

## 0.1 2.1.5

Question 0.2 1. prove by Limit definition:

- (1).  $\lim_{n \rightarrow \infty} \frac{3n^2}{n^2-4} = 3$ .
- (2).  $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$ .
- (3).  $\lim_{n \rightarrow \infty} (1+n)^{\frac{1}{n}} = 0$ .
- (4).  $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0, (a > 0)$ .

Question 0.3 2. Suppose  $a_n, n \in \mathbb{N}_+$ . sequence  $a_n$  converge to  $a$ .

Prove  $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$ .

**Proof**  $n \rightarrow \infty a_n \rightarrow a$ .

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$ .

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$ .  $\square$  (check, not consider the condition  $a = 0$ ) add  $a = 0, \forall \epsilon \in (0, 1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$ . s.t  $a_n < \epsilon^2 < \epsilon, \sqrt{a_n} < \epsilon$ .

Question 0.4 3. If  $\lim_{n \rightarrow \infty} a_n = a$ .

Prove  $\lim_{n \rightarrow \infty} |a_n| = |a|$ . Vice versa?

**Proof**  $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$ .

$$||a_n| - |a|| \leq |a_n - a| < \epsilon$$

$\therefore \lim_{n \rightarrow \infty} |a_n| = |a|$

If We know  $\lim_{n \rightarrow \infty} |a_n| = |a|$ .

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), ||a_n| - |a|| < \epsilon$ . We can't get  $\lim_{n \rightarrow \infty} a_n = a$ . For example:

$a_n = \frac{1}{n} + 1, a = -1, \lim_{n \rightarrow \infty} |a_n| = |a|$  is  $\lim_{n \rightarrow \infty} |\frac{1}{n} + 1| = |-1|$ , but  $\lim_{n \rightarrow \infty} \frac{1}{n} + 1 \neq -1$   $\square$

Question 0.5 (1). Suppose  $p(x)$  is a polynomial of  $x$ , if  $\lim_{n \rightarrow \infty} a_n = a$ , Prove  $\lim_{n \rightarrow \infty} p(a_n) = p(a)$ .

(2). Suppose  $b > 0, \lim_{n \rightarrow \infty} a_n = a$ . Prove  $b^{a_n} = b^a$ .

(3). Suppose  $b > 0, \{a_n\}, a_n > 0, \forall n \in \mathbb{N}$ .  $\lim_{n \rightarrow \infty} a_n = a, a > 0$ . Prove  $\lim_{n \rightarrow \infty} \log_b a_n = \log_b a$ .

(4) Suppose  $b \in \mathbb{R}, \{a_n\}, a_n > 0$  when  $n \in \mathbb{N}$ .  $\lim_{n \rightarrow \infty} a_n = a$ . Prove  $\lim_{n \rightarrow \infty} a_n^b = a^b$ .

(5) Suppose  $\lim_{n \rightarrow \infty} a_n = a$ . Prove  $\lim_{n \rightarrow \infty} \sin a_n = \sin a$ .

**Proof** 4.(1)

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geq N(\epsilon), |a_n - a| < \epsilon$ .

$p(a) = k_m a^m + k_{m-1} a^{m-1} + \dots + k_0 a^0$ .

$\therefore p(a_n) - p(a) = k_m (a_n^m - a^m) + k_{m-1} (a_n^{m-1} - a^{m-1}) + \dots + k_0 (a_n^0 - a^0)$ .

$$|a_n^m - a^m| = |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2} a + \dots + a^{m-1}|$$

$$< \epsilon \cdot |a_n^{m-1} + a_n^{m-2} a + \dots + a^{m-1}|$$

$$< \epsilon(m-1) \dots (a + \delta)^{m-1}$$

$\therefore \lim_{n \rightarrow \infty} p(a_n) = p(a)$ .  $\square$

**Proof** 4.(2)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$ .

If  $b = 1, 1^{a_n} = 1^a = 1$ .

If  $b > 1, b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1) < b^a \cdot (b^\epsilon - 1) \because b > 0, \epsilon \rightarrow 0,$

$\therefore b^\epsilon - 1 \rightarrow 0. \therefore \lim_{n \rightarrow \infty} b^{a_n} = b^a$ .

If  $b < 1, b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$ , we can prove this condition by considering  $\frac{1}{b} > 1$ .

