测试文件

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Chapter 1 第二章数列极限

1.1 数列极限的基本概念

1.1.1 2.1.5 练习题

1. prove by Limit definition:
(1). $\lim_{n\to\infty} \frac{3n^2}{n^2-4} = 3.$
(2). $\lim_{n\to\infty} \frac{\sin n}{n} = 0.$
(3). $\lim_{n\to\infty} (1+n)^{\frac{1}{n}} = 0.$
(4). $\lim_{n\to\infty} \frac{a^n}{n!} = 0, (a > 0).$

2. Suppose $a_n, n \in \mathbb{N}_+$. sequence a_n converge to a.
Prove $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

Proof $n \to \infty a_n \to a$.
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$

$$|\sqrt{a_n} - \sqrt{a}| = \left|\frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}\right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$ □ (check, not consider the condition$a = 0$) add $a = 0, \forall \epsilon \in (0,1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$. s.t $a_n < \epsilon^2 < \epsilon$, $\sqrt{a_n} < \epsilon$.

3. If $\lim_{n\to\infty} a_n = a$.
Prove $\lim_{n\to\infty} |a_n| = |a|$. Vice versa?

Proof $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$

$$\Big||a_n| - |a|\Big| \leqslant |a_n - a| < \epsilon$$

$\therefore \lim_{n\to\infty} |a_n| = |a|$
If We know $\lim_{n\to\infty} |a_n| = |a|$.
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), \Big||a_n| - |a|\Big| < \epsilon$. We can't get $\lim_{n\to\infty} a_n = a$. For example: $a_n = \frac{1}{n} + 1, a = -1$, $\lim_{n\to\infty} |a_n| = |a|$ is $\lim_{n\to\infty} |\frac{1}{n} + 1| = |-1|$, but $\lim_{n\to\infty} \frac{1}{n} + 1 \neq -1$ □

(1). Suppose $p(x)$ is a polynomial of x, if $\lim_{n\to\infty} a_n = a$, Prove $\lim_{n\to\infty} p(a_n) = p(a)$.
(2). Suppose $b > 0$, $\lim\limits_{n\to\infty} a_n = a$. Prove $b^{a_n} = b^a$.
(3). Suppose $b > 0$, $\{a_n\}, a_n > 0, \forall n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a.a > 0$. Prove $\lim\limits_{n\to\infty} \log_b a_n = \log_b a$.
(4) Suppose $b \in \mathbb{R}$, $\{a_n\}, a_n > 0$ when $n \in \mathbb{N}$. $\lim\limits_{n\to\infty} a_n = a$. Prove $\lim\limits_{n\to\infty} {a_n}^b = a^b$.
(5) Suppose $\lim\limits_{n\to\infty} a_n = a$. Prove $\lim\limits_{n\to\infty} \sin a_n = \sin a$.

Proof 4.(1)
$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geqslant N(\epsilon)$, $|a_n - a| < \epsilon$.
$p(a) = k_m a^m + k_{m-1} a^{m-1} + \cdots + k_0 a^0$.
$\therefore p(a_n) - p(a) = k_m(a_n^m - a^m) + k_{m-1}(a_n^{m-1} - a^{m-1}) + \cdots + k_0(a_n^0 - a^0)$.

$$\begin{aligned} |a_n^m - a^m| &= |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2} a + \cdots + a^{m-1}| \\ &< \epsilon \cdot |a_n^{m-1} + a_n^{m-2} a + \cdots + a^{m-1}| \\ &< \epsilon(m-1) \cdots (a+\delta)^{m-1} \end{aligned}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a)$. □

Proof 4.(2)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

If $b = 1$, $1^{a_n} = 1^a = 1$.

If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1)$ $\because b > 0, \epsilon \to 0$, $\therefore b^\epsilon - 1 \to 0$.

$\therefore \lim\limits_{n\to\infty} b_n^a = b^a$.

If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$.

