FDM for wave motion Exercise

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1 Exercise 4: Use ghost cells to implement Neumann conditions

The 1-D wave equation:

$$u_{tt} = c^2 u_{xx} + f(x,t) \tag{1}$$

with $u(x,0) = I(x), u_t(x,0) = V(x)$.

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + f_i^n \tag{2}$$

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + (c\frac{\Delta t}{\Delta x})^2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + (\Delta t)^2 f_i^n \tag{3}$$

For Neumann boundary condition

$$\frac{\partial u}{\partial n} = b, \tag{4}$$

We implement (4) at boundary by central difference:

$$\frac{u_{-1}^n - u_1^n}{2\Delta x} = b \tag{5}$$

$$\frac{u_{N_x+1}^n - u_{N_x-1}^n}{2\Delta x} = b \tag{6}$$

The problem is that $u_{-1}^n, u_{N_x+1}^n$ are not u values which are outside the mesh. But we can obtain:

$$u_{-1}^n = u_1^n + 2b\Delta x \tag{7}$$

$$u_{N_x+1}^n = u_{N_x-1}^n + 2b\Delta x \tag{8}$$

$$u_i^1 = 0.5(2u_i^0 + 2\Delta t V_i + (c\frac{\Delta t}{\Delta x})^2 (u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + (\Delta t)^2 f_i^n) \tag{9}$$

1.1 implementation via ghost cells

Instead of modifying the difference scheme at the boundary, extra points outside the mesh such as u^n_{-1} and $u^n_{N_x+1}$ can be defined. The standard difference scheme can be used when ghost cells are introduced. `Filename:wave1D_dn_ghost.c`

2 Exercise 5: Find a symmetry boundary condition

If we choose the solution is symmetric around $x = 0$, we can simulate the wave only in half of the domain with the Neumann boundary at $x = 0$. It means the wave will be reflected back at $x = 0$. `Filename:wave1D_symmetric.c`

3 Exercise 6: Prove symmetry of a 1D wave problem computationally

We perform simulations of the wave on $[-10, 10]$. Half of the domain is $[-10, 0]$. The point $x = 0$ has Neumann boundary condition. The numerical solutions in half of the domain and whole domain are the same. `Filename:wave1D_symmetric.c`

4 Exercise 8: Send pulse waves through a layered medium

In the homogeneous case with $s_f = 1$, the wave can preserve its shape in $x \in [0.7, 0.9]$. If the wave velocity is decreased by a factor s_f in the above mentioned domain, the wave changes its shape. The experiments show non-physical waves, especially for $s_f = 4$ and the plug pulse and the half a "cohat" pulse. The noise is less visible for the Gaussian pulse. See Figure 1,2 3,4. `Filename:pulse1D.py`

5 Exercise 10: Investigate harmonic averaging in a 1D model

Replace arithmetic mean by harmonic mean

$$q_{i+1} \approx 2(\frac{1}{q_i} + \frac{1}{q_{i+1}})^{-1} \tag{10}$$

From Figure 5, 6, 7, 8, we can see that replacing arithmetic mean by harmonic mean can't give better numerical results.

