Solve $\begin{cases} r_0 = 1 \\ r_n = r_{n-1} + 3n + 5 \end{cases}$

First, get some cases

$r_0 = 1$

$r_1 = 1 + 3\times 1 + 5 = 9$

$r_2 = 9 + 3\times 2 + 5 = 20$

$r_3 = 20 + 3\times 3 + 5 = 34$

n	0	1	2	3
r_n	1	9	20	34

Unsimplified cases

$r_0 = 1$

$r_1 = r_0 + 3\times 1 + 5 \quad = 1 + 3 + 5$

$r_2 = r_1 + 3\times 2 + 5$

$= 1 + 3 + 5 + 3\times 2 + 5$

$= 1 + 3\times 3 + 5\times 2$

$r_3 = r_2 + 3\times 3 + 5$

$= 1 + 3\times 3 + 5\times 2 + 3\times 3 + 5$

$= 1 + 3\times 6 + 5\times 3$

$r_4 = r_3 + 3\times 4 + 5$

$= 1 + 3\times 10 + 5\times 4$

A pattern in unsimplified cases

$$r_n = 1\,A(n) + 3\,B(n) + 5\,C(n)$$

where $A(n)$, $B(n)$, $C(n)$ are simple functions of n

$$\begin{cases} A(n) = 1 \\ B(n) = \frac{n(n+1)}{2} \\ C(n) = n \end{cases}$$

$$\begin{aligned} r_n &= 1\times 1 + 3\times \frac{n(n+1)}{2} + 5\times n \\ &= \frac{3}{2}n^2 + \frac{13}{2}n + 1 \end{aligned}$$

Summarizing

$$\begin{cases} r_0 = 1 \\ r_n = r_{n-1} + 3n + 5 \end{cases}$$

is $r_n = \frac{3}{2}n^2 + \frac{13}{2}n + 1$

Testing

n	0	1	2	3	4	5
r_n	1	9	20	34	51	71
$\frac{3}{2}n^2+\frac{13}{2}n+1$	1	9	20	34	51	71

Prove It by induction

First we generalize :

$$\begin{cases} r_0 = 1 \\ r_n = r_{n-1} + 3n + 5 \end{cases}$$

replace constants 1, 3, 5 by variables α, β, γ

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \end{cases}$$

Cases of our generalized version

n	r_n
0	α
1	$\alpha + \beta + \gamma$
2	$\alpha + \beta + \gamma + \beta \times 2 + \gamma$ $= \alpha + 3\beta + 2\gamma$
3	$\alpha + 3\beta + 2\gamma + \beta \times 3 + \gamma$ $= \alpha + 6\beta + 3\gamma$
4	$\alpha + 10\beta + 4\gamma$

Wild assumption:

Let's assume that there are three fixed functions A, B, C, such that the solution to the above always has this form:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

We don't know this is true, but the evidence suggests it

Can we figure out what A, B, and C are? Yes!

Is this easier than the original problem? Yes!

Here's How

We assume that any recurrence defined by:

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \end{cases}$$

has a solution that looks like:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

no matter what α, β, and γ are.

Different α, β, and γ will define different r_n

☆ but $A(n)$, $B(n)$ and $C(n)$ are the same of all of them!

What does this buy us?

For any α, β, and γ, the equations

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \end{cases}$$

are always solved by

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

If we pick up really simple functions (with really easy values for α, β and γ) we can solve for A, B and C.

And once we have A, B and C, we have a solution to the general recurrence.

Easy Solutions.

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \\ r_n = \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases}$$

First easy solution.

Let try $r_n = 1$

$$\begin{cases} 1 = \alpha \\ 1 = 1 + \beta n + \gamma \\ 1 = \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases}$$

$$\begin{cases} \alpha = 1 \\ \beta = 0 \\ \gamma = 0 \end{cases} \Rightarrow \quad A(n) = 1$$

$r_n = 1$ has consequences

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \\ r_n = \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases} \Rightarrow \begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \\ r_n = \alpha + \beta B(n) + \gamma C(n) \end{cases}$$

Let try $r_n = n$

$$\begin{cases} 0 = \alpha \\ n = n-1 + \beta n + \gamma \\ n = \alpha + \beta B(n) + \gamma C(n) \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 1 \end{cases} \Rightarrow \quad C(n) = n$$

