DigiPen INSTITUTE OF TECHNOLOGY

# CS230
# Game Implementation Techniques

Lecture 8

# Overview

- 2D Transformations
- Homogeneous Coordinates and Matrix Representation
- Composition of 2D Transformations

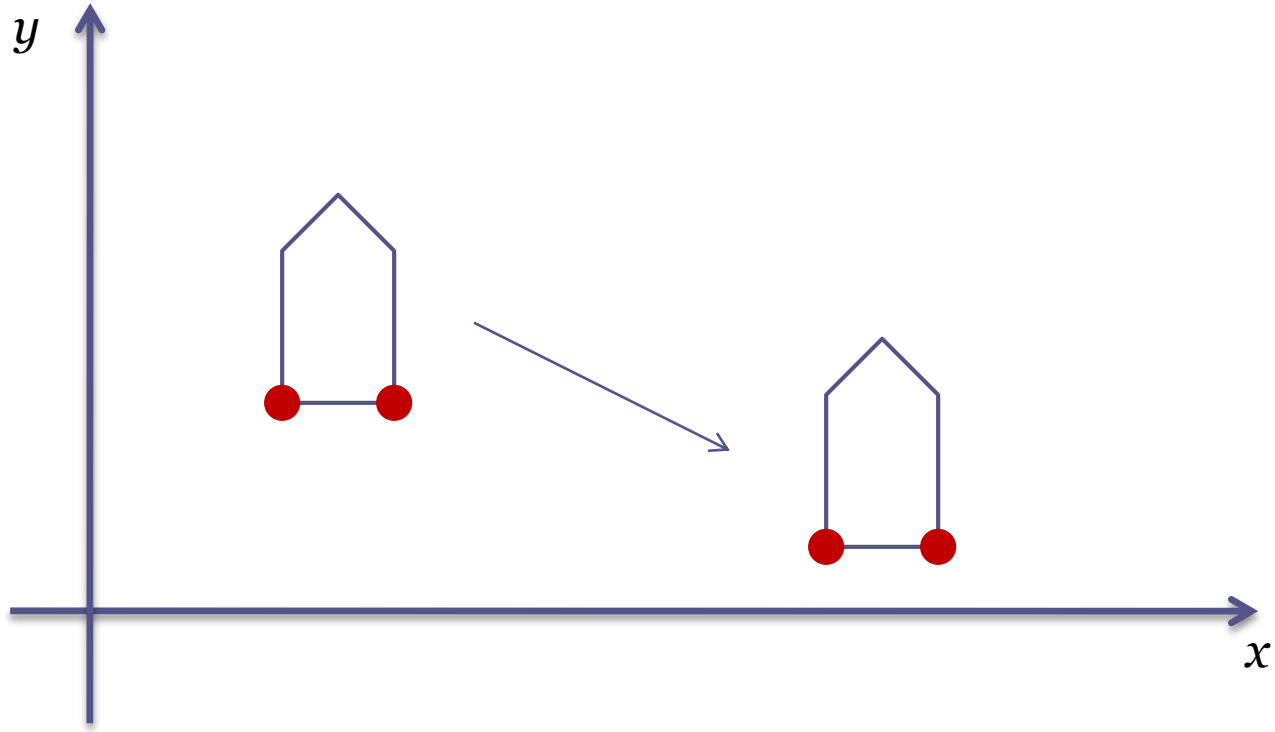
# Translation (1/2)

- For each $P(x, y)$ to be moved by $d_x$, $d_y$ units parallel to the x and y axis we get a new point $P'(x', y')$

$$x' = x + d_x \qquad y' = y + d_y$$

$$P = \begin{bmatrix} x \\ y \end{bmatrix},\ P' = \begin{bmatrix} x' \\ y' \end{bmatrix},\ T = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \quad \Rightarrow \quad P' = P + T$$

