Exercise 15,16,17

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1 Exercise 15

The original equation:

$$u'(t) = -a(t)u(t) + b(t) \tag{1}$$

with $u(0) = I$.

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = -a^{n-1}u^{n-1} + b^{n-1}$$
$$u^{n+1} = (1 - 2\Delta t a^{n-1})u^{n-1} + 2\Delta t b^{n-1}$$

Use Forward Euler scheme to obtain the u^1.

$$u^{n+1} = (1 - 2\Delta t a^n)u^n + 2\Delta t b^n$$
$$u^1 = (1 - 2\Delta t a^0)I + 2\Delta t b^0$$

For $u' = -u + 1, u(0) = 0$, the numerical and exact solutions are plotted in Figure 1.

Convergence rate: consider two consecutive data, $(\Delta t_{i-1}, E_{i-1})$ and $(\Delta t_i, E_i)$. $E_{i-1} = C(\Delta t_{i-1})^r, E_i = C(\Delta t_i)^r$. Solving for r

$$r_{i-1} = \frac{\ln(E_{i-1}/E_i)}{\ln(\Delta t_{i-1}/\Delta t_i)} \tag{2}$$

for $i = 1, 2, 3, \ldots N$. We obtain a sequence r_i, and the last value r_N can be taken as the convergence rate. For $t \in [0, 5]$, I choose 50 different time steps Δt which are between 0.01 and 0.5. The convergence rate $r \approx 1.0$.

2 Exercise 16

I choose 4 different time step: $\Delta t = 0.1, 0.15, 0.2, 0.25$. The results are shown in Figure 2,3,4,5.

3 Exercise 17

Assume an exact solution has the form $u^n = A^n$, the Leapfrog scheme

$$u^{n+1} = (1 - 2\Delta t a)u^{n-1},$$

we get

$$A^2 = (1 - 2\Delta t a)$$
$$A_1 = \sqrt{(1 - 2\Delta t a)}, A_2 = -\sqrt{(1 - 2\Delta t a)}.$$
$$u^n = C_1 A_1^n + C_2 A_2^n$$

