



数学分析习题课讲义上册习题

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$$\begin{aligned}
 I &= \int_{\frac{\pi}{4}}^{\pi} \int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta) r dr d\theta \\
 &= \left[\int_0^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\pi - \arcsin \frac{r}{2}} + \int_{\sqrt{2}}^2 \int_{\arcsin \frac{r}{2}}^{\pi - \arcsin \frac{r}{2}} \right] f(r\cos\theta, r\sin\theta) r dr d\theta
 \end{aligned} \tag{1}$$

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2+3}\right) \cdots \left(1 - \frac{1}{1+2+\cdots+n}\right) = ? \tag{2}$$

$$\begin{aligned}
 1 - \frac{1}{\frac{n(n+1)}{2}} &= 1 - \frac{2}{n(n+1)} \\
 &= \frac{n^2 + n - 2}{n(n+1)} \\
 &= \frac{(n+2)(n-1)}{n(n+1)}
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 I &= \lim_{n \rightarrow +\infty} \frac{1 \times 4}{2 \times 3} \frac{2 \times 5}{3 \times 4} \cdots \frac{(n-2)(n+1)}{(n-1)n} \frac{(n-1)(n+2)}{n(n+1)} \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{3} \frac{4}{2} \frac{2}{3} \frac{5}{4} \frac{3}{5} \frac{6}{4} \cdots \frac{n+2}{n} \\
 &= \lim_{n \rightarrow +\infty} \frac{1}{3} \frac{n+2}{n} \\
 &= \frac{1}{3} \lim_{n \rightarrow +\infty} \frac{n+2}{n} \\
 &= \frac{1}{3}
 \end{aligned} \tag{4}$$

Theorem 0.1. A-G 不等式

意 n 个非负实数 a_1, a_2, \dots, a_n

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \tag{5}$$

其中等号成立 $\iff a_1 = a_2 = \cdots = a_n$



Proof 数学归纳法

$n = 1$ 时结论平凡

$$n = 2 \quad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

$$(a_1 - a_2)^2 = a_1^2 - 2a_1 a_2 + a_2^2 \geq 0$$

$$a_1^2 + 2a_1 a_2 + a_2^2 \geq 4a_1 a_2$$

$$(a_1 + a_2)^2 \geq 4a_1 a_2$$

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

$n = k$ 时, 假设 $\frac{a_1 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 \cdots a_k}$ 成立

$n = k + 1$

$$\begin{aligned} & \frac{a_1 + \cdots + a_k + a_{k+1}}{k+1} - \frac{a_1 + \cdots + a_k}{k} \\ &= \frac{k(a_1 + \cdots + a_{k+1}) - (k+1)(a_1 + \cdots + a_k)}{k(k+1)} \\ &= \frac{ka_{k+1} - (a_1 + \cdots + a_k)}{k(k+1)} \end{aligned} \quad (6)$$

we found

$$\frac{a_1 + \cdots + a_k + a_{k+1}}{k+1} = \frac{a_1 + \cdots + a_k}{k} + \frac{ka_{k+1} - (a_1 + \cdots + a_k)}{k(k+1)}$$

note

$$A := \frac{a_1 + \cdots + a_k}{k}, \quad B := \frac{ka_{k+1} - (a_1 + \cdots + a_k)}{k(k+1)}$$

$$\left(\frac{a_1 + \cdots + a_k + a_{k+1}}{k+1}\right)^{k+1} = (A+B)^{k+1} \geq A^{k+1} + (k+1)A^k B \quad (7)$$

使用二项式展开需要对 a_i 从小到大重排, 而使用 Bernoulli 不等式则只需要 $A \geq 0, (A+B) \geq 0$ 即可

$$A^{k+1} + (k+1)A^k B = A^k(A + (k+1)B) \quad (8)$$

$$\begin{aligned} A^k &= \left(\frac{a_1 + \cdots + a_k + a_{k+1}}{k+1}\right)^{k+1} \geq a_1 \cdots a_k \quad \text{assume at}(n=k) \\ A + (k+1)B &= \frac{a_1 + \cdots + a_k}{k} + \frac{ka_{k+1} - (a_1 + \cdots + a_k)}{k} = a_{k+1} \\ \therefore (A+B)^{k+1} &\geq A^k(A + (k+1)B) \geq a_1 \cdots a_k a_{k+1} \\ \therefore \frac{a_1 + \cdots + a_k + a_{k+1}}{k+1} &\geq \sqrt[k+1]{a_1 \cdots a_k a_{k+1}} \end{aligned} \quad (9)$$

使用二项式展开定理的条件:

在归纳法第二步对 $a_1 \cdots a_{k+1}$ 重编号, 使 a_{k+1} 为其中最大的数 (之一)

这使得分解式右边第二项 $\frac{ka_{k+1} - (a_1 + \cdots + a_k)}{k(k+1)}$ 一定是非负数

Proof Forward and backward (Cauchy, 1897)

Forward Part:

$n = 2$

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \quad (10)$$

$n = 4$

$$\begin{aligned} \frac{a_1 + a_2 + a_3 + a_4}{4} &\geq \sqrt{\frac{a_1 + a_2}{2} \frac{a_3 + a_4}{2}} \\ &\geq \sqrt{\sqrt{a_1 a_2} \sqrt{a_3 a_4}} \\ &\geq \sqrt[4]{a_1 a_2 a_3 a_4} \end{aligned} \quad (11)$$

$n = 2^k$ 假设不等式 $\frac{a_1 + \cdots + a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 \cdots a_{2^k}}$ 成立

$n = 2^{k+1}$

$$\begin{aligned} \frac{a_1 + \cdots + a_{2^k} + \cdots + a_{2^{k+1}}}{2^{k+1}} &\geq \sqrt{\frac{a_1 + \cdots + a_{2^k}}{2^k} \frac{a_{2^k+1} + \cdots + a_{2^{k+1}}}{2^k}} \\ &\geq \sqrt{\sqrt[2^k]{a_1 \cdots a_{2^k}} \sqrt[2^k]{a_{2^k+1} \cdots a_{2^{k+1}}}} \\ &\geq \sqrt[2^{k+1}]{a_1 \cdots a_{2^{k+1}}} \end{aligned} \quad (12)$$

Backward Part: A-G 不等式对某个 $n \geq 2$ 成立, 则它对 $n-1$ 也成立

$$\begin{aligned}\frac{1}{n-1} \sum_{i=1}^{n-1} a_i &= \frac{1}{n} \left(\frac{n}{n-1} \right) \sum_{i=1}^{n-1} a_i \\ &= \frac{1}{n} \left(\sum_{i=1}^{n-1} a_i + \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)\end{aligned}\quad (13)$$

将 $\frac{1}{n-1} \sum_{i=1}^{n-1} a_i$ 看作 a_n

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n]{\left(\prod_{i=1}^{n-1} a_i \right) \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)} \quad (14)$$

$$\left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^n \geq \prod_{i=1}^{n-1} a_i \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \quad (15)$$

$$\left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^{n-1} \geq \prod_{i=1}^{n-1} a_i \quad (16)$$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \quad (17)$$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n]{\left(\prod_{i=1}^{n-1} a_i \right) \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)} \quad (18)$$

$$\left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^n \geq \prod_{i=1}^{n-1} a_i \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \quad (19)$$

$$\left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^{n-1} \geq \prod_{i=1}^{n-1} a_i \quad (20)$$

$$\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i} \quad (21)$$

Theorem 0.2. 柯西-施瓦茨不等式

a_1, \dots, a_n 和 $b_1, \dots, b_n \in \mathbb{R}$, 成立

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \quad (22)$$



Proof

$$\sum_{i=1}^n (a_i - \lambda b_i)^2 = \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2 \geq 0$$

由韦达定理 (视 λ 为未知数), 原方程无解或只有唯一解

$$\begin{aligned}
\Delta &= b^2 - 4ac \leq 0 \\
(-2 \sum_{i=1}^n a_i b_i)^2 - 4 \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 &\leq 0 \\
(\sum_{i=1}^n a_i b_i)^2 &\leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \\
\sum_{i=1}^n a_i b_i &\leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}
\end{aligned} \tag{23}$$

Theorem 0.3. 定积分第一中值定理

函数 $f(x), g(x) \in \mathbb{C}[a, b]$. 且在 $[a, b]$ 上不变号, 则存在 $\zeta \in [a, b]$, 使得 $\int_a^b f(x)g(x)dx = f(\zeta) \int_a^b g(x)dx$



Proof suppose that $g(x) \geq 0$. $f(x)$ continuous on close set, so we can get the maximum and minimum value of f . We note that m is the minimum value of $f(x), x \in [a, b]$, and M is the maximum value of $f(x)$, then we have:

$$\begin{aligned}
mg(x) &\leq f(x)g(x) \leq Mg(x) \\
m \int_a^b g(x)dx &\leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx
\end{aligned}$$

note that we don't know $\int_a^b g(x)dx \neq 0$

When $\int_a^b g(x)dx = 0$, then $g(x) \equiv 0$, So $\forall \zeta \in [a, b]$, the theorem works.

When $\int_a^b g(x)dx \neq 0$, then $m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$

From the Intermediate Value Theorem, $f(x) \in \mathbb{C}[a, b]$ $m \leq f(x) \leq M$

$$\begin{aligned}
\exists \zeta \in [a, b] \quad f(\zeta) &= \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \\
\int_a^b f(x)g(x)dx &= f(\zeta) \int_a^b g(x)dx
\end{aligned}$$

设 $g(x)$ 在 $[a, b]$ 上连续可积, $f(x)$ 在 $[a, b]$ 上连续单调递增, 且 $f'(x) \geq 0$, 并对 $\forall x \in [a, b]$ 有 $f(x) \geq 0$. 则存在 $\zeta \in [a, b]$, 使得

$$\int_a^b f(x)g(x)dx = f(\zeta) \int_a^b g(x)dx$$

Proof set $G(x) = \int_x^b g(t)dt$, $g(x)$ 在 $[a, b]$ 上可积

则 $G(x), x \in [a, b]$ 存在最值, 设最小值和最大值分别为 m, M

$$\begin{aligned}
G(x) &= - \int_b^x g(t)dt, \quad G'(x) = -g(x) \\
\int_a^b f(x)g(x)dx &= - \int_a^b f(x)dG(x) \\
&= -(f(b)G(b) - f(a)G(a)) - \int_a^b G(x)f'(x)dx \\
&= f(a)G(a) + \int_a^b G(x)f'(x)dx
\end{aligned} \tag{24}$$

$$m \int_a^b f'(x) dx \leq \int_a^b G(x) f'(x) dx \leq M \int_a^b f'(x) dx$$

$$m[f(b) - f(a)] \leq \int_a^b G(x) f'(x) dx \leq M[f(b) - f(a)]$$

$$mf(a) \leq f(a)G(a) \leq Mf(a)$$

$$mf(b) \leq \int_a^b f(x)g(x)dx \leq Mf(b)$$

From the Intermediate Value Theorem, $\exists \zeta \in [a, b]$ s.t. $G(\zeta) = \frac{\int_a^b f(x)g(x)dx}{f(b)}$
then we have

$$\int_a^b f(x)g(x)dx = f(b)G(\zeta) = f(b) \int_{\zeta}^b g(x)dx$$

1.3.2 练习题

1. 关于 Bernoulli 不等式的推广:

(1) 证明: 当 $-2 \geq h \geq -1$ 时 Bernoulli 不等式 $(1+h)^n \geq 1+nh$ 仍成立;

