Author: weiyuan

Date: 2022 10 24

Chapter 1

1.1

1.1.1 2.1.5

1. prove by Limit definition:

(1). $\lim_{n\to\infty}\frac{3n^2}{n^2-4}=3$.

(2). $\lim_{n\to\infty}\frac{\sin n}{n}=0$.

(3). $\lim_{n\to\infty}(1+n)^{\frac{1}{n}}=0$.

(4). $\lim_{n\to\infty}\frac{a^n}{n!}=0,(a>0)$.

2. Suppose $a_n, n\in\mathbb{N}_+$. sequence a_n converge to a.

Prove $\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a}$.

Proof $n\to\infty a_n\to a$.

$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$.

$$|\sqrt{a_n}-\sqrt{a}|=\left|\frac{a_n-a}{\sqrt{a_n}+\sqrt{a}}\right|<\frac{\epsilon}{\sqrt{a_n}+\sqrt{a}}$$

$\therefore\lim_{n\to\infty}\sqrt{a_n}=\sqrt{a}$. $\square$ (check, not consider the condition$a=0$) add $a=0,\forall\epsilon\in(0,1),\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$. s.t $a_n<\epsilon^2<\epsilon$, $\sqrt{a_n}<\epsilon$.

3. If $\lim_{n\to\infty}a_n=a$.

Prove $\lim_{n\to\infty}|a_n|=|a|$. Vice versa?

Proof $\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),|a_n-a|<\epsilon$.

$$\Big||a_n|-|a|\Big|\leqslant|a_n-a|<\epsilon$$

$\therefore\lim_{n\to\infty}|a_n|=|a|$

If We know $\lim_{n\to\infty}|a_n|=|a|$.

$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n>N(\epsilon),\big||a_n|-|a|\big|<\epsilon$. We can't get $\lim_{n\to\infty}a_n=a$. For example: $a_n=\frac{1}{n}+1,a=-1$, $\lim_{n\to\infty}|a_n|=|a|$ is $\lim_{n\to\infty}|\frac{1}{n}+1|=|-1|$, but $\lim_{n\to\infty}\frac{1}{n}+1\neq-1$ $\square$

(1). Suppose $p(x)$ is a polynomial of x, if $\lim_{n\to\infty}a_n=a$, Prove $\lim_{n\to\infty}p(a_n)=p(a)$.

(2). Suppose $b>0$, $\lim\limits_{n\to\infty}a_n=a$. Prove $b^{a_n}=b^a$.

(3). Suppose $b>0$, $\{a_n\},a_n>0,\forall n\in\mathbb{N}$. $\lim\limits_{n\to\infty}a_n=a.a>0$. Prove $\lim\limits_{n\to\infty}\log_b a_n=\log_b a$.

(4) Suppose $b\in\mathbb{R}$, $\{a_n\},a_n>0$ when $n\in\mathbb{N}$. $\lim\limits_{n\to\infty}a_n=a$. Prove $\lim\limits_{n\to\infty}{a_n}^b=a^b$.

(5) Suppose $\lim\limits_{n\to\infty}a_n=a$. Prove $\lim\limits_{n\to\infty}\sin a_n=\sin a$.

Proof 4.(1)

$\forall\epsilon>0,\exists N(\epsilon)\in\mathbb{N}^+,\forall n\geqslant N(\epsilon),|a_n-a|<\epsilon$.

$p(a)=k_ma^m+k_{m-1}a^{m-1}+\cdots+k_0a^0$.

$\therefore p(a_n)-p(a)=k_m(a_n^m-a^m)+k_{m-1}(a_n^{m-1}-a^{m-1})+\cdots+k_0(a_n^0-a^0)$.

$$\begin{aligned}|a_n^m-a^m|&=|a_n-a|\cdot|a_n^{m-1}+a_n^{m-2}a+\cdots+a^{m-1}|\\&<\epsilon\cdot|a_n^{m-1}+a_n^{m-2}a+\cdots+a^{m-1}|\\&<\epsilon(m-1)\cdots(a+\delta)^{m-1}\end{aligned}$$

$\therefore \lim_{n\to\infty} p(a_n) = p(a)$. $\square$

Proof 4.(2)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

If $b = 1$, $1^{a_n} = 1^a = 1$.

If $b > 1$, $b^{a_n} - b^a = b^a(b^{a_n - a} - 1) < b^a(b^\epsilon - 1)$ $0 < |b^{a_n} - b^a| < b^a \cdot (b^\epsilon - 1) \because b > 0, \epsilon \to 0, \therefore b^\epsilon - 1 \to 0$.

$\therefore \lim\limits_{n\to\infty} b^a_n = b^a$.

If $b < 1$, $b^{a_n} = \frac{1}{(\frac{1}{b})^{a_n}}$, we can prove this condition by considering $\frac{1}{b} > 1$.

