Exercise 15,16,17

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1 Exercise 15

The original equation:

$$u'(t) = -a(t)u(t) + b(t) \tag{1}$$

with $u(0) = I$.

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = (-(1-\theta)a^n u^n - \theta a^{n+1}u^{n+1} + (1-\theta)b^n + \theta b^{n+1}) * 2\Delta t$$
$$u^{n+1} = \frac{u^{n-1} + (-(1-\theta)a^n u^n - \theta a^{n+1}u^{n+1} + (1-\theta)b^n + \theta b^{n+1}) * 2\Delta t}{1 + 2\Delta t\theta a^{n+1}}$$

Use Forward Euler scheme to obtain the u^1.

$$\frac{u^1 - u^0}{\Delta t} = -a^0 u^0 + b^0$$
$$u^1 = (-a^0 u^0 + b^0)\Delta t + u^0$$

For the case $u' = -u + 1, u(0) = 0$, $u^1 = b^0 \Delta t$.

Convergence rate: consider two consecutive data, $(\Delta t_{i-1}, E_{i-1})$ and $(\Delta t_i, E_i)$. $E_{i-1} = C(\Delta t_{i-1})^r, E_i = C(\Delta t_i)^r$. Solving for r

$$r_{i-1} = \frac{\ln(E_{i-1}/E_i)}{\ln(\Delta t_{i-1}/\Delta t_i)} \tag{2}$$

for $i = 1, 2, 3, \ldots N$. We obtain a sequence r_i, and the last value r_N can be taken as the convergence rate. For $t \in [0, 4], \theta = 0$, I choose 6 different time steps $\Delta t = 0.2, 0.1, 0.05, 0.03, 0.02, 0.01$. The convergence rate $r = 2$, which is the expected value. The choice of the first time step has no impact on the overall accuracy of the Leapfrog scheme.

2 Exercise 16

I choose 4 different time step: $\Delta t = 0.1, 0.05, 0.03, 0.0.01$. The results are shown in Figure 1,2,3,4. Choosing smaller time step can have better results. Large time step show instabilities of the scheme.

3 Exercise 17

Assume an exact solution has the form $u^n = A^n$, the Leapfrog scheme

$$u^{n+1} = (1 - 2\Delta ta)u^{n-1},$$

we get

$$\begin{aligned}
A^2 &= (1 - 2\Delta ta) \\
A_1 &= \sqrt{(1 - 2\Delta ta)} \geq 0 \\
A_2 = -A_1 &= -\sqrt{(1 - 2\Delta ta)} \leq 0 \\
u^n &= C_1 A_1^n + C_2 A_2^n \\
&= [C_1 + C_2(-1)^n]A_1
\end{aligned}$$

Due to $(-1)^n$, there are instabilities with the leapfrog scheme. Here we choose $dt = 0.1$ and $a = 1$. We plot 4 different examples:

- $C_1 = 1, C_2 = 1$. See Figure 5
- $C_1 = -1, C_2 = -1$. See Figure 6
- $C_1 = 1, C_2 = -1$. See Figure 7
- $C_1 = -1, C_2 = 1$. See Figure 8

