Histogram Processing -1

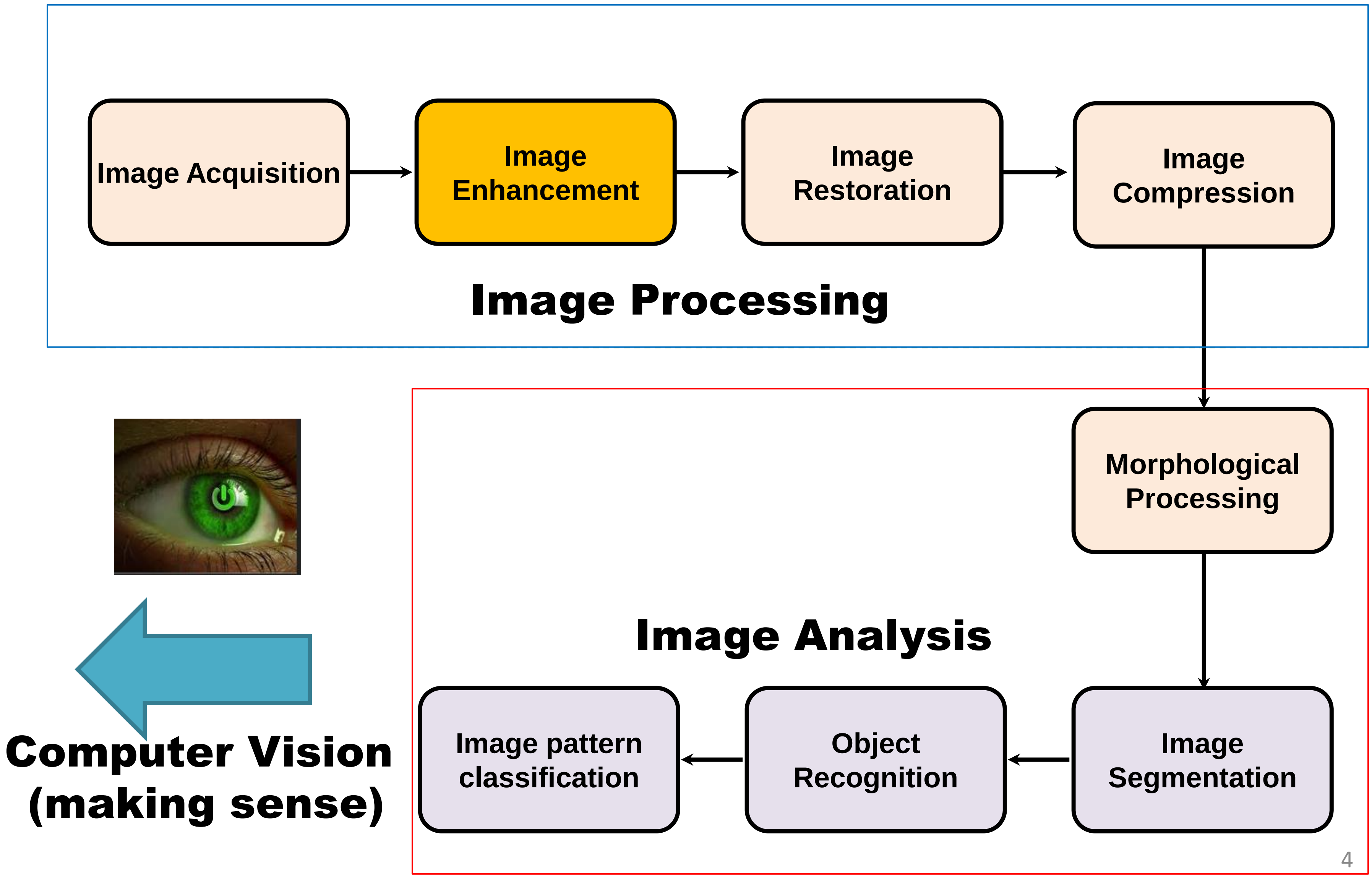
Recap

- Spatial Domain Transformations - preview
- The Transformation Operator
- Intensity Transformations – examples
  - Contrast stretching
  - Image thresholding
- Basic Intensity Transformation Functions
  - Basic transforms
  - Piecewise-linear transformations

Lecture Objectives

- What is a Histogram?
- Histogram Normalization
- What is Random variable
- Histogram Equalization

Key Stages in DIP



What is a Histogram?

histogram

- a bar graph representing frequency distribution for certain ranges or intervals.

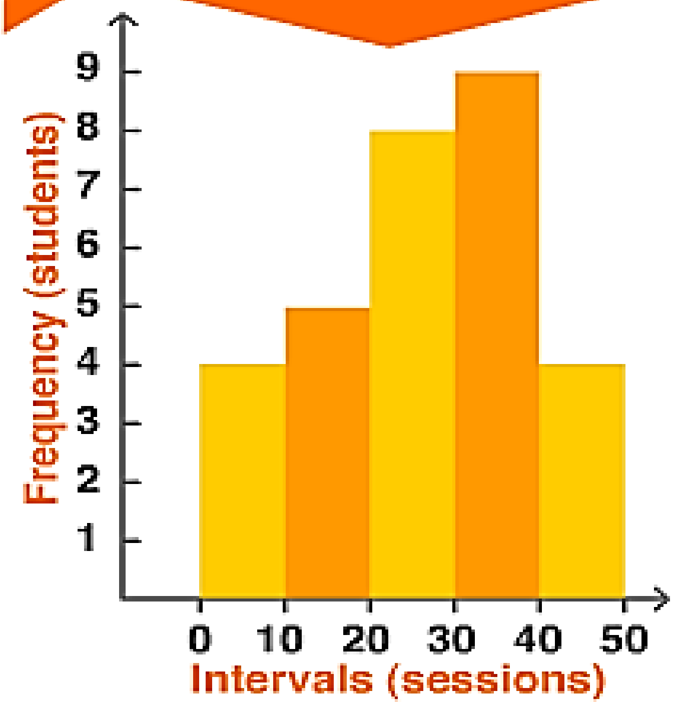
The number of data items in an interval is a frequency.
The bar heights represent these frequencies.

EXAMPLE: A survey of 30 students to see how many times they accessed the internet last week.

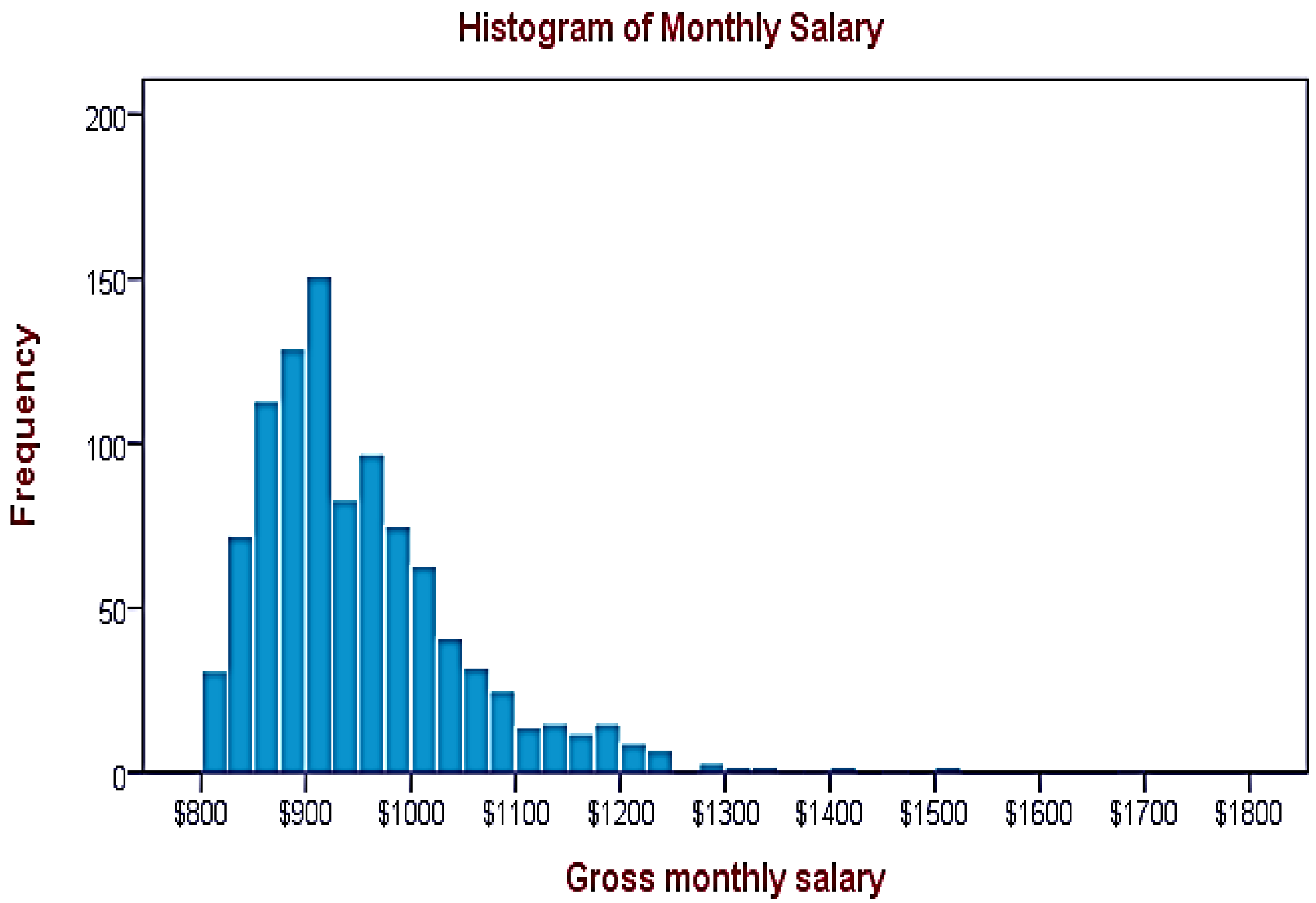
frequency distribution table

Number of Sessions on the Internet (intervals)	Number of Students (frequency)
0 - 10	4
11 - 20	5
21 - 30	8
31 - 40	9
41 - 50	4

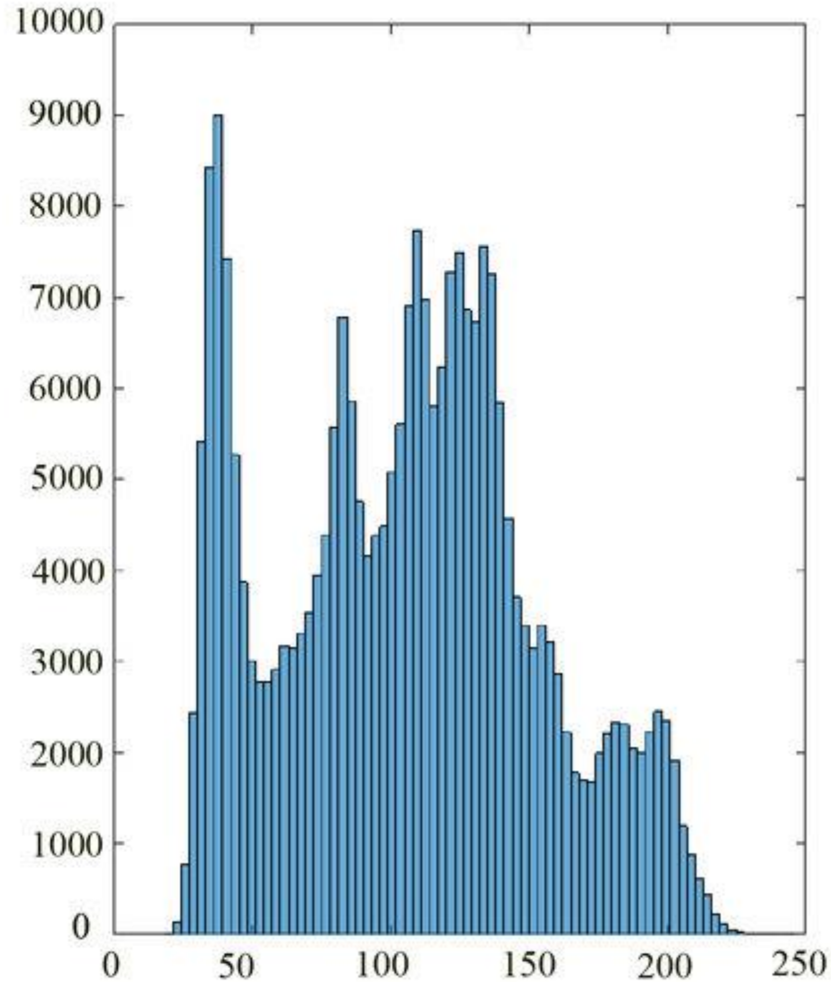
frequency histogram



What is a Histogram?



What is a Histogram?



What is a Histogram?

