

# 谢惠民数学分析习题课讲义上册笔记整理

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# 目录

<b>第一章 引论</b>	<b>1</b>
1.1 关于习题课教案的组织	1
1.1.1 书中常用记号	1
1.1.2 几个常用的初等不等式	1
1.2 1.4 逻辑符号与对偶法则	16
<b>第二章 数列极限</b>	<b>17</b>
2.1 数列极限的基本概念	17
2.1.1 2.1.5 练习题	17
2.2 收敛数列的基本性质	19
2.2.1 思考题	20
2.2.2 练习 2.2.4	21
2.3 2.3 单调数列	28
2.3.1 2.3.2 练习题	28



# 第一章 引论

## 1.1 关于习题课教案的组织

### 1.1.1 书中常用记号

1.  $\mathbf{N}_+$ : 所有正整数组成的集合.
2.  $\mathbf{R}$ : 所有实数组成的集合 (同时也用于表示无限区间  $(-\infty, \infty)$ ).
3.  $\mathbf{Q}$ : 所有有理数组成的集合.
4.  $\mathbf{C}$ : 所有复数组成的集合.
5.  $\iff$  是等价关系的记号.  $A \iff B$  表示  $A$  和  $B$  等价. 例如,  $A$  代表  $x > 3$ ,  $B$  代表  $x - 3 > 0$ , 则  $x > 3 \iff x - 3 > 0$ .
6.  $[x]$  是实数  $x$  的整数部分, 即不超过  $x$  的最大整数. 例如,  $[\sqrt{2}] = 1, [-\sqrt{2}] = -2$ . 关于  $[x]$  的基本不等式是:  $[x] \leq x < [x] + 1$ , 或  $x - 1 < [x] \leq x$
7. 空心方块表示一个证明或解的结束.
8.  $\binom{n}{k} = C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k!}$ .
9. 记号  $\approx$  表示近似值. 例如  $\sqrt{2} \approx 1.4$ .
10. 复合函数  $f(g(x))$  也写成  $(f \circ g)(x)$  或  $f \circ g$ .
11. 若  $A$  和  $B$  为两个集合, 则用记号  $A - B$  或  $A \setminus B$  表示  $A$  与  $B$  的差集, 也就是集合  $\{x | x \in A \text{ 且 } x \notin B\}$ .
12. 用  $O_\delta(a)$  表示以  $a$  为中心, 以  $\delta > 0$  为半径的邻域. 它就是开区间  $(a - \delta, a + \delta)$  (也可以用  $U_\delta(a)$  等记号). 如不必指出半径, 则可简记为  $O(a)$  (或  $U(a)$ ).

### 1.1.2 几个常用的初等不等式

#### 几个初等不等式的证明

A.G 不等式  $a_1, a_2, \dots, a_n$ ,  $n$  个非负实数

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \quad (1.1)$$

$\geq$  in inequation became  $= \iff a_1 = a_2 = \cdots = a_n$

**Proof.** method 1. induction method

$$k = 1 \quad a_1 = a_1$$

$$k = 2 \quad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

$$k = n \quad \text{suppose} \quad \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

$$k = n + 1$$

$$\begin{aligned} & \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} - \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &= \frac{n(a_1 + a_2 + \cdots + a_{n+1}) - (n+1)(a_1 + a_2 + \cdots + a_n)}{n(n+1)} \\ &= \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n(n+1)} \end{aligned}$$

$$\text{Set } A = \frac{a_1 + a_2 + \cdots + a_n}{n}, B = \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n(n+1)}$$

$$\left( \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} \right)^{n+1} = (A + B)^{n+1}$$

$$A > 0, B \geq 0$$

$$(A + B)^{n+1} \geq A^{n+1} + (n+1)A^n B$$

$$A^{n+1} + (n+1)A^n B = A^n(A + (n+1)B)$$

$$A^n = \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^n \geq a_1 a_2 \cdots a_n$$

$$A + (n+1)B = \frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n} = a_{n+1}$$

$$\therefore (A + B)^{n+1} \geq A^n(A + (n+1)B) \geq a_1 a_2 \cdots a_n \cdot a_{n+1}$$

$$\therefore \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} \geq \sqrt[n+1]{a_1 a_2 \cdots a_n a_{n+1}}$$

### 使用二项式展开定理的条件

在归纳法第二步, 将  $a_1, a_2, \cdots, a_{n+1}$  重编号, 使得  $n+1$  为其中最大的数 (之一), 这使得分解式右边第二项  $(na_{n+1} - (a_1 + a_2 + \cdots + a_n))/n(n+1)$  一定是非负数。

method 2. Forward and Backward (Cauchy, 1897)

Forward part

$$k = 2. \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

$$k = 4. \frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt{\left(\frac{a_1 + a_2}{2}\right) \cdot \left(\frac{a_3 + a_4}{2}\right)}.$$

$$\geq \sqrt{\sqrt{a_1 a_2} \sqrt{a_3 a_4}} = \sqrt[4]{a_1 a_2 a_3 a_4}.$$

$$k = 2^n. \text{ Suppose } \frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdots a_{2^n}}$$

$$k = 2^{n+1}.$$

$$\frac{a_1 + a_2 + \cdots + a_{2^n} + \cdots + a_{2^{n+1}}}{2^{n+1}} \geq \sqrt{\left(\frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n}\right) \cdot \left(\frac{a_{2^n+1} + a_{2^n+2} + \cdots + a_{2^{n+1}}}{2^n}\right)}$$

$$I \geq \sqrt{\sqrt[2^n]{a_1 a_2 \cdots a_{2^n}} \sqrt[2^n]{a_{2^n+1} a_{2^n+2} \cdots a_{2^{n+1}}}} = \sqrt[2^{n+1}]{a_1 a_2 \cdots a_{2^{n+1}}}$$

Backward part suppose A.G inequality is valid when  $k = n$ , Consider  $k = n - 1$ .

$$\begin{aligned} \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &= \frac{1}{n} \left( \frac{n}{n-1} \right) \sum_{i=1}^{n-1} a_i \\ \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &= \frac{1}{n} \left( \sum_{i=1}^{n-1} a_i + \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \\ \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &\geq \sqrt[n]{\left( \prod_{i=1}^{n-1} a_i \right) \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)} \\ \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^n &\geq \left( \prod_{i=1}^{n-1} a_i \right) \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \\ \left( \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^{n-1} &\geq \left( \prod_{i=1}^{n-1} a_i \right) \\ \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &\geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \end{aligned}$$

□

**Proposition 1.1.1.** 柯西-施瓦茨不等式对  $a_1, a_2, \cdots, a_n$  和  $b_1, b_2, \cdots, b_n$ , 成立

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

**Proof.**

$$0 \leq \sum_{i=1}^n (a_i - \lambda b_i)^2 = \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2$$

由韦达定理 (视  $\lambda$  为未知数). 原方程无解或只有唯一解。

$$\begin{aligned}\Delta &= b^2 - 4ac \leq 0 \\ (-2 \sum_{i=1}^n a_i b_i)^2 - 4 \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 &\leq 0 \\ (\sum_{i=1}^n a_i b_i)^2 &\leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \\ \sum_{i=1}^n a_i b_i &\leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}\end{aligned}$$

□

### 练习题

**Example 1.1.1.** 关于 Bernoulli 不等式的推广:

- (1) 证明: 当  $-2 \leq h \leq -1$  时 Bernoulli 不等式  $(1+h)^n \geq 1+nh$  仍成立;
- (2) 证明: 当  $h \geq 0$  时成立不等式  $(1+h)^n \geq \frac{n(n-1)h^2}{2}$ , 并推广之;
- (3) 证明: 若  $a_i > -1 (i=1, 2, \dots, n)$  且同号, 则成立不等式

$$\prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$$

**Proof.** (1)

$$-2 \leq h \leq -1$$

$$-1 \leq 1+h \leq 0$$

$$-2n \leq nh \leq -n$$

$$-1 \leq (1+h)^n \leq 0$$

$$1-2n \leq 1+nh \leq 1-n$$

$$n=0.$$

$$(1+h)^0 = 1 = 1 + 0 \times h$$

$$n=1.$$

$$1+h = 1+h$$

$$n \geq 2.$$

$$1-n \leq -2$$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1+nh \geq 1-2n$$

$$(1+h)^n \geq 1+nh$$

(2)

$$h \geq 0$$

$$(1+h)^n = 1+nh + \frac{n(n-1)}{2}h^2 + \dots \geq \frac{n(n-1)}{2}h^2$$



推广:

$$(1+h)^n \geq \binom{n}{3}h^3, \binom{n}{4}h^4, \dots, \binom{n}{k}h^k, 0 \leq k \leq n$$

