具体数学阅读笔记-chap1 exercise

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1 Exercises

1.1 Warmups

练习 1 1 All horses are the same color; we can prove this by induction on the number of horses in a given set. Here's how: "If there's just one horse then it's the same color as itself, so the basis is trivial. For the induction step, assume that there are n horses numbered 1 to n. By the induc- tion hypothesis, horses 1 through n - 1 are the same color, and similarly horses 2 through n are the same color. But the middle horses, 2 through n - 1, can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and n must be the same color as well, by transitivity. Thus all n horses are the same color; QED." What, if anything, is wrong with this reasoning?

题目解答 1 n=1 情况下马有相同颜色

但 n=2 时该假设不一定成立

练习 2 2

题目解答 2 不允许在A B之间直接移动, 求最短的移动序列

$$
\begin{array}{llllr}
k=1 & 1 & A \rightarrowtail C, C \rightarrowtail B & 2 & sum=2 \\
k=2 & 1 & A \rightarrowtail C, C \rightarrowtail B, & 2 & \\
 & 2 & A \rightarrowtail C & 1 & \\
 & 1 & B \rightarrowtail C, C \rightarrowtail A, & 2 & \\
 & 2 & C \rightarrowtail B & 1 & \\
 & 1 & A \rightarrowtail C, C \rightarrowtail B & 2 & sum=8 \\
k=3 & 1 & A \rightarrowtail C, C \rightarrowtail B, & 2 & \\
 & 2 & A \rightarrowtail C & 1 & \\
 & 1 & B \rightarrowtail C, C \rightarrowtail A, & 2 & \\
 & 2 & C \rightarrowtail B & 1 & \\
 & 1 & A \rightarrowtail C, C \rightarrowtail B & 2 & sum=8 \\
 & 3 & A \rightarrowtail C & 1 & \\
 & 1 & B \rightarrowtail C, C \rightarrowtail A, & 2 & \\
 & 2 & B \rightarrowtail C & 1 & \\
 & 1 & A \rightarrowtail C, C \rightarrowtail B, & 2 & \\
 & 2 & C \rightarrowtail A & 1 & \\
 & 1 & B \rightarrowtail C, C \rightarrowtail A, & 2 & \\
 & 3 & C \rightarrowtail B & 1 & sum=18 \\
 & 1 & A \rightarrowtail C, C \rightarrowtail B, & 2 & \\
 & 2 & A \rightarrowtail C & 1 & \\
 & 1 & B \rightarrowtail C, C \rightarrowtail A, & 2 & \\
 & 2 & C \rightarrowtail B & 1 & \\
 & 1 & A \rightarrowtail C, C \rightarrowtail B & 2 & sum=26 \\
\vdots & & & & \\
k=n & 1 & A \rightarrowtail C, C \rightarrowtail B & &
\end{array}
$$

从前面的移动可以看出 $f(n)=3*f(n-1)+2$, 设 $g(n)=f(n)+1, g(1)=f(1)+1=3, g(n)=3g(n-1)$. $g(n)=3^n$, $f(n)=3^n-1$.

练习 3 3

题目解答 3 是的, 以n个圆盘为例正确的叠放方法有3^n种将ABC视为3个序列, 将所有圆盘从大到小依次放置在3个序列中, 每个圆盘放置时有3种选择, 所共有3^n种正确的叠放方法。第二题移动3^n-1次, 再加上移动前所有圆盘都在A柱上的情况, 共有3^n种情况, 所以所有正确的叠放方法均会出现。

我的思考, n个圆盘在3根柱子上任意放的方法有多少种?

练习 4 4

题目解答 4 Are there any starting and ending configurations of n disks on three pegs that are more than 2^n-1 moves apart, under Lucas's original rules?

是否存在 $m>2^n-1$

不存在。根据卢卡斯的规则, 将可能出现的移动情况分为两种:
1. 最大的圆盘不需要移动, 根据归纳法, 最多需要移动$2^{n-1}-1$次。
2. 最大的圆盘需要移动, 根据归纳法, 最多需要移动 $2^{n-1}-1+1+2^{n-1}-1$ 即2^n-1次

练习 5 5

题目解答 5 3个给定集合, 共有8个可能子集。使用 Venn 图表示

[1] $A, B, C,$ 三个集合的所有子集为 $\{\varnothing, A, B, C, A\cap B, A\cap C, B\cap C, A\cap B\cap C,\}$, $\{A\setminus B, A\setminus C, B\setminus A, B\setminus C, C\setminus A, C\setminus B\}, \{A\setminus(B\cup C), B\setminus(C\cup A), C\setminus(A\cup B)\}$, $\{A\cup B, A\cup C, B\cup C, A\cup B\cup C\}. \ldots$

