MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 1

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 1.1: If $r \in \mathbf{Q} \setminus \{0\}$ and $x \in \mathbf{R} \setminus \mathbf{Q}$, prove that $r + x$, $rx \notin \mathbf{Q}$.

Solution: We prove this by contradiction. Let $r \in \mathbf{Q} \setminus \{0\}$, and suppose that $r+x \in \mathbf{Q}$. Then, using the field properties of both $\mathbf{R}$ and $\mathbf{Q}$, we have $x = (r+x) - r \in \mathbf{Q}$. Thus $x \notin \mathbf{Q}$ implies $r + x \notin \mathbf{Q}$.

Similarly, if $rx \in \mathbf{Q}$, then $x = (rx)/r \in \mathbf{Q}$. (Here, in addition to the field properties of $\mathbf{R}$ and $\mathbf{Q}$, we use $r \neq 0$.) Thus $x \notin \mathbf{Q}$ implies $rx \notin \mathbf{Q}$. ∎

Problem 1.2: Prove that there is no $x \in \mathbf{Q}$ such that $x^2 = 12$.

Solution: We prove this by contradiction. Suppose there is $x \in \mathbf{Q}$ such that $x^2 = 12$. Write $x = \frac{m}{n}$ in lowest terms. Then $x^2 = 12$ implies that $m^2 = 12n^2$. Since 3 divides $12n^2$, it follows that 3 divides m^2. Since 3 is prime (and by unique factorization in $\mathbf{Z}$), it follows that 3 divides m. Therefore 3^2 divides $m^2 = 12n^2$. Since 3^2 does not divide 12, using again unique factorization in $\mathbf{Z}$ and the fact that 3 is prime, it follows that 3 divides n. We have proved that 3 divides both m and n, contradicting the assumption that the fraction $\frac{m}{n}$ is in lowest terms. ∎

Alternate solution (Sketch): If $x \in \mathbf{Q}$ satisfies $x^2 = 12$, then $\frac{x}{2}$ is in $\mathbf{Q}$ and satisfies $\left(\frac{x}{2}\right)^2 = 3$. Now prove that there is no $y \in \mathbf{Q}$ such that $y^2 = 3$ by repeating the proof that $\sqrt{2} \notin \mathbf{Q}$. ∎

Problem 1.5: Let $A \subset \mathbf{R}$ be nonempty and bounded below. Set $-A = \{-a \colon a \in A\}$. Prove that $\inf(A) = -\sup(-A)$.

Solution: First note that $-A$ is nonempty and bounded above. Indeed, A contains some element x, and then $-x \in A$; moreover, A has a lower bound m, and $-m$ is an upper bound for $-A$.

We now know that $b = \sup(-A)$ exists. We show that $-b = \inf(A)$. That $-b$ is a lower bound for A is immediate from the fact that b is an upper bound for $-A$. To show that $-b$ is the greatest lower bound, we let $c > -b$ and prove that c is not a lower bound for A. Now $-c < b$, so $-c$ is not an upper bound for $-A$. So there exists $x \in -A$ such that $x > -c$. Then $-x \in A$ and $-x < c$. So c is not a lower bound for A. ∎

Problem 1.6: Let $b \in \mathbf{R}$ with $b > 1$, fixed throughout the problem.

Comment: We will assume known that the function $n \mapsto b^n$, from $\mathbf{Z}$ to $\mathbf{R}$, is strictly increasing, that is, that for $m, n \in \mathbf{Z}$, we have $b^m < b^n$ if and only if $m < n$. Similarly, we take as known that $x \mapsto x^n$ is strictly increasing when n is

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an *integer* with $n > 0$. We will also assume that the usual laws of exponents are known to hold when the exponents are *integers*. We can't assume anything about fractional exponents, except for Theorem 1.21 of the book and its corollary, because the context makes it clear that we are to assume fractional powers have not yet been defined.

(a) Let $m, n, p, q \in \mathbf{Z}$, with $n > 0$ and $q > 0$. Prove that if $\frac{m}{n} = \frac{p}{q}$, then $(b^m)^{1/n} = (b^p)^{1/q}$.

Solution: By the uniqueness part of Theorem 1.21 of the book, applied to the positive integer nq, it suffices to show that

$$\left[(b^m)^{1/n}\right]^{nq} = \left[(b^p)^{1/q}\right]^{nq}.$$

Now the definition in Theorem 1.21 implies that

$$\left[(b^m)^{1/n}\right]^{n} = b^m \quad \text{and} \quad \left[(b^p)^{1/q}\right]^{q} = b^p.$$

Therefore, using the laws of integer exponents and the equation $mq = np$, we get

$$\begin{aligned}\left[(b^m)^{1/n}\right]^{nq} &= \left[\left[(b^m)^{1/n}\right]^{n}\right]^{q} = (b^m)^q = b^{mq} \\ &= b^{np} = (b^p)^n = \left[\left[(b^p)^{1/q}\right]^{q}\right]^{n} = \left[(b^p)^{1/q}\right]^{nq},\end{aligned}$$

as desired. ∎

By Part (a), it makes sense to define $b^{m/n} = (b^m)^{1/n}$ for $m, n \in \mathbf{Z}$ with $n > 0$. This defines b^r for all $r \in \mathbf{Q}$.

(b) Prove that $b^{r+s} = b^r b^s$ for $r, s \in \mathbf{Q}$.

Solution: Choose $m, n, p, q \in \mathbf{Z}$, with $n > 0$ and $q > 0$, such that $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Then $r + s = \frac{mq+np}{nq}$. By the uniqueness part of Theorem 1.21 of the book, applied to the positive integer nq, it suffices to show that

$$\left[b^{(mq+np)/(nq)}\right]^{nq} = \left[(b^m)^{1/n}(b^p)^{1/q}\right]^{nq}.$$

Directly from the definitions, we can write

$$\left[b^{(mq+np)/(nq)}\right]^{nq} = \left[\left[b^{(mq+np)}\right]^{1/(nq)}\right]^{nq} = b^{(mq+np)}.$$

Using the laws of integer exponents and the definitions for rational exponents, we can rewrite the right hand side as

$$\left[(b^m)^{1/n}(b^p)^{1/q}\right]^{nq} = \left[\left[(b^m)^{1/n}\right]^{n}\right]^{q}\left[\left[(b^p)^{1/q}\right]^{q}\right]^{n} = (b^m)^q(b^p)^n = b^{(mq+np)}.$$

This proves the required equation, and hence the result. ∎

(c) For $x \in \mathbf{R}$, define

$$B(x) = \{b^r : r \in \mathbf{Q} \cap (-\infty, x]\}.$$

Prove that if $r \in \mathbf{Q}$, then $b^r = \sup(B(r))$.

Solution: The main point is to show that if $r, s \in \mathbf{Q}$ with $r < s$, then $b^r < b^s$. Choose $m, n, p, q \in \mathbf{Z}$, with $n > 0$ and $q > 0$, such that $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Then also $r = \frac{mq}{nq}$ and $s = \frac{np}{nq}$, with $nq > 0$, so

$$b^r = (b^{mq})^{1/(nq)} \quad \text{and} \quad b^s = (b^{np})^{1/(nq)}.$$

Now $mq < np$ because $r < s$. Therefore, using the definition of $c^{1/(nq)}$,

$$(b^r)^{nq} = b^{mq} < b^{np} = (b^s)^{nq}.$$

Since $x \mapsto x^{nq}$ is strictly increasing, this implies that $b^r < b^s$.

Now we can prove that if $r \in \mathbf{Q}$ then $b^r = \sup(B(r))$. By the above, if $s \in \mathbf{Q}$ and $s \leq r$, then $b^s \leq b^r$. This implies that b^r is an upper bound for $B(r)$. Since $b^r \in B(r)$, obviously no number smaller than b^r can be an upper bound for $B(r)$. So $b^r = \sup(B(r))$. ∎

We now define $b^x = \sup(B(x))$ for every $x \in \mathbf{R}$. We need to show that $B(x)$ is nonempty and bounded above. To show it is nonempty, choose (using the Archimedean property) some $k \in \mathbf{Z}$ with $k < x$; then $b^k \in B(x)$. To show it is bounded above, similarly choose some $k \in \mathbf{Z}$ with $k > x$. If $r \in \mathbf{Q} \cap (-\infty, x]$, then $b^r \in B(k)$ so that $b^r \leq b^k$ by Part (c). Thus b^k is an upper bound for $B(x)$. This shows that the definition makes sense, and Part (c) shows it is consistent with our earlier definition when $r \in \mathbf{Q}$.

(d) Prove that $b^{x+y} = b^x b^y$ for all x, $y \in \mathbf{R}$.

Solution:

In order to do this, we are going to need to replace the set $B(x)$ above by the set

$$B_0(x) = \{b^r : r \in \mathbf{Q} \cap (-\infty, x)\}$$

(that is, we require $r < x$ rather than $r \leq x$) in the definition of b^x. (If you are skeptical, read the main part of the solution first to see how this is used.)

We show that the replacement is possible via some lemmas.

Lemma 1. If $x \in [0, \infty)$ and $n \in \mathbf{Z}$ satisfies $n \geq 0$, then $(1+x)^n \geq 1 + nx$.

Proof: The proof is by induction on n. The statement is obvious for $n = 0$. So assume it holds for some n. Then, since $x \geq 0$,

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x) \geq (1+nx)(1+x) \\ &= 1 + (n+1)x + nx^2 \geq 1 + (n+1)x.\end{aligned}$$

This proves the result for $n+1$. ∎

Lemma 2. $\inf\{b^{1/n} : n \in \mathbf{N}\} = 1$. (Recall that $b > 1$ and $\mathbf{N} = \{1, 2, 3, \ldots\}$.)

Proof: Clearly 1 is a lower bound. (Indeed, $(b^{1/n})^n = b > 1 = 1^n$, so $b^{1/n} > 1$.) We show that $1+x$ is not a lower bound when $x > 0$. If $1+x$ were a lower bound, then $1 + x \leq b^{1/n}$ would imply $(1+x)^n \leq (b^{1/n})^n = b$ for all $n \in \mathbf{N}$. By Lemma 1, we would get $1 + nx \leq b$ for all $n \in \mathbf{N}$, which contradicts the Archimedean property when $x > 0$. ∎

Lemma 3. $\sup\{b^{-1/n} : n \in \mathbf{N}\} = 1$.

Proof: Part (b) shows that $b^{-1/n}b^{1/n} = b^0 = 1$, whence $b^{-1/n} = (b^{1/n})^{-1}$. Since all numbers $b^{-1/n}$ are strictly positive, it now follows from Lemma 2 that 1 is an upper bound. Suppose $x < 1$ is an upper bound. Then x^{-1} is a lower bound for

$\{b^{1/n} \colon n \in \mathbf{N}\}$. Since $x^{-1} > 1$, this contradicts Lemma 2. Thus $\sup\{b^{-1/n} \colon n \in \mathbf{N}\} = 1$, as claimed. ▮

Lemma 4. $b^x = \sup(B_0(x))$ for $x \in \mathbf{R}$.

Proof: If $x \notin \mathbf{Q}$, then $B_0(x) = B(x)$, so there is nothing to prove. If $x \in \mathbf{Q}$, then at least $B_0(x) \subset B(x)$, so $b^x \geq \sup(B_0(x))$. Moreover, Part (b) shows that $b^{x-1/n} = b^x b^{-1/n}$ for $n \in \mathbf{N}$. The numbers $b^{x-1/n}$ are all in $B_0(x)$, and

$$\sup\{b^x b^{-1/n} \colon n \in \mathbf{N}\} = b^x \sup\{b^{-1/n} \colon n \in \mathbf{N}\}$$

because $b^x > 0$, so using Lemma 3 in the last step gives

$$\sup(B_0(x)) \geq \sup\{b^{x-1/n} \colon n \in \mathbf{N}\} = b^x \sup\{b^{-1/n} \colon n \in \mathbf{N}\} = b^x.$$

▮

Now we can prove the formula $b^{x+y} = b^x b^y$. We start by showing that $b^{x+y} \leq b^x b^y$, which we do by showing that $b^x b^y$ is an upper bound for $B_0(x+y)$. Thus let $r \in \mathbf{Q}$ satisfy $r < x+y$. Then there are s_0, $t_0 \in \mathbf{R}$ such that $r = s_0 + t_0$ and $s_0 < x$, $t_0 < y$. Choose s, $t \in \mathbf{Q}$ such that $s_0 < s < x$ and $t_0 < t < y$. Then $r < s + t$, so $b^r < b^{s+t} = b^s b^t \leq b^x b^y$. This shows that $b^x b^y$ is an upper bound for $B_0(x+y)$.

(Note that this does not work using $B(x+y)$. If $x + y \in \mathbf{Q}$ but x, $y \notin \mathbf{Q}$, then $b^{x+y} \in B(x+y)$, but it is not possible to find s and t with $b^s \in B(x)$, $b^t \in B(y)$, and $b^s b^t = b^{x+y}$.)

We now prove the reverse inequality. Suppose it fails, that is, $b^{x+y} < b^x b^y$. Then

$$\frac{b^{x+y}}{b^y} < b^x.$$

The left hand side is thus not an upper bound for $B_0(x)$, so there exists $s \in \mathbf{Q}$ with $s < x$ and

$$\frac{b^{x+y}}{b^y} < b^s.$$

It follows that

$$\frac{b^{x+y}}{b^s} < b^y.$$

Repeating the argument, there is $t \in \mathbf{Q}$ with $t < y$ such that

$$\frac{b^{x+y}}{b^s} < b^t.$$

Therefore

$$b^{x+y} < b^s b^t = b^{s+t}$$

(using Part (b)). But $b^{s+t} \in B_0(x+y)$ because $s + t \in \mathbf{Q}$ and $s + t < x + y$, so this is a contradiction. Therefore $b^{x+y} \leq b^x b^y$. ▮

Problem 1.9: Define a relation on $\mathbf{C}$ by $w < z$ if and only if either $\mathrm{Re}(w) < \mathrm{Re}(z)$ or both $\mathrm{Re}(w) = \mathrm{Re}(z)$ and $\mathrm{Im}(w) < \mathrm{Im}(z)$. (For $z \in \mathbf{C}$, the expressions $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z.) Prove that this makes $\mathbf{C}$ an ordered set. Does this order have the least upper bound property?

Solution: We verify the two conditions in the definition of an order. For the first, let w, $z \in \mathbf{C}$. There are three cases.

Case 1: $\mathrm{Re}(w) < \mathrm{Re}(z)$. Then $w < z$, but $w = z$ and $w > z$ are both false.

Case 2: $\mathrm{Re}(w) > \mathrm{Re}(z)$. Then $w > z$, but $w = z$ and $w < z$ are both false.

Case 3: $\mathrm{Re}(w) = \mathrm{Re}(z)$. This case has three subcases.

Case 3.1: $\mathrm{Im}(w) < \mathrm{Im}(z)$. Then $w < z$, but $w = z$ and $w > z$ are both false.

Case 3.2: $\mathrm{Im}(w) > \mathrm{Im}(z)$. Then $w > z$, but $w = z$ and $w < z$ are both false.

Case 3.3: $\mathrm{Im}(w) = \mathrm{Im}(z)$. Then $w = z$, but $w > z$ and $w < z$ are both false.

These cases exhaust all possibilities, and in each of them exactly one of $w < z$, $w = z$, and $w > z$ is true, as desired.

Now we prove transitivity. Let $s < w$ and $w < z$. If either $\mathrm{Re}(s) < \mathrm{Re}(w)$ or $\mathrm{Re}(w) < \mathrm{Re}(z)$, then clearly $\mathrm{Re}(s) < \mathrm{Re}(z)$, so $s < z$. If $\mathrm{Re}(s) = \mathrm{Re}(w)$ and $\mathrm{Re}(w) = \mathrm{Re}(z)$, then the definition of the order requires $\mathrm{Im}(s) < \mathrm{Im}(w)$ and $\mathrm{Im}(w) < \mathrm{Im}(z)$. We thus have $\mathrm{Re}(s) = \mathrm{Re}(z)$ and $\mathrm{Im}(s) < \mathrm{Im}(z)$, so $s < z$ by definition.

It remains to answer the last question. We show that this order does not have the least upper bound property. Let $S = \{z \in \mathbf{C} \colon \mathrm{Re}(z) < 0\}$. Then $S \neq \varnothing$ because $-1 \in S$, and S is bounded above because 1 is an upper bound for S.

We show that S does not have a least upper bound by showing that if w is an upper bound for S, then there is a smaller upper bound. First, by the definition of the order it is clear that $\mathrm{Re}(w)$ is an upper bound for

$$\{\mathrm{Re}(z) \colon z \in S\} = (-\infty, 0).$$

Therefore $\mathrm{Re}(w) \geq 0$. Moreover, every $u \in \mathbf{C}$ with $\mathrm{Re}(u) \geq 0$ is in fact an upper bound for S. In particular, if w is an upper bound for S, then $w - i < w$ and has the same real part, so is a smaller upper bound. ∎

Note: A related argument shows that the set $T = \{z \in \mathbf{C} \colon \mathrm{Re}(z) \leq 0\}$ also has no least upper bound. One shows that w is an upper bound for T if and only if $\mathrm{Re}(w) > 0$.

Problem 1.13: Prove that if $x,\, y \in \mathbf{C}$, then $\big| |x| - |y| \big| \leq |x - y|$.

Solution: The desired inequality is equivalent to

$$|x| - |y| \leq |x - y| \quad \text{and} \quad |y| - |x| \leq |x - y|.$$

We prove the first; the second follows by exchanging x and y.

Set $z = x - y$. Then $x = y + z$. The triangle inequality gives $|x| \leq |y| + |z|$. Substituting the definition of z and subtracting $|y|$ from both sides gives the result. ∎

Problem 1.17: Prove that if $x,\, y \in \mathbf{R}^n$, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Interpret this result geometrically in terms of parallelograms.

