CS100 #10

IEEE754

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So what is the Problem?

Given the following binary representation:

$37.25_{10} = 100101.01_2$

$7.625_{10} = 111.101_2$

$0.3125_{10} = 0.0101_2$

How we can represent the whole and fraction part of the binary rep. in 4 bytes?

Solution is Normalization

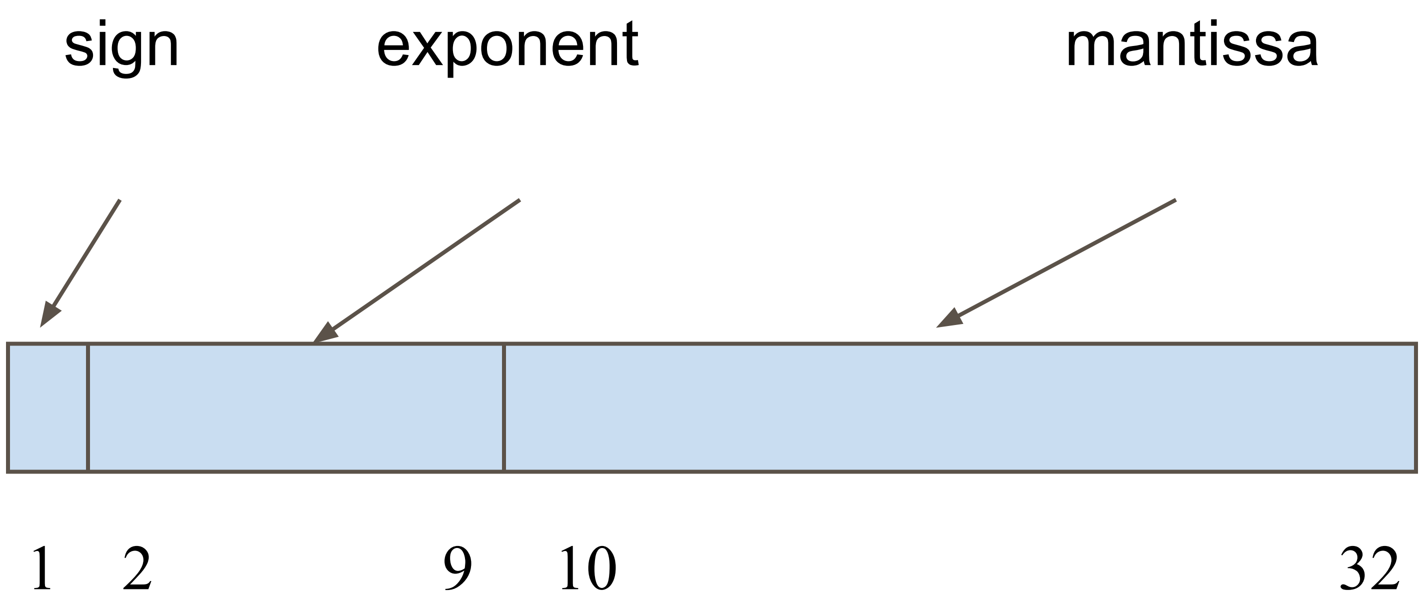
- Every binary number, **except the one corresponding to the number zero**, can be normalized by choosing the exponent so that the radix point falls to the right of the leftmost 1 bit.
- Ex:
  - $37.25_{10} = 100101.01_2 = 1.0010101 \times 2^5$
  - $7.625_{10} = 111.101_2 = 1.11101 \times 2^2$
  - $0.3125_{10} = 0.0101_2 = 1.01 \times 2^{-2}$
- After normalizing, the numbers now have different mantissas and exponents.