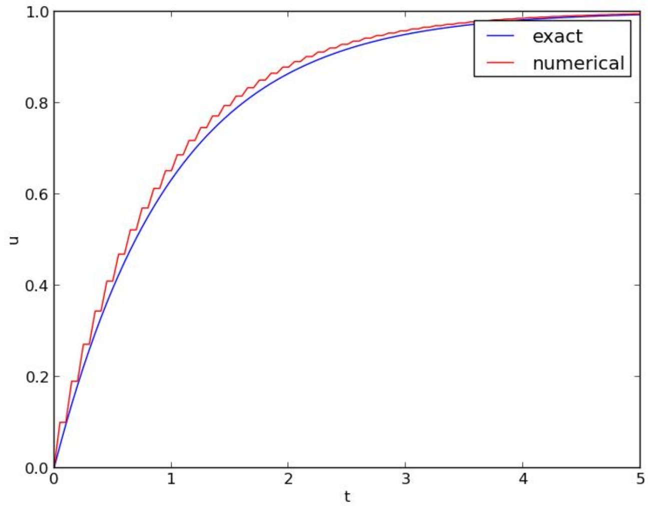

Figure 1: Leapfrog scheme: $t \in [0, 5], \Delta t = 0.05$

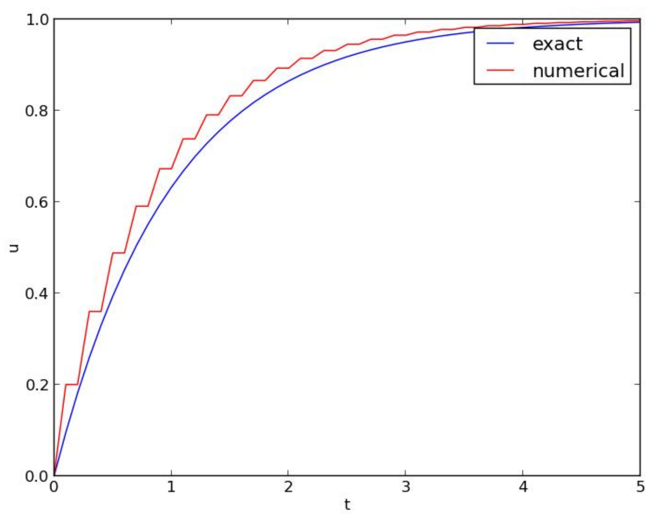

Figure 2: Leapfrog scheme: $t \in [0, 5], \Delta t = 0.1$

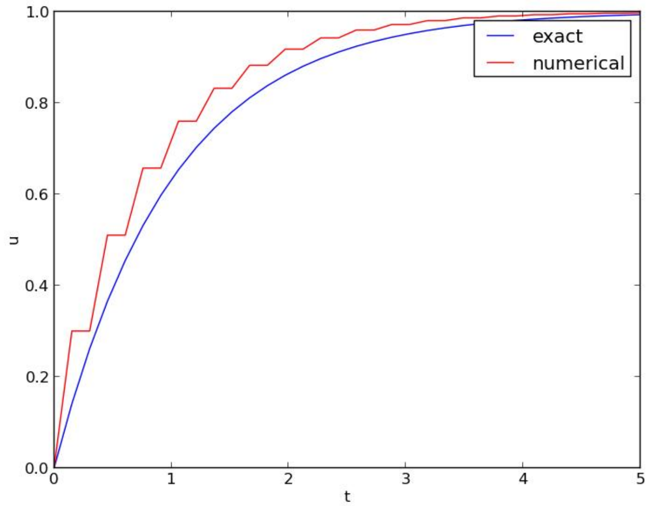

Figure 3: Leapfrog scheme: $t \in [0, 5], \Delta t = 0.15$

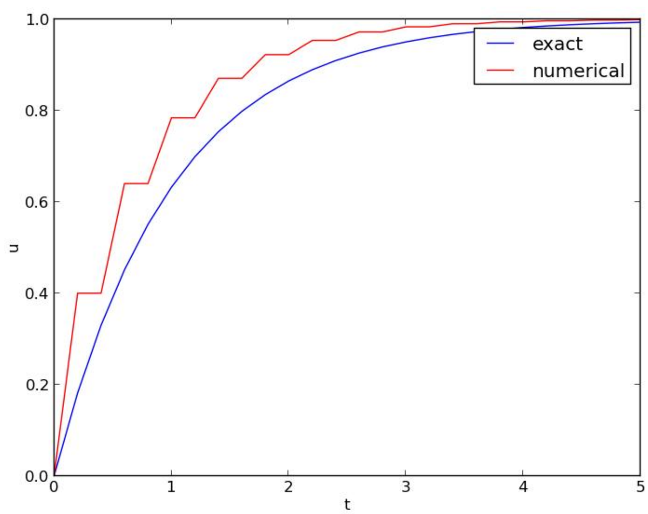

Figure 4: Leapfrog scheme: $t \in [0, 5], \Delta t = 0.2$

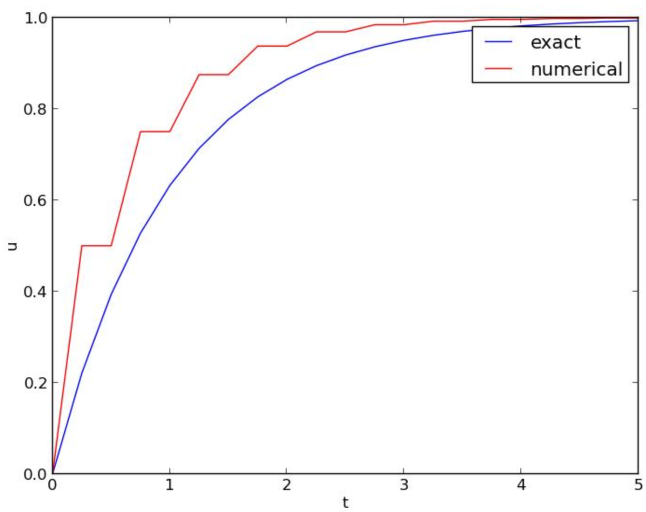

Figure 5: Leapfrog scheme: $t \in [0, 5], \Delta t = 0.25$