- The **unnormalized histogram** **(Raw Histogram)** of a digital image ***f*** with *intensity levels* in the range **[0, L-1]** is defined as a discrete function:

$$h(r_k)=n_k$$

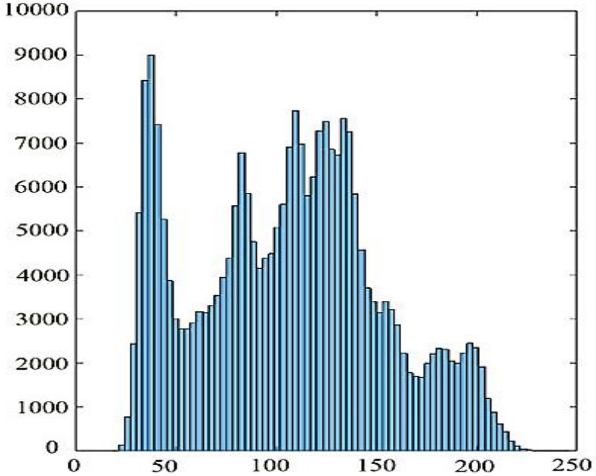
where,

r_k: the k^{th} intensity value for $K=0, 1, 2, \dots, L-1$.

n_k: the number of pixels in the image ***f*** with intensity r_k

histogram bins: are the subdivisions of the intensity scale

- It is basis for numerous spatial domain processing techniques:
  - Intensity transformation
  - Image compression
  - Image segmentation
  - e.t.c



Histogram Normalization

- Image processing commonly use **normalized histogram** instead of *raw histogram*.

$$h(r_k)=n_k$$ → **Raw Histogram**

Divide each $\mathbf{n_k}$ value by the *total number of pixels* in the image, $\mathbf{M \times N}$

$$p(r_k)=\frac{n_k}{(M*N)}$$ → **Normalized Histogram**

- $\mathbf{p(r_k)}$ is then the probability of occurrence of intensity r_k in the image.

- The *sum* of $\mathbf{p(r_k)}$ for *all values* of ***k*** is always **1**.

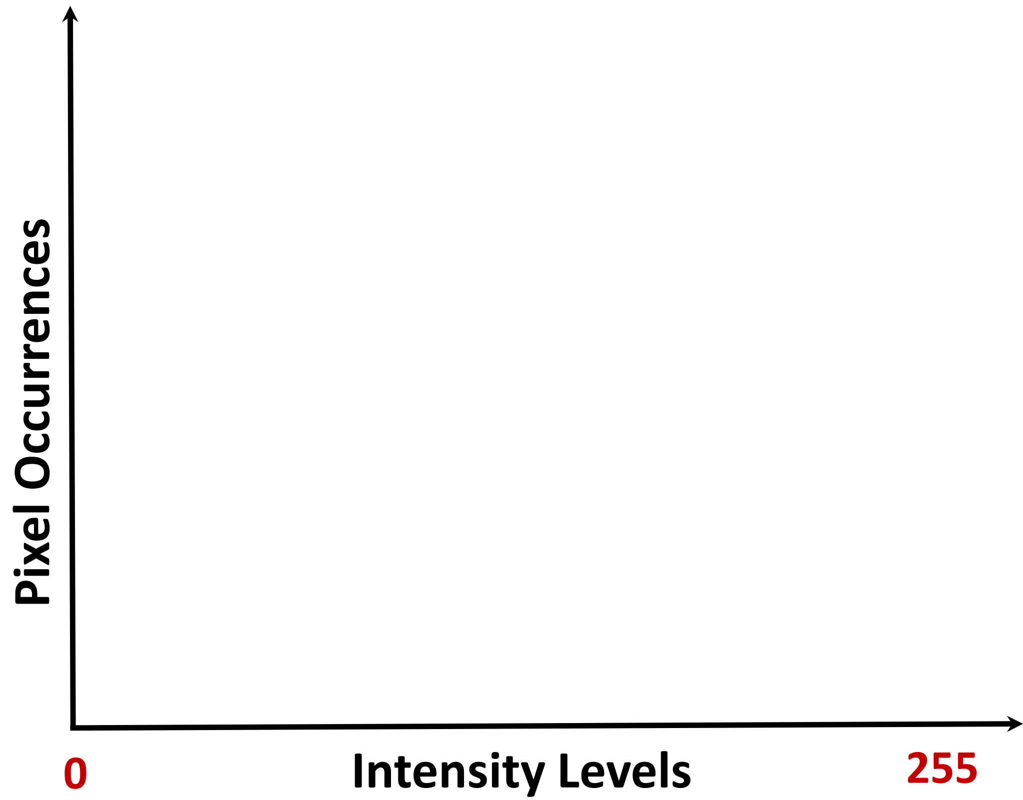
$$\sum_k p(rk) = 1$$

Histogram - shape

- Histogram **shape** is always related to **image appearance**.