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \\ r_n = \alpha A(n) + \beta B(n) + \gamma C(n) \end{cases} \Rightarrow \begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \\ r_n = \alpha + \beta B(n) + \gamma C(n) \end{cases} \Rightarrow \begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \\ r_n = \alpha + \beta B(n) + \gamma \cdot n \end{cases}$$

Let's try $r_n = n^2$

$$\begin{cases} 0 = \alpha \\ n^2 = (n-1)^2 + \beta n + \gamma \\ n^2 = \alpha + \beta B(n) + \gamma n \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 2 \\ \gamma = -1 \end{cases} \Rightarrow \quad \begin{matrix} 2B(n) - n = n^2 \\ B(n) = \frac{n(n+1)}{2} \end{matrix} \quad \Rightarrow \quad r_n = \alpha + \beta \frac{n(n+1)}{2} + \gamma n$$

Let's try it out

$$\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \end{cases}$$

$$r_n = \alpha + \beta \frac{n(n+1)}{2} + \gamma n$$

Testing

$$\begin{cases} r_0 = 1 \\ r_n = r_{n-1} + 3n + 5 \end{cases}$$

$$r_n = 1 + 3 \frac{n^2+n}{2} + 5n = \frac{3}{2}n^2 + \frac{13}{2}n + 1$$

Summations

Recurrence like these $\begin{cases} r_0 = \alpha \\ r_n = r_{n-1} + \beta n + \gamma \end{cases}$

with solution : $r_n = \alpha + \beta \frac{n^2+n}{2} + \gamma n$

Can be used to solve summations like these

$$S_n = \sum_{i=0}^{n} (3i+2)$$

$$\begin{cases} S_0 = 2 \\ S_n = S_{n-1} + 3n + 2 \end{cases}$$

$$\begin{cases} \alpha = 2 \\ \beta = 3 \\ \gamma = 2 \end{cases} \qquad S_n = 2 + 3 \frac{n^2+n}{2} + 2n$$

$$= \frac{3}{2}n^2 + \frac{7}{2}n + 2$$

Testing $\quad S_n = \sum_{i=0}^{n} (3i+2) = 3\sum_{i=0}^{n} i + 2\sum_{i=0}^{n} 1$

$$= 3\,\frac{(0+n)(n+1)}{2} + 2\cdot(n+1)$$

$$= \frac{3}{2}n^2 + \frac{7}{2}n + 2$$

Let's try something harder.

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases}$$

Case

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases}$$

n	0	1	2	3	4	5
r_n	1	3	8	19	42	89

First generalize

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases}$$

$$\begin{cases} r_0 = \alpha \\ r_n = \beta r_{n-1} + \gamma n \end{cases}$$

我的错误想法

$$\begin{cases} r_0 = \alpha \\ r_n = \beta r_{n-1} + \gamma n + \delta \end{cases}$$

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n) + \delta D(n)$$

$r_n = 1$ $\quad \begin{cases} 1 = \alpha \\ 1 = \beta \cdot 1 + \gamma n + \delta \end{cases}$

$$\begin{cases} \alpha = 1 \\ \beta + \delta = 1 \\ \gamma = 0 \end{cases} < \begin{matrix} \alpha = 1,\ \beta = 1,\ \gamma = \delta = 0 & ① \\ \alpha = 1,\ \beta = \gamma = 0\ \ \delta = 1 & ② \end{matrix}$$

① $(\alpha, \beta, \gamma, \delta) = (1, 1, 0, 0)$

$\underline{A(n) + B(n) = 1}$ $\quad B(n) = 1 - A(n)$

② $= (1, 0, 0, 1)$

$\underline{A(n) + D(n) = 1}$ $\quad D(n) = 1 - A(n)$

$r_n = n$ $\quad \begin{cases} 0 = \alpha \\ n = \beta(n-1) + \gamma n + \delta \end{cases}$

$$\begin{cases} \alpha = 0 \\ \beta + \gamma = 1 \\ \delta - \beta = 0 \end{cases} < \begin{matrix} (0, 1, 0, 1) \\ (0, 0, 1, 0) \end{matrix}$$