**Proof** 4.(3)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$ .

$$\begin{aligned} \log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b \left( \frac{a_n - a}{a} + 1 \right) < \log_b \left( \frac{\epsilon}{a} + 1 \right) \end{aligned}$$

$0 < \log_b a_n - \log_b a < \log_b \left( 1 + \frac{\epsilon}{a} \right). \because b > 0, a \neq 0, a_n > 0$  when  $\epsilon \rightarrow 0. \therefore \log_b \left( 1 + \frac{\epsilon}{a} \right) \rightarrow 0$ .

$\therefore \lim_{n \rightarrow \infty} \log_b a_n = \log_b a$

**Proof** 4.(4)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$ .

$a_n^b = e^{b \ln a_n}, a_n^b - a^b = e^{b \ln a_n} - e^{b \ln a}$ .

$$\begin{aligned} e^{b \ln a_n} - e^{b \ln a} &= e^{b \ln a} (e^{b \ln a_n - b \ln a} - 1) \\ &= e^{b \ln a} (e^{b \ln \frac{a_n}{a}} - 1) \end{aligned}$$

$0 < |a_n^b - a^b| < e^{b \ln a} (e^{b \ln(1 + \frac{\epsilon}{a})} - 1)$

$\therefore \lim_{n \rightarrow \infty} a_n^b = a^b$

**Proof** 4.(5)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$ .

$$\begin{aligned} \sin(A + B) - \sin(A - B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B \end{aligned}$$

$$\sin a_n - \sin a = 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}$$

$$|\sin a_n - \sin a| = \left| 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2} \right| < \left| 2 \sin \frac{a_n - a}{2} \right|$$

$$\left| 2 \sin \frac{a_n - a}{2} \right| < \left| 2 \frac{a_n - a}{2} \right| = \epsilon$$

$$|\sin a_n - \sin a| < \epsilon, \therefore \lim_{n \rightarrow \infty} \sin a_n = \sin a$$

Question 0.6 assume  $a > 1$ . Prove  $\lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$

**Proof**  $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$ . We already know that  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \log_a 1 = 0$ .

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geq N, |\sqrt[n]{n} - 1| < \epsilon$ .

$a > 1$ , and  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .  $\therefore$  when  $n \rightarrow \infty, \sqrt[n]{n} < a^\epsilon$ , take logarithm on base of  $a$ , we can get  $\frac{1}{n} \log_a n < \epsilon$

$\therefore \lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$

收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界

3. 收敛数列的比较定理, 包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

## 0.7 2.2.1

## 思考题

- Question 0.8
1.  $\{a_n\}$  收敛,  $\{b_n\}$  发散,  $\{a_n + b_n\}$  发散,  $\{a_n \cdot b_n\}$  可能收敛, 可能发散.
  2.  $\{a_n\}, \{b_n\}$  都发散,  $\{a_n + b_n\}$  可能收敛, 可能发散 (ex:  $n + -n, n + -2n$ ),  $\{a_n \cdot b_n\}$  发散 (?).
  3.  $a_n \leq b_n \leq c_n, n \in \mathbb{N}_+$ . 已知  $\lim_{n \rightarrow \infty} (c_n - a_n) = 0$ . 问数列  $\{b_n\}$  是否收敛?
  4.  $\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n})$
  5.  $a_n \rightarrow a (n \rightarrow \infty)$ .  $\forall n, b < a_n < c$ . 是否成立  $b < a < c$ ?
  6.  $a_n \rightarrow a (n \rightarrow \infty)$ . and  $b \leq a \leq c$ , 是否存在  $N \in \mathbb{N}_+$ , s.t. 当  $n > N$  时, 成立  $b \leq a_n \leq c$
  7. 已知  $\lim_{n \rightarrow \infty} a_n = 0$ , 问: 是否有  $\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0$ . 反之如何?