Histogram - examples

Image Histogram

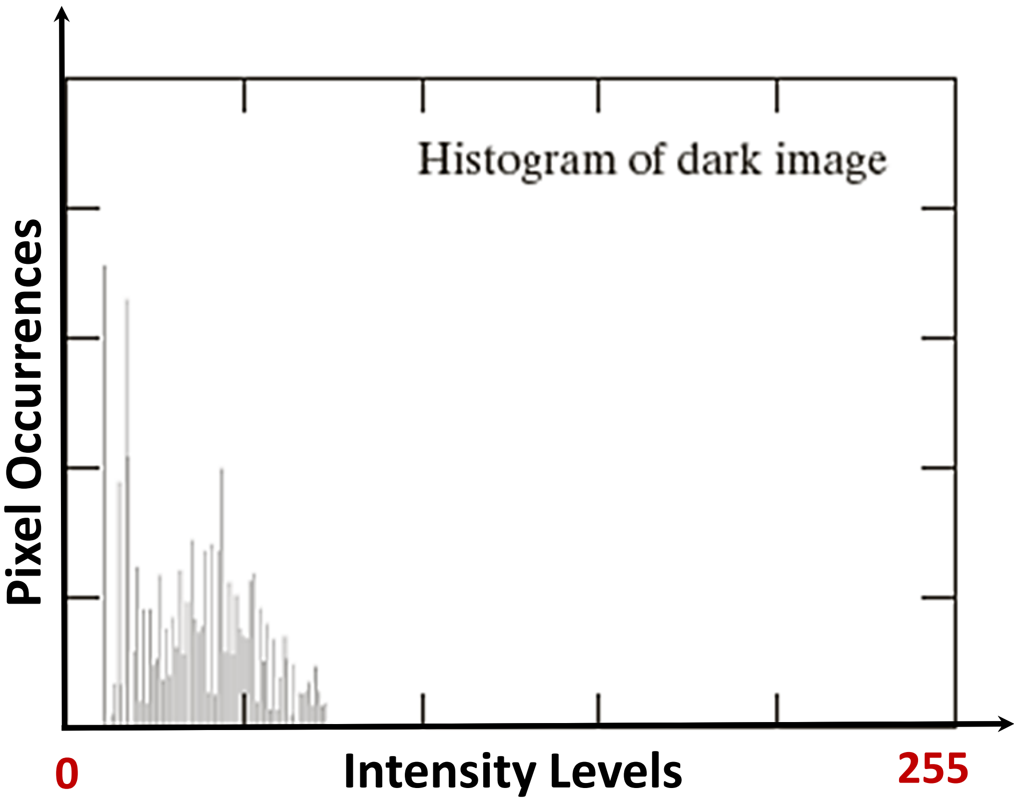


Dark image

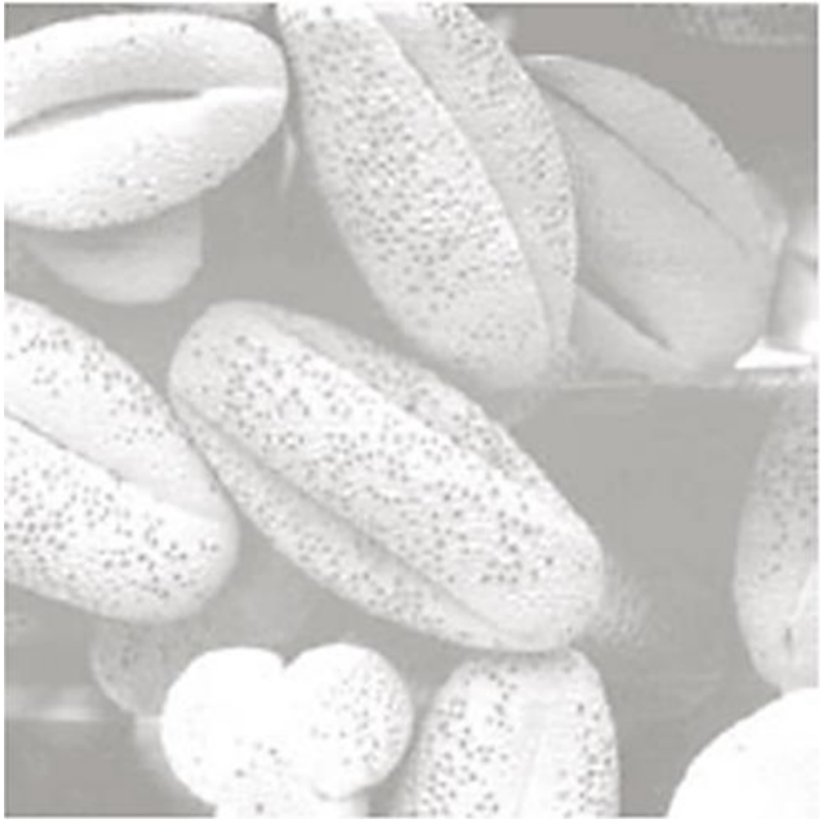
Histogram - examples

Dark image

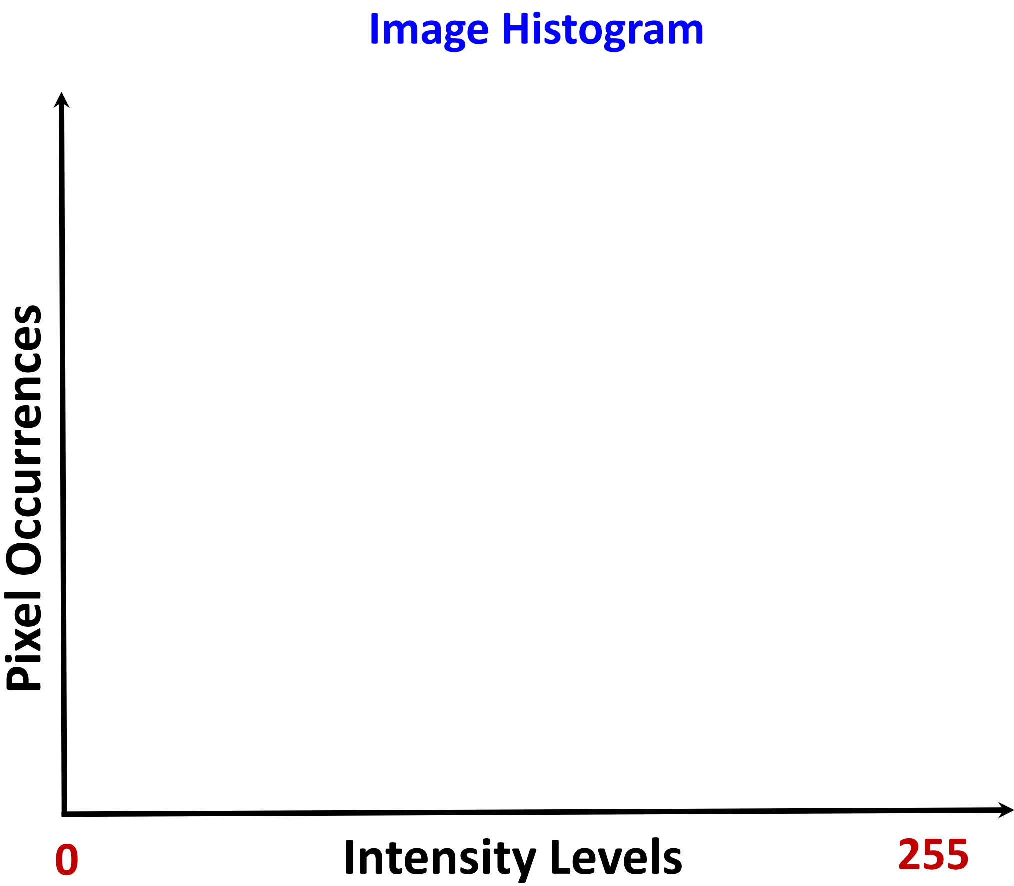
Image Histogram



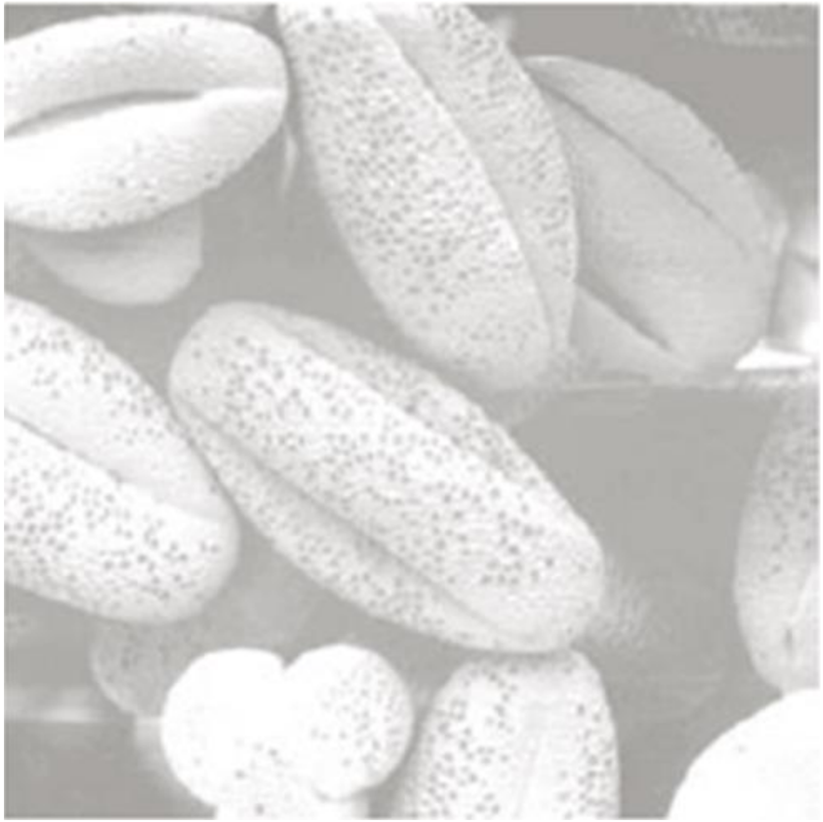
Histogram - examples



Light image

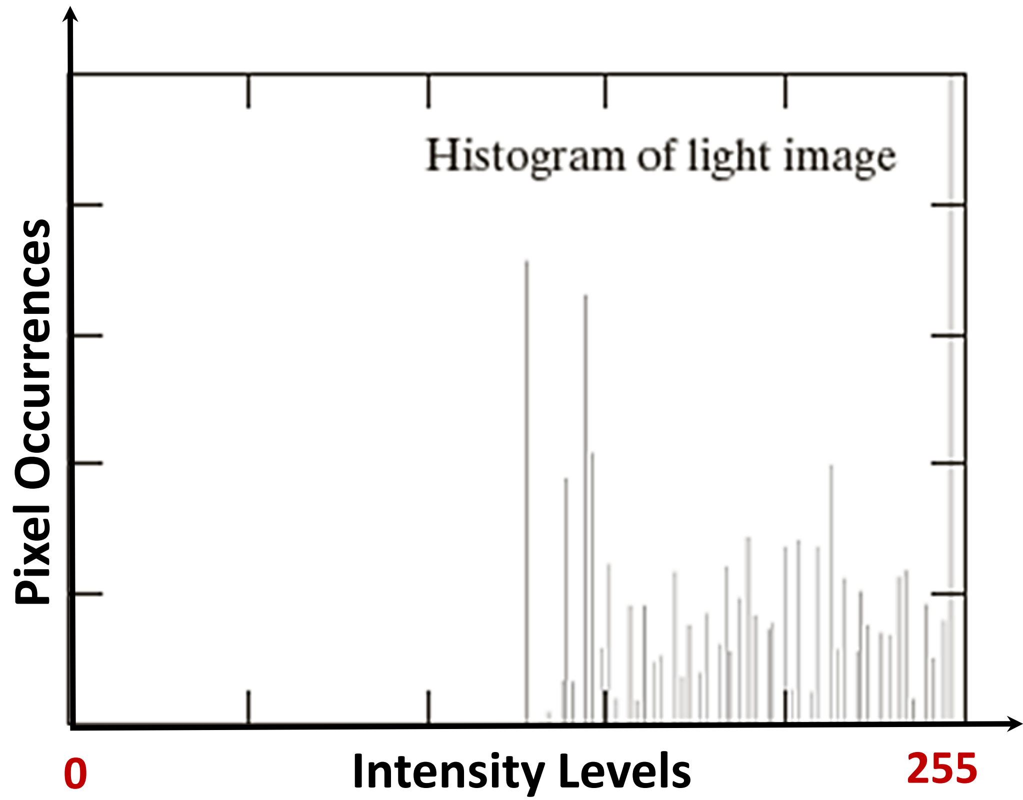


Histogram - examples



Light image

Image Histogram



Histogram - examples

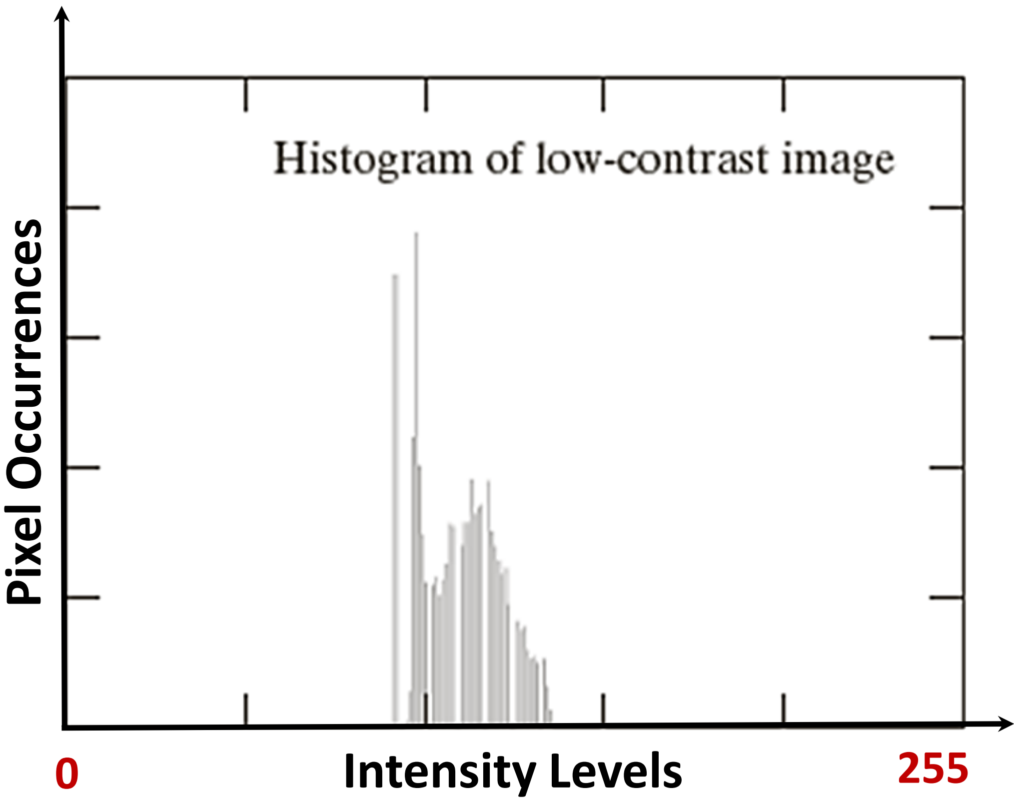
Low-contrast image

Image Histogram

Histogram - examples

Low-contrast image

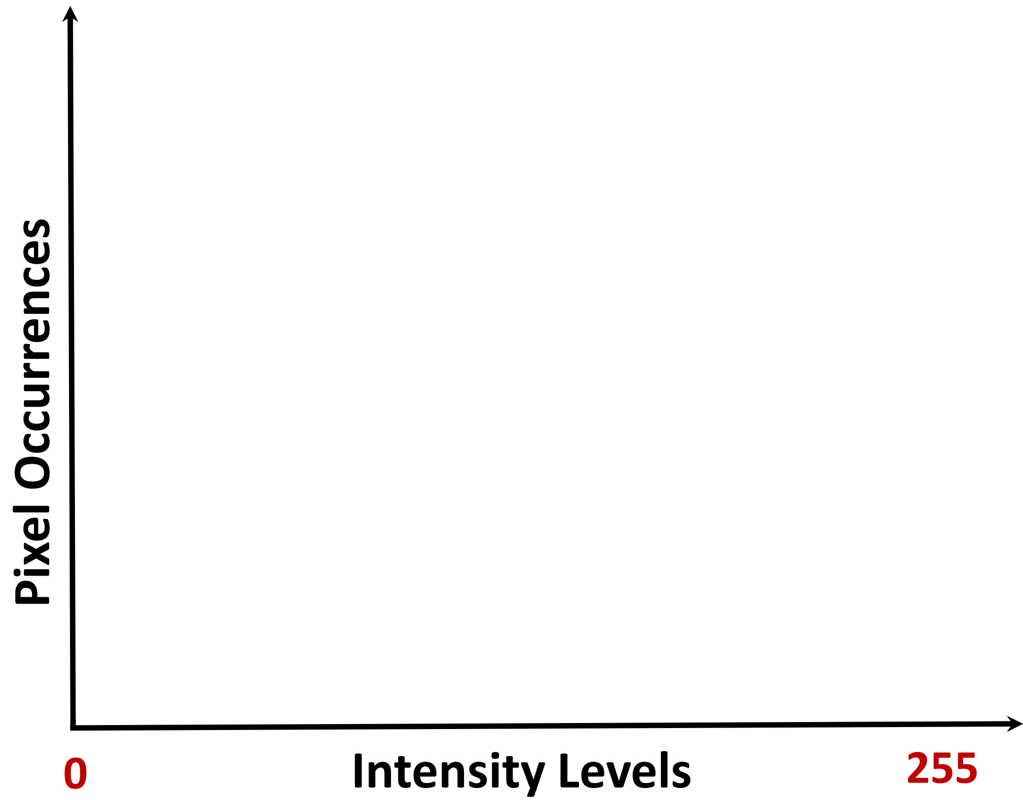
Image Histogram



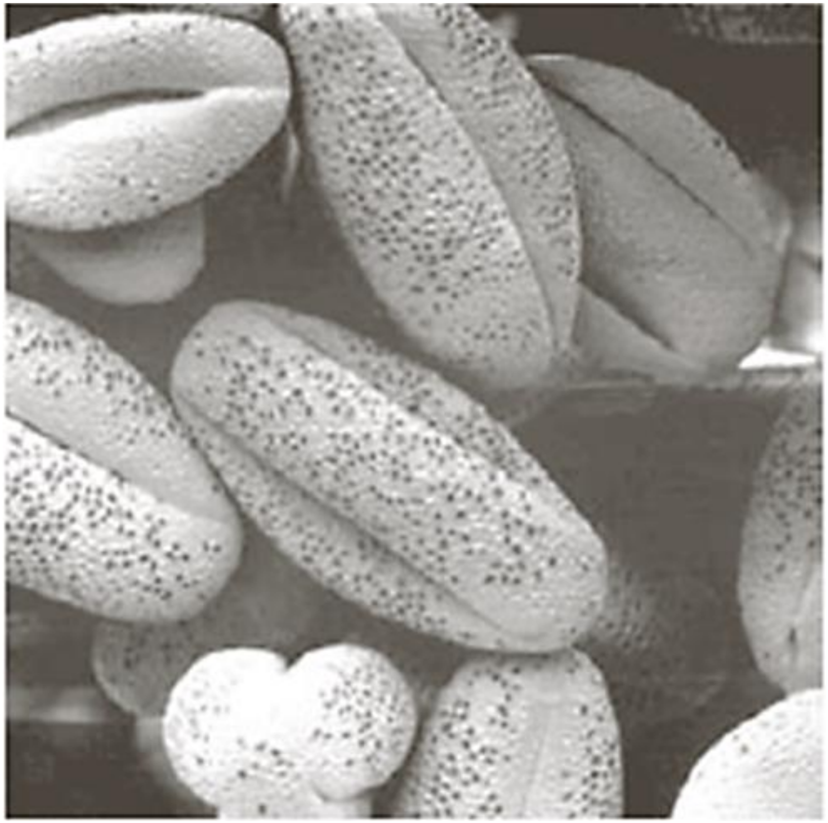
Histogram - examples

High-contrast image

Image Histogram

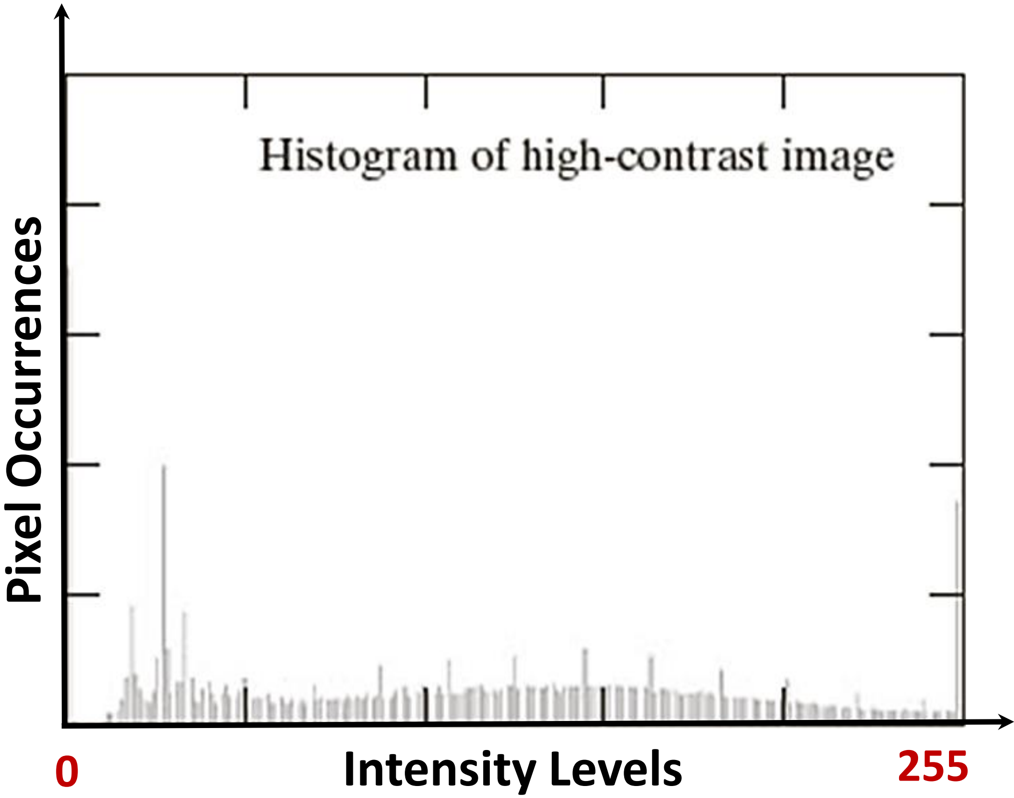


Histogram - examples



High-contrast image

Image Histogram



Random Variable

Random Variable

- A random variable ***X*** is a variable whose value is *unknown* **OR,** it is a *real-valued function* defined on a sample space ***S***. A continuous random variable could have any value (usually *within a certain range*).