Solution: Using the definition of the norm in terms of scalar products, we have:

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y \rangle + \langle x - y, x - y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &\quad + \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= 2\langle x, x \rangle + 2\langle y, y \rangle = 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

The interpretation is that 0, x, y, $x + y$ are the vertices of a parallelogram, and that $\|x + y\|$ and $\|x - y\|$ are the lengths of its diagonals while $\|x\|$ and $\|y\|$ are each the lengths of two opposite sides. Therefore the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the sides. ∎

Note: One can do the proof directly in terms of the formula $\|x\|^2 = \sum_{k=1}^{n} |x_k|^2$. The steps are all the same, but it is more complicated to write. It is also less general, since the argument above applies to any norm that comes from a scalar product.

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 2

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.2: Prove that the set of algebraic numbers is countable.

Solution (sketch): For each fixed integer $n \geq 0$, the set P_n of all polynomials with integer coefficients and degree at most n is countable, since it has the same cardinality as the set $\{(a_0, \ldots, a_n) \colon a_i \in \mathbf{N}\} = \mathbf{N}^{n+1}$. The set of all polynomials with integer coefficients is $\bigcup_{n=0}^{\infty} P_n$, which is a countable union of countable sets and so countable. Each polynomial has only finitely many roots (at most n for degree n), so the set of all possible roots of all polynomials with integer coefficients is a countable union of finite sets, hence countable. ∎

Problem 2.3: Prove that there exist real numbers which are not algebraic.

Solution (Sketch): This follows from Problem 2.2, since $\mathbf{R}$ is not countable. ∎

Problem 2.4: Is $\mathbf{R} \setminus \mathbf{Q}$ countable?

Solution (Sketch): No. $\mathbf{Q}$ is countable and $\mathbf{R}$ is not countable. ∎

Problem 2.5: Construct a bounded subset of $\mathbf{R}$ with exactly 3 limit points.

Solution (Sketch): For example, use

$$\left\{ \tfrac{1}{n} \colon n \in \mathbf{N} \right\} \cup \left\{ 1 + \tfrac{1}{n} \colon n \in \mathbf{N} \right\} \cup \left\{ 2 + \tfrac{1}{n} \colon n \in \mathbf{N} \right\}.$$

∎

Problem 2.6: Let E' denote the set of limit points of E. Prove that E' is closed. Prove that $\overline{E}' = E'$. Is $(E')'$ always equal to E'?

Solution (Sketch): Proving that E' is closed is equivalent to proving that $(E')' \subset E'$. So let $x \in (E')'$ and let $\varepsilon > 0$. Choose $y \in E' \cap (N_\varepsilon(x) \setminus \{x\})$. Choose $\delta = \min(d(x, y), \varepsilon - d(x, y)) > 0$. Choose $z \in E \cap (N_\delta(y) \setminus \{y\})$. The triangle inequality ensures $z \neq x$ and $z \in N_\varepsilon(x)$. This shows x is a limit point of E.

Here is a different way to prove that $(E')' \subset E'$. Let $x \in (E')'$ and $\varepsilon > 0$. Choose $y \in E' \cap (N_{\varepsilon/2}(x) \setminus \{x\})$. By Theorem 2.20 of Rudin, there are infinitely many points in $E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$. In particular there is $z \in E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$ with $z \neq x$. Now $z \in E \cap (N_\varepsilon(x) \setminus \{x\})$.

To prove $\overline{E}' = E'$, it suffices to prove $\overline{E}' \subset E'$. We first claim that if A and B are any subsets of X, then $(A \cup B)' \subset A' \cup B'$. The fastest way to do this is to assume that $x \in (A \cup B)'$ but $x \notin A'$, and to show that $x \in B'$. Accordingly, let $x \in (A \cup B)' \setminus A'$. Since $x \notin A'$, there is $\varepsilon_0 > 0$ such that $N_{\varepsilon_0}(x) \cap A$ contains no

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points except possibly x itself. Now let $\varepsilon > 0$; we show that $N_\varepsilon(x) \cap B$ contains at least one point different from x. Let $r = \min(\varepsilon, \varepsilon_0) > 0$. Because $x \in (A \cup B)'$, there is $y \in N_r(x) \cap (A \cup B)$ with $y \neq x$. Then $y \notin A$ because $r \leq \varepsilon_0$. So necessarily $y \in B$, and thus y is a point different from x and in $N_r(x) \cap B$. This shows that $x \in B'$, and completes the proof that $(A \cup B)' \subset A' \cup B'$.

To prove $\overline{E}' \subset E'$, we now observe that

$$\overline{E}' = (E \cup E')' \subset E' \cup (E')' \subset E' \cup E' = E'.$$

An alternate proof that $\overline{E}' \subset E'$ can be obtained by slightly modifying either of the proofs above that $(E')' \subset E'$.

For the third part, the answer is no. Take

$$E = \{0\} \cup \left\{ \tfrac{1}{n} : n \in \mathbf{N} \right\}.$$

Then $E' = \{0\}$ and $(E')' = \varnothing$. (Of course, you must prove these facts.) ∎

Problem 2.8: If $E \subset \mathbf{R}^2$ is open, is every point of E a limit point of E? What if E is closed instead of open?

Solution (Sketch): Every point of an open set $E \subset \mathbf{R}^2$ is a limit point of E. Indeed, if $x \in E$, then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$, and it is easy to show that x is a limit point of $N_\varepsilon(x)$.

(Warning: This is *not true* in a general metric space.)

Not every point of a closed set need be a limit point. Take $E = \{(0, 0)\}$, which has no limit points. ∎

Problem 2.9: Let E° denote the set of interior points of a set E, that is, the interior of E.

(a) Prove that E° is open.

Solution (sketch): If $x \in E^\circ$, then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$. Since $N_\varepsilon(x)$ is open, every point in $N_\varepsilon(x)$ is an interior point of $N_\varepsilon(x)$, hence of the bigger set E. So $N_\varepsilon(x) \in E^\circ$. ∎

(b) Prove that E is open if and only if $E^\circ = E$.

Solution: If E is open, then $E = E^\circ$ by the definition of E°. If $E = E^\circ$, then E is open by Part (a). ∎

(c) If G is open and $G \subset E$, prove that $G \subset E^\circ$.

Solution (sketch): If $x \in G \subset E$ and G is open, then x is an interior point of G. Therefore x is an interior point of the bigger set E. So $x \in E^\circ$. ∎

(d) Prove that $X \setminus E^\circ = \overline{X \setminus E}$.

Solution (sketch): First show that $X \setminus E^\circ \subset \overline{X \setminus E}$. If $x \notin E$, then clearly $x \in \overline{X \setminus E}$. Otherwise, consider $x \in E \setminus E^\circ$. Rearranging the statement that x fails to be an interior point of E, and noting that x itself is not in $X \setminus E$, one gets exactly the statement that x is a limit point of $X \setminus E$.

Now show that $\overline{X \setminus E} \subset X \setminus E^\circ$. If $x \in X \setminus E$, then clearly $x \notin E^\circ$. If $x \notin X \setminus E$ but x is a limit point of $X \setminus E$, then one simply rearranges the definition of a limit point to show that x is not an interior point of E. ∎

(e) Prove or disprove: $\left(\overline{E}\right)^\circ = \overline{E}$.

Solution (sketch): This is false. Example: take $E = (0,1) \cup (1,2)$. We have $E^\circ = E$, $\bar{E} = [0,2]$, and $\left(\overline{E}\right)^\circ = (0,2)$.

Another example is $\mathbf{Q}$. ∎

(f) Prove or disprove: $\overline{E^\circ} = \overline{E}$.

Solution (sketch): This is false. Example: take $E = (0,1) \cup \{2\}$. Then $\overline{E} = [0,1] \cup \{2\}$, $E^\circ = (0,1)$, and $\overline{E}^\circ = [0,1]$.

The sets $\mathbf{Q}$ and $\{0\}$ are also examples: in both cases, $E^\circ = \varnothing$. ∎

Problem 2.11: Which of the following are metrics on $\mathbf{R}$?

(a) $d_1(x,y) = (x-y)^2$.

Solution (Sketch): No. The triangle inequality fails with $x = 0$, $y = 2$, and $z = 4$. ∎

(b) $d_2(x,y) = \sqrt{|x-y|}$.

Solution (Sketch): Yes. Some work is needed to check the triangle inequality. ∎

(c) $d_3(x,y) = |x^2 - y^2|$.

Solution (Sketch): No. $d_3(1,-1) = 0$. ∎

(d) $d_4(x,y) = |x - 2y|$.

Solution (Sketch): No. $d_4(1,1) \neq 0$. Also, $d_4(1,6) \neq d_4(6,1)$. ∎

(e) $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$.

Solution (Sketch): Yes. Some work is needed to check the triangle inequality. You need to know that $t \mapsto \frac{t}{1+t}$ is nondecreasing on $[0,\infty)$, and that a, $b \geq 0$ implies

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Do the first by algebraic manipulation. The second is

$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

(This is easier than what most people did the last time I assigned this problem.) ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 3

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.14: Give an example of an open cover of the interval $(0, 1) \subset \mathbf{R}$ which has no finite subcover.

Solution (sketch): $\{(1/n, 1) \colon n \in \mathbf{N}\}$. (Note that you must show that this works.) ∎

Problem 2.16: Regard $\mathbf{Q}$ as a metric space with the usual metric. Let $E = \{x \in \mathbf{Q} \colon 2 < x^2 < 3\}$. Prove that E is a closed and bounded subset of $\mathbf{Q}$ which is not compact. Is E an open subset of $\mathbf{Q}$?

Solution (sketch): Clearly E is bounded.

We prove E is closed. The fast way to do this is to note that

$$\mathbf{Q} \setminus E = \mathbf{Q} \cap \left[\left(-\infty, -\sqrt{3}\right) \cup \left(-\sqrt{2}, \sqrt{2}\right) \cup \left(\sqrt{3}, \infty\right) \right],$$

and so is open by Theorem 2.30. To do it directly, suppose $x \in \mathbf{Q}$ is a limit point of E which is not in E. Since we can't have $x^2 = 2$ or $x^2 = 3$, we must have $x^2 < 2$ or $x^2 > 3$. Assume $x^2 > 3$. (The other case is handled similarly.) Let $r = |x| - \sqrt{3} > 0$. Then every $z \in N_r(x)$ satisfies

$$|z| \geq |x| - |x - z| > |x| - r > 0,$$

which implies that $z^2 > (|x| - r)^2 = 3$. This shows that $z \notin E$, which contradicts the assumption that x is a limit point of E.

The fast way to see that E is not compact is to note that it is a subset of $\mathbf{R}$, but is not closed in $\mathbf{R}$. (See Theorem 2.23.) To prove this directly, show that, for example, the sets

$$\left\{ y \in \mathbf{Q} \colon 2 + \tfrac{1}{n} < y^2 < 3 - \tfrac{1}{n} \right\}$$

form an open cover of E which has no finite subcover.

To see that E is open in $\mathbf{Q}$, the fast way is to write

$$E = \mathbf{Q} \cap \left[\left(-\sqrt{3}, -\sqrt{2}\right) \cup \left(\sqrt{2}, -\sqrt{3}\right) \right],$$

which is open by Theorem 2.30. It can also be proved directly. ∎

Problem 2.19: Let X be a metric space, fixed throughout this problem.

(a) If A and B are disjoint closed subsets of X, prove that they are separated.

Solution (Sketch): We have $A \cap \overline{B} = \overline{A} \cap B = A \cap B = \varnothing$ because A and B are closed. ∎

Date: 15 October 2001.

(b) If A and B are disjoint open subsets of X, prove that they are separated.

Solution (Sketch): $X \setminus A$ is a closed subset containing B, and hence containing $\overline{B}$. Thus $A \cap \overline{B} = \varnothing$. Interchanging A and B, it follows that $\overline{A} \cap B = \varnothing$. ∎

(c) Fix $x_0 \in X$ and $\delta > 0$. Set

$$A = \{x \in X \colon d(x, x_0) < \delta\} \quad \text{and} \quad B = \{x \in X \colon d(x, x_0) > \delta\}.$$

Prove that A and B are separated.

Solution (Sketch): Both A and B are open sets (proof!), and they are disjoint. So this follows from Part (b). ∎

(d) Prove that if X is connected and contains at least two points, then X is uncountable.

Solution: Let x and y be distinct points of X. Let $R = d(x, y) > 0$. For each $r \in (0, R)$, consider the sets

$$A_r = \{z \in X \colon d(z, x) < r\} \quad \text{and} \quad B_r = \{z \in X \colon d(z, x) > r\}.$$

They are separated by Part (c). They are not empty, since $x \in A_r$ and $y \in B_r$. Since X is connected, there must be a point $z_r \in X \setminus (A_r \cup B_r)$. Then $d(x, z_r) = r$.

Note that if $r \neq s$, then $d(x, z_r) \neq d(x, z_s)$, so $z_r \neq z_s$. Thus $r \mapsto z_r$ defines an injective map from $(0, R)$ to X. Since $(0, R)$ is not countable, X can't be countable either. ∎

Problem 2.20: Let X be a metric space, and let $E \subset X$ be a connected subset. Is $\overline{E}$ necessarily connected? Is $\operatorname{int}(E)$ necessarily connected?

Solution to the first question (sketch): The set $\operatorname{int}(E)$ need not be connected. The easiest example to write down is to take $X = \mathbf{R}^2$ and

$$E = \{x \in \mathbf{R}^2 \colon \|x - (1, 0)\| \leq 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-1, 0)\| \leq 1\}.$$

Then

$$\operatorname{int}(E) = \{x \in \mathbf{R}^2 \colon \|x - (1, 0)\| < 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-1, 0)\| < 1\}.$$

This set fails to be connected because the point $(0, 0)$ is missing. A more dramatic example is two closed disks joined by a line, say

$$E = \{x \in \mathbf{R}^2 \colon \|x - (2, 0)\| \leq 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-2, 0)\| \leq 1\} \tag{1}$$
$$\cup \{(\alpha, 0) \in \mathbf{R}^2 \colon -3 \leq \alpha \leq 3\}. \tag{2}$$

Then

$$\operatorname{int}(E) = \{x \in \mathbf{R}^2 \colon \|x - (2, 0)\| < 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-2, 0)\| < 1\}.$$

∎

Solution to the second question: If E is connected, then $\overline{E}$ is necessarily connected. To prove this using Rudin's definition, assume $\overline{E} = A \cup B$ for separated sets A and B; we prove that one of A and B is empty. The sets $A_0 = A \cap E$ and $B_0 = B \cap E$ are separated sets such that $E = A_0 \cup B_0$. (They are separated because $\overline{A_0} \subset \overline{A}$ and $\overline{B_0} \subset \overline{B}$.) Because E is connected, one of A_0 and B_0 must be empty; without loss of generality, $A_0 = \varnothing$. Then $A \subset \overline{E} \setminus E$. Therefore $E \subset B$. But then $A \subset \overline{E} \subset \overline{B}$. Because A and B are separated, this can only happen if $A = \varnothing$. ∎

Alternate solution to the second question: If E is connected, we prove that $\overline{E}$ is necessarily connected, using the traditional definition. Thus, assume that $\overline{E} = A \cup B$ for disjoint relatively open sets A and B; we prove that one of A and B is empty. The sets $A_0 = A \cap E$ and $B_0 = B \cap E$ are disjoint relatively open sets in E such that $E = A_0 \cup B_0$. Because E is connected, one of A_0 and B_0 must be empty; without loss of generality, $A_0 = \varnothing$. Then $A \subset \overline{E} \setminus E$ and is relatively open in $\overline{E}$.

Now let $x \in A$. Then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \cap \overline{E} \subset A$. So $N_\varepsilon(x) \cap \overline{E} \subset \overline{E} \setminus E$, which implies that $N_\varepsilon(x) \cap E = \varnothing$. This contradicts the fact that $x \in \overline{E}$. Thus $A = \varnothing$. ∎

Problem 2.22: Prove that $\mathbf{R}^n$ is separable.

Solution (sketch): The subset $\mathbf{Q}^n$ is countable by Theorem 2.13. To show that $\mathbf{Q}^n$ is dense, let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and let $\varepsilon > 0$. Choose $y_1, \ldots, y_n \in \mathbf{Q}$ such that $|y_k - x_k| < \frac{\varepsilon}{n}$ for all k. (Why is this possible?) Then $y = (y_1, \ldots, y_n) \in \mathbf{Q}^n \cap N_\varepsilon(x)$. ∎

Problem 2.23: Prove that every separable metric space has a countable base.

Solution: Let X be a separable metric space. Let $S \subset X$ be a countable dense subset of X. Let

$$\mathcal{B} = \{N_{1/n}(s) \colon s \in S, \, n \in \mathbf{N}\}.$$

Since $\mathbf{N}$ and S are countable, $\mathcal{B}$ is a countable collection of open subsets of X.

Now let $U \subset X$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since S is dense in X, there is $s \in S \cap N_{1/n}(x)$, that is, $s \in S$ and $d(s, x) < \frac{1}{n}$. Then $x \in N_{1/n}(s)$ and $N_{1/n}(s) \in \mathcal{B}$. It remains to show that $N_{1/n}(s) \subset U$. So let $y \in N_{1/n}(s)$. Then

$$d(x, y) \leq d(x, s) + d(s, y) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $y \in N_\varepsilon(x) \subset U$. ∎

Problem 2.25: Let K be a compact metric space. Prove that K has a countable base, and that K is separable.

The easiest way to do this is actually to prove first that K is separable, and then to use Problem 2.23. However, the direct proof that K has a countable base is not very different, so we give it here. We actually give two versions of the proof, which differ primarily in how the indexing is done. The first version is easier to write down correctly, but the second has the advantage of eliminating some of the subscripts, which can be important in more complicated situations. Note that the second proof is shorter, even after the parenthetical remarks about indexing are deleted from the first proof. Afterwards, we give a proof that every metric space with a countable base is separable.