Proof 4.(3)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\log_b a_n - \log_b a &= \log_b \frac{a_n}{a} \\ &= \log_b(\frac{a_n - a}{a} + 1) < \log_b(\frac{\epsilon}{a} + 1)\end{aligned}$$

$0 < \log_b a_n - \log_b a| < \log_b(1 + \frac{\epsilon}{a})$. $\because b > 0, a \neq 0, a_n > 0$ when $\epsilon \to 0$. $\therefore \log_b(1 + \frac{\epsilon}{a}) \to 0$.

$\therefore \lim\limits_{n\to\infty} \log_b a_n = \log_b a$

Proof 4.(4)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$a_n^b = e^{b \ln a_n}$, $a_n^b - a^b = e^{b \ln a_n} - e^{b \ln a}$.

$$\begin{aligned}e^{b \ln a_n} - e^{b \ln a} &= e^{b \ln a}(e^{b \ln a_n - b \ln a} - 1) \\ &= e^{b \ln a}(e^{b \ln \frac{a_n}{a}} - 1)\end{aligned}$$

$0 < |a_n^b - a^b| < e^{b \ln a}(e^{b \ln(1 + \frac{\epsilon}{a})} - 1)$

$\therefore \lim\limits_{n\to\infty} a_n^b = a^b$

Proof 4.(5)

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+ . \forall n \geqslant N, |a_n - a| < \epsilon$.

$$\begin{aligned}\sin(A + B) - \sin(A - B) &= \sin A \cos B + \cos A \sin B \\ &\quad - (\sin A \cos B - \cos A \sin B) \\ &= 2 \cos A \sin B\end{aligned}$$

$$\sin a_n - \sin a = 2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}$$

$|\sin a_n - \sin a| = |2 \cos \frac{a_n + a}{2} \sin \frac{a_n - a}{2}| < |2 \sin \frac{a_n - a}{2}|$

$|2 \sin \frac{a_n - a}{2}| < |2 \frac{a_n - a}{2}| = \epsilon$

$|\sin a_n - \sin a| < \epsilon, \therefore \lim\limits_{n\to\infty} \sin a_n = \sin a$

assume $a > 1$. Prove $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

Proof $\frac{1}{n} \log_a n = \log_a \sqrt[n]{n}$. We already know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$, $\log_a 1 = 0$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}^+, N = \max\{2, [\frac{4}{\epsilon^2}]\} . \forall n \geqslant N, |\sqrt[n]{n} - 1| < \epsilon$.

$a > 1$, and $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\therefore$ when $n \to \infty$, $\sqrt[n]{n} < a^\epsilon$, take logarithm on base of a, we can get $\frac{1}{n} \log_a n < \epsilon$

$\therefore \lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$

1.2

1.

2.
3.
4.
5.

1.2.1

1. $\{a_n\}$, $\{b_n\}$, $\{a_n+b_n\}$, $\{a_n\cdot b_n\}$.
2. $\{a_n\},\{b_n\}$, $\{a_n+b_n\}$ (ex: $n+-n, n+-2n$), $\{a_n\cdot b_n\}$ (?).
3. $a_n\leqslant b_n\leqslant c_n, n\in\mathbb{N}_+$. $\lim\limits_{n\to\infty}(c_n-a_n)=0$. $\{b_n\}$
4. $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$
5. $a_n\to a(n\to 0)$. $\forall n, b<a_n<c$. $b<a<c$?
6. $a_n\to a(n\to 0)$. and $b\leqslant a\leqslant c$, $N\in\mathbb{N}_+$, s.t. $n>N$ $b\leqslant a_n\leqslant c$
7. $\lim\limits_{n\to\infty}a_n=0$, : $\lim\limits_{n\to\infty}(a_1a_2\dots a_n)=0$. ?

Proof 5.4

$$\frac{n}{2n}\leqslant\frac{1}{n+1}+\cdots+\frac{1}{2n}\leqslant\frac{n}{n+1}$$

$\because\lim\limits_{n\to\infty}\frac{n}{2n}=\frac{1}{2}\lim\limits_{n\to\infty}\frac{n}{n+1}=1,\therefore\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})$.

$$\begin{aligned}\frac{1}{n+1}+\cdots+\frac{1}{2n}&=\frac{1}{n}(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots+\frac{1}{1+\frac{n}{n}})\\&=\int_0^1\frac{1}{1+x}\mathrm{d}x\\&=\ln(1+x)|_0^1=\ln 2\end{aligned}$$

$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{2n})=\ln 2$

Proof 5.5

b=0, c=2, $a_n=\frac{1}{n}$, $\lim_{n\to\infty}a_n=0=c$.

Proof 5.6

$a=0,b=0,c=2,a_n=(-1)^n\frac{1}{n}$.

$b\leqslant a\leqslant c$, but $(-1)^{2n+1}\frac{1}{2n+1}<0=b$.

Proof $\lim\limits_{n\to\infty}a_n=0,a_n=\frac{1}{n}.a_1a_2\dots a_n=\frac{1}{n!},\lim\limits_{n\to\infty}\frac{1}{n!}=0$.