$$S \xrightarrow{X} R$$

- The cumulative distribution function (CDF) of a *random variable* **X** is the function:

$$F_X(x) = P(X \leq x), \quad -\infty < x < +\infty$$

OR

The cumulative distribution function (CDF) of a *real-valued random variable* **X** evaluated at x, is the probability that **X** will take a value less than or equal to x.

Discrete Random Variable

- A discrete random variable is a random variable that can take a *finite number of values*.

E.g.: In the case of a fair die, $P(X=1)=\ldots P(X=6)=1/6$

- The probability mass function (PMF) of a discrete random variable $\mathbf{X}$ is the function that gives the probability that a discrete random variable is *exactly equal to some value*:

$$p(k)=P(X=k)$$ i.e., probability of ***X*** at ***k***

Continuous Random Variable

- $\mathbf{X}$ is a continuous random variable if there exists a *nonnegative function* $\mathbf{p}_{\mathrm{X}}(x)$, defined for all real $-\infty < x < +\infty$, having the property that for any set *A* of real numbers,

  $$P(x \in A) = \int_A p_X(x)\,dx$$

- The function $\mathbf{p}_{\mathrm{X}}(x)$ is called the probability density function (PDF) of the random variable $\mathbf{X}$ and is defined by:

  $$p_X(x) = dp_X(x)/dx$$

  – **PDF** is used to specify the probability of the random variable *falling within a particular range of values as opposed to taking any one value*.

- Some properties of **PDF:**

  1. $p(x) \geq 0$ for all x
  2. $\int_{-\infty}^{\infty} p(x)dx = 1$
  3. $F(x) = \int_{-\infty}^{x} p(\alpha)d\alpha$, where α is a dummy variable
  4. $P(x_1 < x \leq x_2) = \int_{x_1}^{x_2} p(x)dx.$

Histogram Equalization

Histogram Equalization

- The approach is to design a transformation function $T(r)$ such that, the **pixel intensity values** in the *output image* is **uniformly distributed in the range $[0, L\text{-}1]$.**

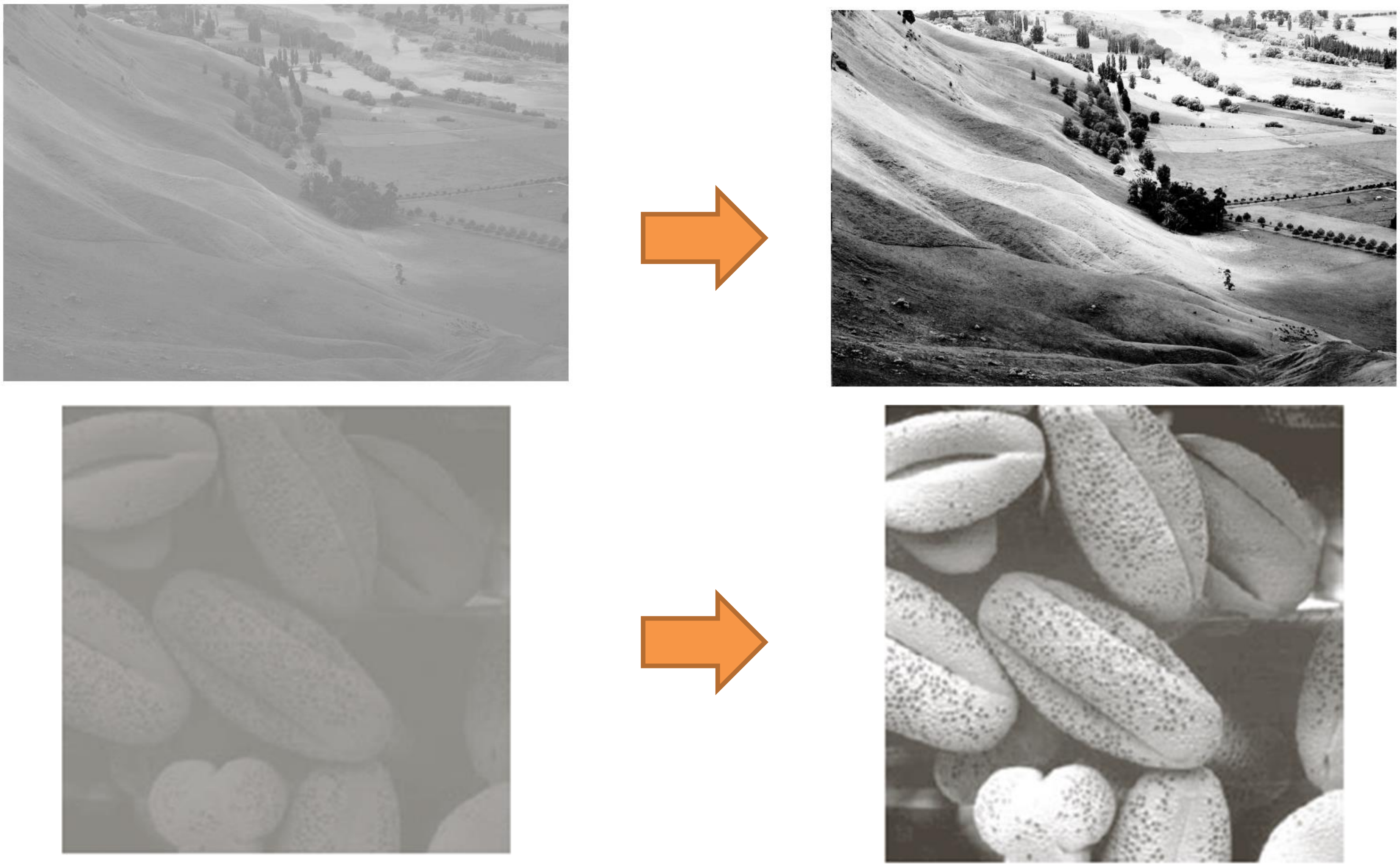
- Let us assume for the moment that the *input image* to be enhanced has continuous gray values, with $r = 0$ representing **black** and $r = L\text{-}1$ representing **white**.

- We focus attention on designing transformations (intensity mappings) of the form:

  $$s=T(r), \quad 0 \leq r \leq L-1$$

  which produce an *output intensity value* ***s***, for a *given intensity value* ***r*** in the input image.

Histogram Equalization

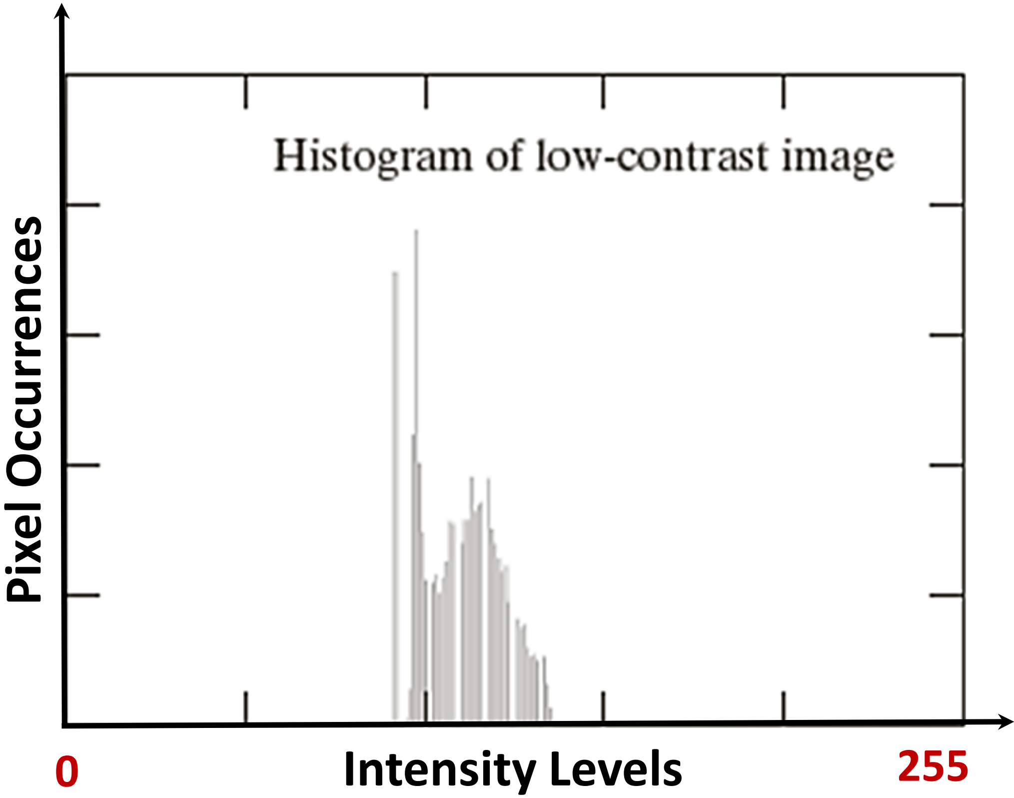


Increases contrast by spreading out the intensity histogram, but **how**?