Solution 1: We prove that K has a countable base. For each $n \in \mathbf{N}$, the open sets $N_{1/n}(x)$, for $x \in K$, form an open cover of K. Since K is compact, this open cover has a finite subcover, say

$$\{N_{1/n}(x_{n,1}), N_{1/n}(x_{n,2}), \ldots, N_{1/n}(x_{n,k_n})\}$$

for suitable $x_{n,1}, x_{n,2}, \ldots, x_{n,k_n} \in K$. (Note: For each n, the collection of x's is different; therefore, they must be labelled independently by both n and a second parameter. The number of them also depends on n, so must be called k_n, $k(n)$, or something similar.)

Now let

$$\mathcal{B} = \{N_{1/n}(x_{n,j}) \colon n \in \mathbf{N},\, 1 \leq j \leq k_n\}.$$

(Note that both subscripts are used here.) Then $\mathcal{B}$ is a countable union of finite sets, hence countable. We show that $\mathcal{B}$ is a base for K.

Let $U \subset K$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since the sets

$$N_{1/n}(x_{n,1}), N_{1/n}(x_{n,2}), \ldots, N_{1/n}(x_{n,k_n})$$

cover K, there is j with $1 \leq j \leq k_n$ such that $x \in N_{1/n}(x_{n,j})$. (Here we see why the double indexing is necessary: the list of centers to choose from *depends on* n, and therefore their names must also depend on n.) By definition, $N_{1/n}(x_{n,j}) \in \mathcal{B}$. It remains to show that $N_{1/n}(x_{n,j}) \subset U$. So let $y \in N_{1/n}(x_{n,j})$. Since x and y are both in $N_{1/n}(x_{n,j})$, we have

$$d(x, y) \leq d(x, x_{n,j}) + d(x_{n,j}, y) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $y \in N_\varepsilon(x) \subset U$. ∎

Solution 2: We again prove that K has a countable base. For each $n \in \mathbf{N}$, the open sets $N_{1/n}(x)$, for $x \in K$, form an open cover of K. Since K is compact, this open cover has a finite subcover. That is, there is a finite set $F_n \subset K$ such that the sets $N_{1/n}(x)$, for $x \in F_n$, still cover K. Now let

$$\mathcal{B} = \{N_{1/n}(x) \colon n \in \mathbf{N},\, x \in F_n\}.$$

Then $\mathcal{B}$ is a countable union of finite sets, hence countable. We show that $\mathcal{B}$ is a base for K.

Let $U \subset K$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since the sets $N_{1/n}(y)$, for $y \in F_n$, cover K, there is $y \in F_n$ such that $x \in N_{1/n}(y)$. By definition, $N_{1/n}(y) \in \mathcal{B}$. It remains to show that $N_{1/n}(y) \subset U$. So let $z \in N_{1/n}(y)$. Since x and z are both in $N_{1/n}(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $z \in N_\varepsilon(x) \subset U$. ∎

It remains to prove the following lemma.

Lemma: Let X be a metric space with a countable base. Then X is separable.

Proof: Let $\mathcal{B}$ be a countable base for X. Without loss of generality, we may assume $\varnothing \notin \mathcal{B}$. For each $U \in \mathcal{B}$, choose an element $x_U \in U$. Let $S = \{x_U \colon U \in \mathcal{B}\}$. Clearly S is (at most) countable. We show it is dense. So let $x \in X$ and let $\varepsilon > 0$. If $x \in S$, there is nothing to prove. Otherwise $N_\varepsilon(x)$ is an open set in X, so there exists $U \in \mathcal{B}$ such that $x \in U \subset N_\varepsilon(x)$. In particular, $x_U \in N_\varepsilon(x)$. Since $x_U \neq x$ and since $\varepsilon > 0$ is arbitrary, this shows that x is a limit point of S. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 4

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.1: Prove that if (s_n) converges, then $(|s_n|)$ converges. Is the converse true?

Solution (sketch): Use the inequality $\big||s_n| - |s|\big| \leq |s_n - s|$ and the definition of the limit. The converse is false. Take $s_n = (-1)^n$. (This requires proof, of course.) ∎

Problem 3.2: Calculate $\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right)$.

Solution (sketch):

$$\sqrt{n^2+1} - n = \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} \to \frac{1}{2}.$$

(Of course, the last step requires proof.) ∎

Problem 3.3: Let $s_1 = \sqrt{2}$, and recursively define

$$s_{n+1} = \sqrt{2+\sqrt{s_n}}$$

for $n \in \mathbf{N}$. Prove that (s_n) converges, and that $s_n < 2$ for all $n \in \mathbf{N}$.

Solution (sketch): By induction, it is immediate that $s_n > 0$ for all n, so that s_{n+1} is always defined.

Next, we show by induction that $s_n < 2$ for all n. This is clear for $n = 1$. The computation for the induction step is

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} \leq \sqrt{2+\sqrt{2}} < 2.$$

To prove convergence, it now suffices to show that (s_n) is nondecreasing. (See Theorem 3.14.) This is also done by induction. To start, observe that

$$s_2 = \sqrt{2+\sqrt{s_1}} = \sqrt{2+\sqrt{\sqrt{2}}} > \sqrt{2}.$$

The computation for the induction step is:

$$s_{n+1} - s_n = \sqrt{2+\sqrt{s_n}} - \sqrt{2+\sqrt{s_{n-1}}} = \frac{\sqrt{s_n}-\sqrt{s_{n-1}}}{\sqrt{2+\sqrt{s_n}}+\sqrt{2+\sqrt{s_{n-1}}}} > 0.$$

∎

Date: 22 October 2001.

Problem 3.4: Let $s_1 = 0$, and recursively define

$$s_{n+1} = \begin{cases} \frac{1}{2} + s_n & n \text{ is even} \\ \frac{1}{2}s_n & n \text{ is odd} \end{cases}.$$

for $n \in \mathbf{N}$. Find $\limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} s_n$.

Solution (sketch): Use induction to show that

$$s_{2m} = \frac{2^{m-1} - 1}{2^m} \quad \text{and} \quad s_{2m+1} = \frac{2^m - 1}{2^m}.$$

It follows that

$$\limsup_{n\to\infty} s_n = 1 \quad \text{and} \quad \liminf_{n\to\infty} s_n = \tfrac{1}{2}.$$

∎

Problem 3.5: Let (a_n) and (b_n) be sequences in $\mathbf{R}$. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided that the right hand side is defined, that is, not of the form $\infty - \infty$ or $-\infty + \infty$.

Four solutions are presented or sketched. The first is what I presume to be the solution Rudin intended. The second is a variation of the first, which minimizes the amount of work that must be done in different cases. The third shows what must be done if one wants to work directly from Rudin's definition. The fourth is the "traditional" proof of the result, and proceeds via the traditional definition.

Solution 1 (sketch): We give a complete solution for the case

$$\limsup_{n\to\infty} a_n \in (-\infty, \infty) \quad \text{and} \quad \limsup_{n\to\infty} b_n \in (-\infty, \infty).$$

One needs to consider several other cases, but the basic method is the same.

Define

$$a = \limsup_{n\to\infty} a_n \quad \text{and} \quad b = \limsup_{n\to\infty} b_n.$$

Let $c > a + b$. We show that c is not a subsequential limit of $(a_n + b_n)$.

Let $\varepsilon = \frac{1}{3}(c - a - b) > 0$. Use Theorem 3.17 (b) of Rudin to choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $a_n < a + \varepsilon$, and also to choose $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $b_n < b + \varepsilon$. For $n \geq \max(N_1, N_2)$, we then have $a_n + b_n < a + b + 2\varepsilon$. It follows that every subsequential limit l of $(a_n + b_n)$ satisfies $l \leq a + b + 2\varepsilon$. Since $c = a + b + 3\varepsilon > a + b + 2\varepsilon$, it follows that c is not a subsequential limit of $(a_n + b_n)$.

We conclude that $a + b$ is an upper bound for the set of subsequential limits of $(a_n + b_n)$. Therefore $\limsup_{n\to\infty}(a_n + b_n) \leq a + b$. ∎

Solution 2: This solution is a variation of Solution 1, designed to handle all cases at once. (You will see, though, that the case breakdown can't be avoided entirely.)

As in Solution 1, define

$$a = \limsup_{n\to\infty} a_n \quad \text{and} \quad b = \limsup_{n\to\infty} b_n,$$

and let $c > a + b$. We show that c is not a subsequential limit of $(a_n + b_n)$.

We first find r, s, $t \in \mathbf{R}$ such that

$$a < r, \quad b < s, \quad c > 0, \quad \text{and} \quad r + s + t \leq c.$$

If $a = \infty$ or $b = \infty$, this is vacuous, since no such c can exist. Next, suppose a and b are finite. If $c = \infty$, then

$$r = a + 1, \quad s = b + 1, \quad \text{and} \quad c = 1$$

will do. Otherwise, let $\varepsilon = \frac{1}{3}(c - a - b) > 0$, and take

$$r = a + \varepsilon, \quad s = b + \varepsilon, \quad \text{and} \quad t = \varepsilon.$$

Finally, suppose at least one of a and b is $-\infty$, but neither is ∞. Exchanging the sequences if necessary, assume that $a = -\infty$. Choose any $s > b$, choose any $t > 0$, and set $r = c - s - t$, which is certainly greater than $-\infty$.

Having r, s, and t, use Theorem 3.17 (b) of Rudin to choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $a_n < r$, and also to choose $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $b_n < s$. For $n \geq \max(N_1, N_2)$, we then have $a_n + b_n < r + s$. It follows that every subsequential limit l of $(a_n + b_n)$ satisfies $l \leq r + s$. Since $c = r + s + t > r + s$, it follows that c is not a subsequential limit of $(a_n + b_n)$.

We conclude that $a + b$ is an upper bound for the set of subsequential limits of $(a_n + b_n)$. Therefore $\limsup_{n\to\infty}(a_n + b_n) \leq a + b$. ∎

Solution 3 (sketch): We only consider the case that both $a = \limsup_{n\to\infty} a_n$ and $b = \limsup_{n\to\infty} b_n$ are finite. Let $s = \limsup_{n\to\infty}(a_n + b_n)$. Then there is a subsequence $(a_{k(n)} + b_{k(n)})$ of $(a_n + b_n)$ which converges to s. Further, $(a_{k(n)})$ is bounded. (This sequence is bounded above by assumption, and it is bounded below because $(a_{k(n)} + b_{k(n)})$ is bounded and $(b_{k(n)})$ is bounded above.) So there is a subsequence $(a_{l(n)})$ of $(a_{k(n)})$ which converges. (That is, there is a strictly increasing function $n \to r(n)$ such that the sequence $(a_{k\circ r(n)})$ converges, and we let $l = k \circ r \colon \mathbf{N} \to \mathbf{N}$. Note that if we used traditional subsequence notation, we would have the subsequence $j \mapsto a_{n_{k_j}}$ at this point.) Let $c = \lim_{n\to\infty} a_{l(n)}$. By similar reasoning to that given above, the sequence $(b_{l(n)})$ is bounded. Therefore it has a convergent subsequence, say $(b_{m(n)})$. (With traditional subsequence notation, we would now have the subsequence $i \mapsto a_{n_{k_{j_i}}}$. You can see why I don't like traditional notation.) Let $d = \lim_{n\to\infty} b_{m(n)}$. Since $(a_{m(n)})$ is a subsequence of $(a_{l(n)})$, we still have $\lim_{n\to\infty} a_{m(n)} = c$. So $\lim_{n\to\infty}(a_{m(n)} + b_{m(n)}) = c + d$. But also $a_{m(n)} + b_{m(n)}$ is a subsequence of $(a_{k(n)} + b_{k(n)})$, and so converges to s. Therefore $s = c + d$. We have $c \leq a$ and $d \leq b$ by the definition of $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$, giving the result. ∎

Solution 4 (sketch): First prove that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

(We will probably prove this result in class; otherwise, see Problem A in Homework 6. This formula is closer to the usual definition of $\limsup_{n\to\infty} x_n$, which is

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbf{N}} \sup_{k\geq n} x_k,$$

using a limit instead of an infimum.)

Then prove that

$$\sup_{k\geq n}(a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k,$$

provided that the right hand side is defined. (For example, if both terms on the right are finite, then the right hand side is clearly an upper bound for $\{a_k + b_k : k \geq n\}$.) Now take limits to get the result. ∎

Remark: It is quite possible to have

$$\limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Problem 3.21: Prove the following analog of Theorem 3.10(b): If

$$E_1 \supset E_2 \supset E_3 \supset \cdots$$

are closed bounded nonempty subsets of a complete metric space X, and if

$$\lim_{n\to\infty} \operatorname{diam}(E_n) = 0,$$

then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

Solution (sketch): It is clear that $\bigcap_{n=1}^{\infty} E_n$ can contain no more than one point, so we need to prove that $\bigcap_{n=1}^{\infty} E_n \neq \varnothing$.

For each n, choose some $x_n \in E_n$. Then, for each n, we have

$$\{x_n, x_{n+1}, \ldots\} \subset E_n,$$

whence

$$\operatorname{diam}(\{x_n, x_{n+1}, \ldots\}) \leq \operatorname{diam}(E_n).$$

Therefore (x_n) is a Cauchy sequence. Since X is complete, $x = \lim_{n\to\infty} x_n$ exists in X. Since E_n is closed, we have $x \in E_n$ for all n. So $x \in \bigcap_{n=1}^{\infty} E_n$.

Problem 3.22: Prove the Baire Category Theorem: If X is a complete metric space, and if (U_n) is a sequence of dense open subsets of X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

Note: In this formulation, the statement is true even if $X = \varnothing$.

Solution (sketch): Let $x \in X$ and let $\varepsilon > 0$. We recursively construct points $x_n \in X$ and numbers $\varepsilon_n > 0$ such that

$$d(x, x_1) < \frac{\varepsilon}{3}, \quad \varepsilon_1 < \frac{\varepsilon}{3}, \quad \varepsilon_n \to 0,$$

and

$$\overline{N_{\varepsilon_{n+1}}(x_{n+1})} \subset U_{n+1} \cap \overline{N_{\varepsilon_n}(x_n)}$$

for all n. Problem 3.21 will then imply that

$$\bigcap_{n=1}^{\infty} N_{\varepsilon_n}(x_n) \neq \varnothing.$$

(Note that $\operatorname{diam}\left(\overline{N_{\varepsilon_n}(x_n)}\right) \leq 2\varepsilon_n$.) One easily checks that

$$\bigcap_{n=1}^{\infty} N_{\varepsilon_n}(x_n) \subset N_{\varepsilon}(x) \cap \bigcap_{n=1}^{\infty} U_n.$$

Thus, we will have shown that $\bigcap_{n=1}^{\infty} U_n$ contains points arbitrarily close to x, proving density.

Since U_1 is dense in X, there is $x_1 \in U_1$ such that $d(x, x_1) < \frac{\varepsilon}{3}$. Choose $\varepsilon_1 > 0$ so small that

$$\varepsilon_1 < 1, \quad \varepsilon_1 < \frac{\varepsilon}{3}, \quad \text{and} \quad N_{2\varepsilon_1}(x_1) \subset U_1.$$

Then also

$$\overline{N_{\varepsilon_1}(x_1)} \subset U_1.$$

Given ε_n and x_n, use the density of U_{n+1} in X to choose

$$x_{n+1} \in U_{n+1} \cap N_{\varepsilon_n/2}(x_n).$$

Choose $\varepsilon_{n+1} > 0$ so small that

$$\varepsilon_{n+1} < \frac{1}{n+1}, \quad \varepsilon_{n+1} < \frac{\varepsilon_n}{2}, \quad \text{and} \quad N_{2\varepsilon_{n+1}}(x_{n+1}) \subset U_{n+1}.$$

Then also

$$\overline{N_{\varepsilon_{n+1}}(x_{n+1})} \subset U_{n+1}.$$

This gives all the required properties. (We have $\varepsilon_n \to 0$ since $\varepsilon_n < \frac{1}{n}$ for all n.) ∎

Note: We don't really need to use Problem 3.21 here. If we always require $\varepsilon_n < 2^{-n}$ in the argument above, we will get $d(x_n, x_{n+1}) < 2^{-n-1}$ for all n. This inequality implies that (x_n) is a Cauchy sequence.

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 5

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.6: Investigate the convergence or divergence of the following series.

(Note: I have supplied lower limits of summation, which I have chosen for maximum convenience. Of course, convergence is independent of the lower limit, provided none of the individual terms is infinite.)

(a)

$$\sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n} \right).$$

Solution: The n-th partial sum is

$$\left(\sqrt{n+1} - \sqrt{n} \right) + \left(\sqrt{n} - \sqrt{n-1} \right) + \cdots + \left(\sqrt{1} - \sqrt{0} \right) = \sqrt{n+1}.$$

We have $\lim_{n \to \infty} \sqrt{n} = \infty$, so $\lim_{n \to \infty} \sqrt{n+1} = \infty$. Therefore the series diverges. ∎

Remark: This sort of series is known as a telescoping series. The more interesting cases of telescoping series are the ones that converge.

Alternate solution: We calculate:

$$\begin{aligned} \sqrt{n+1} - \sqrt{n} &= \left(\sqrt{n+1} - \sqrt{n} \right) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2\sqrt{n+1}}. \end{aligned}$$

Now $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by Theorem 3.28 of Rudin. Therefore $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges, and hence so does $\sum_{n=0}^{\infty} \frac{1}{2\sqrt{n+1}}$. So the comparison test implies that

$$\sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n} \right)$$

diverges. ∎

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}.$$

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Solution (sketch):

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} \leq \frac{1}{n^{3/2}}.$$

Therefore the series converges by the comparison test. ∎

(c)

$$\sum_{n=1}^{\infty}\left(\sqrt[n]{n}-1\right)^{n}.$$

Solution: We use the root test (Theorem 3.33 of Rudin). With $a_n = \left(\sqrt[n]{n}-1\right)^n$, we have $\sqrt[n]{a_n} = \sqrt[n]{n}-1$. Theorem 3.20 (c) of Rudin implies that $\lim_{n\to\infty}\sqrt[n]{n} = 1$. Therefore $\lim_{n\to\infty}\sqrt[n]{a_n} = 0$. Since $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$, convergence follows.

