

# 具体数学阅读笔记-chap2

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## 1 求和

### 1.1 求和符号

### 1.2 求和与递归式 Sums and recurrences

和式

$$S_n = \sum_{k=0}^n a_k \quad (1)$$

等价于递归式

$$\begin{cases} S_0 = a_0 \\ S_n = S_{n-1} + a_n, \quad n > 0. \end{cases} \quad (2)$$

若  $a_n = \text{const.} + k \cdot n$ , 则有

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n, \quad n > 0 \end{cases} \quad (3)$$

$$R_1 = R_0 + \beta + \gamma$$

$$R_2 = R_0 + 2\beta + 3\gamma$$

$$\vdots$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \quad (4)$$

repertoire method 令  $R_n = 1$ , 则  $\alpha = 1, \beta = 0, \gamma = 0$ ,

$$A(n) = 1$$

令  $R_n = n$ , 则  $\alpha = 0, \beta = 1, \gamma = 0$ ,

$$B(n) = n$$

令  $R_n = n^2$ , 则  $\alpha = 0, \beta = -1, \gamma = 2$ ,

$$C(n) = \frac{n(n+1)}{2}$$

### 例 1.1

$$\sum_{k=0}^n (a + bk)$$

### 解 1

$$\begin{cases} R_0 = a \\ R_n = R_{n-1} + a + bn \end{cases}$$

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n \end{cases}$$

$$\alpha = \beta = a, \gamma = b$$

$$\begin{aligned} A(n)\alpha + B(n)\beta + C(n)\gamma &= aA(n) + aB(n) + bC(n) \\ &= a + an + b\frac{n(n+1)}{2} \\ &= a(n+1) + \frac{bn(n+1)}{2} \end{aligned}$$

对上述递归情况进行推广

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n + \delta n^2, \quad n > 0 \end{cases} \quad (5)$$

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta \quad (6)$$

$\delta = 0$  时 (5) 与 (3) 一致, 说明  $A(n), B(n), C(n)$  不变

$$R_n = n^3$$

$$\begin{aligned} R_n - R_{n-1} &= n^3 - (n-1)^3 \\ &= 3n^2 - 3n + 1 \end{aligned}$$

解得  $\alpha = 0, \beta = 1, \gamma = -3, \delta = 3$

$$\begin{aligned} n^3 &= B(n) - 3C(n) + 3D(n) \\ &= n - 3\frac{n(n+1)}{2} + 3D(n) \end{aligned}$$

$$\begin{aligned} 3D(n) &= n^3 - n + 3\frac{n(n+1)}{2} \\ &= n(n+1) \left[ (n-1) + \frac{3}{2} \right] \\ &= n(n+1) \left( n + \frac{1}{2} \right) \\ D(n) &= \frac{1}{3} \left( (n+1) \left( n + \frac{1}{2} \right) n \right) \end{aligned}$$

## 1.3 求和式处理

## 1.4 多重求和

### 1.4.1 Exercise 1

$$A = \begin{bmatrix} a_1 a_1 & a_1 a_2 & \cdots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & \cdots & a_2 a_n \\ \vdots & \vdots & & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n a_n \end{bmatrix} \quad (7)$$

求  $S_{\triangleleft} = \sum_{1 \leq j \leq k \leq n} a_j a_k$  <sup>1</sup>

**解 2**  $\because a_j a_k = a_k a_j$ ,  $\therefore$  矩阵  $A$  沿主对角线对称,  $S_{\triangleleft} = S_{\triangleright}$ .

$$[1 \leq j \leq k \leq n] + [1 \leq k \leq j \leq n] = [1 \leq j, k \leq n] + [1 \leq j = k \leq n]$$

$$\begin{aligned} 2S_{\triangleleft} &= S_{\triangleleft} + S_{\triangleright} = S_A + S_{\text{diag}(A)} \\ &= \sum_{1 \leq j, k \leq n} a_j a_k + \sum_{1 \leq j = k \leq n} a_j a_k \\ &= \left( \sum_{j=1}^n a_j \right) \left( \sum_{k=1}^n a_k \right) + \sum_{k=1}^n a_k^2 \\ &= \left( \sum_{k=1}^n a_k \right)^2 + \sum_{k=1}^n a_k^2 \end{aligned}$$

$$\therefore S_{\triangleleft} = \frac{1}{2} [(\sum_{k=1}^n a_k)^2 + \sum_{k=1}^n a_k^2]$$