(2) 证明: 当 $h \geq 0$ 时成立不等式

$$(1+h)^n \geq \frac{n(n-1)h^2}{2} \quad (25)$$

(3) 证明: 若 $a_i > -1$ ($i = 1, 2, \dots, n$) 且同号, 则成立不等式

solve:

(1)

$$-2 \leq h \leq -1$$

$$-1 \leq 1+h \leq 0$$

$$-1 \leq (1+h)^n \leq 0$$

$$-2n \leq nh \leq -n$$

$$1-2n \leq 1+nh \leq 1-n$$

$$n=0 \quad (1+h)^0 = 1 = 1+0 \cdot h \text{ 结果是平凡的}$$

$$n=1 \quad 1+h = 1+h \text{ 结果是平凡的}$$

$$n \geq 2 \quad \text{此时 } 1-n \leq -2$$

$$0 \geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1-nh \geq 1-2n$$

$$(1+h)^n \geq 1+nh$$

(2)

$$h \geq 0 \quad (1+h)^n \geq \frac{n(n-1)h^2}{2}$$

$$(1+h)^n = 1+nh + \frac{n(n-1)}{2}h^2 + \dots \geq \frac{n(n-1)}{2}h^2$$

推广:

$$(1+h)^n \geq C_n^3 h^3, C_n^4 h^4, \dots, C_n^k h^k, \quad 0 \leq k \leq n$$

(3)

$$\prod_{i=1}^n (1 + a_i) \geq 1 + \sum_{i=1}^n a_i$$

(a) $a_i \geq 0$, 且同号。

$$\prod_{i=1}^n (1 + a_i) = 1 + \sum_{i=1}^n a_i + \sum_{i=1}^n \sum_{i \neq j}^n a_i a_j + \sum_{i=1, i \neq j, k}^n \sum_{j=1, j \neq k}^n \sum_{k=1}^n a_i a_j a_k + \dots$$

$$\prod_{i=1}^n (1 + a_i) \geq \frac{\prod_{i=1}^n (1 + a_i)}{1 + a_k} \quad \forall k \in 1, 2, \dots, n, \quad 1 + a_k \geq 1$$

(b) $0 > a_i > -1$ 此时 $1 > 1 + a_i > 0$

别人的方法: $n = 1$ 时不等式变成等式, 显然成立

设 $n = k$ 时不等式也成立

$$\prod_{i=1}^k (1 + a_i) \geq 1 + \sum_{i=1}^k a_i$$

则 $n = k + 1$ 时, 有

$$\begin{aligned} \prod_{i=1}^{k+1} (1 + a_i) &= \prod_{i=1}^k a_i (1 + a_{k+1}) \geq (1 + \sum_{i=1}^k a_i) (1 + a_{k+1}) \\ (1 + \sum_{i=1}^k a_i) (1 + a_{k+1}) &= 1 + \sum_{i=1}^k a_i + a_{k+1} + \sum_{i=1}^k a_i \cdot a_{k+1} \geq 1 + \sum_{i=1}^{k+1} a_i \\ \therefore \prod_{i=1}^{k+1} (1 + a_i) &\geq 1 + \sum_{i=1}^{k+1} a_i \end{aligned}$$

2. 利用 A-G 不等式求解下列有关阶乘 $n!$ 的不等式

(1) 证明: 当 $n > 1$ 时成立

$$n! < \left(\frac{n+1}{2}\right)^n \quad (26)$$

(2) 利用 $(n!)^2 = (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n)$ 证明: 当 $n > 1$ 时成立

$$n! < \left(\frac{n+2}{\sqrt{6}}\right)^n \quad (27)$$

(3) 比较 (1)(2) 两个不等式的优劣, 并说明原因;

(4) 证明: 对任意实数 r 成立

$$\left(\sum_{k=1}^n k^r\right)^n \geq n^n (n!)^r \quad (28)$$

solve:

(1) when $n > 1$

$$\begin{aligned} n! &= 1 \times 2 \times \dots \times n < \left(\frac{1+2+\dots+n}{n}\right)^n \\ \left(\frac{1+2+\dots+n}{n}\right)^n &= \left(\frac{n(n+1)}{2n}\right)^n = \left(\frac{n+1}{2}\right)^n \end{aligned}$$

(2) when $n > 1$

$$(n!)^2 = (n \cdot 1)((n-1) \cdot 2) \dots (1 \cdot n) < \left(\frac{n \cdot 1 + (n-1) \cdot 2 + \dots + 1 \cdot n}{n}\right)^n$$

$$\begin{aligned}
n \cdot 1 + (n-1) \cdot 2 + \cdots + 1 \cdot n &= \sum_{k=1}^n (n-k+1)k \\
\sum_{k=1}^n (n-k+1)k &= (n+1) \sum_{k=1}^n k - \sum_{k=1}^n k^2 \\
&= (n+1) \frac{n(n+1)}{2} - \frac{n(2n+1)(n+1)}{6} \\
&= \frac{n(n+1)}{6} (3(n+1) - (2n+1)) \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned} \tag{29}$$

$$\begin{aligned}
(n!)^2 &= (n \cdot 1)((n-1) \cdot 2) \cdots (1 \cdot n) \\
&< \left(\frac{n \cdot 1 + (n-1) \cdot 2 + \cdots + 1 \cdot n}{n} \right)^n \\
&= \left(\frac{1}{n} \frac{n(n+1)(n+2)}{6} \right)^n \\
&= \left(\frac{(n+1)(n+2)}{6} \right)^n \\
&< \left(\frac{n+2}{6} \right)^{2n}
\end{aligned} \tag{30}$$

$$\therefore n! < \left(\frac{n+2}{\sqrt{6}} \right)^n \tag{31}$$

(3)

$$\frac{n+1}{2} = \frac{n+2}{\sqrt{6}} \tag{32}$$

解得 $n = 1 + \sqrt{6} > 3$, $n > 3$ 时 (2) 式更精确, 结果比 (1) 式更好。

(4) $\forall r \in \mathbb{R} \quad (n!)^r \leq \frac{1}{n^n} (\sum_{k=1}^n k^r)^n$ 由 A-G 不等式

$$\frac{1}{n} \sum_{k=1}^n k^r \geq \sqrt[n]{\prod_{k=1}^n k^r} \tag{33}$$

$$(n!)^r = \prod_{k=1}^n k^r \leq \left(\frac{1}{n} \sum_{k=1}^n k^r \right)^n = \frac{1}{n^n} \left(\sum_{k=1}^n k^r \right)^n \tag{34}$$

2.(4)

$$\begin{aligned}
&\forall r \in \mathbb{R} \quad \left(\sum_{i=1}^n k^r \right)^n \geq n^n (n!)^r \\
(n!)^r &= \prod_{k=1}^n k^r \leq \left(\frac{1^r + 2^r + \cdots + n^r}{n} \right)^n = \frac{1}{n^n} \left(\sum_{k=1}^n k^r \right)^n \quad \text{A-G inequality} \\
&\therefore \left(\sum_{k=1}^n k^r \right)^n \geq n^n (n!)^r
\end{aligned} \tag{35}$$

3. $a_k > 0, \quad k = 1, 2, \dots, n$ 证明几何-调和平均值不等式

$$\left(\prod_{k=1}^n a_k \right)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}} \tag{36}$$

Proof from A-G inequality

$$\begin{aligned}\frac{\sum_{k=1}^n \frac{1}{a_k}}{n} &\geq \sqrt[n]{\prod_{k=1}^n \frac{1}{a_k}} \\ &= \frac{1}{\sqrt[n]{\prod_{k=1}^n a_k}}\end{aligned}\quad (37)$$

$$\therefore a_k > 0, \quad \sqrt[n]{\prod_{k=1}^n a_k} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}$$

4. $a, b, c \geq 0$, proof that

$$\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3} \quad (38)$$

并推广到 n 个非负数的情况

Proof left:

$$\begin{aligned}\sqrt[3]{abc} &= \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \\ &\leq \sqrt{\frac{ab+bc+ca}{3}}\end{aligned}\quad (39)$$

right:

$$\begin{aligned}\sqrt{\frac{ab+bc+ca}{3}} &\leq \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ &= \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}} \\ &= \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}}\end{aligned}\quad (40)$$

$$\because a, b, c \geq 0 \quad \frac{ab+bc+ca}{3} \leq \frac{a^2+b^2+c^2+ab+bc+ca}{6} \quad (41)$$

需要证明 $\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$

对该式两边平方

$$\frac{ab+bc+ca}{3} \leq \frac{(a+b+c)^2}{9} = \frac{a^2+b^2+c^2+2ab+2bc+2ca}{9} \quad (42)$$

$$\begin{aligned}\frac{ab+bc+ca}{3} &\leq \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6} \\ &\leq \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{3} \\ &= \left(\frac{a+b+c}{3}\right)^2\end{aligned}\quad (43)$$

$$\therefore \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

Proof 推广至 n 个

$$\begin{aligned}[l]n=2 \quad \sqrt{ab} &\leq \frac{a+b}{2} \\ n=3 \quad \sqrt[3]{abc} &\leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3} \\ n=k \quad \sqrt[k]{\prod_{i=1}^k a_i} &\leq \sqrt{\frac{\sum_{i=1}^k -1a_i a_{i+1} + a_k a_1}{k}} \leq \frac{\sum_{i=1}^k a_i}{k}\end{aligned}\quad (44)$$

$$1 \quad \sqrt[k]{a_1 a_2 \dots a_k} = \sqrt{\sqrt[k]{a_1^2 a_2^2 \dots a_k^2}} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k}} \quad (45)$$

$$2 \quad \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k}} \leq \frac{a_1 + \dots + a_k}{k} \quad (46)$$

$$\begin{aligned} \frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k} &\leq \frac{a_1^2 + \dots + a_k^2}{2k} \\ 2 \frac{a_1 a_2 + a_2 a_3 + \dots + a_k a_1}{k} &\leq \frac{(a_1 + \dots + a_k)^2}{2k} \\ \sqrt{\frac{a_1 \dots a_k}{k}} &\leq \frac{a_1 + \dots + a_k}{\sqrt{4k}} \quad \text{wrong!} \end{aligned} \quad (47)$$

Chapter 1 第一章

1.1 引论

1.1.1 关于习题课教案的组织

1.1.2 书中常用记号

1. \mathbf{N}_+ : 所有正整数组成的集合.
2. \mathbf{R} : 所有实数组成的集合 (同时也用于表示无限区间 $(-\infty, \infty)$).
3. \mathbf{Q} : 所有有理数组成的集合.
4. \mathbf{C} : 所有复数组成的集合.
5. \Longleftrightarrow 是等价关系的记号. $A \Longleftrightarrow B$ 表示 A 和 B 等价. 例如, A 代表 $x > 3$, B 代表 $x - 3 > 0$, 则 $x > 3 \Longleftrightarrow x - 3 > 0$.
6. $[x]$ 是实数 x 的整数部分, 即不超过 x 的最大整数. 例如, $[\sqrt{2}] = 1, [-\sqrt{2}] = -2$. 关于 $[x]$ 的基本不等式是: $[x] \leq x < [x] + 1$, 或 $x - 1 < [x] \leq x$.
7. \square 表示一个证明或解的结束.
8. $\binom{n}{k} = C_n^k = \frac{n(n-1)\cdots(n-k+1)}{k!}$.
9. 记号 \approx 表示近似值. 例如 $\sqrt{2} \approx 1.4$.
10. 复合函数 $f(g(x))$ 也写成 $(f \circ g)(x)$ 或 $f \circ g$.
11. 若 A 和 B 为两个集合, 则用记号 $A - B$ 或 $A \setminus B$ 表示 A 与 B 的差集, 也就是集合 $\{x | x \in A \text{ 且 } x \notin B\}$.
12. 用 $O_\delta(a)$ 表示以 a 为中心, 以 $\delta > 0$ 为半径的邻域. 它就是开区间 $(a - \delta, a + \delta)$ (也可以用 $U_\delta(a)$ 等记号). 如不必指出半径, 则可简记为 $O(a)$ (或 $U(a)$).