我认为这里所将的八个子集应当是$\{\varnothing\}, \{A\setminus(B\cup C), B\setminus(C\cup A), C\setminus(A\cup B)\}$, $\{(A\cap B)\setminus C, (C\cap A)\setminus B, (B\cap C)\setminus A\}, \{A\cap B\cap C\}$. 空集和7个互不相交的真子集。

[1]Venn 图之后会补上

对于4个集合, Venn图不能给出可能的16个子集, 因为不同的圆至多交于两点。参考答案中说的卵形 (ovals) 是什么意思?

练习 6 6

题目解答 6 无界区域个数$2n$
所有区域个数$\frac{n(n+1)}{2}+1$
二者相减得到有界区域个数 $\frac{(n-1)(n-2)}{2}$

练习 7 7

题目解答 7 设 $H(n)=J(n+1)-J(n)$.
$H(2n)=2$, 对$n\geqslant 1$有

$$\begin{aligned}H(2n+1)&=J(2n+2)-J(2n+1)\\&=(2J(n+1)-1)-(2J(n)+1)\\&=2H(n)-2\end{aligned}$$

但在$n=0$ 时, 由此推出

$$H(1)=J(2)-J(1)=1-1=0\neq 2$$

1.2 作业题

练习 8

$$\begin{aligned}Q_0&=\alpha\\Q_1&=\beta\\Q_n&=\frac{1+Q_{n-1}}{Q_{n-2}},\quad n>1\end{aligned}$$

(hint: $Q_4=\frac{1+\alpha}{\beta}$)

题目解答 8

$$
\begin{aligned}
Q_0 &= \alpha && && && &&= \alpha \\
Q_1 &= \beta && && && &&= \beta \\
Q_2 &= \frac{1+Q_1}{Q_0} && && && &&= \frac{1+\beta}{\alpha} \\
Q_3 &= \frac{1+Q_2}{Q_1} &&= \frac{1+\dfrac{1+\beta}{\alpha}}{\beta} && && &&= \frac{1+\alpha+\beta}{\alpha\beta} \\
Q_4 &= \frac{1+Q_3}{Q_2} &&= \frac{1+\dfrac{1+\alpha+\beta}{\alpha\beta}}{\dfrac{1+\beta}{\alpha}} &&= \frac{\alpha\beta+1+\alpha+\beta}{\beta(1+\beta)} &&= \frac{1+\alpha}{\beta} \\
Q_5 &= \frac{1+Q_4}{Q_3} &&= \frac{1+\dfrac{1+\alpha}{\beta}}{\dfrac{1+\alpha+\beta}{\alpha\beta}} &&= \frac{\alpha\beta+\alpha(1+\alpha)}{1+\alpha+beta} &&= \alpha \\
Q_6 &= \frac{1+Q_5}{Q_4} &&= \frac{1+\alpha}{\dfrac{1+\alpha}{\beta}} && && &&= \beta
\end{aligned}
$$

因此解得

$$
\begin{array}{rlllllll}
Q_i & = \{ & \alpha, & \beta, & \dfrac{1+\beta}{\alpha}, & \dfrac{1+\alpha+\beta}{\alpha\beta}, & \dfrac{1+\alpha}{\beta} & \} \\
(i\%n) & = \{ & 0, & 1, & 2, & 3, & 4, & \}
\end{array}
$$

练习 9 反向归纳法, 从 n 到 $n-1$ 证明命题

$$
P(n):\ x_1 \dots x_n \leqslant \left(\frac{x_1+\cdots+x_n}{n}\right)^n, \quad x_i \geqslant 0, i=1,\dots,n
$$

$n=2$时为真

$$
(x_1+x_2)^2 - 4x_1x_2 = (x_1-x_2)^2 \geqslant 0
$$

a) $x_n = \frac{x_1+\cdots+x_{n-1}}{n-1}$, 证明只要 $n > 1$ 时 $P(n)$ 蕴含 $P(n-1)$.

