Finite Element Method

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1 Exercise 17: Compute the deflection of a cable with sine functions

The problem is

$$u'' = 1, x \in (0,1), \qquad u(0) = 0, \quad u'(1) = 0 \tag{1}$$

We choose the base functions $\varphi_i = \sin((i+1)\pi x/2)$, $i = 1, \ldots, N$.

1.1 Galerkin method:

$$u \approx \hat{u} = 0 + \sum_{j=1}^{N} u_j N_j(x)$$

with $N_j(0) = 0$. We choose weighting function $W_i = N_i$. This leads to:

$$\sum_{j=1}^{N} \left(\int_0^1 N_i N_j''(x) dx \right) u_j = \int_0^1 N_i dx, \qquad i = 1, \ldots, N. \tag{2}$$

Integrating by parts on the left side

$$-\sum_{j=1}^{N} \left(\int_0^1 N_i' N_j'(x) dx \right) u_j + N_i(1)\hat{u}'(1) - N_i(0)\hat{u}'(0) = \int_0^1 N_i dx, \quad i = 1, \ldots, N. \tag{3}$$

Since $\hat{u}'(1) = 0$, $N_i(0) = 0$, we get

$$\sum_{j=1}^{N} \left(\int_0^1 N_i' N_j'(x) dx \right) u_j = -\int_0^1 N_i dx, \quad i = 1, \ldots, N. \tag{4}$$

We get a linear system $\mathbf{A}u = \mathbf{b}$, with

$$A_{i,j} = \int_0^1 N_i' N_j'(x) dx, \qquad b_i = -\int_0^1 N_i dx. \tag{5}$$

1.2 Least Squares Method:

The residual

$$R = \sum_{j=1}^{N} u_j N_j''(x) - 1 \tag{6}$$

the derivative

$$\frac{\partial R}{\partial u_i} = \sum_{j=1}^{N} \frac{\partial}{\partial u_j} N_j''(x) = N_i''(x) \tag{7}$$

the least squares equation

$$\int_0^1 \left(\sum_{j=1}^{N} u_j N_j''(x) - 1 \right) N_i''(x) dx = 0$$

$$\sum_{j=1}^{N} \left(\int_0^1 N_i''(x) N_j''(x) dx \right) u_j = \int_0^1 N_i'' dx, \qquad i = 1, \ldots, N. \tag{8}$$

We get a linear system $\mathbf{A}u = b$, with

$$A_{i,j} = \int_0^1 N_i'' N_j''(x) dx, \qquad b_i = \int_0^1 N_i'' dx. \tag{9}$$

The results when using one basis function are shown in Figure 1.

If we choose basis functions $\varphi_i = \sin((i+1)\pi x)$, the matrix $\mathbf{A}$ will become diagonal matrix since the basis functions are orthogonal

$$\varphi_i \varphi_j = 0, \qquad i \neq j \tag{10}$$

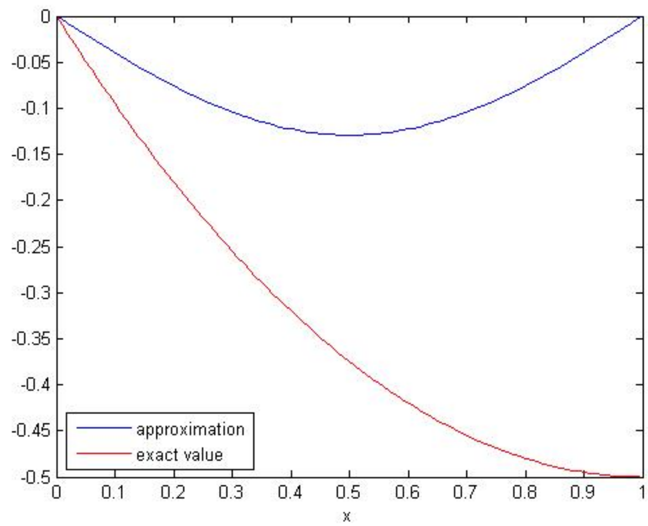

Figure 1: Compute the deflection of a cable with using one basis function