### 1.4.2 Exercise 2

$$S = \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) \quad (8)$$

<sup>1</sup>下三角形矩阵的符号是一个右上部分的直角三角形, 目前我还会不会输入

**解 3** 交换  $j, k$  仍有对称性.

$$\begin{aligned}
S &= \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) = \sum_{1 \leq j < k \leq n} (a_j - a_k)(b_j - b_k) \\
&[1 \leq j < k \leq n] + [1 \leq k < j \leq n] = [1 \leq j, k \leq n] - [1 \leq j = k \leq n] \\
2S &= 2 \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) \\
&= \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j) + \sum_{1 \leq k < j \leq n} (a_k - a_j)(b_k - b_j) \\
&= \sum_{1 \leq j, k \leq n} (a_k - a_j)(b_k - b_j) - \sum_{1 \leq j = k \leq n} (a_k - a_j)(b_k - b_j) \\
&(a_j - a_k = 0, b_j - b_k = 0, [j = k]) = \sum_{1 \leq j, k \leq n} (a_k b_k - a_j b_k - a_k b_j + a_j b_j) \\
&= \sum_{j=1}^n \sum_{k=1}^n a_k b_k - \sum_{j=1}^n \sum_{k=1}^n a_j b_k - \sum_{j=1}^n \sum_{k=1}^n a_k b_j + \sum_{j=1}^n \sum_{k=1}^n a_j b_j \\
&= n \sum_{k=1}^n a_k b_k - \sum_{j=1}^n \sum_{k=1}^n a_j b_k - \sum_{j=1}^n \sum_{k=1}^n a_k b_j + n \sum_{j=1}^n a_j b_j \\
&= 2n \sum_{k=1}^n a_k b_k - 2 \sum_{j=1}^n a_j \sum_{k=1}^n b_k \\
S &= n \sum_{k=1}^n a_k b_k - \left( \sum_{k=1}^n a_k \right) \left( \sum_{k=1}^n b_k \right)
\end{aligned}$$

对上式结果重新排序得

$$\left( \sum_{k=1}^n a_k \right) \left( \sum_{k=1}^n b_k \right) = n \sum_{k=1}^n a_k b_k - \sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j)$$

**定理 1.1** 切比雪夫单调不等式 (Chebyshech's monotonic inequality)

$$\begin{aligned}
(\sum_{k=1}^n a_k)(\sum_{k=1}^n b_k) &\leq n \sum_{k=1}^n a_k b_k & a_1 \leq a_2 \leq \cdots \leq a_n, \text{ and } b_1 \leq b_2 \leq \cdots \leq b_n \\
& & a_1 \geq a_2 \geq \cdots \geq a_n, \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n \\
(\sum_{k=1}^n a_k)(\sum_{k=1}^n b_k) &\geq n \sum_{k=1}^n a_k b_k & a_1 \leq a_2 \leq \cdots \leq a_n, \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n \\
& & a_1 \geq a_2 \geq \cdots \geq a_n, \text{ and } b_1 \leq b_2 \leq \cdots \leq b_n
\end{aligned}$$

一般来说，如果  $a_1 \leq a_2 \leq \cdots \leq a_n$  且  $p$  是  $\{1, \dots, n\}$  的一个排列。

那么不难证明：

当  $b_{p(1)} \leq \cdots \leq b_{p(n)}$  时  $\sum_{k=1}^n a_k b_{p(k)}$  最大。

当  $b_{p(1)} \geq \cdots \geq b_{p(n)}$  时  $\sum_{k=1}^n a_k b_{p(k)}$  最小。

$$\sum_{k \in K} a_k = \sum_{P(k) \in K} a_{P(k)}$$

$P(k)$  为这些整数的任意一个排列。

$$f : J \Rightarrow K, \quad j \in J \quad f(j) \in K$$

$$\sum_{j \in J} a_{f(j)} = \sum_{k \in K} a_k \quad \#f^-(k)$$

式中  $\#f^-(k)$  表示集合  $f^-(k) = \{j | f(j) = k\}$  中元素的个数

$$\sum_{j \in J} [f(j) = k] = \#f^-(k)$$

$$\sum_{j \in J} a_{f(j)} = \sum_{\substack{j \in J \\ k \in K}} a_k [f(j) = k] = \sum_{k \in K} a_k \sum_{j \in J} [f(j) = k]$$