`Filename:pulse1D_harmonic.py`

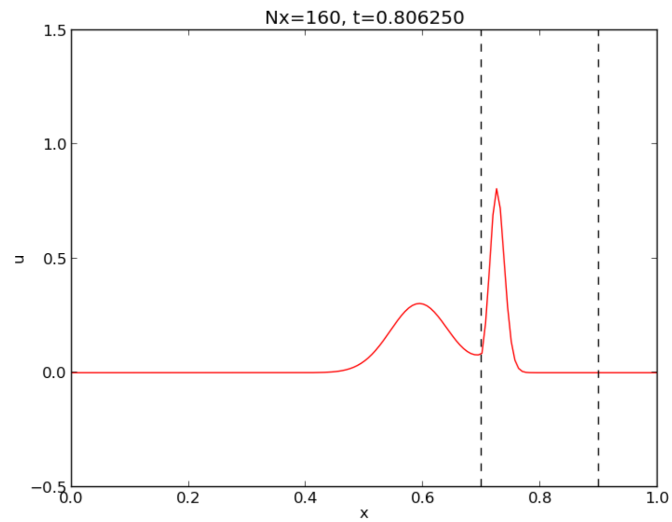

Figure 1: Gaussian pulse, $s_f = 4$, $N_x = 160$,$T = 2$

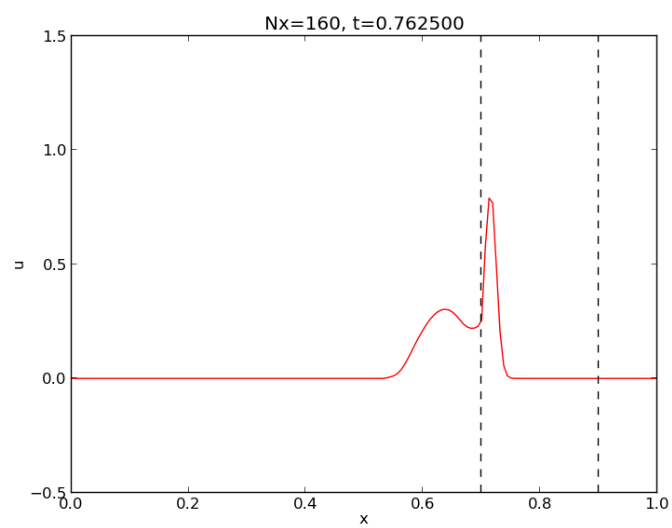

Figure 2: cohat pulse, $s_f = 4$, $N_x = 160$,$T = 2$

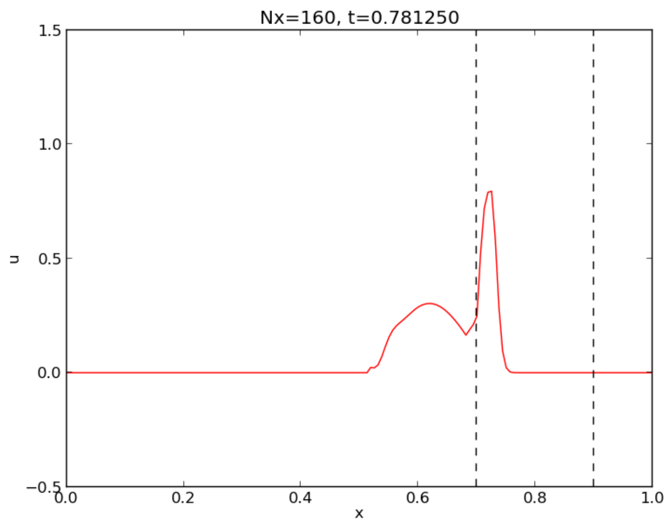

Figure 3: half-cohat pulse, $s_f = 4$, $N_x = 160$,$T = 2$

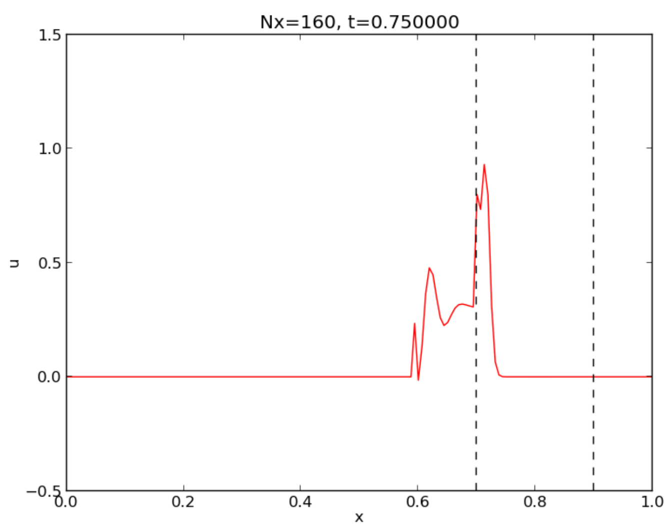

Figure 4: plug pulse, $s_f = 4$, $N_x = 160$,$T = 2$