1.1.3 几个常用的初等不等式

1.1.3.1 几个初等不等式的证明

Theorem 1.1. 1. AG 不等式

个非负实数 a_1, a_2, \dots, a_n

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \quad (1.1)$$

\geq in inequation became $\Longleftrightarrow a_1 = a_2 = \cdots = a_n$



Proof

method 1. induction method

$$\begin{aligned}
k=1 & \quad a_1 = a_1 \\
k=2 & \quad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \\
k=n & \quad \text{suppose} \quad \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \\
k=n+1 & \\
& \quad \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} - \frac{a_1 + a_2 + \cdots + a_n}{n} \\
& = \frac{n(a_1 + a_2 + \cdots + a_{n+1}) - (n+1)(a_1 + a_2 + \cdots + a_n)}{n(n+1)} \\
& = \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n(n+1)}
\end{aligned}$$

$$\text{Set } A = \frac{a_1 + a_2 + \cdots + a_n}{n}, B = \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n(n+1)}$$

$$\left(\frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1}\right)^{n+1} = (A+B)^{n+1}$$

$$A > 0, B \geq 0$$

$$\begin{aligned}
(A+B)^{n+1} & \geq A^{n+1} + (n+1)A^n B \\
A^{n+1} + (n+1)A^n B & = A^n(A + (n+1)B) \\
A^n & = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \geq a_1 a_2 \cdots a_n \\
A + (n+1)B & = \frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n} = a_{n+1} \\
\therefore (A+B)^{n+1} & \geq A^n(A + (n+1)B) \geq a_1 a_2 \cdots a_n \cdot a_{n+1} \\
\therefore \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} & \geq \sqrt[n+1]{a_1 a_2 \cdots a_n a_{n+1}}
\end{aligned}$$

使用二项式展开定理的条件

在归纳法第二步, 将 a_1, a_2, \dots, a_{n+1} 重编号, 使得 $n+1$ 为其中最大的数 (之一), 这使得分解式右边第二项 $(na_{n+1} - (a_1 + a_2 + \cdots + a_n))/n(n+1)$ 一定是非负数。

method 2. Forward and Backward (Cauchy, 1897)

Forward part

$$\begin{aligned}
k=2. & \quad \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}. \\
k=4. & \quad \frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt{\left(\frac{a_1 + a_2}{2}\right) \cdot \left(\frac{a_3 + a_4}{2}\right)}. \\
& \geq \sqrt{\sqrt{a_1 a_2} \sqrt{a_3 a_4}} = \sqrt[4]{a_1 a_2 a_3 a_4}. \\
k=2^n. & \quad \text{Suppose} \quad \frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n} \geq \sqrt[2^n]{a_1 a_2 \cdots a_{2^n}} \\
k=2^{n+1}. & \\
& \quad \frac{a_1 + a_2 + \cdots + a_{2^n} + \cdots + a_{2^{n+1}}}{2^{n+1}} \geq \sqrt{\left(\frac{a_1 + a_2 + \cdots + a_{2^n}}{2^n}\right) \cdot \left(\frac{a_{2^n+1} + a_{2^n+2} + \cdots + a_{2^{n+1}}}{2^n}\right)} \\
I \geq & \quad \sqrt{2^n \sqrt{a_1 a_2 \cdots a_{2^n}} \cdot 2^n \sqrt{a_{2^n+1} a_{2^n+2} \cdots a_{2^{n+1}}}} = \sqrt[2^{n+1}]{a_1 a_2 \cdots a_{2^{n+1}}}
\end{aligned}$$

Backward part suppose A.G inequality is valid when $k = n$, Consider $k = n - 1$.

$$\begin{aligned}\frac{1}{n-1} \sum_{i=1}^{n-1} a_i &= \frac{1}{n} \left(\frac{n}{n-1} \right) \sum_{i=1}^{n-1} a_i \\ \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &= \frac{1}{n} \left(\sum_{i=1}^{n-1} a_i + \frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \\ \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &\geq \sqrt[n]{\left(\prod_{i=1}^{n-1} a_i \right) \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)} \\ \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^n &\geq \left(\prod_{i=1}^{n-1} a_i \right) \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right) \\ \left(\frac{1}{n-1} \sum_{i=1}^{n-1} a_i \right)^{n-1} &\geq \left(\prod_{i=1}^{n-1} a_i \right) \\ \frac{1}{n-1} \sum_{i=1}^{n-1} a_i &\geq \sqrt[n-1]{\prod_{i=1}^{n-1} a_i}\end{aligned}$$

Proposition 1.1. 1.3.5 柯西-施瓦茨不等式

a_1, a_2, \dots, a_n 和 b_1, b_2, \dots, b_n , 成立



$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

Proof

$$0 \leq \sum_{i=1}^n (a_i - \lambda b_i)^2 = \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2$$

由韦达定理 (视 λ 为未知数). 原方程无解或只有唯一解。

$$\Delta = b^2 - 4ac \leq 0$$

$$\left(-2 \sum_{i=1}^n a_i b_i \right)^2 - 4 \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq 0$$

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

$$\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

1.1.3.2 练习题

Example 1.1 关于 Bernoulli 不等式的推广:

- (1) 证明: 当 $-2 \leq h \leq -1$ 时 Bernoulli 不等式 $(1+h)^n \geq 1+nh$ 仍成立;
- (2) 证明: 当 $h \geq 0$ 时成立不等式 $(1+h)^n \geq \frac{n(n-1)h^2}{2}$, 并推广之;
- (3) 证明: 若 $a_i > -1 (i=1, 2, \dots, n)$ 且同号, 则成立不等式

$$\prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$$

Proof (1)

$$\begin{aligned}
 -2 &\leq h \leq -1 \\
 -1 &\leq 1+h \leq 0 & -1 &\leq (1+h)^n \leq 0 \\
 -2n &\leq nh \leq -n & 1-2n &\leq 1+nh \leq 1-n \\
 n &= 0. & (1+h)^0 &= 1 = 1+0 \times h \\
 n &= 1. & 1+h &= 1+h \\
 n &\geq 2. & 1-n &\leq -2 \\
 0 &\geq (1+h)^n \geq -1 \geq -2 \geq 1-n \geq 1+nh \geq 1-2n \\
 & & (1+h)^n &\geq 1+nh
 \end{aligned}$$

(2)

$$\begin{aligned}
 h &\geq 0 \\
 (1+h)^n &= 1+nh + \frac{n(n-1)}{2}h^2 + \dots \geq \frac{n(n-1)}{2}h^2
 \end{aligned}$$

推广:

$$(1+h)^n \geq \binom{n}{3}h^3, \binom{n}{4}h^4, \dots, \binom{n}{k}h^k, 0 \leq k \leq n$$

(3) $k=1$ 时显然成立. 使用归纳法证明. 假设 $k=n$ 时不等式 $\prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i$ 成立, 证明 $k=n+1$ 时 $\prod_{i=1}^{n+1} (1+a_i) \geq 1 + \sum_{i=1}^{n+1} a_i$ 成立.

$$\begin{aligned}
 k=n+1 \quad \prod_{i=1}^{n+1} (1+a_i) &= \prod_{i=1}^n (1+a_i)(1+a_{n+1}) \\
 &\because \prod_{i=1}^n (1+a_i) \geq 1 + \sum_{i=1}^n a_i \\
 \prod_{i=1}^n (1+a_i)(1+a_{n+1}) &\geq (1 + \sum_{i=1}^n a_i)(1+a_{n+1}) \\
 (1 + \sum_{i=1}^n a_i)(1+a_{n+1}) &= 1 + \sum_{i=1}^n a_i + a_{n+1} + a_{n+1} \sum_{i=1}^n a_i \\
 &= 1 + \sum_{i=1}^{n+1} a_i + a_{n+1} \sum_{i=1}^n a_i \\
 &\geq 1 + \sum_{i=1}^{n+1} a_i
 \end{aligned}$$

Example 1.2 利用 A.G. 不等式求解:

(1). $n! \leq (\frac{n+1}{2})^n$, while $n > 1$

(2). $(n!)^2 = (n \cdot 1)[(n-1) \cdot 2] \dots (1 \cdots n)$. 证明: 当 $n > 1$ 时成立

$$n! < (\frac{n+2}{6})^n$$

(3). 比较上述两个不等式的优劣

(4). 证明: 对任意实数 r 成立:

$$(n!)^r \leq \frac{1}{n^n} \left(\sum_{k=1}^n k^r \right)^n \quad (1.2)$$

Proof (1).

$$n > 1 \quad n! = 1 \times 2 \times \cdots \times n < \left(\frac{1+2+\cdots+n}{n}\right)^n = \left(\frac{(1+n)n}{2n}\right)^n = \left(\frac{n+1}{2}\right)^n$$

$\because 1 \neq 2 \neq \cdots n$, 所以不会有等号出现的情况

(2). $n > 1$

$$\begin{aligned} (n!)^2 &= (n \cdot 1)[(n-1) \cdot 2] \cdots (1 \cdots n) \\ &< \left(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n}\right)^n \end{aligned}$$

Consider this equation

$$\left(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n}\right)^n \quad (1.3)$$

$$\begin{aligned} \sum_{k=1}^n (n-k+1)k &= (n+1) \sum_{k=1}^n k - \sum_{k=1}^n k^2 \\ &= (n+1) \frac{(n+1)n}{2} - \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{6} (3(n+1) - (2n+1)) \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

$$\begin{aligned} (n!)^2 &< \left(\frac{n \times 1 + (n-1) \times 2 + \cdots + 1 \times n}{n}\right)^n \\ &= \left(\frac{(n+1)(n+2)}{6}\right)^n \end{aligned}$$

$\because n+1 < n+2, \therefore n! < \left(\frac{n+2}{\sqrt{6}}\right)^n$

(3). $n > 3$ 时, $\frac{n+2}{\sqrt{6}} < \frac{n+1}{2}$ (2) 的结果较好.

(4). $\forall r \in \mathbb{R}$, prove formula 1.2

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n k^r &\geq \sqrt[n]{\prod_{k=1}^n k^r} \\ (n!)^r &= \prod_{k=1}^n k^r \leq \left(\frac{1}{n} \sum_{k=1}^n k^r\right)^n = \frac{1}{n^n} \left(\sum_{k=1}^n k^r\right)^n \end{aligned}$$

my answer

$$\begin{aligned} \forall r \in \mathbb{R}, \quad \left(\sum_{k=1}^n k^r\right)^n &\geq n^n (n!)^r \\ (n!)^r &= \sum_{k=1}^n k^r \leq \left(\frac{1^r + 2^r + \cdots + n^r}{n}\right)^n = \frac{1}{n^n} \left(\sum_{k=1}^n k^r\right)^n \\ \therefore \left(\sum_{k=1}^n k^r\right)^n &\geq n^n (n!)^r \end{aligned}$$

Example 1.3 $a_k > 0, k = 1, 2, \dots, n$ 证明几何-调和平均值不等式

$$\left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}$$

Proof from A.G inequality

$$\frac{\sum_{k=1}^n \frac{1}{a_k}}{n} \geq \sqrt[n]{\prod_{k=1}^n \frac{1}{a_k}} = \frac{1}{\sqrt[n]{\prod_{k=1}^n a_k}}$$

$$a_k > 0, \quad \sqrt[n]{\prod_{k=1}^n a_k} \geq \frac{n}{\sum_{k=1}^n \frac{1}{a_k}}$$

Example 1.4 $a, b, c \geq 0$. prove $\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$. 并推广到 n 个非负数的情况

Proof 1. $\sqrt[3]{abc} = \sqrt{\sqrt[3]{ab \cdot bc \cdot ca}} \leq \sqrt{\frac{ab+bc+ca}{3}}$.