# Translation (2/2)



# Scaling (1/2)

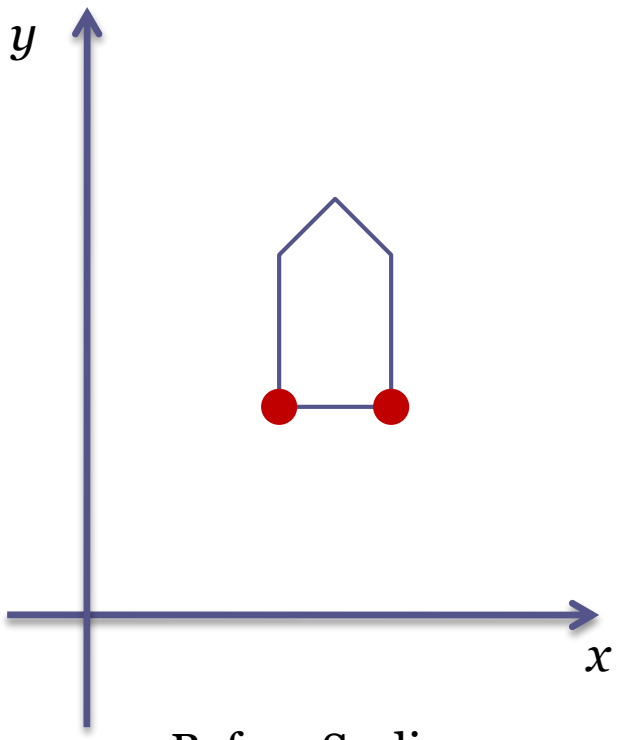
- Points can be stretched by $S_x$ and $S_y$ along the *x-axis* and *y-axis* respectively into new points by the multiplications

$$x' = S_x \cdot x \qquad y' = S_y \cdot y$$

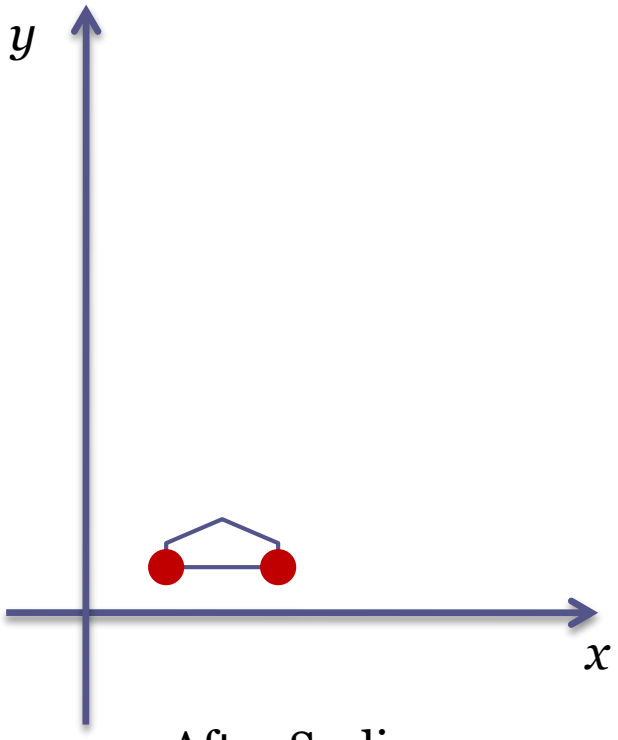
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad P' = S \cdot P$$