Proof 5.4

$$\frac{n}{2n} \leq \frac{1}{n+1} + \dots + \frac{1}{2n} \leq \frac{n}{n+1}$$

$\therefore \lim_{n \rightarrow \infty} \frac{n}{2n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{1}{2}$ ,  $\therefore \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n})$  收敛.

$$\begin{aligned} \frac{1}{n+1} + \dots + \frac{1}{2n} &= \frac{1}{n} \left( \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \dots + \frac{1}{1+\frac{n}{n}} \right) \\ &= \int_0^1 \frac{1}{1+x} dx \\ &= \ln(1+x)|_0^1 = \ln 2 \end{aligned}$$

$$\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n}) = \ln 2$$

Proof 5.5

不成立, 应当为小于等于号.  $b=0, c=2, a_n = \frac{1}{n}, \lim_{n \rightarrow \infty} a_n = 0 = c$ .

Proof 5.6

不成立.  $a = 0, b = 0, c = 2, a_n = (-1)^n \frac{1}{n}$ .

$b \leq a \leq c$ , but  $(-1)^{2n+1} \frac{1}{2n+1} < 0 = b$ .

Proof  $\lim_{n \rightarrow \infty} a_n = 0, a_n = \frac{1}{n} \cdot a_1 a_2 \dots a_n = \frac{1}{n!}, \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$ .

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0 \quad \checkmark$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0 \rightarrow \lim_{n \rightarrow \infty} a_n = 0 \quad \times$$

$$|a_n| < \epsilon, |a_{N+1} \dots a_n| < \epsilon^{n-N} < \epsilon, a_n < \sqrt[n]{\epsilon}.$$

for example,  $a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \dots, a_n = \frac{n}{n+1}$ .

$$a_1 a_2 \dots a_n = \frac{1}{2} \cdot \frac{2}{3} \dots \frac{n}{n+1} = \frac{1}{n+1}.$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

but  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

**Example 0.1** 2.2.2  $\lim_{n \rightarrow \infty} \frac{x_n - 1}{x_n + a} = 0$ , prove  $\lim_{n \rightarrow \infty} x_n = a$

**Proof**  $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |\frac{x_n - 1}{x_n + a} - 0| < \epsilon$ .

$$|x_n - 1| < \epsilon |x_n + a| < 4a \cdot \epsilon. (\text{这个 } 4 \text{ 是怎么取得的?})$$

$$|x_n - a| < \epsilon |x_n + a| = \epsilon |(x_n - a) + 2a| \leq \epsilon (|x_n - a| + 2a).$$

$$\text{限制 } \epsilon < 1, |x_n - a| < 2\epsilon |a| / (1 - \epsilon).$$

$$\text{限制 } \epsilon < \frac{1}{2}, |x_n - a| < 2\epsilon |a| / (1 - \epsilon) < 4|a|\epsilon.$$

$$\text{Let } \epsilon' = 4a\epsilon, |x_n - 1| < \epsilon'. \therefore \lim_{n \rightarrow \infty} x_n = a.$$

**Example 0.2** 2.2.3  $a > 0, b > 0$ , 计算  $\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}}$ .

**Proof** Suppose  $a \leq b$ .

$$b = (b^b)^{\frac{1}{b}} < (a^n + b^n)^{\frac{1}{n}} \leq (2b^n)^{\frac{1}{n}}.$$

$$b < (a^n + b^n)^{\frac{1}{n}} \leq \sqrt[n]{2b}, \lim_{n \rightarrow \infty} = 1. \text{ 夹逼定理.}$$

$$\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

两数  $n$  次方之和和再开  $n$  次根号的结果由较大的值决定,  $a, b$  中较大的值为这个数的主要部分.

**Example 0.3** 2.2.4  $a_n = \frac{1! + 2! + \dots + n!}{n!}, n \in \mathbb{N}^+$

$$\lim_{n \rightarrow \infty} a_n = 1$$

**Example 0.4**  $\lim_{n \rightarrow \infty} \frac{n^3 + n - 7}{n + 3} = +\infty$

**Example 0.5**  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

调和级数  $H_n$  发散.

## 0.8.1 练习 2.2.4

**Proof** 1.

$\{a_n\}$  收敛于  $a$ ,  $\rightarrow$  两个子列  $\{a_{2n}\}, \{a_{2n+1}\}$  均收敛于  $a$ .

两个子列  $\{a_{2n}\}, \{a_{2n+1}\}$  均收敛于  $a$ ,  $\rightarrow \{a_n\}$  收敛于  $a$ .

### 2. 应用夹逼定理

(1). 给定  $p$  个正数  $a_1, a_2, \dots, a_p$ . 求  $\lim_{n \rightarrow \infty} \sqrt[n]{a_1^n + a_2^n + \dots + a_p^n}$ .