IEEE Standard 754

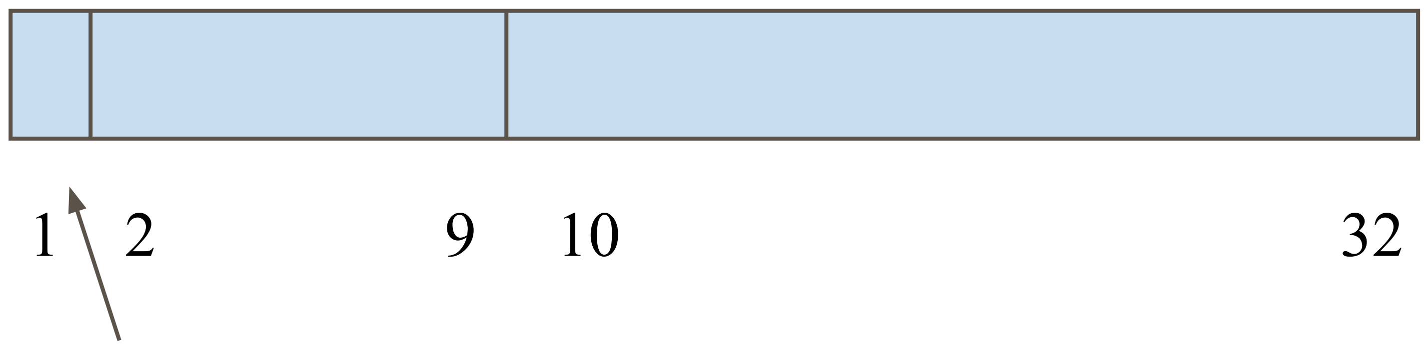
- Established in 1985 as uniform standard for floating point arithmetic
- Supported by all major CPUs
- Variations
  - Single precision: 8 exp bits, 23 frac bits
    - 32 bits total
  - Double precision: 11 exp bits, 52 frac bits
    - 64 bits total

IEEE Floating Point Representation

- Floating point numbers can be represented by binary codes by dividing them into three parts:

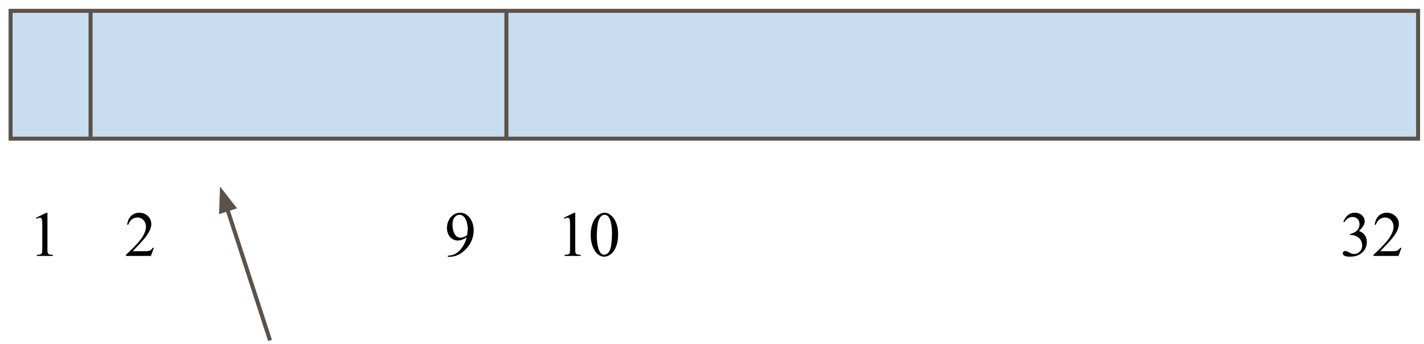


IEEE Floating Point Representation



- The first, or leftmost, field of our floating point representation will be the sign bit:
  - 0 for a positive number,
  - 1 for a negative number.

