Frequency Domain

(Part-1)

The terms ***Function*** and ***Signal*** are used interchangeably in the context of Frequency domain processing

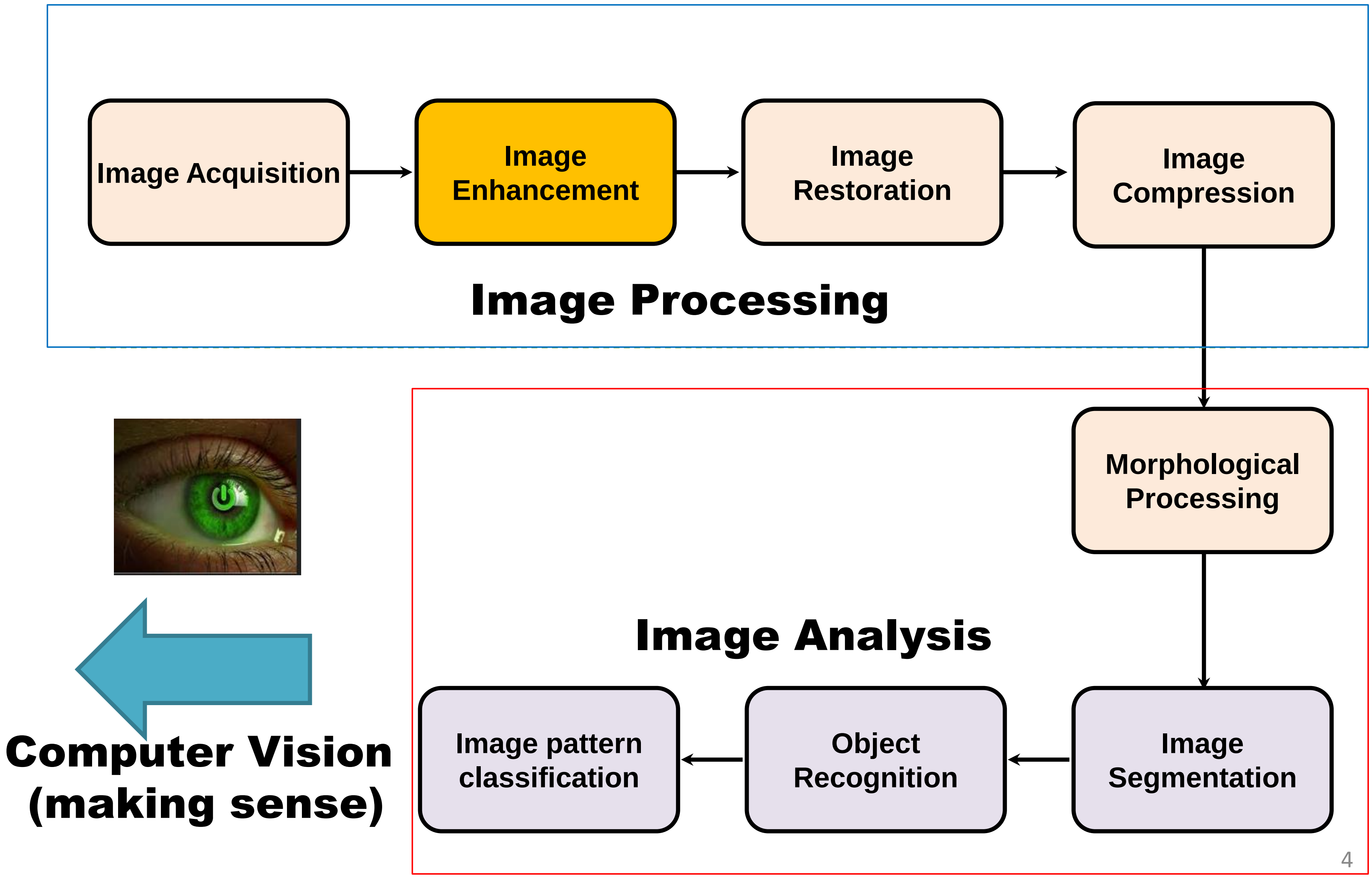
Recap

- Sharpening spatial filters
  - Foundation of image sharpening
  - Using second-order derivative for image sharpening (Laplacian)
  - Unsharp masking and high boost filtering
  - Using first-order derivative for image sharpening (Gradient)
  - Highpass, Bandreject, and Bandpass filters from lowpass filters
  - Combining spatial enhancement methods

Lecture Objectives

- Introduction to Frequency Domain
  - Background
  - Sinusoidal Waves
  - Complex Numbers
- Fourier Series
- Impulse
- Fourier Transform
- Convolution of Continuous Functions

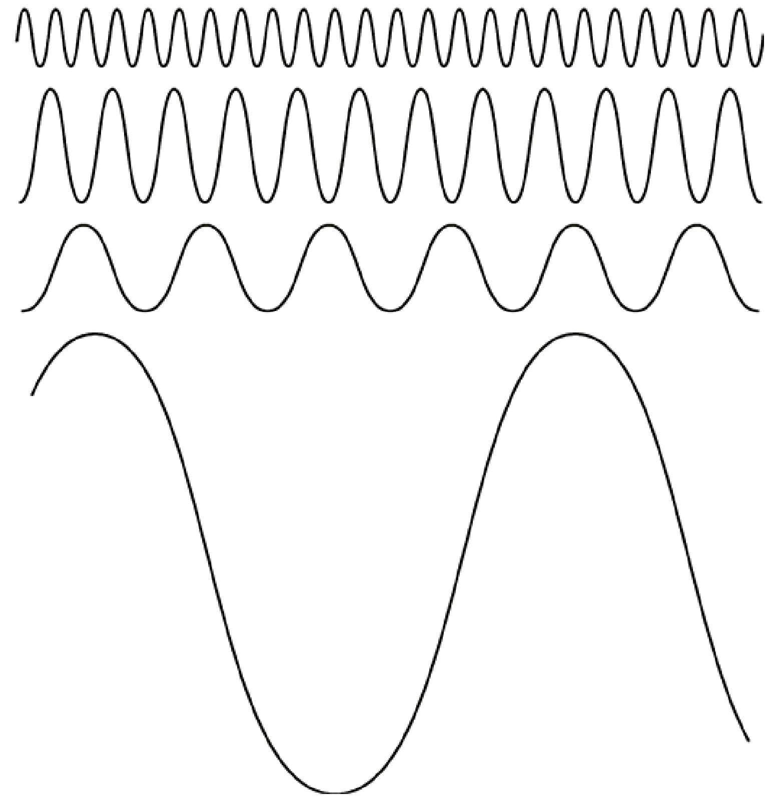
Key Stages in DIP



Background

Representing Periodic Function as Sines & Cosines

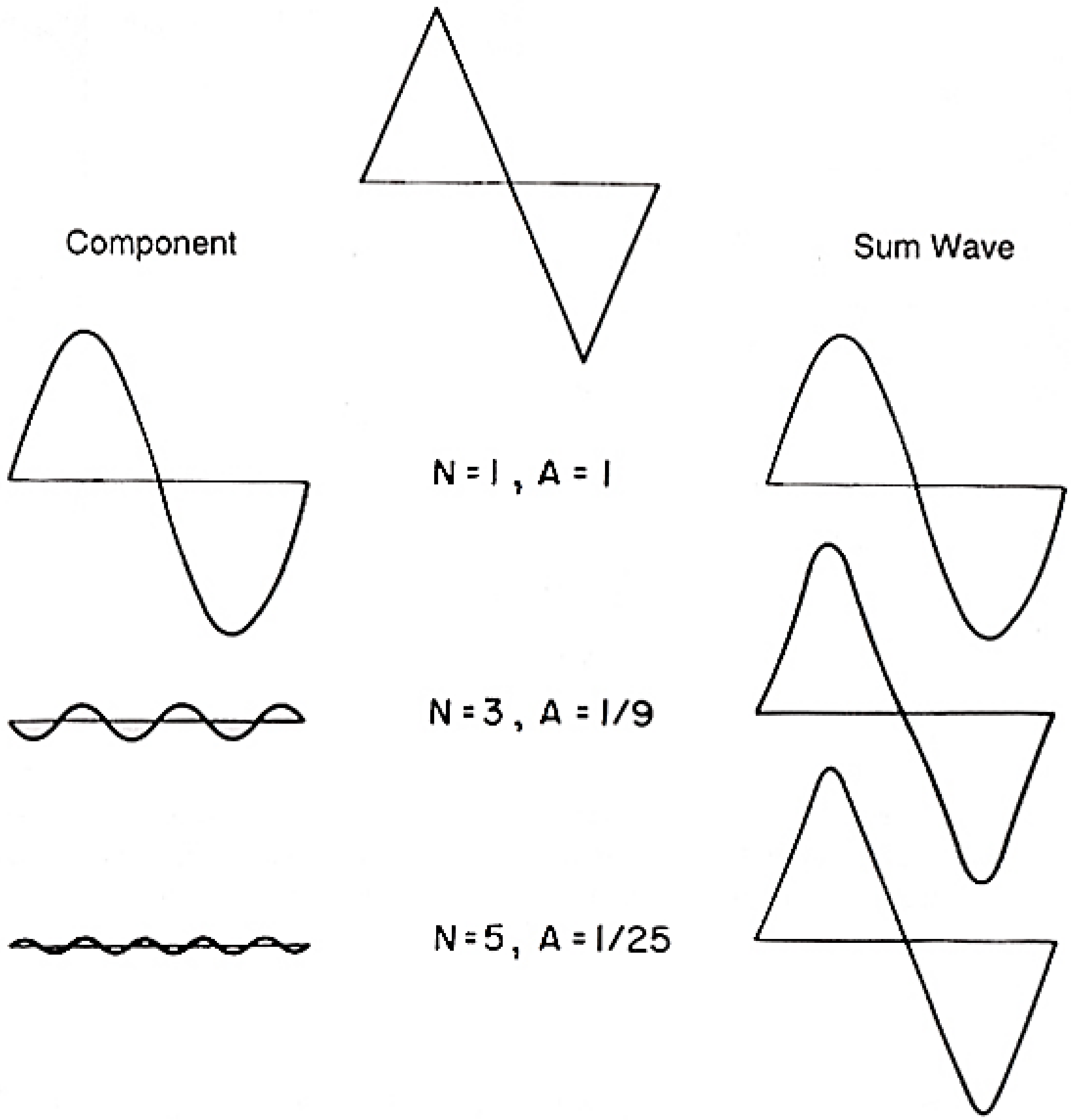
Periodic function



Sines and Cosines

Fourier series **- Any *periodic function* can be expressed as the sum of sines and/or cosines of *different frequencies*, each multiplied by a *different coefficient*.**

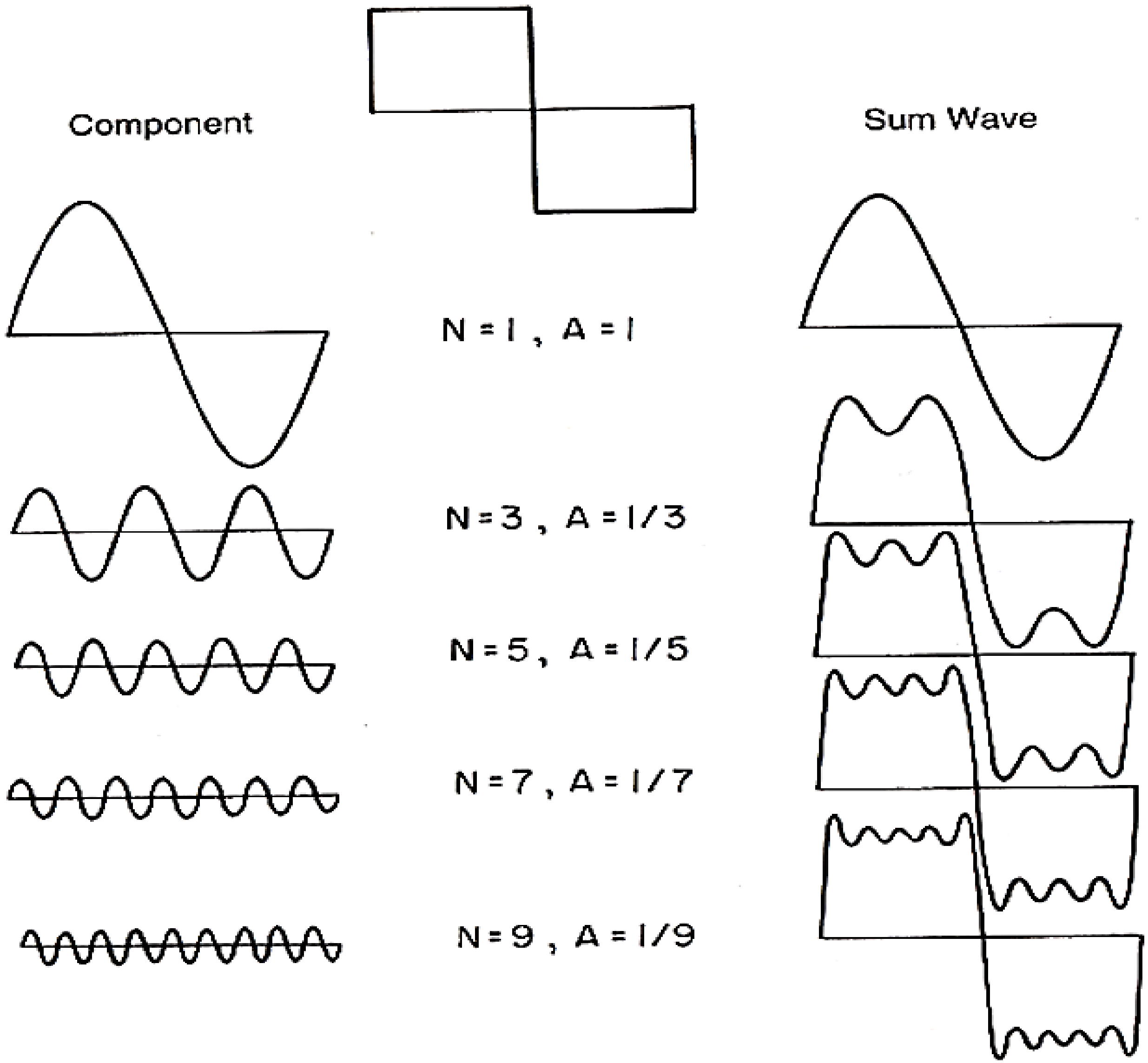
Representing Periodic Function as Sines & Cosines



Making a triangle wave with a sum of harmonics.

Adding in higher frequencies makes the triangle tips sharper and sharper

Representing Periodic Function as Sines & Cosines

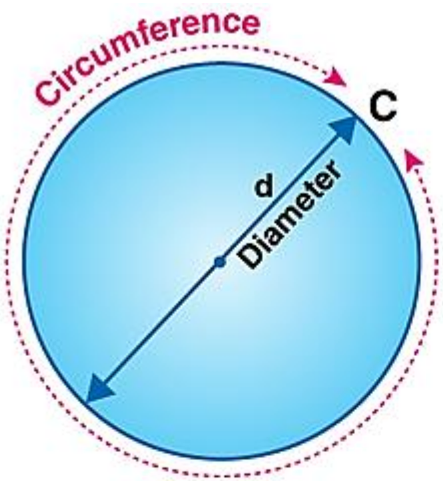


Sinusoidal Waves

Remember !!! **π** radians = **180°**

Why **π** value is 3.14159….

- The value of **Pi (π)** is the ***ratio*** of the *circumference of a circle to its diameter.*

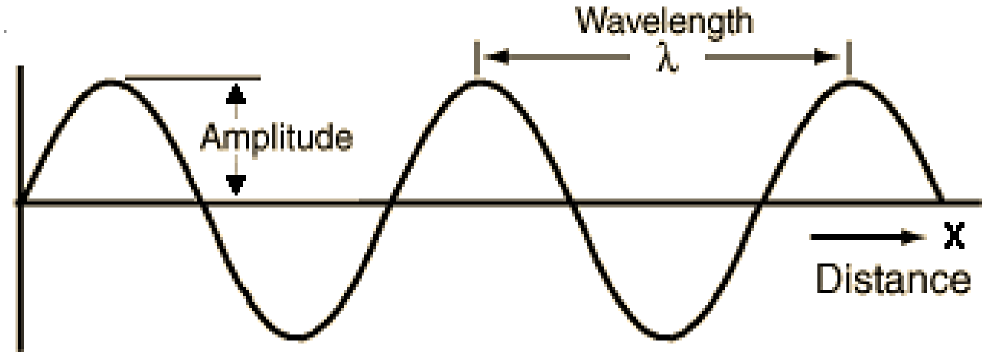


$$\frac{\text{Circumference}}{\text{Diameter}} = \pi = 3.14159.....$$

Properties of a Wave

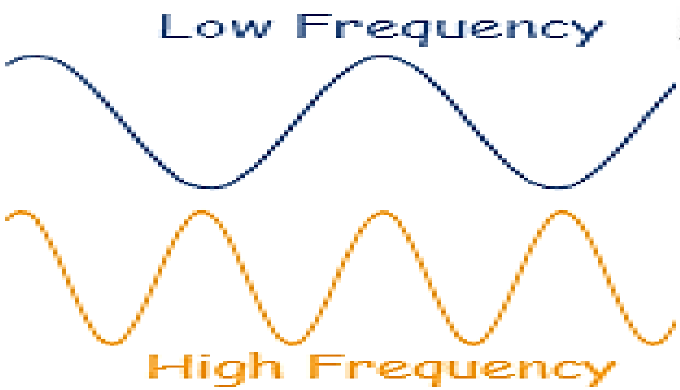
- **Wavelength (λ)** is the length from one peak to the next (or from any point to the next matching point).

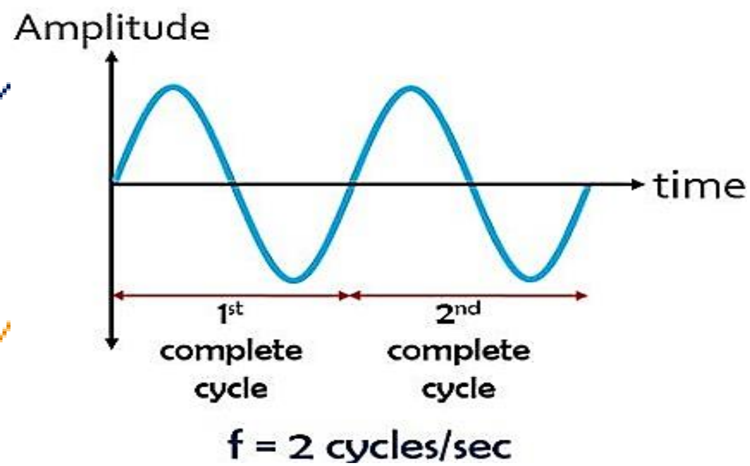
- **Amplitude (*A*)** is the height from the center line to the peak (or to the trough). **OR,** we can measure the height from highest to lowest points and divide that by 2.



Properties of a Wave

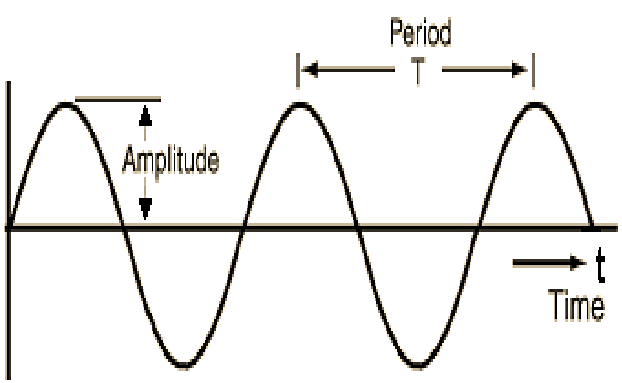
- **Frequency (*f*)** describes the number of waves that pass a fixed place in a given amount of time.