Alternate solution (sketch): Let $x_n = \sqrt[n]{n}-1$, so that

$$\left(\sqrt[n]{n}-1\right)^n = x_n^n \quad \text{and} \quad (1+x_n)^n = n.$$

The binomial formula implies that

$$n = (1+x_n)^n = 1 + nx_n + \frac{n(n-1)}{2}\cdot x_n^2 + \cdots \geq \frac{n(n-1)}{2}\cdot x_n^2,$$

from which it follows that

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$$

for $n \geq 2$. Hence, for $n \geq 4$,

$$\left(\sqrt[n]{n}-1\right)^n = x_n^n \leq \left(\frac{2}{n-1}\right)^{n/2} \leq \left(\left(\tfrac{2}{3}\right)^{1/2}\right)^n.$$

Since $\left(\frac{2}{3}\right)^{1/2} < 1$, the series converges by the comparison test. ∎

(d)

$$\sum_{n=0}^{\infty}\frac{1}{1+z^n},$$

for $z \in \mathbf{C}$ arbitrary.

Solution: We show that the series converges if and only if $|z| > 1$.

If $z = \exp(2\pi i r)$, with $r = k/l$, with k an odd integer and l an even integer, then $z^n = -1$ for infinitely many values of n, so that infinitely many of the terms of the series are undefined. Convergence is therefore clearly impossible.

In all other cases with $|z| \leq 1$, we have

$$|1+z^n| \leq 1 + |z^n| \leq 1+1 = 2,$$

which implies that

$$\left|\frac{1}{1+z^n}\right| \geq \frac{1}{2}.$$

The terms thus don't converge to 0, and again the series diverges.

Now let $|z| > 1$. Then $|1 + z^n| \geq |z^n| - 1 = |z|^n - 1$. Choose N such that if $n \geq N$ then $|z|^n > 2$. For such n, we have

$$\frac{|z|^n}{2} > 1,$$

whence

$$|1 + z^n| \geq |z|^n - 1 = \frac{|z|^n}{2} + \left(\frac{|z|^n}{2} - 1\right) > \frac{|z|^n}{2}.$$

So

$$\left|\frac{1}{1 + z^n}\right| \leq 2 \cdot \frac{1}{|z|^n}$$

for all $n \geq N$. Since $|z| > 1$, the comparison test implies that

$$\sum_{n=0}^{\infty} \frac{1}{1 + z^n}$$

converges. ∎

Problem 3.7: Let $a_n \geq 0$ for $n \in \mathbf{N}$. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{a_n}$ converges.

Solution (sketch): Using the inequality $2ab \leq a^2 + b^2$, we get

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right).$$

Since both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge, $\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{a_n}$ converges by the comparison test. ∎

Problem 3.8: Let (b_n) be a bounded monotone sequence in $\mathbf{R}$, and let $a_n \in \mathbf{C}$ be such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution (sketch): We first reduce to the case $\lim_{n \to \infty} b_n = 0$. Since (b_n) is a bounded monotone sequence, it follows that $b = \lim_{n \to \infty} b_n$ exists. Set $c_n = b_n - b$. Then (c_n) is a bounded monotone sequence with $\lim_{n \to \infty} c_n = 0$. Since $a_n b_n = a_n c_n + a_n b$ and $\sum_{n=1}^{\infty} a_n b$ converges, it suffices to prove that $\sum_{n=1}^{\infty} a_n c_n$ converges. That is, we may assume that $\lim_{n \to \infty} b_n = 0$.

With this assumption, if $b_1 \geq 0$, then $b_1 \geq b_2 \geq \cdots \geq 0$, so $\sum_{n=1}^{\infty} a_n b_n$ converges by Theorem 3.42 in the book. Otherwise, replace b_n by $-b_n$. ∎

Problem 3.9: Find the radius of convergence of each of the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} n^3 z^n$.

Solution 1: Use Theorem 3.20 (c) of Rudin in the second step to get

$$\limsup_{n \to \infty} \sqrt[n]{n^3} = \left(\lim_{n \to \infty} \sqrt[n]{n}\right)^3 = 1.$$

It now follows from Theorem 3.39 of Rudin that the radius of convergence is 1. ∎

Solution 2: We show that the series converges for $|z| < 1$ and diverges for $|z| > 1$.

For $|z| = 0$, convergence is trivial.

For $0 < |z| < 1$, we use the ratio test (Theorem 3.34 of Rudin). We have

$$\lim_{n\to\infty} \frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} = \lim_{n\to\infty} |z| \left(\frac{n+1}{n}\right)^3 = |z| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3$$
$$= |z| \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^3 = |z|.$$

For $|z| < 1$ the hypotheses of Theorem 3.34 (a) of Rudin are therefore satisfied, so that the series converges.

For $|z| > 1$, we use the ratio test. The same calculation as in the case $0 < |z| < 1$ gives

$$\lim_{n\to\infty} \frac{|(n+1)^3 z^{n+1}|}{n^3 z^n|} = |z|.$$

Since $|z| > 1$, it follows that there is N such that for all $n \geq N$ we have

$$\left| \frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} - |z| \right| < \tfrac{1}{2}(|z| - 1).$$

In particular,

$$\frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} > 1$$

for $n \geq N$. The hypotheses of Theorem 3.34 (b) of Rudin are therefore satisfied, so that the series diverges.

Theorem 3.39 of Rudin implies that there is some number $R \in [0, \infty]$ such that the series converges for $|z| < R$ and diverges for $|z| > R$. We have therefore shown that $R = 1$. ∎

Solution 3: We calculate

$$\lim_{n\to\infty} \frac{|(n+1)^3|}{|n^3|} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^3 = 1.$$

According to Theorem 3.37 of Rudin, we have

$$\liminf_{n\to\infty} \frac{|(n+1)^3|}{|n^3|} \leq \liminf_{n\to\infty} \sqrt[n]{|n^3|} \leq \limsup_{n\to\infty} \sqrt[n]{|n^3|} \leq \limsup_{n\to\infty} \frac{|(n+1)^3|}{|n^3|}.$$

Therefore $\lim_{n\to\infty} \sqrt[n]{|n^3|}$ exists and is equal to 1. It now follows from Theorem 3.39 of Rudin that the radius of convergence is 1. ∎

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot z^n.$

Solution (sketch): Use the ratio test to show that the series converges for all z. (See Solution 2 to Part (a).) So the radius of convergence is ∞. ∎

Remark: Note that $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot z^n = e^{2z}.$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^2} \cdot z^n.$

Solution (sketch): Either the root or ratio test gives radius of convergence equal to $\frac{1}{2}$. (Use the methods of any of the three solutions to Part (a).) ∎

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n} \cdot z^n$.

Solution (sketch): Either the root or ratio test gives radius of convergence equal to 3. (Use the methods of any of the three solutions to Part (a).) ∎

Problem 3.10: Suppose the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ are integers, infinitely many of which are nonzero. Prove that the radius of convergence is at most 1.

Solution 1: For infinitely many n, the numbers a_n are nonzero integers, and therefore satisfy $|a_n| \geq 1$. So, if $|z| \geq 1$, then infinitely many of the terms $a_n z^n$ have absolute value $|a_n z^n| \geq |a_n| \geq 1$, and the terms of the series $\sum_{n=0}^{\infty} a_n z^n$ don't approach zero. This shows that $\sum_{n=0}^{\infty} a_n z^n$ diverges for $|z| \geq 1$, and therefore that its radius of convergence is at most 1. ∎

Solution 2 (Sketch): There is a subsequence $(a_{k(n)})$ of (a_n) such that $|a_{k(n)}| \geq 1$ for all n. So $\sqrt[k(n)]{|a_{k(n)}|} \geq 1$ for all n, whence $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$. ∎

Remarks: (1) It is quite possible that infinitely many of the a_n are zero, so that $\liminf_{n\to\infty} \sqrt[n]{|a_n|}$ could be zero. For example, we could have $a_n = 0$ for all odd n.

(2) In Solution 2, the expression $\sqrt[n]{|a_{k(n)}|}$, and its possible limit as $n \to \infty$, have no relation to the radius of convergence.

Problem 3.16: Fix $\alpha > 0$. Choose $x_1 > \sqrt{\alpha}$, and recursively define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that (x_n) is nonincreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

Solution (sketch): Using the inequality $a^2 + b^2 \geq 2ab$, and assuming $x_n > 0$, we get

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \sqrt{x_n} \cdot \sqrt{\frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

This shows (using induction) that $x_n > \sqrt{\alpha}$ for all n. Next,

$$x_n - x_{n+1} = \frac{x_n^2 - \alpha}{2x_n} > 0.$$

Thus (x_n) is nonincreasing. We already know that this sequence is bounded below (by α), so $x = \lim_{n\to\infty} x_n$ exists. Letting $n \to \infty$ in the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

gives

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right).$$

This equation implies $x = \pm\sqrt{\alpha}$, and we must have $x = \sqrt{\alpha}$ because (x_n) is bounded below by $\sqrt{\alpha} > 0$. ∎

(b) Set $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}.$$

Further show that, with $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}.$$

Solution (sketch): Prove all the relations at once by induction on n, together with the statement $\varepsilon_{n+1} > 0$. For $n = 1$, the relation $\varepsilon_{n+1} = \dfrac{\varepsilon_n^2}{2x_n}$ is just algebra, the inequality $\dfrac{\varepsilon_n^2}{2x_n} < \dfrac{\varepsilon_n^2}{2\sqrt{\alpha}}$ follows from $x_1 > \sqrt{\alpha}$, and the inequality $\varepsilon_{n+1} < \beta \left(\dfrac{\varepsilon_1}{\beta} \right)^{2^n}$ is just a rewritten form of $\dfrac{\varepsilon_n^2}{2x_n} < \dfrac{\varepsilon_n^2}{2\sqrt{\alpha}}$. The statement $\varepsilon_{n+1} > 0$ is clear from $\varepsilon_{n+1} = \dfrac{\varepsilon_n^2}{2x_n}$ and $x_1 > 0$.

Now assume all this is known for some value of n. As before, the relation $\varepsilon_{n+1} = \dfrac{\varepsilon_n^2}{2x_n}$ is just algebra, and implies that $\varepsilon_{n+1} > 0$. (We know that $x_n = \sqrt{\alpha} + \varepsilon_n > \sqrt{\alpha} > 0$.) Since $\varepsilon_n > 0$, we have $x_n > \sqrt{\alpha}$, so the inequality $\dfrac{\varepsilon_n^2}{2x_n} < \dfrac{\varepsilon_n^2}{2\sqrt{\alpha}}$ follows. To get the other inequality, write

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} = \frac{\varepsilon_n^2}{\beta} < \left(\frac{1}{\beta} \right) \cdot \left[\beta \cdot \left(\frac{\varepsilon_1}{\beta} \right)^{2^{n-1}} \right]^2 = \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}.$$

∎

(c) Specifically take $\alpha = 3$ and $x_1 = 2$. show that

$$\frac{\varepsilon_1}{\beta} < \frac{1}{10}, \quad \varepsilon_5 < 4 \cdot 10^{-16}, \quad \text{and} \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution (sketch): This is just calculation. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 6

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.23: Let X be a metric space, and let $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ be Cauchy sequences in X. Prove that $\lim_{n \to \infty} d(x_n, y_n)$ exists.

Solution (sketch): Since $\mathbf{R}$ is complete, it suffices to show that $(d(x_n, y_n))_{n \in \mathbf{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$. Choose N so large that if $m, n \geq N$, then both $d(x_m, x_n) < \frac{\varepsilon}{2}$ and $d(y_m, y_n) < \frac{\varepsilon}{2}$. Then check that, for such m and n,

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n) < \varepsilon.$$

∎

Problem 3.24: Let X be a metric space.

(a) Let $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ be Cauchy sequences in X. We say they are equivalent, and write $(x_n)_{n \in \mathbf{N}} \sim (y_n)_{n \in \mathbf{N}}$, if $\lim_{n \to \infty} d(x_n, y_n) = 0$. Prove that this is an equivalence relation.

Solution (sketch): That $(x_n)_{n \in \mathbf{N}} \sim (x_n)_{n \in \mathbf{N}}$, and that $(x_n)_{n \in \mathbf{N}} \sim (y_n)_{n \in \mathbf{N}}$ implies $(y_n)_{n \in \mathbf{N}} \sim (x_n)_{n \in \mathbf{N}}$, are obvious. For transitivity, assume $(x_n)_{n \in \mathbf{N}} \sim (y_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}} \sim (z_n)_{n \in \mathbf{N}}$. Then $(x_n)_{n \in \mathbf{N}} \sim (z_n)_{n \in \mathbf{N}}$ follows by taking limits in the inequality

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n).$$

∎

(b) Let X^* be the set of equivalence classes from Part (a). Denote by $[(x_n)_{n \in \mathbf{N}}]$ the equivalence class in X^* of the Cauchy sequence $(x_n)_{n \in \mathbf{N}}$. If $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ are Cauchy sequences in X, set

$$\Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}}) = \lim_{n \to \infty} d(x_n, y_n).$$

Prove that $\Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}})$ only depends on $[(x_n)_{n \in \mathbf{N}}]$ and $[(y_n)_{n \in \mathbf{N}}]$. Moreover, show that the formula

$$\Delta([(x_n)_{n \in \mathbf{N}}], [(y_n)_{n \in \mathbf{N}}]) = \Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}})$$

defines a metric on X^*.

Solution (sketch): It is easy to check that Δ_0 is a semimetric, that is, it satisfies all the conditions for a metric except that possibly

$$\Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}}) = 0$$

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without having $(x_n)_{n\in\mathbf{N}} = (y_n)_{n\in\mathbf{N}}$. (For example,

$$\begin{aligned}\Delta_0((x_n)_{n\in\mathbf{N}},\, (z_n)_{n\in\mathbf{N}}) &= \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &= \Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}) + \Delta_0((y_n)_{n\in\mathbf{N}},\, (z_n)_{n\in\mathbf{N}})\end{aligned}$$

because $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$; the other properties are proved similarly.) We further note that, by definition, $(x_n)_{n\in\mathbf{N}} \sim (y_n)_{n\in\mathbf{N}}$ if and only if $\Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}) = 0$.

Now we prove that $\Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}})$ only depends on

$$[(x_n)_{n\in\mathbf{N}}] \quad \text{and} \quad [(y_n)_{n\in\mathbf{N}}].$$

Let $(x_n)_{n\in\mathbf{N}} \sim (r_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}} \sim (s_n)_{n\in\mathbf{N}}$. Then, by the previous paragraph,

$$\begin{aligned}&\Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}) \\ &\leq \Delta_0((x_n)_{n\in\mathbf{N}},\, (r_n)_{n\in\mathbf{N}}) + \Delta_0((r_n)_{n\in\mathbf{N}},\, (s_n)_{n\in\mathbf{N}}) + \Delta_0((s_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}) \\ &= 0 + \Delta_0((r_n)_{n\in\mathbf{N}},\, (s_n)_{n\in\mathbf{N}}) + 0 = \Delta_0((r_n)_{n\in\mathbf{N}},\, (s_n)_{n\in\mathbf{N}});\end{aligned}$$

similarly

$$\Delta_0((r_n)_{n\in\mathbf{N}},\, (s_n)_{n\in\mathbf{N}}) \leq \Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}).$$

Thus

$$\Delta_0((r_n)_{n\in\mathbf{N}},\, (s_n)_{n\in\mathbf{N}}) = \Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}).$$

The previous paragraph implies that Δ is well defined. It is now easy to check that Δ satisfies all the conditions for a metric except that possibly

$$\Delta([(x_n)_{n\in\mathbf{N}}], [(y_n)_{n\in\mathbf{N}}]) = 0$$

without having $[(x_n)_{n\in\mathbf{N}}] = [(y_n)_{n\in\mathbf{N}}]$. (For example,

$$\begin{aligned}\Delta([(x_n)_{n\in\mathbf{N}}],\, [(z_n)_{n\in\mathbf{N}}]) &= \Delta_0((x_n)_{n\in\mathbf{N}},\, (z_n)_{n\in\mathbf{N}}) \\ &\leq \Delta_0((x_n)_{n\in\mathbf{N}},\, (y_n)_{n\in\mathbf{N}}) + \Delta_0((y_n)_{n\in\mathbf{N}},\, (z_n)_{n\in\mathbf{N}}) \\ &= \Delta([(x_n)_{n\in\mathbf{N}}],\, [(y_n)_{n\in\mathbf{N}}]) + \Delta_0([(y_n)_{n\in\mathbf{N}}],\, [(z_n)_{n\in\mathbf{N}}]);\end{aligned}$$

the other properties are proved similarly.)

Finally, if $\Delta([(x_n)_{n\in\mathbf{N}}],\, [(y_n)_{n\in\mathbf{N}}]) = 0$ then it follows from the definition of $(x_n)_{n\in\mathbf{N}} \sim (y_n)_{n\in\mathbf{N}}$ that we actually do have $[(x_n)_{n\in\mathbf{N}}] = [(y_n)_{n\in\mathbf{N}}]$. So Δ is a metric. ∎

(c) Prove that X^* is complete in the metric Δ.

The basic idea is as follows. We start with a Cauchy sequence in X^*, which is a sequence of (equivalence classes of) Cauchy sequences in X. The limit is supposed to be (the equivalence class of) another Cauchy sequence in X. This sequence is constructed by taking suitable terms from the given sequences. The choices get a little messy. Afterwards, we will give a different proof.

Solution (sketch): Let $(a_k)_{k\in\mathbf{N}}$ be a Cauchy sequence in X^*; we show that it converges. Each a_k is an equivalence class of Cauchy sequences in X. We may therefore write $a_k = [(x_n^{(k)})_{n\in\mathbf{N}}]$, where each $(x_n^{(k)})_{n\in\mathbf{N}}$ is a Cauchy sequence in X. The limit we construct in X^* will have the form $a = [(x_n^{(f(n))})]$ for a suitable function $f\colon \mathbf{N} \to \mathbf{N}$.