① $(0, 1, 0, 1)$

$B(n) + D(n) = n$ $\quad 2 - 2A(n) = n$

② $(0, 0, 1, 0)$ $\quad \underline{A(n) = 1 - \frac{n}{2}}$

$\underline{C(n) = n}$

$$r_n = \alpha\left(1 - \frac{n}{2}\right) + \beta\frac{n}{2} + \gamma n + \delta\frac{n}{2}$$

$$\begin{cases} r_0 = \alpha \\ r_n = \beta r_{n-1} + \gamma n \end{cases}$$

n	r_n
0	α
1	$\beta\alpha + \gamma$
2	$\beta(\beta\alpha + \gamma) + 2\gamma$
3	$\beta(\beta(\beta\alpha + \gamma) + 2\gamma) + 3\gamma$

α. β γ mixed up?

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases} \Longrightarrow \begin{cases} r_0 = \alpha \\ r_n = 2r_{n-1} + \beta n + \gamma \end{cases}$$

n	r_n
0	α
1	$2\alpha + \beta + \gamma$
2	$2(2\alpha + \beta + \gamma) + 2\beta + \gamma$
	$= 4\alpha + 4\beta + 3\gamma$
3	$2(4\alpha + 4\beta + 3\gamma) + 3\beta + \gamma$
	$= 8\alpha + 11\beta + 7\gamma$

没有 γ 的情况

$$\begin{cases} r_0 = \alpha \\ r_n = 2r_{n-1} + \beta n \end{cases}$$

n	r_n
0	α
1	$2\alpha + \beta$
2	$2(2\alpha+\beta) + 2\beta$
	$= 4\alpha + 4\beta$
3	$2(4\alpha + 4\beta) + 3\beta$
	$= 8\alpha + 11\beta$

$$r_n = \alpha A(n) + \beta B(n)$$

$r_n = 1$

$$\begin{cases} 1 = \alpha \\ \underline{1 = 2\cdot 1 + \beta n} \end{cases} \longrightarrow \text{不可能!}$$

所以必须有 γ

$$r_n = \alpha A n + \beta B(n) + \gamma C(n)$$

$r_n = 1$

$$\begin{cases} 1 = \alpha \\ 1 = 2\cdot 1 + \beta n + \gamma \end{cases}$$

$(\alpha, \beta, \gamma) = (1, 0, -1)$

$\underline{A(n) - C(n) = 1}$ $\qquad C(n) = A(n) - 1$

$r_n = n$

$$\begin{cases} 0 = \alpha \\ n = 2(n-1) + \beta n + \gamma \end{cases}$$

$(\alpha, \beta, \gamma) = (0, -1, 2)$

$-B(n) + 2C(n) = n$ $\qquad$ $B(n) = 2C(n) - n$
$\qquad = 2A(n) - 2 - n$

$r_n = 2^n$ $\qquad$ $\begin{cases} 1 = \alpha \\ 2^n = 2\, 2^{n-1} + \beta n + \gamma \end{cases}$

$(\alpha, \beta, \gamma) = (1, 0, 0)$ $\qquad$ $A(n) = 2^n$

$$\begin{cases} A(n) = 2^n \\ B(n) = 2^{n+1} - 2 - n \\ C(n) = 2^n - 1 \end{cases}$$

$$r_n = \alpha \cdot 2^n + \beta(2^{n+1} - 2 - n) + \gamma(2^n - 1)$$

$(\alpha, \beta, \gamma) = (1, 1, 0)$

$$r_n = 2^n + 2^{n+1} - 2 - n = 3 \cdot 2^n - 2 - n$$

Reprise

$$\begin{cases} r_0 = 1 \\ r_n = 2r_{n-1} + n \end{cases} \xrightarrow{\text{general}} \begin{cases} r_0 = \alpha \\ r_n = 2r_{n-1} + \beta n + \gamma \end{cases}$$

$$r_n = \alpha 2^n + \beta(2^{n+1} - 2 - n) + \gamma(2^n - 1)$$

$$\begin{cases} 1 = A(n) - C(n) \\ n = -B(n) + 2C(n) \\ 2^n = A(n) \end{cases}$$

$\longrightarrow$ 这份ppt非常详细

我以后总结知识也应如此

将来回顾可以快速看懂.