若有  $\#f^-(k) = 1$  (一一对应)<sup>2</sup>

$$\sum_{j \in J} a_{f(j)} = \sum_{f(j) \in K} a_{f(j)} = \sum_{k \in K} a_k$$

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<sup>2</sup>这里还不太理解

### 1.4.3 Exercise 3

$$S_n = \sum_{1 \leq j < k \leq n} \frac{1}{k-j}$$

首先写出前几项，尝试寻找规律：

$$S_1 = 0$$

$$S_2 = \frac{1}{2-1} = 1$$

$$S_3 = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{3-2} = \frac{5}{2}$$

$$S_4 = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{3-2} + \frac{1}{4-2} + \frac{1}{4-3} = \frac{13}{3}$$

**解 4** 1. 先对  $j$  求和

$$\begin{aligned} S_n &= \sum_{1 \leq k \leq n} \sum_{1 \leq j < k} \frac{1}{k-j} \\ &= \sum_{1 \leq k \leq n} \sum_{1 \leq (k-j) < k} \frac{1}{k - (k-j)} \quad j \Rightarrow (k-j) \\ &= \sum_{1 \leq k \leq n} \sum_{0 < j \leq k-1} \frac{1}{j} \\ &= \sum_{1 \leq k \leq n} H_{k-1} \quad (H_k \text{ 为调和级数}) \\ &= \sum_{1 \leq k+1 \leq n} H_k \quad k \Rightarrow k+1 \\ &= \sum_{0 \leq k < n} H_k \end{aligned}$$

## 2. 先对 $k$ 求和

$$\begin{aligned}
 S_n &= \sum_{1 \leq j \leq n} \sum_{j < k \leq n} \frac{1}{k-j} \\
 &= \sum_{1 \leq j \leq n} \sum_{j < (k+j) \leq n} \frac{1}{(k+j)-j} \quad k \Rightarrow (k+j) \\
 &= \sum_{1 \leq j \leq n} \sum_{0 < k \leq n-j} \frac{1}{k} \\
 &= \sum_{1 \leq j \leq n} H_{n-j} \quad (H_k \text{ 为调和级数}) \\
 &= \sum_{1 \leq n-j \leq n} H_k \quad j \Rightarrow n-j \\
 &= \sum_{0 \leq j < n} H_j
 \end{aligned}$$

以上两种常用的求和顺序都无法得到这个多重求和的结果，我们需要转换思路。

## 3. 先用 $k+j$ 替换 $k$ (先换元，再求和)

$$\begin{aligned}
 S_n &= \sum_{1 \leq j < (k+j) \leq n} \frac{1}{(k+j)-j} \quad k \Rightarrow k+j \\
 &= \sum_{1 \leq j < (k+j) \leq n} \frac{1}{k} \\
 &= \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq n-k} \frac{1}{k} \quad \text{首先对 } j \text{ 求和} \\
 &= \sum_{1 \leq k \leq n} \frac{n-k}{k} \\
 &= \sum_{1 \leq k \leq n} \left( \frac{n}{k} - 1 \right) = nH_n - n
 \end{aligned}$$

综上所述可得  $\sum_{1 \leq k \leq n} H_k = nH_n - n$

代数：  $k + f(j)$ ,  $f$  为任意函数.

用  $k + f(j)$  替换  $k$ ，并对  $j$  先求和较好。



几何:  $S_n$  ( $n = 4$ )

|         | $k = 1$ | $k = 2$       | $k = 3$        | $k = 4$        |
|---------|---------|---------------|----------------|----------------|
| $j = 1$ |         | $\frac{1}{1}$ | $+\frac{1}{2}$ | $+\frac{1}{3}$ |
| $j = 2$ |         |               | $+\frac{1}{1}$ | $+\frac{1}{2}$ |
| $j = 3$ |         |               |                | $+\frac{1}{1}$ |
| $j = 4$ |         |               |                |                |

