Finite Element Method

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1 Exercise 10: Finite Element Algorithm by hand

The nodes: $[0, \pi/2, \pi]$, the elements $[0,1], [1,2]$.

$$\varphi_0 = 1 - \frac{x}{\pi/2}, (0 \leq x \leq \frac{\pi}{2})$$

$$\varphi_1 = \begin{cases} \frac{x}{\pi/2} & 0 \leq x \leq \frac{\pi}{2} \\ 1 - \frac{x - \pi/2}{\pi/2} & \frac{\pi}{2} < x \leq \pi \end{cases}$$

$$\varphi_2 = \frac{x - \pi/2}{\pi/2}, (\frac{\pi}{2} \leq x \leq \pi)$$

$$\hat{u} = \begin{cases} c_0\varphi_0 + c_1\varphi_1 & 0 \leq x \leq \frac{\pi}{2} \\ c_1\varphi_1 + c_2\varphi_2 & \frac{\pi}{2} < x \leq \pi \end{cases} \tag{1}$$

$$A^{(0)}_{i,j} = \int_0^{\pi/2} \varphi_i\varphi_j dx,$$

$$A^{(1)}_{i,j} = \int_{\pi/2}^{\pi} \varphi_i\varphi_j dx$$

$$A^{(0)} = A^{(1)} = \begin{pmatrix} \pi/6 & \pi/12 \\ \pi/12 & \pi/6 \end{pmatrix}$$

$$b^{(0)}_0 = \int_0^{\pi/2} \varphi_0 \sin(x) dx = 1 - \frac{2}{\pi},$$

$$b^{(0)}_1 = \int_0^{\pi/2} \varphi_1 \sin(x) dx = \frac{2}{\pi},$$

$$b_0^{(1)} = \int_{\pi/2}^{\pi} \varphi_0 \sin(x) dx = \frac{2}{\pi},$$
$$b_1^{(1)} = \int_{\pi/2}^{\pi} \varphi_0 \sin(x) dx = 1 - \frac{2}{\pi},$$

We assemble the elements

$$\begin{pmatrix} \pi/6 & \pi/12 & 0 \\ \pi/12 & \pi/6 + \pi/6 & \pi/12 \\ 0 & \pi/12 & \pi/6 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_0^{(0)} \\ b_1^{(0)} + b_0^{(1)} \\ b_1^{(1)} \end{pmatrix}$$

$$c_0 = 0.1148, c_1 = 1.1585, c_2 = 0.1148 \tag{2}$$

$$\hat{u} = \begin{cases} 0.1148(1 - \dfrac{x}{\pi/2}) + 1.1585\dfrac{x}{\pi/2} & 0 \leq x \leq \dfrac{\pi}{2} \\ 1.1585(1 - \dfrac{x - \pi/2}{\pi/2}) + 0.1148\dfrac{x - \pi/2}{\pi/2} & \dfrac{\pi}{2} < x \leq \pi \end{cases} \tag{3}$$

The results are plotted in Figure 1.

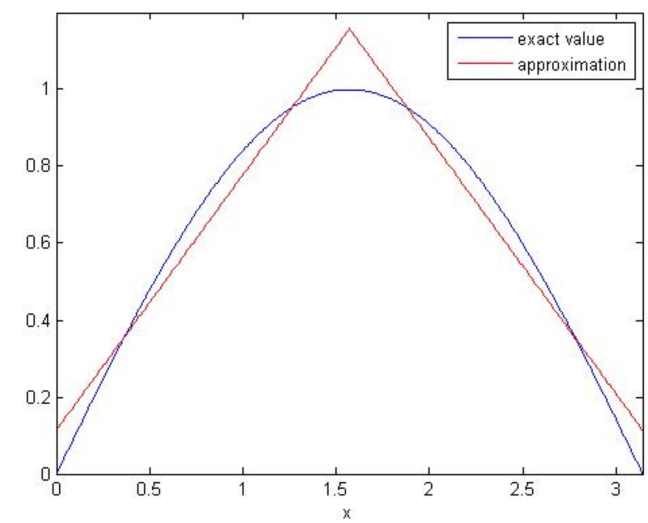

Figure 1: Finite element approximation for $f = sin(x)$ using 2 P1 elements