IEEE Floating Point Representation

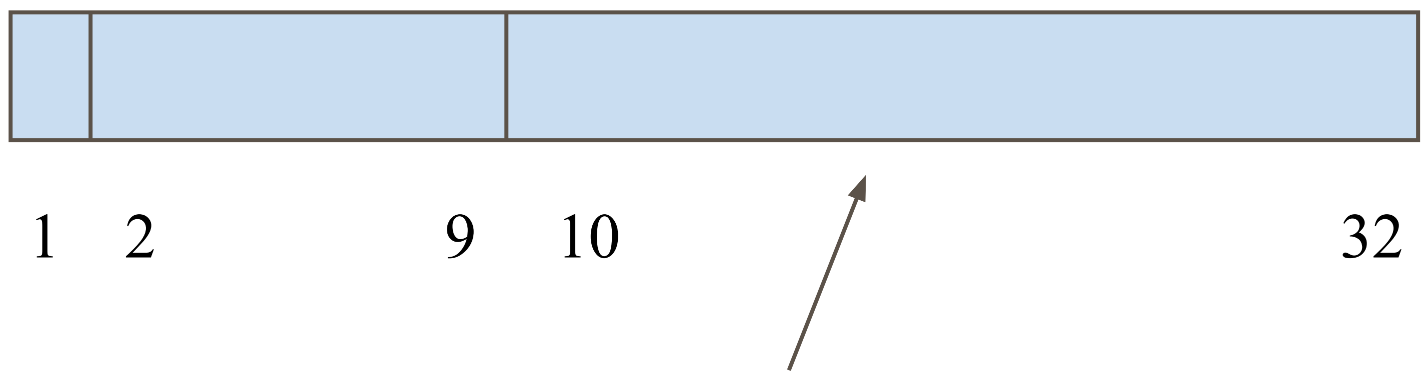


- Exponent
- Since we must be able to represent both positive and negative exponents, we will use a convention which uses a value known as a **bias of 127 (**or **excess 127)** to determine the representation of the exponent.
    - $127 = 2^{8-1}-1$

IEEE Floating Point Representation

- An exponent of 5 is stored as 127 + 5 or 132
- An exponent of -5 is stored as 127 + (-5) OR 122
- The biased exponent, the value actually stored, will range from 0 through 255. This is the range of values that can be represented by 8-bit, unsigned binary numbers.

IEEE Floating Point Representation



- Mantissa in 23 bit field
- The mantissa is the set of 0's and 1's to the left of the radix point of the **normalized** binary number.
  - Ex: if 1.00101 X 2^3 then mantissa is 00101
  - 1 (as in 1.01*21) is always there, so we don't need to waste one of our precious bits on it. Just assume it is always there.

Zero

- The number zero is represented specially:
  - sign = 0 for positive zero, 1 for negative zero.
  - biased exponent = 0.
  - fraction = 0.

Positive and negative infinity

- Positive and negative infinity are represented thus:
  - sign = 0 for positive infinity, 1 for negative infinity.
  - biased exponent = all 1 bits.
  - fraction = all 0 bits.

NaN

- Some operations of floating-point arithmetic are invalid, such as taking the square root of a negative number.
- The act of reaching an invalid result is called a floating-point exception. An exceptional result is represented by a special code called a NaN, for "Not a Number".
- All NaNs in IEEE 754 have this format:
  - sign = either 0 or 1.
  - biased exponent = all 1 bits.
  - fraction = anything except all 0 bits (since all 0 bits represents infinity).

Double Precision

- Double precision uses more space, allows greater magnitude and greater precision
- Other than that, it behaves just like single precision.
- We will use only single precision in following examples, but any could easily be expanded to double precision.

DEC to IEEE: 40.15625

Step 1.

Compute the binary equivalent of the whole part and the fractional part

$40 = 1*32+0*16+1*8+0*4+0*2+0*1$

$\rightarrow 101000$

$.15625 = 0*0.5+0*0.25+1*0.125+0*0.0625+1*0.03125$

$\rightarrow .00101$

DEC to IEEE: 40.15625

Step 2.

Normalize the number by moving the decimal point to the right of the leftmost one.

$$101000.00101 = 1.0100000101 \times 2^5$$

Step 3.

Convert the exponent to a biased exponent

$$127 + 5 = 132 \quad \Rightarrow \quad 132_{10} = 10000100_2$$

Step 4.