先对  $j$  求和 (按列)  $H_1 + H_2 + H_3$  先对  $k$  求和 (按行)  $H_3 + H_2 + H_1$   $k \Rightarrow k + j$  按对角线求和

$$\sum_{k=1}^n \frac{n-k}{k} = n \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n 1$$

$$nH_n - n, n = 4$$

$$\begin{aligned} \frac{4}{1} + \frac{3}{2} + \frac{2}{3} + \frac{1}{4} &= \sum_{k=1}^4 \frac{4-k}{k} \\ &= 4 \sum_{k=1}^4 \frac{1}{k} - \sum_{k=1}^4 1 \\ &= 4H_4 - 4 \end{aligned}$$

$$4 \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) - 4 = \frac{4}{2} + \frac{4}{3} + \frac{4}{4}$$

|         | $k - j = 0$ | $k - j = 1$   | $k - j = 2$    | $k - j = 3$    |
|---------|-------------|---------------|----------------|----------------|
| $j = 1$ |             | $\frac{1}{1}$ | $+\frac{1}{2}$ | $+\frac{1}{3}$ |
| $j = 2$ |             | $\frac{1}{1}$ | $+\frac{1}{2}$ |                |
| $j = 3$ |             | $\frac{1}{1}$ |                |                |
| $j = 4$ |             |               |                |                |

## 1.5 General methods

### 1.5.1 Exercise 4

求  $\square_n = \sum_{0 \leq k \leq n} k^2$ ,  $n \geq 0$  的封闭形式

$$\begin{aligned}\sum_{k=0}^n k^2 &= \sum_{k=0}^n [(k+1)^2 - 2k - 1] \\ &= \sum_{k=1}^{n+1} k^2 - 2 \sum_{k=0}^n k - \sum_{k=0}^n 1\end{aligned}$$

$$\begin{aligned}0^2 - (n+1)^2 &= -2 \sum_{k=0}^n k - (n+1) \\ 2 \sum_{k=0}^n k &= (n+1)^2 - (n+1) \\ \sum_{k=0}^n k &= \frac{(n+1)n}{2}\end{aligned}$$

上述运算没有告诉我们  $\square_n$  的值，但却能推导出  $\sum_{k=0}^n k$  的值。我们可以利用这种思路求解  $\square_n$ 。

$$\begin{aligned}\sum_{k=0}^n [(k+1)^3 - k^3] &= \sum_{k=0}^n [3k^2 + 3k + 1] \\ (n+1)^3 - 0^3 &= 3 \sum_{k=0}^n k^2 + 3 \sum_{k=0}^n k + \sum_{k=0}^n 1 \\ (n+1)^3 &= 3 \sum_{k=0}^n k^2 + 3 \frac{n(n+1)}{2} + (n+1)\end{aligned}$$

$$\begin{aligned}
3 \sum_{k=0}^n k^2 &= (n+1)^3 - 3 \frac{n(n+1)}{2} - (n+1) \\
\sum_{k=0}^n k^2 &= \frac{1}{3}(n+1) \left( (n+1)^2 - \frac{3}{2}n - 1 \right) \\
\sum_{k=0}^n k^2 &= \frac{1}{3}(n+1) \left( n + \frac{1}{2} \right) n
\end{aligned}$$

reference book list:

1. (CRC Tables) CRC Standard Mathematical Tables
2. Handbook of Mathematical Functions
3. Sloane. Handbook of Integer Sequences

software:

Axiom MACSYMA Maple Mathematica

my: Octave maxima 熟悉标准的信息源

方法 3: 建立成套方法

参考第二节的内容

方法 4: 用积分替换和式  $\sum \Rightarrow \int$

$$\square_n = 1 \times 1 + 1 \times 4 + 1 \times 9 + \cdots + 1 \times n^2$$

该式近似等于 0 到  $n$  之间曲线  $f(x) = x^2$  下的面积

$$\begin{aligned}
S &= \int_0^n x^2 dx \\
&= \frac{n^3}{3}
\end{aligned}$$

$\square_n$  近似等于  $\frac{n^3}{3}$ 。近似的误差  $E_n = \square_n - \frac{n^3}{3}$

## 1. 近似误差项递归式

$$\begin{aligned}\square_n &= \square_{n-1} + n^2 \\ E_n &= \square_n - \frac{n^3}{3} = \square_{n-1} + n^2 - \frac{n^3}{3} \\ E_{n-1} &= \square_{n-1} - \frac{(n-1)^3}{3} \\ E_n &= E_{n-1} + \frac{(n-1)^3}{3} + n^2 - \frac{n^3}{3} \\ &= E_{n-1} + \frac{-3n^2 + 3n - 1}{3} + n^2 \\ &= E_{n-1} + n - \frac{1}{3}\end{aligned}$$