We recursively construct

$$M(1) < M(2) < M(3) < \cdots \quad \text{and} \quad N(1) < N(2) < N(3) < \cdots$$

such that

(1) $\Delta(a_k, a_l) < 2^{-r-2}$ for $k, l \geq M(r)$.
(2) For all $k, l \leq M(r)$ and $m \geq N(r)$, we have $d(x_m^{(k)}, x_m^{(l)}) < \Delta(a_k, a_l) + 2^{-r-2}$.
(3) For all $k \leq M(r)$ and $m, n \geq N(r)$, we have $d(x_m^{(k)}, x_n^{(k)}) < 2^{-r-2}$.

To do this, first use the fact that $(a_k)_{k \in \mathbf{N}}$ is a Cauchy sequence to find $M(1)$. Then choose $N(1)$ large enough to satisfy (2) and (3) for $r = 1$; this can be done because $\lim_{m \to \infty} d(x_m^{(k)}, x_m^{(l)}) = \Delta(a_k, a_l)$ (for (2)) and because $(x_n^{(k)})_{n \in \mathbf{N}}$ is Cauchy (for (3)), and using the fact that there are only finitely many pairs (k, l) to consider in (2) and only finitely many k to consider in (3). Next, use the fact that $(a_k)_{k \in \mathbf{N}}$ is a Cauchy sequence to find $M(2)$, and also require $M(2) > M(1)$. Choose $N(2) > N(1)$ by the same reasoning as used to get $N(1)$. Proceed recursively.

We now take a to be the equivalence class of the sequence

$$x_1^{(1)}, \ldots, x_{N(1)-1}^{(1)}, x_{N(1)}^{(M(1))}, \ldots, x_{N(2)-1}^{(M(1))}, x_{N(2)}^{(M(2))}, \ldots, x_{N(3)-1}^{(M(2))}, x_{N(3)}^{(M(3))}, \ldots.$$

That is, the function f above is given by $f(n) = M(r)$ for $N(r) \leq n \leq N(r+1) - 1$.

We show that $(x_n^{(f(n))})_{n \in \mathbf{N}}$ is Cauchy. First, estimate:

$$\begin{aligned} d\left(x_{N(r)}^{(M(r))}, x_{N(r+1)}^{(M(r+1))}\right) &\leq d\left(x_{N(r)}^{(M(r))}, x_{N(r+1)}^{(M(r))}\right) + d\left(x_{N(r+1)}^{(M(r))}, x_{N(r+1)}^{(M(r+1))}\right) \\ &< 2^{-r-2} + \Delta(a_{M(r)}, a_{M(r+1)}) + 2^{-r-3} \\ &< 2^{-r-2} + 2^{-r-2} + 2^{-r-3} < 3 \cdot 2^{-r-2}. \end{aligned}$$

The first term on the second line is gotten from (2) above, because $M(r) \leq M(r)$ and $N(r), N(r+1) \geq N(r)$. The other two terms on the second line are gotten from (3) above (for $r+1$)), because $M(r), M(r+1) \leq M(r+1)$ and $N(r+1) \geq N(r+1)$. The estimate used to get the third line comes from (1) above. Then use induction to show that $s \geq r$ implies

$$d\left(x_{N(r)}^{(M(r))}, x_{N(s)}^{(M(s))}\right) \leq 3[2^{-r-2} + 2^{-r-3} + \cdots + 2^{-s-1}].$$

Now let $n \geq N(r)$ be arbitrary. Choose $s \geq r$ such that $N(s) \leq n \leq N(s+1) - 1$. Then $f(n) = M(s)$, so

$$d\left(x_n^{(f(n))}, x_{N(s)}^{(M(s))}\right) = d\left(x_n^{(M(s))}, x_{N(s)}^{(M(s))}\right) < 3 \cdot 2^{-s-2},$$

using (3) above with $r = s$ and $k = M(s)$. Therefore

$$d\left(x_{N(r)}^{(M(r))}, x_n^{(f(n))}\right) < 3[2^{-r-2} + 2^{-r-3} + \cdots + 2^{-s-1} + 2^{-s-2}] < 3 \cdot 2^{-r-1}.$$

Finally, if $m, n \geq N(r)$ are arbitrary, then

$$d\left(x_m^{(f(m))}, x_n^{(f(n))}\right) \leq d\left(x_m^{(f(m))}, x_{N(r)}^{(M(r))}\right) + d\left(x_{N(r)}^{(M(r))}, x_n^{(f(n))}\right) < 3 \cdot 2^{-r}.$$

This is enough to prove that $(x_n^{(f(n))})_{n \in \mathbf{N}}$ is Cauchy.

It remains to show that $\Delta(a_k, a) \to 0$. Fix k, choose r with $M(r-1) < k \leq M(r)$, and let $n \geq N(r)$. From the previous paragraph we have

$$d\left(x_{N(r)}^{(M(r))}, x_n^{(f(n))}\right) < 3 \cdot 2^{-r-1}.$$

Since $n, N(r) \geq N(r)$ and $k \leq M(r)$, condition (3) above gives

$$d\left(x_n^{(k)}, x_{N(r)}^{(k)}\right) < 2^{-r-2}.$$

Furthermore,

$$d\left(x_{N(r)}^{(k)},\, x_{N(r)}^{(M(r))}\right) < \Delta(a_k,\, a_{M(r)}) + 2^{-r-2} < 2^{-r-1} + 2^{-r-2},$$

where the first step uses (2) above and the inequalities $k,\, M(r) \leq M(r)$ and $N(r) \geq N(r)$, while the second step uses (1) above and the inequality $r \geq M(r-1)$. Combining these estimates using the triangle inequality, we get

$$d(x_n^{(k)},\, x_n^{(f(n))}) < 2^{-r-2} + [2^{-r-1} + 2^{-r-2}] + 3 \cdot 2^{-r-1} < 2^{-r+2}.$$

Therefore

$$\Delta(a_k, a) = \lim_{n \to \infty} d(x_n^{(k)},\, x_n^{(f(n))}) \leq 2^{-r+2}$$

for $M(r-1) < k \leq M(r)$. Since $M(r) \to \infty$, this implies that $\Delta(a_k, a) \to 0$. ∎

Here is a perhaps slicker way to do the same thing, although it isn't any shorter. Essentially, by passing to suitable subsequences, we can take the representative of the limit to be the diagonal sequence, that is, $f(n) = n$ in the proof above. The construction requires the following lemmas.

Lemma 1. Let $(x_n)_{n \in \mathbf{N}}$ be a Cauchy sequence in a metric space Y. Then there is a subsequence $(x_{k(n)})_{n \in \mathbf{N}}$ of $(x_n)_{n \in \mathbf{N}}$ such that $d(x_{k(n+1)},\, x_{k(n)}) < 2^{-n}$ for all n.

Proof (sketch): Choose $k(n)$ recursively to satisfy $k(n+1) > k(n)$ and $d(x_l, x_m) < 2^{-n}$ for all $l,\, m \geq k(n)$. ∎

Lemma 2. Let $(x_n)_{n \in \mathbf{N}}$ be a sequence in a metric space Y. Suppose

$$\sum_{k=1}^{\infty} d(x_k,\, x_{k+1})$$

converges. Then $(x_n)_{n \in \mathbf{N}}$ is Cauchy. Moreover, if $n > m$ then

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k,\, x_{k+1}).$$

Proof (sketch): The Cauchy criterion for convergence of a series implies that for all $\varepsilon > 0$, there is N such that if $n > m \geq N$, then $\sum_{k=m}^{n-1} d(x_k,\, x_{k+1}) < \varepsilon$. But the triangle inequality gives $d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k,\, x_{k+1})$. Thus if $n > m \geq N$ then $d(x_m, x_n) < \varepsilon$. The case $m > n \geq N$ is handled by symmetry, and the case $n = m \geq N$ is trivial. ∎

Lemma 3. Let $(x_n)_{n \in \mathbf{N}}$ be a Cauchy sequence in a metric space Y, and let $(x_{k(n)})_{n \in \mathbf{N}}$ be a subsequence. Then $\lim_{n \to \infty} d(x_n,\, x_{k(n)}) = 0$.

Proof (sketch): Let $\varepsilon > 0$. Choose N such that if $m,\, n \geq N$ then $d(x_m, x_n) < \varepsilon$. If $n \geq N$, then $k(n) \geq n \geq N$, so $d(x_n,\, x_{k(n)}) < \varepsilon$. ∎

Lemma 4. Let $(x_n)_{n \in \mathbf{N}}$ be a Cauchy sequence in a metric space Y. If $(x_n)_{n \in \mathbf{N}}$ has a convergent subsequence, then $(x_n)_{n \in \mathbf{N}}$ converges.

Proof (sketch): Let $(x_{k(n)})_{n \in \mathbf{N}}$ be a subsequence with limit x. Then, using Lemma 3,

$$d(x_n, x) \leq d(x_n,\, x_{k(n)}) + d(x_{k(n)},\, x) \to 0.$$

∎

Proof of the result: Let (a_k) be a Cauchy sequence in X^*; we show that it converges. Use Lemma 1 to choose a subsequence $(a_{r(k)})_{k \in \mathbf{N}}$ such that

$$\Delta(a_{r(k)},\, a_{r(k+1)}) < 2^{-k}$$

for all k. By Lemma 4, it suffices to show that $(a_{r(k)})_{k \in \mathbf{N}}$ converges. Without loss of generality, therefore, we may assume the original sequence $(a_k)_{k \in \mathbf{N}}$ satisfies $\Delta(a_k,\, a_{k+1}) < 2^{-k}$ for all k.

Each a_k is an equivalence class of Cauchy sequences in X. By Lemmas 1 and 3, we may write $a_k = [(x_n^{(k)})_{n \in \mathbf{N}}]$, with $d(x_n^{(k)},\, x_{n+1}^{(k)}) < 2^{-n}$ for all n.

We now estimate $d(x_n^{(k)},\, x_n^{(k+1)})$. For $\varepsilon > 0$, we can find $m > n$ such that

$$\begin{aligned} d(x_m^{(k)},\, x_m^{(k+1)}) &< \Delta([(x_n^{(k)})_{n \in \mathbf{N}}],\, [(x_n^{(k+1)})_{n \in \mathbf{N}}]) + \varepsilon \\ &= \Delta(a_k,\, a_{k+1}) + \varepsilon < 2^{-k} + \varepsilon. \end{aligned}$$

Now, using Lemma 2,

$$d(x_n^{(k)},\, x_m^{(k)}) < 2^{-n} + 2^{-n-1} + \cdots + 2^{-m+1} < 2^{-n+1}.$$

The same estimate holds for $d(x_n^{(k+1)},\, x_m^{(k+1)})$. Therefore

$$\begin{aligned} d(x_n^{(k)},\, x_n^{(k+1)}) &\leq d(x_n^{(k)},\, x_m^{(k)}) + d(x_m^{(k)},\, x_m^{(k+1)}) + d(x_m^{(k+1)},\, x_n^{(k+1)}) \\ &< 2^{-n+1} + \varepsilon + 2^{-n+1}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives

$$d(x_n^{(k)},\, x_n^{(k+1)}) \leq 2^{-n+2}$$

for all n and k.

Now define $y_n = x_n^{(n)}$. First, observe that

$$\begin{aligned} d(y_n,\, y_{n+1}) &\leq d(x_n^{(n)},\, x_{n+1}^{(n)}) + d(x_{n+1}^{(n)},\, x_{n+1}^{(n+1)}) \\ &\leq 2^{-n} + 2^{-n+1} < 2^{-n+2}. \end{aligned}$$

Therefore $(y_n)_{n \in \mathbf{N}}$ is Cauchy, by Lemma 2. So $a = [(y_n)_{n \in \mathbf{N}}] \in X^*$.

It remains to show that $\Delta(a_n, a) \to 0$. If $m > n$, then we use the estimates $d(x_n^{(n)},\, x_m^{(n)}) < 2^{-n+1}$ (as above) and $d(y_n, y_m) < 2^{-n+3}$ (obtained similarly, using Lemma 2 again) to get

$$d(x_m^{(n)},\, y_m) \leq d(x_m^{(n)},\, x_n^{(n)}) + d(y_n, y_m) < 2^{-n+4}.$$

In particular,

$$\Delta(a_n, a) = \lim_{m \to \infty} d(x_m^{(n)},\, y_m) \leq 2^{-n+4}.$$

Thus $\Delta(a_n, a) \to 0$, as desired. ∎

(d) Define $f \colon X \to X^*$ by $f(x) = [(x, x, x, \dots)]$. Prove that f is isometric, that is, that $\Delta(f(x), f(y)) = d(x, y)$ for all $x,\, y \in X$.

Solution (sketch): This is immediate. ∎

(e) Prove that $f(X)$ is dense in X^*, and that $f(X) = X^*$ if X is complete.

Solution (sketch): To prove density, let $[(x_n)_{n \in \mathbf{N}}] \in X^*$, and let $\varepsilon > 0$. Choose N such that if $m,\, n \geq N$ then $d(x_m, x_n) < \frac{\varepsilon}{2}$. Then

$$\Delta(f(x_N),\, [(x_n)_{n \in \mathbf{N}}]) = \lim_{n \to \infty} d(x_N, x_n),$$

which is at most $\frac{\varepsilon}{2}$ because $d(x_N, x_n) < \frac{\varepsilon}{2}$ for $n \geq N$. (Note that the limit exists by Problem 3.23.) In particular, $\Delta(f(x_N), [(x_n)_{n \in \mathbf{N}}]) < \varepsilon$.

Now assume X is complete. Then $f(X)$ is complete, because f is isometric. Therefore it suffices to prove that a complete subset of a metric space is closed. (A subset of a metric space which is both closed and dense must be equal to the whole space.)

Accordingly, let Y be a metric space, and let $E \subset Y$ be a complete subset. Let $(x_n)_{n \in \mathbf{N}}$ be a sequence in E which converges to some point $y \in Y$; we show $y \in E$. (By a theorem proved in class, this is sufficient to verify that E is closed.) Now $(x_n)_{n \in \mathbf{N}}$ converges, and is therefore Cauchy. Since E is complete, there is $x \in E$ such that $x_n \to x$. By uniqueness of limits, we have $x = y$. Thus $y \in E$, as desired. ∎

Problem A. Prove the equivalence of four definitions of the lim sup of a sequence. That is, prove the following theorem.

Theorem. Let (a_n) be a sequence in $\mathbf{R}$. Let E be the set of all subsequential limits of (a_n) in $[-\infty, \infty]$. Define numbers r, s, t, and $u \in [-\infty, \infty]$ as follows:

(1) $r = \sup(E)$.
(2) $s \in E$ and for every $x > s$, there is $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n < x$.
(3) $t = \inf_{n \in \mathbf{N}} \sup_{k \geq n} a_k$.
(4) $u = \lim_{n \to \infty} \sup_{k \geq n} a_k$.

Prove that s is uniquely determined by (2), that the limit in (4) exists in $[-\infty, \infty]$, and that $r = s = t = u$.

Note: You do not need to repeat the part that is done in the book (Theorem 3.17).

Solution: Theorem 3.17 of Rudin implies that s is uniquely determined by (2) and that $s = r$.

Define $b_n = \sup_{k \geq n} a_k$, which exists in $(-\infty, \infty]$. We clearly have

$$\{a_k : k \geq n+1\} \subset \{a_k : k \geq n\},$$

so that $\sup_{k \geq n+1} a_k \leq \sup_{k \geq n} a_k$. This shows that the sequence in (4), which has values in $(-\infty, \infty]$, is nonincreasing. Therefore it has a limit $u \in [-\infty, \infty]$, and moreover

$$u = \inf_{n \in \mathbf{N}} b_n = \inf_{n \in \mathbf{N}} \sup_{k \geq n} a_k = t.$$

We finish the proof by showing that $s \leq t$ and $t \leq r$.

To show that $s \leq t$, let $x > s$; we show that $x > t$. Choose y with $x > y > s$. By the definition of s, there is $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n < y$. This implies that $y \geq \sup_{n \geq N} a_n$, so that $x > \sup_{n \geq N} a_n$. It follows that x is not a lower bound for $\{\sup_{n \geq N} a_n : N \in \mathbf{N}\}$. So $x > t$ by the definition of a greatest lower bound.

To show that $t \geq r$, let $x > t$; we show that $x \geq r$. Since $x > t$, it follows that x is not a lower bound for the set $\{\sup_{n \geq N} a_n : N \in \mathbf{N}\}$. Accordingly, there is $N_0 \in \mathbf{N}$ such that $\sup_{n \geq N_0} a_n < x$. In particular, $n \geq N_0$ implies $a_n < x$. Now let $(a_{k(n)})$ be any convergent subsequence of (a_n). Choose $N \in \mathbf{N}$ such that $n \geq N$ implies $k(n) \geq N_0$. Then $n \geq N$ implies $a_{k(n)} < x$, from which it follows that $\lim_{n \to \infty} a_{k(n)} \leq x$. This shows that x is an upper bound for the set E, so that $x \geq \sup(E) = r$. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 7

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.1: Let $f \colon \mathbf{R} \to \mathbf{R}$ satisfy $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ for all $x \in \mathbf{R}$. Is f necessarily continuous?

Solution (Sketch): No. The simplest counterexample is

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

More generally, let $f_0 \colon \mathbf{R} \to \mathbf{R}$ be continuous. Fix $x_0 \in \mathbf{R}$, and fix $y_0 \in \mathbf{R}$ with $y_0 \neq f_0(x_0)$. Then the function given by

$$f(x) = \begin{cases} f_0(x) & x \neq x_0 \\ y_0 & x = x_0 \end{cases}$$

is a counterexample. There are even examples with a nonremovable discontinuity, such as

$$f(x) = \begin{cases} \frac{1}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

∎

Problem 4.3: Let X be a metric space, and let $f \colon X \to \mathbf{R}$ be continuous. Let $Z(f) = \{x \in X \colon f(x) = 0\}$. Prove that $Z(f)$ is closed.