2.

$$\begin{aligned} \sqrt{\frac{ab+bc+ca}{3}} &\leq \sqrt{\frac{(\frac{a+b}{2})^2 + (\frac{b+c}{2})^2 + (\frac{c+a}{2})^2}{3}} \\ &= \sqrt{\frac{2(a^2+b^2+c^2) + 2(ab+bc+ca)}{12}} \\ &= \sqrt{\frac{a^2+b^2+c^2+ab+bc+ca}{6}} \end{aligned}$$

$a, b, c \geq 0$, 希望证明

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

$$\begin{aligned} \frac{ab+bc+ca}{3} &\leq \frac{a^2+b^2+c^2}{6} + \frac{ab+bc+ca}{6} \\ \frac{ab+bc+ca}{2} &\leq \frac{a^2+b^2+c^2}{6} + 2 \frac{ab+bc+ca}{6} \quad (\text{add } \frac{ab+bc+ca}{6}) \\ \frac{ab+bc+ca}{3} &\leq \frac{ab+bc+ca}{2} \leq \left(\frac{a+b+c}{3}\right)^2 \\ \sqrt{\frac{ab+bc+ca}{3}} &\leq \frac{a+b+c}{3} \end{aligned}$$

推广至 n 个

$$\begin{aligned} [l]n=2 \quad \sqrt{ab} &\leq \frac{a+b}{2} \\ n=3 \quad \sqrt[3]{abc} &\leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3} \\ n=4 \quad \sqrt[4]{abcd} &\leq \sqrt[3]{\frac{abc+bcd+cda+dab}{4}} \leq \sqrt{\frac{a+b+c}{3}} \leq \frac{a+b+c+d}{4} \\ k=n \quad \sqrt[k]{a_1 a_2 \dots a_n} &\leq \sqrt{\frac{a_1+a_2+\dots+a_n}{n}} \leq \frac{a_1+a_2+\dots+a_n}{n} \end{aligned}$$

This is

$$\sqrt[n]{\sum_{k=1}^n a_k} \leq \sqrt{\frac{\sum_{k=1}^n a_k}{k}} \leq \frac{\sum_{k=1}^n a_k}{k}$$

$$\begin{aligned} 1. \quad \sqrt[n]{a_1 a_2 \dots a_n} &= \sqrt[n]{\sqrt[n]{a_1^2 a_2^2 \dots a_n^2}} \leq \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_n a_1}{n}} \\ 2. \quad \sqrt{\frac{a_1 a_2 + a_2 a_3 + \dots + a_n a_1}{n}} &\leq \sqrt{\frac{a_1 + a_2 + \dots + a_n}{n}}? \end{aligned}$$

Example 1.5 (1) $|\alpha + \beta| \leq |\alpha| + |\beta|$

Proof let $\alpha = a - b, \beta = b$, the identity became $|(a - b) + b| \leq |a - b| + |b|$. This is $|a - b| \geq |a| - |b|$.

$$||a| - |b|| = \begin{cases} |a| - |b|, & a \geq b \\ |b| - |a|, & a < b \end{cases}$$

When $a \geq b$, $||a| - |b|| = |a| - |b|$. There is $|a - b| \geq |a| - |b| = ||a| - |b||$

When $a < b$, $|a - b| = |b - a| \geq |b| - |a| = ||a| - |b||$.

\therefore , We have $|a - b| \geq ||a| - |b||$

$$(2) \sum |a_k| \geq |\sum a_k|$$

Proof We can prove this statement by induction.

$$k = 2, \quad |a_1| + |a_2| \geq |a_1 + a_2|$$

$$k = 3, \quad |a_1| + |a_2| + |a_3| \geq |a_1 + a_2 + a_3|$$

$$\text{Suppose } k = n, \quad \sum_{k=1}^n |a_k| \geq \left| \sum_{k=1}^n a_k \right|$$

$$k = n + 1, \quad \text{prove } \sum_{k=1}^{n+1} |a_k| \geq \left| \sum_{k=1}^{n+1} a_k \right|$$

$$\begin{aligned} \sum_{k=1}^{n+1} |a_k| &= \sum_{k=1}^n |a_k| + |a_{n+1}| \\ &\geq \left| \sum_{k=1}^n a_k \right| + |a_{n+1}| \\ &\geq \left| \sum_{k=1}^{n+1} a_k \right| \end{aligned}$$

$$k = 2, \quad |a_1| - |a_2| \leq |a_1 - a_2|$$

$$\text{Suppose } k = n, \quad |a_1| - \sum_{k=2}^n |a_k| \leq \left| \sum_{k=1}^n a_k \right|$$

$$k = n + 1, \quad \text{prove } |a_1| - \sum_{k=2}^{n+1} |a_k| \leq \left| \sum_{k=1}^{n+1} a_k \right|$$

$$\begin{aligned} |a_1| - \sum_{k=2}^{n+1} |a_k| &= |a_1| - \sum_{k=2}^n |a_k| - |a_{n+1}| \\ &\leq \left| \sum_{k=1}^n a_k \right| - |a_{n+1}| \\ &\leq \left| \sum_{k=1}^{n+1} a_k \right| \end{aligned}$$

Can left side became $|a_1| - \sum_{k=2}^n |a_k|$?

$$\left| |a_1| - \sum_{k=2}^n |a_k| \right| = |a_1| - \sum_{k=2}^n |a_k| \quad |a_1| \geq \sum_{k=2}^n |a_k| \quad (1.4)$$

$$\left| |a_1| - \sum_{k=2}^n |a_k| \right| = \sum_{k=2}^n |a_k| - |a_1| \quad |a_1| \geq \sum_{k=2}^n |a_k| \quad (1.5)$$

in eq1.4, the inequality is still valid. However in eq1.5, $\sum_{k=2}^n |a_k| - |a_1|$ and $|a_1|$

$$(3). \quad \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Proof

$$\begin{aligned}\frac{|a+b|}{1+|a+b|} &\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \\ \frac{|a+b|}{1+|a+b|} &\leq \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\ 1 - \frac{|a+b|}{1+|a+b|} &\geq 1 - \frac{|a|+|b|+2|a||b|}{(1+|a|)(1+|b|)} \\ \frac{1}{1+|a+b|} &\geq \frac{1-|a||b|}{(1+|a|)(1+|b|)}\end{aligned}$$

$$1+|a|+|b|+|a||b| \geq 1+|a+b|-|a||b|-|a||b||a+b|$$

$$|a|+|b|+2|a||b|+|a||b||a+b| > 0, \text{ Since } +2|a||b|+|a||b||a+b| \geq |a+b|$$

Therefore $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

Example 1.6 (4). $|(a+b)^n - a^n| \leq (|a|+|b|)^n - |a|^n$

$$\begin{aligned}(a+b)^n - a^n &= \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0b^n \\ (|a|+|b|)^n - |a|^n &= \binom{n}{1}|a|^{n-1}|b|^1 + \binom{n}{2}|a|^{n-2}|b|^2 + \cdots + \binom{n}{n}|a|^0|b|^n \\ &\because |a|^j|b|^k \geq a^jb^k \\ &\therefore \sum |a|^j|b|^k \geq |\sum a^jb^k|\end{aligned}$$

$$|(a+b)^n - a^n| = \begin{cases} (a+b)^n - a^n, & a+b \geq a; b \geq 0 \\ a^n - (a+b)^n, & a+b < a; b < 0 \end{cases}$$

$$|(a+b)^n - a^n| \leq (|a|+|b|)^n - |a|^n. \quad (1.6)$$

Proposition 1.2. 1.3.5(Cauchy inequality)

or a_1, a_2, \dots, a_n . and b_1, b_2, \dots, b_n . $a_i, b_i \in \mathbb{R}$, There is

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \quad (1.7)$$

Proof Let's prove eq1.7

First way on book:

Use variable λ , change the inequality into nonnegative binomial.

$$\begin{aligned}0 &\leq \sum_{i=1}^n (a_i - \lambda b_i)^2 &= \sum_{i=1}^n a_i^2 - 2\lambda \sum_{i=1}^n a_i b_i + \lambda^2 \sum_{i=1}^n b_i^2 \\ \Delta &= B^2 - 4AC &= (-2 \sum_{i=1}^n a_i b_i)^2 - 4(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) \leq 0\end{aligned}$$

$$(\sum_{i=1}^n a_i b_i)^2 \leq (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$$

sqrt on both side of the inequality above, we can get

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

6. Cauchy 不等式的不同证明

(1). 数学归纳法.

$$k = 1, \quad |ab| = \sqrt{a^2}\sqrt{b^2}$$

$$k = 1, \quad |a_1b_1 + a_2b_2| = \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$$

$$\text{Suppose } k = n, \quad \left| \sum_{i=1}^n a_i b_i \right| = \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

$$k = n + 1, \quad \left| \sum_{i=1}^{n+1} a_i b_i \right| = \left| \sum_{i=1}^n a_i b_i + a_{n+1} b_{n+1} \right|$$

$$\begin{aligned} \left| \sum_{i=1}^{n+1} a_i b_i \right| &= \left| \sum_{i=1}^n a_i b_i + a_{n+1} b_{n+1} \right| \\ &\leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} + a_{n+1} b_{n+1} \end{aligned}$$

Note that $A = \sqrt{\sum_{i=1}^n a_i^2}$, $B = \sqrt{\sum_{i=1}^n b_i^2}$

$$\begin{aligned} \left| \sum_{i=1}^{n+1} a_i b_i \right| &\leq |AB + a_{n+1} b_{n+1}| \\ &\leq \sqrt{A^2 + a_{n+1}^2} \sqrt{B^2 + b_{n+1}^2} \\ &= \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \end{aligned}$$

(2) Lagrange 恒等式

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n |a_i b_i| \right)^2 = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n (|a_k||b_i| - |a_i||b_k|)^2 \quad (1.8)$$

$$\begin{aligned} (|a_k||b_i| - |a_i||b_k|)^2 &= |a_k|^2 |b_i|^2 - 2|a_i||a_k||b_i||b_k| + |b_k|^2 |a_i|^2 \\ &= a_k^2 b_i^2 + b_k^2 a_i^2 - 2|a_i a_k b_i b_k| \end{aligned}$$

$$\sum_{i=1}^n \sum_{k=1}^n (|a_k||b_i| - |a_i||b_k|)^2 = 2 \sum_{i=1}^n a_i^2 \sum_{k=1}^n b_k^2 - 2 \sum_{i=1}^n \sum_{k=1}^n |a_i a_k b_i b_k|$$

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n |a_i b_i| \right)^2 = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n (|a_k||b_i| - |a_i||b_k|)^2 \geq 0$$

$$\therefore \left(\sum_{i=1}^n |a_i b_i| \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

$$\therefore \left| \sum_{i=1}^n a_i b_i \right| \leq \sum_{i=1}^n |a_i b_i|$$

$$\therefore \left(\left| \sum_{i=1}^n a_i b_i \right| \right)^2 \leq \left(\sum_{i=1}^n |a_i b_i| \right)^2$$

$$\therefore \left(\left| \sum_{i=1}^n a_i b_i \right| \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$$