## 2. 对楔形误差项面积求和

$$\begin{aligned}\square_n - \int_0^n x^2 dx &= \sum_{k=1}^n \left( k^2 - \int_{k-1}^k x^2 dx \right) \\ &= \sum_{k=1}^n \left( k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\ &= \sum_{k=1}^n \left( k - \frac{1}{3} \right) \\ E_n &= \sum_{k=1}^n \left( k - \frac{1}{3} \right) = \frac{n(n+1)}{2} - \frac{n}{3} = \frac{n(3n+1)}{6} \\ \square_n &= \frac{n^3}{3} + E_n \\ &= \frac{n^3}{3} + \frac{n(3n+1)}{6} \\ &= \frac{n(2n^2 + 3n - 1)}{6} \\ &= \frac{n(n + \frac{1}{2})(n+1)}{3}\end{aligned}$$

## 方法 5: 展开和收缩

$$\begin{aligned}
\Box_n &= \sum_{1 \leq k \leq n} k^2 = \sum_{1 \leq j \leq k \leq n} k \\
&= \sum_{1 \leq j \leq n} \sum_{j \leq k \leq n} k \\
&= \sum_{1 \leq j \leq n} \left( \frac{(j+n)(n-j+1)}{2} \right) \\
&= \frac{1}{2} \sum_{1 \leq j \leq n} (n(n+1) + j - j^2) \\
&= \frac{1}{2} \left[ n^2(n+1) + \frac{n(n+1)}{2} - \Box_n \right] \\
&= \frac{1}{2} n(n + \frac{1}{2})(n+1) - \frac{1}{2} \Box_n \\
\\
\frac{3}{2} \Box_n &= \frac{1}{2} n(n + \frac{1}{2})(n+1) \\
\Box_n &= \frac{1}{3} n(n + \frac{1}{2})(n+1)
\end{aligned}$$

## 方法 6: 使用有限微积分

## 方法 7: 用生成函数

## 1.6 有限微积分和无限微积分 Finite and infinite calculus

**表 1:** 有限微积分和无限微积分中的运算对比

| 无限微积分  | 有限微积分  |
|--------|--------|
| 微分算子 D | 差分算子 Δ |

$$Df(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \Delta f(x) = f(x+1) - f(x)$$

$$D(x^m) = mx^{m-1} \quad \Delta(x^3) = 3x^2 + 3x + 1$$

为使差分运算在形式上与微分运算类似，引入下降阶乘幂和上升阶乘幂。  
下降阶乘幂 (falling factorial power),  $x^{\underline{m}}$ , 读作  $x$  直降  $m$  次。

$$x^{\underline{m}} = \underbrace{x(x-1) \dots (x-m+1)}_{m \text{ 个因子}}, \quad (m \leq 0, m \in \mathbb{N})$$

上升阶乘幂 (rising factorial power),  $x^{\overline{m}}$ , 读作  $x$  直升  $m$  次。

$$x^{\overline{m}} = \underbrace{x(x+1) \dots (x+m-1)}_{m \text{ 个因子}}, \quad (m \leq 0, m \in \mathbb{N})$$

$$\begin{aligned} \Delta(x^{\underline{m}}) &= (x+1)^{\underline{m}} - x^{\underline{m}} \\ &= (x+1)x \dots (x+1-m+1) - x(x-1) \dots (x-m+1) \\ &= (x+1 - (x-m+1))x(x-1) \dots (x-m+2) \\ &= mx(x-1) \dots (x-m+2) \\ &= mx^{\underline{m-1}} \end{aligned}$$

$$g(x) = Df(x), \iff \underbrace{\int g(x)dx}_{\text{不定积分}} = f(x) + \underbrace{C}_{\text{任意常数}} \quad (9)$$

$$g(x) = \Delta f(x), \iff \underbrace{\sum g(x)\delta(x)}_{\text{不定和式}} = f(x) + \underbrace{C}_{\text{满足 } p(x+1)=p(x) \text{ 的任意函数}} \quad (10)$$