# Scaling (2/2)

- Scaling about the origin




Before Scaling




After Scaling

# Rotation (1/5)

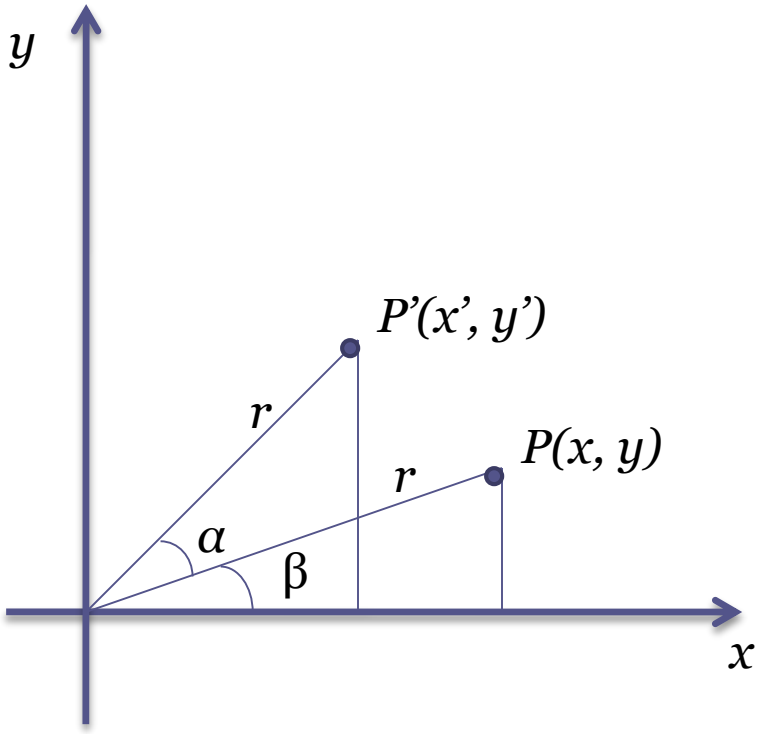
- Points can be rotated through an angle $\alpha$ about the origin, defined as

$$x' = x \cdot \cos\alpha - y \cdot \sin\alpha$$

$$y' = x \cdot \sin\alpha + y \cdot \cos\alpha$$

- These two equations are derived from the following

# Rotation (2/5)



# Rotation (3/5)

- Rotation by $\alpha$ transforms $P(x, y)$ into $P'(x', y')$
- Rotation along the origin, meaning the distances from the origin to $P$ and $P'$ is $r$

$$x = r \cdot \cos\beta, \qquad y = r \cdot \sin\beta$$

$$x' = r \cdot \cos(\alpha + \beta) = r \cdot \cos\beta \cdot \cos\alpha - r \cdot \sin\beta \cdot \sin\alpha$$

$$y' = r \cdot \sin(\alpha + \beta) = r \cdot \cos\beta \cdot \sin\alpha + r \cdot \sin\beta \cdot \cos\alpha$$