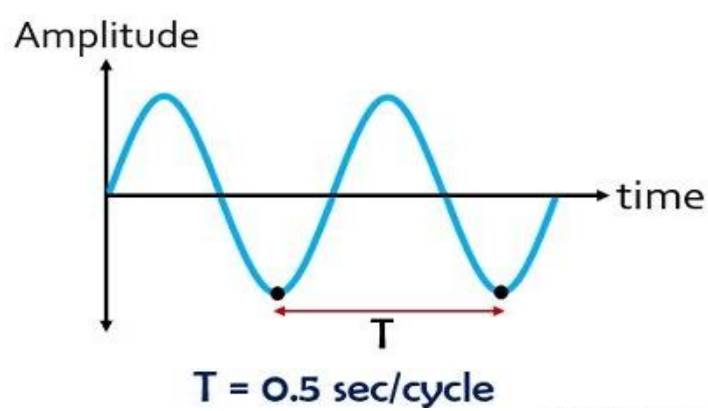




$$\text{Frequency} = \frac{1}{\text{Period}}$$

- **Period (*T*)** is the time it takes to complete one cycle of the wave.

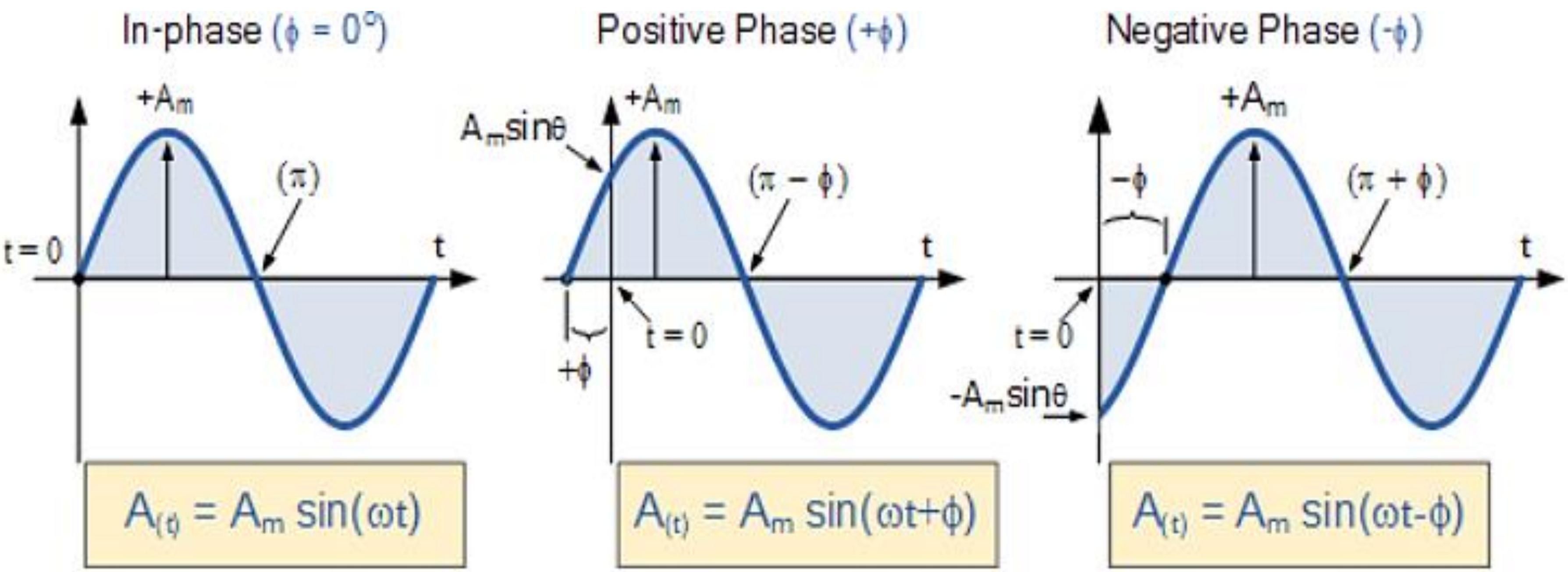




$$\text{Period} = \frac{1}{\text{Frequency}}$$

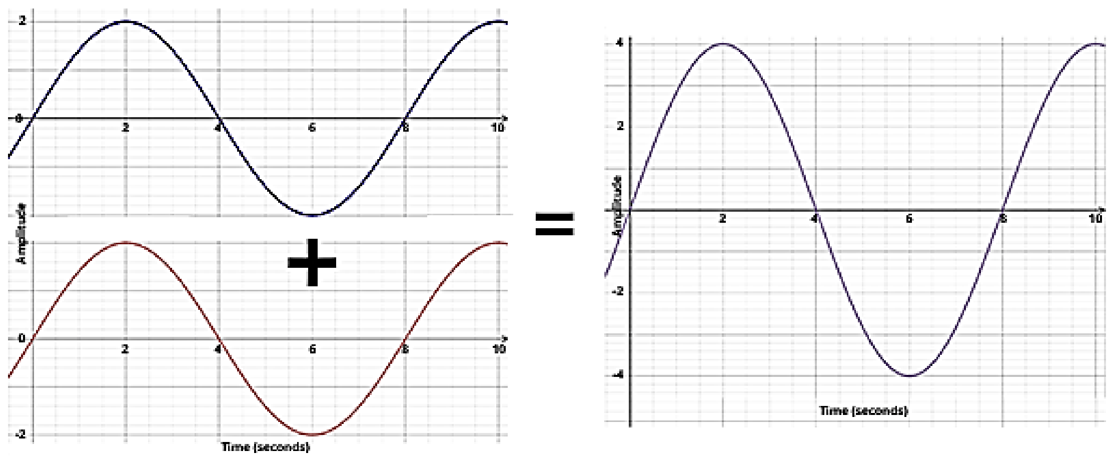
Properties of a Wave

- **Phase (Ø)** represents an angular shift of a wave(s) and is measured in radians (or degrees).



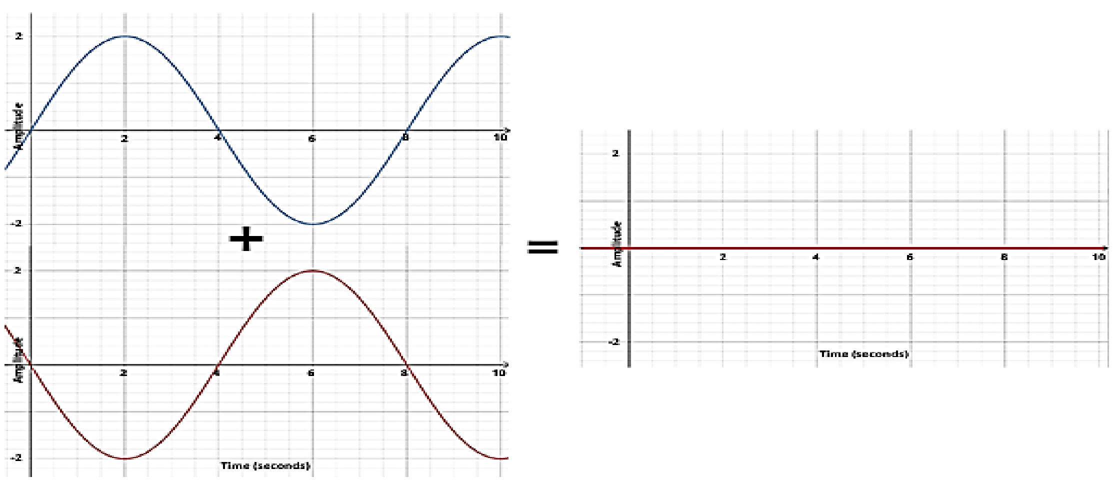
Properties of a Wave

- **Phase Difference - "in-phase"**



Waves that are **in-phase** add to produce a wave with an amplitude equal to the sum of the amplitudes of the two waves.

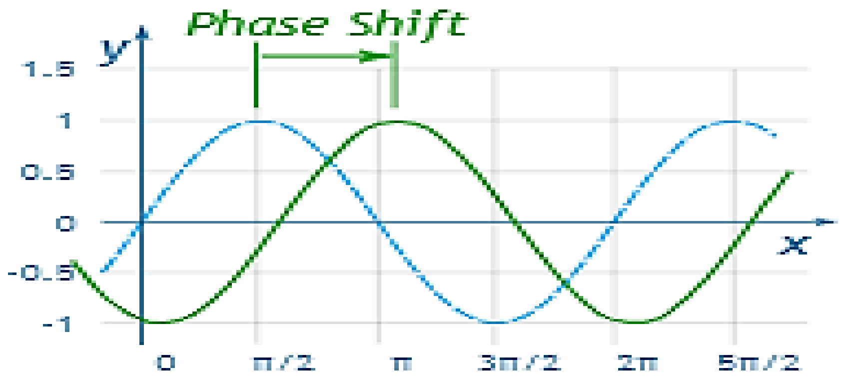
- **Phase Difference - "out-of-phase"**



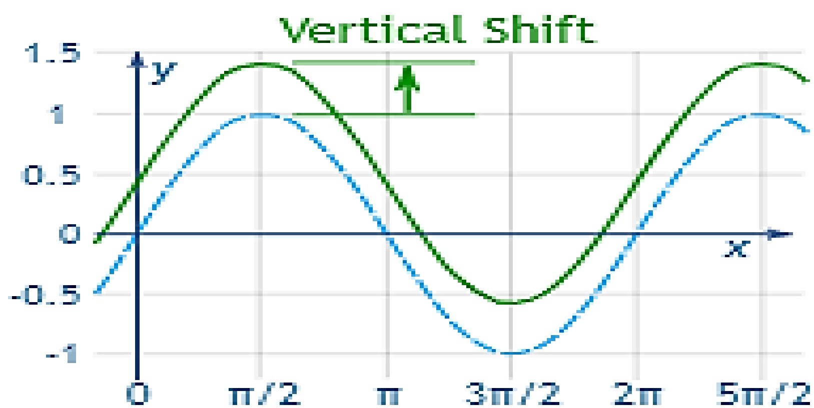
Waves that are **out-of-phase** exactly cancel each other when added together.

Properties of a Wave

- **Phase Shift** is how far one wave is shifted **horizontally** from the other wave.



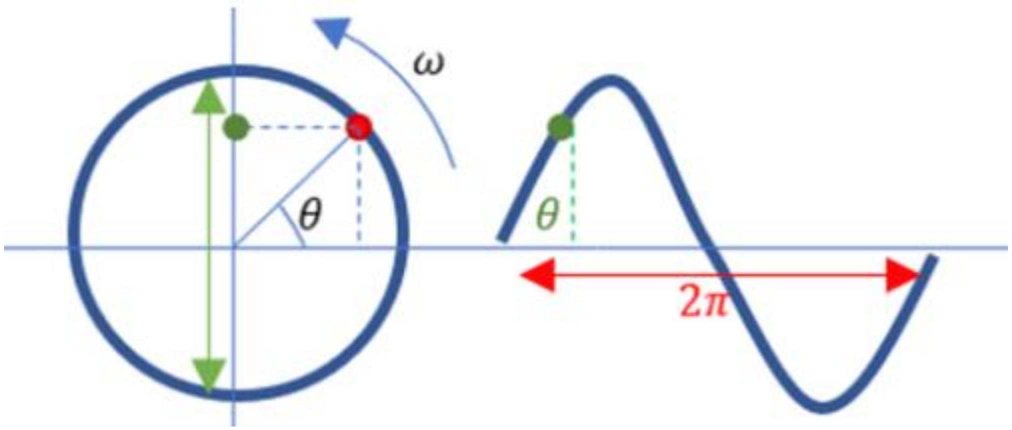
- **Vertical Shift** is how far one wave is shifted **vertically** from the other wave.



Properties of a Wave

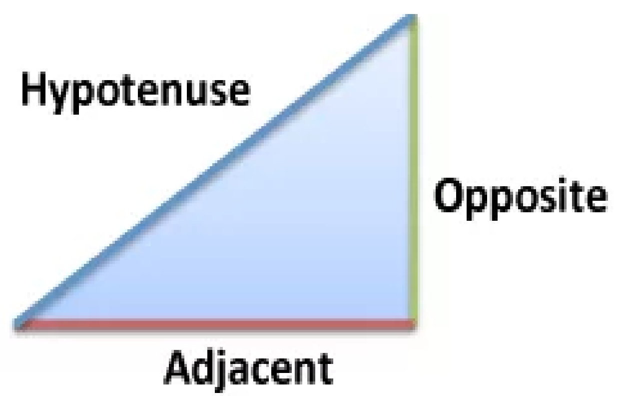
- **[Angular velocity (ω)](#)** refers to the *angular displacement per unit time* (for example, in rotation) **OR** the rate of change of the phase of a sinusoidal waveform measured in degrees (or radians) per second.

$$\omega = \frac{2\pi}{T} = 2\pi f$$



Why we need sine, cosine, tangent ?

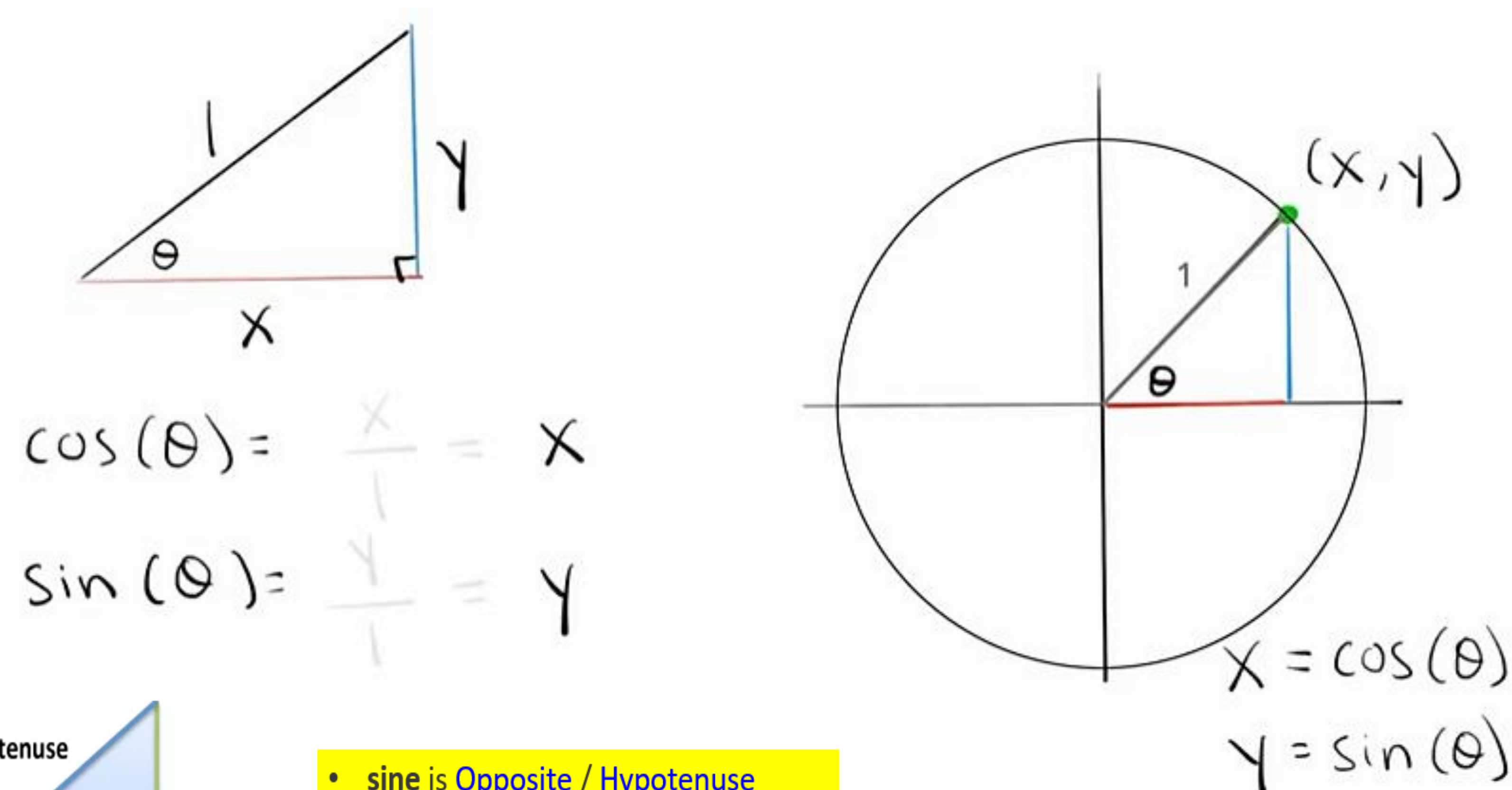
Angles (In Degrees)	0°	30°	45°	60°	90°	180°	270°	360°
Angles (In Radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not Defined	0	Not Defined	1



- **sine** is Opposite / Hypotenuse
- **cosine** is Adjacent / Hypotenuse
- **tangent** is Opposite / Adjacent

Why we need Sine, Cosine, Tangent ?

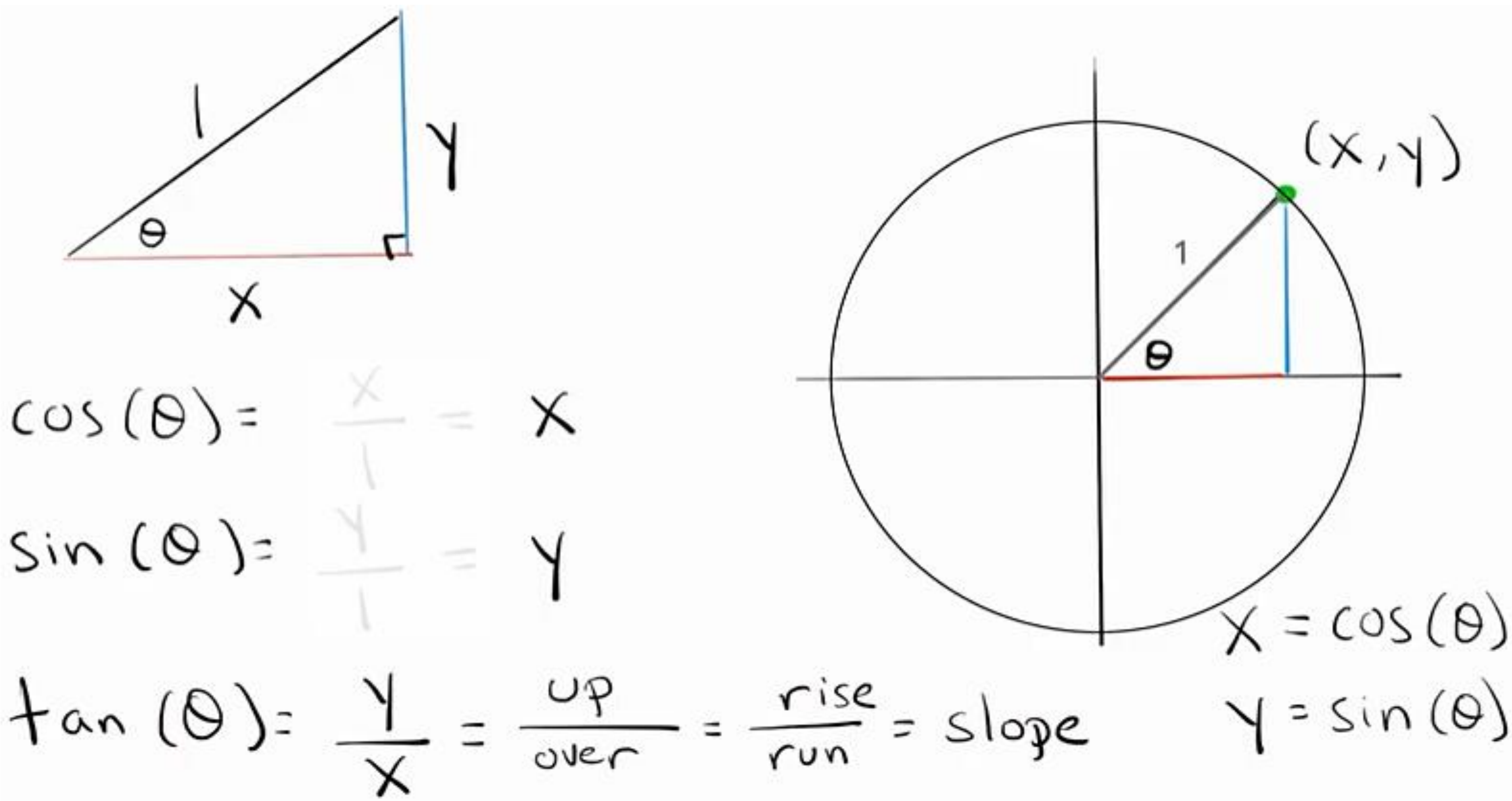
- **sine and cosine give the value of a point (x, y) on the circumference of a circle.**

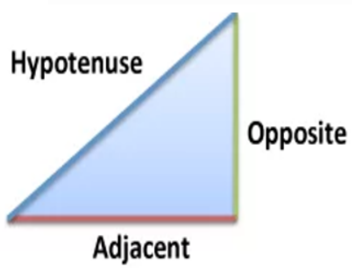


- **sine** is Opposite / Hypotenuse
- **cosine** is Adjacent / Hypotenuse
- **tangent** is Opposite / Adjacent

Why we need Sine, Cosine, Tangent ?