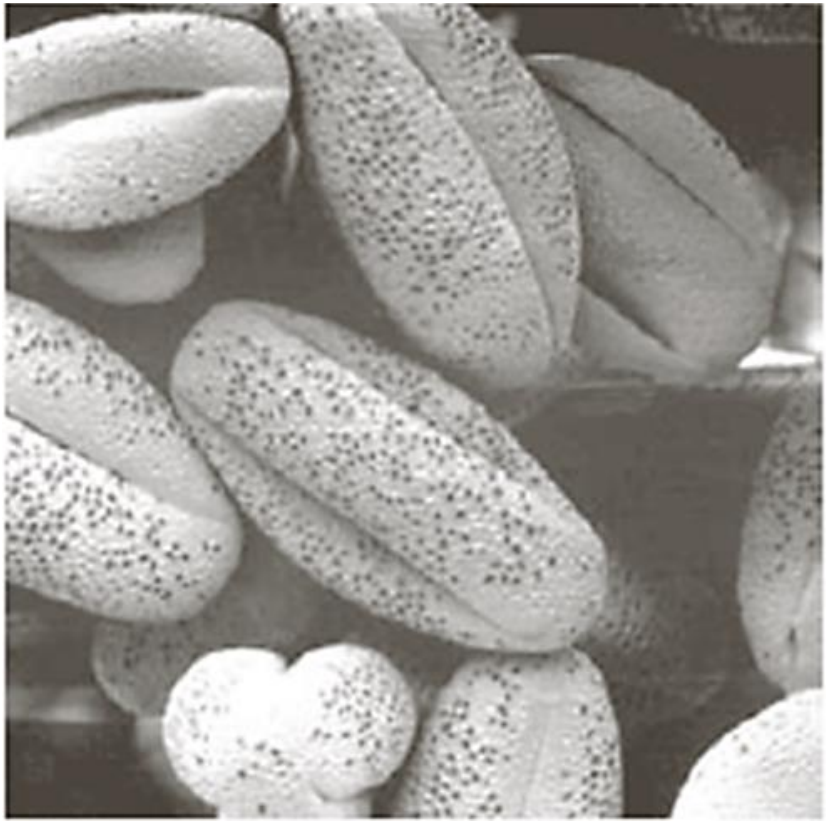
Histogram - examples

Low-contrast image

Image Histogram

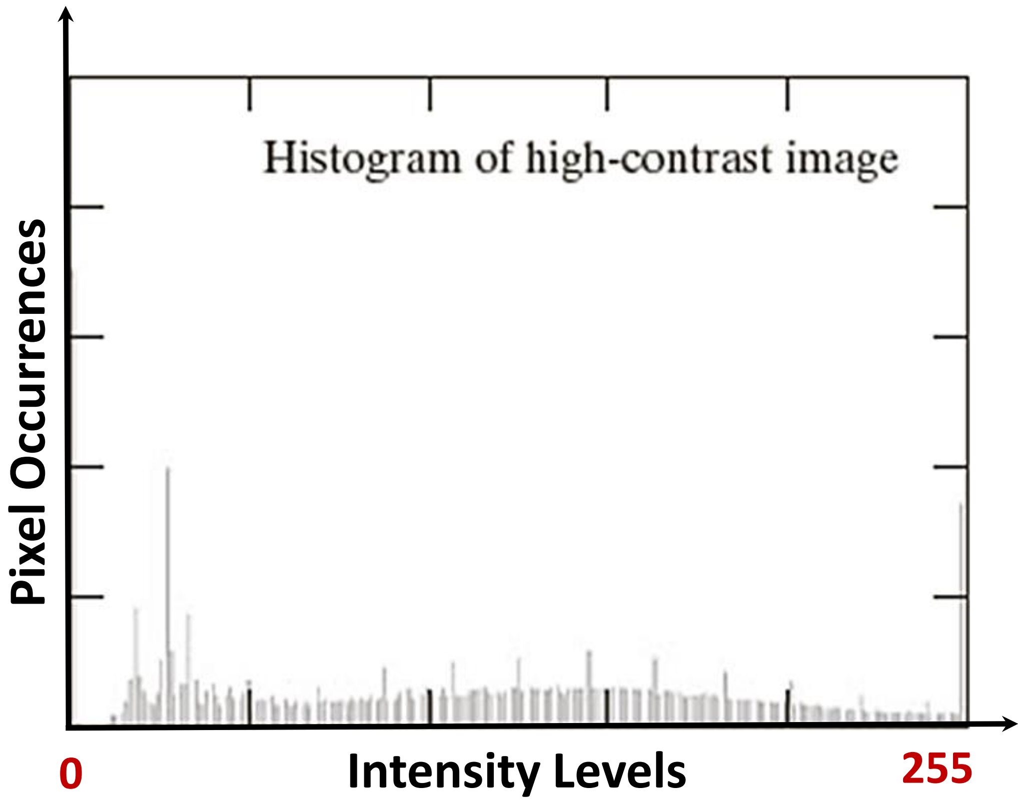


Histogram - examples



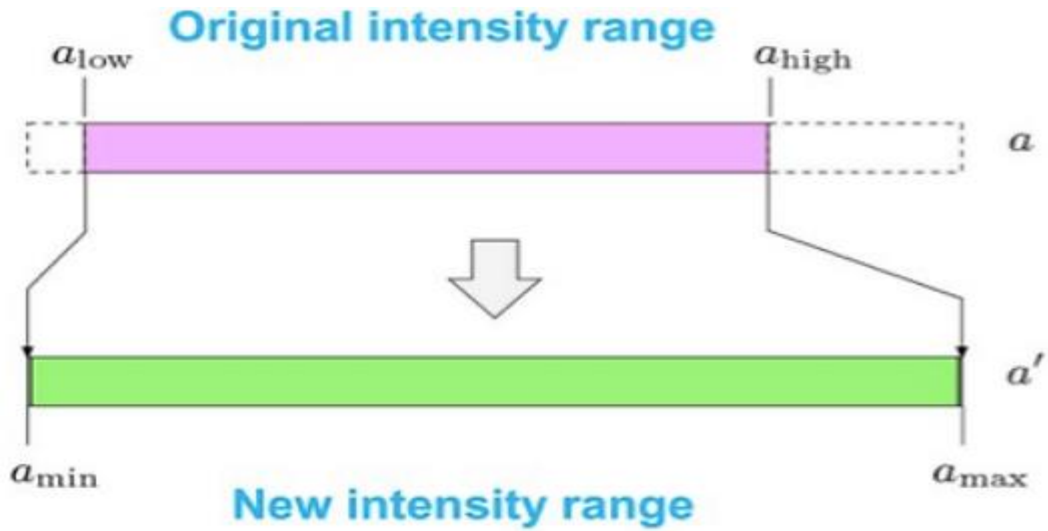
High-contrast image

Image Histogram

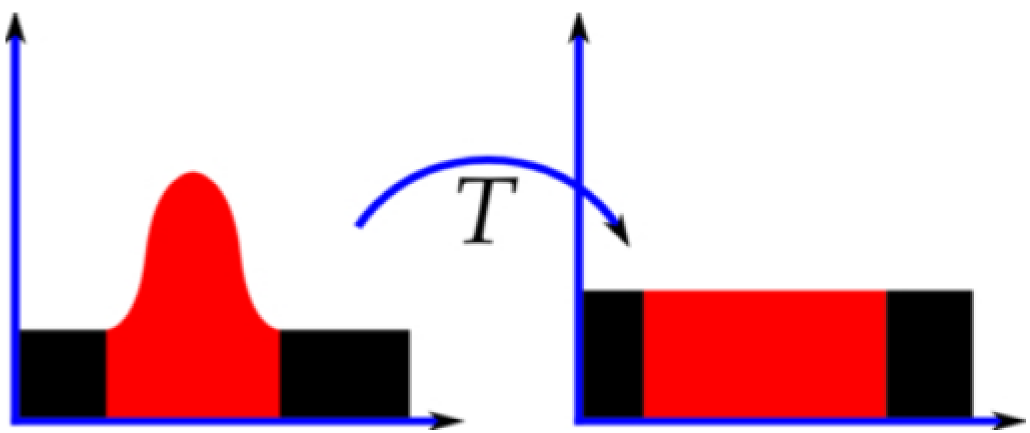


Contrast Stretching Vs. Histogram Equalization

- In **Contrast stretching**, you manipulate the entire range of intensity values. Like what you do in Normalization.



- In **Histogram equalization**, you want to flatten the histogram into a uniform distribution.



Histogram Equalization Transformation - T

- **Transformation: $s=T(r)$** such that:
  a) **T(r)** is monotonically increasing function for $0 \leq r \leq L-1$
  b) $T(r) \in [0, L-1]$ for $0 \leq r \leq L-1$

– The condition in **(a)** that $T(r)$ be monotonically increasing guarantees that *output intensity values will never be less than corresponding input values*, thus preventing artifacts created by reversals of intensity.

– Condition **(b)** guarantees that the *range of output intensities is the same as the input*.

- A function $T(r)$ is a ***monotonic increasing*** function if $T(r_2) \geq T(r_1)$ for $r_2 > r_1$.
- $T(r)$ is a ***strictly monotonic increasing*** function if $T(r_2) > T(r_1) >$ for $r_2 > r_1$.

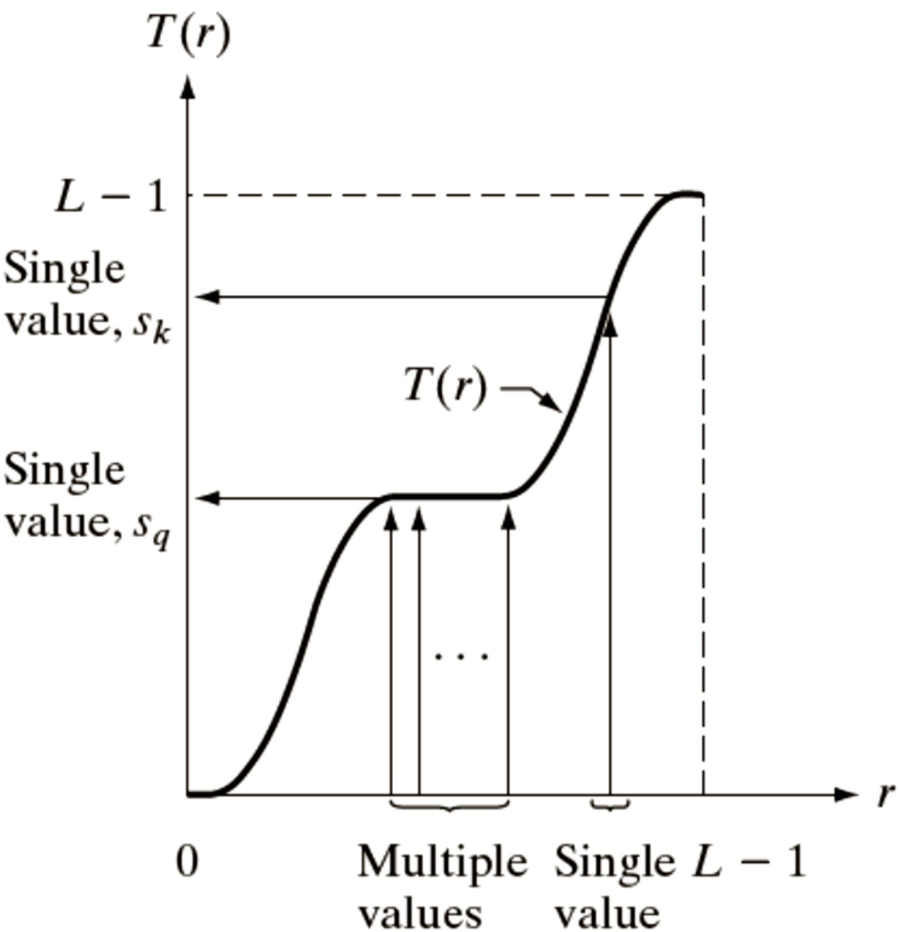
Histogram Equalization Transformation - T

- **Transformation:** $s=T(r)$

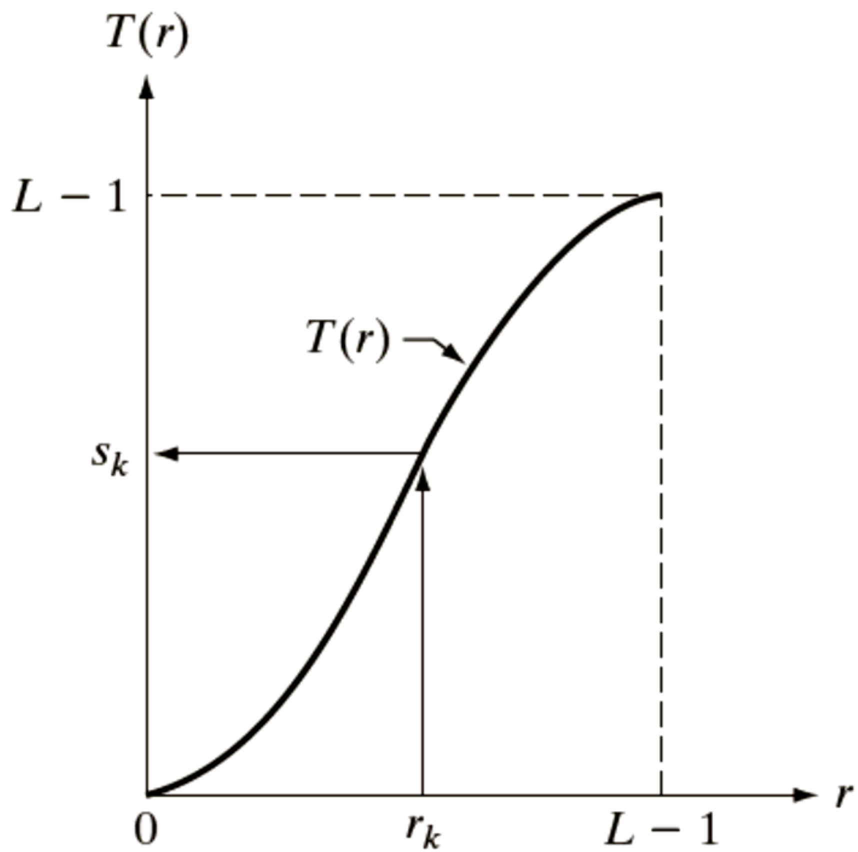
- **Inverse Transformation:** $r=T^{-1}(s)$
  - Requirement of a more strict condition to *avoid ambiguity*: $\mathbf{T(r)}$ is strictly monotonic increasing function in the interval $\mathbf{0 \leq r \leq L\text{-}1}$.
  - This strict condition guarantees that the *mappings from **s** back to **r** will be one-to-one*, thus preventing ambiguities.

- A function $T(r)$ is a ***monotonic increasing*** function if $T(r_2) \geq T(r_1)$ for $r_2 > r_1$.
- $T(r)$ is a ***strictly monotonic increasing*** function if $T(r_2) > T(r_1) >$ for $r_2 > r_1$.

Histogram Equalization



Non-strictly monotonically increasing function



Strictly monotonically increasing function

Intensity Level as a Random Variable

- The intensity of an image may be viewed as a *random variable* in the interval **[0, *L* – 1]**.

- So, in the transformation $\mathbf{s=T(r)}$**,** there are two random variables **r** and **s**.
  - Let $\mathbf{p_R(r)}$ and $\mathbf{p_S(s)}$ be the PDFs of **r** and **s** respectively.