Let  $a_s = \max_{1 \leq i \leq p} \{a_1, a_2, \dots, a_p\}$ .

Solve 0.9 (1).

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} \leq (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$$n \rightarrow \infty, p^{\frac{1}{n}} \rightarrow 1. \lim_{n \rightarrow \infty} (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} = a_s$$

$$(2). x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}, n \in \mathbb{N}_+. \text{ 求 } \lim_{n \rightarrow \infty} x_n$$

Solve 0.10 (2).

$$\frac{2n+1}{(n+1)} \leq x_n \leq \frac{2n+1}{\sqrt{n^2+1}}$$

$$\lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = 2, \lim_{n \rightarrow \infty} \frac{2n+1}{\sqrt{n^2+1}} = 2. \therefore \lim_{n \rightarrow \infty} x_n = 2$$

$$(3). a_n = (1 + \frac{1}{2} + \dots + \frac{1}{n})^{\frac{1}{n}}, n \in \mathbb{N}_+. \text{ 求 } \lim_{n \rightarrow \infty} a_n$$

Solve 0.11 (3).

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leq (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \therefore \lim_{n \rightarrow \infty} a_n = 1$$

(4).  $a_n > 0$ .  $\lim_{n \rightarrow \infty} a_n = a$ ,  $a > 0$ . 证明  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$

**Proof**  $\lim_{n \rightarrow \infty} a_n = a$

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$ .

$$0 < a - \epsilon < a_n < a + \epsilon$$

$$\therefore \sqrt[n]{a - \epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a + \epsilon}.$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a - \epsilon} = 1, \lim_{n \rightarrow \infty} \sqrt[n]{a + \epsilon} = 1. \therefore \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1.$$

$$3. (1). \lim_{n \rightarrow \infty} (1+x)(1+x^2) \dots (1+x^{2^n}) = \lim_{n \rightarrow \infty} \prod_{i=1}^{2^n} (1+x^i), |x| < 1.$$

Solve 0.12 3.(1).

$$|x| < 1, \quad 1 > x^2 > x^4 > \dots > x^{2^n} > 0$$

$$x \in (0, 1) \quad 1 < (1+x)(1+x^2) \dots (1+x^{2^n}) < (1+x)^{n+1} \quad \lim_{n \rightarrow \infty} (1+x)^{n+1} = 1$$

$$x \in (-1, 0) \quad 0 < (1+x)(1+x^2) \dots (1+x^{2^n}) < (1+x)(1+x^2)^n \quad \lim_{n \rightarrow \infty} (1+x)(1+x^2)^n = 1$$

Solve 0.13 3.(1). another way

$$\begin{aligned} & \lim_{n \rightarrow \infty} (1+x)(1+x^2) \dots (1+x^n) \\ &= \lim_{n \rightarrow \infty} \frac{(1-x)(1+x)(1+x^2) \dots (1+x^n)}{1-x} \\ &= \lim_{n \rightarrow \infty} \frac{(1-x^{n+1})}{1-x} \\ &= \frac{1}{1-x} \end{aligned}$$

Solve 0.14 3. (2).

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \dots \left(1 - \frac{1}{n^2}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \dots \frac{n-1}{n} \cdot \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \end{aligned}$$

Solve 0.15 3. (3).

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2+3}\right) \dots \left(1 - \frac{1}{1+2+\dots+n}\right) \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{2}{3 \times 2}\right) \left(1 - \frac{2}{4 \times 3}\right) \dots \left(1 - \frac{2}{(n+1) \times n}\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{3 \times 2 - 2}{3 \times 2}\right) \left(\frac{4 \times 3 - 2}{4 \times 3}\right) \dots \left(\frac{(n+1) \times n - 2}{(n+1) \times n}\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{4}{3 \times 2}\right) \left(\frac{10}{4 \times 3}\right) \dots \left(\frac{n^2 + n - 2}{(n+1) \times n}\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1 \times 4}{3 \times 2}\right) \left(\frac{2 \times 5}{4 \times 3}\right) \dots \left(\frac{(n-2) \times (n+1)}{n \times (n-1)}\right) \left(\frac{(n-1) \times (n+2)}{(n+1) \times n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{3} \times \frac{n+2}{n} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \right] \\
&= \lim_{n \rightarrow \infty} \left[ \frac{1}{1} - \frac{1}{n+1} \right] \\
&= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\
&= 1
\end{aligned}$$