$$\sum_a^b g(x)\delta x = f(x)|_a^b = f(b) - f(a) \quad (11)$$

表 2: 有限微积分和无限微积分中的运算对比

| 无限微积分   | 有限微积分   |
|---|---|
| D 逆运算 $\int$ (积分算子, 逆微分算子)  | $\Delta$ 逆运算 $\sum$ (求和算子, 逆差分算子)   |
| 微积分基本定理   |   |
| $g(x) = Df(x) \iff \int g(x)dx = f(x) + C$  | $g(x) = \Delta f(x) \iff \sum g(x)\delta(x) = f(x) + C$   |
| 定积分   | 和式  |
| 若 $g(x) = Df(x)$ 那么   | 若 $g(x) = \Delta f(x)$ 那么   |
| $\int_a^b g(x)dx = f(x) _a^b = f(b) - f(a)$   | $\sum_a^b g(x)\delta x = f(x) _a^b = f(b) - f(a)$   |
| $\int_b^a g(x)dx = -\int_a^b g(x)dx$  | $\sum_b^a g(x)\delta x = -\sum_a^b g(x)\delta x$  |
| $\int_a^b + \int_b^c = \int_a^c$  | $\sum_a^b + \sum_b^c = \sum_a^c$  |
| $\int_0^n x^m = \frac{x^{m+1}}{m+1}\Big _0^n = \frac{n^{m+1}}{m+1}, m \neq -1$                                    | $\sum_0^n k^{\overline{m}} = \frac{k^{\overline{m+1}}}{m+1}\Big _0^n = \frac{n^{\overline{m+1}}}{m+1}, m \neq -1$ |
| $\sum_0^n k^{\overline{m}} = \frac{k^{\overline{m+1}}}{m+1}\Big _0^n = \frac{n^{\overline{m+1}}}{m+1}, m \neq -1$ |   |
| $(x+y)^2 = x^2 + 2xy + y^2$   | $(x+y)^{\underline{2}} = x^{\underline{2}} + 2x^{\underline{1}}y^{\underline{1}} + y^{\underline{2}}$             |
| $(x+y)^{\overline{2}} = x^{\overline{2}} + 2x^{\overline{1}}y^{\overline{1}} + y^{\overline{2}}$                  |   |
| $m = -1, \int_a^b x^{-1} = \ln x \Big _a^b$   | $m = -1, \sum_a^b k^{\overline{-1}} = H_k \Big _a^b$  |
| $\int_a^b x^m = \frac{x^{m+1}}{m+1}\Big _a^b, m \neq -1$  | $\sum_a^b k^{\overline{m}} = \frac{k^{\overline{m+1}}}{m+1}\Big _a^b, m \neq -1$                                  |
| $\ln n \Big _a^b, m = -1$   | $H_k \Big _a^b, m = -1$   |
|   | $\sum_a^b k^{\overline{m}} = \frac{k^{\overline{m+1}}}{m+1}\Big _a^b, m \neq -1$                                  |
|   | $H_{(k+1)} \Big _a^b, m \neq -1$  |
| 连续性问题的解中会出现自然对数   | 快速排序这样的问题中会出现调和数的原因   |
| $e^x$ , 性质 $De^x = e^x$   | $\Delta f(x) = f(x), f(x) = 2^x$ 离散指数函数   |

设  $g(x) = \Delta f(x) = f(x+1) - f(x)$

如果  $b = a$ , 我们就有

$$\sum_a^a g(x)\delta x = f(a) - f(a) = 0 \quad (12)$$

如果  $b = a + 1$ , 我们就有

$$\sum_a^{a+1} g(x)\delta x = f(a+1) - f(a) = g(a) \quad (13)$$

$$\begin{aligned} \sum_a^{b+1} g(x)\delta x - \sum_a^b g(x)\delta x &= [f(b+1) - f(a)] - [f(b) - f(a)] \\ &= f(b+1) - f(b) = g(b) \end{aligned}$$

由数学归纳法  $a, b \in \mathbb{N}$  且  $b \leq a$  时,  $\sum_a^b g(x)\delta x$  的确切含义是

$$\sum_a^b g(x)\delta x = \sum_{k=1}^{b-1} g(k) = \sum_{a \leq k < b} g(k), \quad (b \geq a, a, b \in \mathbb{N}) \quad (14)$$