(3)  $k=1$  时显然成立. 使用归纳法证明. 假设  $k=n$  时不等式  $\prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$  成立, 证明  $k=n+1$  时  $\prod_{i=1}^{n+1} (1+a_i) \geq 1 + \sum_{i=1}^{n+1} a_i$  成立.

$$\begin{aligned} k=n+1 \quad \prod_{i=1}^{n+1} (1+a_i) &= \prod_{i=1}^n (1+a_i)(1+a_{n+1}) \\ &\because \prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i \\ \prod_{i=1}^n (1+a_i)(1+a_{n+1}) &\geq (1 + \sum_{i=1}^n a_i)(1+a_{n+1}) \end{aligned}$$

$$\begin{aligned} (1 + \sum_{i=1}^n a_i)(1+a_{n+1}) &= 1 + \sum_{i=1}^n a_i + a_{n+1} + a_{n+1} \sum_{i=1}^n a_i \\ &= 1 + \sum_{i=1}^{n+1} a_i + a_{n+1} \sum_{i=1}^n a_i \\ &\geq 1 + \sum_{i=1}^{n+1} a_i \end{aligned}$$

□

**Example 1.1.2.** 利用 A.G. 不等式求解:

(1).  $n! \leq (\frac{n+1}{2})^n$ , while  $n > 1$

(2).  $(n!)^2 = (n \cdot 1)[(n-1) \cdot 2] \dots (1 \dots n)$ . 证明: 当  $n > 1$  时成立

$$n! < (\frac{n+2}{6})^n$$

(3). 比较上述两个不等式的优劣

(4). 证明: 对任意实数  $r$  成立:

$$(n!)^r \leq \frac{1}{n^n} (\sum_{k=1}^n k^r)^n \quad (1.2)$$

**Proof.** (1).

$$n > 1 \quad n! = 1 \times 2 \times \dots \times n < (\frac{1+2+\dots+n}{n})^n = (\frac{(1+n)n}{2n})^n = (\frac{n+1}{2})^n$$

$\because 1 \neq 2 \neq \dots n$ , 所以不会有等号出现的情况

(2).  $n > 1$

$$\begin{aligned} (n!)^2 &= (n \cdot 1)[(n-1) \cdot 2] \dots (1 \dots n) \\ &< (\frac{n \times 1 + (n-1) \times 2 + \dots + 1 \times n}{n})^n \end{aligned}$$

Consider this equation

$$\left(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n}\right)^n \quad (1.3)$$

$$\begin{aligned} \sum_{k=1}^n (n-k+1)k &= (n+1) \sum_{k=1}^n k - \sum_{k=1}^n k^2 \\ &= (n+1) \frac{(n+1)n}{2} - \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{6} (3(n+1) - (2n+1)) \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

$$\begin{aligned} (n!)^2 &< \left(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n}\right)^n \\ &= \left(\frac{(n+1)(n+2)}{6}\right)^n \end{aligned}$$

$\therefore n+1 < n+2, \therefore n! < \left(\frac{n+2}{\sqrt{6}}\right)^n$

(3).  $n > 3$  时,  $\frac{n+2}{\sqrt{6}} < \frac{n+1}{2}$  (2) 的结果较好.

(4).  $\forall r \in \mathbb{R}$ , prove formula 1.2

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n k^r &\geq \sqrt[n]{\prod_{k=1}^n k^r} \\ (n!)^r &= \prod_{k=1}^n k^r \leq \left(\frac{1}{n} \sum_{k=1}^n k^r\right)^n = \frac{1}{n^n} \left(\sum_{k=1}^n k^r\right)^n \end{aligned}$$

my answer

$$\begin{aligned} \forall r \in \mathbb{R}, \quad \left(\sum_{k=1}^n k^r\right)^n &\geq n^n (n!)^r \\ (n!)^r &= \sum_{k=1}^n k^r \leq \left(\frac{1^r + 2^r + \cdots + n^r}{n}\right)^n = \frac{1}{n^n} \left(\sum_{k=1}^n k^r\right)^n \\ \therefore \left(\sum_{k=1}^n k^r\right)^n &\geq n^n (n!)^r \end{aligned}$$

□

**Example 1.1.3.**  $a_k > 0, k = 1, 2, \dots, n$  证明几何-调和平均值不等式

$$\left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}$$

**Proof.** from A.G inequality

$$\frac{\sum_{k=1}^n \frac{1}{a_k}}{n} \geq \sqrt[n]{\prod_{k=1}^n \frac{1}{a_k}} = \frac{1}{\sqrt[n]{\prod_{k=1}^n a_k}}$$

$$a_k > 0, \quad \sqrt[n]{\prod_{k=1}^n a_k} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}$$

□

**Example 1.1.4.**  $a, b, c \geq 0$ . prove  $\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$ . 并推广到  $n$  个非负数的情况

**Proof.** 1.  $\sqrt[3]{abc} = \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \leq \sqrt{\frac{ab+bc+ca}{3}}$ .

2.

$$\begin{aligned} \sqrt{\frac{ab+bc+ca}{3}} &\leq \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ &= \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}} \\ &= \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}} \end{aligned}$$

$a, b, c \geq 0$ , 希望证明

$$\begin{aligned} \sqrt{\frac{ab+bc+ca}{3}} &\leq \frac{a+b+c}{3} \\ \frac{ab+bc+ca}{3} &\leq \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6} \\ \frac{ab+bc+ca}{2} &\leq \frac{a^2+b^2+c^2}{6} + 2\frac{ab+bc+ca}{6} \quad (\text{add } \frac{ab+bc+ca}{6}) \\ \frac{ab+bc+ca}{3} &\leq \frac{ab+bc+ca}{2} \leq \left(\frac{a+b+c}{3}\right)^2 \\ \sqrt{\frac{ab+bc+ca}{3}} &\leq \frac{a+b+c}{3} \end{aligned}$$

推广至  $n$  个

$$\begin{aligned} [l] n=2 \quad \sqrt{ab} &\leq \frac{a+b}{2} \\ n=3 \quad \sqrt[3]{abc} &\leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3} \\ n=4 \quad \sqrt[4]{abcd} &\leq \sqrt[3]{\frac{abc+bcd+cda+dab}{4}} \leq \sqrt{\frac{a+b+c}{3}} \leq \frac{a+b+c+d}{4} \\ k=n \quad \sqrt[n]{a_1 a_2 \dots a_n} &\leq \sqrt{\frac{a_1+a_2+\dots+a_n}{n}} \leq \frac{a_1+a_2+\dots+a_n}{n} \end{aligned}$$

This is

$$\sqrt[n]{\sum_{k=1}^n a_k} \leq \sqrt{\frac{\sum_{k=1}^n a_k}{k}} \leq \frac{\sum_{k=1}^n a_k}{k}$$

$$1. \sqrt[n]{a_1 a_2 \dots a_n} = \sqrt[n]{a_1^2 a_2^2 \dots a_n^2} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_n a_1}{n}}$$

$$2. \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_n a_1}{n}} \leq \sqrt{\frac{a_1 + a_2 + \dots + a_n}{n}} ?$$

□

**Example 1.1.5.** (1)  $|\alpha + \beta| \leq |\alpha| + |\beta|$

**Proof.** let  $\alpha = a - b, \beta = b$ , the identity became  $|(a - b) + b| \leq |a - b| + |b|$ . This is  $|a - b| \geq |a| - |b|$ .

$$||a| - |b|| = \begin{cases} |a| - |b|. & a \geq b \\ |b| - |a|. & a < b \end{cases}$$

When  $a \geq b$ ,  $||a| - |b|| = |a| - |b|$ . There is  $|a - b| \geq |a| - |b| = ||a| - |b||$

When  $a < b$ ,  $|a - b| = |b - a| \geq |b| - |a| = ||a| - |b||$ .