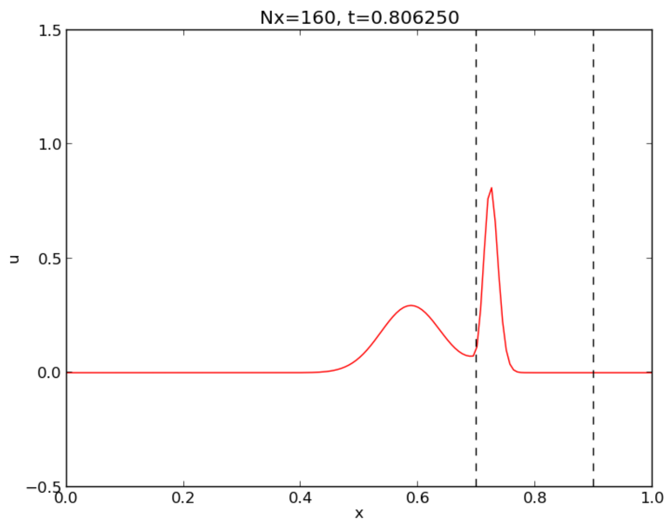

Figure 5: Gaussian pulse(harmonic mean) with, $s_f = 4$, $N_x = 160$,$T = 2$

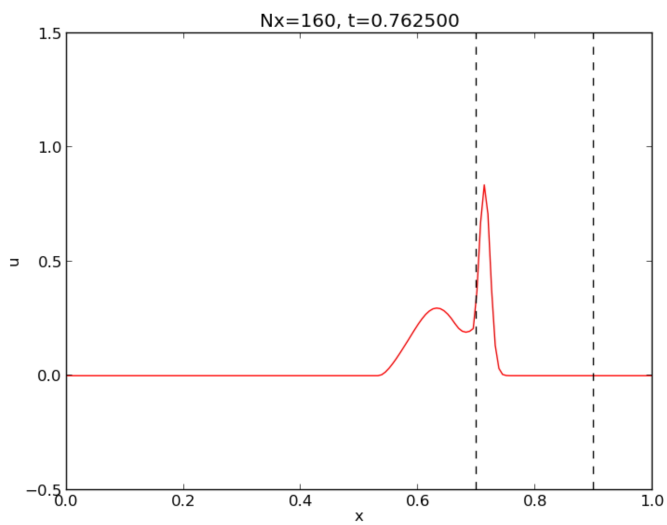

Figure 6: cohat pulse(harmonic mean), $s_f = 4$, $N_x = 160$,$T = 2$

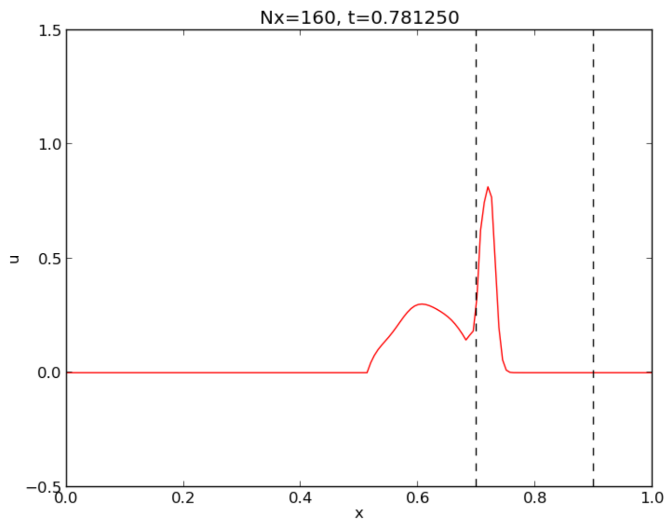

Figure 7: half-cohat pulse(harmonic mean), $s_f = 4$, $N_x = 160$,$T = 2$

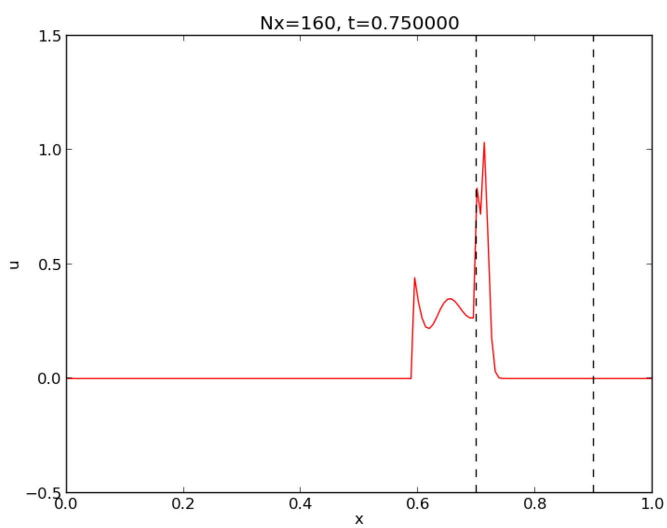

Figure 8: plug pulse(harmonic mean), $s_f = 4$, $N_x = 160$,$T = 2$