Proof 4.(3)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1)\end{aligned}$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a})$. $\because b > 0, a \neq 0$, $a_n > 0$ when $\epsilon \to 0$. $\therefore \log_b(1 + \frac{\epsilon}{a}) \to 0$.

$\therefore \lim\limits_{n\to\infty} \log_b a_n = \log_b a$

Proof 4.(4)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$a_n^b = e^{b \ln a_n}$, $a_n^b - a^b = e^{b \ln a_n} - e^{b \ln a}$.

$$\begin{aligned}e^{b \ln a_n} - e^{b \ln a} &= e^{b \ln a}(e^{b \ln a_n - b \ln a} - 1) \\ &= e^{b \ln a}(e^{b \ln \frac{a_n}{a}} - 1)\end{aligned}$$

$0 < |a_n^b - a^b| < e^{b \ln a}(e^{b \ln(1 + \frac{\epsilon}{a})} - 1)$

$\therefore \lim\limits_{n\to\infty} a_n^b = a^b$

Proof 4.(5)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\sin(A + B) - \sin(A - B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B \\ \sin a_n - \sin a &= 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}\end{aligned}$$

$|\sin a_n - \sin a| = |2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}| < |2 \sin \frac{a_n - a}{2}|$

$|2 \sin \frac{a_n - a}{2}| < |2 \frac{a_n - a}{2}| = \epsilon$

$|\sin a_n - \sin a| < \epsilon$, $\therefore \lim\limits_{n\to\infty} \sin a_n = \sin a$

assume $a > 1$. Prove $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\log_a 1 = 0$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\} . \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon$.

$a > 1$, and $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\therefore$ when $n \to \infty$, $\sqrt[n]{n} < a^\epsilon$, take logarithm on base of a, we can get $\frac{1}{n} \log_a n < \epsilon$

$\therefore \lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

1.2 收敛数列的基本性质

收敛数列的性质

1. 收敛数列的极限是唯一的

2. 收敛数列一定有界
3. 收敛数列的比较定理，包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

1.2.1 思考题

1. $\{a_n\}$ 收敛, $\{b_n\}$ 发散, $\{a_n+b_n\}$ 发散, $\{a_n\cdot b_n\}$ 可能收敛，可能发散.
2. $\{a_n\},\{b_n\}$ 都发散, $\{a_n+b_n\}$ 可能收敛，可能发散 (ex: $n+-n$, $n+-2n$), $\{a_n\cdot b_n\}$ 发散 (?).
3. $a_n\leqslant b_n\leqslant c_n$, $n\in\mathbb{N}_+$. 已知 $\lim\limits_{n\to\infty}(c_n-a_n)=0$. 问数列 $\{b_n\}$ 是否收敛?
4. $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$
5. $a_n\to a(n\to 0)$. $\forall n, b<a_n<c$. 是否成立 $b<a<c$?
6. $a_n\to a(n\to 0)$. and $b\leqslant a\leqslant c$, 是否存在 $N\in\mathbb{N}_+$, s.t. 当 $n>N$ 时，成立 $b\leqslant a_n\leqslant c$
7. 已知 $\lim\limits_{n\to\infty}a_n=0$, 问: 是否有 $\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0$. 反之如何?

Proof 5.4

$$\frac{n}{2n}\leqslant\frac{1}{n+1}+\cdots+\frac{1}{2n}\leqslant\frac{n}{n+1}$$

$\because \lim\limits_{n\to\infty}\frac{n}{2n}=\frac{1}{2}\lim\limits_{n\to\infty}\frac{n}{n+1}=1$, $\therefore \lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$ 收敛.

$$\begin{aligned}\frac{1}{n+1}+\cdots+\frac{1}{2n}&=\frac{1}{n}(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}})\\&=\int_0^1\frac{1}{1+x}\mathrm{d}x\\&=\ln(1+x)|_0^1=\ln 2\end{aligned}$$

$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})=\ln 2$

Proof 5.5

不成立，应当为小于等于号。b=0, c=2, $a_n=\frac{1}{n}$, $\lim_{n\to\infty}a_n=0=c$.