$\therefore$ , We have  $|a - b| \geq ||a| - |b||$

□

$$(2) \sum |a_k| \geq |\sum a_k|$$

**Proof.** We can prove this statement by induction.

$$k = 2, \quad |a_1| + |a_2| \geq |a_1 + a_2|$$

$$k = 3, \quad |a_1| + |a_2| + |a_3| \geq |a_1 + a_2 + a_3|$$

$$\text{Suppose } k = n, \quad \sum_{k=1}^n |a_k| \geq \left| \sum_{k=1}^n a_k \right|$$

$$k = n + 1, \quad \text{prove } \sum_{k=1}^{n+1} |a_k| \geq \left| \sum_{k=1}^{n+1} a_k \right|$$

$$\begin{aligned} \sum_{k=1}^{n+1} |a_k| &= \sum_{k=1}^n |a_k| + |a_{n+1}| \\ &\geq \left| \sum_{k=1}^n a_k \right| + |a_{n+1}| \\ &\geq \left| \sum_{k=1}^{n+1} a_k \right| \end{aligned}$$

$$k = 2, \quad |a_1| - |a_2| \leq |a_1 - a_2|$$

$$\text{Suppose } k = n, \quad |a_1| - \sum_{k=2}^n |a_k| \leq \left| \sum_{k=1}^n a_k \right|$$

$$k = n + 1, \quad \text{prove } |a_1| - \sum_{k=2}^{n+1} |a_k| \leq \left| \sum_{k=1}^{n+1} a_k \right|$$

$$\begin{aligned}
|a_1| - \sum_{k=2}^{n+1} |a_k| &= |a_1| - \sum_{k=2}^n |a_k| - |a_{n+1}| \\
&\leq \left| \sum_{k=1}^n a_k \right| - |a_{n+1}| \\
&\leq \left| \sum_{k=1}^{n+1} a_k \right|
\end{aligned}$$

Can left side became  $||a_1| - \sum_{k=2}^n |a_k||$ ?

$$\left| |a_1| - \sum_{k=2}^n |a_k| \right| = |a_1| - \sum_{k=2}^n |a_k| \quad |a_1| \geq \sum_{k=2}^n |a_k| \quad (1.4)$$

$$\left| |a_1| - \sum_{k=2}^n |a_k| \right| = \sum_{k=2}^n |a_k| - |a_1| \quad |a_1| \geq \sum_{k=2}^n |a_k| \quad (1.5)$$

in eq1.4, the inequality is still vaild. However in eq1.5,  $\sum_{k=2}^n |a_k| - |a_1|$  and  $|a_1|$

□

$$(3). \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

**Proof.**

$$\begin{aligned}
\frac{|a+b|}{1+|a+b|} &\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \\
\frac{|a+b|}{1+|a+b|} &\leq \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
1 - \frac{|a+b|}{1+|a+b|} &\geq 1 - \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\
\frac{1}{1+|a+b|} &\geq \frac{1-|a||b|}{(1+|a|)(1+|b|)}
\end{aligned}$$

$$1 + |a| + |b| + |a||b| \geq 1 + |a+b| - |a||b| - |a||b||a+b|$$

$$|a| + |b| + 2|a||b| + |a||b||a+b| > 0, \text{ Since } +2|a||b| + |a||b||a+b| \geq |a+b|$$

$$\text{Therefore } \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

□

**Example 1.1.6.** (4).  $|(a+b)^n - a^n| \leq (|a| + |b|)^n - |a|^n$

$$\begin{aligned}
(a+b)^n - a^n &= \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} a^0 b^n \\
(|a| + |b|)^n - |a|^n &= \binom{n}{1} |a|^{n-1} |b|^1 + \binom{n}{2} |a|^{n-2} |b|^2 + \cdots + \binom{n}{n} |a|^0 |b|^n
\end{aligned}$$

$$\therefore |a|^j |b|^k \geq a^j b^k$$

$$\therefore \sum |a|^j |b|^k \geq \sum a^j b^k$$

$$|(a+b)^n - a^n| = \begin{cases} (a+b)^n - a^n, & a+b \geq a; b \geq 0 \\ a^n - (a+b)^n, & a+b < a; b < 0 \end{cases}$$

$$|(a+b)^n - a^n| \leq (|a| + |b|)^n - |a|^n. \quad (1.6)$$

**Proposition 1.1.2.** 1.3.5(Cauchy inequality)

For  $a_1, a_2, \dots, a_n$ . and  $b_1, b_2, \dots, b_n$ .  $a_i, b_i \in \mathbb{R}$ , There is

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \quad (1.7)$$

**Proof.** Let's prove eq1.7

First way on book:

Use variable  $\lambda$ , change the inequality into nonnegative binomial.

$$\begin{aligned} 0 &\leq \sum_{i=1}^n (a_i - \lambda b_i)^2 &&= \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2 \\ \Delta &= B^2 - 4AC &&= (-2 \sum_{i=1}^n a_i b_i)^2 - 4 \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) \leq 0 \end{aligned}$$

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right)$$

sqrt on both side of the inequality above, we can get

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

□

## 6. Cauchy 不等式的不同证明

(1). 数学归纳法.

$$k = 1, \quad |ab| = \sqrt{a^2} \sqrt{b^2}$$

$$k = 1, \quad |a_1 b_1 + a_2 b_2| = \sqrt{a_1^2 + a_2^2} \sqrt{b_1^2 + b_2^2}$$

$$\text{Suppose } k = n, \quad \left| \sum_{i=1}^n a_i b_i \right| = \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

$$k = n + 1, \quad \left| \sum_{i=1}^{n+1} a_i b_i \right| = \left| \sum_{i=1}^n a_i b_i + a_{n+1} b_{n+1} \right|$$

$$\begin{aligned}
\left| \sum_{i=1}^{n+1} a_i b_i \right| &= \left| \sum_{i=1}^n a_i b_i + a_{n+1} b_{n+1} \right| \\
&\leq \left| \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} + a_{n+1} b_{n+1} \right|
\end{aligned}$$

Note that  $A = \sqrt{\sum_{i=1}^n a_i^2}$ ,  $B = \sqrt{\sum_{i=1}^n b_i^2}$

$$\begin{aligned}
\left| \sum_{i=1}^{n+1} a_i b_i \right| &\leq |AB + a_{n+1} b_{n+1}| \\
&\leq \sqrt{A^2 + a_{n+1}^2} \sqrt{B^2 + b_{n+1}^2} \\
&= \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}
\end{aligned}$$

(2) Lagrange 恒等式

$$\sum_{i=1}^n a_k^2 \sum_{i=1}^n b_k^2 - \left( \sum_{i=1}^n |a_k b_k| \right)^2 = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n (|a_k| |b_i| - |a_i| |b_k|)^2 \quad (1.8)$$

$$\begin{aligned}
(|a_k| |b_i| - |a_i| |b_k|)^2 &= |a_k|^2 |b_i|^2 - 2|a_i| |a_k| |b_i| |b_k| + |b_k|^2 |a_i|^2 \\
&= a_k^2 b_i^2 + b_k^2 a_i^2 - 2|a_i a_k b_i b_k|
\end{aligned}$$

$$\sum_{i=1}^n \sum_{k=1}^n (|a_k| |b_i| - |a_i| |b_k|)^2 = 2 \sum_{i=1}^n a_i^2 \sum_{k=1}^n b_k^2 - 2 \sum_{i=1}^n \sum_{k=1}^n |a_i a_k b_i b_k|$$

$$\sum_{i=1}^n a_k^2 \sum_{i=1}^n b_k^2 - \left( \sum_{i=1}^n |a_k b_k| \right)^2 = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n (|a_k| |b_i| - |a_i| |b_k|)^2 \geq 0$$

$$\therefore \left( \sum_{i=1}^n |a_i b_i| \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

$$\therefore \left| \sum_{i=1}^n a_i b_i \right| \leq \sum_{i=1}^n |a_i b_i|$$

$$\therefore \left( \left| \sum_{i=1}^n a_i b_i \right| \right)^2 \leq \left( \sum_{i=1}^n |a_i b_i| \right)^2$$

$$\therefore \left( \left| \sum_{i=1}^n a_i b_i \right| \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

不等式两边开平方，得到：

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

(3). 用不等式  $|AB| \leq \frac{A^2+B^2}{2}$

$$\begin{aligned} |a_i b_i| &\leq \frac{a_i^2 + b_i^2}{2} \\ \left| \sum_{i=1}^n a_i b_i \right| &\leq \sum_{i=1}^n |a_i b_i| \leq \frac{\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2}{2} \\ \frac{\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2}{2} &\geq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \quad ?? \end{aligned}$$

如何用均值不等式证明 Cauchy 不等式?