# Rotation (4/5)

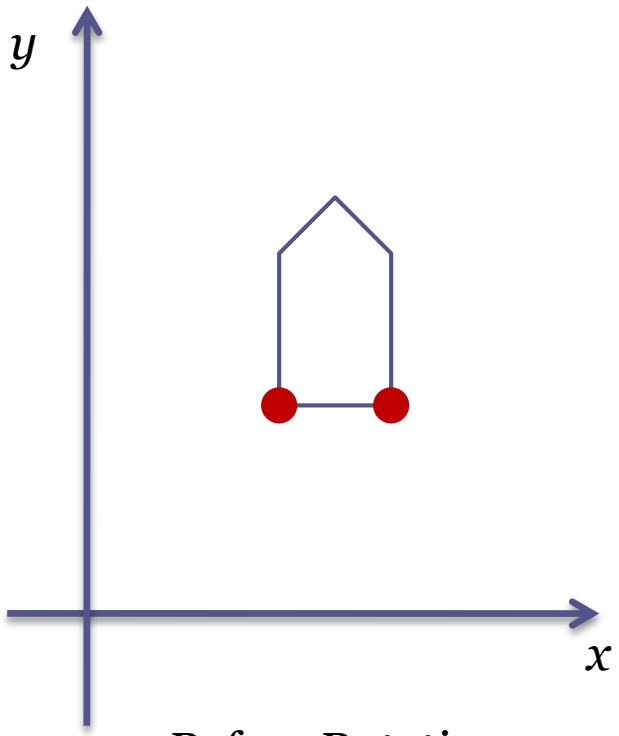
- The rotation matrix would be

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad P' = R \cdot P$$

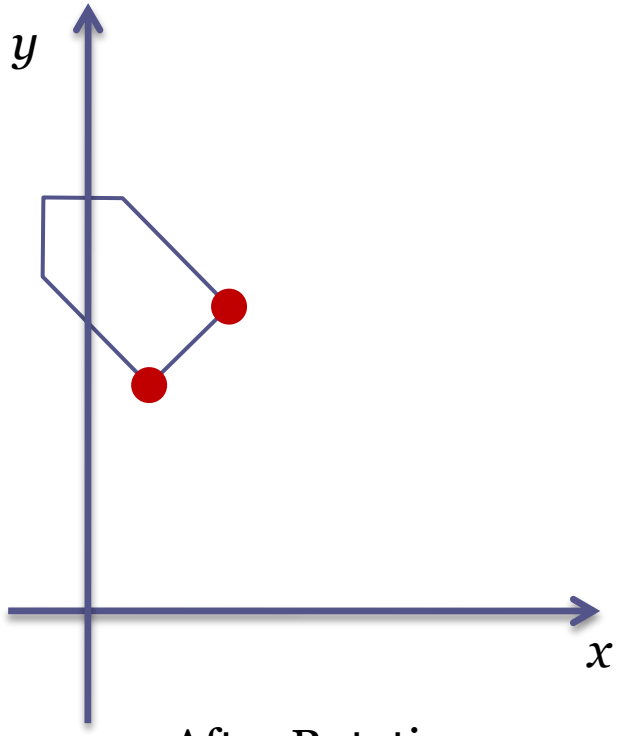
- Where positive angles are measured counterclockwise from $x$ towards $y$

$$\cos(-\alpha) = \cos\alpha \qquad \sin(-\alpha) = -\sin\alpha$$

# Rotation (5/5)




Before Rotation




After Rotation

# Overview

- 2D Transformations
- Homogeneous Coordinates and Matrix Representation
- Composition of 2D Transformations

# Homogeneous (1/3)

$$P' = P + T$$

$$P' = S \cdot P$$

$$P' = R \cdot P$$

- Problem: Translation can not be represented as 2x2 matrix
- Solution: Use **homogeneous coordinates** by adding a third coordinate $w$

# Homogeneous (2/3)

- Instead of points represented by pairs *(x, y)* we use triples *(x, y, w)*
- Points *(x, y, w)* and *(x', y', w')* are considered equal (represent the same 2D point) if one is scalar multiple of the other:

$$\textbf{e.g. } (4, 2, 1) \equiv (8, 4, 2) \equiv (16, 8, 4)$$

# Homogeneous (3/3)

- Divide by $w$ to obtain a 2D point:

$$(24, 16, 4) \equiv (\frac{24}{4}, \frac{16}{4}, \frac{4}{4}) = (6, 4, 1)$$

- For **points** we use $w = 1$

$$(6, 4) \mapsto (6, 4, 1)$$

- For **direction vectors** we use $w = 0$ (points at $\infty$)

# Translation

- In the 3x3 matrix form for homogeneous coordinates, the translation equation is:

$$P' = T(d_x, d_y) \cdot P$$

$$T(d_x, d_y) = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

# Exercise

- What happens if a point $P$ is translated by $T(d_{x1},d_{y1})$ to $P'$ and then translated by $T(d_{x2}, d_{y2})$ to $P''$ ?

# Scaling

$$P' = S(S_x, S_y) \cdot P$$

$$S(S_x, S_y) = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- What about successive scaling?

# Rotation

$$P' = R(\alpha) \cdot P$$

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- What about successive rotations?