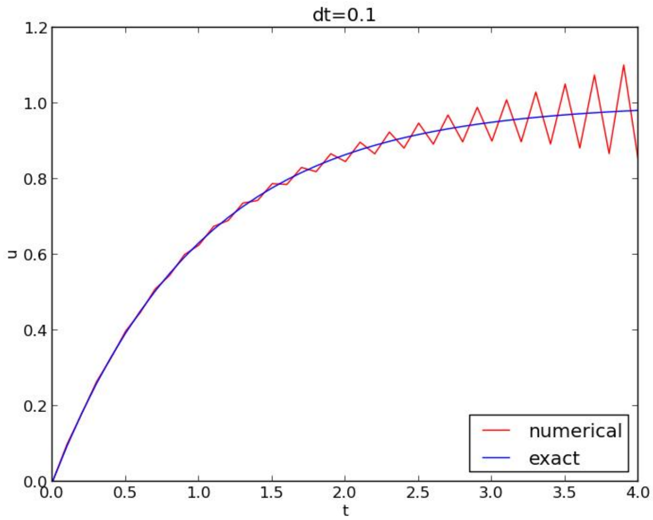

Figure 1: Leapfrog scheme $dt = 0.1$

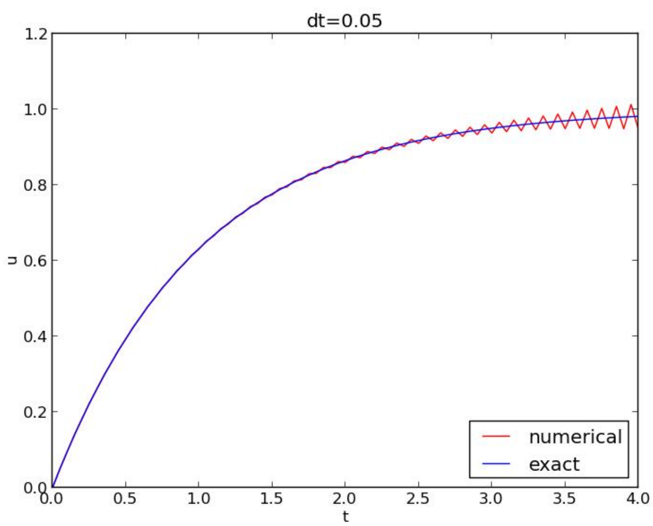

Figure 2: Leapfrog scheme $dt = 0.05$

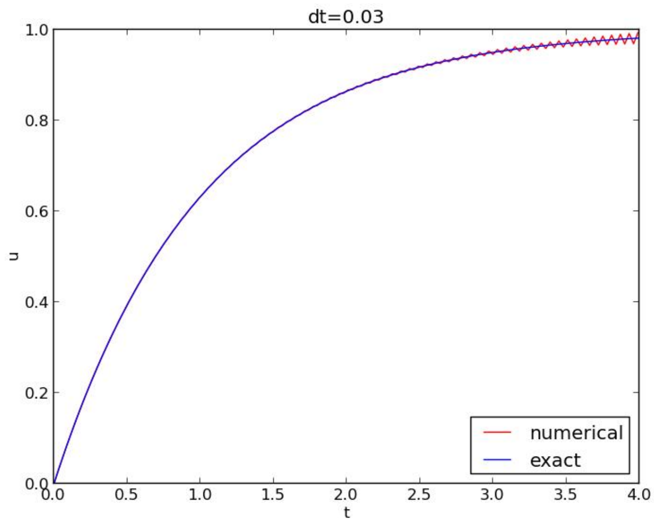

Figure 3: Leapfrog scheme $dt = 0.03$

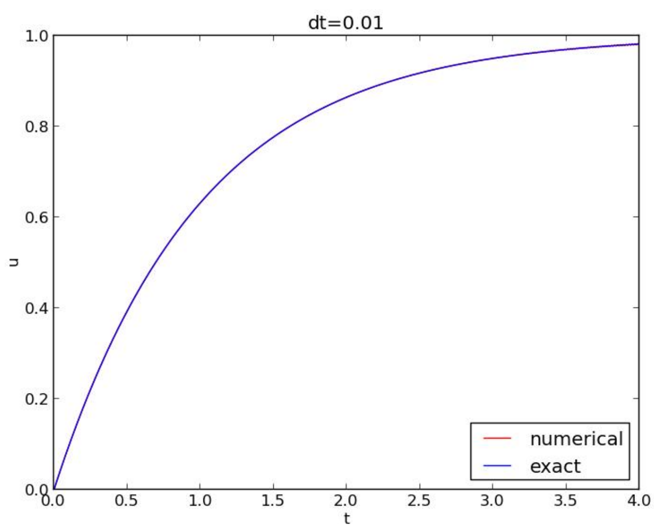

Figure 4: Leapfrog scheme $dt = 0.01$

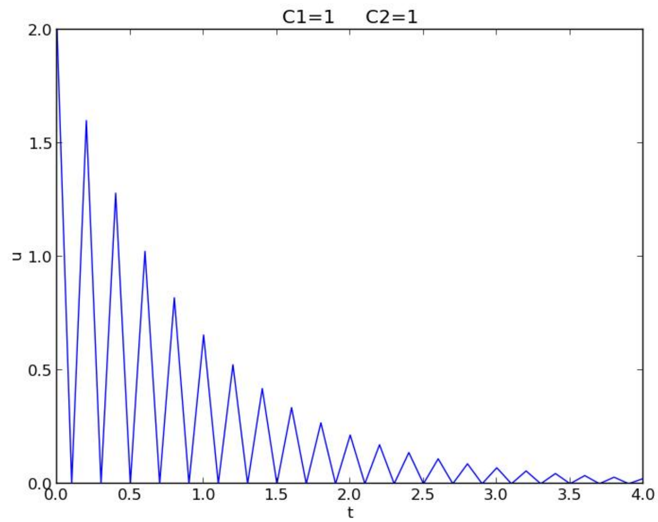