Proof 5.6

不成立。$a=0,b=0,c=2$, $a_n=(-1)^n\frac{1}{n}$.

$b\leqslant a\leqslant c$, but $(-1)^{2n+1}\frac{1}{2n+1}<0=b$.

Proof $\lim\limits_{n\to\infty}a_n=0, a_n=\frac{1}{n}.a_1a_2\ldots a_n=\frac{1}{n!}$, $\lim\limits_{n\to\infty}\frac{1}{n!}=0$.

$\lim\limits_{n\to\infty}a_n=0\to\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0$ ✓

$\lim\limits_{n\to\infty}(a_1a_2\ldots a_n)=0\to\lim\limits_{n\to\infty}a_n=0$ ×

$|a_n|<\epsilon$, $|a_{N+1}\ldots a_n|<\epsilon^{n-N}<\epsilon$, $a_n<\sqrt[n]{\epsilon}$.

for example, $a_n=\frac{n}{n+1}, a_1=\frac{1}{2}, a_2=\frac{2}{3},\ldots,a_n=\frac{n}{n+1}$.

$$\begin{aligned}a_1a_2\ldots a_n=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}&=\frac{1}{n+1}.\\ \lim_{n\to\infty}(a_1a_2\ldots a_n)&\\ =\lim_{n\to\infty}\frac{1}{n}&=0\end{aligned}$$

but $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{n}{n+1}=1\neq 0$

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

Example 1.1 2.2.2 $\lim_{n\to\infty}\frac{x_n-1}{x_n+a}=0$, prove $\lim_{n\to\infty} x_n = a$

Proof $\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |\frac{x_n-1}{x_n+a} - 0| < \epsilon$.

$|x_n - 1| < \epsilon|x_n + a| < 4a \cdot \epsilon$.(这个 4 是怎么取得的?)

$|x_n - a| < \epsilon|x_n + a| = \epsilon|(x_n - a) + 2a| \leqslant \epsilon(|x_n - a| + 2a)$.

限制 $\epsilon < 1$, $|x_n - a| < 2\epsilon|a|/(1-\epsilon)$.

限制 $\epsilon < \frac{1}{2}$, $|x_n - a| < 2\epsilon|a|/(1-\epsilon) < 4|a|\epsilon$.

Let $\epsilon' = 4a\epsilon$, $|x_n - 1| < \epsilon'$. $\therefore \lim\lim_{n\to\infty} x_n = a$.

Example 1.2 2.2.3 $a > 0, b > 0$, 计算 $\lim\limits_{n\to\infty}(a_n + b_n)^{\frac{1}{n}}$.

Proof Suppose $a \leqslant b$.

$b = (b^b)^{\frac{1}{n}} < (a^n + b^n)^{\frac{1}{n}} \leqslant (2b^n)^{\frac{1}{n}}$.

$b < (a^n + b^n)^{\frac{1}{n}} \leqslant \sqrt[n]{2}b$, $\lim\limits_{n\to\infty} = 1$. 夹逼定理.

$\lim\limits_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}$.

两数 n 次方之和再开 n 次根号的结果由较大的值决定, a,b 中较大的值为这个数的主要部分.

Example 1.3 2.2.4 $a_n = \frac{1!+2!+\cdots+n!}{n!}, n \in \mathbb{N}^+$

$\lim\limits_{n\to\infty} a_n = 1$

Example 1.4 $\lim\limits_{n\to\infty} \frac{n^3+n-7}{n+3} = +\infty$

Example 1.5 $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

调和级数 H_n 发散.

1.2.2 练习 2.2.4

Proof 1.

$\{a_n\}$ 收敛于 a, $\to$ 两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 a.

两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 a, $\to$ $\{a_n\}$ 收敛于 a.

2. 应用夹逼定理

(1). 给定 p 个正数 $a_1, a_2, \ldots, a_p$. 求 $\lim\limits_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \ldots a_p^n}$.

