

具体数学阅读笔记-chap2

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$$A = \begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & & \vdots \\ a_na_1 & a_na_2 & \cdots & a_na_n \end{bmatrix} \quad (1)$$

求 $S_{\triangleleft} = \sum_{1 \leq j \leq k \leq n} a_j a_k$ ¹

解 1 $\because a_j a_k = a_k a_j$, \therefore 矩阵 A 沿主对角线对称, $S_{\triangleleft} = S_{\triangleright}$.

$$[1 \leq j \leq k \leq n] + [1 \leq k \leq j \leq n] = [1 \leq j, k \leq n] + [1 \leq j = k \leq n]$$

$$\begin{aligned} 2S_{\triangleleft} &= S_{\triangleleft} + S_{\triangleright} = S_A + S_{diag(A)} \\ &= \sum_{1 \leq j, k \leq n} a_j a_k + \sum_{1 \leq j=k \leq n} a_j a_k \\ &= \left(\sum_{j=1}^n a_j \right) \left(\sum_{k=1}^n a_k \right) + \sum_{k=1}^n a_k^2 \\ &= \left(\sum_{k=1}^n a_k \right)^2 + \sum_{k=1}^n a_k^2 \end{aligned}$$

$$\therefore S_{\triangleleft} = \frac{1}{2} [(\sum_{k=1}^n a_k)^2 + \sum_{k=1}^n a_k^2]$$

¹下三角形矩阵的符号是一个右上部分的直角三角形, 目前我还会不会输入

$$S = \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) \quad (2)$$

解 2 交换 j, k 仍有对称性.

$$S = \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) = \sum_{1 \leq j < k \leq n} (a_j - a_k)(b_j - b_k)$$

$$[1 \leq j < k \leq n] + [1 \leq k < j \leq n] = [1 \leq j, k \leq n] - [1 \leq j = k \leq n]$$

$$2S = 2 \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j)$$

$$= \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) + \sum_{1 \leq k < j \leq n} (a_k - a_j)(b_k - b_j)$$

$$= \sum_{1 \leq j, k \leq n} (a_k - a_j)(b_k - b_j) - \sum_{1 \leq j = k \leq n} (a_k - a_j)(b_k - b_j)$$

$$(a_j - a_k = 0, b_j - b_k = 0, [j = k]) = \sum_{1 \leq j, k \leq n} (a_k b_k - a_j b_k -$$

$$= \sum_{j=1}^n \sum_{k=1}^n a_k b_k - \sum_{j=1}^n \sum_{k=1}^n a_j b_k - \sum_{j=1}^n \sum_{k=1}^n a_k b_j + \sum_{j=1}^n \sum_{k=1}^n a_j b_j$$

$$= n \sum_{k=1}^n a_k b_k - \sum_{j=1}^n \sum_{k=1}^n a_j b_k - \sum_{j=1}^n \sum_{k=1}^n a_k b_j + n \sum_{j=1}^n a_j b_j$$

$$= 2n \sum_{k=1}^n a_k b_k - 2 \sum_{j=1}^n a_j \sum_{k=1}^n b_k$$

$$S = n \sum_{k=1}^n a_k b_k - \left(\sum_{k=1}^n a_k \right) \left(\sum_{k=1}^n b_k \right)$$

对上式结果重新排序得

$$\left(\sum_{k=1}^n a_k \right) \left(\sum_{k=1}^n b_k \right) = n \sum_{k=1}^n a_k b_k - \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j)$$

定理 0.1 切比雪夫单调不等式 (Chebyshech's monotonic inequality)

$$\begin{array}{ll} (\sum_{k=1}^n a_k) (\sum_{k=1}^n b_k) & \leq n \sum_{k=1}^n a_k b_k & a_1 \leq a_2 \leq \cdots \leq a_n, \text{ and } b_1 \leq b_2 \leq \cdots \leq b_n \\ & & a_1 \geq a_2 \geq \cdots \geq a_n, \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n \\ (\sum_{k=1}^n a_k) (\sum_{k=1}^n b_k) & \geq n \sum_{k=1}^n a_k b_k & a_1 \leq a_2 \leq \cdots \leq a_n, \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n \\ & & a_1 \geq a_2 \geq \cdots \geq a_n, \text{ and } b_1 \leq b_2 \leq \cdots \leq b_n \end{array}$$