由切比雪夫不等式, 有

$$\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \leq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \left( \frac{b_1 + b_2 + \cdots + b_n}{n} \right) \quad (1.9)$$

由均值不等式, 有

$$\begin{aligned} \frac{a_1 + a_2 + \cdots + a_n}{n} &\leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \\ \frac{b_1 + b_2 + \cdots + b_n}{n} &\leq \sqrt{\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}} \\ \therefore \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} &\leq \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right) \left( \frac{b_1 + b_2 + \cdots + b_n}{n} \right) \\ &\leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \sqrt{\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}} \\ &= \frac{1}{n} \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \end{aligned}$$

This is

$$\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

Square on both side of the inequality, The calculate square root. We can get eq1.9:

(4). 构造复的辅助数列

$$c_k = a_k^2 - b_k^2 + 2|a_k b_k|, \quad k = 1, 2, \dots, n$$

Then we use

$$\left| \sum_{k=1}^n c_k \right| \leq \sum_{k=1}^n |c_k|$$

**Solve 1.**

$$\begin{aligned} c_k &= (|a_k| + |b_k|)^2 = a_k^2 + b_k^2 + 2|a_k b_k| \\ \sum_{k=1}^n c_k &= \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 + 2 \sum_{k=1}^n |a_k b_k| \\ |c_k| &= \sqrt{\Re^2 c_k + \Im^2 c_k} = \sqrt{(a_k^2 - b_k^2)^2 + (2a_k b_k)^2} = a_k^2 + b_k^2 \end{aligned}$$



$$\begin{aligned}
& \therefore \left| \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 + 2 \sum_{k=1}^n |a_k b_k| \right| = \sqrt{\Re^2 \sum_{k=1}^n c_k + \Im^2 \sum_{k=1}^n c_k} \\
& = \sqrt{\left( \sum_{k=1}^n (a_k^2 - b_k^2) \right)^2 + \sum_{k=1}^n (2a_k b_k)^2} \\
& = \sqrt{\left( \sum_{k=1}^n a_k^2 \right)^2 + \left( \sum_{k=1}^n b_k^2 \right)^2 - 2 \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right) + 4 \sum_{k=1}^n (a_k b_k)^2} \\
& \therefore \left| \sum_{k=1}^n c_k \right| \leq \sum_{k=1}^n |c_k| \\
& \therefore \left( \sum_{k=1}^n a_k^2 \right)^2 + \left( \sum_{k=1}^n b_k^2 \right)^2 - 2 \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right) + 4 \sum_{k=1}^n (a_k b_k)^2 \leq \left( \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 \right)^2 \\
& \therefore 4 \left( \sum_{k=1}^n a_k b_k \right)^2 \leq 4 \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right) \\
& \text{extracting both side: } \left| \sum_{k=1}^n a_k b_k \right| \leq \sqrt{\sum_{k=1}^n a_k^2} \sqrt{\sum_{k=1}^n b_k^2}
\end{aligned}$$

**Example 1.1.7.** 7. Suppose  $0 < x_i \leq \frac{1}{2}, i = 1, 2, \dots, n$ , then

$$\frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^n} \leq \frac{\prod_{i=1}^n (1 - x_i)}{(\sum_{i=1}^n (1 - x_i))^n} \quad (1.10)$$

**Proof.** Let's prove eq1.10 by induction method.

$$n = 2, \quad \frac{x_1 x_2}{(x_1 + x_2)^2} \leq \frac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2}$$

$$\begin{aligned}
& \frac{(x_1 x_2)}{(x_1^2 + 2x_1 x_2 + x_2^2)} \leq \frac{1 - x_1 - x_2 + x_1 x_2}{(1 - x_1)^2 + 2(1 - x_1)(1 - x_2) + (1 - x_2)^2} \\
& \frac{(x_1 + x_2)^2}{(x_1 x_2)} \geq \frac{((1 - x_1)(1 - x_2))^2}{1 - x_1 - x_2 + x_1 x_2} \\
& \frac{x_1}{x_2} + 2 + \frac{x_2}{x_1} \geq \frac{1 - x_1}{1 - x_2} + 2 \frac{1 - x_2}{1 - x_1} \\
& \frac{x_1}{x_2} - \frac{1 - x_1}{1 - x_2} \geq \frac{1 - x_2}{1 - x_1} - \frac{x_2}{x_1} \\
& \frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_2(1 - x_2)} \geq \frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_1(1 - x_1)} \\
& \frac{x_1 - x_2}{x_2(1 - x_2)} \geq \frac{x_1 - x_2}{x_1(1 - x_1)}
\end{aligned}$$

$f(x) = x - x^2, f'(x) = 1 - 2x > 0$ , while  $x \in (0, \frac{1}{2})$

When  $x_1 > x_2, 0 < x_2 < x_1 \leq \frac{1}{2}, x_1 - x_1^2 \geq x_2 - x_2^2, x_1 - x_2 > 0$

When  $x_1 < x_2, 0 < x_1 < x_2 \leq \frac{1}{2}, x_1 - x_1^2 \leq x_2 - x_2^2, x_1 - x_2 < 0$

$$\frac{x_1 - x_2}{x_2(1 - x_2)} \geq \frac{x_1 - x_2}{x_1(1 - x_1)}$$

$$k = 2, \quad \frac{x_1 x_2}{(x_1 + x_2)^2} \leq \frac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2}$$

$$k = 4, \quad \frac{x_1 x_2 x_3 x_4}{(x_1 + x_2 + x_3 + x_4)^2} \leq \frac{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)}{((1 - x_1) + (1 - x_2) + (1 - x_3) + (1 - x_4))^2}$$

Use Cauchy's forward and backward method, We can prove this equation

$$\text{Suppose } k = n, \quad \frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} \leq \frac{\prod_{i=1}^n (1 - x_i)}{(\sum_{i=1}^n (1 - x_i))^2}$$

$$k = n - 1, \quad \text{prove } \frac{\prod_{i=1}^{n-1} x_i}{(\sum_{i=1}^{n-1} x_i)^2} \leq \frac{\prod_{i=1}^{n-1} (1 - x_i)}{(\sum_{i=1}^{n-1} (1 - x_i))^2}$$

todo! need to complete!

□

**Proposition 1.1.3.** 1.3.1 Bernoulli inequality Suppose that  $h > -1, n \in \mathbb{N}$ , Then:

$$(1 + h)^n \geq 1 + nh \quad (1.11)$$

When  $n > 1$ , the inequality became equation iff  $h = 0$ .

**Proof.** When  $n = 1, 1 + h = 1 + h$

$$h = 0, 1^n = 1$$

Let's consider the condition  $n > 1, h \neq 0$ .

$$\text{i). } h > 0, (1 + h)^n = \binom{n}{0}h^0 + \binom{n}{1}h^1 + \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n.$$

$$\because \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n > 0, \therefore (1 + h)^n > 1 + nh$$

$$\text{ii). } -1 < h < 0, 0 < 1 + h < 1.$$

$$\begin{aligned} (1 + h)^n - 1 &= (1 + h - 1) \left( 1 + (1 + h) + (1 + h)^2 + \cdots + (1 + h)^{n-1} \right) \\ &= h \left( 1 + (1 + h) + (1 + h)^2 + \cdots + (1 + h)^{n-1} \right) \end{aligned}$$

$$\because 1 + (1 + h) + (1 + h)^2 + \cdots + (1 + h)^{n-1} < n \text{ when } h < 0$$

$$\therefore (1 + h)^n > 1 + nh$$

Two variable extension of the Bernoulli inequality, Suppose  $h = \frac{B}{A}, A > 0, A + B > 0$ , Then  $1 + h > 0$  is established.  $\square$

**Proposition 1.1.4.** 1.3.2 Suppose  $A > 0, A + B > 0, n \in \mathbb{N}$ , Then the inequality is true:

$$(A + B)^n \geq A^n + nA^{n-1}B \quad (1.12)$$

The inequality became equation iff  $B = 0$ .

**Proof.** divide  $A^n$  on both side of the inequality 1.12. Set  $h = \frac{B}{A} (A > 0)$ , Then the inequality became Eq 1.11. So we can prove Eq 1.12 by prove Eq 1.11. Eq 1.11 is true when  $h > -1$ .  $\therefore 1 + h > 0, 1 + \frac{B}{A} > 0, \because A > 0, \therefore A + B > 0$ . And when  $n > 1$  the equation is true iff  $h = 0, \frac{B}{A} = 0, \therefore B = 0$ .  $\square$

**Example 1.1.8.** Ex 1.3.2 exercise 8

$$a, c, t, g \geq 0, a + c + t + g = 1. \text{ Prove that } a^2 + c^2 + t^2 + g^2 \geq \frac{1}{4}.$$

The inequality became equation iff  $a = c = t = g = \frac{1}{4}$ .