Let$a_s = \max_{1\leqslant i\leqslant p}\{a_1, a_2, \ldots, a_p\}$.

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \ldots a_p^n)^{\frac{1}{n}} \leqslant (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$n \to \infty, p^{\frac{1}{n}} \to 1$. $\lim\limits_{n\to\infty}(a_1^n + a_2^n + \ldots a_p^n)^{\frac{1}{n}} = a_s$

(2). $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} x_n$

$$\frac{2n+1}{(n+1)} \leqslant x_n \leqslant \frac{2n+1}{\sqrt{n^2+1}}$$

$\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2$, $\lim\limits_{n\to\infty} \frac{2n+1}{\sqrt{n^2+1}} = 2$. $\therefore \lim\limits_{n\to\infty} x_n = 2$

(3). $a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}$, $n \in \mathbb{N}_+$. 求 $\lim\limits_{n\to\infty} a_n$

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leqslant (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\therefore \lim\limits_{n\to\infty} a_n = 1$

(4). $a_n > 0$. $\lim\limits_{n\to\infty} a_n = a$, $a > 0$. 证明 $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$

Proof $\lim\limits_{n\to\infty} a_n = a$

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$0 < a - \epsilon < a_n < a + \epsilon$$

$\therefore \sqrt[n]{a-\epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a+\epsilon}$.

$\lim\limits_{n\to\infty} \sqrt[n]{a-\epsilon} = 1$, $\lim\limits_{n\to\infty} \sqrt[n]{a+\epsilon} = 1$. $\therefore \lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$.

3. (1). $\lim\limits_{n\to\infty}(1+x)(1+x^2)\dots(1+x^{2^n}) = \lim\limits_{n\to\infty}\prod_{i=1}^{2^n}(1+x^i)$, $|x|<1$.

$|x|<1, \quad 1 > x^2 > x^4 > \cdots > x^{2^n} > 0$

$x\in(0,1) \quad 1 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)^{n+1} \qquad \lim\limits_{n\to\infty}(1+x)^{n+1} = 1$

$x\in(-1,0) \quad 0 < (1+x)(1+x^2)\dots(1+x^{2^n}) < (1+x)(1+x^2)^n \qquad \lim\limits_{n\to\infty}(1+x)(1+x^2)^n = 1$

$$
\begin{aligned}
&\lim_{n\to\infty}(1+x)(1+x^2)\dots(1+x^n)\\
=&\lim_{n\to\infty}\frac{(1-x)(1+x)(1+x^2)\dots(1+x^n)}{1-x}\\
=&\lim_{n\to\infty}\frac{(1-x^{2^{n+1}})}{1-x}\\
=&\frac{1}{1-x}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}(1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{n^2})\\
=&\lim_{n\to\infty}\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\dots\frac{n-1}{n}\cdot\frac{n+1}{n}\\
=&\lim_{n\to\infty}\frac{1}{2}\frac{n+1}{n}\\
=&\lim_{n\to\infty}\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}\left(1-\frac{1}{1+2}\right)\left(1-\frac{1}{1+2+3}\right)\dots\left(1-\frac{1}{1+2+\dots+n}\right)\\
=&\lim_{n\to\infty}\left(1-\frac{2}{3\times 2}\right)\left(1-\frac{2}{4\times 3}\right)\dots\left(1-\frac{2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{3\times 2-2}{3\times 2}\right)\left(\frac{4\times 3-2}{4\times 3}\right)\dots\left(\frac{(n+1)\times n-2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{4}{3\times 2}\right)\left(\frac{10}{4\times 3}\right)\dots\left(\frac{n^2+n-2}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\left(\frac{1\times 4}{3\times 2}\right)\left(\frac{2\times 5}{4\times 3}\right)\dots\left(\frac{(n-2)\times(n+1)}{n\times(n-1)}\right)\left(\frac{(n-1)\times(n+2)}{(n+1)\times n}\right)\\
=&\lim_{n\to\infty}\frac{1}{3}\times\frac{n+2}{n}\\
=&\frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n\cdot(n+1)}]\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots+\frac{1}{n}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{1}{1}-\frac{1}{n+1}\\
=&\lim_{n\to\infty}\frac{n}{n+1}\\
=&1
\end{aligned}
$$