b) 证明 $P(n)$和 $P(2)$蕴含 $P(2n)$

c) 由 a), b) 说明这就蕴含了$P(n)$对所有n为真

题目解答 9 a) $P(n)$成立, $\forall n > 1$

给定 $x_n = \frac{x_1+\cdots+x_{n-1}}{n-1}$, 则有

$$
\begin{aligned}
x_1 \dots x_{n-1} \cdot \frac{x_1+\cdots+x_{n-1}}{n-1} &\leqslant \left(\frac{x_1+x_{n-1}+\dfrac{x_1+\cdots+x_{n-1}}{n-1}}{n} \right)^n \\
x_1 \dots x_{n-1} \cdot \frac{x_1+\cdots+x_{n-1}}{n-1} &\leqslant \left(\frac{x_1+\cdots+x_{n-1}}{n-1} \right)^n \\
x_1 \dots x_{n-1} &\leqslant \left(\frac{x_1+\cdots+x_{n-1}}{n-1} \right)^{n-1}
\end{aligned}
$$

$P(n)$成立

b) 由 $P(n)$可得

$$
x_1 \dots x_n \cdot x_{n+1} \dots x_{2n} \leqslant \left(\frac{x_1+\cdots+x_n}{n} \right)^n \cdots \left(\frac{x_{n+1}+\cdots+x_{2n}}{n} \right)^n
$$

记$A = \left(\frac{x_1+\cdots+x_n}{n} \right), B = \left(\frac{x_{n+1}+\cdots+x_{2n}}{n} \right)$

由$P(2)$可得

$$
\begin{aligned}
AB &\leqslant \left(\frac{A+B}{2}\right)^2 \\
A^nB^n = (AB)^n &\leqslant \left(\frac{A+B}{2}\right)^{2n} \\
x_1 \ldots x_{2n} &\leqslant \left(\frac{x_1 + \cdots + x_{2n}}{2n}\right)^{2n}
\end{aligned}
$$

由此推知$P(2n)$成立。

c) Cauchy 向前-向后方法。

1. $P(2) \to P(4) \to \cdots P(2^n)$.
2. $P(n) \to P(n-1)$.

$\therefore \forall n \geqslant 1, P(n)$成立

练习 10 圆盘只能在ABC三根柱子上按照顺时针方向移动。记:

Q_n为n个盘从A到B最少移动的次数。

R_n为n个盘从B到A最少移动的次数。

题目解答 10 先列出两种移动方式各自的迭代式:

$$
Q_n = \begin{cases} 0, & n = 0 \\ 2R_{n-1} + 1, & n > 0 \end{cases} \qquad R_n = \begin{cases} 0, & n = 0 \\ Q_n + Q_{n-1} + 1, & n > 0 \end{cases}
$$

这两个公式是如何得到的？

$$
\begin{array}{lll}
k=0 & Q_0=0 & R_0=0 \\
k=1 & A\to B & B\to C\to A \\
 & Q_1=1 & R_1=2 \\
k=2 & 1:A\to B\to C & 1:B\to C\to A \\
 & 2:A\to B & 2:B\to C \\
 & 1:C\to A\to B & 1:A\to B \\
 & & 2æC\to A \\
 & & 1:B\to C\to A \\
 & Q_2=5 & R_2=7 \\
k=n & A\to B\ Q_n & B\to A\ R_n \\
 & (n-1)A\to C\ R_{n-1} & (n-1)B\to A\ R_{n-1} \\
 & n:A\to B\ 1 & n:B\to C\ 1 \\
 & (n-1)C\to B\ R_{n-1} & (n-1)A\to B\ Q_{n-1} \\
 & & n:C\to A\ 1 \\
 & & (n-1)B\to A\ R_{n-1} \\
 & Q_n=2R_{n-1}+1 & R_n=Q_{n-1}+2R_{n-1}+2=Q_n+Q_{n-1}+1
\end{array}
$$

练习 11 双重河内塔 2n个圆盘，第$2k-1$个与第$2k$个大小相同。

题目解答 11 a) 不区分相同尺寸

$$
\begin{array}{ll}
n=0S_0=0 & \\
n=1S_1=2 & A\to B, A\to B \\
n=2S_2=6 & A\to C, A\to C \\
\multicolumn{2}{c}{A\to B, A\to B} \\
\multicolumn{2}{c}{C\to B, C\to B}
\end{array}
$$