**Proof.** from A.G inequality,

$$\frac{a + c + t + g}{4} \geq \sqrt[4]{actg}, \quad a + c + t + g = 1 \quad (1.13)$$

$$\therefore \sqrt[4]{actg} \leq \frac{1}{4}$$

$$a + c + t + g = 1, (a + c + t + g)^2 = 1$$

$$(a + c + t + g)^2 = a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg = 1 \quad (1.14)$$

$$a^2 + c^2 \geq 2acc^2 + t^2 \geq 2ct \quad (1.15)$$

$$a^2 + t^2 \geq 2atc^2 + g^2 \geq 2cg \quad (1.16)$$

$$a^2 + g^2 \geq 2agt^2 + g^2 \geq 2tg \quad (1.17)$$

substitute  $2ac, 2ag, \dots$  in equation 1.14, we can get

$$4(a^2 + c^2 + t^2 + g^2) \geq a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg$$

Then we get the inequality 1.13. □

## 1.2 1.4 逻辑符号与对偶法则

The law of duality:  $\forall(\exists) \rightarrow \exists(\forall)$  with negative statement

Inverse proposition?

1. A have upper limit,  $\exists M > 0, \forall x \in A, x \leq M$ .

It's negative statement is 'A don't have upper limit'.  $\forall M > 0, \exists x \in A, x > M$ .

2. the minum item in A is b,  $b \in A, \forall x \in A, x \geq b$ .

It's negative statement is 'b is not the minum item in A'.  $b \in A, \exists x \in A, x < b$ .

3.  $f \in (a, b)$  is a monotonic augmentation function,  $\forall x, y \in (a, b), x < y, f(x) \leq f(y)$ . (or  $f(x) < f(y)$ , depends on monotonic function's definition)

It's negative statement is 'f  $\in (a, b)$  isn't a monotonic augmentation function'.  $\exists x, y \in (a, b), x < y, f(x) > f(y)$  (or  $f(x) \geq f(y)$ ).

4.  $f \in (a, b)$  is a monotonic function,  $\forall x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) \geq 0$ .

It's negative statement is 'f  $\in (a, b)$  isn't a monotonic function'.  $\exists x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) < 0$ .

(Another way  $\forall x, y \in (a, b), x < y, f(x) - f(y) \geq 0$  or  $f(x) - f(y) \leq 0$ .)

5.  $A \subset B, \forall x \in A, x \in B$ .

It's negative statement is  $A \not\subset B, \exists x \in A, x \notin B$ .

6.  $A - B \neq \emptyset, \exists x \in A, x \in B$ .

It's negative statement is  $A - B = \emptyset, \forall x \in A, x \notin B$ .

7.  $x_n$  is an infinitesimal amounts,  $\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \forall n > N, |x_n| < \epsilon$ .

It's negative statement is ' $x_n$  is not an infinitesimal amounts',  $\exists \epsilon > 0, \forall N \in \mathbb{N}^+, \exists n > N, |x_n| \geq \epsilon$ .

8.  $x_n$  is infinitely large,  $\forall M > 0, \exists N \in \mathbb{N}^+, \forall n > N, x_n > M$ .

It's negative statement is ' $x_n$  is not infinitely large',  $\exists M > 0, \forall N \in \mathbb{N}^+, \exists n > N, x_n \leq M$ .

## 第二章 数列极限

### 2.1 数列极限的基本概念

#### 2.1.1 2.1.5 练习题

2021.5.5 1. prove by Limit definition:

(1).  $\lim_{n \rightarrow \infty} \frac{3n^2}{n^2-4} = 3.$

(2).  $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0.$

(3).  $\lim_{n \rightarrow \infty} (1+n)^{\frac{1}{n}} = 0.$

(4).  $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0, (a > 0).$

2. Suppose  $a_n, n \in \mathbb{N}_+$ . sequence  $a_n$  converge to  $a$ .

Prove  $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}.$

**Proof.**  $n \rightarrow \infty, a_n \rightarrow a.$

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}.$   $\square$  (check, not consider the condition  $a = 0$ ) add  $a = 0, \forall \epsilon \in (0, 1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$  s.t  $a_n < \epsilon^2 < \epsilon, \sqrt{a_n} < \epsilon.$   $\square$

3. If  $\lim_{n \rightarrow \infty} a_n = a.$

Prove  $\lim_{n \rightarrow \infty} |a_n| = |a|.$  Vice versa?

**Proof.**  $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon.$

$$||a_n| - |a|| \leq |a_n - a| < \epsilon$$

$\therefore \lim_{n \rightarrow \infty} |a_n| = |a|$

If We know  $\lim_{n \rightarrow \infty} |a_n| = |a|.$

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), ||a_n| - |a|| < \epsilon.$  We can't get  $\lim_{n \rightarrow \infty} a_n = a.$  For example:

$a_n = \frac{1}{n} + 1, a = -1, \lim_{n \rightarrow \infty} |a_n| = |a|$  is  $\lim_{n \rightarrow \infty} |\frac{1}{n} + 1| = |-1|,$  but  $\lim_{n \rightarrow \infty} \frac{1}{n} + 1 \neq -1$   $\square$

$\square$

- (1). Suppose  $p(x)$  is a polynomial of  $x$ , if  $\lim_{n \rightarrow \infty} a_n = a$ , Prove  $\lim_{n \rightarrow \infty} p(a_n) = p(a)$ .  
 (2). Suppose  $b > 0$ ,  $\lim_{n \rightarrow \infty} a_n = a$ . Prove  $b^{a_n} = b^a$ .  
 (3). Suppose  $b > 0$ ,  $\{a_n\}$ ,  $a_n > 0, \forall n \in \mathbb{N}$ .  $\lim_{n \rightarrow \infty} a_n = a, a > 0$ . Prove  $\lim_{n \rightarrow \infty} \log_b a_n = \log_b a$ .  
 (4) Suppose  $b \in \mathbb{R}$ ,  $\{a_n\}$ ,  $a_n > 0$  when  $n \in \mathbb{N}$ .  $\lim_{n \rightarrow \infty} a_n = a$ . Prove  $\lim_{n \rightarrow \infty} a_n^b = a^b$ .  
 (5) Suppose  $\lim_{n \rightarrow \infty} a_n = a$ . Prove  $\lim_{n \rightarrow \infty} \sin a_n = \sin a$ .

**Proof. 4.(1)**

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geq N(\epsilon), |a_n - a| < \epsilon.$$

$$p(a) = k_m a^m + k_{m-1} a^{m-1} + \cdots + k_0 a^0.$$

$$\therefore p(a_n) - p(a) = k_m (a_n^m - a^m) + k_{m-1} (a_n^{m-1} - a^{m-1}) + \cdots + k_0 (a_n^0 - a^0).$$

$$\begin{aligned} |a_n^m - a^m| &= |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon(m-1) \cdots (a + \delta)^{m-1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} p(a_n) = p(a). \quad \square$$

**Proof. 4.(2)**

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$\text{If } b = 1, 1^{a_n} = 1^a = 1.$$

$$\text{If } b > 1, b^{a_n} - b^a = b^a (b^{a_n - a} - 1) < b^a (b^\epsilon - 1) \quad 0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1) \therefore b > 0, \epsilon \rightarrow 0,$$

$$\therefore b^\epsilon - 1 \rightarrow 0. \therefore \lim_{n \rightarrow \infty} b_n^a = b^a.$$

$$\text{If } b < 1, b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}, \text{ we can prove this condition by considering } \frac{1}{b} > 1. \quad \square$$

**Proof. 4.(3)**

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$\begin{aligned} \log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b \left( \frac{a_n - a}{a} + 1 \right) < \log_b \left( \frac{\epsilon}{a} + 1 \right) \end{aligned}$$

$$0 < \log_b a_n - \log_b a < \log_b \left( 1 + \frac{\epsilon}{a} \right). \therefore b > 0, a \neq 0, a_n > 0 \text{ when } \epsilon \rightarrow 0. \therefore \log_b \left( 1 + \frac{\epsilon}{a} \right) \rightarrow 0.$$

$$\therefore \lim_{n \rightarrow \infty} \log_b a_n = \log_b a \quad \square$$

**Proof. 4.(4)**

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$a_n^b = e^{b \ln a_n}, a_n^b - a^b = e^{b \ln a_n} - e^{b \ln a}.$$

$$\begin{aligned} e^{b \ln a_n} - e^{b \ln a} &= e^{b \ln a} (e^{b \ln a_n - b \ln a} - 1) \\ &= e^{b \ln a} (e^{b \ln \frac{a_n}{a}} - 1) \end{aligned}$$

$$0 < |a_n^b - a^b| < e^{b \ln a} (e^{b \ln(1 + \frac{\epsilon}{a})} - 1)$$

$$\therefore \lim_{n \rightarrow \infty} a_n^b = a^b$$

□

**Proof.** 4.(5)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$\begin{aligned} \sin(A+B) - \sin(A-B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B \end{aligned}$$

$$\sin a_n - \sin a = 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}$$

$$|\sin a_n - \sin a| = |2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}| < |2 \sin \frac{a_n - a}{2}|$$

$$|2 \sin \frac{a_n - a}{2}| < |2 \frac{a_n - a}{2}| = \epsilon$$

$$|\sin a_n - \sin a| < \epsilon, \therefore \lim_{n \rightarrow \infty} \sin a_n = \sin a$$

□

assume  $a > 1$ . Prove  $\lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$

**Proof.**  $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$ . We already know that  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \log_a 1 = 0$ .