3.(4).

$$\begin{aligned}
&\lim_{n\to\infty}\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)}\right]\\
=&\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{1\cdot2}-\frac{1}{2\cdot3}\Big)+\frac{1}{2}\Big(\frac{1}{2\cdot3}-\frac{1}{3\cdot4}\Big)+\cdots+\frac{1}{2}\Big(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\Big)\\
=&\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{1\cdot2}-\frac{1}{(n+1)(n+2)}\Big)\\
=&\frac{1}{2}\times\frac{1}{2}\\
=&\frac{1}{4}
\end{aligned}$$

3.(5).

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma)},\qquad \text{其中}\gamma\text{为正整数}\\
=&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\gamma}\Big[\frac{1}{k(k+1)\dots(k+\gamma-1)}-\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\sum_{k=1}^{n}\frac{1}{k(k+1)\dots(k+\gamma-1)}-\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)\dots(k+\gamma)}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma^{\underline{\gamma}}}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]\\
=&\lim_{n\to\infty}\frac{1}{\gamma}\Big[\frac{1}{\gamma!}-\frac{1}{(n+\gamma)^{\underline{\gamma}}}\Big]\\
=&\frac{1}{\gamma}\cdot\frac{1}{\gamma!}
\end{aligned}$$

其中 $x^{\underline{n}}=x(x-1)(x-2)\dots(x-n+1)$, 称为下阶乘. 而 $x^{\overline{n}}=x(x+1)(x+2)\dots(x+n-1)$, 称为上阶乘.

2.2.4-4 $S_n=a+3a^2+\cdots+(2n-1)a^n$, $|a|<1$. 求 $\{S_n\}$ 的极限.

$$\begin{aligned}
S_n-aS_n&=a+3a^2+\cdots+(2n-1)a^n\\
&\quad-\ \ a^2-\cdots+(2n-3)a^n-(2n-1)a^n+1\\
&=a+2a^2+\cdots+2aa^n-(2n-1)a^{n+1}\\
&=2(a+a^2+\cdots+a^n)-a-(2n-1)a^{n+1}\\
&=2\cdot a\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}
\end{aligned}$$

$|a|<1$, $\lim\limits_{n\to\infty}a_n=0$

$\lim\limits_{n\to\infty}(S_n-aS_n)=(1-a)\lim\lim_{n\to\infty}S_n$

$$\begin{aligned}
\lim_{n\to\infty}(S_n-aS_n)&=\lim_{n\to\infty}2a\cdot\frac{1-a^{n+1}}{1-a}-a-(2n-1)a^{n+1}\\
&=2a\cdot\frac{1}{1-a}-a\\
&=a\Big(\frac{2}{1-a}-a\Big)\\
&=a\frac{1+a}{1-a}
\end{aligned}$$

$\therefore\lim\limits_{n\to\infty}S_n=\frac{a(a+1)}{(1-a)^2}$

2.2.4-5 设 $\lim\limits_{n\to\infty}x_n=A>0$. 取 $\epsilon=\frac{A}{2}$, 则 $\exists N\in\mathbb{N}_+$. $\forall n>N$. 成立 $|x_n-A|<\frac{A}{2}$

$$A-\frac{A}{2}<x_n<A+\frac{A}{2},\ \frac{A}{2}<x_n<\frac{3A}{2}$$

即 $x_n>\frac{A}{2}$.

令 $m=\min\{x_1,x_2,\dots,x_N,\frac{A}{2}\}>0$. 则 m 为 $\{x_n\}$ 的正下界.
不一定有最小数的例子 $x_n=1+\frac{1}{n}$. $\lim\limits_{n\to\infty}x_n=1$, 下界 $m=\frac{1}{2}$. 但 $\{x_n\}$ 取不到下界.