- **Question:** if we know $\mathbf{p_R(r)}$ and $\mathbf{T(r)}$, how shall we determine $\mathbf{p_S(s)}$?

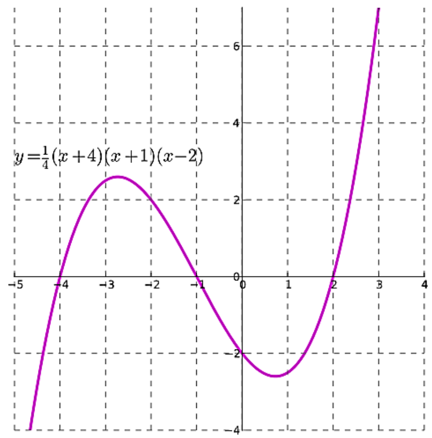
– The probability density function (PDF) is used to specify the probability of the random variable falling within a particular range of values as opposed to taking any one value.

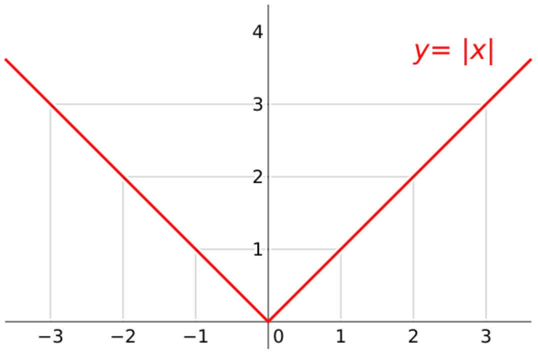
Intensity Level as a Random Variable

- From the fundamental result from probability theory, if we know $\mathbf{p_R(r)}$ and $\mathbf{T(r)}$ and if $\mathrm{T(r)}$ is continuous and differentiable over the range of values of interest, then:

$$p_S(s) = p_R(r)\left|\frac{dr}{ds}\right|$$

- Recall: differentiable/non-differentiable functions





Differentiable means that a function has a derivative. In simple terms, it means there is a slope (one that you can calculate).

Intensity Level as a Random Variable

- A transformation function of particular importance in image processing is:

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw \longrightarrow \boxed{\text{CDF of r}}$$

Where,

- w is a dummy variable of integration that can be replaced by any suitable variable name for the calculation.
- $P_r(\ )$ denote the PDF.
- L is the intensity range of L-level digital image.
- r denote the intensities of the image in the range [0, L-1]

The cumulative distribution function (CDF) of a *real-valued random variable* **X** evaluated at x, is the probability that **X** will take a value less than or equal to x.

Intensity Level as a Random Variable

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$ → CDF of r

– PDFs are ***positive***, hence T(r) is monotonically increasing (condition **a**)

– When the ***upper-limit*** in this equation is r=(L-1), the integral evaluates to 1 (i.e., PDF is 1). Thus the maximum value of **s** is ***L-1*** (condition **b**)

$$p_S(s) = p_R(r)\left|\frac{dr}{ds}\right|$$

To compute $p_s(s)$, we first compute $\left|\frac{dr}{ds}\right|$,

Intensity Level as a Random Variable

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = (L-1)\frac{d}{dr}\left[\int_o^r p_r(w)dw\right] = (L-1)p_r(r) \text{ (by Leibniz's rule)}$$

Leibniz's Rule

- The derivative of a definite integral with respective to its upper limit is the integrand evaluated at the limit.

$$\frac{d}{dx}\int_0^x f(t)dt = f(x)$$

Intensity Level as a Random Variable

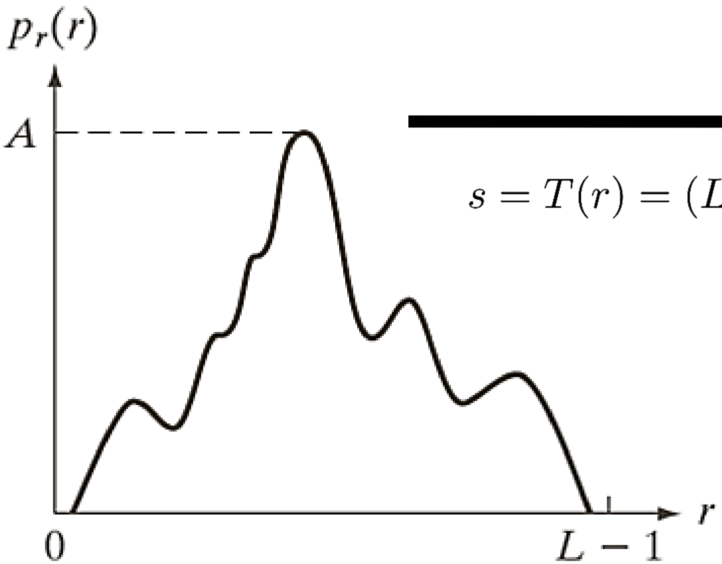
$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = (L-1)\frac{d}{dr}\left[\int_o^r p_r(w)dw\right] = (L-1)p_r(r) \text{ (by Leibniz's rule)}$$

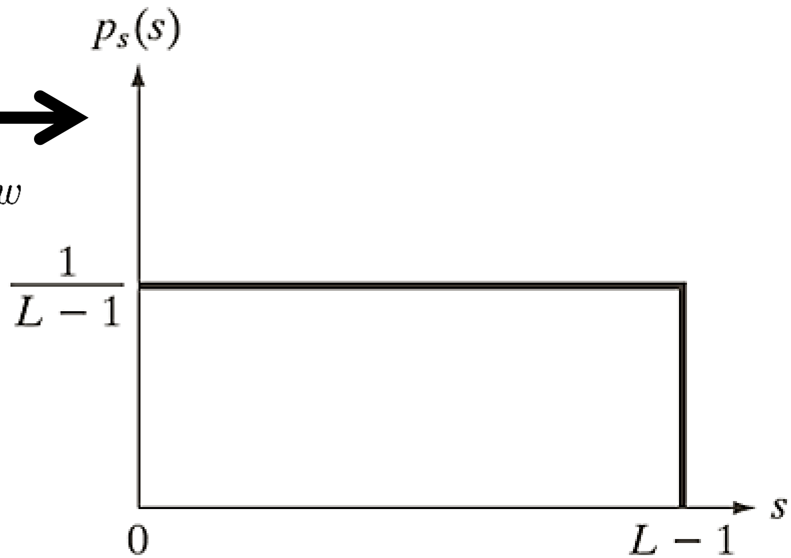
$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = p_r(r)\left|\frac{1}{(L-1)p_r(r)}\right| = \boxed{\frac{1}{L-1}}$$

- **$p_S(s)$** in the equation will <u>always be uniform</u>, *independently* of the form of **$p_r(r)$**.

Intensity Level as a Random Variable



$$s = T(r) = (L-1)\int_0^r P_R(w)dw$$



Arbitrary PDF

Resulting PDF

Intensity Level as a Random Variable - example

- Suppose that the (continuous) intensity values in an image have the PDF:

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2} & \text{for } 0 \le r \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

Find **s** and $\mathbf{p_S(s)}$ **?**

Intensity Level as a Random Variable - example

Finding **s**:

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2} & \text{for } 0 \le r \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{2}{L-1}\int_0^r r\,dr$$

$$= \frac{2}{L-1}\int_0^r \frac{r^2}{2}$$

$$\boldsymbol{S = \frac{r^2}{L-1}}$$

Intensity Level as a Random Variable - example

Finding $p_S(s)$:

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2} & \text{for } 0 \le r \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

$$s = \frac{r^2}{L-1}$$

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| = \frac{2r}{(L-1)^2}\left|\left[\frac{ds}{dr}\right]^{-1}\right|$$

$$= \frac{2r}{(L-1)^2}\left|\left[\frac{d}{dr}\frac{r^2}{L-1}\right]^{-1}\right| = \frac{2r}{(L-1)^2}\left|\frac{(L-1)}{2r}\right| = \frac{1}{L-1}$$

Histogram Equalization

- **Recall** that the probability of occurrence of intensity level r_k in a digital image is approximated by:

$$p_r(r_k) = \frac{n_k}{MN}$$

  - where ***MN*** is the total number of pixels in the image, and n_k denotes the number of pixels that have intensity r_k .
  - $p_r(r_k)$ with $r_k \in [0, L-1]$, is commonly referred to as a normalized image histogram.

Histogram Equalization

- The discrete form of the transformation

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

is given by:

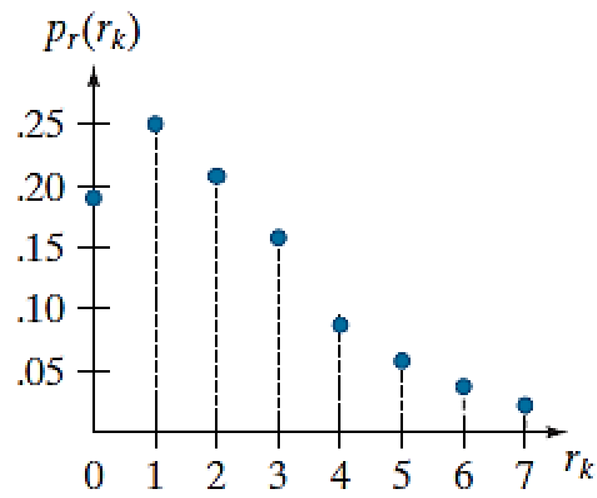
$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) \qquad k = 0,1,2,\ldots,L-1$$

Thus, a processed **output image** is obtained by mapping each pixel in the **input image** with intensity r_k into a corresponding pixel with level s_k in the output image, This is called a ***histogram equalization*** or ***histogram linearization*** transformation.

Histogram Equalization Illustration - Example

- Suppose that a *3-bit image* (***L* = 8**) of size *64 × 64* pixels (***MN* = 4096**) has the following intensity distribution where the intensity levels are integers in the range [**0, *L* – 1**] = [**0, 7**].

r_k	n_k	$p_r(r_k) = n_k / MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02



Histogram Equalization Illustration - Example

- Values of the histogram equalization transformation function are obtained using:

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) \quad k = 0,1,2,\dots,L-1$$

- So the values of the **equalized histogram** are:

$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7p_r(r_0) = 1.33 \approx \mathbf{1}$

$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_1) = 3.08 \approx \mathbf{3}$

$s_2 = T(r_2) = 7\sum_{j=0}^{2} p_r(r_j) = 7p_r(r_2) = 4.55 \approx \mathbf{5}$

$s_3 = T(r_3) = 7\sum_{j=0}^{3} p_r(r_j) = 7p_r(r_3) = 5.67 \approx \mathbf{6}$

$s_4 = T(r_4) = 7\sum_{j=0}^{4} p_r(r_j) = 7p_r(r_4) = 6.23 \approx \mathbf{6}$

$s_5 = T(r_5) = 7\sum_{j=0}^{5} p_r(r_j) = 7p_r(r_5) = 6.65 \approx \mathbf{7}$

$s_6 = T(r_6) = 7\sum_{j=0}^{6} p_r(r_j) = 7p_r(r_6) = 6.86 \approx \mathbf{7}$

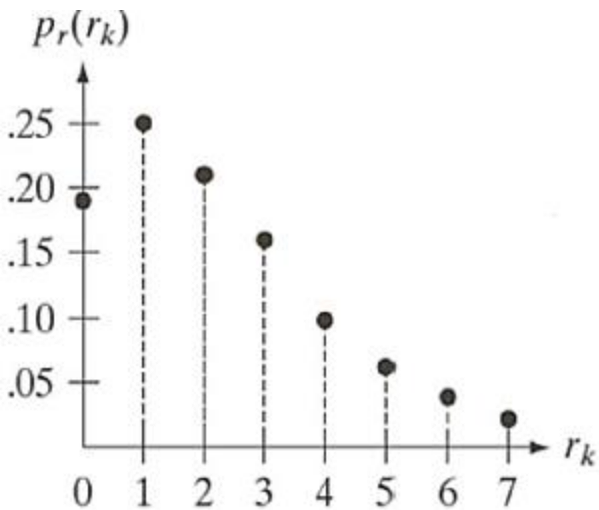
$s_7 = T(r_7) = 7\sum_{j=0}^{7} p_r(r_j) = 7p_r(r_7) = 7.00 \approx \mathbf{7}$

