} \frac{1}{k}$(调和级数) 发散, $S_n > H_n$,$\therefore \{S_n\}$ 也发散. □

ex2.3.5 $0 < b < a$ 令 $a_0 = a, b_0 = b$ 递推公式

$$
a_n = \frac{a_{n-1} + b_{n-1}}{2}, b_n = \sqrt{a_{n-1}b_{n-1}}, \quad n \in \mathbb{N}_+ \tag{1}
$$

定义数列a_n,b_n. 证明这两个数列收敛于同一个极限 $AG(a, b)$.

由 AG 不等式 $a > \frac{a+b}{2} > \sqrt{ab} > b > 0$, 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a,b) = \frac{\pi}{2G} \tag{2}$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$. 则

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta + b_1^2\sin^2\theta}} \tag{3}$$

上面这个公式是怎么得到的:
参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \qquad (a > b > 0) \tag{4}$$

这里令

$$\sin\phi = \frac{2a\sin\theta}{(a+b) + (a-b)\sin^2\theta} \tag{5}$$

$\theta \in [0, \frac{\pi}{2}] \to \phi \in [0, \frac{\pi}{2}]$, 取微分

$$\cos\phi\mathrm{d}\phi = 2a\frac{(a+b) - (a-b)\sin^2\theta}{[(a+b) + (a-b)\sin^2\theta]^2}\cos\theta\mathrm{d}\theta \tag{6}$$

但是

$$\cos\phi = \frac{\sqrt{(a+b)^2 - (a-b)^2\sin^2\theta}}{(a+b) + (a-b)\sin^2\theta}\cos\theta. \tag{7}$$

(6) / (7), 两式相除, 得到

$$\mathrm{d}\phi = 2a\frac{(a+b) - (a-b)\sin^2\theta}{(a+b) + (a-b)\sin^2\theta}\frac{\mathrm{d}\theta}{\sqrt{(a+b)^2 - (a-b)^2\sin^2\theta}} \tag{8}$$

另一方面

$$\sqrt{a^2\cos\phi + b^2\sin^2\phi} = a\frac{(a+b) - (a-b)\sin^2\theta}{(a+b) + (a-b)\sin^2\theta} \tag{9}$$

因而

$$\frac{\mathrm{d}\phi}{\sqrt{a^2\cos\phi + b^2\sin^2\phi}} = \frac{\mathrm{d}\theta}{\sqrt{(\frac{a+b}{2})^2\cos^2\theta + ab\sin^2\theta}}. \tag{10}$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$, 则

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta + b_1^2\sin^2\theta}} \tag{11}$$

反复应用该公式, 得到

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a_n^2\cos^2\phi + b_n^2\sin^2\phi}}, \qquad (n = 1, 2, 3, \dots) \tag{12}$$

$$\frac{\pi}{2a_n} < G < \frac{\pi}{2b_n} \tag{13}$$

积分 G 可以归结到第一类全椭圆积分 $K(k) = (1+k_1)K(k_1) = \frac{\pi}{2}(1+k_1)(1+k_2)\ldots(1+k_n)$

$$\int_0^{\frac{pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}} = (1+k_1)\int_0^{\frac{pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k_1^2\sin^2\theta}} \tag{14}$$

其中
$a_1 = \frac{1+\sqrt{1-k^2}}{2} = \frac{1+k'}{2}, b_1 = \sqrt{k'}$
$k_1 = \frac{\sqrt{a_1^2-b_1^2}}{a_1} = \frac{1-k'}{1+k'}, \frac{1}{a_1} = 1+k_1$