Store the results from above

Sign	Exponent	Mantissa
0	10000100	01000001010000000000000

DEC to IEEE: –24.75

Step 1. Convert

$24_{10} = 11000_2 \quad .75_{10} = .11_2$

So: $-24.75_{10} = -11000.11_2$

Step 2. Normalize

$-11000.11 = -1.100011 \times 2^4$

Step 3. Convert the exponent to a biased exponent

$127 + 4 = 131 \quad \Rightarrow \quad 131_{10} = 10000011_2$

Step 4. Store the results from above

Sign Exponent Mantissa

1 10000011 10001100000000000000000

IEEE to DEC

- Do the steps in reverse order
- In reversing the normalization step move the radix point the number of digits equal to the exponent. if exponent is positive move to the right, if negative move to the left.

IEEE to DEC:
1 01111101 01000000000000000000000

Step 1

Extract exponent (unbias exponent)

biased exponent = $01111101_2 = 125_{10}$

exponent: 125 - 127 = -2

Step 2

Write normalized number

-1.01×2^{-2}

IEEE to DEC:
1 01111101 01000000000000000000000

Step 3:

Write the binary number (denormalize value from step 2)

-0.0101_2

Step 4:

Convert binary number to floating-point equivalent

$-0.0101_2 = -(0.25 + 0.0625) = -0.3125$

IEEE to DEC:
0 10000011 11010100000000000000000

Step 1

Extract exponent (unbias exponent)

biased exponent = 10000011 = 131

exponent: 131 - 127 = 4

Step 2

Write Normalized number

1.110101×2^{4}

IEEE to DEC:
0 10000011 11010100000000000000000

Step 3:

Write the binary number (denormalize value from step 2)

$$11101.01_2$$

Step 4:

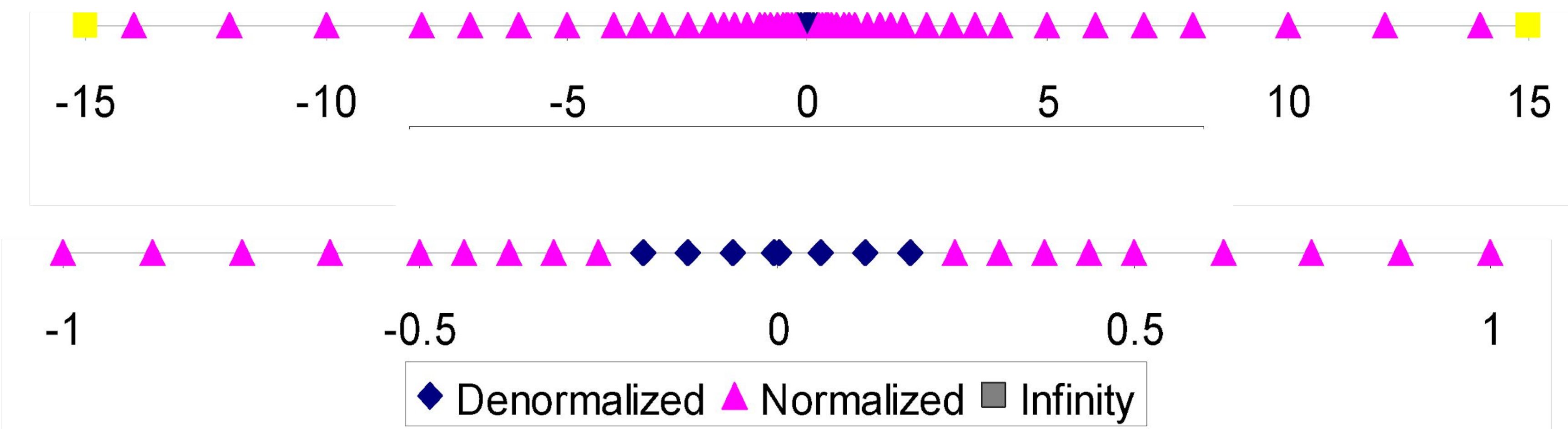
Convert binary number to FP equivalent (add column values)

$$11101.01_2 = 16 + 8 + 4 + 1 + 0.25 = 29.25_{10}$$

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

- Notice how the distribution gets denser toward zero.