Solution 1: The set $Z(f)$ is equal to $f^{-1}(\{0\})$. Since $\{0\}$ is a closed subset of $\mathbf{R}$ and f is continuous, it follows from the Corollary to Theorem 4.8 of Rudin that $Z(f)$ is closed in X. ∎

Solution 2: We show that $X \setminus Z(f)$ is open. Let $x \in X \setminus Z(f)$. Then $f(x) \neq 0$. Set $\varepsilon = \frac{1}{2}|f(x)| > 0$. Choose $\delta > 0$ such that $y \in X$ and $d(x, y) < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Then $f(y) \neq 0$ for $y \in N_\delta(x)$. Thus $N_\delta(x) \subset X \setminus Z(f)$ with $\delta > 0$. This shows that $X \setminus Z(f)$ is open. ∎

Note that we really could have taken $\varepsilon = |f(x)|$. Also, there is no need to do anything special if $Z(f)$ is empty, or even to mention the that case separately: the argument works (vacuously) just as well in that case.

Solution 3 (sketch): We show $Z(f)$ contains all its limit points. Let x be a limit point of $Z(f)$. Then there is a sequence (x_n) in $Z(f)$ such that $x_n \to x$. Since f is continuous and $f(x_n) = 0$ for all n, we have

$$f(x) = \lim_{n \to \infty} f(x_n) = 0.$$

So $x \in Z(f)$. ∎

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Again, there is no need to treat separately the case in which $Z(f)$ has no limit points.

Problem 4.4: Let X and Y be metric spaces, and let $f, g\colon X \to Y$ be continuous functions. Let $E \subset X$ be dense. Prove that $f(E)$ is dense in $f(X)$. Prove that if $f(x) = g(x)$ for all $x \in E$, then $f = g$.

Solution: We first show that $f(E)$ is dense in $f(X)$. Let $y \in f(X)$. Choose $x \in X$ such that $f(x) = y$. Since E is dense in X, there is a sequence (x_n) in E such that $x_n \to x$. Since f is continuous, it follows that $f(x_n) \to f(x)$. Since $f(x_n) \in f(E)$ for all n, this shows that $x \in \overline{f(E)}$.

Now assume that $f(x) = g(x)$ for all $x \in E$; we prove that $f = g$. It suffices to prove that $F = \{x \in X\colon f(x) = g(x)\}$ is closed in X, and we prove this by showing that $X \setminus F$ is open. Thus, let $x_0 \in F$. Set $\varepsilon = \frac{1}{2} d(f(x_0), g(x_0)) > 0$. Choose $\delta_1 > 0$ such that if $x \in X$ satisfies $d(x, x_0) < \delta$, then $d(f(x), f(x_0)) < \varepsilon$. Choose $\delta_2 > 0$ such that if $x \in X$ satisfies $d(x, x_0) < \delta$, then $d(g(x), g(x_0)) < \varepsilon$. Set $\delta = \min(\delta_1, \delta_2)$. If $d(x, x_0) < \delta$, then (using the triangle inequality several times)

$$\begin{aligned} d(f(x), g(x)) &\geq d(f(x_0), g(x_0)) - d(f(x), f(x_0)) - d(g(x), g(x_0)) \\ &> d(f(x_0), g(x_0)) - \varepsilon - \varepsilon = 0. \end{aligned}$$

So $f(x) \neq g(x)$. This shows that $N_\delta(x_0) \subset X \setminus F$, so that $X \setminus F$ is open. ∎

The second part is closely related to Problem 4.3. If $Y = \mathbf{R}$ (or $\mathbf{C}^n$, or $\ldots$), then $\{x \in X\colon f(x) = g(x)\} = Z(f - g)$, and $f - g$ is continuous when f and g are. For general Y, however, this solution fails, since $f - g$ won't be defined. The argument given is the analog of Solution 2 to Problem 4.3. The analog of Solution 3 to Problem 4.3 also works the same way: F is closed because if $x_n \to x$ and $f(x_n) = g(x_n)$ for all n, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n)$. The analog of Solution 1 can actually be patched in the following way (using the fact that the product of two metric spaces is again a metric space): Define $h\colon X \to Y \times Y$ by $h(x) = (f(x), g(x))$. Then h is continuous and $D = \{(y, y)\colon y \in Y\} \subset Y \times Y$ is closed, so $\{x \in X\colon f(x) = g(x)\} = h^{-1}(D)$ is closed.

Problem 4.6: Let $E \subset \mathbf{R}$ be compact, and let $f\colon E \to \mathbf{R}$ be a function. Prove that f is continuous if and only if the graph $G(f) = \{(x, f(x))\colon x \in E\} \subset \mathbf{R}^2$ is compact.

Remark: This statement is my interpretation of what was intended. Normally one would assume that E is supposed to be a compact subset of an arbitrary metric space X, and that f is supposed to be a function from E to some other metric space Y. (In fact, one might as well assume $E = X$.) The proofs are all the same (with one exception, noted below), but require the notion of the product of two metric spaces. We make $X \times Y$ into a metric space via the metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2};$$

there are other choices which are easier to deal with and work just as well.

We give several solutions for each direction. We first show that if f is continuous then $G(f)$ is compact.

Solution 1 (Sketch): The map $x \mapsto (x, f(x))$ is easily checked to be continuous, and $G(f)$ is the image of the compact set E under this map, so $G(f)$ is compact by Theorem 4.14 of Rudin. ∎

Solution 2 (Sketch): The graph of a continuous function is closed, as can be verified by arguments similar to those of Solutions 2 and 3 to Problem 4.3. The graph is a subset of $E \times f(E)$. This set is bounded (clear) and closed (check this!) in $\mathbf{R}^2$, and is therefore compact. (Note: This does not work for general metric spaces. However, it is true in general that the product of two compact sets, with the product metric, is compact.) Therefore the closed subset $G(f)$ is compact. ∎

Now we show that if $G(f)$ is compact then f is continuous.

Solution 1 (Sketch): We know that the function $g_0 \colon E \times \mathbf{R} \to E$, given by $g_0(x, y) = x$, is continuous. (See Example 4.11 of Rudin.) Therefore $g = g_0|_{G(f)} \colon G(f) \to E$ is continuous. Also g is bijective (because f is a function). Since $G(f)$ is compact, it follows (Theorem 4.17 of Rudin) that $g^{-1} \colon E \to G(f)$ is continuous. Furthermore, the function $h \colon E \times \mathbf{R} \to \mathbf{R}$, given by $h(x, y) = y$, is continuous, again by Example 4.11 of Rudin. Therefore $f = h \circ g^{-1}$ is continuous. ∎

Solution 2 (Sketch): Let (x_n) be a sequence in E with $x_n \to x$. We show that $f(x_n) \to f(x)$. We do this by showing that every subsequence of $(f(x_n))$ has in turn a subsubsequence which converges to $f(x)$. (To see that this is sufficient, let (y_n) be a sequence in some metric space Y, let $y \in Y$, and suppose that (y_n) does not converge to y. Find a subsequence $(y_{k(n)})$ of (y_n) such that $\inf_{n \in \mathbf{N}} d(y_{k(n)}, y) > 0$. Then no subsequence of $(y_{k(n)})$ can converge to y.)

Accordingly, let $(f(x_{k(n)}))$ be a subsequence of $(f(x_n))$. Let $(x_{k(n)})$ be the corresponding subsequence of (x_n). If $(x_{k(n)})$ is eventually constant, then already $f(x_{k(n)}) \to f(x)$. Otherwise, $\{x_{k(n)} \colon n \in \mathbf{N}\}$ is an infinite set, whence so is $\{(x_{k(n)}, f(x_{k(n)})) \colon n \in \mathbf{N}\} \subset G(f)$. Since $G(f)$ is compact, this set has a limit point, say (a, b). It is easy to check that a must equal x. Since $G(f)$ is compact, it is closed, so $b = f(x)$. Since (a, b) is a limit point of $G(f)$, there is a subsequence of $\big((x_{k(n)}, f(x_{k(n)}))\big)$ which converges to (a, b). Using continuity of projection onto the second coordinate, we get a subsequence of $\big(f(x_{k(n)})\big)$ which converges to $f(x)$. ∎

Solution 3: We first observe that the range $Y = \{f(x) \colon x \in E\}$ of f is compact. Indeed, Y is the image of $G(f)$ under the map $(x, y) \mapsto y$, which is continuous by Example 4.11 of Rudin. So Y is compact by Theorem 4.14 of Rudin. It suffices to prove that f is continuous as a function from E to Y, as can be seen, for example, from the sequential criterion for limits (Theorem 4.2 of Rudin).

Now let $x_0 \in E$ and let $V \subset Y$ be an open set containing $f(x_0)$. We must find an open set $U \subset E$ containing x_0 such that $f(U) \subset V$. For each $y \in Y \setminus V$, the point (x_0, y) is not in the closed set $G \subset E \times \mathbf{R}$. Therefore there exist open sets $R_y \subset E$ containing x_0 and $S_y \subset Y$ containing y such that $(R_y \times S_y) \cap G = \varnothing$. Since $Y \setminus V$ is compact, there are n and $y(1), \ldots, y(n) \in Y \setminus V$ such that the sets $S_{y(1)}, \ldots, S_{y(n)}$ cover $Y \setminus V$. Set $U = R_{y(1)} \cap \cdots \cap R_{y(n)}$ to obtain $f(U) \subset V$. ∎

Problem 4.7: Define f, $g \colon \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Prove:

(1) f is bounded.
(2) f is not continuous at $(0,0)$.
(3) The restriction of f to every straight line in $\mathbf{R}^2$ is continuous.
(4) g is not bounded on any neighborhood of $(0,0)$.
(5) g is not continuous at $(0,0)$.
(6) The restriction of g to every straight line in $\mathbf{R}^2$ is continuous.

Solution (Sketch): (1) We use the inequality $2ab \leq a^2 + b^2$ (which follows from $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$). Taking $a = |x|$ and $b = y^2$, we get $2|x|y^2 \leq x^2 + y^4$, which implies $|f(x,y)| \leq \frac{1}{2}$ for all $(x,y) \in \mathbf{R}^2$.

(2) Set $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$. Then $(x_n, y_n) \to (0,0)$, but $f(x_n, y_n) = \frac{1}{2} \not\to 0 = f(0,0)$.

(3) Clearly f is continuous on $\mathbf{R}^2 \setminus \{(0,0)\}$, so the restriction of f to every straight line in $\mathbf{R}^2$ not going through $(0,0)$ is clearly continuous. Furthermore, the restriction of f to the y-axis is given by $(0,y) \mapsto 0$, which is clearly continuous.

Every other line has the form $y = ax$ for some $a \in \mathbf{R}$. We have

$$f(x, ax) = \frac{a^2 x}{1 + a^4 x^2}$$

for all $x \in \mathbf{R}$, so the restriction of f to this line is given by the continuous function

$$(x,y) \mapsto \frac{a^2 x}{1 + a^4 x^2}.$$

(4) Set $x_n = \frac{1}{n^3}$ and $y_n = \frac{1}{n}$. Then $(x_n, y_n) \to (0,0)$, but $g(x_n, y_n) = n \to \infty$.

(5) This is immediate from (4).

(6) Clearly g is continuous on $\mathbf{R}^2 \setminus \{(0,0)\}$, so the restriction of g to every straight line in $\mathbf{R}^2$ not going through $(0,0)$ is clearly continuous. Furthermore, the restriction of g to the y-axis is given by $(0,y) \mapsto 0$, which is clearly continuous.

Every other line has the form $y = ax$ for some $a \in \mathbf{R}$. We have

$$g(x, ax) = \frac{a^3 x}{1 + a^6 x^4}$$

for all $x \in \mathbf{R}$, so the restriction of g to this line is given by the continuous function

$$(x,y) \mapsto \frac{a^3 x}{1 + a^6 x^4}.$$

∎

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Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.8: Let $E \subset \mathbf{R}$ be bounded, and let $f \colon E \to \mathbf{R}$ be uniformly continuous. Prove that f is bounded. Show that a uniformly continuous function on an unbounded subset of $\mathbf{R}$ need not be bounded.

Solution (Sketch): Choose $\delta > 0$ such that if $x, y \in E$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \delta$. Since E is a bounded subset of $\mathbf{R}$, there are finitely many closed intervals $\left[a_k, \, a_k + \frac{1}{n}\right]$ whose union contains the closed interval $[\inf(E), \sup(E)]$ and hence also E. Let S be the finite set of those k for which $E \cap \left[a_k, \, a_k + \frac{1}{n}\right] \neq \varnothing$. Thus $E \subset \bigcup_{k \in S} \left[a_k, \, a_k + \frac{1}{n}\right]$. Choose $b_k \in E \cap \left[a_k, \, a_k + \frac{1}{n}\right]$. Set $M = 1 + \max_{k \in S} |f(b_k)|$.

We show that $|f(x)| \leq M$ for all $x \in E$. For such x, choose $k \in S$ such that $x \in \left[a_k, \, a_k + \frac{1}{n}\right]$. Then $|x - b_k| \leq \frac{1}{n} < \delta$, so $|f(x) - f(b_k)| < 1$. Thus $|f(x)| \leq |f(x) - f(b_k)| + |f(b_k)| < 1 + M$.

As a counterexample with E unbounded, take $E = \mathbf{R}$ and $f(x) = x$ for all x. ∎

Problem 4.9: Let X and Y be metric spaces, and let $f \colon X \to Y$ be a function. Prove that f is uniformly continuous if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $E \subset X$ satisfies $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$.

Solution: Let f be uniformly continuous, and let $\varepsilon > 0$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1), \, f(x_2)) < \frac{1}{2}\varepsilon$. Let $E \subset X$ satisfy $\operatorname{diam}(E) < \delta$. We show that $\operatorname{diam}(f(E)) < \varepsilon$. Let $y_1, y_2 \in f(E)$. Choose $x_1, x_2 \in E$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $d(x_1, x_2) < \delta$, so $d(f(x_1), \, f(x_2)) < \frac{1}{2}\varepsilon$. This shows that $d(y_1, y_2) < \frac{1}{2}\varepsilon$ for all $y_1, y_2 \in f(E)$. Therefore

$$\operatorname{diam}(f(E)) = \sup_{y_1, y_2 \in E} d(y_1, y_2) \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

Now assume that for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $E \subset X$ satisfies $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$. We prove that f is uniformly continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ as in the hypotheses. Let $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$. Set $E = \{x_1, x_2\}$. Then $\operatorname{diam}(E) < \delta$. So $d(f(x_1), \, f(x_2)) = \operatorname{diam}(f(E)) < \varepsilon$. ∎

Problem 4.10: Use the fact that infinite subsets of compact sets have limit points to give an alternate proof that if X and Z are metric spaces with X compact, and $f \colon X \to Z$ is continuous, then f is uniformly continuous.

Date: 19 Nov. 2001.

Solution: Assume that f is not uniformly continuous. Choose $\varepsilon > 0$ for which the definition of uniform continuity fails. Then for every $n \in \mathbf{N}$ there are $x_n, y_n \in X$ such that $d(x_n, y_n) < \frac{1}{n}$ and $d(f(x_n), f(y_n)) \geq \varepsilon$. Since X is compact, the sequence (x_n) has a convergent subsequence. (See Theorem 3.6 (a) of Rudin.) Let $x = \lim_{n\to\infty} x_{k(n)}$. Since $d(x_{k(n)}, y_{k(n)}) < \frac{1}{k(n)} \leq \frac{1}{n}$, we also have $\lim_{n\to\infty} y_{k(n)} = x$.

If f were continuous at x, we would have

$$\lim_{n\to\infty} f(x_{k(n)}) = \lim_{n\to\infty} f(y_{k(n)}) = f(x).$$

This contradicts $d(f(x_n), f(y_n)) \geq \varepsilon$ for all n. To see this, choose $N \in \mathbf{N}$ such that $n \geq N$ implies

$$d(f(x_{k(n)}), f(x)) < \tfrac{1}{3}\varepsilon \quad \text{and} \quad d(f(y_{k(n)}), f(x)) < \tfrac{1}{3}\varepsilon.$$

Then

$$d(f(x_{k(N)}), f(y_{k(N)}) \leq d(f(x_{k(n)}), f(x)) + d(f(x), f(y_{k(n)})) < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \tfrac{2}{3}\varepsilon,$$

but by construction $d(f(x_{k(N)}), f(y_{k(N)}) \geq \varepsilon$. ∎

Remark: It is not correct to simply claim that the sequence (x_n) has a convergent subsequence $(x_{k(n)})$ and the sequence (y_n) has a convergent subsequence $(y_{k(n)})$. If one chooses convergent subsequences of (x_n) and (y_n), they must be called, say, $(x_{k(n)})$ and $(y_{l(n)})$ for *different* functions $k, l \colon \mathbf{N} \to \mathbf{N}$.

It is nevertheless possible to carry out a proof by passing to convergent subsequences of (x_n) and (y_n). The following solution shows how it can be done. This solution is not recommended here, but in other situations it may be the only way to proceed.

Alternate solution: Assume that f is not uniformly continuous. Choose $\varepsilon > 0$ for which the definition of uniform continuity fails. Then for every $n \in \mathbf{N}$ there are $x_n, y_n \in X$ such that $d(x_n, y_n) < \frac{1}{n}$ and $d(f(x_n), f(y_n)) \geq \varepsilon$. Since X is compact, the sequence (x_n) has a convergent subsequence $(x_{k(n)})$. (See Theorem 3.6 (a) of Rudin.) Then $(y_{k(n)})$ is a sequence in a compact metric space, and therefore, again by Theorem 3.6 (a) of Rudin, it has a convergent subsequence $(y_{k(r(n))})$. Let $l = k \circ r$. Then $(x_{l(n)})$ is a subsequence of the convergent sequence $(x_{k(n)})$, and therefore converges.

Let $x = \lim_{n\to\infty} x_{l(n)}$ and $y = \lim_{n\to\infty} y_{l(n)}$. Now $d(x_{l(n)}, y_{l(n)}) < \frac{1}{l(n)} \leq \frac{1}{n}$. It follows that $d(x, y) = 0$. (To see this, let $\varepsilon > 0$, and choose $N_1, N_2, N_3 \in \mathbf{N}$ so large that $n \geq N_1$ implies $d(x_{l(n)}, x) < \frac{1}{3}\varepsilon$, so large that $n \geq N_2$ implies $d(y_{l(n)}, y) < \frac{1}{3}\varepsilon$, and so large that $n \geq N_3$ implies $\frac{1}{n} < \frac{1}{3}\varepsilon$. Then with $n = \max(N_1, N_2, N_3)$, we get

$$d(x, y) \leq d(x, x_{l(n)}) + d(x_{l(n)}, y_{l(n)}) + d(y_{l(n)}, y) < \tfrac{1}{3}\varepsilon + \tfrac{1}{n} + \tfrac{1}{3}\varepsilon < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it follows that $d(x, y) = 0$.)

