# K-degree linear homogeneous recurrence with constant coefficients

CS330 – Lecture 03 Efficiency Analysis of Recursive Algorithm

# Definition

- A linear recurrence relation of k-degree:

$$T(n) = a_1T(n-1) + a_2T(n-2) + \cdots + a_kT(n-k) + b$$

  - If $b == 0$ → The recurrence is homogeneous
  - If $a_1 \ldots a_k$ are constants → The recurrence is with constant coefficients

K-degree linear homogeneous recurrence with constant coefficients

$$T(n) = a_1T(n-1) + a_2T(n-2) + \cdots + a_kT(n-k)$$

# Solving Steps

$$T(n) = a_1T(n-1) + a_2T(n-2) + \cdots + a_kT(n-k)$$

- 1. Characteristic Equation

$$r^k - a_1r^{k-1} - a_2r^{k-2} - \cdots - a_k = 0$$

- 2. Solve the equation, the roots are $r_1 \ldots r_k$

# Solving Steps – cont.

$T(n) = a_1 T(n-1) + a_2 T(n-2) + \cdots + a_k T(n-k)$

- 3. Time function
  - Case 1, when the roots of the characteristic equation are real and distinct
    $T(n) = \alpha_1 r_1^n + \cdots + \alpha_k r_k^n$
  - Case 2, when the same roots occur multiple times, the terms in the formula corresponding to the second and later occurrences of the same root are multiplied by increasing power of $n$

    $\alpha_1 r_1^n \cdot n^0$ $\alpha_2 \cdot r_2^n \cdot n^1$ $\alpha_3 \cdot r_3^n \cdot n^2$

    - Example1: K=3, $r_1 = r_2 = r_3$, $T(n) = \alpha_1 r_1^n + \alpha_2 n r_2^n + \alpha_3 n^2 r_3^n$
    - Example2: K=3, $r_1 = r_2 <> r_3$, $T(n) = \alpha_1 r_1^n + \alpha_2 n r_2^n + \alpha_3 r_3^n$

    $\alpha_1 r_1^n \cdot n^0$ $\alpha_2 r_2^n \cdot n^1$ $\alpha_3 \cdot r_3^n \cdot n^0$

# Solving Steps – cont.

- 4. Using the initial conditions to get equations.
- 5. Solve the above equations for $\alpha$s.

$$T(n) = \alpha_1 r_1^n + \cdots + \alpha_k r_k^n$$

- 6. Get the solution to the recurrence.

find out its family.

($O$, $\Omega$, $\Theta$)

$T(n) = 2T(n-1) - T(n-2)$ 2-degree

$T(0) = 1, T(1) = 2$

1. $r^2 - 2r - (-1) = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

2. $r_1 = r_2 = 1$

3. $T(n) = \alpha_1 r_1^n \cdot n^0 + \alpha_2 \cdot r_2^n \cdot n^1 = \alpha_1 \times 1^n \times 1 + \alpha_2 \times 1^n \times n$

$= \alpha_1 + \alpha_2 n$ .

4. $T(0) = \alpha_1 + \alpha_2 \times 0 = \alpha_1 = 1$

$T(1) = \alpha_1 + \alpha_2 \times 1 = \alpha_1 + \alpha_2 = 2$

5. $\alpha_1 = 1$

$\alpha_2 = 1$

6. $T(n) = 1 + n \in \Theta(n)$

$T(n) = T(n-1) + 1$ → constant term ⇒ non-homogeneous

$T(1) = 1$

subtraction

$T(n) = T(n-1) + 1$

$T(n+1) = T(n) + 1$

2-degree

$T(n+1) - T(n) = T(n) + 1 - T(n-1) - 1$

$T(n+1) = 2T(n) - T(n-1)$

1 $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

2 $r_1 = r_2 = 1$

3 $T(n) = \alpha_1 r_1^n + \alpha_2 \cdot n \cdot r_2^n$

$= \alpha_1 + \alpha_2 \cdot n$

4 $T(1) = \alpha_1 + \alpha_2 = 1$

$T(2) = \alpha_1 + 2\alpha_2 = 2$

5 $\alpha_1 = 0$

$\alpha_2 = 1$

6 $T(n) = n$

$\in \Theta(n)$

$$T(n+1) = T(n) + 2n + 1$$

$$T(1) = 1$$

$$T(n+1) = T(n) + 2n + 1$$

$$T(n+2) = T(n+1) + 2(n+1) + 1$$

Subtracting $T(n+1)$ from $T(n+2)$

$$T(n+2) - T(n+1) = T(n+1) + 2(n+1) + 1 - (T(n) + 2n + 1)$$

$$T(n+2) = 2T(n+1) \cancel{+ 2n} + 2 \cancel{+1} - T(n) \cancel{- 2n} \cancel{-1}$$

$$= 2T(n+1) - T(n) + 2$$

$$T(n+2) = 2T(n+1) - T(n) + 2$$

$$T(n+3) = 2T(n+2) - T(n+1) + 2.$$

subtracting $T(n+2)$ from $T(n+3)$

$$T(n+3) - T(n+2) = 2T(n+2) - T(n+1) \cancel{+2} - 2T(n+1) + T(n) \cancel{-2}$$

$$T(n+3) = 3T(n+2) - 3T(n+1) + T(n)$$

3-degree

$T(n+3) = 3T(n+2) - 3T(n+1) + T(n)$ 3- degree

① $r^3 - 3r^2 + 3r - 1 = 0 \Rightarrow (r-1)^3 = 0$

$T(1) = 1$

$\boxed{⑥\ T(n) = n^2 \in \Theta(n^2)}$

② $r_1 = r_2 = r_3 = 1$

③ $T(n) = \alpha_1 r_1^n + \alpha_2 \cdot n \cdot r_2^n + \alpha_3 \cdot n^2 \cdot r_3^n$ $[r_1 = r_2 = r_3 = 1]$

$T(n) = \alpha_1 + \alpha_2 n + \alpha_3 \cdot n^2$

④ $T(1) = \alpha_1 + \alpha_2 \cdot 1 + \alpha_3 \cdot 1^2 = \alpha_1 + \alpha_2 + \alpha_3 = 1$

$T(2) = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 2^2 = \alpha_1 + 2\alpha_2 + 4\alpha_3 = 4$

$T(n+1) = T(n) + 2n + 1 \Rightarrow T(1+1) = T(1) + 2 \times 1 + 1 = 1 + 2 + 1 = 4$ $(n=1)$

$T(3) = \alpha_1 + \alpha_2 \cdot 3 + \alpha_3 \cdot 3^2 = \alpha_1 + 3\alpha_2 + 9\alpha_3 = 9$

$T(3) = T(2) + 2 \times 2 + 1 = 4 + 4 + 1 = 9$

⑤ $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 1$