Proof 2.2.4-6 $\because \lim\limits_{n\to\infty}a_n=+\infty$. $\forall M>0,\exists N\in\mathbb{N}_+,\forall n>N,a_n>M$.
$m=\min\{a_1,a_2,\dots,a_N,M\}$, 但 M 在数列 $\{a_n\}$ 中不一定取的到!
$M=a_1+1$, $\exists N_1\in\mathbb{N}_+,\forall n>N_1,a_n>M>a_1$
则 $m=\min\{a_1,a_2,\dots,a_{N_1}\}$ 为数列的最小数.

Proof 2.2.4-7 构造数列
不妨设无界数列 $\{a_n\}$ 无上界.
$\forall M\in\mathbb{R},\exists N\in\mathbb{N}_+,\forall n_k>N,a_{n_k}>M$.
取 $M_1=1$, 则 $\exists n_1\in\mathbb{N}_+$ s.t. $a_{n_1}>M_1$.
取 $M_2=\max\{a_n,2\}$, $\exists n_2\in\mathbb{N}_+$ s.t. $a_{n_2}>M_2$.
以此类推，构造数列 $\{a_{n_k}\}$. s.t.$a_{n_k}>k$. 即 a_{n_k} 为无穷大量.

Proof 2.2.4-8 证明 $\{a_n\},a_n=\tan n$ 发散.
构造 a_n 的发散子列即可. 已知 $\tan\frac{\pi}{2}=\infty$, π 是一个无理数, 因此存在数列 $\{b_n\}$, $\lim\limits_{n\to\infty}b_n=\frac{\pi}{2}$.

Proof 2.2.4-8 证明 $\{a_n\},a_n=\tan n$ 发散. 参考别人的答案
由于 $\{\sin 2n\}$ 极限不存在, 又

$$\begin{aligned}\sin 2n=2\sin n\cos n&=\frac{2\sin n\cos n}{\sin^2 n+\cos^2 n}\\&=\frac{2\tan n}{\tan^n+1}\end{aligned}$$

若 $\{\tan n\}$ 极限存在 $\to\{\sin 2n\}$ 极限存在, 矛盾.
故 $\{\tan n\}$ 极限不存在.

2.2.4-9 $S_n=\frac{1}{1^p}+\frac{1}{2^p}+\dots+\frac{1}{n^p}$, $\quad n\in\mathbb{N}_+$. S_n 在 1. $p\leqslant 0$, 2. $0<p<1$ 情况下均发散

Proof 1. $p\leqslant 0$. $\lim\limits_{n\to\infty}n^{-p}=\infty$, S_n 发散.
2. $0<p<1$. $\frac{1}{n^p}>\frac{1}{n}$. $\because H_n=\sum_{k=1}^n\frac{1}{k}$(调和级数) 发散, $S_n>H_n$,$\therefore\{S_n\}$ 也发散.

ex2.3.5 $0<b<a$ 令 $a_0=a,b_0=b$ 递推公式

$$a_n=\frac{a_{n-1}+b_{n-1}}{2},b_n=\sqrt{a_{n-1}b_{n-1}},\quad n\in\mathbb{N}_+ \tag{1.1}$$

定义数列a_n,b_n. 证明这两个数列收敛于同一个极限 $AG(a,b)$.