We now know that $\lim_{n\to\infty} y_{l(n)} = x = \lim_{n\to\infty} x_{l(n)}$.

If f were continuous at x, we would have

$$\lim_{n\to\infty} f(x_{l(n)}) = \lim_{n\to\infty} f(y_{l(n)}) = f(x).$$

This contradicts $d(f(x_n), f(y_n)) \geq \varepsilon$ for all n. To see this, choose $N \in \mathbf{N}$ such that $n \geq N$ implies

$$d(f(x_{l(n)}), f(x)) < \tfrac{1}{3}\varepsilon \quad \text{and} \quad d(f(y_{l(n)}), f(x)) < \tfrac{1}{3}\varepsilon.$$

Then

$$d(f(x_{l(N)}),\, f(y_{l(N)}) \leq d(f(x_{l(n)}),\, f(x)) + d(f(x),\, f(y_{l(n)})) < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \tfrac{2}{3}\varepsilon,$$

but by construction $d(f(x_{l(N)}),\, f(y_{l(N)}) \geq \varepsilon$. ∎

Problem 4.11: Let X and Y be metric spaces, and let $f\colon X \to Y$ be uniformly continuous. Prove that if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y. Use this result to prove that if Y is complete, $E \subset X$ is dense, and $f_0\colon E \to Y$ is uniformly continuous, then there is a unique continuous function $f\colon X \to Y$ such that $f|_E = f_0$.

Solution: We prove the first statement. Let (x_n) be a Cauchy sequence in X. Let $\varepsilon > 0$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(s_1, s_2) < \delta$, then $d(f(s_1),\, f(s_2)) < \varepsilon$. Choose $N \in \mathbf{N}$ such that if $m, n \in \mathbf{N}$ satisfy $m, n \geq N$, then $d(x_m, x_n) < \delta$. Then whenever $m, n \in \mathbf{N}$ satisfy $m, n \geq N$, we have $d(x_m, x_n) < \delta$, so that $d(f(x_m),\, f(x_n)) < \varepsilon$. This shows that $(f(x_n))$ is a Cauchy sequence.

Now we prove the second statement. The neatest arrangement I can think of is to prove the following lemmas first.

Lemma 1. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f\colon E \to Y$ be uniformly continuous. Let (x_n) be a sequence in E which converges to some point in X. Then $\lim_{n\to\infty} f(x_n)$ exists in Y.

Proof: We know that convergent sequences are Cauchy. This is therefore immediate from the first part of the problem and the definition of completeness. ∎

Lemma 2. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f\colon E \to Y$ be uniformly continuous. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, and whenever (r_n) and (s_n) are sequences in E such that $r_n \to x_1$ and $s_n \to x_2$, then

$$d\left(\lim_{n\to\infty} f(r_n),\, \lim_{n\to\infty} f(s_n)\right) < \varepsilon.$$

Note that the limits exist by Lemma 1.

Proof of Lemma 2: Let $\varepsilon > 0$. Choose $\rho > 0$ such that whenever $x_1, x_2 \in E$ satisfy $d(x_1, x_2) < \rho$, then $d(f(x_1),\, f(x_2)) < \frac{1}{2}\varepsilon$. Set $\delta = \frac{1}{2}\rho > 0$. Let $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, and let (r_n) and (s_n) be sequences in E such that $r_n \to x_1$ and $s_n \to x_2$. Let $y_1 = \lim_{n\to\infty} f(r_n)$ and $y_2 = \lim_{n\to\infty} f(s_n)$. (These exist by Lemma 1.) Choose N so large that for all $n \in \mathbf{N}$ with $n \geq N$, the following four conditions are all satisfied:

- $d(r_n, x_1) < \frac{1}{4}\rho$.
- $d(s_n, x_2) < \frac{1}{4}\rho$.
- $d(f(r_n),\, y_1) < \frac{1}{4}\varepsilon$.
- $d(f(s_n),\, y_2) < \frac{1}{4}\varepsilon$.

We then have

$$d(r_N, s_N) \leq d(r_N, x_1) + d(x_1, x_2) + d(x_2, s_N) < \tfrac{1}{4}\rho + \tfrac{1}{2}\rho + \tfrac{1}{4}\rho = \rho.$$

Therefore $d(f(r_N),\, f(s_N)) < \frac{1}{2}\varepsilon$. So

$$d(y_1, y_2) \leq d(y_1,\, f(r_N)) + d(f(r_N),\, f(s_N)) + d(f(s_N),\, y_2) < \tfrac{1}{4}\varepsilon + \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon = \varepsilon,$$

as desired. ∎

Theorem. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f \colon E \to Y$ be uniformly continuous. Then there is a unique continuous function $f \colon X \to Y$ such that $f|_E = f_0$.

Proof: If f exists, then it is unique by Problem 4.4, which was in the previous assignment. So we prove existence. For $x \in X$, we want to define $f(x)$ by choosing a sequence (r_n) in E with $\lim_{n\to\infty} r_n = x$ and then setting $f(x) = \lim_{n\to\infty} f_0(r_n)$. We know that such a sequence exists because E is dense in X. We know that $\lim_{n\to\infty} f_0(r_n)$ exists, by Lemma 1. However, we must show that $\lim_{n\to\infty} f_0(r_n)$ only depends on x, not on the sequence (r_n).

To prove this, let (r_n) and (s_n) be sequences in E with

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = x.$$

Let $\varepsilon > 0$; we show that

$$d\left(\lim_{n\to\infty} f_0(r_n), \ \lim_{n\to\infty} f_0(s_n)\right) < \varepsilon.$$

(Since ε is arbitrary, this will give $\lim_{n\to\infty} f_0(r_n) = \lim_{n\to\infty} f_0(s_n)$.) To do this, choose $\delta > 0$ according to Lemma 2. We certainly have $d(x, x) < \delta$. Therefore the conclusion of Lemma 2 gives

$$d\left(\lim_{n\to\infty} f_0(r_n), \ \lim_{n\to\infty} f_0(s_n)\right) < \varepsilon,$$

as desired.

We now get a well defined function $f \colon X \to Y$ by setting $f(x) = \lim_{n\to\infty} f_0(r_n)$, where (r_n) is any sequence in E with $\lim_{n\to\infty} r_n = x$. By considering the constant sequence $x_n = x$ for all n, we see immediately that $f(x) = f_0(x)$ for $x \in E$. We show that f is continuous, in fact uniformly continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ according to Lemma 2. For $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, choose (by density of E, as above) sequences (r_n) and (s_n) in E such that $r_n \to x_1$ and $s_n \to x_2$. Then $d(\lim_{n\to\infty} f_0(r_n), \lim_{n\to\infty} f_0(s_n)) < \varepsilon$. By construction, we have $f(x_1) = \lim_{n\to\infty} f_0(r_n)$ and $f(x_2) = \lim_{n\to\infty} f_0(s_n)$. Therefore we have shown that $d(f(x_1), f(x_2)) < \varepsilon$, as desired. ∎

The point of stating Lemma 2 separately is that the proof that f is well defined, and the proof that f is continuous, use essentially the same argument. By putting that argument in a lemma, we avoid repeating it.

Problem 4.12: State precisely and prove the following: "A uniformly continuous function of a uniformly continuous function is uniformly continuous."

Solution: Here is the precise statement:

Proposition. Let X, Y, and Z be metric spaces. Let $f \colon X \to Y$ and $g \colon Y \to Z$ be uniformly continuous functions. Then $g \circ f$ is uniformly continuous.

Proof: Let $\varepsilon > 0$. Choose $\rho > 0$ such that if $y_1, y_2 \in Y$ satisfy $d(y_1, y_2) < \rho$, then $d(g(y_1), g(y_2)) < \varepsilon$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \rho$. Then whenever $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, we have $d(f(x_1), f(x_2)) < \rho$, so that $d(g(f(x_1)), g(f(x_2))) < \varepsilon$. ∎

Problem 4.14: Let $f \colon [0, 1] \to [0, 1]$ be continuous. Prove that there is $x \in [0, 1]$ such that $f(x) = x$.

Solution: Define $g\colon [0,1] \to \mathbf{R}$ by $g(x) = x - f(x)$. Then g is continuous. Since $f(0) \in [0,1]$, we have $g(0) = -f(0) \leq 0$, while since $g(1) \in [0,1]$, we have $g(1) = 1 - f(1) \geq 0$. If $g(0) = 0$ then $x = 0$ satisfies the conclusion, while if $g(1) = 0$ then $x = 1$ satisfies the conclusion. Otherwise, $g(0) < 0$ and $g(1) > 0$, so Theorem 4.23 of Rudin provides $x \in (0,1)$ such that $g(x) = 0$. This x satisfies $f(x) = x$. ∎

Something much more general is true, namely the Brouwer Fixed Point Theorem:

Theorem. Let $n \geq 1$, and let $B = \{x \in \mathbf{R}^n \colon \|x\| \leq 1\}$. Let $f\colon B \to B$ be continuous. Then there is $x \in B$ such that $f(x) = x$.

The proof requires higher orders of connectedness, and is best done with algebraic topology.

Problem 4.16: For $x \in \mathbf{R}$, define $[x]$ by the relations $[x] \in \mathbf{Z}$ and $x - 1 < [x] \leq x$ (this is called the "integer part of x" or the "greatest integer function"), and define $(x) = x - [x]$ (this is called the "fractional part of x", but the notation (x) is not standard). What discontinuities do the functions $x \mapsto [x]$ and $x \mapsto (x)$ have?

Solution (Sketch): Both functions are continuous at all noninteger points, since $x \in (n,\, n+1)$ implies $[x] = n$ and $(x) = x - n$; both expressions are continuous on the interval $(n,\, n+1)$.

Both functions have jump discontinuities at all integers: for $n \in \mathbf{Z}$, we have

$$\lim_{x \to n^+} [x] = \lim_{x \to n^+} n = n = f(n) \quad \text{and} \quad \lim_{x \to n^-} [x] = \lim_{x \to n^-} (n-1) = n - 1 \neq f(n),$$

and also

$$\lim_{x \to n^+} (x) = \lim_{x \to n^+} (x - n) = 0 = f(n)$$

and

$$\lim_{x \to n^-} (x) = \lim_{x \to n^-} [x - (n-1)] = 1 \neq f(n).$$

Problem 4.18: Define $f\colon \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & x \in \mathbf{R} \setminus \mathbf{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}.$$

(By definition, we require $q > 0$. If $x = 0$ we take $p = 0$ and $q = 1$.) Prove that f is continuous at each $x \in \mathbf{R} \setminus \mathbf{Q}$, and that f has a simple discontinuity at each $x \in \mathbf{Q}$.

Solution: We show that $\lim_{x \to 0} f(x) = 0$ for all $x \in \mathbf{R}$. This immediately implies that f is continuous at all points x for which $f(x) = 0$ and has a removable discontinuity at every x for which $f(x) \neq 0$.

Let $x \in \mathbf{R}$, and let $\varepsilon > 0$. Choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \varepsilon$. For $1 \leq n \leq N$, let

$$S_n = \left\{ \frac{a}{n} \colon a \in \mathbf{Z} \text{ and } 0 < \left| \frac{a}{n} - x \right| < 1 \right\}.$$

Then S_n is finite; in fact, $\operatorname{card}(S_n) \leq 2n$. Set

$$S = \bigcup_{n=1}^{N} S_n,$$

which is a finite union of finite sets and hence finite. Note that $x \notin S$. Set

$$\delta = \min \left(1, \ \min_{y \in S} |y - x| \right).$$

Then $\delta > 0$ because $x \notin S$ and S is finite.

Let $0 < |y - x| < \delta$. If $y \notin \mathbf{Q}$, then $|f(y) - 0| = 0 < \varepsilon$. Otherwise, because $y \notin S$, $|y - x| < 1$, and $y \neq x$, it is not possible to write $y = \frac{p}{q}$ with $q \leq N$. Thus, when we write $y = \frac{p}{q}$ in lowest terms, we have $q > N$, so $f(y) = \frac{1}{q} < \frac{1}{N} < \varepsilon$. This shows that $|f(y) - 0| < \varepsilon$ in this case also. ∎

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Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.15: Prove that every continuous open map $f\colon \mathbf{R} \to \mathbf{R}$ is monotone.

Sketches of two solutions are presented. The second is what I expect people to have done. The first is essentially a careful rearrangement of the ideas of the second, done so as to minimize the number of cases. (You will see when reading the second solution why this is desirable.)

Solution (Sketch):

Lemma 1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. Then $f(a) \neq f(b)$.

Proof (sketch): Suppose $f(a) = f(b)$. Let m_1 and m_2 be the minimum and maximum values of f on $[a, b]$. (These exist because f is continuous and $[a, b]$ is compact.) If $m_1 = m_2$, then $m_1 = m_2 = f(a) = f(b)$, and $f((a, b)) = \{m_1\}$ is not an open set. Since (a, b) is open, this is a contradiction. So suppose $m_1 < m_2$. If $m_1 \neq f(a)$, choose $c \in [a, b]$ such that $f(c) = m_1$. Then actually $c \in (a, b)$. So $f((a, b))$ contains $f(c)$ but contains no real numbers smaller than $f(c)$. This is easily seen to contradict the assumption that $f((a, b))$ is open. The case $m_2 \neq f(a)$ is handled similarly, or by considering $-f$ in place of f. ∎

Note: The last part of this proof is the only place where I would expect a submitted solution to be more complete than what I have provided.

Lemma 2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbf{R}$ satisfies $x < a$, we have $f(x) < f(a)$.

Proof: We can't have $f(x) = f(a)$, by Lemma 1. If $f(x) = f(b)$, we again have a contradiction by Lemma 1. If $f(x) > f(b)$, then the Intermediate Value Theorem provides $z \in (x, a)$ such that $f(z) = f(b)$. Since $z < b$, this contradicts Lemma 1. If $f(a) < f(x) < f(b)$, then the Intermediate Value Theorem provides $z \in (a, b)$ such that $f(z) = f(x)$. Since $x < z$, this again contradicts Lemma 1. The only remaining possibility is $f(x) < f(a)$. ∎

Lemma 3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbf{R}$ satisfies $b < x$, we have $f(b) < f(x)$.

Proof: Apply Lemma 2 to the function $x \mapsto -f(-x)$. ∎

Lemma 4. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbf{R}$ satisfies $a < x < b$, we have $f(a) < f(x) < f(b)$.

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Proof: Lemma 1 implies that $f(x)$ is equal to neither $f(a)$ nor $f(b)$. If $f(x) < f(a)$, we apply Lemma 3 to $-f$, with b and x interchanged, to get $f(b) < f(x)$. This implies $f(b) < f(a)$, which contradicts the hypotheses. If $f(x) > f(b)$, we apply Lemma 2 to $-f$, with a and x interchanged, to get $f(a) > f(x)$. This again contradicts the hypotheses. ∎

Corollary 5. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$, and suppose that $f(a) < f(b)$. Let $x \in \mathbf{R}$. Then:

(1) If $x \leq a$ then $f(x) \leq f(a)$.
(2) If $x \leq b$ then $f(x) \leq f(b)$.
(3) If $x \geq a$ then $f(x) \geq f(a)$.
(4) If $x \geq b$ then $f(x) \geq f(b)$.

Proof: If we have equality ($x = a$ or $x = b$), the conclusion is obvious. With strict inequality, Part (1) follows from Lemma 2, and Part (4) follows from Lemma 3. Part (2) follows from Lemma 4 if $x > a$, from Lemma 2 if $x < a$, and is trivial if $x = a$. Part (3) follows from Lemma 4 if $x < b$, from Lemma 3 if $x > b$, and is trivial if $x = b$. ∎

I won't actually use Part (2); it is included for symmetry.

Now we prove the result. Choose arbitrary $c, d \in \mathbf{R}$ with $c < d$. We have $f(c) \neq f(d)$ by Lemma 1. Suppose first that $f(c) < f(d)$. Let $r, s \in \mathbf{R}$ satisfy $r \leq s$. We prove that $f(r) \leq f(s)$, and there are several cases. I will try to arrange this to keep the number of cases as small as possible.

Case 1: $r \leq c \leq s$. Then $f(r) \leq f(c) \leq f(s)$ by Parts (1) and (3) of Corollary 5, taking $a = c$ and $b = d$.

Case 2: $r \leq s \leq a$. Then $f(s) \leq f(a)$ by Part (1) of Corollary 5, taking $a = c$ and $b = d$. Further, $f(r) \leq f(s)$ by Part (1) of Corollary 5, taking $a = s$ and $b = d$.

Case 3: $a \leq r \leq s$. Then $f(a) \leq f(r)$ by Part (3) of Corollary 5, taking $a = c$ and $b = d$. Further, $f(r) \leq f(s)$ by Part (4) of Corollary 5, taking $a = c$ and $b = r$.