- **tangent gives the slope.**





- **sine** is Opposite / Hypotenuse
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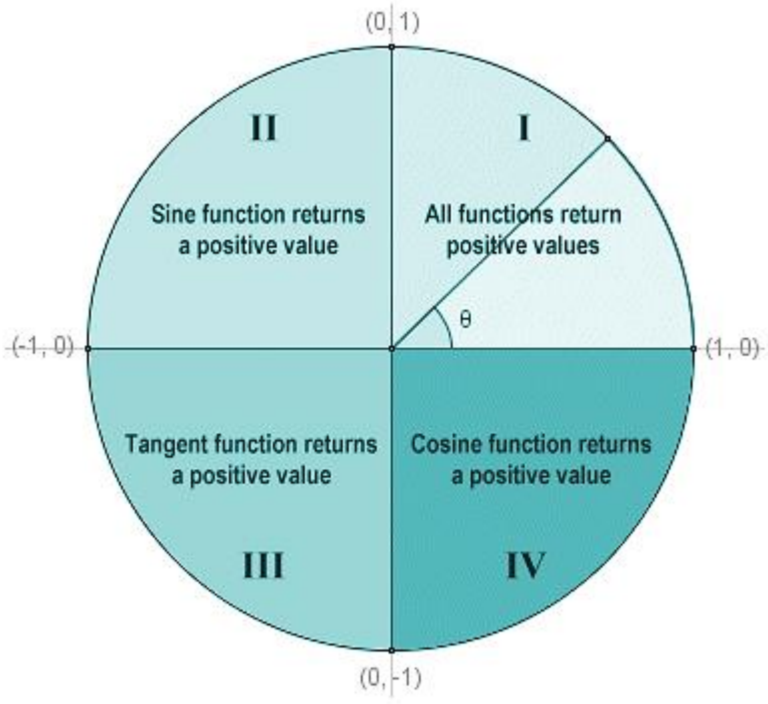
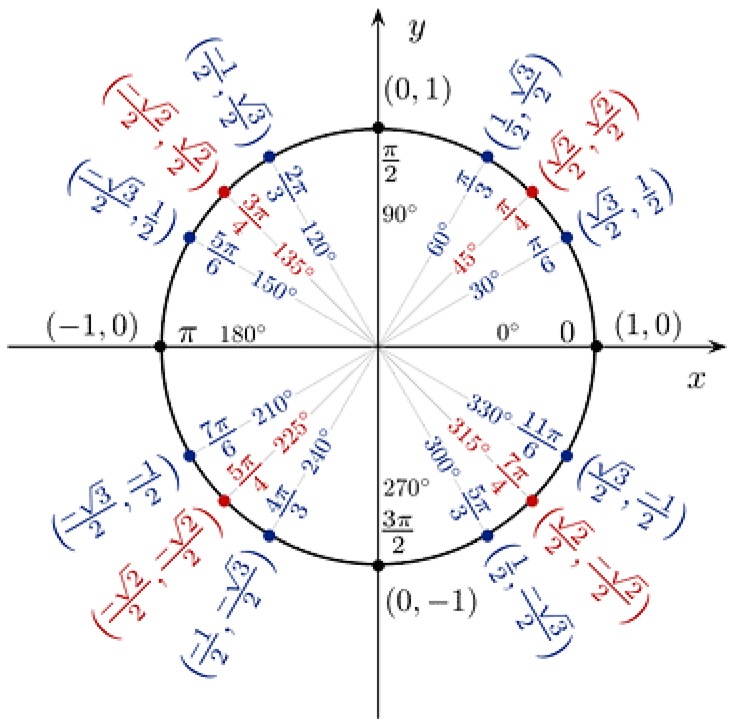
Why we need Sine, Cosine, Tangent ?



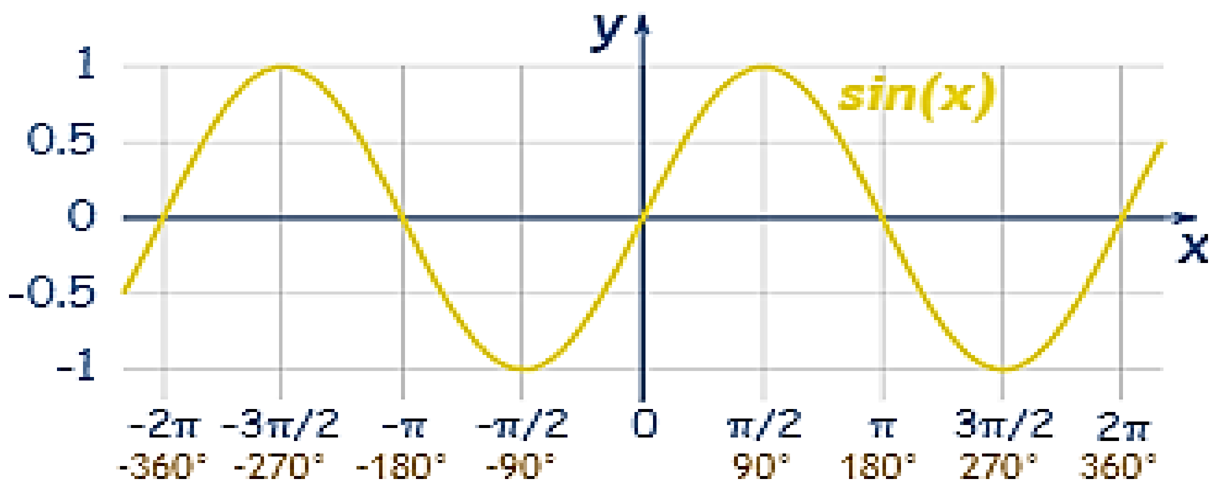
(cos, sin)



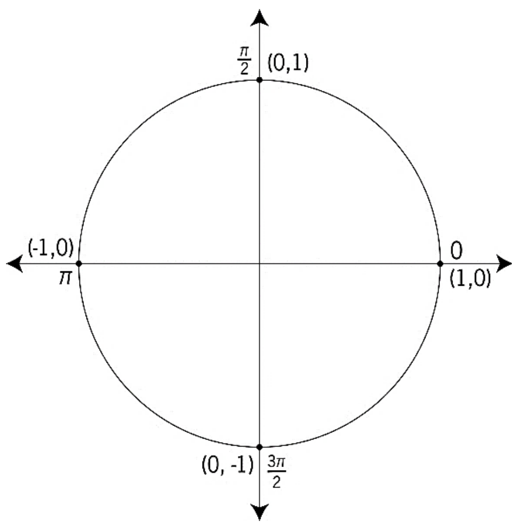
What is a Sine Wave ?

- A basic **sine** wave is an *S-shaped waveform* defined by the mathematical function **y = sin(x)**.

- The **sine** function has up-down curve (which repeats every **2π** radians, or **360°**).



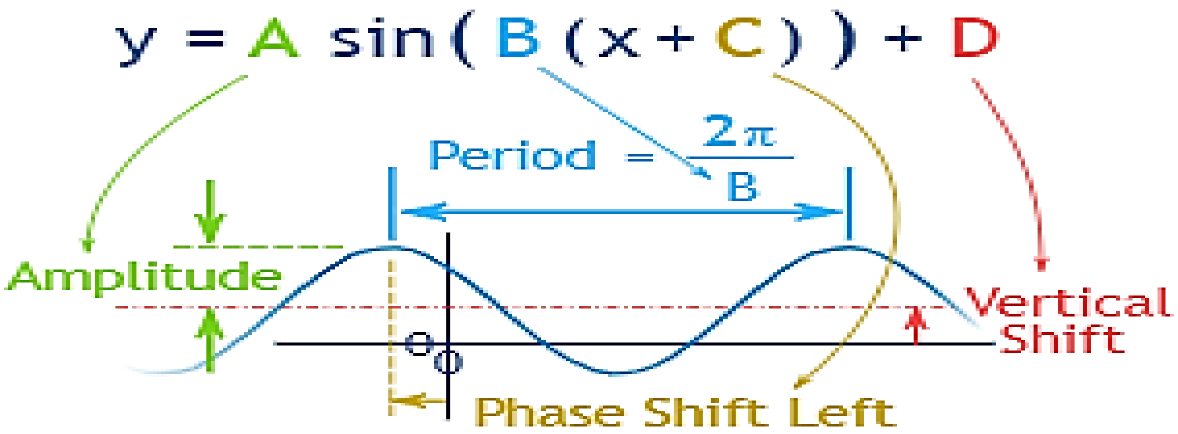
(cos, sin)



- For every cycle, it starts at **0**, heads up to **1** by **π**/2 radians (90°) and then heads down to **–1** and finally heads up to **0.**

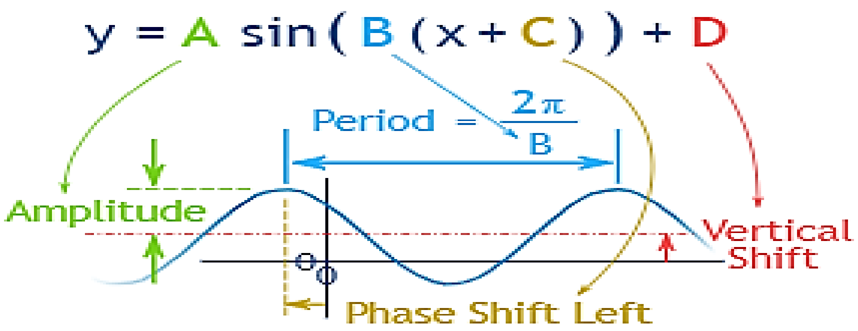
What is Sine Wave ?

- The general equation of a **sine** wave is given by:



- amplitude is **A**
- period is **2π/B** (defines **periodicity**)
- phase shift is **C** (positive is to the **left**)
- vertical shift is **D**

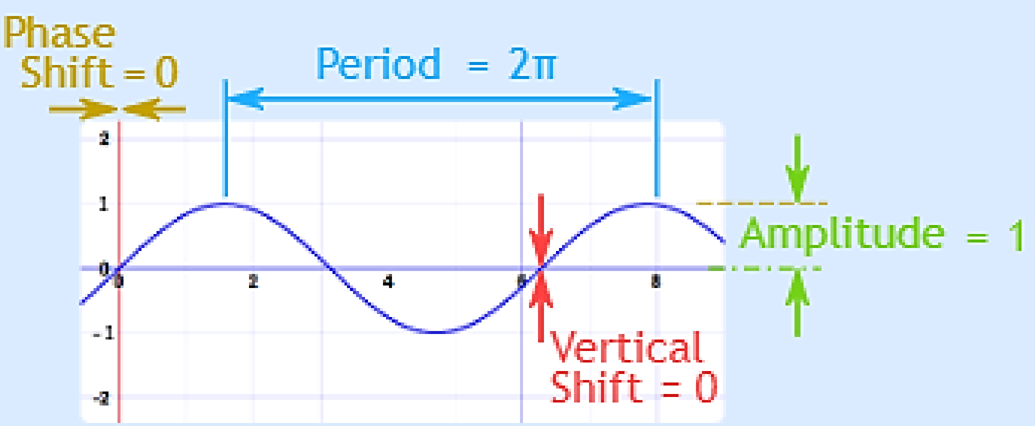
Sine wave - examples



Example: sin(x)

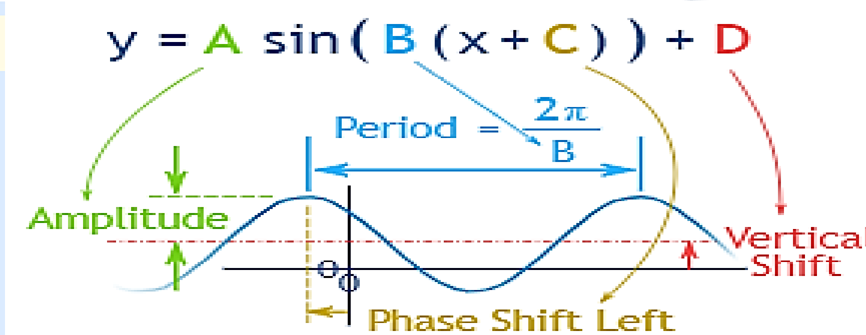
This is the basic unchanged sine formula. **A = 1, B = 1, C = 0 and D = 0**

So amplitude is **1**, period is **2π**, there is no phase shift or vertical shift:

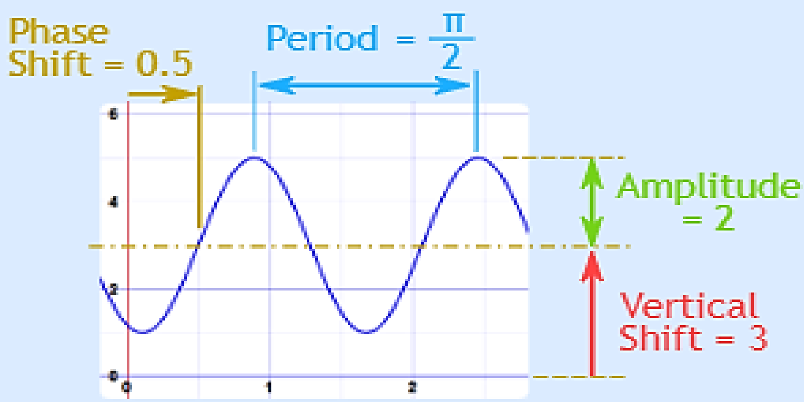


Sine wave - examples

Example: $2\sin(4(x - 0.5)) + 3$



- amplitude $\mathbf{A = 2}$
- period $\mathbf{2\pi/B} = \mathbf{2\pi/4} = \mathbf{\pi/2}$
- phase shift = $\mathbf{-0.5}$ (or $\mathbf{0.5}$ to the right)
- vertical shift $\mathbf{D} = \mathbf{3}$



In words:

- the **2** tells us it will be 2 times taller than usual, so Amplitude = 2
- the usual period is 2π, but in our case that is "sped up" (made shorter) by the **4** in 4x, so Period = $\mathbf{\pi/2}$
- and the **−0.5** means it will be shifted to the **right** by **0.5**
- lastly the **+3** tells us the center line is y = +3, so Vertical Shift = 3

Sine wave - examples

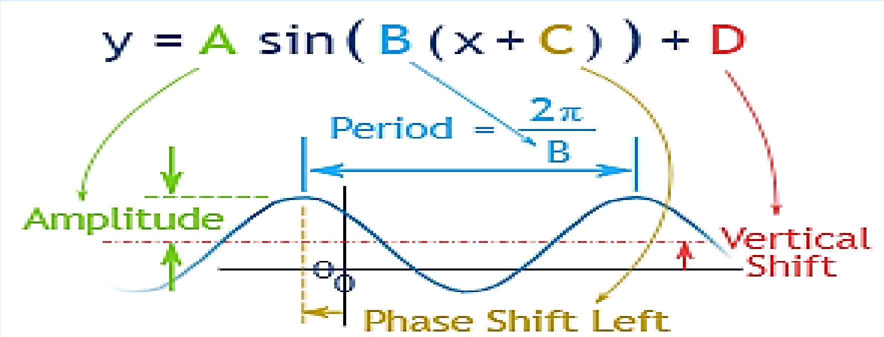
Example: $3 \sin(100(t + 0.01))$

First we need brackets around the (t+1), so we can start by dividing the 1 by 100:

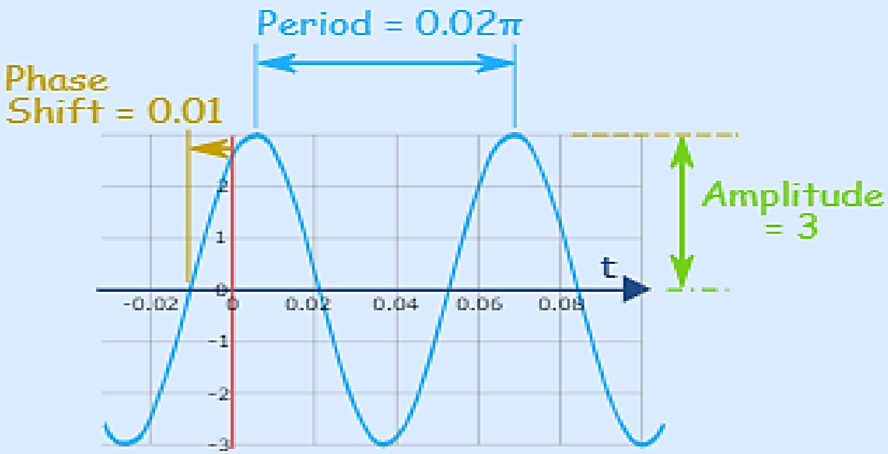
$3 \sin(100t + 1) = \mathbf{3 \sin(100(t + 0.01))}$

Now we can see:

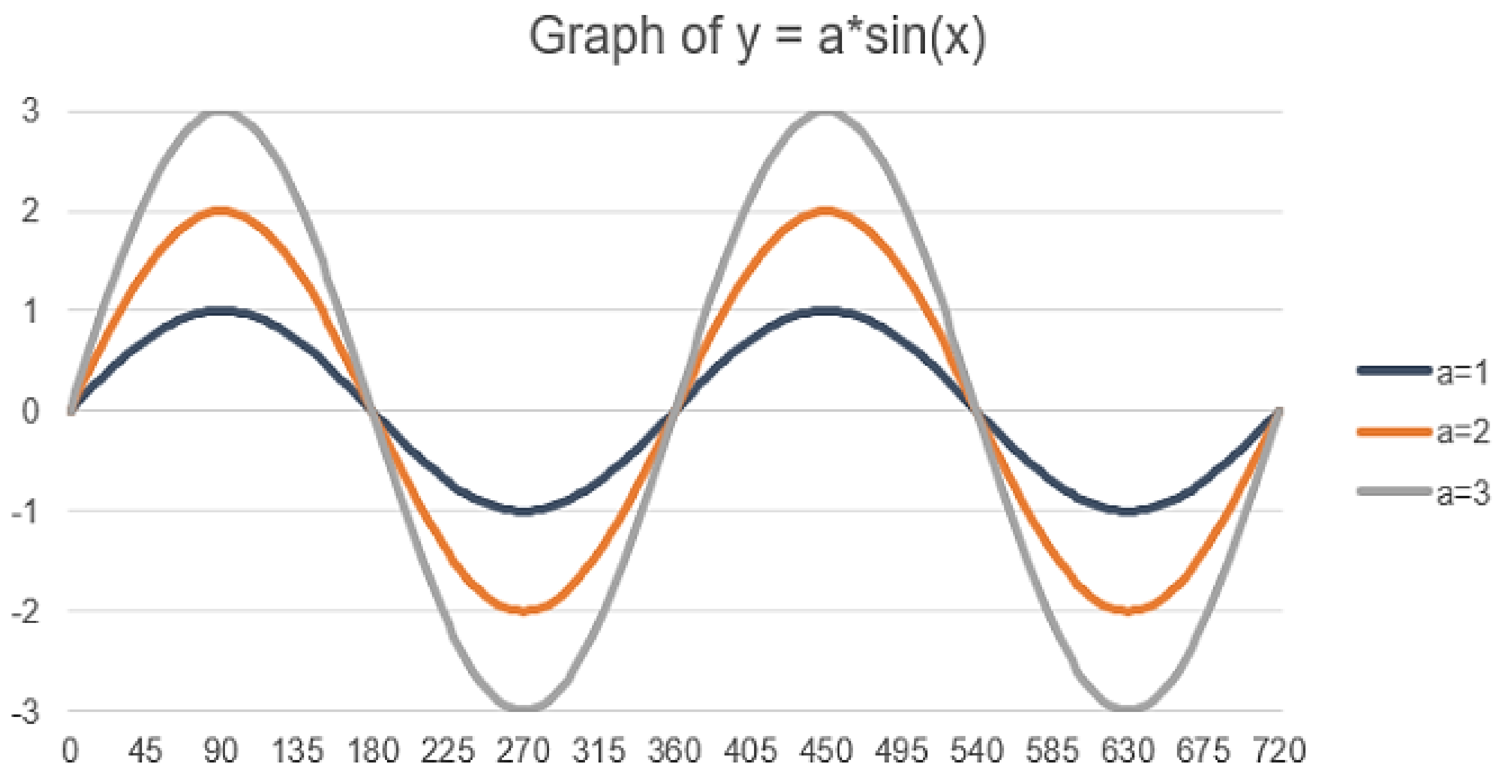
- amplitude is $\mathbf{A = 3}$
- period is $\mathbf{2\pi/100 = 0.02\,\pi}$
- phase shift is $\mathbf{C = 0.01}$ (to the left)
- vertical shift is $\mathbf{D = 0}$



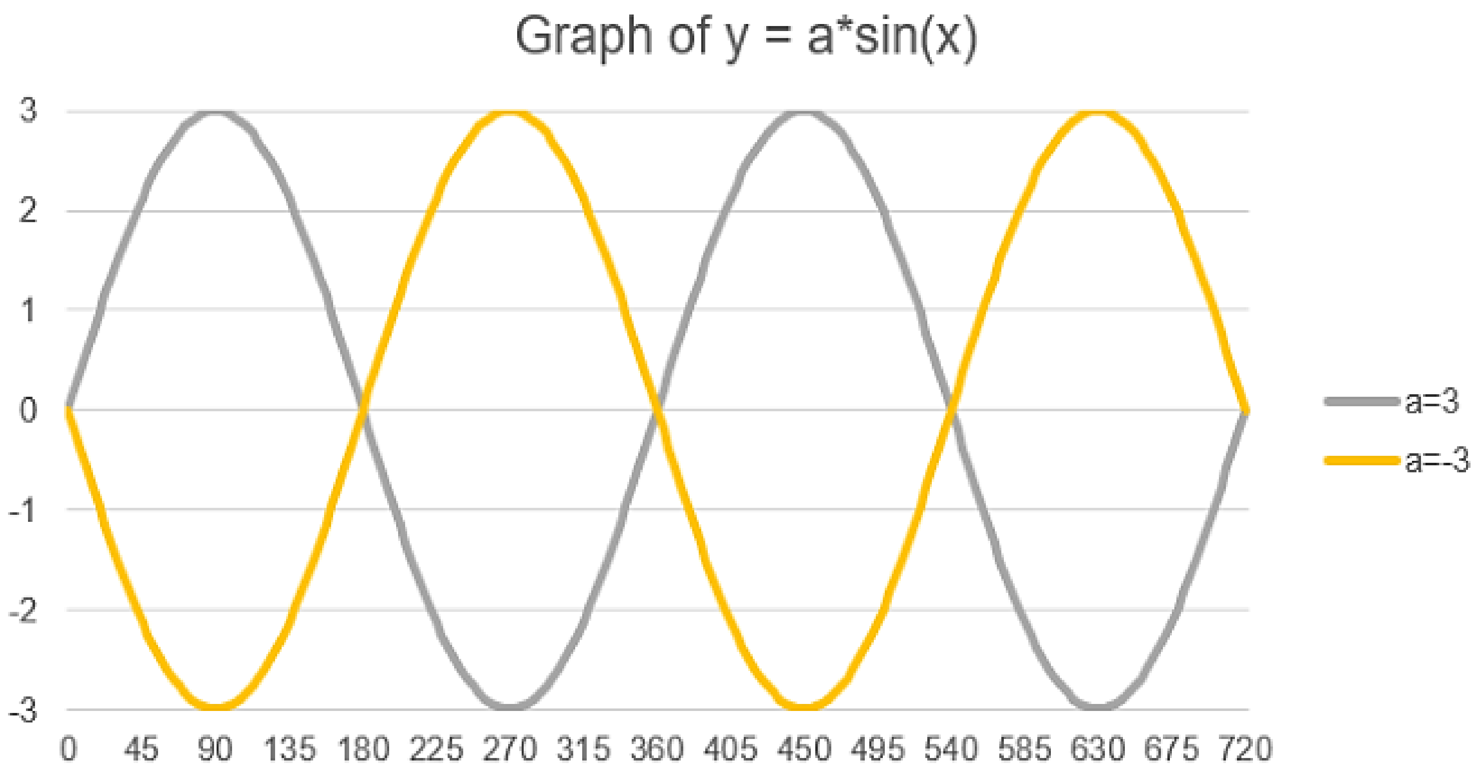
And we get:



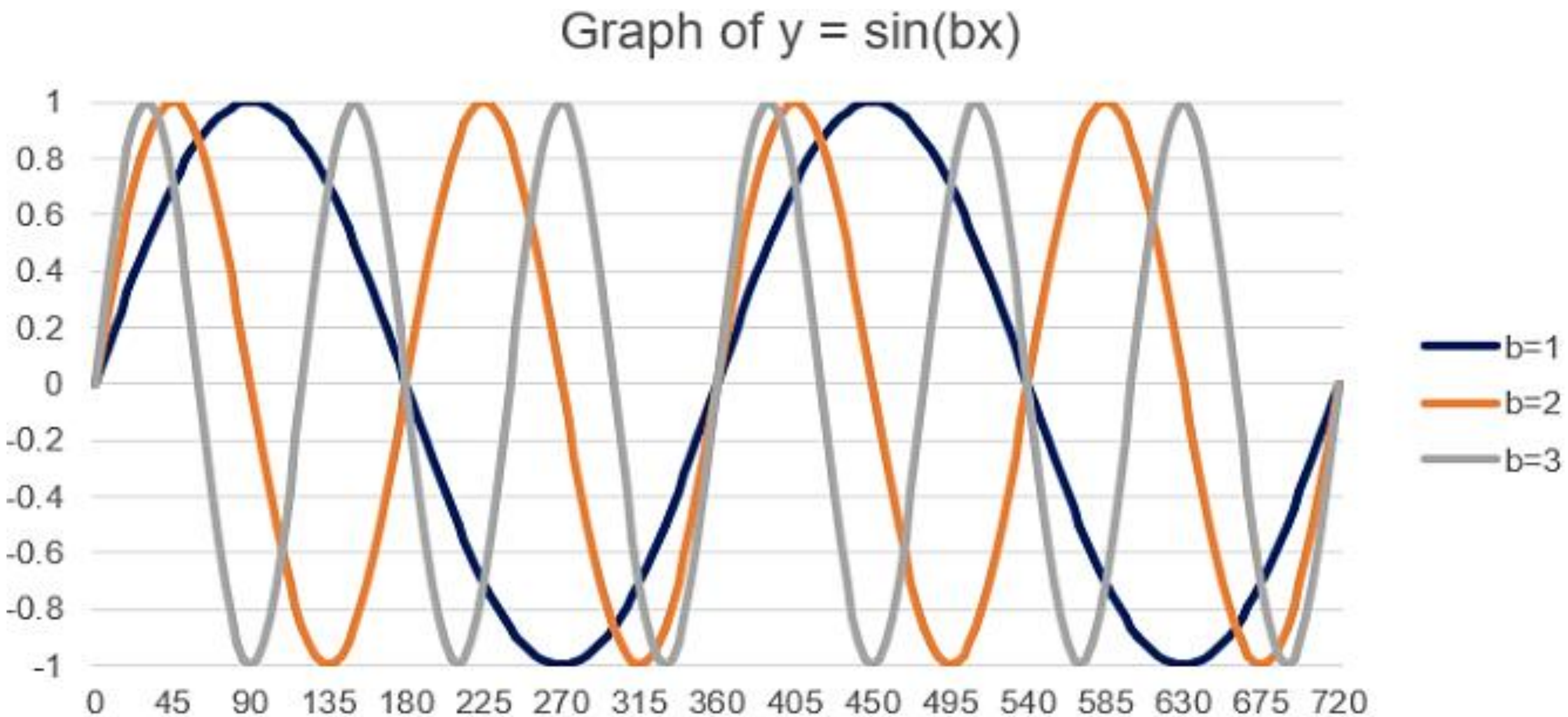
Sine wave - Variation in Amplitude



Sine wave - Variation in Amplitude



Sine wave - Variation in Periodicity



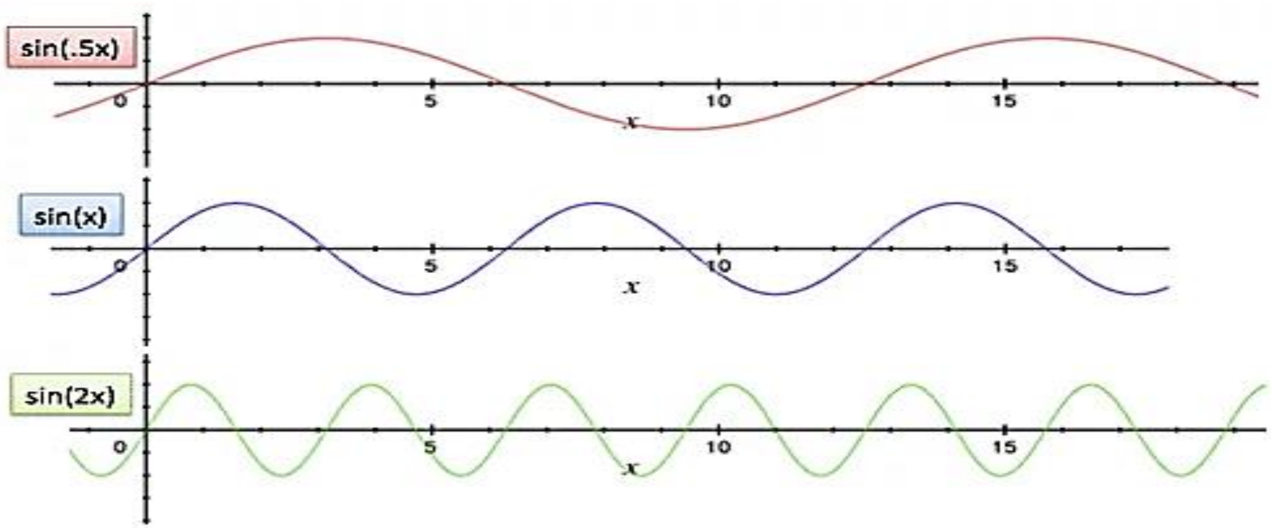
How fast is Sine Wave?

- **sin(x)** is the *default* sine wave, that indeed takes **π** units of time to go from **0** to **1** and back to **0** (or **2π** for a complete cycle).

- **sin(2x)** is a wave that moves twice as fast.

- **sin(0.5x)** is a wave that moves twice as slow.

- So, we use **sin(n*x)** to get a sine wave cycling as fast as we need.



What is Cosine Wave ?

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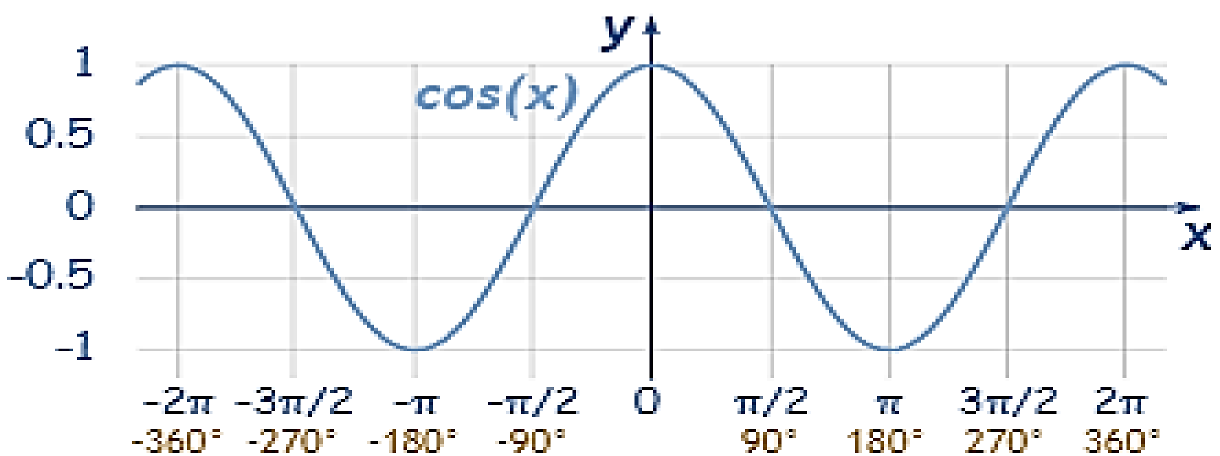
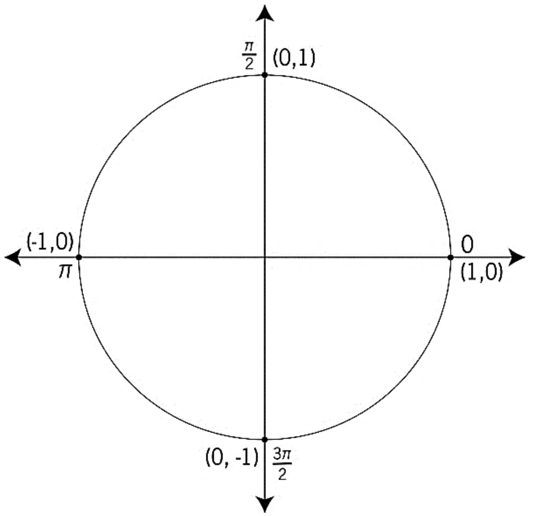
- The **cosine** function has up-down curve (which repeats every **2π** radians, or **360°**).



(cos, sin)



- For every cycle, it starts at **1**, heads down to **0** by **π/2** radians (90°) and again heads down to **–1** and finally heads up to **0.**

What is Cosine Wave ?

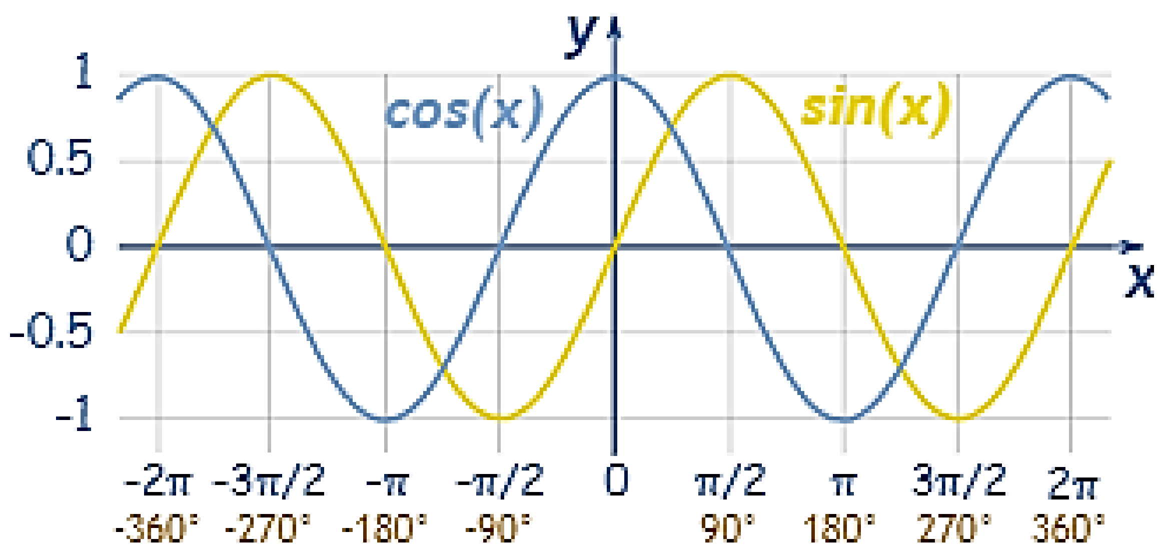
- The general equation of a **cosine** wave is given by:

$$Y = A\cos(B(x + C)) + D$$

  - amplitude is **A**
  - period is **2π/B**
  - phase shift is **C** (positive is to the **left**)
  - vertical shift is **D**

Plot of Sine and Cosine Waves

- Sine and Cosine are **good friends**: they follow each other, exactly **π/2** radians (**90°**) apart.



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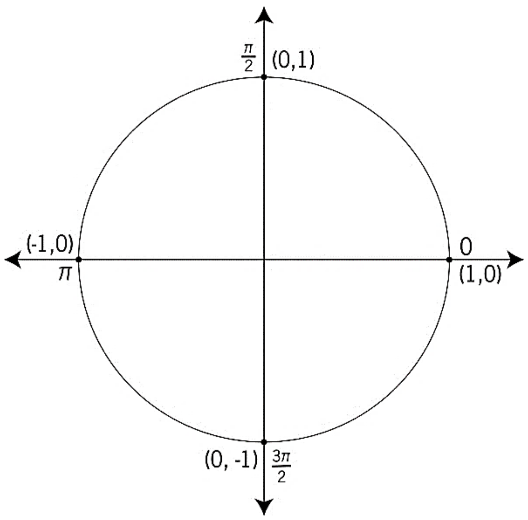
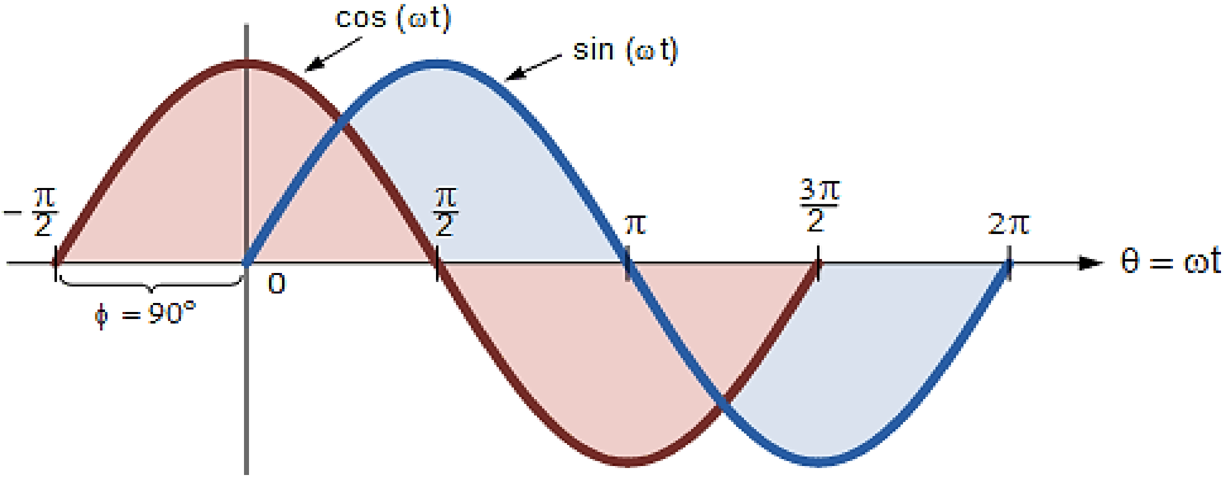


Plot of Sine and Cosine Waves



- A sine wave and a cosine wave are **90°** (***π/2*** radians) out of phase with each other.