不等式两边开平方，得到：

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

(3). 用不等式 $|AB| \leq \frac{A^2+B^2}{2}$

$$\begin{aligned} |a_i b_i| &\leq \frac{a_i^2 + b_i^2}{2} \\ \left| \sum_{i=1}^n a_i b_i \right| &\leq \sum_{i=1}^n |a_i b_i| \leq \frac{\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2}{2} \\ \frac{\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2}{2} &\geq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} \quad ?? \end{aligned}$$

如何用均值不等式证明 Cauchy 不等式？

由切比雪夫不等式，有

$$\frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right) \left(\frac{b_1 + b_2 + \cdots + b_n}{n} \right) \quad (1.9)$$

由均值不等式，有

$$\begin{aligned} \frac{a_1 + a_2 + \cdots + a_n}{n} &\leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \\ \frac{b_1 + b_2 + \cdots + b_n}{n} &\leq \sqrt{\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}} \\ \therefore \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{n} &\leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right) \left(\frac{b_1 + b_2 + \cdots + b_n}{n} \right) \\ &\leq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \sqrt{\frac{b_1^2 + b_2^2 + \cdots + b_n^2}{n}} \\ &= \frac{1}{n} \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \end{aligned}$$

This is

$$\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}$$

Square on both side of the inequality, The calculate square root. We can get eq1.9:

(4). 构造复的辅助数列

$$c_k = a_k^2 - b_k^2 + 2i|a_k b_k|, \quad k = 1, 2, \dots, n$$

Then we use

$$\left| \sum_{k=1}^n c_k \right| \leq \sum_{k=1}^n |c_k|$$

Solve 1

$$\begin{aligned} c_k &= (|a_k| + i|b_k|)^2 = a_k^2 + b_k^2 + 2i|a_k b_k| \\ \sum_{k=1}^n c_k &= \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 + 2i \sum_{k=1}^n |a_k b_k| \\ |c_k| &= \sqrt{\Re^2 c_k + \Im^2 c_k} = \sqrt{(a_k^2 - b_k^2)^2 + (2a_k b_k)^2} = a_k^2 + b_k^2 \end{aligned}$$

$$\begin{aligned}
& \therefore \left| \sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 + 2i \sum_{k=1}^n |a_k b_k| \right| = \sqrt{\Re^2 \sum_{k=1}^n c_k + \Im^2 \sum_{k=1}^n c_k} \\
& = \sqrt{\left(\sum_{k=1}^n (a_k^2 - b_k^2) \right)^2 + \sum_{k=1}^n (2a_k b_k)^2} \\
& = \sqrt{\left(\sum_{k=1}^n a_k^2 \right)^2 + \left(\sum_{k=1}^n b_k^2 \right)^2 - 2 \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) + 4 \sum_{k=1}^n (a_k b_k)^2} \\
& \therefore \left| \sum_{k=1}^n c_k \right| \leq \sum_{k=1}^n |c_k| \\
& \therefore \left(\sum_{k=1}^n a_k^2 \right)^2 + \left(\sum_{k=1}^n b_k^2 \right)^2 - 2 \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) + 4 \sum_{k=1}^n (a_k b_k)^2 \leq \left(\sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2 \right)^2 \\
& \therefore 4 \left(\sum_{k=1}^n a_k b_k \right)^2 \leq 4 \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) \\
& \text{extracting both side: } \left| \sum_{k=1}^n a_k b_k \right| \leq \sqrt{\sum_{k=1}^n a_k^2} \sqrt{\sum_{k=1}^n b_k^2}
\end{aligned}$$

Example 1.7 7. Suppose $0 < x_i \leq \frac{1}{2}, i = 1, 2, \dots, n$, then

$$\frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^n} \leq \frac{\prod_{i=1}^n (1 - x_i)}{(\sum_{i=1}^n (1 - x_i))^n} \quad (1.10)$$

Proof Let's prove eq1.10 by induction method.

$$\begin{aligned}
n = 2, \quad & \frac{x_1 x_2}{(x_1 + x_2)^2} \leq \frac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2} \\
& \frac{(x_1 x_2)}{(x_1^2 + 2x_1 x_2 + x_2^2)} \leq \frac{1 - x_1 - x_2 + x_1 x_2}{(1 - x_1)^2 + 2(1 - x_1)(1 - x_2) + (1 - x_2)^2} \\
& \frac{(x_1 + x_2)^2}{(x_1 x_2)} \geq \frac{((1 - x_1)(1 - x_2))^2}{1 - x_1 - x_2 + x_1 x_2} \\
& \frac{x_1}{x_2} + 2 + \frac{x_2}{x_1} \geq \frac{1 - x_1}{1 - x_2} + 2 \frac{1 - x_2}{1 - x_1} \\
& \frac{x_1}{x_2} - \frac{1 - x_1}{1 - x_2} \geq \frac{1 - x_2}{1 - x_1} - \frac{x_2}{x_1} \\
& \frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_2(1 - x_2)} \geq \frac{x_1(1 - x_2) - x_2(1 - x_1)}{x_1(1 - x_1)} \\
& \frac{x_1 - x_2}{x_2(1 - x_2)} \geq \frac{x_1 - x_2}{x_1(1 - x_1)}
\end{aligned}$$

$f(x) = x - x^2, f'(x) = 1 - 2x > 0$, while $x \in (0, \frac{1}{2})$

When $x_1 > x_2, 0 < x_2 < x_1 \leq \frac{1}{2}, x_1 - x_1^2 \geq x_2 - x_2^2, x_1 - x_2 > 0$

When $x_1 < x_2, 0 < x_1 < x_2 \leq \frac{1}{2}, x_1 - x_1^2 \leq x_2 - x_2^2, x_1 - x_2 < 0$

$$\frac{x_1 - x_2}{x_2(1 - x_2)} \geq \frac{x_1 - x_2}{x_1(1 - x_1)}$$

$$k = 2, \quad \frac{x_1 x_2}{(x_1 + x_2)^2} \leq \frac{(1 - x_1)(1 - x_2)}{((1 - x_1) + (1 - x_2))^2}$$

$$k = 4, \quad \frac{x_1 x_2 x_3 x_4}{(x_1 + x_2 + x_3 + x_4)^2} \leq \frac{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)}{((1 - x_1) + (1 - x_2) + (1 - x_3) + (1 - x_4))^2}$$

Use Cauchy's forward and backward method, We can prove this equation

$$\text{Suppose } k = n, \quad \frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} \leq \frac{\prod_{i=1}^n (1 - x_i)}{(\sum_{i=1}^n (1 - x_i))^2}$$

$$k = n - 1, \quad \text{prove } \frac{\prod_{i=1}^{n-1} x_i}{(\sum_{i=1}^{n-1} x_i)^2} \leq \frac{\prod_{i=1}^{n-1} (1 - x_i)}{(\sum_{i=1}^{n-1} (1 - x_i))^2}$$

We already know that

$$\frac{\sum_{i=1}^{n-1} x_i}{n-1} = \frac{1}{n} \left(\sum_{i=1}^{n-1} x_i + \frac{1}{n-1} \sum_{i=1}^{n-1} x_i \right)$$

This trick always use in (n-1) terms tranfer to (n) terms

When the inequality holds for $n > 2$, fork $k = n$, we have:

$$\begin{aligned} \frac{\prod_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^n} &\leq \frac{\prod_{i=1}^n (1 - x_i)}{(\sum_{i=1}^n (1 - x_i))^n} \\ \frac{(\sum_{i=1}^n (1 - x_i))^n}{(\sum_{i=1}^n x_i)^n} &\leq \frac{\prod_{i=1}^n (1 - x_i)}{\prod_{i=1}^n x_i} \\ \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1 - x_i)} \right)^n &\geq \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1 - x_i)} \end{aligned}$$

for $k = n - 1$, Let $M = x_n = \frac{\sum_{i=1}^{n-1} x_i}{n-1}$. The inequality 1.10 left side:

$$\begin{aligned} &\left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1 - x_i)} \right)^n \\ &= \left(\frac{x_1 + \cdots + x_n}{(1 - x_1) + \cdots + (1 - x_n)} \right)^n \\ &= \left(\frac{x_1 + \cdots + x_{n-1} + M}{(1 - x_1) + \cdots + (1 - x_{n-1}) + (1 - M)} \right)^n \\ &= \left(\frac{x_1 + \cdots + x_{n-1} + \frac{\sum_{i=1}^{n-1} x_i}{n-1}}{(1 - x_1) + \cdots + (1 - x_{n-1}) + (1 - \frac{\sum_{i=1}^{n-1} x_i}{n-1})} \right)^n \\ &= \left(\frac{\frac{n}{n-1}(x_1 + \cdots + x_{n-1})}{\frac{n}{n-1}((1 - x_1) + \cdots + (1 - x_{n-1}))} \right)^n \\ &= \left(\frac{M}{1 - M} \right)^n \end{aligned}$$

while the right side become

$$\begin{aligned} &\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1 - x_i)} \\ &= \frac{\prod_{i=1}^{n-1} x_i \cdot M}{\prod_{i=1}^{n-1} (1 - x_i) \cdot (1 - M)} \\ &= \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1} (1 - x_i)} \frac{M}{1 - M} \end{aligned}$$

$$\begin{aligned}\left(\frac{M}{1-M}\right)^n &\geq \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1} (1-x_i)} \frac{M}{1-M} \\ \left(\frac{M}{1-M}\right)^{n-1} &\geq \frac{\prod_{i=1}^{n-1} x_i}{\prod_{i=1}^{n-1} (1-x_i)}\end{aligned}$$

Proposition 1.3. 1.3.1 Bernoulli inequality

suppose that $h > -1, n \in \mathbb{N}$, Then:

$$(1+h)^n \geq 1+nh \quad (1.11)$$

When $n > 1$, the inequality became equation iff $h = 0$.



Proof When $n = 1, 1+h = 1+h$
 $h = 0, 1^n = 1$

Let's consider the condition $n > 1, h \neq 0$.

i). $h > 0, (1+h)^n = \binom{n}{0}h^0 + \binom{n}{1}h^1 + \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n$.

$\because \binom{n}{2}h^2 + \cdots + \binom{n}{n}h^n > 0, \therefore (1+h)^n > 1+nh$

ii). $-1 < h < 0, 0 < 1+h < 1$.

$$\begin{aligned} (1+h)^n - 1 &= (1+h-1)\left(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\right) \\ &= h\left(1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1}\right) \end{aligned}$$

$\because 1 + (1+h) + (1+h)^2 + \cdots + (1+h)^{n-1} < n$ when $h < 0$

$\therefore (1+h)^n > 1+nh$

Two variable extension of the Bernoulli inequality, Suppose $h = \frac{B}{A}, A > 0, A+B > 0$, Then $1+h > 0$ is established.

Proposition 1.4. 1.3.2

suppose $A > 0, A+B > 0, n \in \mathbb{N}$, Then the inequality is true:

$$(A+B)^n \geq A^n + nA^{n-1}B \quad (1.12)$$

The inequality became equation iff $B = 0$.



Proof divide A^n on both side of the inequality 1.12. Set $h = \frac{B}{A} (A > 0)$, Then the inequality became Eq 1.11. So we can prove Eq 1.12 by prove Eq 1.11. Eq 1.11 is true when $h > -1$.
 $\therefore 1+h > 0, 1+\frac{B}{A} > 0, \because A > 0, \therefore A+B > 0$. And when $n > 1$ the equation is true iff $h = 0, \frac{B}{A} = 0, \therefore B = 0$.

Example 1.8 Ex 1.3.2 exercise 8

$a, c, t, g \geq 0, a+c+t+g=1$. Prove that $a^2+c^2+t^2+g^2 \geq \frac{1}{4}$.