由 AG 不等式 $a>\frac{a+b}{2}>\sqrt{ab}>b>0$, 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a,b)=\frac{\pi}{2G} \tag{1.2}$$

如果令 $a_1=\frac{a+b}{2},b_1=\sqrt{ab}$. 则

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta+b_1^2\sin^2\theta}} \tag{1.3}$$

上面这个公式是怎么得到的:
参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}\qquad(a>b>0) \tag{1.4}$$

这里令

$$\sin\phi=\frac{2a\sin\theta}{(a+b)+(a-b)\sin^2\theta} \tag{1.5}$$

$\theta \in [0, \frac{\pi}{2}] \to \phi \in [0, \frac{\pi}{2}]$, 取微分

$$\cos\phi \mathrm{d}\phi = 2a\frac{(a+b)-(a-b)\sin^2\theta}{[(a+b)+(a-b)\sin^2\theta]^2}\cos\theta\mathrm{d}\theta \tag{1.6}$$

但是

$$\cos\phi = \frac{\sqrt{(a+b)^2-(a-b)^2\sin^2\theta}}{(a+b)+(a-b)\sin^2\theta}\cos\theta. \tag{1.7}$$

(2.6) / (2.7), 两式相除, 得到

$$\mathrm{d}\phi = 2a\frac{(a+b)-(a-b)\sin^2\theta}{(a+b)+(a-b)\sin^2\theta}\frac{\mathrm{d}\theta}{\sqrt{(a+b)^2-(a-b)^2\sin^2\theta}} \tag{1.8}$$

另一方面

$$\sqrt{a^2\cos}$$

因而

$$\frac{\mathrm{d}\phi}{\sqrt{a^2\cos}}$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$, 则

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a_1^2\cos^2\theta + b_1^2\sin^2\theta}} \tag{1.11}$$

反复应用该公式, 得到

$$G = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{a_n^2\cos^2\phi + b_n^2\sin^2\phi}}, \qquad (n = 1, 2, 3, \dots) \tag{1.12}$$

$$\frac{\pi}{2a_n} < G < \frac{\pi}{2b_n} \tag{1.13}$$

积分 G 可以归结到第一类全椭圆积分 $K(k) = (1+k_1)K(k_1) = \frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}} = (1+k_1)\int_0^{\frac{pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k_1^2\sin^2\theta}} \tag{1.14}$$

其中

$a_1 = \frac{1+\sqrt{1-k^2}}{2} = \frac{1+k'}{2}, b_1 = \sqrt{k'}$

$k_1 = \frac{\sqrt{a_1^2-b_1^2}}{a_1} = \frac{1-k'}{1+k'}, \frac{1}{a_1} = 1 + k_1$

1.3 2.3 单调数列

Example 1.6 2.3.6

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\cdots+n!}{n!}} \\ &= \frac{1}{n+1}\frac{1!+2!+\cdots+(n+1)!}{1!+2!+\cdots+n!} \\ &= \frac{3+3!+\cdots+(n+1)!}{(n+1)1!+(n+1)2!+\cdots+(n+1)!}\end{aligned}$$

$n > 2$ 时, 分母每一项大于等于分子对应项.. $n > 2$ 后a_n 单调减少. 由于 0 是下界, 因此 a_n 单调有界, 数列

收敛.

$$a_{n+1} = \frac{1! + 2! + \cdots + (n+1)!}{(n+1)!}$$
$$= \frac{1! + 2! + \cdots + n!}{n!} \frac{1}{n+1} + 1$$
$$= 1 + \frac{a_n}{n+1}$$

设 $n \to \infty$ 时, $a_n \to a$

$$a = 1 + \left(\frac{1}{n+1} \to 0\right) = 1 + 0, \qquad \therefore a = 1$$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1!+2!+\cdots+n!}{n!} = 1$

1.3.1 2.3.2 练习题

证明, 若x_n 单调, 则$|x_n|$ 至少从某项开始后单调, 又问: 反之如何?

Proof 分类讨论, 不妨设 $x_1 \geqslant 0$

1. x_n 单调递增, $|x_n|$ 从第一项开始单调.
2. x_n 单调递减, 且 $|x_n| \geqslant 0$. $|x_n|$ 从第一项开始单调.
3. x_n 单调递减, 且 $\exists N$ s.t. $x_n < 0$(第一个负数项). 则$|x_n|$ 从第 N 项 (x_N) 开始单调.

反之该结论不成立.