# Identity

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Rigid-Body Transformations

- Has the form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & d_x \\ \sin\alpha & \cos\alpha & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

- The upper 3x3 matrix is orthogonal
  - Preserves angles and lengths
- A unit square after transformation will remain a unit square

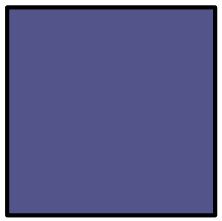
# Affine Transformations (1/2)

- What about the product of an arbitrary sequence of rotation, translation and scale matrices?

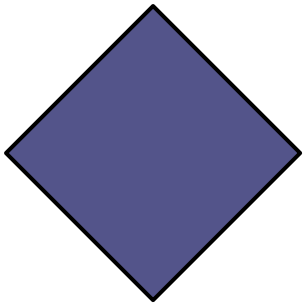
  Answer: Affine transformations

- Affine transformations, have the property of preserving parallelism of lines but **not** lengths and angles
  - That also goes for a sequence of rotation, scale and translation operations

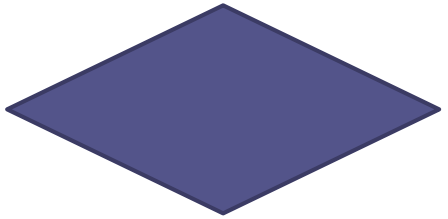
# Affine Transformations (2/2)



Unit Cube



Rotation about 45°



Scale in $x$, not in $y$

# Overview

- 2D Transformations
- Homogeneous Coordinates and Matrix Representation
- Composition of 2D Transformations

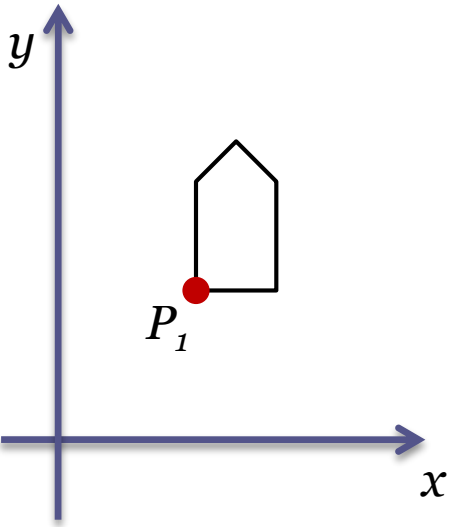
# Composition of 2D Transformations

- Purpose:
  - Gain efficiency by applying a single composed transformation to a point, rather than a series of them

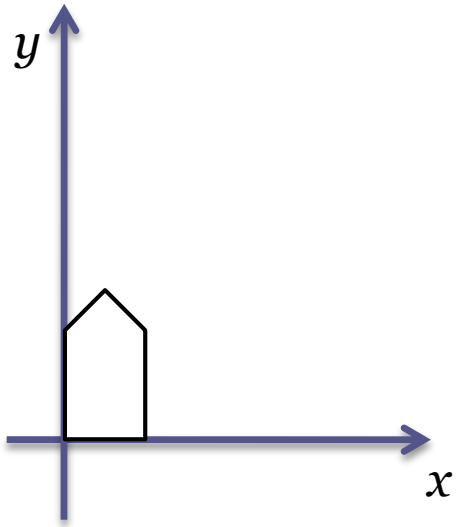
# Rotation About Some Arbitrary Point (1/3)

- To rotate about an arbitrary point $P_1(x_1, y_1)$ we need a sequence of transformations:
  1. Translate $P_1$ by $(-x_1, -y_1)$ (i.e. move to origin)
  2. Rotate by $\alpha$
  3. Translate $P_1$ by $(x_1, y_1)$ (i.e. translate back)