$\lim\limits_{n\to\infty}a_n=0\to\lim\limits_{n\to\infty}(a_1a_2\dots a_n)=0$ ✓

$\lim\limits_{n\to\infty}(a_1a_2\dots a_n)=0\to\lim\limits_{n\to\infty}a_n=0$ ×

$|a_n|<\epsilon, |a_{N+1}\dots a_n|<\epsilon^{n-N}<\epsilon, a_n<\sqrt[n]{\epsilon}$.

for example, $a_n=\frac{n}{n+1},a_1=\frac{1}{2},a_2=\frac{2}{3},\dots,a_n=\frac{n}{n+1}$.

$$a_1a_2\dots a_n=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}=\frac{1}{n+1}.$$

$$\begin{aligned}&\lim_{n\to\infty}(a_1a_2\dots a_n)\\&=\lim_{n\to\infty}\frac{1}{n}=0\end{aligned}$$

but $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{n}{n+1}=1\neq 0$

:

1. .
2. .

Example 1.1 2.2.2 $\lim_{n\to\infty}\frac{x_n-1}{x_n+a}=0$, prove $\lim_{n\to\infty} x_n = a$

Proof $\forall \epsilon > 0, \exists N \in \mathbb{N}^+. \forall n \geqslant N, |\frac{x_n - 1}{x_n + a} - 0| < \epsilon.$

$|x_n - 1| < \epsilon |x_n + a| < 4a \cdot \epsilon.$(4 ?)

$|x_n - a| < \epsilon |x_n + a| = \epsilon |(x_n - a) + 2a| \leqslant \epsilon(|x_n - a| + 2a).$

$\epsilon < 1, |x_n - a| < 2\epsilon |a|/(1-\epsilon).$

$\epsilon < \frac{1}{2}, |x_n - a| < 2\epsilon |a|/(1-\epsilon) < 4|a|\epsilon.$

Let $\epsilon' = 4a\epsilon, |x_n - 1| < \epsilon'. \therefore \lim \lim_{n\to\infty} x_n = a.$

Example 1.2 2.2.3 $a > 0, b > 0$, $\lim\limits_{n\to\infty}(a_n + b_n)^{\frac{1}{n}}$.

Proof Suppose $a \leqslant b$.

$b = (b^b)^{\frac{1}{n}} < (a^n + b^n)^{\frac{1}{n}} \leqslant (2b^n)^{\frac{1}{n}}.$

$b < (a^n + b^n)^{\frac{1}{n}} \leqslant \sqrt[n]{2}b, \lim\limits_{n\to\infty} = 1.$.

$\lim\limits_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$

n n , a,b .

Example 1.3 2.2.4 $a_n = \frac{1!+2!+\cdots+n!}{n!}, n \in \mathbb{N}^+$

$\lim\limits_{n\to\infty} a_n = 1$

Example 1.4 $\lim\limits_{n\to\infty} \frac{n^3+n-7}{n+3} = +\infty$

Example 1.5 $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

H_n .

1.2.2 2.2.4

Proof 1.

$\{a_n\}$ $a, \to$ $\{a_{2n}\}, \{a_{2n+1}\}$ $a.$

$\{a_{2n}\}, \{a_{2n+1}\}$ $a, \to \{a_n\}$ $a.$

2.

(1). p $a_1, a_2, \ldots, a_p$. $\lim\limits_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \ldots a_p^n}$.

Let$a_s = \max_{1\leqslant i\leqslant p}\{a_1, a_2, \ldots, a_p\}$.

$$a_s = (a_s^n)^{\frac{1}{n}} < (a_1^n + a_2^n + \ldots a_p^n)^{\frac{1}{n}} \leqslant (pa_s^n)^{\frac{1}{n}} = p^{\frac{1}{n}} a_s$$

$n \to \infty, p^{\frac{1}{n}} \to 1. \lim\limits_{n\to\infty}(a_1^n + a_2^n + \ldots a_p^n)^{\frac{1}{n}} = a_s$

(2). $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}, n \in \mathbb{N}_+$. $\lim\limits_{n\to\infty} x_n$

$$\frac{2n+1}{(n+1)} \leqslant x_n \leqslant \frac{2n+1}{\sqrt{n^2+1}}$$

$\lim\limits_{n\to\infty} \frac{2n+1}{n+1} = 2, \lim\limits_{n\to\infty} \frac{2n+1}{\sqrt{n^2+1}} = 2. \therefore \lim\limits_{n\to\infty} x_n = 2$

(3). $a_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n})^{\frac{1}{n}}, n \in \mathbb{N}_+$. $\lim\limits_{n\to\infty} a_n$

$$1 = (\frac{n}{n})^{\frac{1}{n}} < a_n \leqslant (1 \cdot n)^{\frac{1}{n}} = \sqrt[n]{n}$$

$\lim\limits_{n\to\infty} \sqrt[n]{n} = 1, \therefore \lim\limits_{n\to\infty} a_n = 1$