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geq N, |\sqrt[n]{n} - 1| < \epsilon.$$

$a > 1$ , and  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .  $\therefore$  when  $n \rightarrow \infty$ ,  $\sqrt[n]{n} < a^\epsilon$ , take logarithm on base of  $a$ , we can get

$$\frac{1}{n} \log_a n < \epsilon$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$$

□

## 2.2 收敛数列的基本性质

### 2021.5.6 收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界
3. 收敛数列的比较定理, 包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

## 2.2.1 思考题

1.  $\{a_n\}$  收敛,  $\{b_n\}$  发散,  $\{a_n + b_n\}$  发散,  $\{a_n \cdot b_n\}$  可能收敛, 可能发散.
2.  $\{a_n\}, \{b_n\}$  都发散,  $\{a_n + b_n\}$  可能收敛, 可能发散 (ex:  $n + -n, n + -2n$ ),  $\{a_n \cdot b_n\}$  发散 (?).
3.  $a_n \leq b_n \leq c_n, n \in \mathbb{N}_+$ . 已知  $\lim_{n \rightarrow \infty} (c_n - a_n) = 0$ . 问数列  $\{b_n\}$  是否收敛?
4.  $\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \cdots + \frac{1}{2n})$
5.  $a_n \rightarrow a (n \rightarrow 0), \forall n, b < a_n < c$ . 是否成立  $b < a < c$ ?
6.  $a_n \rightarrow a (n \rightarrow 0)$ . and  $b \leq a \leq c$ , 是否存在  $N \in \mathbb{N}_+$ , s.t. 当  $n > N$  时, 成立  $b \leq a_n \leq c$
7. 已知  $\lim_{n \rightarrow \infty} a_n = 0$ , 问: 是否有  $\lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n) = 0$ . 反之如何?

**Proof. 5.4**

$$\frac{n}{2n} \leq \frac{1}{n+1} + \cdots + \frac{1}{2n} \leq \frac{n}{n+1}$$

$\therefore \lim_{n \rightarrow \infty} \frac{n}{2n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{1}{2}, \therefore \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \cdots + \frac{1}{2n})$  收敛.

$$\begin{aligned} \frac{1}{n+1} + \cdots + \frac{1}{2n} &= \frac{1}{n} \left( \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}} \right) \\ &= \int_0^1 \frac{1}{1+x} dx \\ &= \ln(1+x)|_0^1 = \ln 2 \end{aligned}$$

$$\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \cdots + \frac{1}{2n}) = \ln 2$$

□

**Proof. 5.5**

不成立, 应当为小于等于号。b=0, c=2,  $a_n = \frac{1}{n}, \lim_{n \rightarrow \infty} a_n = 0 = c$ .

□

**Proof. 5.6**

不成立。  $a = 0, b = 0, c = 2, a_n = (-1)^n \frac{1}{n}$ .

$b \leq a \leq c$ , but  $(-1)^{2n+1} \frac{1}{2n+1} < 0 = b$ .

□

**Proof.**  $\lim_{n \rightarrow \infty} a_n = 0, a_n = \frac{1}{n} \cdot a_1 a_2 \cdots a_n = \frac{1}{n!}, \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$ .

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n) = 0 \quad \checkmark$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n) = 0 \rightarrow \lim_{n \rightarrow \infty} a_n = 0 \quad \times$$

$$|a_n| < \epsilon, |a_{N+1} \cdots a_n| < \epsilon^{n-N} < \epsilon, a_n < \sqrt[n]{\epsilon}.$$

for example,  $a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \dots, a_n = \frac{n}{n+1}$ .

$$a_1 a_2 \cdots a_n = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}.$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$



but  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$

□

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

**Example 2.2.1.** 2.2.2  $\lim_{n \rightarrow \infty} \frac{x_n - 1}{x_n + a} = 0$ ,  
prove  $\lim_{n \rightarrow \infty} x_n = a$

**Proof.**  $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |\frac{x_n - 1}{x_n + a} - 0| < \epsilon$ .  
 $|x_n - 1| < \epsilon |x_n + a| < 4a \cdot \epsilon$ . (这个 4 是怎么取得的?)  
 $|x_n - a| < \epsilon |x_n + a| = \epsilon |(x_n - a) + 2a| \leq \epsilon (|x_n - a| + 2a)$ .  
 限制  $\epsilon < 1, |x_n - a| < 2\epsilon |a| / (1 - \epsilon)$ .  
 限制  $\epsilon < \frac{1}{2}, |x_n - a| < 2\epsilon |a| / (1 - \epsilon) < 4|a|\epsilon$ .  
 Let  $\epsilon' = 4a\epsilon, |x_n - 1| < \epsilon'. \therefore \lim_{n \rightarrow \infty} x_n = a$ .

□

**Example 2.2.2.** 2.2.3  $a > 0, b > 0$ , 计算  $\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}}$ .

**Proof.** Suppose  $a \leq b$ .  
 $b = (b^b)^{\frac{1}{b}} < (a^n + b^n)^{\frac{1}{n}} \leq (2b^n)^{\frac{1}{n}}$ .  
 $b < (a^n + b^n)^{\frac{1}{n}} \leq \sqrt[n]{2}b, \lim_{n \rightarrow \infty} = 1$ . 夹逼定理.  
 $\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}$ .  
 两数  $n$  次方之和再开  $n$  次根号的结果由较大的值决定,  $a, b$  中较大的值为这个数的主要部分.

□

**Example 2.2.3.** 2.2.4  $a_n = \frac{1! + 2! + \dots + n!}{n!}, n \in \mathbb{N}^+$

$$\lim_{n \rightarrow \infty} a_n = 1$$

**Example 2.2.4.**  $\lim_{n \rightarrow \infty} \frac{n^3 + n - 7}{n + 3} = +\infty$

**Example 2.2.5.**  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

调和级数  $H_n$  发散.

## 2.2.2 练习 2.2.4

**Proof.** 1.

$\{a_n\}$  收敛于  $a, \rightarrow$  两个子列  $\{a_{2n}\}, \{a_{2n+1}\}$  均收敛于  $a$ .  
 两个子列  $\{a_{2n}\}, \{a_{2n+1}\}$  均收敛于  $a, \rightarrow \{a_n\}$  收敛于  $a$ .

□

2. 应用夹逼定理

(1). 给定  $p$  个正数  $a_1, a_2, \dots, a_p$ . 求  $\lim_{n \rightarrow \infty} \sqrt[n]{a_1^n + a_2^n + \dots + a_p^n}$ .

$$\text{Let } a_s = \max_{1 \leq i \leq p} \{a_1, a_2, \dots, a_p\}.$$

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} \leq (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$$n \rightarrow \infty, p^{\frac{1}{n}} \rightarrow 1. \lim_{n \rightarrow \infty} (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} = a_s$$

$$(2). x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}, n \in \mathbb{N}_+. \text{ 求 } \lim_{n \rightarrow \infty} x_n$$

$$\frac{2n+1}{(n+1)} \leq x_n \leq \frac{2n+1}{\sqrt{n^2+1}}$$

$$\lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = 2, \lim_{n \rightarrow \infty} \frac{2n+1}{\sqrt{n^2+1}} = 2. \therefore \lim_{n \rightarrow \infty} x_n = 2$$

$$(3). a_n = (1 + \frac{1}{2} + \dots + \frac{1}{n})^{\frac{1}{n}}, n \in \mathbb{N}_+. \text{ 求 } \lim_{n \rightarrow \infty} a_n$$

$$1 = \left(\frac{n}{n}\right)^{\frac{1}{n}} < a_n \leq (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \therefore \lim_{n \rightarrow \infty} a_n = 1$$

(4).  $a_n > 0$ .  $\lim_{n \rightarrow \infty} a_n = a, a > 0$ . 证明  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$