Solve 0.16 3.(4).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1) \cdot (n+2)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \frac{1}{2} \left( \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right) + \cdots + \frac{1}{2} \left( \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)} \right) \\
&= \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

Solve 0.17 3.(5).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1) \cdots (k+\gamma)}, \quad \text{其中 } \gamma \text{ 为正整数} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\gamma} \left[ \frac{1}{k(k+1) \cdots (k+\gamma-1)} - \frac{1}{(k+1)(k+2) \cdots (k+\gamma)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[ \sum_{k=1}^n \frac{1}{k(k+1) \cdots (k+\gamma-1)} - \sum_{k=1}^n \frac{1}{(k+1)(k+2) \cdots (k+\gamma)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[ \frac{1}{\gamma!} - \frac{1}{(n+\gamma)!} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[ \frac{1}{\gamma!} - \frac{1}{(n+\gamma)!} \right] \\
&= \frac{1}{\gamma} \cdot \frac{1}{\gamma!}
\end{aligned}$$

其中  $x^n = x(x-1)(x-2) \cdots (x-n+1)$ , 称为下阶乘. 而  $x^{\overline{n}} = x(x+1)(x+2) \cdots (x+n-1)$ , 称为上阶乘.

Question 0.18 2.2.4-4  $S_n = a + 3a^2 + \cdots + (2n-1)a^n$ ,  $|a| < 1$ . 求  $\{S_n\}$  的极限.

Solve 0.19

$$\begin{aligned}
S_n - aS_n &= a + 3a^2 + \cdots + (2n-1)a^n \\
&\quad - a^2 - \cdots + (2n-3)a^n - (2n-1)a^n + 1 \\
&= a + 2a^2 + \cdots + 2aa^n - (2n-1)a^{n+1} \\
&= 2(a + a^2 + \cdots + a^n) - a - (2n-1)a^{n+1} \\
&= 2 \cdot a \frac{1 - a^{n+1}}{1 - a} - a - (2n-1)a^{n+1}
\end{aligned}$$

$|a| < 1, \lim_{n \rightarrow \infty} a_n = 0$

$\lim_{n \rightarrow \infty} (S_n - aS_n) = (1-a) \lim_{n \rightarrow \infty} S_n$

$$\begin{aligned}\lim_{n \rightarrow \infty} (S_n - aS_n) &= \lim_{n \rightarrow \infty} 2a \cdot \frac{1 - a^{n+1}}{1 - a} - a - (2n-1)a^{n+1} \\ &= 2a \cdot \frac{1}{1-a} - a \\ &= a \left( \frac{2}{1-a} - a \right) \\ &= a \frac{1+a}{1-a}\end{aligned}$$

$\therefore \lim_{n \rightarrow \infty} S_n = \frac{a(a+1)}{(1-a)^2}$

Solve 0.20 2.2.4-5 设  $\lim_{n \rightarrow \infty} x_n = A > 0$ . 取  $\epsilon = \frac{A}{2}$ , 则  $\exists N \in \mathbb{N}_+, \forall n > N$ . 成立  $|x_n - A| < \frac{A}{2}$

$$A - \frac{A}{2} < x_n < A + \frac{A}{2}, \frac{A}{2} < x_n < \frac{3A}{2}$$

即  $x_n > \frac{A}{2}$ .

令  $m = \min\{x_1, x_2, \dots, x_N, \frac{A}{2}\} > 0$ . 则  $m$  为  $\{x_n\}$  的正下界.

不一定有最小数的例子  $x_n = 1 + \frac{1}{n}$ .  $\lim_{n \rightarrow \infty} x_n = 1$ , 下界  $m = \frac{1}{2}$ . 但  $\{x_n\}$  取不到下界.

**Proof** 2.2.4-6  $\because \lim_{n \rightarrow \infty} a_n = +\infty, \forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M$ .

$m = \min\{a_1, a_2, \dots, a_N, M\}$ , 但  $M$  在数列  $\{a_n\}$  中不一定取的到!

$M = a_1 + 1, \exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$

则  $m = \min\{a_1, a_2, \dots, a_{N_1}\}$  为数列的最小数.