The inequality became equation iff $a=c=t=g=\frac{1}{4}$.

Proof from A.G inequality,

$$\frac{a+c+t+g}{4} \geq \sqrt[4]{actg}, \quad a+c+t+g=1 \quad (1.13)$$

$\therefore \sqrt[4]{actg} \leq \frac{1}{4}$

$$a+c+t+g=1, (a+c+t+g)^2=1$$

$$(a+c+t+g)^2 = a^2+c^2+t^2+g^2+2ac+2at+2ag+2ct+2cg+2tg=1 \quad (1.14)$$

$$a^2+c^2 \geq 2acc^2+t^2 \geq 2ct \quad (1.15)$$

$$a^2+t^2 \geq 2atc^2+g^2 \geq 2cg \quad (1.16)$$

$$a^2+g^2 \geq 2agt^2+g^2 \geq 2tg \quad (1.17)$$

substitute $2ac, 2ag, \dots$ in equation 1.14, we can get

$$4(a^2 + c^2 + t^2 + g^2) \geq a^2 + c^2 + t^2 + g^2 + 2ac + 2at + 2ag + 2ct + 2cg + 2tg$$

Then we get the inequality 1.13.

1.2 1.4 逻辑符号与对偶法则

The law of duality: $\forall(\exists) \rightarrow \exists(\forall)$ with negative statement

Inverse proposition?

1. A have upper limit, $\exists M > 0, \forall x \in A, x \leq M$.

It's negative statement is 'A don't have upper limit'. $\forall M > 0, \exists x \in A, x > M$.

2. the minum item in A is b, $b \in A, \forall x \in A, x \geq b$.

It's negative statement is 'b is not the minum item in A'. $b \in A, \exists x \in A, x < b$.

3. $f \in (a, b)$ is a monotonic augmentation function, $\forall x, y \in (a, b), x < y, f(x) \leq f(y)$. (or $f(x) < f(y)$, depends on monotonic function's definition)

It's negative statement is ' $f \in (a, b)$ isn't a monotonic augmentation function'. $\exists x, y \in (a, b), x < y, f(x) > f(y)$ (or $f(x) \geq f(y)$).

4. $f \in (a, b)$ is a monotonic function, $\forall x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) \geq 0$.

It's negative statement is ' $f \in (a, b)$ isn't a monotonic function'. $\exists x, y, z \in (a, b), x < y < z, (f(x) - f(y))(f(y) - f(z)) < 0$.

(Another way $\forall x, y \in (a, b), x < y, f(x) - f(y) \geq 0$ or $f(x) - f(y) \leq 0$.)

5. $A \subset B, \forall x \in A, x \in B$.

It's negative statement is $A \subsetneq B, \exists x \in A, x \notin B$.

6. $A - B \neq \emptyset, \exists x \in A, x \in B$.

It's negative statement is $A - B = \emptyset, \forall x \in A, x \notin B$.

7. x_n is an infinitesimal amounts, $\forall \epsilon > 0, \exists N \in \mathbb{N}^+, \forall n > N, |x_n| < \epsilon$.

It's negative statement is ' x_n is not an infinitesimal amounts', $\exists \epsilon > 0, \forall N \in \mathbb{N}^+, \exists n > N, |x_n| \geq \epsilon$.

8. x_n is infinitely large, $\forall M > 0, \exists N \in \mathbb{N}^+, \forall n > N, x_n > M$.

It's negative statement is ' x_n is not infinitely large', $\exists M > 0, \forall N \in \mathbb{N}^+, \exists n > N, x_n \leq M$.

Chapter 2 第二章数列极限

2.1 数列极限的基本概念

2.1.1 2.1.5 练习题

Question 1 1. prove by Limit definition:

- (1). $\lim_{n \rightarrow \infty} \frac{3n^2}{n^2-4} = 3$.
- (2). $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$.
- (3). $\lim_{n \rightarrow \infty} (1+n)^{\frac{1}{n}} = 0$.
- (4). $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0, (a > 0)$.

Question 2 2. Suppose $a_n, n \in \mathbb{N}_+$. sequence a_n converge to a .

Prove $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$.

Proof $n \rightarrow \infty, a_n \rightarrow a$.

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \frac{\epsilon}{\sqrt{a_n} + \sqrt{a}}$$

$\therefore \lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$. \square (check, not consider the condition $a = 0$) add $a = 0, \forall \epsilon \in (0, 1), \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$. s.t $a_n < \epsilon^2 < \epsilon, \sqrt{a_n} < \epsilon$.

Question 3 3. If $\lim_{n \rightarrow \infty} a_n = a$.

Prove $\lim_{n \rightarrow \infty} |a_n| = |a|$. Vice versa?

Proof $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), |a_n - a| < \epsilon$.

$$||a_n| - |a|| \leq |a_n - a| < \epsilon$$

$\therefore \lim_{n \rightarrow \infty} |a_n| = |a|$

If We know $\lim_{n \rightarrow \infty} |a_n| = |a|$.

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n > N(\epsilon), ||a_n| - |a|| < \epsilon$. We can't get $\lim_{n \rightarrow \infty} a_n = a$. For example:
 $a_n = \frac{1}{n} + 1, a = -1, \lim_{n \rightarrow \infty} |a_n| = |a|$ is $\lim_{n \rightarrow \infty} |\frac{1}{n} + 1| = |-1|$, but $\lim_{n \rightarrow \infty} \frac{1}{n} + 1 \neq -1$ \square

Question 4 (1). Suppose $p(x)$ is a polynomial of x , if $\lim_{n \rightarrow \infty} a_n = a$, Prove $\lim_{n \rightarrow \infty} p(a_n) = p(a)$.

- (2). Suppose $b > 0, \lim_{n \rightarrow \infty} a_n = a$. Prove $b^{a_n} = b^a$.
- (3). Suppose $b > 0, \{a_n\}, a_n > 0, \forall n \in \mathbb{N}$. $\lim_{n \rightarrow \infty} a_n = a, a > 0$. Prove $\lim_{n \rightarrow \infty} \log_b a_n = \log_b a$.
- (4) Suppose $b \in \mathbb{R}, \{a_n\}, a_n > 0$ when $n \in \mathbb{N}$. $\lim_{n \rightarrow \infty} a_n = a$. Prove $\lim_{n \rightarrow \infty} a_n^b = a^b$.
- (5) Suppose $\lim_{n \rightarrow \infty} a_n = a$. Prove $\lim_{n \rightarrow \infty} \sin a_n = \sin a$.

Proof 4.(1)

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}^+, \forall n \geq N(\epsilon), |a_n - a| < \epsilon$.

$$p(a) = k_m a^m + k_{m-1} a^{m-1} + \cdots + k_0 a^0.$$

$$\therefore p(a_n) - p(a) = k_m (a_n^m - a^m) + k_{m-1} (a_n^{m-1} - a^{m-1}) + \cdots + k_0 (a_n^0 - a^0).$$

$$\begin{aligned} |a_n^m - a^m| &= |a_n - a| \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon \cdot |a_n^{m-1} + a_n^{m-2}a + \cdots + a^{m-1}| \\ &< \epsilon(m-1) \cdots (a + \delta)^{m-1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} p(a_n) = p(a). \quad \square$$

Proof 4.(2)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$\text{If } b = 1, 1^{a_n} = 1^a = 1.$$

$$\text{If } b > 1, b^{a_n} - b^a = b^a (b^{a_n - a} - 1) < b^a (b^\epsilon - 1) \quad 0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1) \because b > 0, \epsilon \rightarrow 0,$$

$$\therefore b^\epsilon - 1 \rightarrow 0. \therefore \lim_{n \rightarrow \infty} b^{a_n} = b^a.$$

$$\text{If } b < 1, b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}, \text{ we can prove this condition by considering } \frac{1}{b} > 1.$$

Proof 4.(3)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$\begin{aligned} \log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b \left(\frac{a_n - a}{a} + 1 \right) < \log_b \left(\frac{\epsilon}{a} + 1 \right) \end{aligned}$$

$$0 < \log_b a_n - \log_b a < \log_b \left(1 + \frac{\epsilon}{a} \right). \because b > 0, a \neq 0, a_n > 0 \text{ when } \epsilon \rightarrow 0. \therefore \log_b \left(1 + \frac{\epsilon}{a} \right) \rightarrow 0.$$

$$\therefore \lim_{n \rightarrow \infty} \log_b a_n = \log_b a$$

Proof 4.(4)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$a_n^b = e^{b \ln a_n}, a_n^b - a^b = e^{b \ln a_n} - e^{b \ln a}.$$

$$\begin{aligned} e^{b \ln a_n} - e^{b \ln a} &= e^{b \ln a} (e^{b \ln a_n - b \ln a} - 1) \\ &= e^{b \ln a} (e^{b \ln \frac{a_n}{a}} - 1) \end{aligned}$$

$$0 < |a_n^b - a^b| < e^{b \ln a} (e^{b \ln(1 + \frac{\epsilon}{a})} - 1)$$

$$\therefore \lim_{n \rightarrow \infty} a_n^b = a^b$$

Proof 4.(5)

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$\begin{aligned} \sin(A+B) - \sin(A-B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B \end{aligned}$$

$$\sin a_n - \sin a = 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}$$

$$|\sin a_n - \sin a| = \left| 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2} \right| < \left| 2 \sin \frac{a_n - a}{2} \right|$$

$$\left| 2 \sin \frac{a_n - a}{2} \right| < \left| 2 \frac{a_n - a}{2} \right| = \epsilon$$

$$|\sin a_n - \sin a| < \epsilon, \therefore \lim_{n \rightarrow \infty} \sin a_n = \sin a$$

Question 5 assume $a > 1$. Prove $\lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$

Proof $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \log_a 1 = 0$.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\}. \forall n \geq N, |\sqrt[n]{n} - 1| < \epsilon.$$

$a > 1$, and $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$. \therefore when $n \rightarrow \infty$, $\sqrt[n]{n} < a^\epsilon$, take logarithm on base of a , we can get

$$\frac{1}{n} \log_a n < \epsilon$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\log_a n}{n} = 0$$

2.2 收敛数列的基本性质

收敛数列的性质

1. 收敛数列的极限是唯一的
2. 收敛数列一定有界
3. 收敛数列的比较定理，包括保号性定理
4. 收敛数列满足一定的四则运算规则
5. 收敛数列的每一个子列一定收敛于同一极限

2.2.1 思考题

- Question 6
1. $\{a_n\}$ 收敛, $\{b_n\}$ 发散, $\{a_n + b_n\}$ 发散, $\{a_n \cdot b_n\}$ 可能收敛, 可能发散.
 2. $\{a_n\}, \{b_n\}$ 都发散, $\{a_n + b_n\}$ 可能收敛, 可能发散 (ex: $n + -n, n + -2n$), $\{a_n \cdot b_n\}$ 发散 (?).
 3. $a_n \leq b_n \leq c_n, n \in \mathbb{N}_+$. 已知 $\lim_{n \rightarrow \infty} (c_n - a_n) = 0$. 问数列 $\{b_n\}$ 是否收敛?
 4. $\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n})$
 5. $a_n \rightarrow a (n \rightarrow \infty)$. $\forall n, b < a_n < c$. 是否成立 $b < a < c$?
 6. $a_n \rightarrow a (n \rightarrow \infty)$. and $b \leq a \leq c$, 是否存在 $N \in \mathbb{N}_+$, s.t. 当 $n > N$ 时, 成立 $b \leq a_n \leq c$
 7. 已知 $\lim_{n \rightarrow \infty} a_n = 0$, 问: 是否有 $\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0$. 反之如何?

