具体数学阅读笔记-chap1 repertoire method 参考

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1 Solve

$$\begin{cases} r_0 & = 1 \\ r_n & = r_{n-1} + 3n + 5 \end{cases}$$

First, get some cases

$$\begin{aligned} r_0 &= 1 \\ r_1 &= 1 + 3 \times 1 + 5 && = 9 \\ r_2 &= 9 + 3 \times 2 + 5 && = 20 \\ r_3 &= 20 + 3 \times 3 + 5 && = 34 \end{aligned}$$

Unsimplified cases

$$\begin{aligned} r_0 &= 1 \\ r_1 &= r_0 + 3 \times 1 + 5 && = 9 \\ r_2 &= r_1 + 3 \times 2 + 5 && = 20 \\ r_3 &= r_2 + 3 \times 3 + 5 && = 34 \end{aligned}$$

A pattern in unsimplified cases

$$r_n = 1A(n) + 3B(n) + 5C(n)$$

where $A(n), B(n), C(n)$ are simple functions of n

$$\begin{cases} A(n) & = 1 \\ B(n) & = \frac{n(n+1)}{2} \\ C(n) & = n \end{cases}$$

$$\begin{aligned} r_n &= 1 \times 3 \times \frac{n(n+1)}{2} + 5 \times n \\ &= \frac{3}{2}n^2 + \frac{13}{2} + 1 \end{aligned}$$

Summarizing

$$\begin{cases} r_0 & = 1 \\ r_n & = r_{n-1} + 3n + 5 \end{cases}$$

is $r_n = \frac{3}{2}n^2 + \frac{13}{2}n + 1$.

Testing

表 1: r(n) 与 n 之间的关系

n	0	1	2	3	4	5
r_n	1	9	20	34	51	71
$\frac{3}{2}n^2 + \frac{13}{2}n + 1$	1	9	20	34	51	71

Prove it by induction.

First we generalize:

$$\begin{cases} r_0 & = 1 \\ r_n & = r_{n-1} + 3n + 5 \end{cases}$$

replace constants by variables α, β, γ

$$\begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{cases}$$

Cases of our generalized version

表 2: Cases of our generalized version

n	r_n
0	α
1	$\alpha + \beta + \gamma$
2	$\alpha + \beta + \gamma + 2 \times \beta + \gamma$
	$= \alpha + 3\beta + 2\gamma$
3	$\alpha + 3\beta + 2\gamma + 3 \times \beta + \gamma$
	$= \alpha + 6\beta + 3\gamma$
4	$\alpha + 10\beta + 4\gamma$

Wild assumption:

Let's assume that there are three-fixed functions A, B, C such that the solution to the above always has this form:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

We don't know this is true but the evidence suggests it

Can we figure out what A, B, and C are? Yes!

Is this easier tha the original problem? Yes!

Here's How

We assume that any recurrence defined by:

$$\begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{cases}$$

has a soluntion that looks like:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

no matter what α, β and γ are.

Different α, β and γ will define Different r_n. But $A(n)$, $B(n)$ and $C(n)$ are the same of all of them!

What does this buy us?

For any α, β and γ , the equations

$$\begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{cases}$$

are always solved by

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

If we pick up really simple functions (with really easy values for α, β and γ) we can solve for A, B and C.

And once we have A, B,and C, we have a solution to the general recurrence.

2 Easy Solutions

$$\begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases}$$

First easy solution.

Let's try $r_n = 1$

$$\begin{cases} 1 &= \alpha \\ 1 &= 1 + \beta n + \gamma \\ 1 &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases}$$

$$\begin{cases} \alpha = 1 \\ \beta = 0 \\ \gamma = 0 \end{cases} \rightarrow A(n) = 1$$

$r_n = 1$ has consequences

$$\begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases} \rightarrow \begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma C(n) \end{cases}$$

Let's try $r_n = n$

$$\begin{cases} 0 &= \alpha \\ n &= n - 1 + \beta n + \gamma \\ n &= \alpha + \beta B(n) + \gamma C(n) \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 1 \end{cases} \rightarrow C(n) = n$$

$r_n = 1$ has consequences

$$\begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma C(n) \end{cases} \rightarrow \begin{cases} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma n \end{cases}$$

Let's try $r_n = n^2$

$$\begin{cases} 0 &= \alpha \\ n^2 &= (n-1)^2 + \beta n + \gamma \\ n^2 &= \alpha + \beta B(n) + \gamma n \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 2 \\ \gamma = -1 \end{cases} \to B(n) = \frac{n(n+1)}{2}$$

$r_n = 1$ has consequences

$$\begin{cases} r_0 & = \alpha \\ r_n & = r_{n-1} + \beta n + \gamma \\ r_n & = \alpha + \beta B(n) + \gamma n \end{cases} \to \begin{cases} r_0 & = \alpha \\ r_n & = r_{n-1} + \beta n + \gamma \\ r_n & = \alpha + \beta \frac{n(n+1)}{2} + \gamma n \end{cases}$$

Let's try it out

$$\begin{cases} r_0 & = \alpha \\ r_n & = r_{n-1} + \beta n + \gamma \\ r_n & = \alpha + \beta \frac{n(n+1)}{2} + \gamma n \end{cases}$$

Testing

$$\begin{cases} r_0 & = 1 \\ r_n & = r_{n-1} + 3n + 5 \end{cases}$$

$\alpha = 1, \beta = 3, \gamma = 5$

$$r_n = 1 + 3\frac{n^2+n}{2}5n = \frac{3}{2}n^2 + \frac{13}{2}n + 1$$

Summations Recurrence like these $\begin{cases} r_0 & = \alpha \\ r_n & = r_{n-1} + \beta n + \gamma \end{cases}$

with solution: $r_n = \alpha + \beta\frac{n^2+n}{2} + \gamma n$.