Histogram Equalization Illustration - Example

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

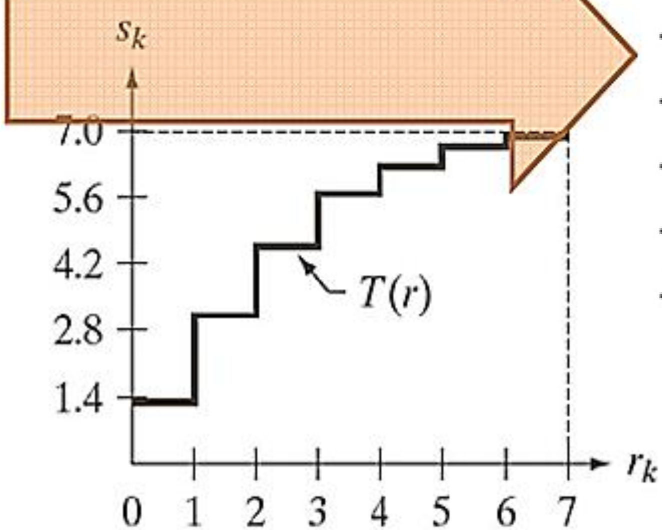
r_k	->	s_k
0	->	1
1	->	3
2	->	5
3	->	6
4	->	6
5	->	7
6	->	7
7	->	7

s	n_s	$p_s(s)$
s_0=0	0	0.00
s_1=1	790	0.19
s_2=2	0	0.00
s_3=3	1023	0.25
s_4=4	0	0.00
s_5=5	850	0.21
s_6=6	656+329	0.24
s_7=7	245+122+81	0.1093

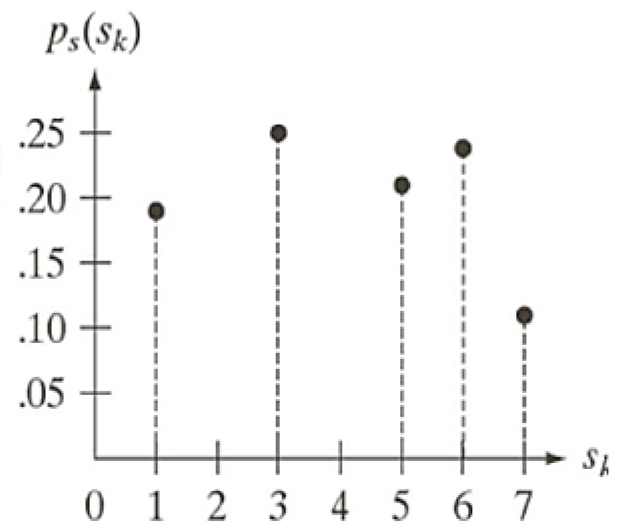


Original histogram

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} P_R(r_j)$$

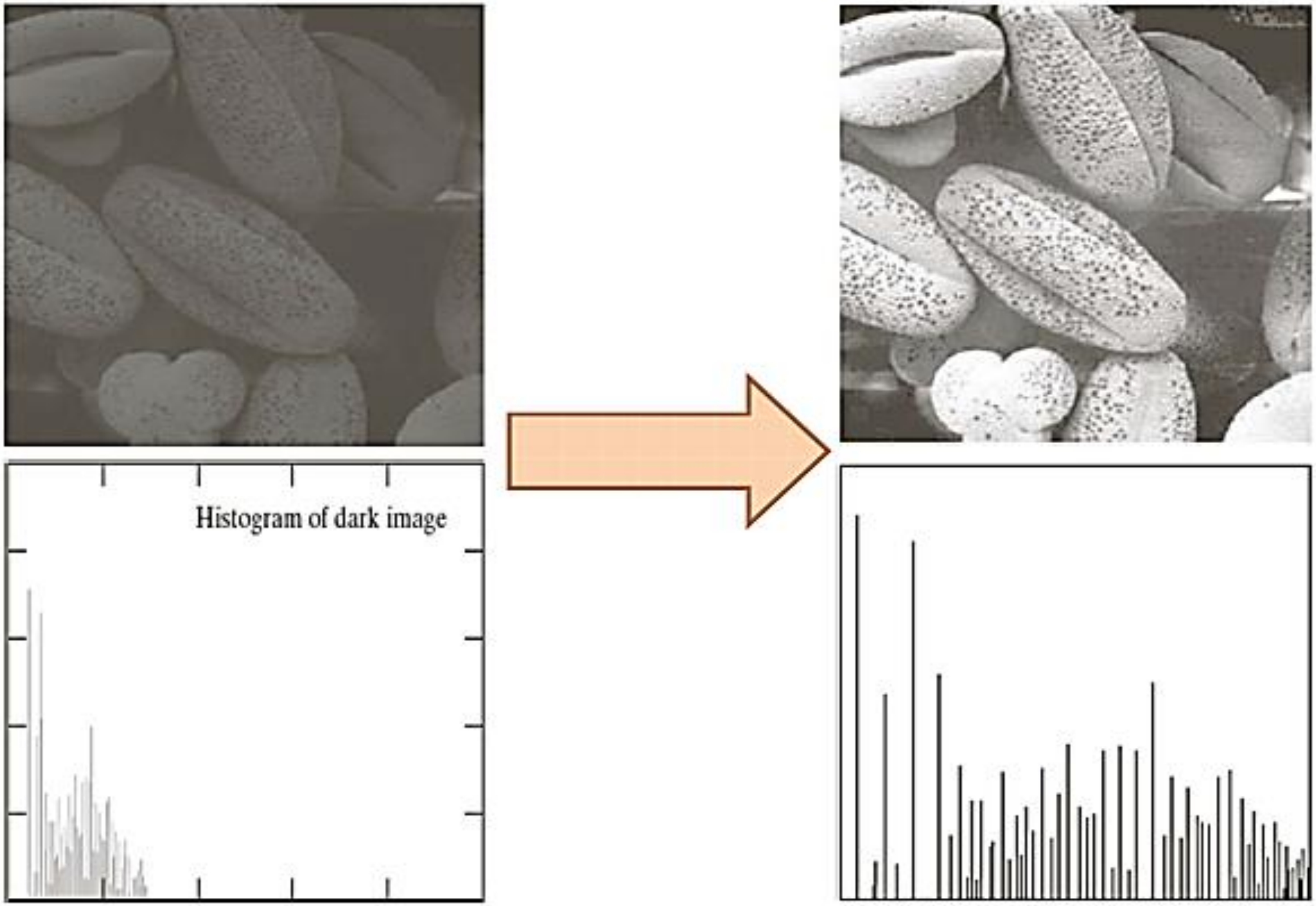


Contrast Enhancement!

