Algorithm Analysis and Design

LECTURE 2

Asymptotic Notations and Complexity

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Asymptotic Notation

In this section, we discuss the various asymptotic notations used when referring to complexity[1] of algorithms. The main notations used will be O, Θ and Ω.

Big O-notation

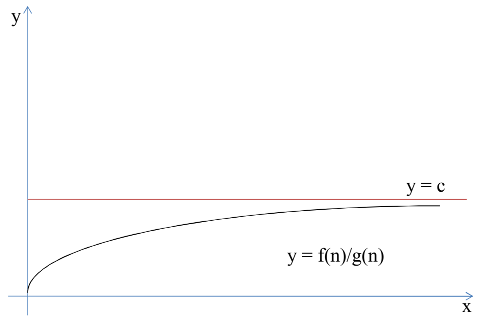
We give below a rigourous definition of the big O notation. In general, a function $f(n) = O(g(n))$ means that there are constants $c > 0, n_0 > 0$ such that

$$f(n) \leq cg(n) \quad \text{for all} \quad n \geq n_0.$$

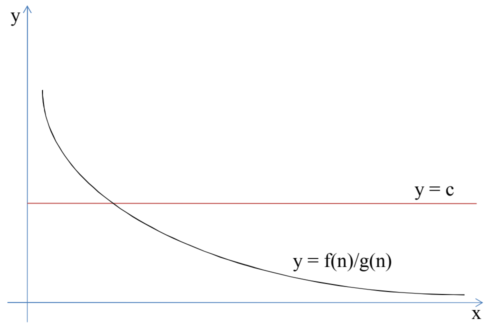
One should note that 1) $f(n)$ is not a C function but a math function and 2) $f(n)$ and $g(n)$ here can be any member of the function space, so long as the conditions above are met. $f(n)$ and $g(n)$ here may or may not represent any measurements in real life.

The big O definition is first and foremost a mathematical definition before it is applied in the realm of algorithms. It also represents a set of functions. Roughly speaking, what this function expresses is that given a large enough n i.e., $n > n_0$, the ratio $f(n)$ over $g(n)$ is capped at some constant c i.e., $f(n)$ is gauranteed to be bigger than $g(n)$ by at

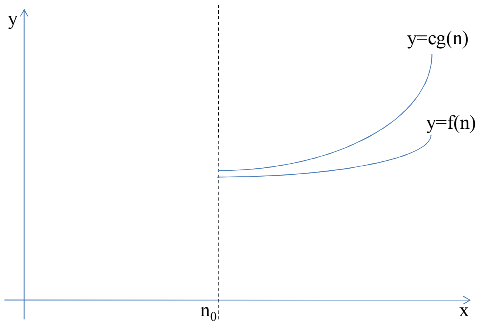
[1]Complexity here really means a measure of the number of basic steps needed to perform the algorithms. In general, if the number of steps scale well with increasing input size e.g., a tenfold increase in input size does not result in an exponential increase in number of steps, it is considered “ simple”.



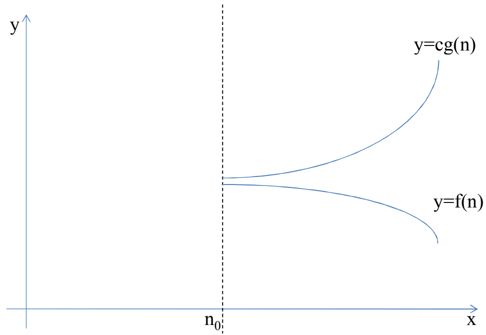
The ratio of $\frac{f(n)}{g(n)}$ tends towards a constant c.



The ratio of $\frac{f(n)}{g(n)}$ tends towrds zero. Still fulfills the definition of a big O relationship. Note that there's even a portion where $\frac{f(n)}{g(n)}$ exceeds the constant c.



Beyond $n = n_0$, $cg(n) \geq f(n)$



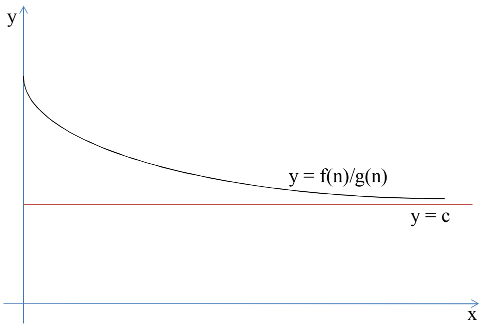
Beyond $n = n_0$, still $cg(n) \geq f(n)$. The point is that O is simply an upper bound relationship. T

Figure 1. Graphs showing possible different relationships between $f(n)$ and $g(n)$ when $f(n) = O(g(n))$.

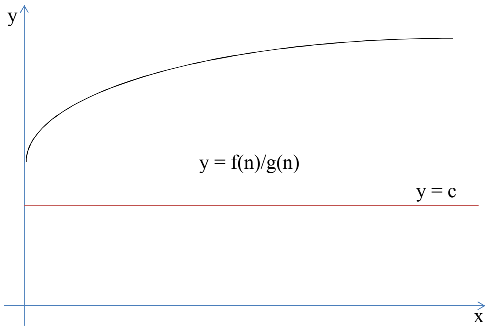
most c times if for all $n > n_0$. Figure 2 shows the possible relationships between $f(n)$ and $g(n)$ when $f(n) = O(g(n))$. Therefore, another way to conceive the relationship between $f(n)$ and $g(n)$ would be:

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} \leq c, \quad c > 0$$

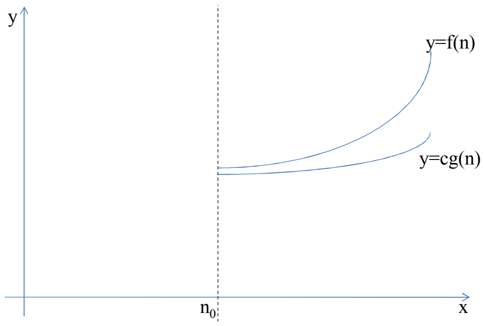
Now it does not mean that the ratio between $f(n)$ and $g(n)$ will be a constant in the long run. For example, let $f(n)$ be $n + 5$ and $g(n)$ be



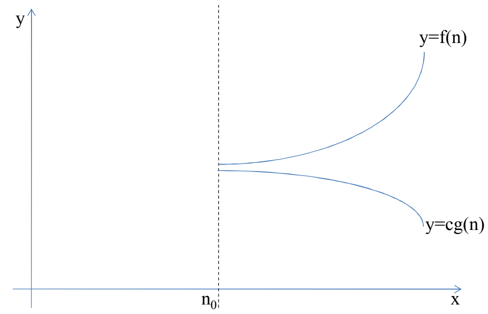
The ratio of $\frac{f(n)}{g(n)}$ tends towards a constant c from it's upside.



The ratio of $\frac{f(n)}{g(n)}$ tends towards away from $y = c$. Still fulfills the definition of a big-Ω relationship.



Beyond $n = n_0$, $cg(n) \geq f(n)$



Beyond $n = n_0$, still $f(n) \geq cg(n)$. The point is that Ω is simply an lower-bound relationship.

Figure 2. Graphs showing possible different relationships between $f(n)$ and $g(n)$ when $f(n) = \Omega(g(n))$.

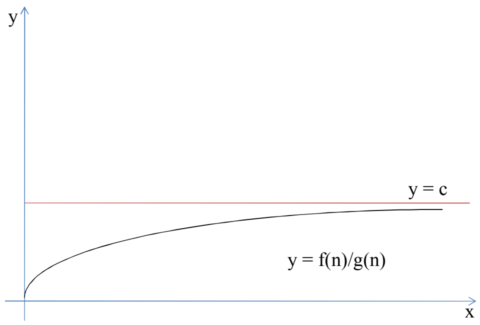
$(n+5)^2$. Clearly $n + 5 \in O(\frac{1}{100}(n+5)^2)$ since,

$$\begin{aligned} say \quad n &> 1 \\ so \quad (n+5) &> 1 \\ \therefore \quad (n+5)^2 &> (n+5), for \quad n > 1 \\ \therefore n_0 = 1, \quad c = 100 & \end{aligned}$$

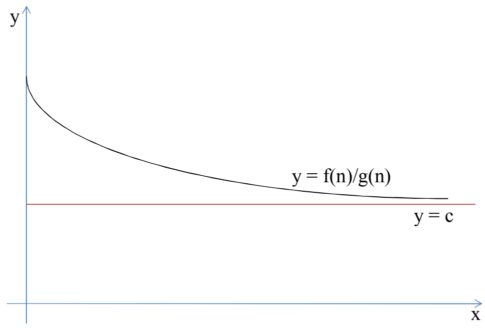
Big Ω notation

We give below a rigourous definition of the big Ω notation. In general, a function $f(n) = \Omega(g(n))$ means that there are constants $c > 0, n_0 > 0$ such that

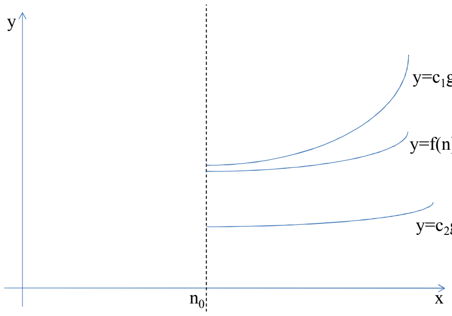
$$f(n) \geq cg(n) \quad \text{for all} \quad n \geq n_0.$$



The ratio of $\frac{f(n)}{g(n)}$ tends towards $y = c$ from the downside.



The ratio of $\frac{f(n)}{g(n)}$ tends towards $y = c$ from the upside.



$f(n)$ remains firmly between $c_1 g(n)$ and $c_2 g(n)$ after $n = n_0$

Figure 3. Graphs showing the only possible relationships between $f(n)$ and $g(n)$ when $f(n) = \Theta(g(n))$.

The big Ω notation is the opposite of the big O. Instead of $g(n)$ being a constant factor larger or equals to $f(n)$ for large enough n, it is gauranteed to be a constant factor smaller or equals to $f(n)$. Roughly speaking, this means that the ratio $f(n)$ over $g(n)$ is at least some constant c. Figure 2 shows the possible relationships between $f(n)$ and $g(n)$ when $f(n) = \Omega(g(n))$

Therefore, another way to conceive the relationship between $f(n)$ and $g(n)$ would be:

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} \geq c, \quad c > 0$$

Big Θ notation

We give below a rigorous definition of the big Θ notation. In general, a function $f(n) = \Theta(g(n))$ means that there are constants $c_1 > 0, c_2 >$

$0, n_0 > 0$ such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for all} \quad n \geq n_0.$$

The Big Θ describes a relationship between $f(n)$ and $g(n)$ that is both big O and big Ω. Hence the relationship between the two is a strict one. Figure 3 shows the only possible relationships that $f(n)$ and $g(n)$ must have. Note how the first two diagrams show different curves that may satisfy the Ω and O but yet still can be Θ.

Therefore, another way to conceive the relationship between $f(n)$ and $g(n)$ if $f(n) \in \Theta(g(n))$ would be:

$$\lim_{n \to +\infty} \frac{f(n)}{g(n)} = c, \quad c > 0.$$