Figure 5: Analyze Leapfrog scheme $C_1 = 1, C_2 = 1$

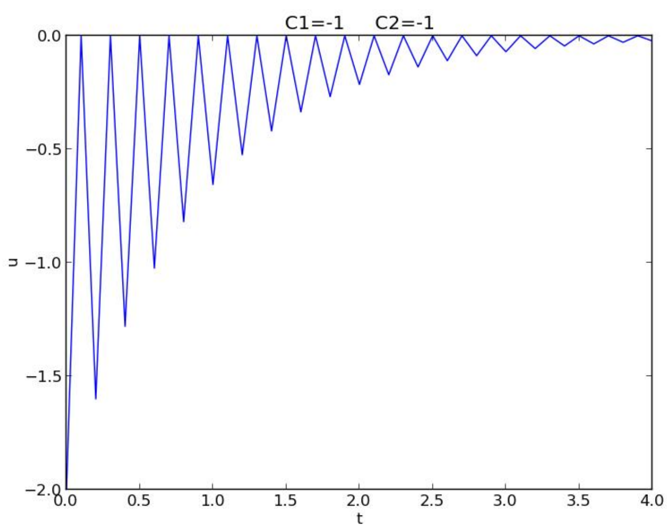

Figure 6: Analyze Leapfrog scheme $C_1 = -1, C_2 = -1$

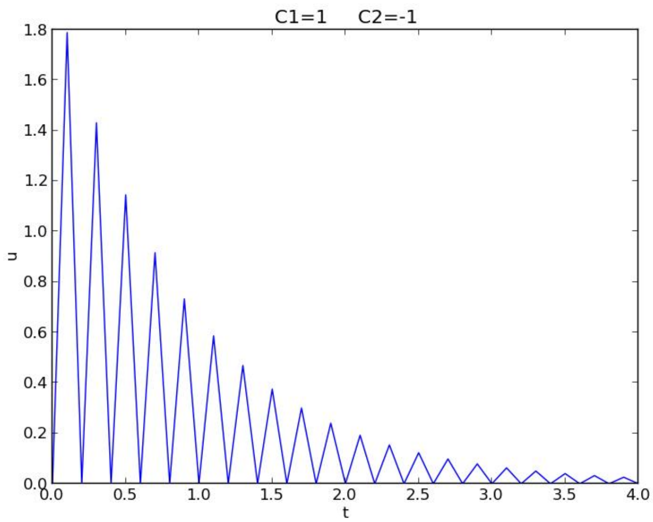

Figure 7: Analyze Leapfrog scheme $C_1 = 1, C_2 = -1$

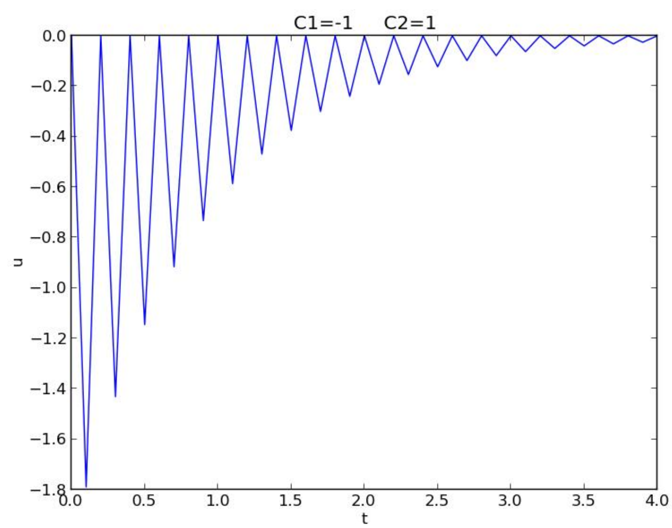

Figure 8: Analyze Leapfrog scheme $C_1 = -1, C_2 = 1$