Can be used to solve summations like these

$$S_n = \sum_{i=0}^{n} (3i + 2)$$

$$\begin{cases} S_0 & = 2 \\ S_n & = S_{n-1} + 3n + 2 \end{cases}$$

$$\begin{cases} \alpha = 2 \\ \beta = 3 \\ \gamma = 2 \end{cases}$$

$$\begin{aligned} S_n &= 2 + 3\frac{n^2+n}{2} + 2n \\ &= \frac{3}{2}n^2 + \frac{7}{2}n + 2 \end{aligned}$$

3 Let's try something harder.

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases}$$

Case

表 3: harder cases

n	0	1	2	3	4	5
r_n	1	3	8	19	42	89

First generalize

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases}$$

$$\begin{cases} r_0 = \alpha \\ r_n = \beta r_{n-1} + \gamma \end{cases}$$

In this case, α, β, γ mixed up

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases} \to \begin{cases} r_0 = \alpha \\ r_n = 2r_{n-1} + \beta n + \gamma \end{cases}$$

表 4: generalize the harder cases

n	r_n
0	α
1	$\beta\alpha+\gamma$
2	$\beta(\beta\alpha+\gamma)+2\gamma$
3	$\beta(\beta(\beta\alpha+\gamma)+2\gamma)+3\gamma$

没有 γ 的情况

表 5: generalize the harder cases(change)

n	r_n
0	α
1	$2\alpha+\beta+\gamma$
2	$2(2\alpha+\beta+\gamma)+2\beta+\gamma$
	$4\alpha+4\beta+3\gamma$
3	$2(4\alpha+4\beta+3\gamma)+3\beta+\gamma$
	$8\alpha+11\beta+7\gamma$

$$\begin{cases} r_0=\alpha \\ r_n=2r_{n-1}+\beta n \end{cases}$$

$$r_n=\alpha A(n)+\beta B(n)$$

$r_n=1$, $\begin{cases} 1=\alpha \\ 1=2\cdot 1+\beta n \end{cases}$ 这是不可能的。因此 $\gamma\neq 0$

$r_n=1$

$$\begin{cases} 1=\alpha \\ 1=2\cdot 1+\beta n+\gamma \end{cases}$$

$(\alpha,\beta,\gamma)=(1,0,-1)$

$$A(n)-C(n)=1$$

表 6: generalize the harder cases($\gamma = 0$)

n	r_n
0	α
1	$2\alpha + \beta$
2	$2(2\alpha + \beta) + 2\beta$
	$4\alpha + 4\beta$
3	$2(4\alpha + 4\beta) + 3\beta$
	$8\alpha + 11\beta$

$$C(n) = A(n) - 1$$

$r_n = n$

$$\begin{cases} 0 = \alpha \\ n = 2(n-1) + \beta n + \gamma \end{cases}$$

$(\alpha, \beta, \gamma) = (0, -1, 2)$

$$-B(n) + 2C(n) = n$$

$$B(n) = 2C(n) - n = 2A(n) - n - 2$$

$r_n = n^2$ 不能推出有效信息, $n^2 = 2(n-1)^2 + \beta n + \gamma$, 推不出合理的解.

$r_n = 2^n$

$$\begin{cases} 2^0 = \alpha \\ 2^n = 2 \times 2^{n-1} + \beta n + \gamma \end{cases}$$

$(\alpha, \beta, \gamma) = (1, 0, 0)$

$$A(n) = 2^n$$

$$\begin{cases} A(n) &= 2^n \\ B(n) &= 2^{n+1} - 2 - n \\ C(n) &= 2^n - 1 \end{cases}$$

$$r_n = \alpha 2^n + \beta(2^{n+1} - n - 2) + \gamma(2^n - 1)$$

Example: $(\alpha, \beta, \gamma) = (1, 1, 0)$

$$r_n = 1 \times 2^n + 1 \times (2^{n+1} - n - 2) = 3 \cdot 2^n - n - 2$$

Reprise

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases} \to \text{(general)} \begin{cases} r_0 = \alpha \\ r_n = 2r_{n-1} + \beta n + \gamma \end{cases}$$

$$r_n = \alpha 2^n + \beta(2^{n+1} - n - 2) + \gamma(2^n - 1)$$

$$\left\{ \begin{array}{lllll} 1 & = A(n) & & -C(n) \\ n & = & -B(n) & +2C(n) \\ 2^n & = A(n) & & \end{array} \right.$$

[1]

[1] 这份 ppt 非常详细, 我以后总结知识也应如此. 将来回顾可以快速看懂.