Proof 5.4

$$\frac{n}{2n} \leq \frac{1}{n+1} + \dots + \frac{1}{2n} \leq \frac{n}{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{n}{2n} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{1}{2}, \therefore \lim_{n \rightarrow \infty} (\frac{1}{n+1} + \dots + \frac{1}{2n}) \text{ 收敛.}$$

$$\begin{aligned} \frac{1}{n+1} + \dots + \frac{1}{2n} &= \frac{1}{n} \left(\frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + \dots + \frac{1}{1 + \frac{n}{n}} \right) \\ &= \int_0^1 \frac{1}{1+x} dx \\ &= \ln(1+x) \Big|_0^1 = \ln 2 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \dots + \frac{1}{2n} \right) = \ln 2$$

Proof 5.5

不成立, 应当为小于等于号. $b=0, c=2, a_n = \frac{1}{n}, \lim_{n \rightarrow \infty} a_n = 0 = c$.

Proof 5.6

不成立. $a=0, b=0, c=2, a_n = (-1)^n \frac{1}{n}$.

$b \leq a \leq c$, but $(-1)^{2n+1} \frac{1}{2n+1} < 0 = b$.

Proof $\lim_{n \rightarrow \infty} a_n = 0, a_n = \frac{1}{n} \cdot a_1 a_2 \dots a_n = \frac{1}{n!}, \lim_{n \rightarrow \infty} \frac{1}{n!} = 0$.

$$\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0 \quad \checkmark$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = 0 \rightarrow \lim_{n \rightarrow \infty} a_n = 0 \quad \times$$

$$|a_n| < \epsilon, |a_{N+1} \dots a_n| < \epsilon^{n-N} < \epsilon, a_n < \sqrt[n]{\epsilon}.$$

for example, $a_n = \frac{n}{n+1}, a_1 = \frac{1}{2}, a_2 = \frac{2}{3}, \dots, a_n = \frac{n}{n+1}$.

$$a_1 a_2 \dots a_n = \frac{1}{2} \cdot \frac{2}{3} \dots \frac{n}{n+1} = \frac{1}{n+1}.$$

$$\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n) = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

but $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$

研究数列收敛方面的两个基本工具:

1. 夹逼定理.
2. 单调有界数列的收敛定理.

Example 2.1 2.2.2 $\lim_{n \rightarrow \infty} \frac{x_n - 1}{x_n + a} = 0$,

prove $\lim_{n \rightarrow \infty} x_n = a$

Proof $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |\frac{x_n - 1}{x_n + a} - 0| < \epsilon$.

$|x_n - 1| < \epsilon |x_n + a| < 4a \cdot \epsilon$. (这个 4 是怎么取得的?)

$|x_n - a| < \epsilon |x_n + a| = \epsilon |(x_n - a) + 2a| \leq \epsilon (|x_n - a| + 2a)$.

限制 $\epsilon < 1, |x_n - a| < 2\epsilon |a| / (1 - \epsilon)$.

限制 $\epsilon < \frac{1}{2}, |x_n - a| < 2\epsilon |a| / (1 - \epsilon) < 4|a|\epsilon$.

Let $\epsilon' = 4a\epsilon, |x_n - 1| < \epsilon' \therefore \lim_{n \rightarrow \infty} x_n = a$.

Example 2.2 2.2.3 $a > 0, b > 0$, 计算 $\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}}$.

Proof Suppose $a \leq b$.

$b = (b^b)^{\frac{1}{b}} < (a^n + b^n)^{\frac{1}{n}} \leq (2b^n)^{\frac{1}{n}}$.

$b < (a^n + b^n)^{\frac{1}{n}} \leq \sqrt[n]{2} b, \lim_{n \rightarrow \infty} \sqrt[n]{2} = 1$. 夹逼定理.

$\lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}$.

两数 n 次方之和再开 n 次根号的结果由较大的值决定, a, b 中较大的值为这个数的主要部分.

Example 2.3 2.2.4 $a_n = \frac{1! + 2! + \dots + n!}{n!}, n \in \mathbb{N}^+$

$\lim_{n \rightarrow \infty} a_n = 1$

Example 2.4 $\lim_{n \rightarrow \infty} \frac{n^3 + n - 7}{n + 3} = +\infty$

Example 2.5 $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

调和级数 H_n 发散.

2.2.2 练习 2.2.4

Proof 1.

$\{a_n\}$ 收敛于 a, \rightarrow 两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 a .

两个子列 $\{a_{2n}\}, \{a_{2n+1}\}$ 均收敛于 $a, \rightarrow \{a_n\}$ 收敛于 a .

2. 应用夹逼定理

(1). 给定 p 个正数 a_1, a_2, \dots, a_p . 求 $\lim_{n \rightarrow \infty} \sqrt[p]{a_1^n + a_2^n + \dots + a_p^n}$.

Let $a_s = \max_{1 \leq i \leq p} \{a_1, a_2, \dots, a_p\}$.

Solve 2 (1).

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} \leq (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$n \rightarrow \infty, p^{\frac{1}{n}} \rightarrow 1, \lim_{n \rightarrow \infty} (a_1^n + a_2^n + \dots + a_p^n)^{\frac{1}{n}} = a_s$

(2). $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}, n \in \mathbb{N}_+$. 求 $\lim_{n \rightarrow \infty} x_n$

Solve 3 (2).

$$\frac{2n+1}{(n+1)} \leq x_n \leq \frac{2n+1}{\sqrt{n^2+1}}$$

$$\lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = 2, \lim_{n \rightarrow \infty} \frac{2n+1}{\sqrt{n^2+1}} = 2. \therefore \lim_{n \rightarrow \infty} x_n = 2$$

$$(3). a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}, n \in \mathbb{N}_+. \text{ 求 } \lim_{n \rightarrow \infty} a_n$$

Solve 4 (3).

$$1 = \left(\frac{n}{n}\right)^{\frac{1}{n}} < a_n \leq (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1, \therefore \lim_{n \rightarrow \infty} a_n = 1$$

$$(4). a_n > 0. \lim_{n \rightarrow \infty} a_n = a, a > 0. \text{ 证明 } \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$$

$$\text{Proof } \lim_{n \rightarrow \infty} a_n = a$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geq N, |a_n - a| < \epsilon.$$

$$0 < a - \epsilon < a_n < a + \epsilon$$

$$\therefore \sqrt[n]{a - \epsilon} < \sqrt[n]{a_n} < \sqrt[n]{a + \epsilon}.$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{a - \epsilon} = 1, \lim_{n \rightarrow \infty} \sqrt[n]{a + \epsilon} = 1. \therefore \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1.$$

$$3. (1). \lim_{n \rightarrow \infty} (1+x)(1+x^2) \cdots (1+x^{2^n}) = \lim_{n \rightarrow \infty} \prod_{i=1}^{2^n} (1+x^i), |x| < 1.$$

Solve 5 3.(1).

$$|x| < 1, \quad 1 > x^2 > x^4 > \cdots > x^{2^n} > 0$$

$$x \in (0, 1) \quad 1 < (1+x)(1+x^2) \cdots (1+x^{2^n}) < (1+x)^{n+1}$$

$$\lim_{n \rightarrow \infty} (1+x)^{n+1} = 1$$

$$x \in (-1, 0) \quad 0 < (1+x)(1+x^2) \cdots (1+x^{2^n}) < (1+x)(1+x^2)^n$$

$$\lim_{n \rightarrow \infty} (1+x)(1+x^2)^n = 1$$

Solve 6 3.(1). another way

$$\begin{aligned} & \lim_{n \rightarrow \infty} (1+x)(1+x^2) \cdots (1+x^n) \\ &= \lim_{n \rightarrow \infty} \frac{(1-x)(1+x)(1+x^2) \cdots (1+x^n)}{1-x} \\ &= \lim_{n \rightarrow \infty} \frac{(1-x^{2^{n+1}})}{1-x} \\ &= \frac{1}{1-x} \end{aligned}$$

Solve 7 3. (2).

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdots \frac{n-1}{n} \cdot \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{n+1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \end{aligned}$$

Solve 8 3. (3).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left(1 - \frac{1}{1+2}\right) \left(1 - \frac{1}{1+2+3}\right) \cdots \left(1 - \frac{1}{1+2+\cdots+n}\right) \\
&= \lim_{n \rightarrow \infty} \left(1 - \frac{2}{3 \times 2}\right) \left(1 - \frac{2}{4 \times 3}\right) \cdots \left(1 - \frac{2}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{3 \times 2 - 2}{3 \times 2}\right) \left(\frac{4 \times 3 - 2}{4 \times 3}\right) \cdots \left(\frac{(n+1) \times n - 2}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{4}{3 \times 2}\right) \left(\frac{10}{4 \times 3}\right) \cdots \left(\frac{n^2 + n - 2}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \left(\frac{1 \times 4}{3 \times 2}\right) \left(\frac{2 \times 5}{4 \times 3}\right) \cdots \left(\frac{(n-2) \times (n+1)}{n \times (n-1)}\right) \left(\frac{(n-1) \times (n+2)}{(n+1) \times n}\right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{3} \times \frac{n+2}{n} \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\
&= \lim_{n \rightarrow \infty} \frac{1}{1} - \frac{1}{n+1} \\
&= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\
&= 1
\end{aligned}$$

Solve 9 3.(4).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1) \cdot (n+2)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right) + \frac{1}{2} \left(\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right) + \cdots + \frac{1}{2} \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)} \right) \\
&= \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

Solve 10 3.(5).

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1) \cdots (k+\gamma)}, \quad \text{其中 } \gamma \text{ 为正整数} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\gamma} \left[\frac{1}{k(k+1) \cdots (k+\gamma-1)} - \frac{1}{(k+1)(k+2) \cdots (k+\gamma)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[\sum_{k=1}^n \frac{1}{k(k+1) \cdots (k+\gamma-1)} - \sum_{k=1}^n \frac{1}{(k+1)(k+2) \cdots (k+\gamma)} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[\frac{1}{\gamma!} - \frac{1}{(n+\gamma)!} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{\gamma} \left[\frac{1}{\gamma!} - \frac{1}{(n+\gamma)!} \right] \\
&= \frac{1}{\gamma} \cdot \frac{1}{\gamma!}
\end{aligned}$$

其中 $x^n = x(x-1)(x-2)\dots(x-n+1)$, 称为下阶乘. 而 $x^{\overline{n}} = x(x+1)(x+2)\dots(x+n-1)$, 称为上阶乘.

Question 7 2.2.4-4 $S_n = a + 3a^2 + \dots + (2n-1)a^n$, $|a| < 1$. 求 $\{S_n\}$ 的极限.

Solve 11

$$\begin{aligned} S_n - aS_n &= a + 3a^2 + \dots + (2n-1)a^n \\ &\quad - a^2 - \dots + (2n-3)a^n - (2n-1)a^n + 1 \\ &= a + 2a^2 + \dots + 2aa^n - (2n-1)a^{n+1} \\ &= 2(a + a^2 + \dots + a^n) - a - (2n-1)a^{n+1} \\ &= 2 \cdot a \frac{1-a^{n+1}}{1-a} - a - (2n-1)a^{n+1} \end{aligned}$$

$$|a| < 1, \lim_{n \rightarrow \infty} a_n = 0$$

$$\lim_{n \rightarrow \infty} (S_n - aS_n) = (1-a) \lim_{n \rightarrow \infty} S_n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} (S_n - aS_n) &= \lim_{n \rightarrow \infty} 2a \cdot \frac{1-a^{n+1}}{1-a} - a - (2n-1)a^{n+1} \\ &= 2a \cdot \frac{1}{1-a} - a \\ &= a \left(\frac{2}{1-a} - a \right) \\ &= a \frac{1+a}{1-a} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \frac{a(a+1)}{(1-a)^2}$$

Solve 12 2.2.4-5 设 $\lim_{n \rightarrow \infty} x_n = A > 0$. 取 $\epsilon = \frac{A}{2}$, 则 $\exists N \in \mathbb{N}_+$. $\forall n > N$. 成立 $|x_n - A| < \frac{A}{2}$

$$A - \frac{A}{2} < x_n < A + \frac{A}{2}, \frac{A}{2} < x_n < \frac{3A}{2}$$

即 $x_n > \frac{A}{2}$.