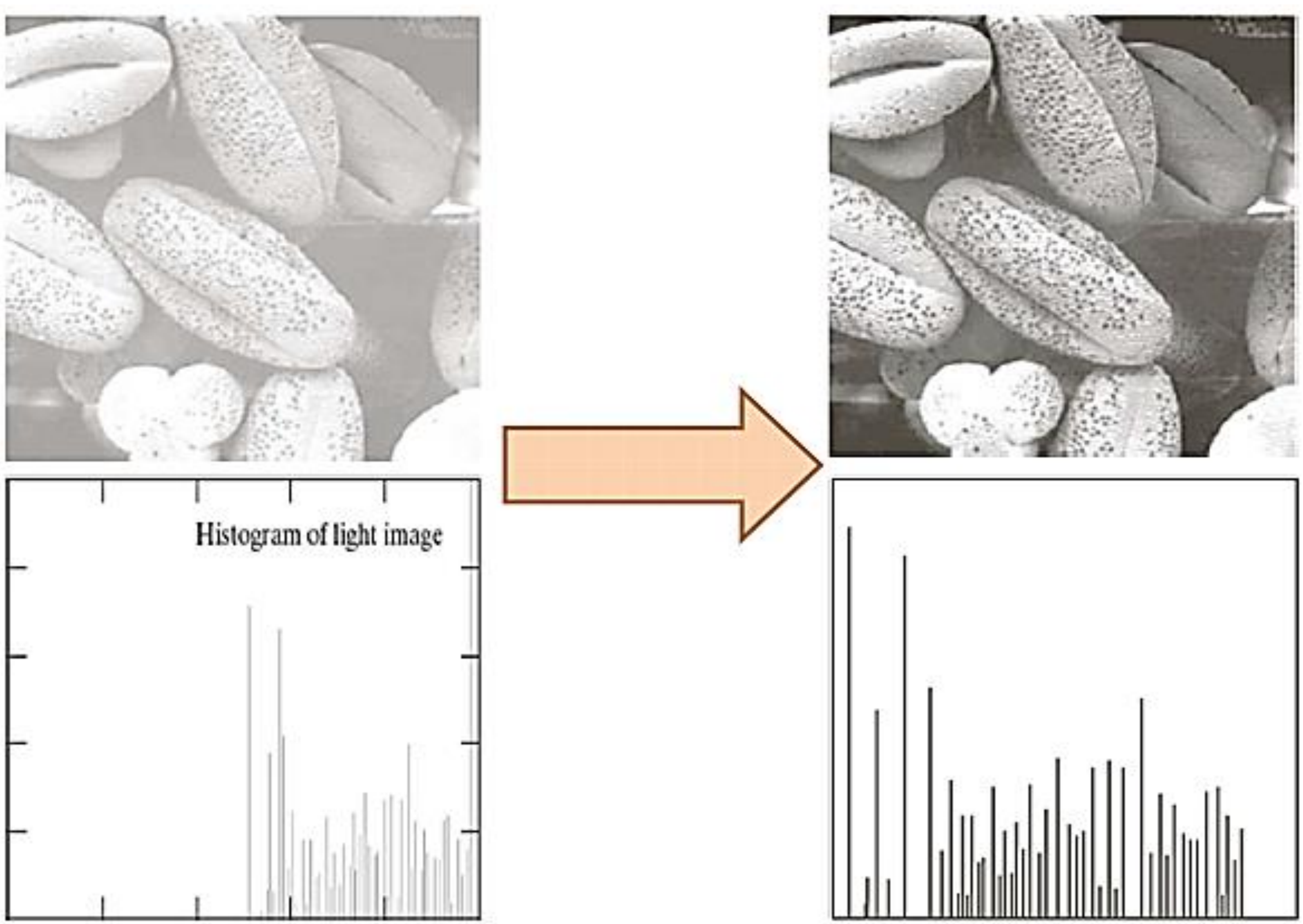


Equalized histogram

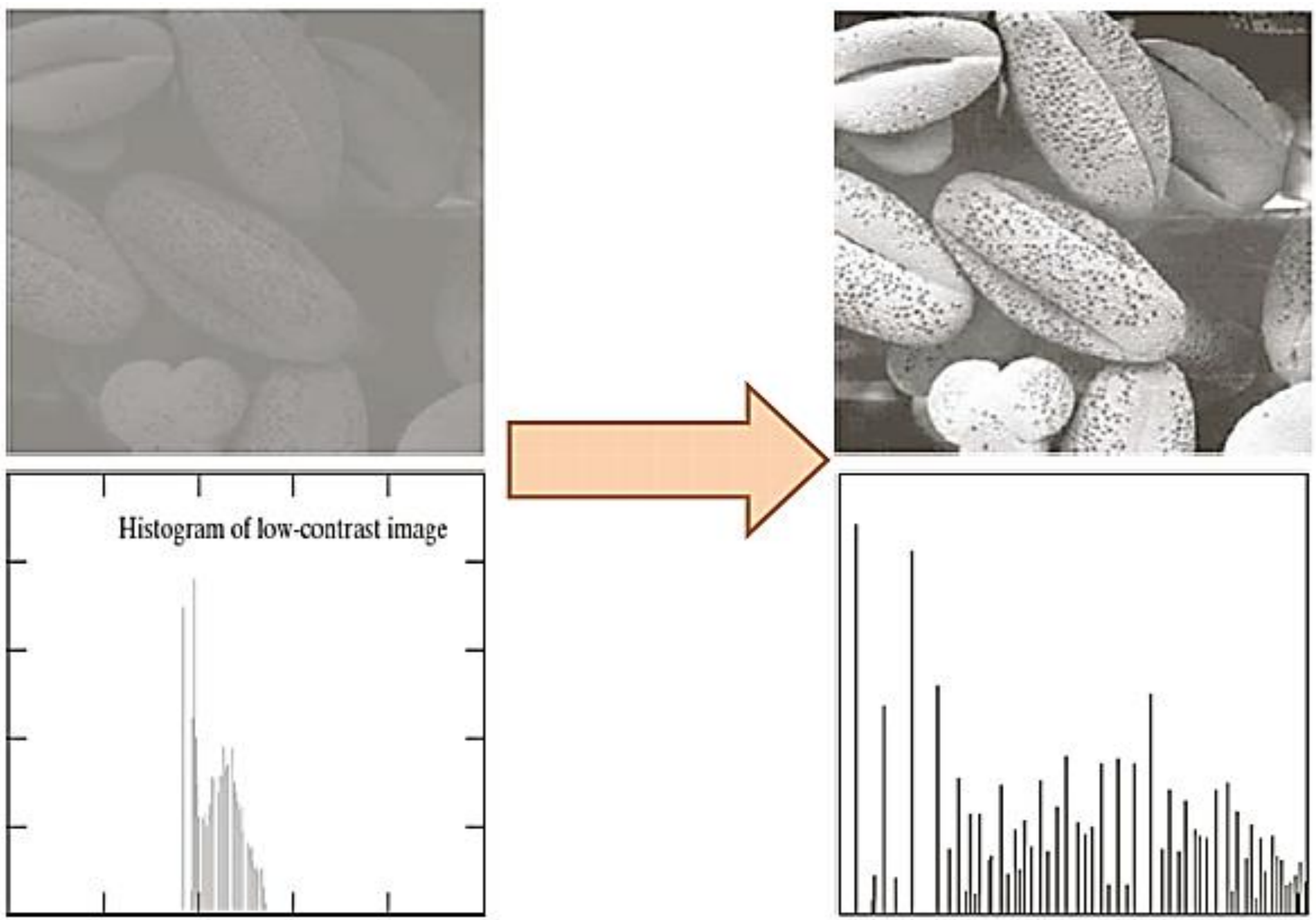
Result of Histogram Equalization



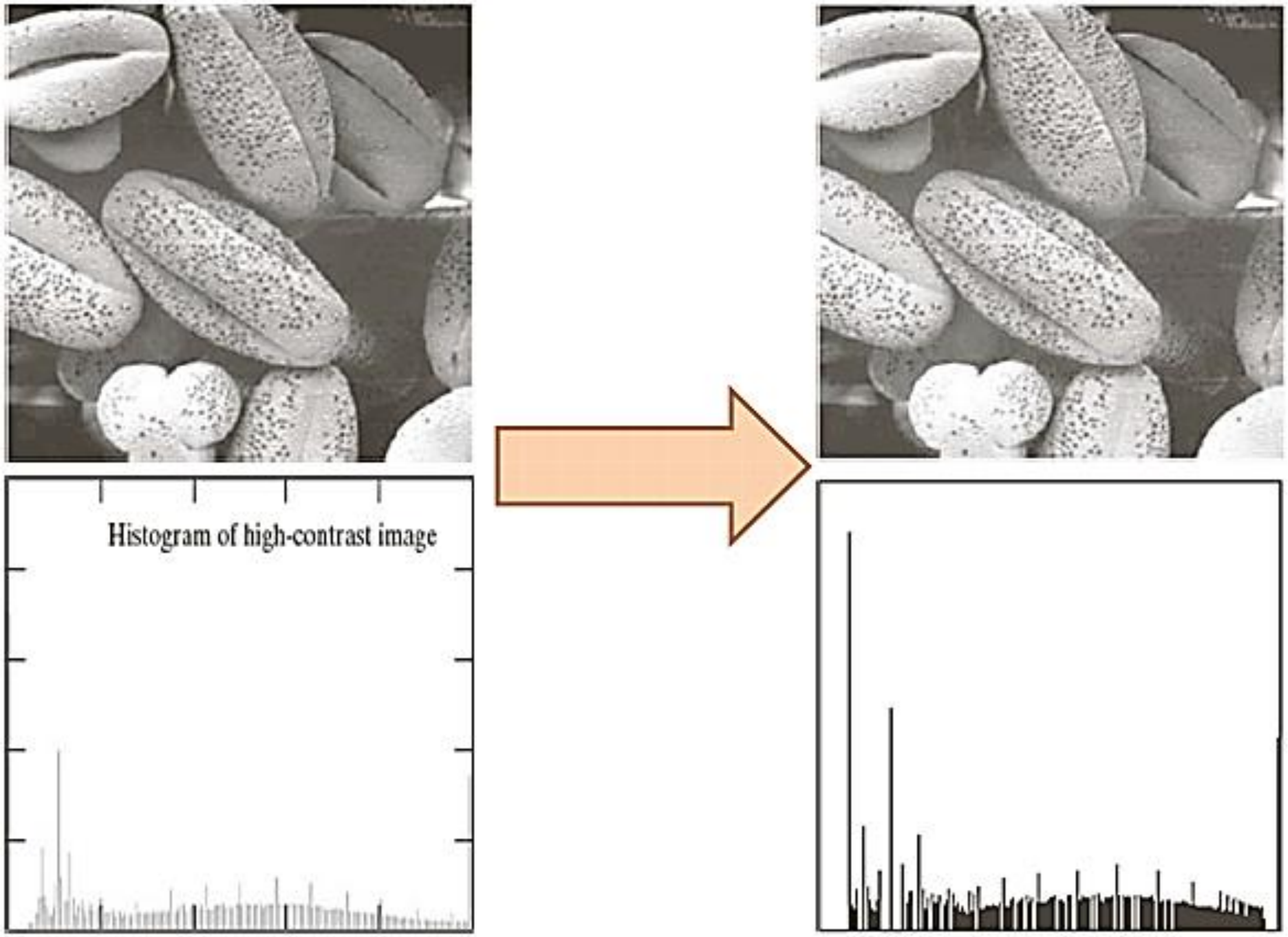
Result of Histogram Equalization



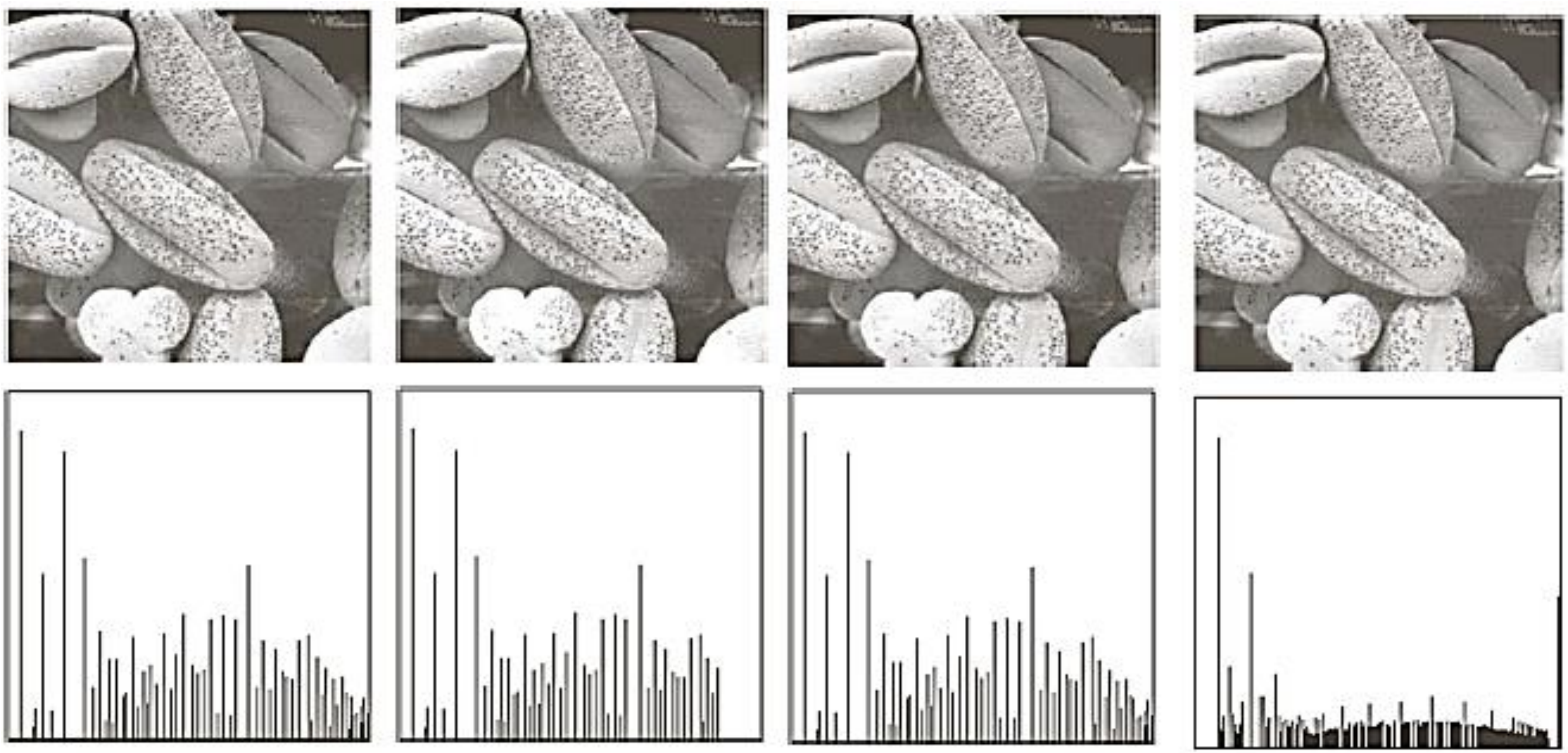
Result of Histogram Equalization



Result of Histogram Equalization



Result of Histogram Equalization



Dark Image | Bright Image | Low Contrast Image | High Contrast Image

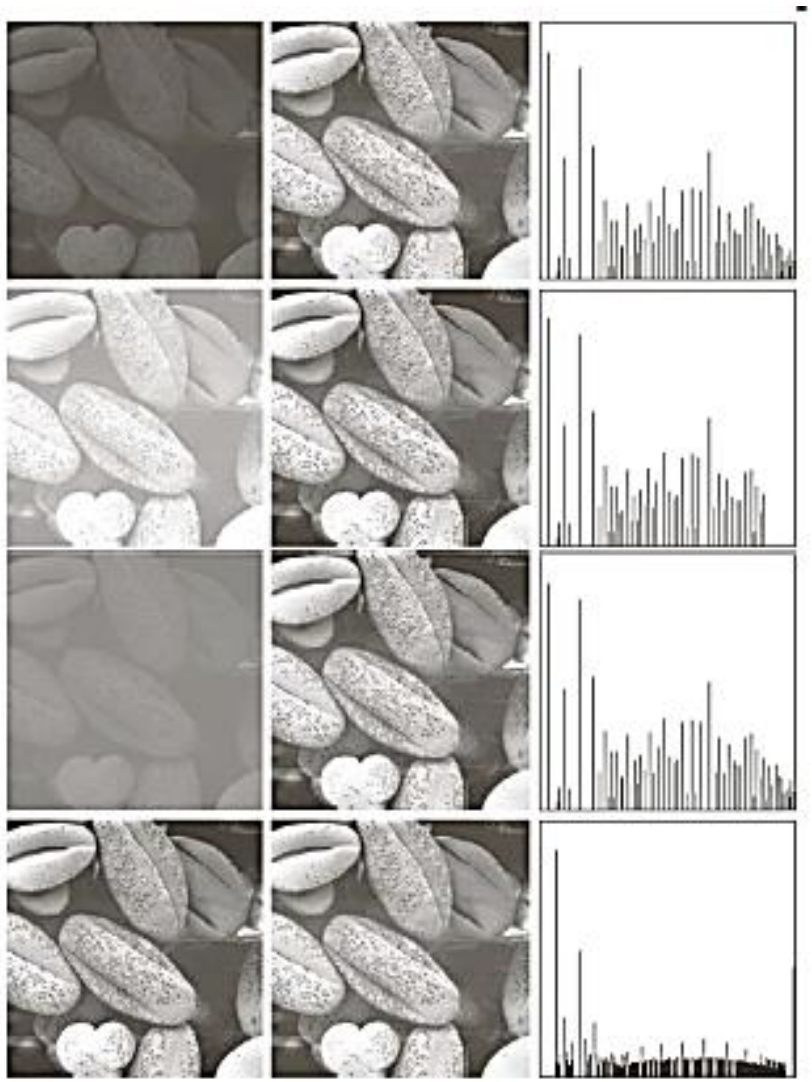
Result of Histogram Equalization

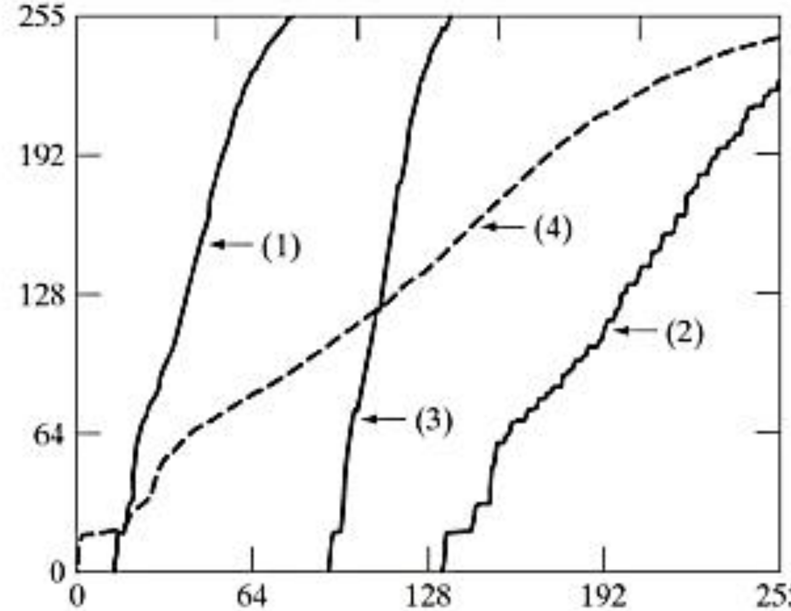
(1)

(2)

(3)

(4)





Properties of Equalized Histogram

- Intensities are integers -> rounding off operation
- Resulting histogram is an approximation to the continuous case
- No new values of intensities are created

  $0 \leq \mathbf{r} \leq L\text{-}1$ and $0 \leq \mathbf{s} \leq L\text{-}1$
- Net result: expansion/spreading out of the intensity values
  – Contrast enhancement!

Next Lecture

- Histogram matching / Histogram specification
- Local histogram processing