The case $f(c) > f(d)$ follows by applying the preceding argument to $-f$. ∎

Alternate solution (Brief sketch):

Suppose f is not monotone; we prove that f is not open. Since f isn't nondecreasing, there exist $a, b \in \mathbf{R}$ such that $a < b$ and $f(a) > f(b)$; and since f isn't nonincreasing, there exist $c, d \in \mathbf{R}$ such that $c < d$ and $f(c) < f(d)$. Now there are various cases depending on how a, b, c, and d are arranged in $\mathbf{R}$, and depending on how $f(a)$ and $f(b)$ relate to $f(c)$ and $f(d)$. Specifically, there are 13 possible ways for a, b, c, and d to be arranged in $\mathbf{R}$, namely:

$$a < b < c < d \tag{1}$$

$$a < b = c < d \tag{2}$$

$$a < c < b < d \tag{3}$$

$$a < c < b = d \tag{4}$$

$$a < c < d < b \tag{5}$$

$$a = c < b < d \tag{6}$$

$$a = c < b = d \tag{7}$$

$$a = c < d < b \tag{8}$$

$$c < a < b < d \tag{9}$$

$$c < a < b = d \tag{10}$$

$$c < a < d < b \tag{11}$$

$$c < a = d < b \tag{12}$$

$$c < d < a < b \tag{13}$$

Of these, the arrangement (7) gives an immediate contradiction. For each of the others, we find $x < y < z$ such that $f(y) < f(x)$, $f(z)$ (so that f is not open by Lemma 2 of the previous solution), or such that $f(y) > f(x)$, $f(z)$ (so that f is not open by Lemma 3 of the previous solution). Many cases break down into subcases depending on how the values of f are arranged. We illustrate by treating the arrangement (1).

Suppose $a < b < c < d$ and $f(b) < f(c)$. Set

$$x = a, \quad y = b, \quad \text{and} \quad z = d.$$

Then $x < y < z$ and $f(y) < f(x)$, $f(z)$, so Lemma 2 applies. Suppose, on the other hand, that $a < b < c < d$ and $f(b) \geq f(c)$. Set

$$x = a, \quad y = c, \quad \text{and} \quad z = d.$$

Then again $x < y < z$ and $f(y) < f(x)$, $f(z)$, so Lemma 2 applies. ∎

Problem 5.1: Let $f \colon \mathbf{R} \to \mathbf{R}$ be a function such that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbf{R}$. Prove that f is constant.

Solution: We first prove that $f'(x) = 0$ for all $x \in \mathbf{R}$. For $h \neq 0$,

$$\left| \frac{f(x+h) - f(x)}{h} \right| = \frac{|f(x+h) - f(x)|}{|h|} \leq \frac{|h^2|}{|h|} = |h|.$$

It follows immediately that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

It now follows that f is constant. (See Theorem 5.11 (b) of Rudin's book.) ∎

The solution above is the intended solution. However, there is another solution which is nearly as easy and does not use calculus.

Alternate solution: Let $x, y \in \mathbf{R}$ and let $\varepsilon > 0$; we prove that $|f(x) - f(y)| < \varepsilon$. It will clearly follow that f is constant.

Choose $N \in \mathbf{N}$ with $N > \varepsilon^{-1}(x-y)^2$. The hypothesis implies that, for any k, we have

$$\left| f\left(x + (k-1) \cdot \tfrac{1}{N}(y-x)\right) - f\left(x + k \cdot \tfrac{1}{N}(y-x)\right) \right| \leq \left(\tfrac{1}{N}(y-x)\right)^2 = \left(\frac{1}{N}\right)\left(\frac{(y-x)^2}{N}\right) < \frac{1}{N} \cdot \varepsilon.$$

Therefore

$$|f(x) - f(y)| \leq \sum_{k=1}^{N} \left| f\left(x + (k-1) \cdot \tfrac{1}{N}(y-x)\right) - f\left(x + k \cdot \tfrac{1}{N}(y-x)\right) \right| < N \cdot \frac{1}{N} \cdot \varepsilon = \varepsilon.$$

∎

Problem 5.2: Let $f \colon (a,b) \to \mathbf{R}$ satisfy $f'(x) > 0$ for all $x \in (a,b)$. Prove that f is strictly increasing, that its inverse function g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for all $x \in (a,b)$.

Solution: That f is strictly increasing on (a,b) follows from the Mean Value Theorem and the fact that $f'(x) > 0$ for all $x \in (a,b)$.

Define

$$c = \inf_{x \in (a,b)} f(x) \quad \text{and} \quad d = \sup_{x \in (a,b)} f(x).$$

(Note that c could be $-\infty$ and d could be ∞.) Our next step is to prove that f is a bijection from (a,b) to (c,d). Clearly f is injective, and has range contained in $[c,d]$. If $c = f(x)$ for some $x \in (a,b)$, then there is $q \in (a,b)$ with $q < x$. This would imply $f(q) < c$, contradicting the definition of c. So c is not in the range of f. Similarly d is not in the range of f. So the range of f is contained in (c,d). For surjectivity, let $y_0 \in (c,d)$. By the definitions of inf and sup, there are $r, s \in (a,b)$ such that $f(r) < y_0 < f(s)$. Clearly $r < s$. The Intermediate Value Theorem provides $x_0 \in (r,s)$ such that $f(x_0) = y_0$. This shows that the range of f is all of (c,d), and completes the proof that f is a bijection from (a,b) to (c,d).

Now we show that $g \colon (c,d) \to (a,b)$ is continuous. Again, let $y_0 \in (c,d)$, and choose r and s as in the previous paragraph. Since f is strictly increasing, and again using the Intermediate Value Theorem, we see that $f|_{[r,s]}$ is a continuous bijection from $[r,s]$ to $[f(r), f(s)]$. Since $[r,s]$ is compact, the function $\left(f|_{[r,s]}\right)^{-1} = g|_{[f(r), f(s)]}$ is continuous. Since $y_0 \in (f(r), f(s))$, it follows that g is continuous at y_0. Thus g is continuous.

Now we find g'. Fix $x_0 \in (a,b)$, and set $y_0 = f(x_0)$. For $y \in (c,d) \setminus \{y_0\}$, we write

$$\frac{g(y) - g(y_0)}{y - y_0} = \left(\frac{f(g(y)) - f(x_0)}{g(y) - x_0}\right)^{-1}.$$

(Note that $g(y) \neq x_0$ because g is injective.) Since g is continuous, we have $\lim_{y \to y_0} g(y) = x_0$. Therefore

$$\lim_{y \to y_0} \frac{f(g(y)) - f(x_0)}{g(y) - x_0} = f'(x_0).$$

Hence

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

That is, $g'(y_0)$ exists and is equal to $\dfrac{1}{f'(x_0)}$, as desired. ∎

Note: I believe, but have not checked, that further use of the Intermediate Value Theorem can be substituted for the use of compactness in the proof that g is continuous.

Problem 5.3: Let $g \colon \mathbf{R} \to \mathbf{R}$ be a differentiable function such that g' is bounded. Prove that there is $r > 0$ such that the function $f(x) = x + \varepsilon g(x)$ is injective whenever $0 < \varepsilon < r$.

Solution: Set $M = \max(0, \sup_{x \in \mathbf{R}}(-g'(x)))$. Set $r = \frac{1}{M}$. (Take $r = \infty$ if $M = 0$.) Suppose $0 < \varepsilon < r$, and define $f(x) = x + \varepsilon g(x)$ for $x \in \mathbf{R}$. For $x \in \mathbf{R}$, we have

$$f'(x) = 1 + \varepsilon g'(x) = 1 - \varepsilon(-g'(x)) \geq 1 - \varepsilon M > 1 - rM = 0$$

(except that $1 - rM = 1$ if $M = 0$). Thus $f'(x) > 0$ for all x, so the Mean Value Theorem implies that f is strictly increasing. In particular, f is injective. ∎

Note: The problem as stated in Rudin's book is slightly ambiguous: it could be interpreted as asking that $f(x) = x + \varepsilon g(x)$ be injective whenever $-r < \varepsilon < r$. To prove this version, take $M = \sup_{x \in \mathbf{R}} |g'(x)|$, and estimate

$$f'(x) = 1 + \varepsilon g'(x) \geq 1 - |\varepsilon| M > 1 - rM = 0.$$

Problem 5.4: Let $C_0, C_1, \ldots, C_n \in \mathbf{R}$. Suppose

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0.$$

Prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real solution in $(0, 1)$.

Solution (Sketch): Define $f \colon \mathbf{R} \to \mathbf{R}$ by

$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_{n-1} x^n}{n} + \frac{C_n x^{n+1}}{n+1}$$

for $x \in \mathbf{R}$. Then $f(0) = 0$ (this is trivial) and $f(1) = 0$ (this follows from the hypothesis). Since f is differentiable on all of $\mathbf{R}$, the Mean Value Theorem provides $x \in (0, 1)$ such that $f'(x) = 0$. Since

$$f'(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n,$$

this is the desired conclusion. ∎

Problem 5.5: Let $f \colon (0, \infty) \to \mathbf{R}$ be differentiable and satisfy $\lim_{x \to \infty} f'(x) = 0$. Prove that $\lim_{x \to \infty}[f(x+1) - f(x)] = 0$.

Solution: Let $\varepsilon > 0$. Choose $M \in \mathbf{R}$ such that $x > M$ implies $|f'(x)| < \varepsilon$. Let $x > M$. By the Mean Value Theorem, there is $z \in (x, x+1)$ such that $f(x+1) - f(x) = f'(z)$. Then $|f(x+1) - f(x)| = |f'(z)| < \varepsilon$. This shows that $\lim_{x \to \infty}[f(x+1) - f(x)] = 0$. ∎

Problem 5.9: Let $f \colon \mathbf{R} \to \mathbf{R}$ be continuous. Assume that $f'(x)$ exists for all $x \neq 0$, and that $\lim_{x \to 0} f'(x) = 3$. Does it follow that $f'(0)$ exists?

Solution: We prove that $f'(0) = 3$. Define $g(x) = x$. Then

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 3$$

by assumption. Therefore Theorem 5.13 of Rudin (L'Hospital's rule) applies to the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{g(x)}$$

(because $f - f(0)$ vanishes at 0 and has derivative f'). Thus

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 3.$$

In particular, $f'(0)$ exists. ∎

Note: It is mathematically bad practice (although it is tolerated in freshman calculus courses) to write

$$\lim_{x \to 0} \frac{f(x) - f(0)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 3$$

before checking that

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

exists, because the equality

$$\lim_{x \to 0} \frac{f(x) - f(0)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

is only known to hold when the second limit exists.

Problem 5.11: Let f be a real valued function defined on a neighborhood of $x \in \mathbf{R}$. Suppose that $f''(x)$ exists. Prove that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by example that the limit might exist even if $f''(x)$ does not exist.

Solution (Sketch): Check using algebra that

$$\begin{aligned} \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} &= \lim_{h \to 0} \left(\frac{f'(x+h) - f'(x)}{2h} + \frac{f'(x) - f'(x-h)}{2h} \right) \\ &= f''(x). \end{aligned}$$

Now use Theorem 5.13 of Rudin (L'Hospital's rule) to show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

For the counterexample, take

$$f(t) = \begin{cases} 1 & t > x \\ 0 & t = x \\ -1 & t < x \end{cases}.$$

Then

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = 0$$

for all $h \neq 0$. This shows that the limit can exist even if f isn't continuous at x. ∎

Note 1: I gave a counterexample for an arbitrary value of x, but it suffices to give one at a single value of x, such as $x = 0$.

Note 2: A legitimate counterexample must be defined at x, since it must satisfy all the hypotheses except for the existence of $f''(x)$.

Note 3: Another choice for the counterexample is

$$f(t) = \begin{cases} (t-x)^2 & t \geq x \\ -(t-x)^2 & t < x \end{cases}.$$

This function is continuous at x, and even has a continuous derivative on $\mathbf{R}$, but $f''(x)$ doesn't exist. One can also construct examples which are continuous nowhere on $\mathbf{R}$.

Note 4: It is tempting to use L'Hospital's rule a second time, to get

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2}.$$

This reasoning is not valid, since the second limit need not exist. (We do not assume that f'' is continuous.)

Problem 5.13: Let a and c be fixed real numbers, with $c > 0$, and define $f = f_{a,c} \colon [-1, 1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} |x|^a \sin\left(|x|^{-c}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove the following statements. (You may use the standard facts about the functions $\sin(x)$ and $\cos(x)$.)

Note: The book has

$$f(x) = \begin{cases} x^a \sin\left(|x|^{-c}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

However, unless a is a rational number with odd denominator, this function will not be defined for $x < 0$.

(a) f is continuous if and only if $a > 0$.

Solution (Sketch): Since $x \mapsto \sin(x)$ is continuous, we need only consider continuity at 0. If $a > 0$, then $\lim_{x \to 0} f(x) = 0$ since $|f(x)| \leq |x|^a$ and $\lim_{x \to 0} |x|^a = 0$.

Now define sequences (x_n) and (y_n) by

$$x_n = \frac{1}{\left[\left(2n + \frac{1}{2}\right)\pi\right]^{1/c}} \quad \text{and} \quad y_n = \frac{1}{\left[\left(2n + \frac{3}{2}\right)\pi\right]^{1/c}}.$$

Note that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$$

and

$$\sin\left(|x_n|^{-c}\right) = 1 \quad \text{and} \quad \sin\left(|y_n|^{-c}\right) = -1$$

for all n. (We will use these sequences in other parts of the problem.)

If now $a = 0$, then

$$\lim_{n\to\infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = -1,$$

so $\lim_{x\to 0} f(x)$ does not exist, and f is not continuous at 0. If $a < 0$, then

$$\lim_{n\to\infty} f(x_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = -\infty,$$

with the same result. ∎

Note: Since $f(0)$ is defined to be 0, we actually need only consider $\lim_{n\to\infty} f(x_n)$. The conclusion $\lim_{x\to 0} f(x)$ does not exist is stronger, and will be useful later.

(b) $f'(0)$ exists if and only if $a > 1$.

Solution (Sketch): We test for existence of

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} f_{a-1,\,c}(h),$$

which we saw in Part (a) exists if and only if $a - 1 > 0$. Moreover (for use below), note that if the limit does exist then it is equal to 0. ∎

(c) f' is bounded if and only if $a \geq 1 + c$.

Solution (Sketch): Boundedness does not depend on $f'(0)$ (or even on whether $f'(0)$ exists). So we use the formula

$$f'(x) = ax^{a-1}\sin\left(x^{-c}\right) + cx^{a-c-1}\cos\left(x^{-c}\right)$$

for $x > 0$, and for $x < 0$ we use

$$f'(x) = -f'(-x) = -f'(|x|) = -a|x|^{a-1}\sin\left(|x|^{-c}\right) - c|x|^{a-c-1}\cos\left(|x|^{-c}\right).$$

If $a - c - 1 \geq 0$, then also $a - 1 \geq 0$ (recall that $c > 0$), and f' is bounded (by $c + a$).

Otherwise, we consider the sequences (w_n) and (z_n) given by

$$w_n = \frac{1}{[2n\pi]^{1/c}} \quad \text{and} \quad y_n = \frac{1}{[(2n+1)\,\pi]^{1/c}}.$$

Since

$$\sin\left(w_n^{-c}\right) = \sin\left(z_n^{-c}\right) = 0$$

and

$$\cos\left(w_n^{-c}\right) = 1 \quad \text{and} \quad \cos\left(z_n^{-c}\right) = -1,$$

arguments as in Part (a) show that

$$\lim_{n\to\infty} f'(w_n) = -\infty \quad \text{and} \quad \lim_{n\to\infty} f'(z_n) = \infty,$$

so f' is not bounded. ∎

(d) f' is continuous on $[-1, 1]$ if and only if $a > 1 + c$.

Solution (Sketch): If $a < 1 + c$, then f' is not bounded on $[-1, 1] \setminus \{0\}$ by Part (c), and therefore can't be the restriction of a continuous function on $[-1, 1]$. If $a = 1+c$, then the sequences of Part (c) satisfy

$$\lim_{n\to\infty} f'(w_n) = -c \quad \text{and} \quad \lim_{n\to\infty} f'(z_n) = c,$$

so again f' can't be the restriction of a continuous function on $[-1, 1]$. If $a > 1 + c$, then also $a > 1$, and $\lim_{x\to 0} f'(x) = 0$ by reasoning similar to that of Part (a). Moreover $f'(0) = 0$ by the extra conclusion in the proof of Part (b). So f' is continuous at 0, hence continuous. ∎

$f''(0)$ exists if and only if $a > 2 + c$.

Solution (Sketch): This is reduced to Part (d) in the same way Part (b) was reduced to Part (a). As there, note also that $f''(0) = 0$ if it exists. ∎

(f) f'' is bounded if and only if $a \geq 2 + 2c$.

Solution (Sketch): For $x \neq 0$, we have

$$\begin{aligned} f''(x) = a(a-1)|x|^{a-2} \sin\left(|x|^{-c}\right) + (2ac - c^2 - c)x^{a-c-2} \cos\left(|x|^{-c}\right) \\ - c^2 |x|^{a-2c-2} \sin\left(|x|^{-c}\right). \end{aligned}$$

(One handles the cases $x > 0$ and $x < 0$ separately, as in Part (c), but this time the resulting formula is the same for both cases.) Since $c > 0$, if $a - 2c - 2 \geq 0$ then also $a - c - 2 \geq 0$ and $a - 2 \geq 0$, so f'' is bounded on $[-1, 1] \setminus \{0\}$. For $a - 2c - 2 < 0$, consider

$$f''(x_n) = a(a-1)x_n^{a-2} - c^2 x_n^{a-2c-2}.$$

Since $x_n \to 0$ and $a - 2c - 2 < \min(0, a - 2)$, one checks that the term $-c^2 x_n^{a-2c-2}$ dominates and $f''(x_n) \to -\infty$. So f'' is not bounded. ∎

(g) f'' is continuous on $[-1, 1]$ if and only if $a > 2 + 2c$.

Solution (Sketch): Recall from the extra conclusion in Part (e) that $f''(0) = 0$ if it exists. If $a - 2c - 2 > 0$, then also $a - c - 1 > 0$ and $a - 2 > 0$, so $\lim_{x\to 0} f''(x) = 0$ by a more complicated version of the arguments used in Parts (a) and (d). If $a - 2c - 2 < 0$, then f'' isn't bounded on $[-1, 1] \setminus \{0\}$, so f'' can't be continuous on $[-1, 1]$. If $a - 2c - 2 = 0$, then $a - 2 > 0$. Therefore

$$f''(x_n) = a(a-1)x_n^{a-2} - c^2 x_n^{a-2c-2} = a(a-1)x_n^{a-2} - c^2 \to c^2 \neq 0$$

as $n \to \infty$. So f'' is not continuous at 0. ∎