令 $m = \min\{x_1, x_2, \dots, x_N, \frac{A}{2}\} > 0$. 则 m 为 $\{x_n\}$ 的正下界.

不一定有最小数的例子 $x_n = 1 + \frac{1}{n}$. $\lim_{n \rightarrow \infty} x_n = 1$, 下界 $m = \frac{1}{2}$. 但 $\{x_n\}$ 取不到下界.

Proof 2.2.4-6 $\because \lim_{n \rightarrow \infty} a_n = +\infty$. $\forall M > 0, \exists N \in \mathbb{N}_+, \forall n > N, a_n > M$.

$m = \min\{a_1, a_2, \dots, a_N, M\}$, 但 M 在数列 $\{a_n\}$ 中不一定取的到!

$M = a_1 + 1, \exists N_1 \in \mathbb{N}_+, \forall n > N_1, a_n > M > a_1$

则 $m = \min\{a_1, a_2, \dots, a_{N_1}\}$ 为数列的最小数.

Proof 2.2.4-7 构造数列

不妨设无界数列 $\{a_n\}$ 无上界.

$\forall M \in \mathbb{R}, \exists N \in \mathbb{N}_+, \forall n_k > N, a_{n_k} > M$.

取 $M_1 = 1$, 则 $\exists n_1 \in \mathbb{N}_+$ s.t. $a_{n_1} > M_1$.

取 $M_2 = \max\{a_{n_1}, 2\}$, $\exists n_2 \in \mathbb{N}_+$ s.t. $a_{n_2} > M_2$.

以此类推, 构造数列 $\{a_{n_k}\}$. s.t. $a_{n_k} > k$. 即 a_{n_k} 为无穷大量.

Proof 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散.

构造 a_n 的发散子列即可. 已知 $\tan \frac{\pi}{2} = \infty$, π 是一个无理数, 因此存在数列 $\{b_n\}, \lim_{n \rightarrow \infty} b_n = \frac{\pi}{2}$.

Proof 2.2.4-8 证明 $\{a_n\}, a_n = \tan n$ 发散. 参考别人的答案

由于 $\{\sin 2n\}$ 极限不存在, 又

$$\begin{aligned}\sin 2n &= 2 \sin n \cos n = \frac{2 \sin n \cos n}{\sin^2 n + \cos^2 n} \\ &= \frac{2 \tan n}{\tan^2 n + 1}\end{aligned}$$

若 $\{\tan n\}$ 极限存在 $\rightarrow \{\sin 2n\}$ 极限存在, 矛盾.

故 $\{\tan n\}$ 极限不存在.

Question 8 2.2.4-9 $S_n = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$, $n \in \mathbb{N}_+$. S_n 在 1. $p \leq 0$, 2. $0 < p < 1$ 情况下均发散

Proof 1. $p \leq 0$. $\lim_{n \rightarrow \infty} n^{-p} = \infty$, S_n 发散.
2. $0 < p < 1$. $\frac{1}{n^p} > \frac{1}{n}$. $\therefore H_n = \sum_{k=1}^n \frac{1}{k}$ (调和级数) 发散, $S_n > H_n$, $\therefore \{S_n\}$ 也发散.
ex2.3.5 $0 < b < a$ 令 $a_0 = a, b_0 = b$ 递推公式

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, b_n = \sqrt{a_{n-1}b_{n-1}}, \quad n \in \mathbb{N}_+ \quad (2.1)$$

定义数列 a_n, b_n . 证明这两个数列收敛于同一个极限 $AG(a, b)$.

由 AG 不等式 $a > \frac{a+b}{2} > \sqrt{ab} > b > 0$, 利用单调有界数列收敛原则可以证明上述结论.

$$AG(a, b) = \frac{\pi}{2G} \quad (2.2)$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$. 则

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \quad (2.3)$$

上面这个公式是怎么得到的:

参考菲赫金哥尔茨 - 微积分学教程. 第二卷 315 小节的高斯公式, 蓝登变换.

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} \quad (a > b > 0) \quad (2.4)$$

这里令

$$\sin \phi = \frac{2a \sin \theta}{(a+b) + (a-b) \sin^2 \theta} \quad (2.5)$$

$\theta \in [0, \frac{\pi}{2}] \rightarrow \phi \in [0, \frac{\pi}{2}]$, 取微分

$$\cos \phi d\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{[(a+b) + (a-b) \sin^2 \theta]^2} \cos \theta d\theta \quad (2.6)$$

但是

$$\cos \phi = \frac{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}}{(a+b) + (a-b) \sin^2 \theta} \cos \theta. \quad (2.7)$$

(2.6) / (2.7), 两式相除, 得到

$$d\phi = 2a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \frac{d\theta}{\sqrt{(a+b)^2 - (a-b)^2 \sin^2 \theta}} \quad (2.8)$$

另一方面

$$\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a \frac{(a+b) - (a-b) \sin^2 \theta}{(a+b) + (a-b) \sin^2 \theta} \quad (2.9)$$

因而

$$\frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \frac{d\theta}{\sqrt{\left(\frac{a+b}{2}\right)^2 \cos^2 \theta + ab \sin^2 \theta}}. \quad (2.10)$$

如果令 $a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}$, 则

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_1^2 \cos^2 \theta + b_1^2 \sin^2 \theta}} \quad (2.11)$$

反复应用该公式, 得到

$$G = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}}, \quad (n = 1, 2, 3, \dots) \quad (2.12)$$

$$\frac{\pi}{2a_n} < G < \frac{\pi}{2b_n} \quad (2.13)$$

积分 G 可以归结到第一类全椭圆积分 $K(k) = (1+k_1)K(k_1) = \frac{\pi}{2}(1+k_1)(1+k_2)\dots(1+k_n)$

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = (1+k_1) \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k_1^2 \sin^2 \theta}} \quad (2.14)$$

其中

$$a_1 = \frac{1+\sqrt{1-k^2}}{2} = \frac{1+k'}{2}, b_1 = \sqrt{k'}$$

$$k_1 = \frac{\sqrt{a_1^2 - b_1^2}}{a_1} = \frac{1-k'}{1+k'}, \frac{1}{a_1} = 1+k_1$$

2.3 2.3 单调数列

Example 2.6 2.3.6

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{1!+2!+\dots+(n+1)!}{(n+1)!}}{\frac{1!+2!+\dots+n!}{n!}} \\ &= \frac{1}{n+1} \frac{1!+2!+\dots+(n+1)!}{1!+2!+\dots+n!} \\ &= \frac{3+3!+\dots+(n+1)!}{(n+1)1!+(n+1)2!+\dots+(n+1)!} \end{aligned}$$

Solve 13 $n > 2$ 时, 分母每一项大于等于分子对应项.. $n > 2$ 后 a_n 单调减少. 由于 0 是下界, 因此 a_n 单调有界, 数列收敛.

$$\begin{aligned} a_{n+1} &= \frac{1!+2!+\dots+(n+1)!}{(n+1)!} \\ &= \frac{1!+2!+\dots+n!}{n!} \frac{1}{n+1} + 1 \\ &= 1 + \frac{a_n}{n+1} \end{aligned}$$

设 $n \rightarrow \infty$ 时, $a_n \rightarrow a$

$$a = 1 + \left(\frac{1}{n+1} \rightarrow 0 \right) = 1 + 0, \quad \therefore a = 1$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1!+2!+\dots+n!}{n!} = 1$$

2.3.1 2.3.2 练习题

Question 9 证明, 若 x_n 单调, 则 $|x_n|$ 至少从某项开始后单调, 又问: 反之如何?

Proof 分类讨论, 不妨设 $x_1 \geq 0$

1. x_n 单调递增, $|x_n|$ 从第一项开始单调.

2. x_n 单调递减, 且 $|x_n| \geq 0$. $|x_n|$ 从第一项开始单调.

3. x_n 单调递减, 且 $\exists N$ s.t. $x_n < 0$ (第一个负数项). 则 $|x_n|$ 从第 N 项 (x_N) 开始单调.

反之该结论不成立.

反例: $x_n = \frac{(-1)^n}{n}$, $|x_n|$ 单调递减. 但 $x_{2k} = \frac{1}{2k} > 0 > x_{2k-1} = \frac{-1}{2k-1}$

Question 10 设 a_n 单调增加, b_n 单调减少, 且有 $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$.

证明: 数列 a_n 和 b_n 都收敛, 且极限相等.

Proof $\lim_{n \rightarrow \infty} (a_n - b_n) = 0, \forall \epsilon > 0, \exists N \in \mathbb{N}_+, \text{s.t.} \forall n > N, |a_n - b_n - 0| < \epsilon$.

$b_n - \epsilon < a_n < b_n + \epsilon$, 同时有 $a_n - \epsilon < b_n < a_n + \epsilon$.

b_n 单调减少, $\therefore \exists N_2, \forall m < N_2, b_m > b_n + \epsilon$.

使用反证法证明 b_m 是 a_n 的上界.

假设 b_m 不是 a_n 的上界, 则存在 $a_n > b_m > b_n + \epsilon$, 这与 $|a_n - b_n| < \epsilon$ 矛盾.

$\therefore b_m$ 是 a_n 的上界, 根据单调有界收敛准则, a_n 收敛. 同理可证 b_n 收敛. $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$.

$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

Question 11 按照极限定义证明:

1. 单调增加有上界的数列的极限不小于数列中的任何一项.

2. 单调减少有下界的数列的极限不大于数列中的任何一项.

Question 12 设 $x_n = \frac{2}{3} \cdot \frac{3}{5} \cdots \frac{n+1}{2n+1}, n \in \mathbb{N}_+$, 求数列 x_n 的极限.

Solve 14

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)+1}{2(n+1)+1} = \frac{n+2}{2n+3} < 1. \quad (n > 0) \quad (2.15)$$

x_n 单调递减. $\therefore x_n > 0, \therefore x_n$ 有下界, x_n 收敛.

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \lim_{n \rightarrow \infty} \frac{n+2}{2n+3} = \frac{1}{2}$$

$\left(\frac{1}{2}\right)^n < x_n < \left(\frac{2}{3}\right)^n$, 由夹逼定理, $\lim_{n \rightarrow \infty} x_n = 0$

Question 13 6. 在例题 2.2.6 的基础上证明: 当 $p > 1$ 时, 数列 S_n 收敛. 其中

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p}, \quad n \in \mathbb{N}_+$$

(S_n 就是 p 级数, 当 $p = 1$ 时为调和级数.)

Proof S_n 单调递增, 记 $\frac{1}{2^{p-1}} = r$, 则 $0 < r < 1$.

$$\begin{aligned} \frac{1}{2^p} + \frac{1}{3^p} &< \frac{1}{2^p} + \frac{1}{2^p} &= \frac{1}{2^{p-1}} = r \\ \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} &< \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} &= \frac{1}{4^{p-1}} = r^2 \\ \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} &< \frac{1}{(2^k)^p} + \frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^k)^p} &= \frac{1}{(2^k)^{p-1}} = r^k \end{aligned}$$