反例: $x_n = \frac{(-1)^n}{n}$, $|x_n|$ 单调递减. 但 $x_{2k} = \frac{1}{2k} > 0 > x_{2k-1} = \frac{-1}{2k-1}$

设a_n 单调增加, b_n 单调减少, 且有 $\lim\limits_{n\to\infty}(a_n - b_n) = 0$.

证明: 数列a_n 和b_n 都收敛, 且极限相等.

Proof $\lim\limits_{n\to\infty}(a_n - b_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N}_+, \text{s.t.} \forall n > N, |a_n - b_n - 0| < \epsilon$.

$b_n - \epsilon < a_n < b_n + \epsilon$, 同时有 $a_n - \epsilon < b_n < a_n + \epsilon$.

b_n 单调减少, $\therefore \exists N_2, \forall m < N_2, b_m > b_n + \epsilon$.

使用反证法证明 b_m 是a_n 的上界.

假设 b_m 不是a_n 的上界，则存在 $a_n > b_m > b_n + \epsilon$, 这与 $|a_n - b_n| < \epsilon$ 矛盾.

$\therefore b_m$ 是a_n 的上界, 根据单调有界收敛准则, a_n 收敛. 同理可证b_n 收敛. $\lim\limits_{n\to\infty}(a_n - b_n) = 0$. $\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.
2. 单调减少有下界的数列的极限不大于数列中的任何一项.

设 $x_n = \frac{2}{3} \cdot \frac{3}{5} \cdots \frac{n+1}{2n+1}$, $n \in \mathbb{N}_+$, 求数列x_n 的极限.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)+1}{2(n+1)+1} = \frac{n+2}{2n+3} < 1. \qquad (n > 0) \tag{1.15}$$

x_n 单调递减. $\because x_n > 0$, $\therefore x_n$ 有下界, x_n 收敛.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$, 由夹逼定理, $\lim\limits_{n\to\infty} x_n = 0$

6. 在例题 2.2.6 的基础上证明: 当 $p > 1$ 时，数列S_n 收敛. 其中

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$$

(S_n 就是 p 级数, 当 $p = 1$ 时为调和级数.)

Proof　S_n 单调递增, 记 $\frac{1}{2^{p-1}}=r$, 则 $0<r<1$.

$$\begin{aligned}
\frac{1}{2^p}+\frac{1}{3^p}&<\frac{1}{2^p}+\frac{1}{2^p} &&=\frac{1}{2^{p-1}}=r\\
\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}&<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} &&=\frac{1}{4^{p-1}}=r^2\\
\frac{1}{(2^k)^p}+\cdots+\frac{1}{(2^{k+1}-1)^p}&<\frac{1}{(2^k)^p}+\frac{1}{(2^k)^p}+\cdots+\frac{1}{(2^k)^p} &&=\frac{1}{(2^k)^{p-1}}=r^k
\end{aligned}$$

由此可知

$$S_n\leqslant S_{2^n-1}<1+r+r^2+\cdots+r^{n-1}=\frac{1-r^n}{1-r}<\frac{1}{1-r}$$

S_n 单调递增有上界，由单调有界收敛准则知S_n 收敛。

7. 设 $0<x_0<\frac{\pi}{2}$, $x_n=\sin x_{n-1}$. $n\in\mathbb{N}_+$.

证明x_n 收敛，并求其极限。

Proof　$x_0\in(0,\frac{\pi}{2})$, $\sin x$,

$0<x_1=\sin x_0<x_0<\frac{\pi}{2}$.

$0<x_2=\sin x_1<x_1<\frac{\pi}{2}$.

$0<\cdots<x_n<x_{n-1}<\cdots<x_2<x_1<\frac{\pi}{2}$.

x_n 单调递减有下界，x_n 收敛。

$a=\sin a,\quad a\in[0,\frac{\pi}{2}]$

解得 $a=0$，$\therefore\lim\limits_{n\to\infty}x_n=0$.