**Proof.**  $\lim_{n \rightarrow \infty} a_n = a$   
 $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon$ .

$$0 < a - \epsilon < a_n < a + \epsilon$$

$$\therefore \sqrt[n]{a - \epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a + \epsilon}.$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a - \epsilon} = 1, \lim_{n \rightarrow \infty} \sqrt[n]{a + \epsilon} = 1. \therefore \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1.$$

□

$$\lim_{n \rightarrow \infty} (1+x)(1+x^2) \dots (1+x^{2^n}) = \lim_{n \rightarrow \infty} \prod_{i=1}^{2^n} (1+x^i), |x| < 1.$$

$$|x| < 1, \quad 1 > x^2 > x^4 > \dots > x^{2^n} > 0$$

$$x \in (0, 1) \quad 1 < (1+x)(1+x^2) \dots (1+x^{2^n}) < (1+x)^{n+1}$$

$$\lim_{n \rightarrow \infty} (1+x)^{n+1} = 1$$

$$x \in (-1, 0) \quad 0 < (1+x)(1+x^2) \dots (1+x^{2^n}) < (1+x)(1+x^2)^n$$

$$\lim_{n \rightarrow \infty} (1+x)(1+x^2)^n = 1$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} (1+x)(1+x^2) \dots (1+x^n) \\ &= \lim_{n \rightarrow \infty} \frac{(1-x)(1+x)(1+x^2) \dots (1+x^n)}{1-x} \\ &= \lim_{n \rightarrow \infty} \frac{(1-x^{2^{n+1}})}{1-x} \\ &= \frac{1}{1-x} \end{aligned}$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{n^2}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \dots \cdot \frac{n-1}{n} \cdot \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2+3}\right) \cdots \left(1 - \frac{1}{1+2+\cdots+n}\right) \\
&= \lim_{n \rightarrow \infty} \left(1 - \frac{2}{3 \times 2}\right) \left(1 - \frac{2}{4 \times 3}\right) \cdots \left(1 - \frac{2}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{3 \times 2 - 2}{3 \times 2}\right) \left(\frac{4 \times 3 - 2}{4 \times 3}\right) \cdots \left(\frac{(n+1) \times n - 2}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{4}{3 \times 2}\right) \left(\frac{10}{4 \times 3}\right) \cdots \left(\frac{n^2 + n - 2}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{1 \times 4}{3 \times 2}\right) \left(\frac{2 \times 5}{4 \times 3}\right) \cdots \left(\frac{(n-2) \times (n+1)}{n \times (n-1)}\right) \left(\frac{(n-1) \times (n+2)}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{3} \times \frac{n+2}{n} \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\
&= \lim_{n \rightarrow \infty} \frac{1}{1} - \frac{1}{n+1} \\
&= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\
&= 1
\end{aligned}$$

3.(4).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1) \cdot (n+2)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \frac{1}{2} \left( \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right) + \cdots + \frac{1}{2} \left( \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)} \right) \\
&= \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

3.(5).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1) \dots (k+\gamma)}, \quad \text{其中 } \gamma \text{ 为正整数} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\gamma} \left[ \frac{1}{k(k+1) \dots (k+\gamma-1)} - \frac{1}{(k+1)(k+2) \dots (k+\gamma)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[ \sum_{k=1}^n \frac{1}{k(k+1) \dots (k+\gamma-1)} - \sum_{k=1}^n \frac{1}{(k+1)(k+2) \dots (k+\gamma)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[ \frac{1}{\gamma!} - \frac{1}{(n+\gamma)!} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[ \frac{1}{\gamma!} - \frac{1}{(n+\gamma)!} \right] \\
&= \frac{1}{\gamma} \cdot \frac{1}{\gamma!}
\end{aligned}$$

其中  $x^n = x(x-1)(x-2) \dots (x-n+1)$ , 称为下阶乘. 而  $x^{\bar{n}} = x(x+1)(x+2) \dots (x+n-1)$ , 称为上阶乘.

2.2.4-4  $S_n = a + 3a^2 + \dots + (2n-1)a^n$ ,  $|a| < 1$ . 求  $\{S_n\}$  的极限.

$$\begin{aligned}
S_n - aS_n &= a + 3a^2 + \dots + (2n-1)a^n \\
&\quad - a^2 - \dots + (2n-3)a^n - (2n-1)a^n + 1 \\
&= a + 2a^2 + \dots + 2aa^n - (2n-1)a^{n+1} \\
&= 2(a + a^2 + \dots + a^n) - a - (2n-1)a^{n+1} \\
&= 2 \cdot a \frac{1 - a^{n+1}}{1 - a} - a - (2n-1)a^{n+1}
\end{aligned}$$

$|a| < 1$ ,  $\lim_{n \rightarrow \infty} a^n = 0$

$$\lim_{n \rightarrow \infty} (S_n - aS_n) = (1-a) \lim_{n \rightarrow \infty} S_n$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} (S_n - aS_n) &= \lim_{n \rightarrow \infty} 2a \cdot \frac{1 - a^{n+1}}{1 - a} - a - (2n-1)a^{n+1} \\
&= 2a \cdot \frac{1}{1-a} - a \\
&= a \left( \frac{2}{1-a} - a \right) \\
&= a \frac{1+a}{1-a}
\end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \frac{a(a+1)}{(1-a)^2}$$

2.2.4-5 设  $\lim_{n \rightarrow \infty} x_n = A > 0$ . 取  $\epsilon = \frac{A}{2}$ , 则  $\exists N \in \mathbb{N}_+$ .  $\forall n > N$ . 成立  $|x_n - A| < \frac{A}{2}$

$$A - \frac{A}{2} < x_n < A + \frac{A}{2}, \quad \frac{A}{2} < x_n < \frac{3A}{2}$$

即  $x_n > \frac{A}{2}$ .

令  $m = \min\{x_1, x_2, \dots, x_N, \frac{A}{2}\} > 0$ . 则  $m$  为  $\{x_n\}$  的正下界.

不一定有最小数的例子  $x_n = 1 + \frac{1}{n}$ .  $\lim_{n \rightarrow \infty} x_n = 1$ , 下界  $m = \frac{1}{2}$ . 但  $\{x_n\}$  取不到下界.

**Proof.** 2.2.4-6  $\because \lim_{n \rightarrow \infty} a_n = +\infty, \forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M.$

$m = \min\{a_1, a_2, \dots, a_N, M\}$ , 但  $M$  在数列  $\{a_n\}$  中不一定取的到!

$M = a_1 + 1, \exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$

则  $m = \min\{a_1, a_2, \dots, a_{N_1}\}$  为数列的最小数. □

**Proof.** 2.2.4-7 构造数列

不妨设无界数列  $\{a_n\}$  无上界.

$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M.$

取  $M_1 = 1$ , 则  $\exists n_1 \in \mathbb{N}_+ \text{ s.t. } a_{n_1} > M_1.$

取  $M_2 = \max\{a_{n_1}, 2\}, \exists n_2 \in \mathbb{N}_+ \text{ s.t. } a_{n_2} > M_2.$

以此类推, 构造数列  $\{a_{n_k}\}$ . s.t.  $a_{n_k} > k$ . 即  $a_{n_k}$  为无穷大量. □

**Proof.** 2.2.4-8 证明  $\{a_n\}, a_n = \tan n$  发散.

构造  $a_n$  的发散子列即可. 已知  $\tan \frac{\pi}{2} = \infty, \pi$  是一个无理数, 因此存在数列  $\{b_n\}, \lim_{n \rightarrow \infty} b_n = \frac{\pi}{2}.$  □

**Proof.** 2.2.4-8 证明  $\{a_n\}, a_n = \tan n$  发散. 参考别人的答案

由于  $\{\sin 2n\}$  极限不存在, 又

$$\begin{aligned} \sin 2n &= 2 \sin n \cos n = \frac{2 \sin n \cos n}{\sin^2 n + \cos^2 n} \\ &= \frac{2 \tan n}{\tan^2 n + 1} \end{aligned}$$

若  $\{\tan n\}$  极限存在  $\rightarrow \{\sin 2n\}$  极限存在, 矛盾.