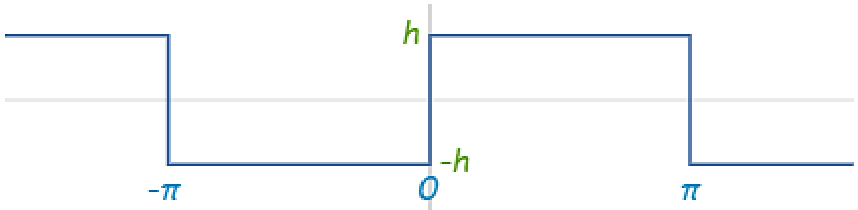
$$\cos(\omega t+\phi) = \sin\left(\omega t+\phi+90^\circ\right)$$

$$\sin(\omega t+\phi) = \cos\left(\omega t+\phi-90^\circ\right)$$

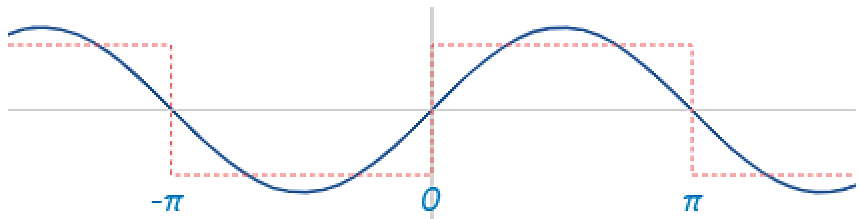
Forming a Square Wave

https://www.geogebra.org/graphing

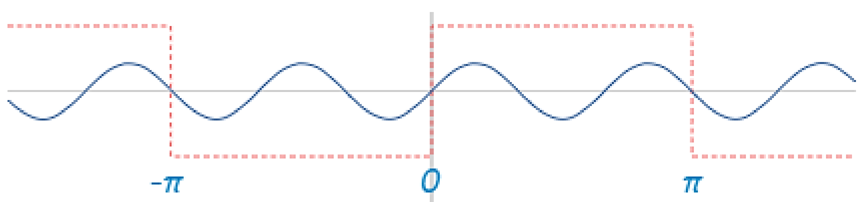
- Can we **add sine** waves to make a **square wave**?



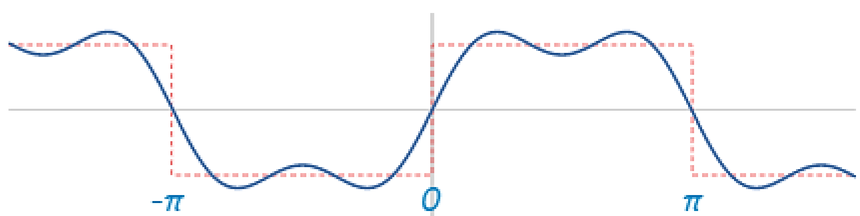
sin(x):



sin(3x)/3:

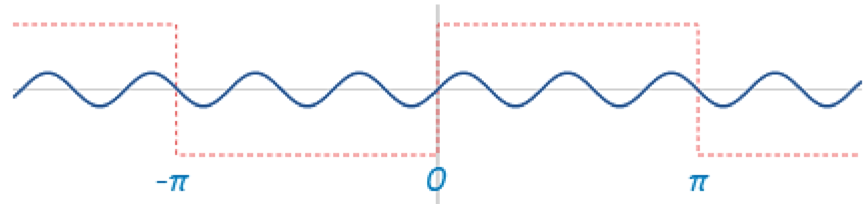


sin(x)+sin(3x)/3:

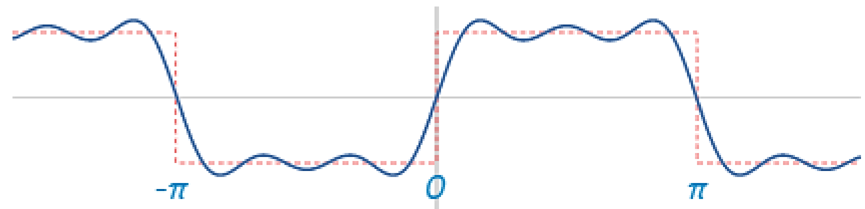


Forming a Square Wave

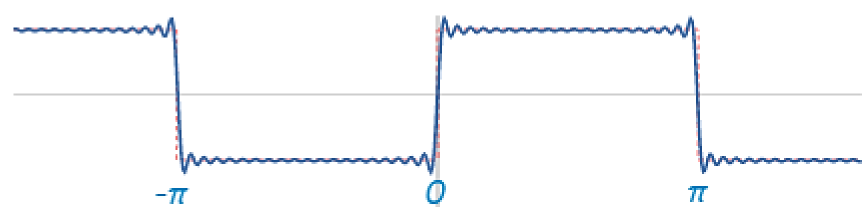
sin(5x)/5:



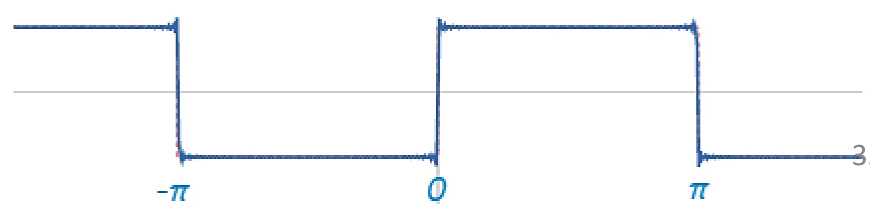
sin(x)+sin(3x)/3+sin(5x)/5:



sin(x)+sin(3x)/3+sin(5x)/5 + ... + sin(39x)/39:



sin(x)+sin(3x)/3+sin(5x)/5 + ... + sin(199x)/199:



Complex Number

Complex Number

- A Complex Number is a combination of a **Real Number** and an **Imaginary Number.**

Real Numbers are numbers like:

1	12.38	−0.8625	3/4	√2	1998

Nearly any number you can think of is a Real Number!

Imaginary Numbers when **squared** give a **negative** result.

Normally this doesn't happen, because:

- when we square a positive number we get a positive result, and
- when we square a negative number we also get a positive result (because a negative times a negative gives a positive), for example **−2 × −2 = +4**

Complex Number

Imaginary number

The "unit" imaginary number (like 1 for Real Numbers) is i, which is the square root of −1

$$i = \sqrt{-1}$$

Because when we square i we get −1

$$i^2 = -1$$

Examples of Imaginary Numbers:

3i	1.04i	−2.8i	3i/4	(√2)i	1998i

And we keep that little "i" there to remind us we need to multiply by $\sqrt{-1}$

Complex Number

When we combine a Real Number and an Imaginary Number we get a **Complex Number**:

$$a + bi$$

Real Part

Imaginary Part

$\sqrt{-1}$

Examples:

$1 + i$	$39 + 3i$	$0.8 - 2.2i$	$-2 + \pi i$	$\sqrt{2} + i/2$

Complex Number

Either Part Can Be Zero

So, a Complex Number has a real part and an imaginary part.

But either part can be **0**, so all Real Numbers and Imaginary Numbers are also Complex Numbers.

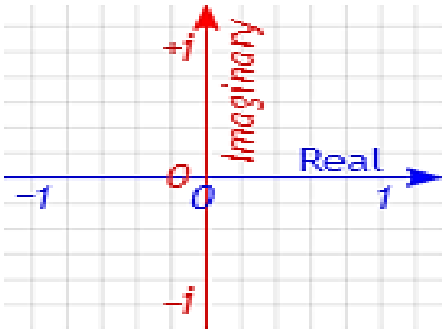
Complex Number	Real Part	Imaginary Part	
3 + 2i	3	2	
5	5	**0**	Purely Real
−6i	**0**	−6	Purely Imaginary

Complex Number

A Visual Explanation

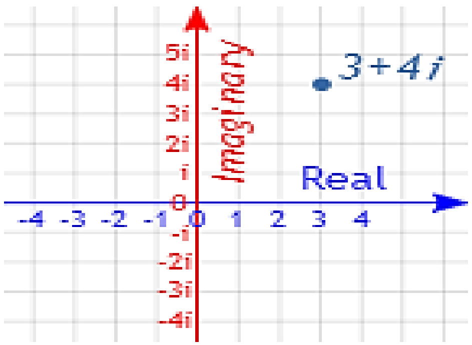
You know how the number line goes **left-right**?

Well let's have the imaginary numbers go **up-down**:



And we get the Complex Plane

A complex number can now be shown as a point:



The complex number 3 + 4**i**

Complex Number

Adding

To add two complex numbers we add each part separately:

$$(a+b\mathbf{i}) + (c+d\mathbf{i}) = (a+c) + (b+d)\mathbf{i}$$

Example: add the complex numbers **3 + 2*i*** and **1 + 7*i***

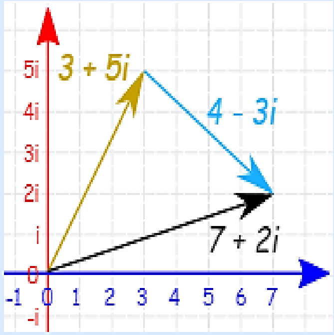
- add the real numbers, and
- add the imaginary numbers:

$$(3 + 2i) + (1 + 7i)$$
$$= 3 + 1 + (2 + 7)i$$
$$= 4 + 9i$$

Example: add the complex numbers **3 + 5*i*** and **4 − 3*i***

$$(3 + 5i) + (4 - 3i)$$
$$= 3 + 4 + (5 - 3)i$$
$$= 7 + 2i$$

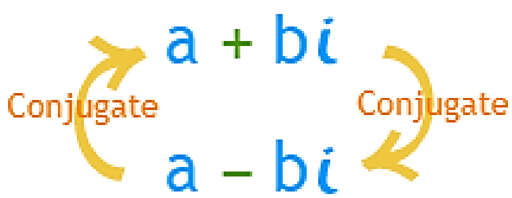
On the complex plane it is:



Complex Number

Conjugates

A conjugate is where we **change the sign in the middle** like this:



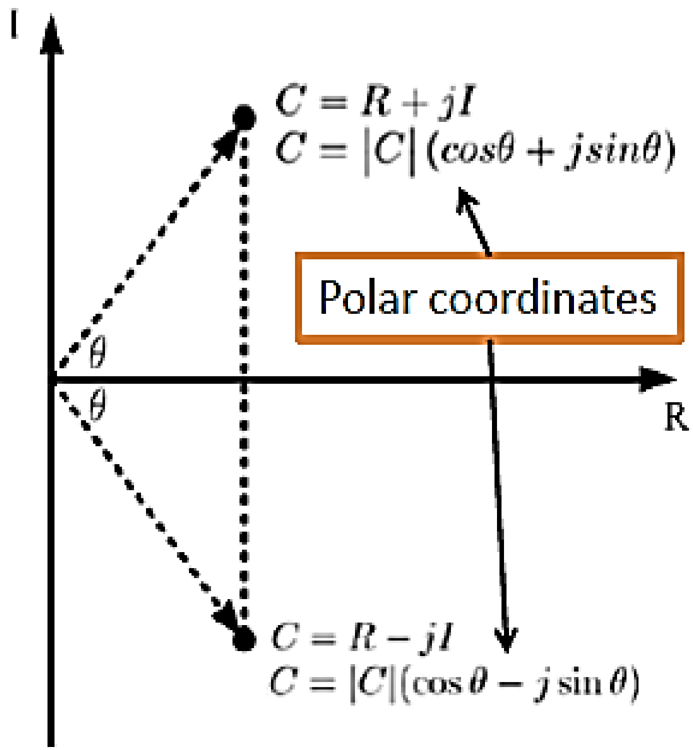
A conjugate is often written with a bar over it:

Example:

$$\overline{5 - 3i} = 5 + 3i$$

Complex Number Representations (important)

In technical disciplines, we represent a complex number as **j** instead of **i** to avoid the confusion for the electric current representation(**i**).



Complex Plane

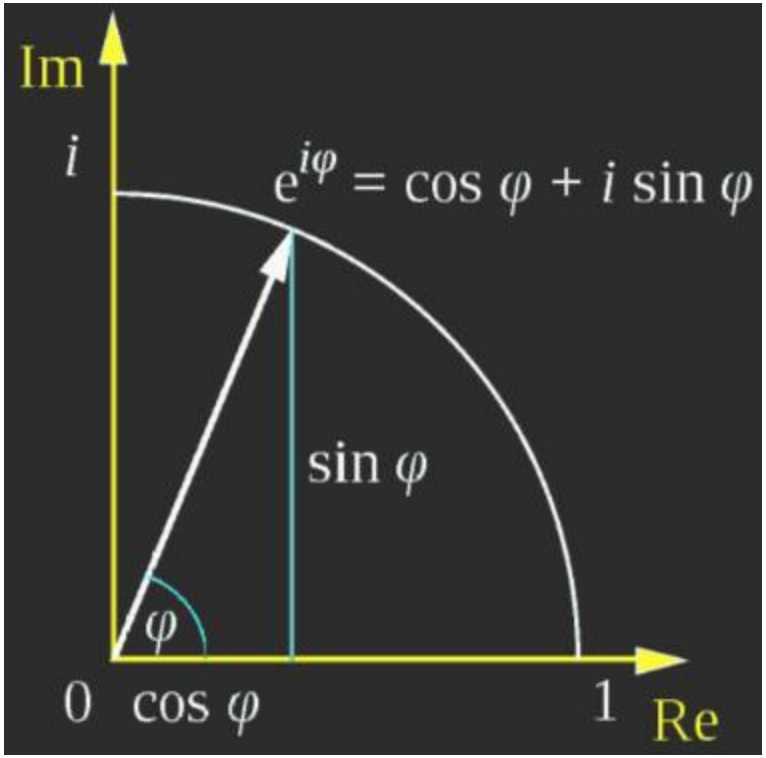
Cartesian Form: $C = R + jI$
Polar Form: $C = |C|(\cos\theta + j\sin\theta) = |C|e^{j\theta}$
Euler's Relation: $e^{j\theta} = \cos\theta + j\sin\theta$

$$|C| = \sqrt{R^2 + I^2}$$ Magnitude

$$\theta = \arctan\left(\frac{I}{R}\right) \approx tan^{-1}\left(\frac{I}{R}\right)$$

Euler's Formula

- On of the applications of Euler's formula is for complex analysis.
- **Euler's formula allows us to express complex numbers as exponentials.**



$\boldsymbol{\varphi}$ = any real number
i = imaginary unit (i.e.,$\sqrt{-1}$)
e = base of the natural logarithm

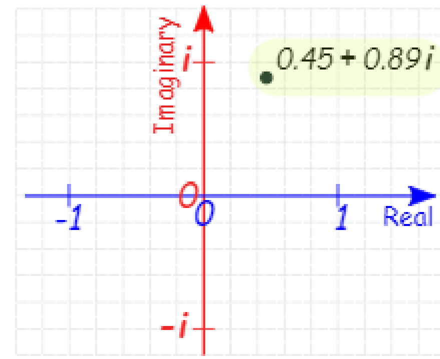
Euler's Formula

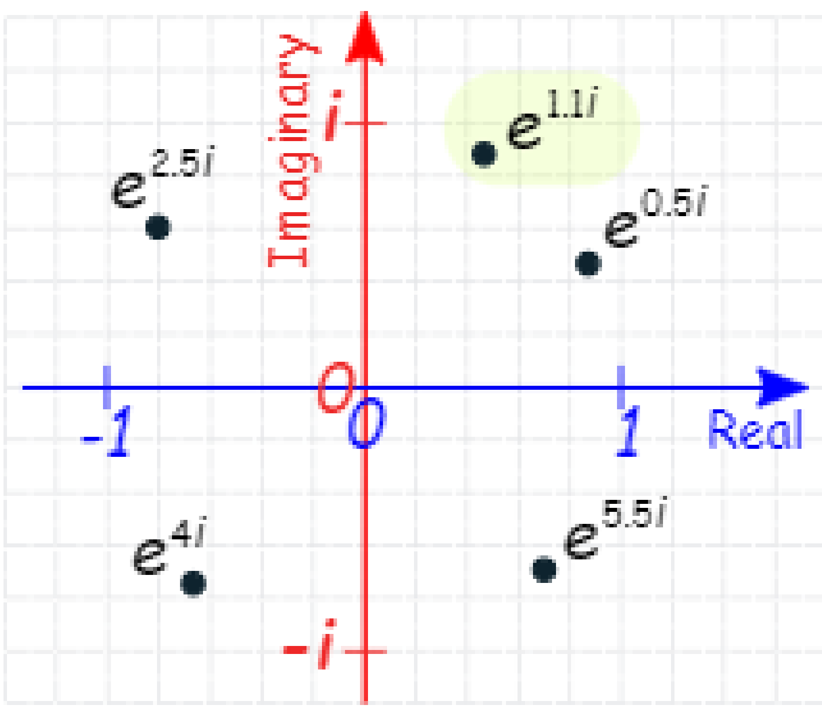
Example: when x = 1.1

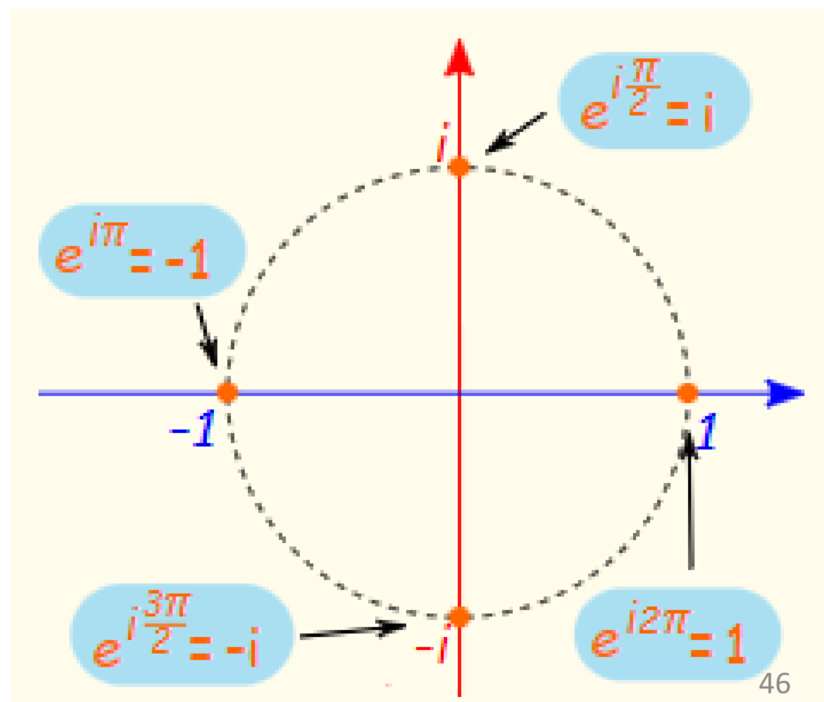
$e^{ix} = \cos x + i \sin x$

$e^{1.1i} = \cos 1.1 + i \sin 1.1$

$e^{1.1i} = 0.45 + 0.89\,i$ (to 2 decimals)