**Proof** 2.2.4-7 构造数列

不妨设无界数列  $\{a_n\}$  无上界.

$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M$ .

取  $M_1 = 1$ , 则  $\exists n_1 \in \mathbb{N}_+$  s.t.  $a_{n_1} > M_1$ .

取  $M_2 = \max\{a_{n_1}, 2\}$ ,  $\exists n_2 \in \mathbb{N}_+$  s.t.  $a_{n_2} > M_2$ .

以此类推, 构造数列  $\{a_{n_k}\}$ . s.t.  $a_{n_k} > k$ . 即  $a_{n_k}$  为无穷大量.

**Proof** 2.2.4-8 证明  $\{a_n\}, a_n = \tan n$  发散.

构造  $a_n$  的发散子列即可. 已知  $\tan \frac{\pi}{2} = \infty$ ,  $\pi$  是一个无理数, 因此存在数列  $\{b_n\}, \lim_{n \rightarrow \infty} b_n = \frac{\pi}{2}$ .

**Proof** 2.2.4-8 证明  $\{a_n\}, a_n = \tan n$  发散. 参考别人的答案

由于  $\{\sin 2n\}$  极限不存在, 又

$$\begin{aligned}\sin 2n &= 2 \sin n \cos n = \frac{2 \sin n \cos n}{\sin^2 n + \cos^2 n} \\ &= \frac{2 \tan n}{\tan^2 n + 1}\end{aligned}$$

若  $\{\tan n\}$  极限存在  $\rightarrow \{\sin 2n\}$  极限存在, 矛盾.

故  $\{\tan n\}$  极限不存在.

Quetion 0.21 2.2.4-9  $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p}, n \in \mathbb{N}_+.$   $S_n$  在 1.  $p \leq 0$ , 2.  $0 < p < 1$  情况下均发散

**Proof** 1.  $p \leq 0. \lim_{n \rightarrow \infty} n^{-p} = \infty, S_n$  发散.

2.  $0 < p < 1. \frac{1}{n^p} > \frac{1}{n}. \because H_n = \sum_{k=1}^n \frac{1}{k}$  (调和级数) 发散,  $S_n > H_n, \therefore \{S_n\}$  也发散.

ex2.3.5  $0 < b < a$  令  $a_0 = a, b_0 = b$  递推公式

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, b_n = \sqrt{a_{n-1}b_{n-1}}, n \in \mathbb{N}_+ \quad (1)$$

定义数列  $a_n, b_n$ . 证明这两个数列收敛于同一个极限  $AG(a, b)$ .

由 AG 不等式  $a > \frac{a+b}{2} > \sqrt{ab} > b > 0$ , 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a, b) = \frac{\pi}{2G} \quad (2)$$

如果令  $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$ . 则

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \quad (3)$$

上面这个公式是怎么得到的:

参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} \quad (a > b > 0) \quad (4)$$

这里令

$$\sin \phi = \frac{2a \sin \theta}{(a+b) + (a-b) \sin^2 \theta} \quad (5)$$

$\theta \in [0, \frac{\pi}{2}] \rightarrow \phi \in [0, \frac{\pi}{2}]$ , 取微分

$$\cos \phi d\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{[(a+b) + (a-b) \sin^2 \theta]^2} \cos \theta d\theta \quad (6)$$

但是

$$\cos \phi = \frac{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}}{(a+b) + (a-b) \sin^2 \theta} \cos \theta. \quad (7)$$

(6) / (7), 两式相除, 得到

$$d\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \frac{d\theta}{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}} \quad (8)$$

另一方面

$$\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \quad (9)$$

因而

$$\frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \frac{d\theta}{\sqrt{(\frac{a+b}{2})^2 \cos^2 \theta + ab \sin^2 \theta}}. \quad (10)$$

如果令  $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$ , 则

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \quad (11)$$

反复应用该公式, 得到

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}}, \quad (n = 1, 2, 3, \dots) \quad (12)$$

$$\frac{\pi}{2a_n} < G < \frac{\pi}{2b_n} \quad (13)$$

积分  $G$  可以归结到第一类全椭圆积分  $K(k) = (1+k_1)K(k_1) = \frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = (1+k_1) \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k_1^2 \sin^2 \theta}} \quad (14)$$