故  $\{\tan n\}$  极限不存在. □

2.2.4-9  $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p}, n \in \mathbb{N}_+.$   $S_n$  在 1.  $p \leq 0, 2. 0 < p < 1$  情况下均发散

**Proof.** 1.  $p \leq 0. \lim_{n \rightarrow \infty} n^{-p} = \infty, S_n$  发散.

2.  $0 < p < 1. \frac{1}{n^p} > \frac{1}{n}. \because H_n = \sum_{k=1}^n \frac{1}{k}$  (调和级数) 发散,  $S_n > H_n, \therefore \{S_n\}$  也发散. □

2021.5.11 ex2.3.5  $0 < b < a$  令  $a_0 = a, b_0 = b$  递推公式

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, b_n = \sqrt{a_{n-1} b_{n-1}}, n \in \mathbb{N}_+ \quad (2.1)$$

定义数列  $a_n, b_n$ . 证明这两个数列收敛于同一个极限  $AG(a, b)$ .

由 AG 不等式  $a > \frac{a+b}{2} > \sqrt{ab} > b > 0$ , 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a, b) = \frac{\pi}{2G} \quad (2.2)$$

如果令  $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$ . 则

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \quad (2.3)$$

上面这个公式是怎么得到的:

参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} \quad (a > b > 0) \quad (2.4)$$

这里令

$$\sin \phi = \frac{2a \sin \theta}{(a+b) + (a-b) \sin^2 \theta} \quad (2.5)$$

$\theta \in [0, \frac{\pi}{2}] \rightarrow \phi \in [0, \frac{\pi}{2}]$ , 取微分

$$\cos \phi d\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{[(a+b) + (a-b) \sin^2 \theta]^2} \cos \theta d\theta \quad (2.6)$$

但是

$$\cos \phi = \frac{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}}{(a+b) + (a-b) \sin^2 \theta} \cos \theta. \quad (2.7)$$

(2.6) / (2.7), 两式相除, 得到

$$d\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \frac{d\theta}{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}} \quad (2.8)$$

另一方面

$$\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \quad (2.9)$$

因而

$$\frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \frac{d\theta}{\sqrt{(\frac{a+b}{2})^2 \cos^2 \theta + ab \sin^2 \theta}}. \quad (2.10)$$

如果令  $a_1 = \frac{a+b}{2}$ ,  $b_1 = \sqrt{ab}$ , 则

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \quad (2.11)$$

反复应用该公式, 得到

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}}, \quad (n = 1, 2, 3, \dots) \quad (2.12)$$

$$\frac{\pi}{2a_n} < G < \frac{\pi}{2b_n} \quad (2.13)$$

积分  $G$  可以归结到第一类全椭圆积分  $K(k) = (1+k_1)K(k_1) = \frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = (1+k_1) \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k_1^2 \sin^2 \theta}} \quad (2.14)$$

其中

$$a_1 = \frac{1+\sqrt{1-k^2}}{2} = \frac{1+k'}{2}, b_1 = \sqrt{k'}$$

$$k_1 = \frac{\sqrt{a_1^2 - b_1^2}}{a_1} = \frac{1-k'}{1+k'}, \frac{1}{a_1} = 1+k_1$$

## 2.3 2.3 单调数列

2021.05.12

### Example 2.3.1. 2.3.6

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{1!+2!+\cdots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\cdots+n!}{n!}} \\ &= \frac{1}{n+1} \frac{1!+2!+\cdots+(n+1)!}{1!+2!+\cdots+n!} \\ &= \frac{3+3!+\cdots+(n+1)!}{(n+1)1!+(n+1)2!+\cdots+(n+1)!}\end{aligned}$$

$n > 2$  时, 分母每一项大于等于分子对应项..  $n > 2$  后  $a_n$  单调减少. 由于 0 是下界, 因此  $a_n$  单调有界, 数列收敛.

$$\begin{aligned}a_{n+1} &= \frac{1!+2!+\cdots+(n+1)!}{(n+1)!} \\ &= \frac{1!+2!+\cdots+n!}{n!} \frac{1}{n+1} + 1 \\ &= 1 + \frac{a_n}{n+1}\end{aligned}$$

设  $n \rightarrow \infty$  时,  $a_n \rightarrow a$

$$a = 1 + \left( \frac{1}{n+1} \rightarrow 0 \right) = 1 + 0, \quad \therefore a = 1$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1!+2!+\cdots+n!}{n!} = 1$$

### 2.3.1 2.3.2 练习题

证明, 若  $x_n$  单调, 则  $|x_n|$  至少从某项开始后单调, 又问: 反之如何?

**Proof.** 分类讨论, 不妨设  $x_1 \geq 0$

1.  $x_n$  单调递增,  $|x_n|$  从第一项开始单调.
2.  $x_n$  单调递减, 且  $|x_n| \geq 0$ .  $|x_n|$  从第一项开始单调.
3.  $x_n$  单调递减, 且  $\exists N$  s.t.  $x_n < 0$  (第一个负数项). 则  $|x_n|$  从第  $N$  项 ( $x_N$ ) 开始单调.

反之该结论不成立.

反例:  $x_n = \frac{(-1)^n}{n}$ ,  $|x_n|$  单调递减. 但  $x_{2k} = \frac{1}{2k} > 0 > x_{2k-1} = \frac{-1}{2k-1}$  □

设  $a_n$  单调增加,  $b_n$  单调减少, 且有  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ .

证明: 数列  $a_n$  和  $b_n$  都收敛, 且极限相等.

**Proof.**  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N}_+, \text{s.t.} \forall n > N, |a_n - b_n - 0| < \epsilon$ .

$b_n - \epsilon < a_n < b_n + \epsilon$ , 同时有  $a_n - \epsilon < b_n < a_n + \epsilon$ .

$b_n$  单调减少,  $\therefore \exists N_2, \forall m < N_2, b_m > b_n + \epsilon$ .



使用反证法证明  $b_m$  是  $a_n$  的上界.

假设  $b_m$  不是  $a_n$  的上界, 则存在  $a_n > b_m > b_n + \epsilon$ , 这与  $|a_n - b_n| < \epsilon$  矛盾.

$\therefore b_m$  是  $a_n$  的上界, 根据单调有界收敛准则,  $a_n$  收敛. 同理可证  $b_n$  收敛.  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ .

$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ . □

按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.

2. 单调减少有下界的数列的极限不大于数列中的任何一项.

设  $x_n = \frac{2}{3} \cdot \frac{3}{5} \cdots \frac{n+1}{2n+1}$ ,  $n \in \mathbb{N}_+$ , 求数列  $x_n$  的极限.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)+1}{2(n+1)+1} = \frac{n+2}{2n+3} < 1. \quad (n > 0) \quad (2.15)$$

$x_n$  单调递减.  $\therefore x_n > 0$ ,  $\therefore x_n$  有下界,  $x_n$  收敛.

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \lim_{n \rightarrow \infty} \frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$ , 由夹逼定理,  $\lim_{n \rightarrow \infty} x_n = 0$

6. 在例题 2.2.6 的基础上证明: 当  $p > 1$  时, 数列  $S_n$  收敛. 其中

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$$

( $S_n$  就是  $p$  级数, 当  $p = 1$  时为调和级数.)

**Proof.**  $S_n$  单调递增, 记  $\frac{1}{2^{p-1}} = r$ , 则  $0 < r < 1$ .

$$\begin{aligned} \frac{1}{2^p} + \frac{1}{3^p} &< \frac{1}{2^p} + \frac{1}{2^p} &&= \frac{1}{2^{p-1}} = r \\ \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} &< \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} &&= \frac{1}{4^{p-1}} = r^2 \\ \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} &< \frac{1}{(2^k)^p} + \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^k)^p} &&= \frac{1}{(2^k)^{p-1}} = r^k \end{aligned}$$

□

由此可知

$$S_n \leq S_{2^n-1} < 1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} < \frac{1}{1-r}$$

$S_n$  单调递增有上界, 由单调有界收敛准则知  $S_n$  收敛.

7. 设  $0 < x_0 < \frac{\pi}{2}$ ,  $x_n = \sin x_{n-1}$ .  $n \in \mathbb{N}_+$ .

证明  $x_n$  收敛, 并求其极限.

**Proof.**  $x_0 \in (0, \frac{\pi}{2})$ ,  $\sin x$ ,

$$0 < x_1 = \sin x_0 < x_0 < \frac{\pi}{2}.$$

$$0 < x_2 = \sin x_1 < x_1 < \frac{\pi}{2}.$$

$$0 < \cdots < x_n < x_{n-1} < \cdots < x_2 < x_1 < \frac{\pi}{2}.$$

$x_n$  单调递减有下界,  $x_n$  收敛。

$$a = \sin a, \quad a \in [0, \frac{\pi}{2}]$$

解得  $a = 0$ ,  $\therefore \lim_{n \rightarrow \infty} x_n = 0$ .

□