若有  $g(x) = f(x+1) - f(x)$

$$\sum_{a \leq k < b} g(l) = \sum_{a \leq k < b} (f(k+1) - f(k)) = f(b) - f(a)$$

$$\sum_a^b g(x)\delta x = f(b) - f(a) \quad (b < a)$$

$$= -(f(a) - f(b)) = -\sum_a^b g(x)\delta x$$

$$\sum_a^b + \sum_b^c = \sum_a^c$$



应用：计算下降幂和式的简单方法

$$\sum_{0 \leq k < n} k^m = \frac{k^{m+1}}{m+1} \Big|_0^n = \frac{n^{m+1}}{m+1}, \quad (m, n \geq 0 \quad m, n \in \mathbb{N}^+) \quad (15)$$

$m = 1$  时,  $k^1 = k$

$$\sum_{0 \leq k < n} k = \frac{n^2}{2} = \frac{n(n-1)}{2}$$

$$k^2 = k(k-1) + k = k^2 + k^1$$

$$\begin{aligned} \sum_{0 \leq k < n} k^2 &= \frac{n^3}{3} + \frac{n^2}{2} \\ &= \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} \\ &= \frac{1}{3}n(n - \frac{1}{2})(n-1) \end{aligned}$$

$$k^3 = k(k-1)(k-2) + 3k(k-1) + k = k^3 + 3k^2 + k^1$$

$$\begin{aligned} \sum_{0 \leq k < n} k^3 &= \frac{n^4}{4} + 3\frac{n^3}{3} + \frac{n^2}{2} \\ &= \frac{n(n-1)(n-2)(n-3)}{4} + 3\frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2} \\ &= \left(\frac{1}{2}n(n-1)\right)^2 = \left(\sum_{0 \leq k < n} k\right)^2 \end{aligned}$$

## 负指数下降幂

$$x^{\underline{3}} = x(x-1)(x-2)$$

$$x^{\underline{2}} = x(x-1)$$

$$x^{\underline{1}} = x$$

$$x^{\underline{0}} = 1$$

$$x^{\overline{-1}} = \frac{1}{x+1}$$

$$x^{\overline{-2}} = \frac{1}{(x+1)(x+2)}$$

$$\vdots$$

$$x^{\overline{-m}} = \frac{1}{(x+1)(x+2)\dots(x+m)}$$

$$x^{\underline{3}} \cdot \frac{1}{x-3+1} = x^{\underline{2}} \Rightarrow x^{\underline{0}} \cdot \frac{1}{x-0+1} = x^{\overline{-1}}$$

为什么选用  $x^{\overline{-1}} = \frac{1}{x+1}$  而不是  $x^{\underline{-1}} = \frac{1}{x+1}$  作为下降阶乘幂的拓展定义?<sup>3</sup>

通常幂法则  $x^{m+n} = x^m x^n$ , 推广:

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}, (m, n \in \mathbb{N}^+)$$

$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}}, (m, n \in \mathbb{N}^+)$$

---

<sup>3</sup>当一个原有的记号被拓展包含更多种情形时, 以一种使得一般性法则继续成立的方式来表述它的定义, 这永远是最佳选择

my 推广至正指数下降幂

$$x^{\overline{3}} = x(x+1)(x+2)$$

$$x^{\overline{2}} = x(x+1)$$

$$x^{\overline{1}} = x$$

$$x^{\overline{0}} = 1$$

$$x^{\overline{-1}} = \frac{1}{x-1}$$

$$x^{\overline{-2}} = \frac{1}{(x-1)(x-2)}$$

$\vdots$

$$x^{\overline{-m}} = \frac{1}{(x-1)(x-2)\dots(x-m)}$$

$$x^{\overline{3}} \cdot \frac{1}{x+3} = x^{\overline{2}} \Rightarrow x^{\overline{0}} \cdot \frac{1}{x+0-1} = x^{\overline{-1}}$$

例如

$$x^{\overline{2+3}} = x^{\overline{2}}(x-2)^{\overline{3}} = x^{\overline{3}}(x-3)^{\overline{2}}$$

$$x^{\overline{2-3}} = x^{\overline{2}}(x-2)^{\overline{-3}}$$

$$= x(x-1) \frac{1}{(x-2+1)(x-2+2)(x-2+3)}$$

$$= x(x-1) \frac{1}{(x-1)x(x+1)}$$

$$= \frac{1}{x+1}$$

$m < 0$  时,  $\Delta x^{\underline{m}} = mx^{\underline{m-1}}$  是否仍成立?