解得$S_n=2T_n=2(2^n-1)=2^{n+1}-2$

b) 在最后排列中将圆盘恢复次序需要移动几次？

$$
\begin{array}{llll}
k=0 & R_0=0 & & \\
k=1 & R_1=3 & 1.1:\ A\to C & \\
 & & 1.2:\ A\to B & \\
 & & 1.1:\ C\to B & \\
k=2 & R_2=11 & 1.1:\ A\to B & \\
 & & 1.2:\ A\to B & \\
 & & 2.1:\ A\to C & \\
 & & 1.2:\ B\to C & \\
 & & 1.1:\ B\to C & \\
 & & 2.2:\ A\to B & \\
 & & 1.1:\ C\to A & \\
 & & 1.2:\ C\to A & \\
 & & 2.1:\ C\to B & \\
 & & 1.2:\ A\to B & \\
 & & 1.1:\ A\to B & \\
k=n & R_n & n-1\ A\to B & S_{n-1} \\
 & & n.1:\ A\to C & 1 \\
 & & n-1\ B\to C & S_{n-1} \\
 & & n.2:\ A\to B & 1 \\
 & & n-1\ C\to A & S_{n-1} \\
 & & n.1:\ C\to B & 1 \\
 & & n-1\ A\to B & S_{n-1}
\end{array}
$$

$$R_n=4S_{n-1}+3=2^{n+2}-5 \quad (n\geqslant 1)$$

练习 12 12 11推广, m_k个尺寸为k的圆盘, 不区分相同尺寸的圆盘移动一个塔最少次数 $A(m_1,\ldots,m_n)$

题目解答 12

$$
\begin{aligned}
F(0) &= 0 \\
F(1) &= m_1 \\
F(2) &= 2F(1) + m_2 = 2m_1 + m_2 \\
&\vdots \\
F(n) &= 2F(n-1) + m_n
\end{aligned}
$$

$$
\begin{aligned}
A(m_1, \ldots, m_n) &= F(n) = 2F(n-1) + m_n \\
&= 2^{n-1}m_1 + 2^{n-2}m_2 + \cdots + m_n \\
&= \sum_{k=1}^{n} 2^{n-k} m_k
\end{aligned}
$$

练习 13 13

题目解答 13

$$
\begin{aligned}
k &= 1 \quad ZZ_1 = 2 + 0 = 2 \\
k &= 2 \quad ZZ_2 = 4 + 8 = 12 \\
k &= 3 \quad ZZ_3 = 6 + 25 = 31
\end{aligned}
$$

对于定义了L_n个区域的n条直线, 可以用极狭窄的Z形线来代替。

例如, 每一对Z形线间有9个交点

$$
\begin{aligned}
ZZ_n &= ZZ_{n-1} + 9n - 8, \quad (n > 0) \\
ZZ_n &= 9S_n - 8n + 1 \\
&= 9\frac{n(n+1)}{2} - 8n + 1 \\
&= \tfrac{9}{2}n^2 - \tfrac{7}{2}n + 1
\end{aligned} \tag{1}
$$

练习 14 14

题目解答 14

$$
\begin{aligned}
n &= 0 \quad P_0 = 1 \\
n &= 1 \quad P_1 = 2 \\
n &= 2 \quad P_2 = 4 \\
n &= 3 \quad P_3 = 8 \\
n &= 4 \quad P_4 = 8 + 6 = 14
\end{aligned}
$$

$$P_n = P_{n-1} + L_{n-1}$$

其中

$$L_n = 1 + S_n, \quad S_n = \frac{n(n+1)}{2}$$

$\therefore P_n = P_{n-1} + 1 + \frac{n(n+1)}{2}$

$$
\begin{aligned}
&P_0 && = 1 \\
&P_1 && = P_0 + L_0 && = 1 + 1 + \tfrac{0 \cdot 1}{2} && = 2 \\
&P_2 && = P_1 + L_1 && = 2 + 1 + \tfrac{1 \cdot 2}{2} && = 4 \\
&P_3 && = P_2 + L_2 && = 4 + 1 + \tfrac{2 \cdot 3}{2} && = 8 \\
&P_4 && = P_3 + L_3 && = 8 + 1 + \tfrac{3 \cdot 4}{2} && = 15 \\
&P_5 && = P_4 + L_4 && = 15 + 1 + \tfrac{4 \cdot 5}{2} && = 26
\end{aligned}
$$

$$
\begin{aligned}
P_n &= P_{n-1} + L_{n-1} \\
&= 0 + \sum_{k=0}^{n-1} \left(1 + \frac{k(k+1)}{2}\right) \\
&= n + \frac{(n-1)n(n+1)}{6} \\
&= \frac{n(n^2+5)}{6}
\end{aligned}
$$

练习 15 15 约瑟夫问题, 倒数第二个 $I(n)$

表 1: 约瑟夫问题J(n)与I(n)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1	3	5
I(n)	∼	2	1	3	5	1	3	5	7	9	11	1	3	5	7	9	11	13