Euler's Formula - Cartesian to Polar conversion

Example: the number **3 + 4i**

To turn **3 + 4i** into re^{ix} form we do a Cartesian to Polar conversion:

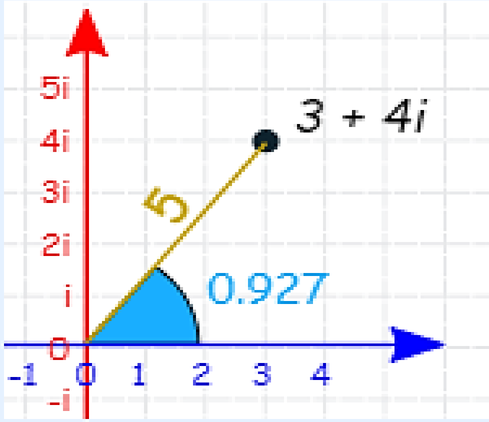
- r = √(3² + 4²) = √(9+16) = √25 = **5**
- x = tan⁻¹ (4 / 3) = **0.927** (to 3 decimals)

$$|C|e^{j\theta}$$

$$|C| = \sqrt{R^2 + I^2}$$ Magnitude

$$\theta = \arctan\left(\frac{I}{R}\right) \approx tan^{-1}\left(\frac{I}{R}\right)$$

So **3 + 4i** can also be $\mathbf{5e^{0.927\,i}}$

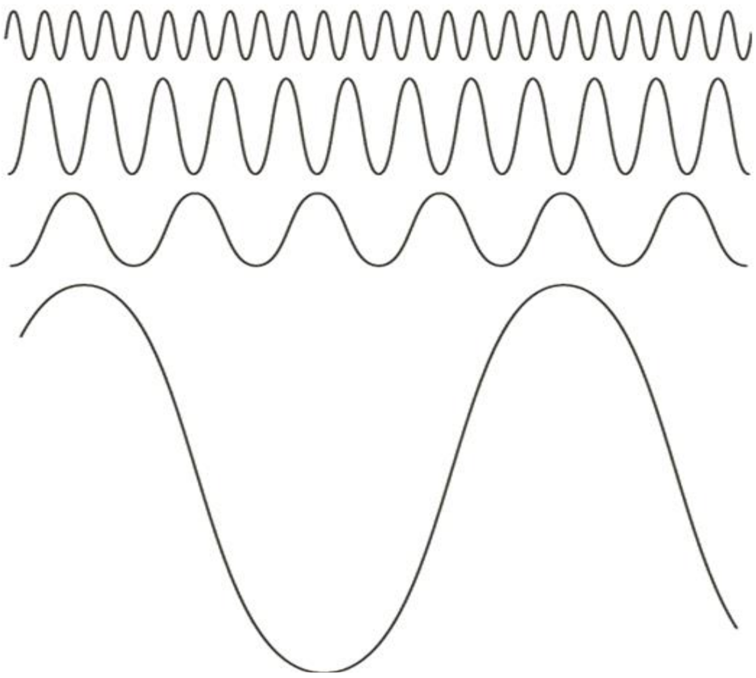


Fourier Series

Background

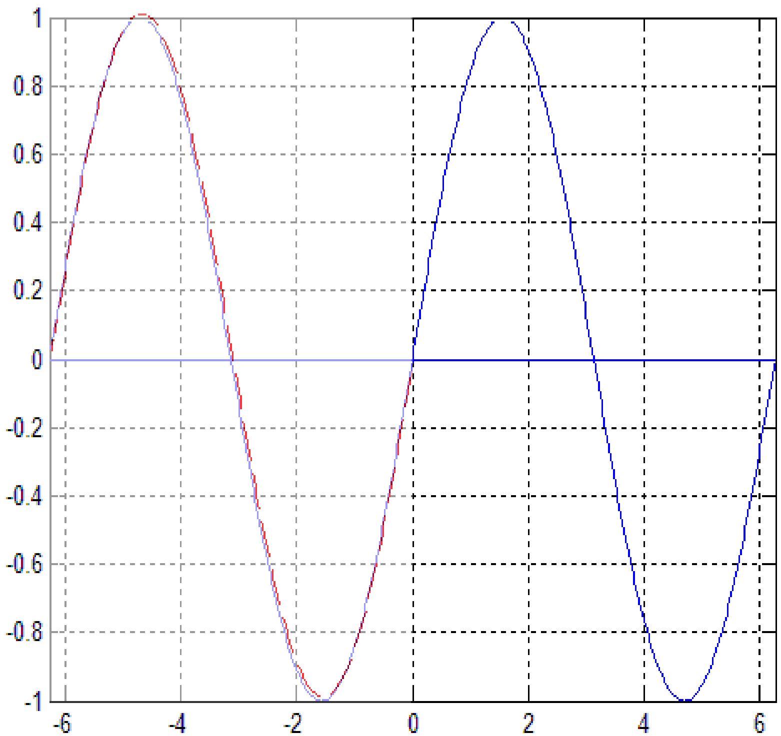
- **Jean Baptiste Joseph Fourier** (in 1807) stated that: any periodic function can be decomposed in to a sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier Series).

Periodic function



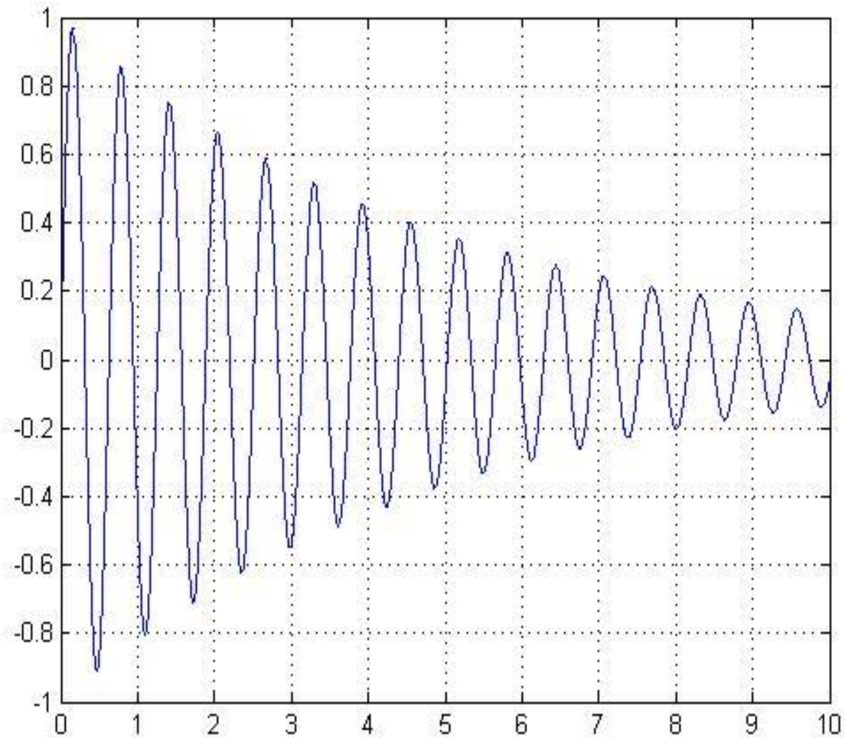
Sines and Cosines

Background



Periodic function

A periodic function remain self-similar for all integer multiples of its period.



Non-periodic function

A non-periodic function does not remain self-similar for all integer multiples of its period.

Background

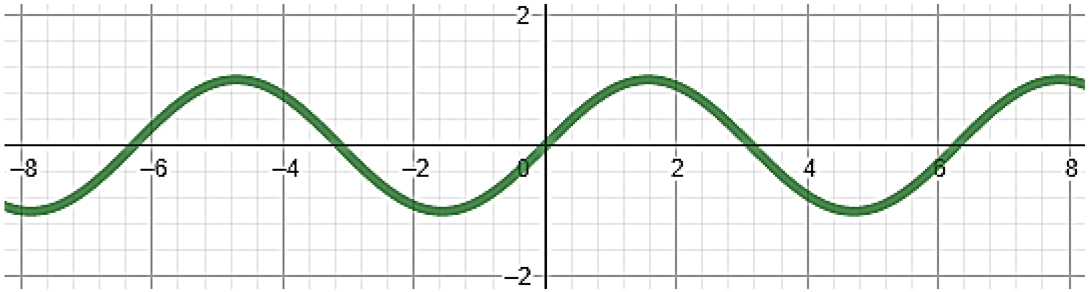
- Fourier Series
  - Any **periodic functions**
  - Expressed as sum of sine and/or cosine functions of different frequencies, each multiplied by a different coefficient

- Fourier Transform
  - Any **Non-periodic functions**, with area under the curve being **finite** (bounded)
  - Expressed as integral of sine and/or cosine functions, multiplied by a weighing function

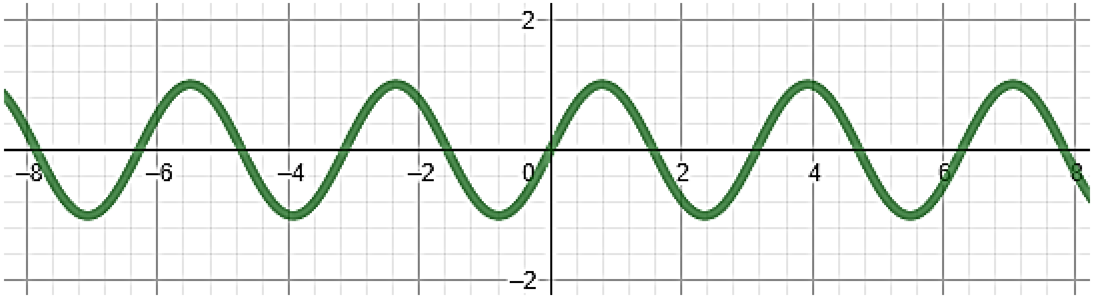
Adding periodic functions - example

- sine and cosine waves can make other periodic functions.

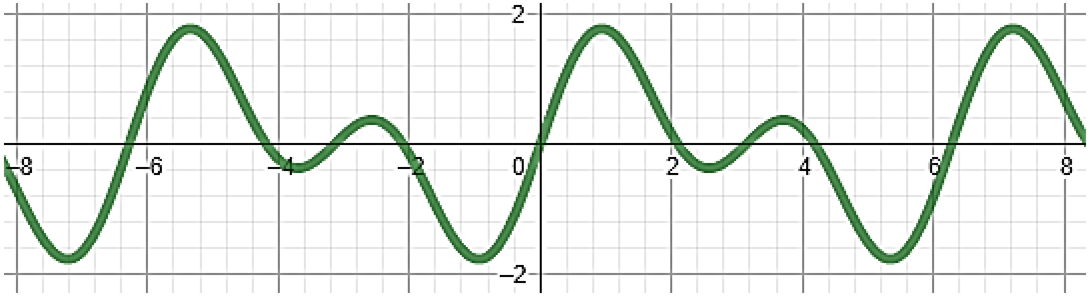
sin(x)



sin(2x)



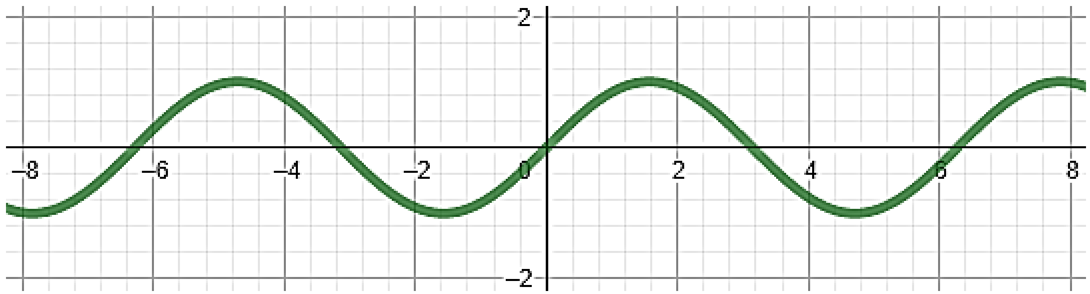
sin(x)+sin(2x)



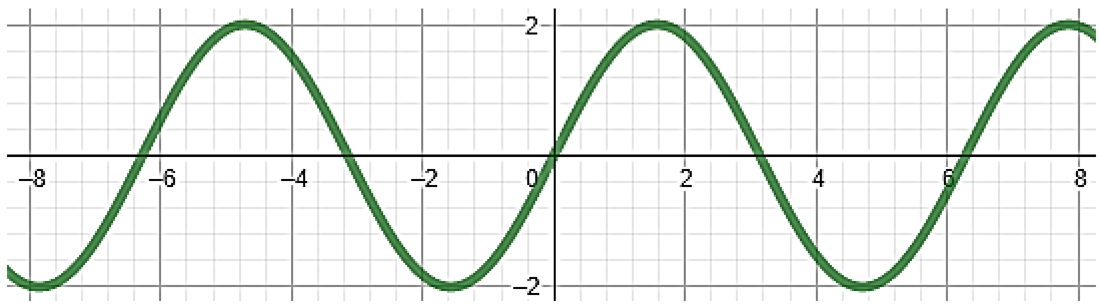
Adding periodic functions - example

- sine and cosine waves can make other periodic functions.

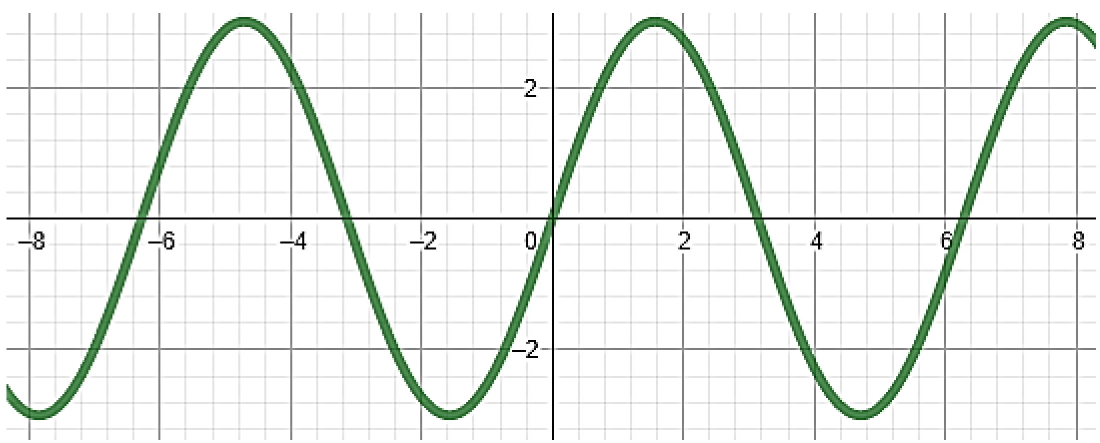
sin(x)



2 sin(x)



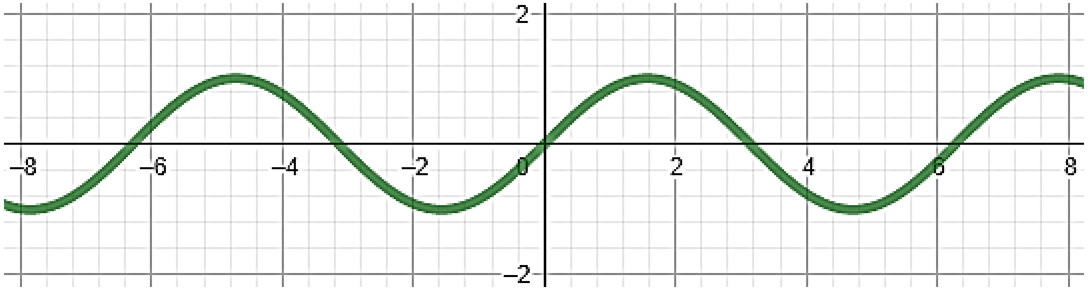
sin(x)+ 2 sin(x)



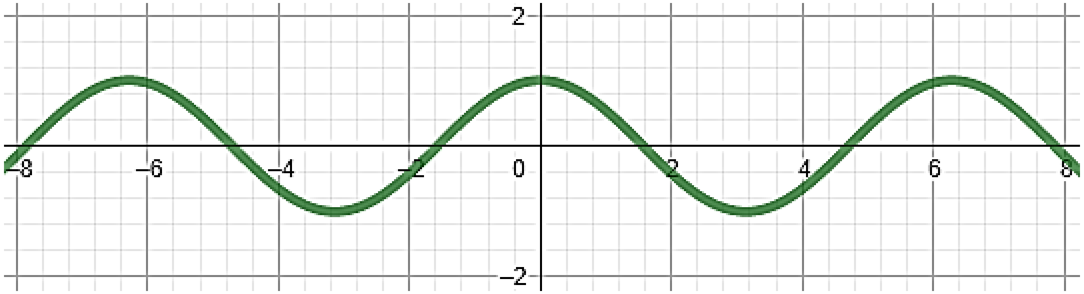
Adding periodic functions - example

- sine and cosine waves can make other periodic functions.

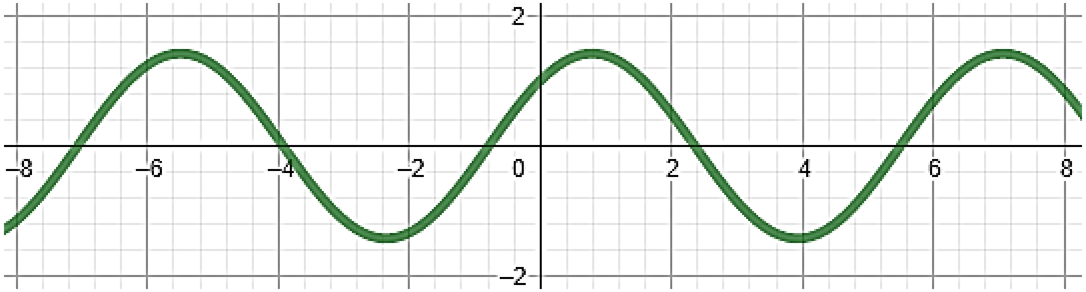
sin(x)



cos(x)



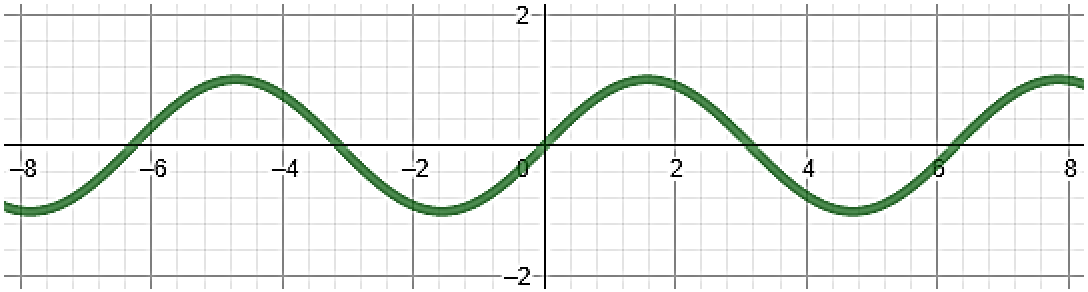
sin(x)+cos(x)



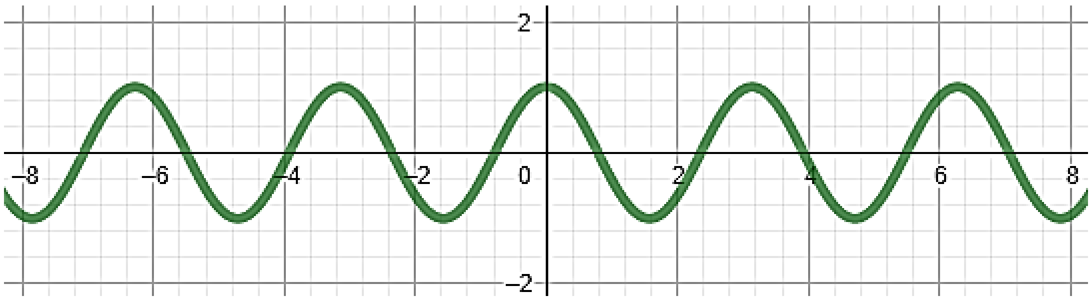
Adding periodic functions - example

- sine and cosine waves can make other periodic functions.