$$\begin{aligned}\Delta x^{\underline{-2}} &= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+1)(x+2)} \\ &= \frac{(x+1) - (x+3)}{(x+1)(x+2)(x+3)} \\ &= -2x^{\underline{-3}}\end{aligned}$$

通常幂法则对负指数下降阶乘幂仍然成立。

离散指数函数  $2^x$

$$\begin{aligned}\Delta(c^x) &= c^{x+1} - c^x = (c-1)c^x \\ c \neq 1 \quad \frac{c^x}{c-1} &\stackrel{\Delta}{\Rightarrow} c^x \\ \sum_{a \leq k < b} c^k &= \sum_a^b c^x \delta x = \left. \frac{c^x}{c-1} \right|_a^b = \frac{c^b - c^a}{c-1}, \quad c \neq 1\end{aligned}$$

$$D(uv) = uDv + vDu \quad (16)$$

$$\int uDv = uv - \int vDu \quad (17)$$

$$\begin{aligned}\Delta(u(x)v(x)) &= u(x+1)v(x+1) - u(x)v(x) \\ &= u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) \\ &= \Delta u(x)v(x+1) + u(x)\Delta v(x) \\ &= u\Delta v + Ev\Delta u\end{aligned} \quad (18)$$

其中  $E$  被称为移位算子.

在无限微积分中, 令  $x+1 \rightarrow x$  无限细分, 避开了  $E$

$$\sum u\Delta v = uv - \sum Ev\Delta u \quad (19)$$

表 3: Table 55(1994), What's difference

| $f = \sum g$                   | $\Delta f = g$                        |
|--------------------------------|---------------------------------------|
| $x^0 = 1$                      | 0                                     |
| $x^1 = x$                      | 1                                     |
| $x^2 = x(x-1)$                 | $2x$                                  |
| $x^m$                          | $mx^{m-1}$                            |
| $x^{m+1}$                      | $(m+1)x^m$                            |
| $H_x$                          | $x^{-1} = \frac{1}{x+1}$              |
| $2^x$                          | $2^x$                                 |
| $c^x$                          | $(c-1)c^x$                            |
| $\frac{c^x}{c-1}$              | $c^x$                                 |
| $cu(x), c \text{ is constant}$ | $c\Delta u(x)$                        |
| $u+v$                          | $\Delta u + \Delta v$                 |
| $uv$                           | $u\Delta v + Ev\Delta u, Ev = v(x+1)$ |

例 1.2  $\int xe^x dx \xrightarrow{\text{离散模拟}} \sum x2^x \delta x \quad (\sum_{k=0}^n k2^k)$

解 5 令  $u(x) = x, \delta v(x) = 2^x$ ,

可得  $\delta u(x) = 1, v(x) = 2^x, Ev = 2^{x+1}$

$$\begin{aligned} \sum x2^x \delta x &= x \cdot 2^x - \sum 2^{x+1} \cdot 1 \delta x \\ &= x \cdot 2^x - 2^{x+1} + C \end{aligned}$$

$$\begin{aligned} \sum_0^n k2^k &= \sum_0^{n+1} x2^x \delta x \\ &= x \cdot 2^x - 2^{x+1} \Big|_0^{n+1} \end{aligned}$$

关于第二组等式的推导, 我一开始没有完全掌握, 主要是对求和符号  $\sum$  的上下标范围存在误解.

$$\text{if } \sum_{0 \leq k \leq n} k2^k = S_n$$

$$\begin{aligned}
\sum_{0 \leq k \leq n} k2^k + (n+1)2^{n+1} &= \sum_{0 \leq k \leq n+1} k2^k \\
&= \sum_{1 \leq k \leq n+1} (k-1)2^k + \sum_{1 \leq k \leq n+1} 2^k \\
&= \sum_{0 \leq k \leq n} k2^{k+1} + \sum_{1 \leq k \leq n+1} 2^k \\
&= 2S_n + \sum_{1 \leq k \leq n+1} 2^k
\end{aligned}$$

$$\begin{aligned}
S_n + (n+1)2^{n+1} &= 2S_n + \frac{2^{n+1} - 2^1}{2 - 1} \\
S_n &= (n+1)2^{n+1} - (2^{n+1} - 2) \\
&= (n-1)2^{n+1} + 2
\end{aligned}$$

$$\sum_{0 \leq k < n} = nH_n - n \tag{20}$$