题目解答 15 $n>1$时, $J(n), I(n)$有相同递归式

$$\begin{aligned} &I(2)=2, I(1)=1 \\ &n=2^m+2^{m-1}+k, \quad 0\leqslant k\leqslant 2^m+2^{m-1} \\ &I(n)=2k+1 \end{aligned}$$

$$n=2^m+l, \quad I(n)=\begin{cases} J(n)+2^{m-1}, & 0\leqslant l<2^{m-1} \\ J(n)-2^m, & 2^{m-1}\leqslant l<2^m \end{cases}$$

练习 16

$$\begin{cases} g(1) & =\alpha \\ g(2n+j) & =3g(n)+\gamma n+\beta_j, \quad j=0,1, n\leqslant 1 \end{cases}$$

(提示, 用$g(n)=n$)

题目解答 16 Suppose $g(n)=n$

$$g(1)=1=\alpha,$$

$$g(2n+j)=2n+j=3n+\gamma n+\beta_j.$$

解得 $\alpha=1, \gamma=-1, \beta_j=\begin{cases} 0, & j=0 \\ 1, & j=1 \end{cases}$

(题解)

$$g(n)=a(n)\alpha+b(n)\beta_0+c(n)\beta_1+d(n)\gamma$$

$n=(1b_{m-1}\ldots b_1b_0)_2$ 将n以基数2展开(写成二进制)。

$$a(n)\alpha+b(n)\beta_0+c(n)\beta_1=(\alpha\beta_{m-1}\beta_{m-2}\ldots\beta_{b_1}\beta_{b_0})_3$$

$g(n)=n.\quad (\alpha=1,\beta_0=0,\beta_1=1,\gamma=-1)$

$$a(n)+c(n)-d(n)=n$$

$g(n)=1.\quad (\alpha=1,\beta_0=-2,\beta_1=2,\gamma=0)$

$$a(n)-2b(n)-c(n)=1$$

$$\begin{aligned} d(n)&=a(n)+c(n)-n\\ b(n)&=\frac{1}{2}a(n)-\frac{1}{2}c(n)-\frac{1}{2}\end{aligned}$$

$$\begin{aligned} g(n)=a(n)\alpha+(\frac{1}{2}a(n)-\frac{1}{2}c(n)-\frac{1}{2})\beta_0\\ +c(n)\beta_1+(a(n)+c(n)-n)\gamma\end{aligned}$$

若$\beta_i=0\ (i=0,1),\gamma=0$

$$\begin{cases} g(1)=\alpha\\ g(2n+j)=3g(n)\end{cases}$$

$g(1)=\alpha, g(2)=g(3)=3\alpha, g(4)=g(5)=g(6)=g(7)=9\alpha, g(8)=3g(4)=27\alpha.$
由此推知

$$g(n)=g(2^m+k)=3^m,\quad a(n)=3^m$$

继续计算$d(n)$ 遇到困难

若$\alpha=0,\beta_0=0,\gamma=0, g(n)=\beta_1 c(n)$

$$\begin{cases} g(1)=0\\ g(2n)=3g(n)\\ g(2n+1)=3g(n)+1\end{cases}$$

表 2: m,k变化规律

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	0	0	1	0	1	3	4	0	1	3	4	9	10	12	13	0	1	3	4
m	0	1		2				3								4			
k			1		1	2	3		1	2	3	4	5	6	7		1	2	3

$g(2) = 0$, $g(3) = 1$, $g(4) = 0$, $g(5) = 1$, $g(6) = 3g(3) = 3$, $g(7) = 3g(3) + 1 = 4$, $g(8) = 0$, $g(9) = 1 \ldots$

$$g(2^m + k) = g((1b_{m-1}b_{m-2} \cdots_1 b_0)_2)$$

复习和重做16题的部分暂不录入

练习 17

$$W_{n(n+1)/2} \leqslant 2W_{n(n-1)/2} + T_n, \quad n > 0$$

题目解答 17 In general we have $W_m \leqslant 2W_{m-k} + T_k,\ 0 \leqslant k \leqslant M$ (This relation corresponds to transferring the top $m - k$. then using only three pegs to move the bottom $k \to T_k$, then finishing with the top $m - k \to 2 \cdot W_{m-k}$)

The stated relation turns out to be based on the unique value of k that minimizes the right-hand side of this general inequality， when $m\frac{n(n+1)}{2}$.

(However, we cannot conclude that equality holds. Many other strategies for transferring the tower are conceivale.)

If we set $Y_n = (W_{n(n+1)/2} - 1)/2^n$

we find that $Y_n \leqslant Y_{n-1} + 1$. hence $W_{n(n+1)/2} \leqslant 2^n(n-1) + 1$