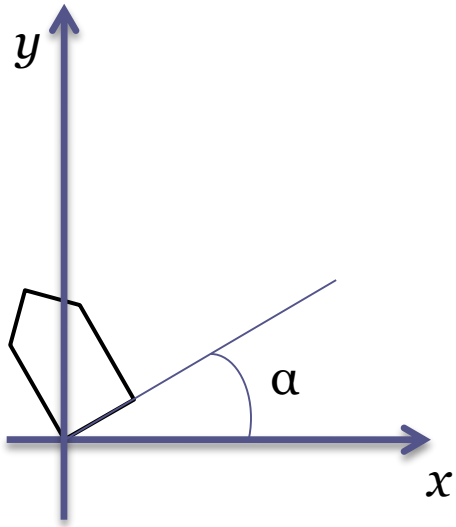
# Rotation About Some Arbitrary Point (2/3)



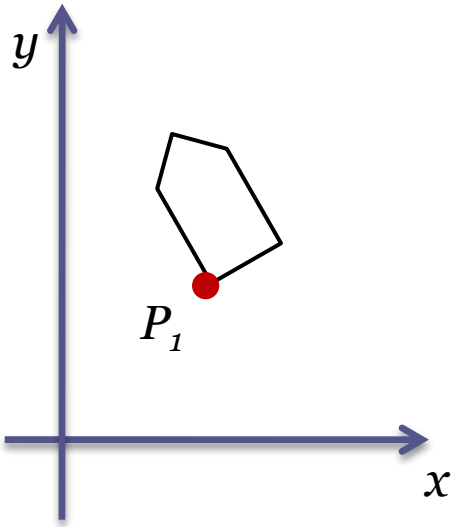

Original House




Move House to Origin




Rotate Around the Origin




Translate Back

# Rotation About Some Arbitrary Point (3/3)

- The transformation is

$$T(x_1, y_1) \cdot R(\alpha) \cdot T(-x_1, -y_1) =$$

$$\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & x_1(1-\cos\alpha) + y_1 \sin\alpha \\ \sin\alpha & \cos\alpha & y_1(1-\cos\alpha) - x_1 \sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

# Scale About Some Arbitrary Point

- The transformation is

$$T(x_1, y_1) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1) =$$

$$\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} S_x & 0 & x_1(1-S_x) \\ 0 & S_y & y_1(1-S_y) \\ 0 & 0 & 1 \end{bmatrix}$$

# Matrix Multiplications (review)

- Matrix multiplication is associative:

$$C(B(Ax)) = C((BA)x) = (C(BA))x$$
$$= ((CB)A)x$$
$$= (CB)(Ax)$$

- Matrix multiplication is **not** commutative

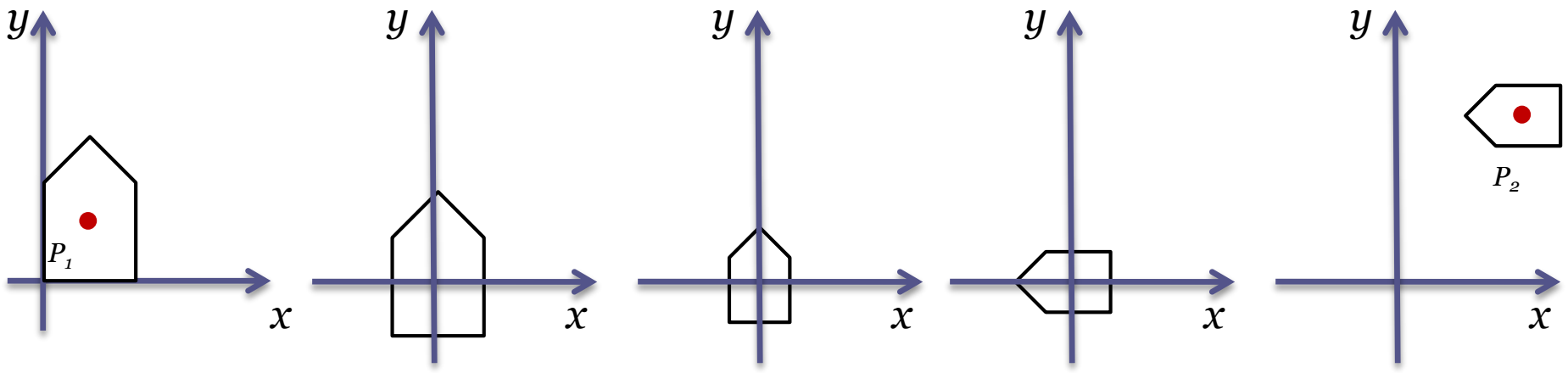
$$AB \neq BA$$

A rotation followed by a translation is not the same as a translation followed by a rotation