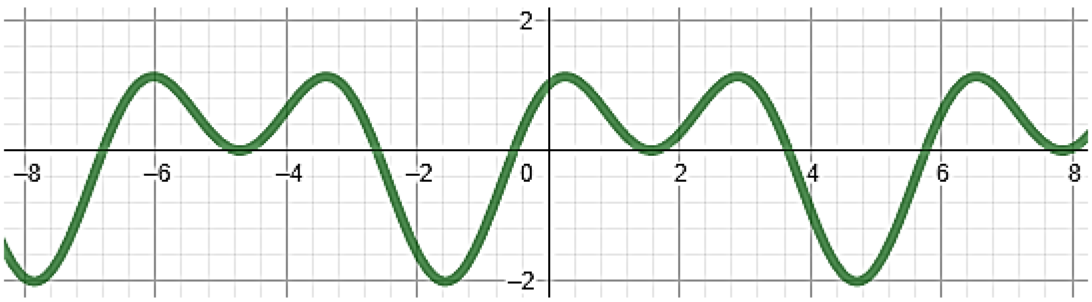
sin(x)



cos(2x)



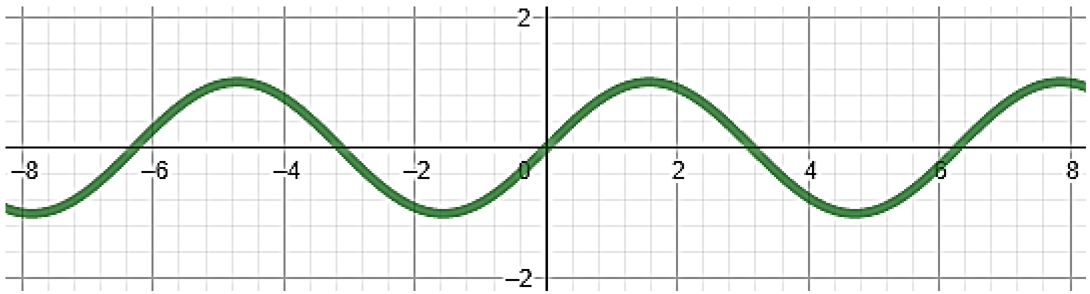
sin(x)+cos(2x)



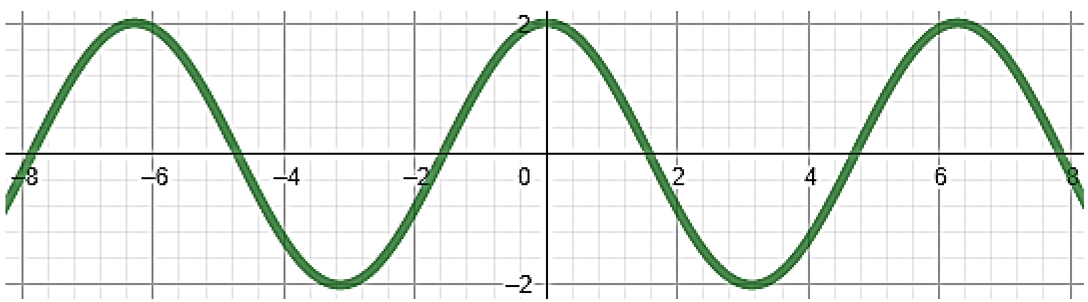
Adding periodic functions - example

- sine and cosine waves can make other periodic functions.

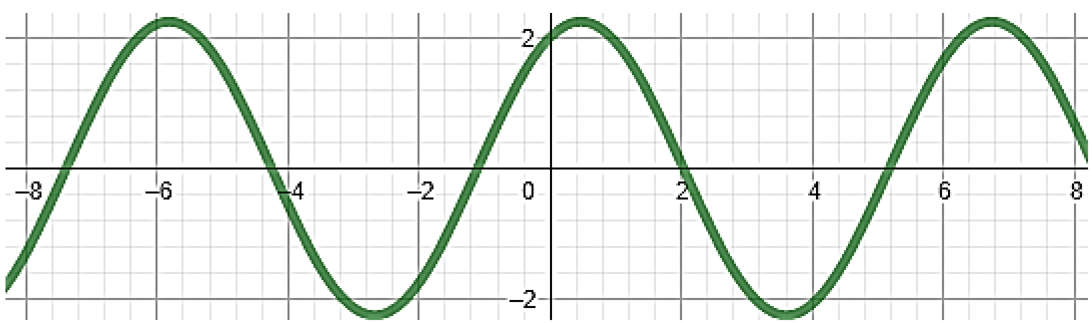
sin(x)



2 cos(x)



sin(x)+ 2 cos(x)



Fourier Series

- A function *f*(*t*) of a *continuous variable* *t*, that is *periodic* with a period *T*, can be expressed as the **sum** of *sines* and *cosines* *multiplied by appropriate coefficients*.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

Where,

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

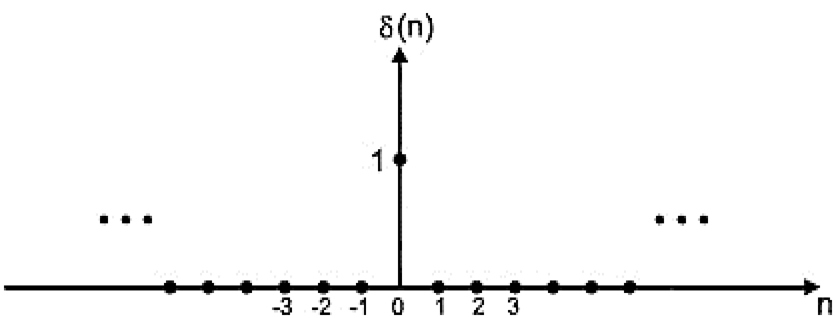
are the coefficients

Small *n* → Longer period and vise versa

Recall

- Discrete unit impulse signal **$\delta(n)$**: signal contains a single **1** with the rest being **0**s.

unit impulse

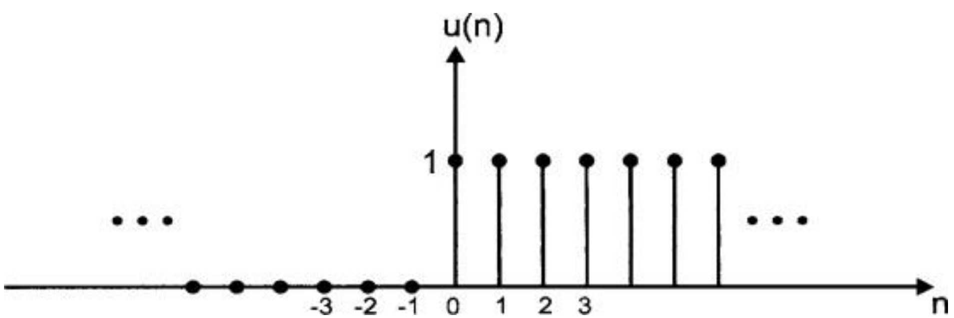


$$\delta(n) = \begin{cases} 1, & \mathrm{n} = 0 \\ 0, & \mathrm{n} \neq 0 \end{cases}$$

or

$$\delta(n) = \{\ldots, 0, 0, 1, 0, 0, \ldots\}$$

unit step



$$u(n) = \begin{cases} 0, & \mathrm{n} < 0 \\ 1, & \mathrm{n} \geq 0 \end{cases}$$

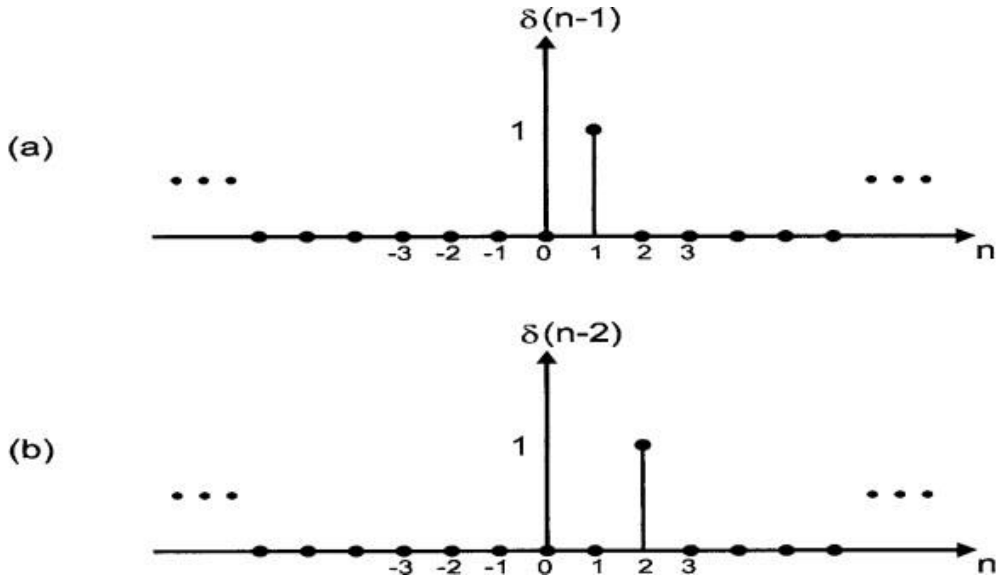
or

$$u(n) = \{\ldots, 0, 0, 1, 1, 1, 1, \ldots\}$$

Recall

- Discrete unit impulse signal **δ(n)**: signal contains a single **1** with the rest being **0**s.

unit impulse



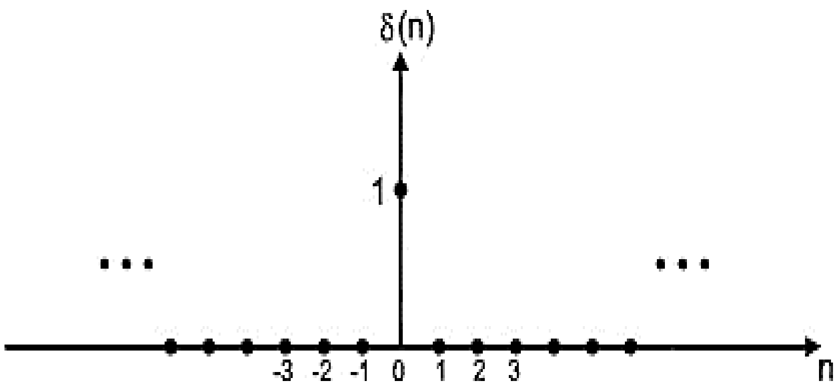
Delayed/shifted versions of the unit impulse sequence

Recall

Difference between an unit impulse signal in discrete time domain and continuous time domain

- <u>**Discrete time Unit Impulse signal**</u>

  $\delta(n) = 1$ for **n = 0**
  0 otherwise

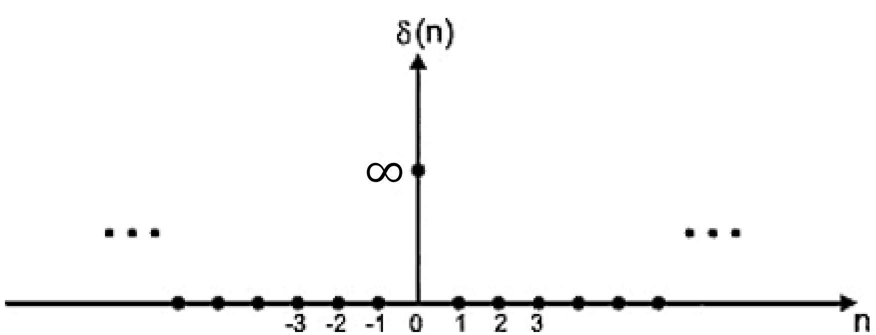


- <u>**Continuous time Unit Impulse signal**</u>

  $\delta(n) = \infty$ for **n = 0**
  0 otherwise

  and

  $\int_{-\infty}^{\infty} \delta(n)dn = 1$ { its area is unity}



Recall

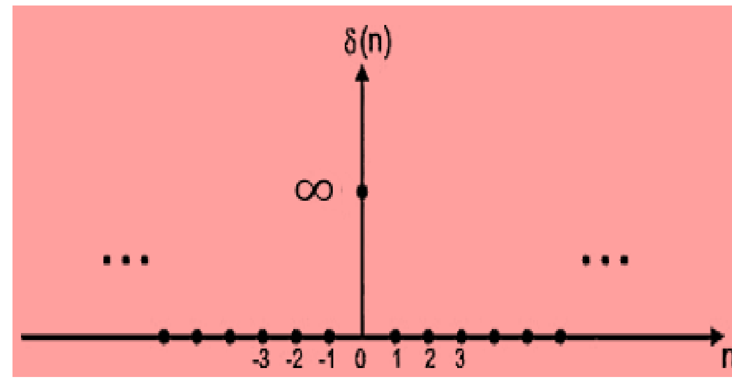
- Correlation of a signal with a discrete unit impulse
  - Outputs a ***<u>rotated version of the signal</u>***, centered at the impulse location.

- Convolution of a signal with a discrete unit impulse
  - Outputs a ***<u>copy of the same signal</u>***, centered at the impulse location.

Sifting Property of Impulses (important)

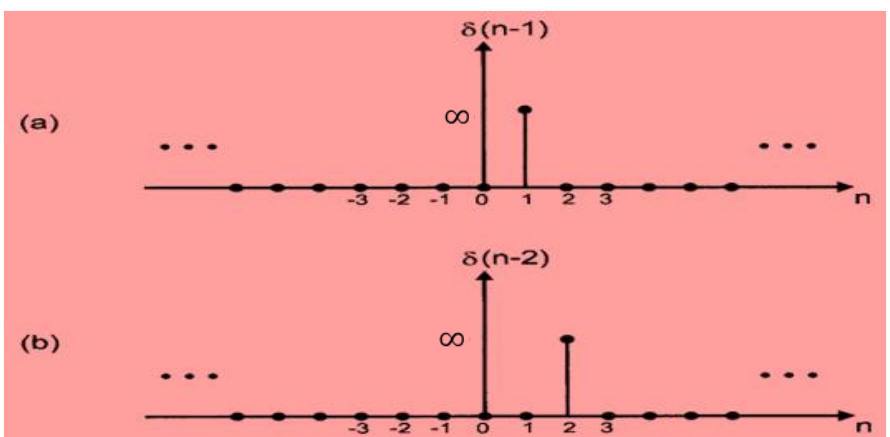
$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

provided $f(t)$ is continuous at $t = 0$



Generalize

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$



Sifting Property of Impulses (important)

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

Sifting yields the *value of the function* at the *location of the impulse.*

For example, if ***f*(*t*) = cos(*t*)**, using the impulse ***δ*(*t* -*π*)** yields the result:
$f(\pi) = \cos(\pi) = $ **−1**.

Sifting Property of Impulses (important)

A **unit impulse** of a discrete variable x, located at $x = 0$, denoted $\delta(x)$ is defined as:

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \text{and subject to:} \quad \sum_{x=-\infty}^{\infty} \delta(x) = 1$$

Sifting Property in discrete variable

$$\int_{-\infty}^{\infty} \delta(n)\, dn = 1$$

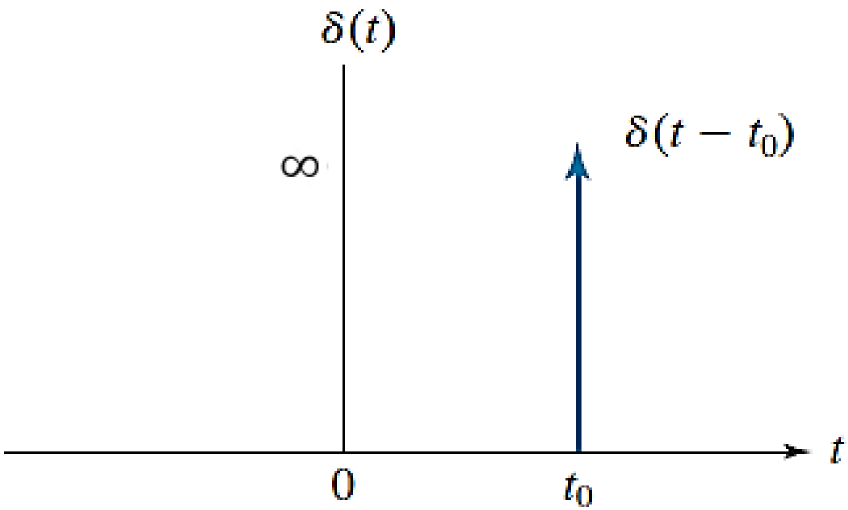
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

Generalize

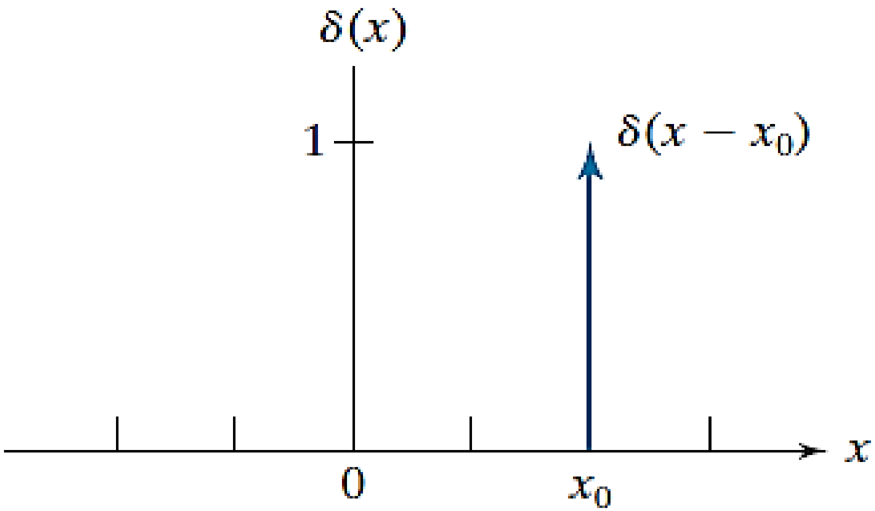
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Sifting yields the *value of the function* at the *location of the impulse.*

Sifting Property of Impulses (important)



Continuous impulse located at $t = t_0$

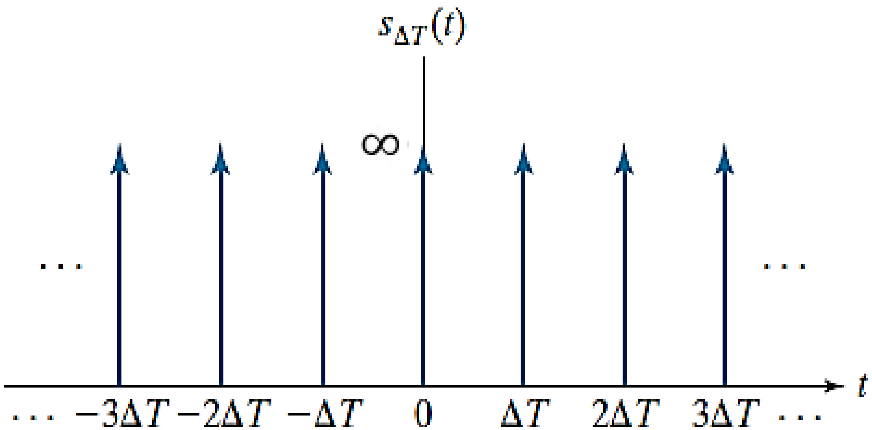


Discrete impulse located at $x = x_0$

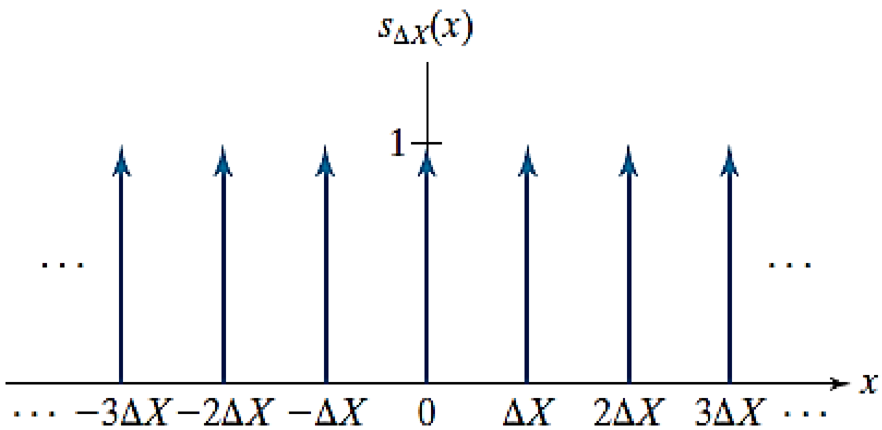
Impulse Train

The ***impulse train*** **$s_{\Delta T}$ (t)** is defined as the *sum of infinitely many impulses* which are ***T*** units apart:

$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$



Continuous impulse train



Discrete impulse train

1D Continuous Fourier Transform

- The ***Fourier transform*** of a continuous function $\boldsymbol{f(t)}$ of a continuous variable $\boldsymbol{t}$, denoted $\Im\{f(t)\}$ is defined by the equation:

$$\Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\,dt$$

  where $\boldsymbol{\mu}$ is a _continuous variable_ (defines *frequency*) also.