Denormalized Values

- Normalized means represented in the normal, or standard, notation. Some numbers do not fit into that scheme and have a separate definition.
- Consider the smallest normalized value: $1.000...000 \times 2^{-126}$
    - How would we represent half of that number?
    - $1.000...000 \times 2^{-127}$ (But we cannot fit 127 into the exponent field, bcs result is all 0, so represents 0)
    - $0.100...000 \times 2^{-126}$ (But we are stuck with that implied 1 before the implied point)
- So, there are a lot of potentially useful values that don't fit into the scheme. The solution: special rules when the exponent has value 0 (which represents -126).

Rounding

- The general rule when rounding binary fractions to the n-th place prescribes to check the digit following the n-th place in the number.
- If it’s 0, then the number should always be rounded down.
- If, instead, the digit is 1 and any of the following digits are also 1, then the number should be rounded up.
- If, however, all of the following digits are 0’s, then a tie breaking rule must be applied and usually it’s the ‘ties to even’. This rule says that we should round to the number that has 0 at the n-th place.

Rounding

- Let's round some numbers to 2 places
  - 0.11001 — rounds down to 0.11, because the digit at the 3-rd place is 0
  - 0.11101 — rounds up to 1.00, because the digit at the 3-rd place is 1 and there are following digits of 1 (5-th place)
  - 0.11100—apply the 'ties to even' tie breaker rule and round up because the digit at 3-rd place is 1 and the following digits are all 0's.

Summary

Item	Single precision	Double precision
Bits in sign	1	1
Bits in exponent	8	11
Bits in fraction	23	52
Bits total	32	64
Exponent system	Excess 127	Excess 1023
Exponent Range	-126 to +127	-1022 to 1023
Smallest normalized number	2^{-126}	2^{-1022}
Largest normalized number	approx. 2^{128}	approx. 2^{1024}
Decimal range	approx. 10^{-38} to 10^{38}	approx. 10^{-308} to 10^{308}
Smallest denormalized number	approx. 10^{-45}	approx. 10^{-324}

Floating Point Puzzles

- Given

```
int x = …;
float f = …;
double d = …;
```

- Assume neither d nor f is NaN

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

Floating Point Puzzles

```
 1.  x == (int)(float) x
 2.  x == (int)(double) x
 3.  f == (float)(double) f
 4.  d == (float) d
 5.  f == -(-f);
 6.  2/3 == 2/3.0
 7.  d < 0.0 ⇒ ((d*2) < 0.0)
 8.  d > f ⇒ -f > -d
 9.  d * d >= 0.0
10.  (d+f)-d == f
```

Floating Point Puzzles

1. `x == (int)(float) x` **No: 24 bit significand**
2. `x == (int)(double) x` **Yes: 53 bit significand**

```
unsigned int n = 0xffffffff;
printf("%u\n", n);
printf("%f\n", (float)n);
printf("%f\n", (double)n);
```

Output:

4294967295

4294967296.000000

4294967295.000000

Answers To Floating Point Puzzles

3. `f == (float)(double) f`	**Yes: increases precision**
4. `d == (float) d`	**No: loses precision**
5. `f == -(-f);`	**Yes: Just change sign bit**
6. `2/3 == 2/3.0`	**No: 2/3 == 0**
7. `d < 0.0` ⇒ `((d*2) < 0.0)`	**Yes!**
8. `d > f` ⇒ `-f > -d`	**Yes!**
9. `d * d >= 0.0`	**Yes!**
10. `(d+f)-d == f`	**No: Not associative**

Comparison

```c
#include <stdio.h>

void main() {
   float a = 1.345f, b = 1.123f, c = a + b;

   if (c == 2.468f)
      printf("They are equal.\n");
   else
      printf("They are not equal!");
}
```

Comparison

```
#include <stdio.h>
#define EPSILON 0.0001  // Define tolerance
int equal(float x, float v) {
   return (((v - EPSILON) < x) &&
               (x < ( v + EPSILON))); }
void main() {
   float a = 1.345f, b = 1.123f, c = a + b;
   if (equal(c, 2.468f))
      printf("They are equal.\n");
   else
      printf("They are not equal!");
}
```