# Applying Transformation Matrix (1/3)

- General form:

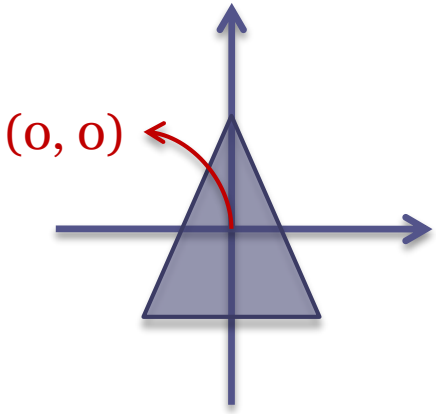
$$T(x_2, y_2) \cdot R(\alpha) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$



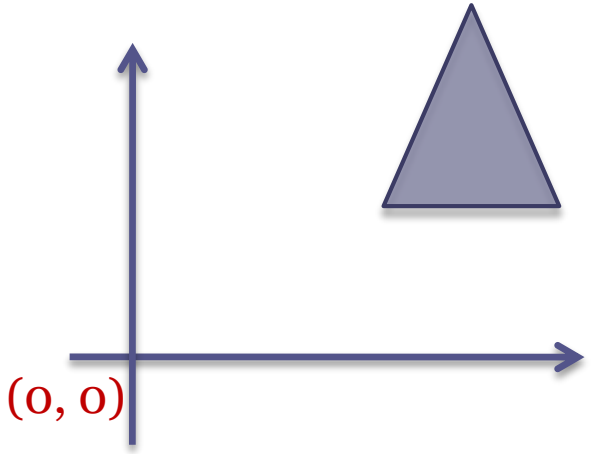

Original House | Translate to Origin | Scale | Rotate | Translate to Final Position $P_2$

# Applying Transformation Matrix (2/3)

- Having our object in its local coordinate system




Sprite's Local Coordinate System




World Coordinate System

# Applying Transformation Matrix (3/3)

- We will apply

$$\begin{bmatrix} posX' \\ posY' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} posX \\ posY \\ 1 \end{bmatrix}$$

# Applying Transformation Matrix (3/3)

- We will apply

$$\begin{bmatrix} posX' \\ posY' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} posX \\ posY \\ 1 \end{bmatrix}$$

- In this formulation, we see how the graphics transforms each vertex (***posX, posY***) that belongs to a mesh from it's local space to a global-world space

# Applying Transformation Matrix (3/3)

- We will apply

$$\begin{bmatrix} posX' \\ posY' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} posX \\ posY \\ 1 \end{bmatrix}$$

- Application in Alpha Engine:
  - AEMtx33 trans, rot, scale;//to store the resultant matrices
  - AEMtx33Trans(&trans, $d_x$ , $d_y$ );//***dx, dy*** are the position of the game object
  - AEMtx33Rot(&rot, $\alpha$ );//***Alpha*** is the orientation angle of the game object
  - AEMtx33Scale(&scale, $S_x$, $S_y$);//***sx, sy*** are the scale of the game object

# Applying Transformation Matrix (3/3)

- Application in Alpha Engine:
  - Concatenation of matrices (matrix multiplication)
  - AEMtx33 result, m1, m2;
  - AEMtx33Concat(&result, m1, m2);//this means: result = m1*m2

  - Sending the result matrix to Alpha Engine graphics system
  - AEGfxSetTransform(&result.m);//m is a matrix data representation