- Because $\boldsymbol{t}$ is integrated out, $\Im\{f(t)\}$ is a function only of $\boldsymbol{\mu}$. Therefore,

$$\Im\{f(t)\} = F(\boldsymbol{\mu}) \quad \text{so,}$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}\,dt$$

1D Continuous Fourier Transform

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} \, d\mu$$

Fourier Transform Pair

$$f(t) \Leftrightarrow F(\mu)$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, dt = \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi\mu t) - j \sin(2\pi\mu t) \right] dt$$

- Here, $\boldsymbol{f(t)}$ is real and $\boldsymbol{F(\mu)}$ in general have complex terms.
- **Fourier transform $F(\mu)$ is an expansion of $f(t)$ multiplied by sinusoidal terms whose frequencies are determined by the values of μ.**
- As $\boldsymbol{\mu}$ (*frequency*) is the only variable left after integration, the domain of $\boldsymbol{F(\mu)}$ is called the ***frequency domain***.

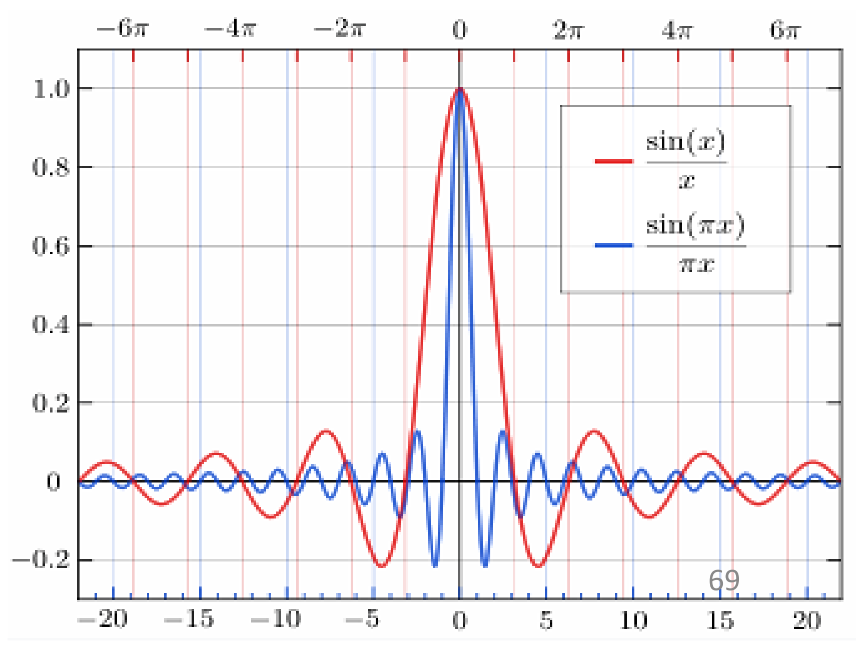
Cardinal Sine Function – sinc( )

- The **unnormalized** ***sinc*** function is given as:

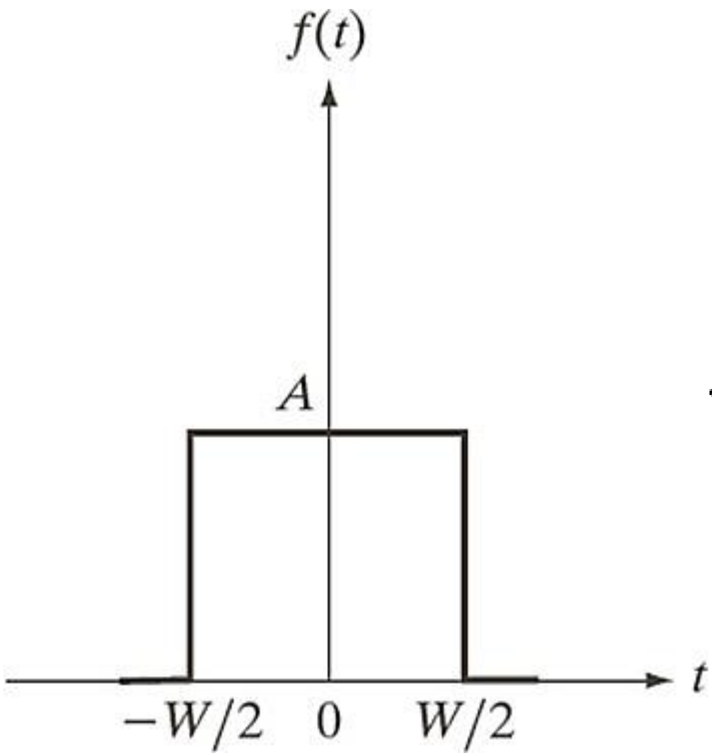
$$\operatorname{sinc}(x) = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin(x)}{x} & \text{otherwise} \end{cases}$$

- The **normalized** ***sinc*** function used in the context of *digital signal processing* is given as:

$$\operatorname{sinc}(\pi x) = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin(\pi x)}{\pi x} & \text{otherwise} \end{cases}$$



Computing Fourier Transform – of a box function



A box function

$$f(t) = \begin{cases} A & \text{if } -\frac{W}{2} \leq t \leq \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Compute Fourier Transform for the box function

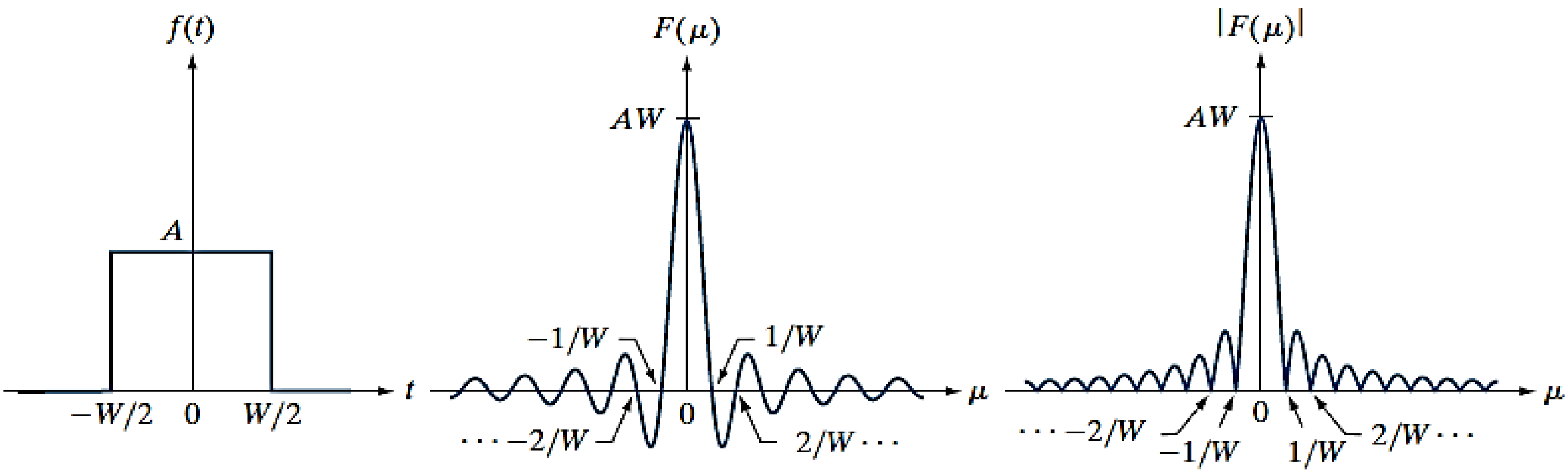
Compute Fourier Transform – of a box function

$$F(\mu)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt=\int_{-W/2}^{W/2} A\,e^{-j2\pi\mu t}dt$$, since $\int_a^b e^{kt}\,dt = \frac{e^{kt}}{k}\Big]_a^b$

$$=\frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2}=\frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W}-e^{j\pi\mu W}\right]$$

$$=\frac{A}{j2\pi\mu}\left[e^{j\pi\mu W}-e^{-j\pi\mu W}\right]$$

$$=AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$ where the trigonometric identity $\sin\theta=(e^{j\theta}-e^{-j\theta})/2j$



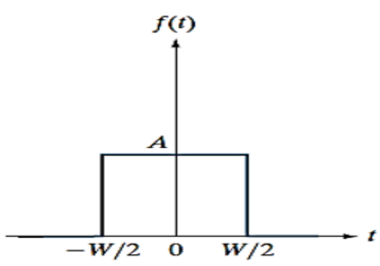
A box function **its Fourier transform** **its Spectrum**

Compute Fourier Transform – of a box function

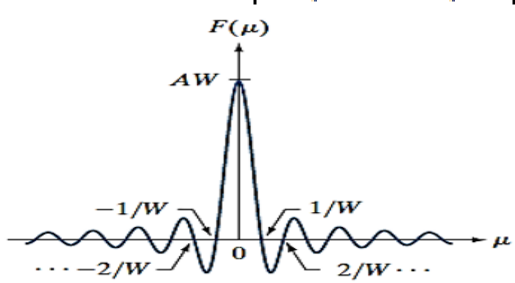
Some Observations:

- Locations of zeroes in **F(μ)** and **|F(μ)|** are inversely proportional to **W**, the width of the "box" function.
- The height of the function decreases away from the origin.
- The function extends to infinity in both directions for variable $\boldsymbol{\mu}$.
- The Fourier transform contains complex terms, and for display purposes, we work with the magnitude of the transform (a real quantity), which is called the *Fourier spectrum* or the *frequency spectrum*:

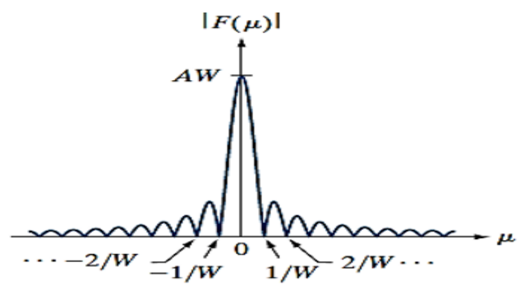
$$|F(\mu)| = AW\left|\frac{\sin(\pi\mu W)}{(\pi\mu W)}\right|$$



A box function



its Fourier transform



its Spectrum

Compute Fourier Transform – of an impulse

- The Fourier transform of a unit impulse located at the origin is:

$$\Im\{\delta(t)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t) dt = e^{-j2\pi\mu}$$

where we used the sifting property $\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$

- Similarly, the Fourier transform of an impulse located at ***t* = *t*$_0$** is:

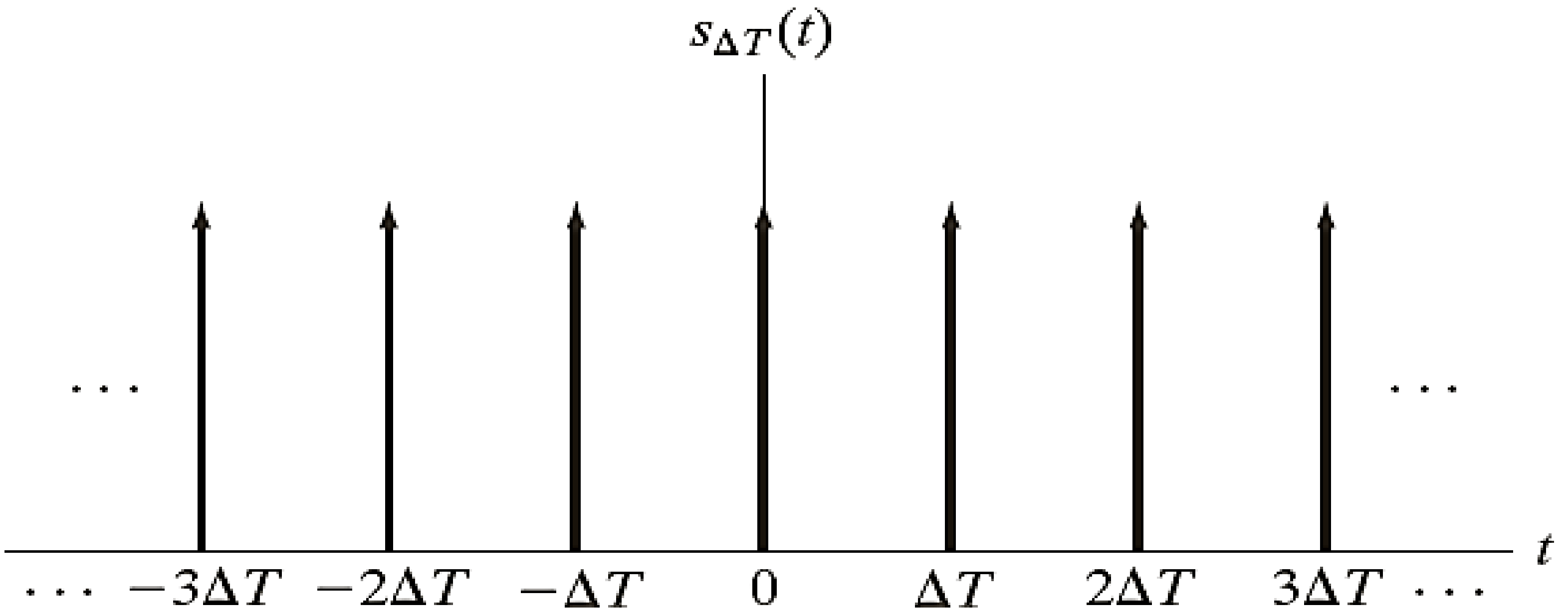
$$\Im\{\delta(t - t_0)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t - t_0) dt = e^{-j2\pi\mu t_0}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

The Fourier transform of an **impulse** located at the **origin** of the *spatial domain* is a **constant** in the *frequency domain*.

Compute Fourier Transform – of a periodic impulse train

Recall: Impulse Train



$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

Compute Fourier Transform – of a periodic impulse train

Fourier Series is given by:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

Where,

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

The impulse train $s_{\Delta T}(t)$ can be expressed as a the following Fourier series:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

Where,

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t)e^{-j\frac{2\pi n}{\Delta T}t}dt$$

Compute Fourier Transform – of a periodic impulse train

Simplify c_n and we have: $c_n = \dfrac{1}{\Delta T}$

The Fourier Series then becomes:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

Finally, the Fourier transform of this Fourier series becomes:

$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

Compute Fourier Transform – of a periodic impulse train

<u>Compare</u> **Impulse train** with its **Fourier transform**:

$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

Vs.

$$\mathfrak{F}\{S_{\Delta T}\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

- The Fourier Transform of an impulse train with period **ΔT** **is also an impulse train**, whose period is **1/ΔT**
- This *inverse proportionality* between the periods of $\boldsymbol{s_{\Delta T}(t)}$ and $\mathbf{s}(\boldsymbol{\mu})$ is analogous to the “box” function transform where *Zero-crossings were* inversely proportional to W/2.

Compute Fourier Transform – of a periodic impulse train

Compare **Impulse train** with its **Fourier transform**:

$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

Vs.

$$S(\mu) = \mathfrak{F}\{S_{\Delta T}\} = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

- The Fourier Transform of an impulse train with period **ΔT** **is also an impulse train**, whose period is **1/ΔT**
- There is *inverse proportionality* between the periods of $\boldsymbol{s_{\Delta T}(t)}$and $\boldsymbol{s(\mu)}$.

Convolution of Continuous Functions

Convolution

- Convolution is a sliding window representation in spatial domain
  - Rotate by 180^0 (flip) *one function* (**kernel**) and slide it over the *second function*.
  - **At each displacement in the sliding process, *compute the sum of products in the local neighborhood* and replace the value at the origin of the kernel with this new value.**
- We are now interested in the **convolution** of *two **spatial domain** continuous functions of one continuous variable*, **t**, say ***f(t)*** and ***h(t)*** ***and its equivalent in the frequency domain***.

Convolution

- Let ***f(t)*** and ***h(t)*** be the *two spatial domain continuous functions of one continuous variable*, **t**.

- The convolution of these two functions is defined as:

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

where the **minus sign** accounts for the *flipping* of kernel function, ***t*** is the *displacement* needed to slide one function past the other, and $\boldsymbol{\tau}$ is a *dummy variable* that is integrated out.

- What is the Fourier transform of $(f \star h)(t)$?

Convolution

$$\Im\{(f \star h)(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j2\pi\mu t} dt \right] d\tau$$

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

claim: $\Im\{h(t-\tau)\} = H(\mu) e^{-j2\pi\mu\tau}$ where $H(\boldsymbol{\mu})$ is the Fourier transform of $h(t)$.

$$\Im\{(f \star h)(t)\} = \int_{-\infty}^{\infty} f(\tau) \left[H(\mu) e^{-j2\pi\mu\tau} \right] d\tau$$

$$= H(\mu) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi\mu\tau} d\tau$$

$$= H(\mu) F(\mu)$$

$$= (H \bullet F)(\mu)$$

$$\boxed{\Im\{(f \star h)(t)\} = (H \bullet F)(\mu)}$$

where **dot (.)** indicates **multiplication**

Convolution

<u>Note:</u> **The Fourier Transform of the convolution of two functions in the spatial domain is equal to the product in the frequency domain of the Fourier transforms of these two functions. The converse statement is also true.**

The Convolution Theorem:

$$(f \star h)(t) \Leftrightarrow (H \bullet F)(\mu)$$

$$(f \bullet h)(t) \Leftrightarrow (H \star F)(\mu)$$

Next Lecture

- 1-D Sampling
  - Sampling Revisited
  - Sampling Theorem
  - Signal Recovery
- 2-D Sampling
- Aliasing
- Aliasing in Images
  - How to reduce the effects of spatial aliasing?
  - Moiré Patterns
  - Halftoning