其中

$$a_1 = \frac{1+\sqrt{1-k^2}}{2} = \frac{1+k'}{2}, b_1 = \sqrt{k'}$$



$$k_1 = \frac{\sqrt{a_1^2 - b_1^2}}{a_1} = \frac{1-k'}{1+k'}, \frac{1}{a_1} = 1 + k_1$$

Example 0.6 2.3.6

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\cdots+n!}{n!}} \\ &= \frac{1}{n+1} \frac{1!+2!+\cdots+(n+1)!}{1!+2!+\cdots+n!} \\ &= \frac{3+3!+\cdots+(n+1)!}{(n+1)1!+(n+1)2!+\cdots+(n+1)!} \end{aligned}$$

Solve 0.22  $n > 2$  时, 分母每一项大于等于分子对应项..  $n > 2$  后  $a_n$  单调减少. 由于 0 是下界, 因此  $a_n$  单调有界, 数列收敛.

$$\begin{aligned} a_{n+1} &= \frac{1!+2!+\cdots+(n+1)!}{(n+1)!} \\ &= \frac{1!+2!+\cdots+n!}{n!} \frac{1}{n+1} + 1 \\ &= 1 + \frac{a_n}{n+1} \end{aligned}$$

设  $n \rightarrow \infty$  时,  $a_n \rightarrow a$

$$a = 1 + \left( \frac{1}{n+1} \rightarrow 0 \right) = 1 + 0, \quad \therefore a = 1$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1!+2!+\cdots+n!}{n!} = 1$$

### 0.22.1 2.3.2 练习题

Question 0.23 证明, 若  $x_n$  单调, 则  $|x_n|$  至少从某项开始后单调, 又问: 反之如何?

**Proof** 分类讨论, 不妨设  $x_1 \geq 0$

1.  $x_n$  单调递增,  $|x_n|$  从第一项开始单调.
2.  $x_n$  单调递减, 且  $|x_n| \geq 0$ .  $|x_n|$  从第一项开始单调.
3.  $x_n$  单调递减, 且  $\exists n$  s.t.  $x_n < 0$  (第一个负数项). 则  $|x_n|$  从第  $N$  项 ( $x_N$ ) 开始单调.

反之该结论不成立.

反例:  $x_n = \frac{(-1)^n}{n}$ ,  $|x_n|$  单调递减. 但  $x_{2k} = \frac{1}{2k} > 0 > x_{2k-1} = \frac{-1}{2k-1}$

Question 0.24 设  $a_n$  单调增加,  $b_n$  单调减少, 且有  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ .

证明: 数列  $a_n$  和  $b_n$  都收敛, 且极限相等.

**Proof**  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N}_+, \text{s.t. } \forall n > N, |a_n - b_n - 0| < \epsilon$ .

$b_n - \epsilon < a_n < b_n + \epsilon$ , 同时有  $a_n - \epsilon < b_n < a_n + \epsilon$ .

$b_n$  单调减少,  $\therefore \exists N_2, \forall m < N_2, b_m > b_n + \epsilon$ .

使用反证法证明  $b_m$  是  $a_n$  的上界.

假设  $b_m$  不是  $a_n$  的上界, 则存在  $a_n > b_m > b_n + \epsilon$ , 这与  $|a_n - b_n| < \epsilon$  矛盾.

$\therefore b_m$  是  $a_n$  的上界, 根据单调有界收敛准则,  $a_n$  收敛. 同理可证  $b_n$  收敛.  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ .

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

Question 0.25 按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.
2. 单调减少有下界的数列的极限不大于数列中的任何一项.

Question 0.26 设  $x_n = \frac{2}{3} \cdot \frac{3}{5} \cdots \frac{n+1}{2n+1}$ ,  $n \in \mathbb{N}_+$ , 求数列  $x_n$  的极限.

Solve 0.27

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)+1}{2(n+1)+1} = \frac{n+2}{2n+3} < 1. \quad (n > 0) \quad (15)$$

$x_n$  单调递减.  $\because x_n > 0$ ,  $\therefore x_n$  有下界,  $x_n$  收敛.

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \lim_{n \rightarrow \infty} \frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$ , 由夹逼定理,  $\lim_{n \rightarrow \infty} x_n = 0$

Question 0.28 内容...