Frequency Domain

Part 2

The terms ***Function*** and ***Signal*** are used interchangeably in the context of Frequency domain processing

Recap

- Introduction to Frequency Domain
  - Background
  - Sinusoidal Waves
  - Complex Numbers
- Fourier Series
- Impulse
- Fourier Transform
- Convolution of Continuous Functions

Lecture Objectives

- 1-D Sampling
  - Sampling Revisited
  - Sampling Theorem
  - Signal Recovery

- 2-D Sampling

- Aliasing

- Aliasing in Images
  - How to reduce the effects of spatial aliasing?

  - Moiré Patterns

  - Halftoning

Key Stages in DIP

Image Acquisition → Image Enhancement → Image Restoration → Image Compression

Image Processing

Morphological Processing

Image Analysis

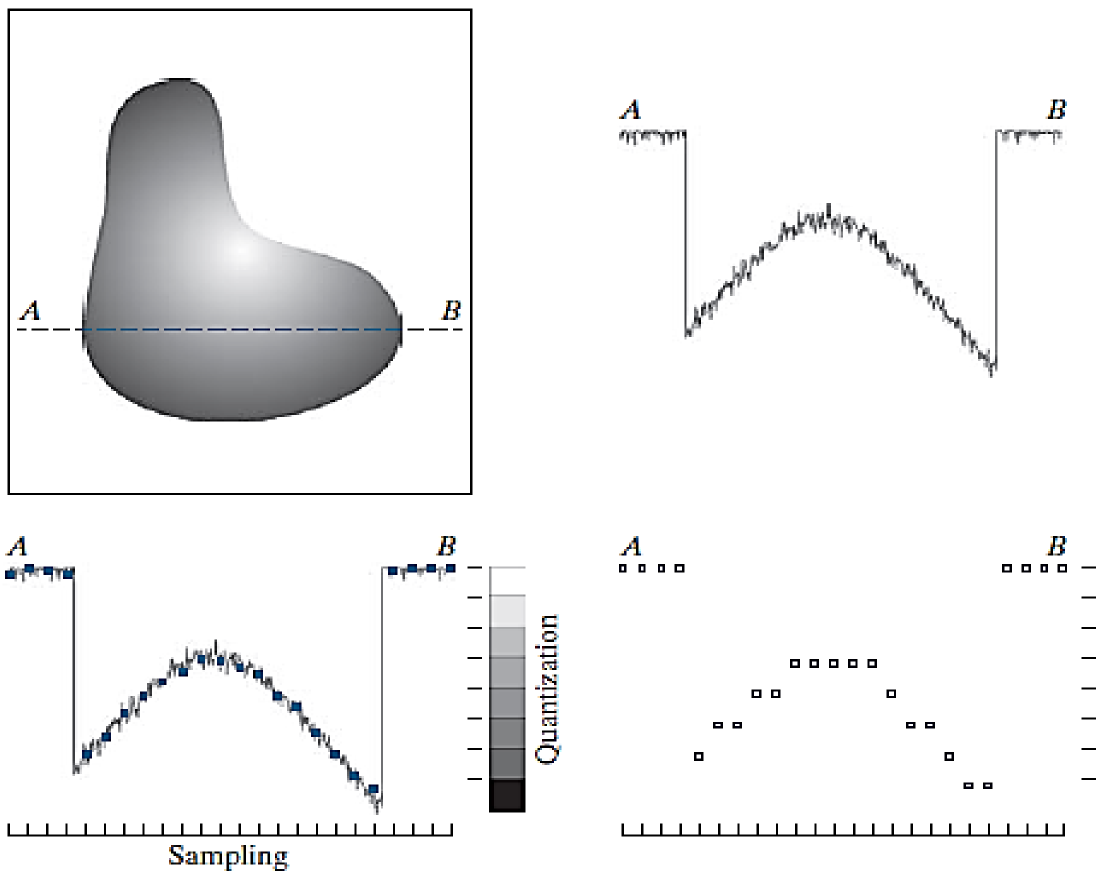
Image Segmentation → Object Recognition → Image pattern classification

Computer Vision (making sense)

1-D Sampling

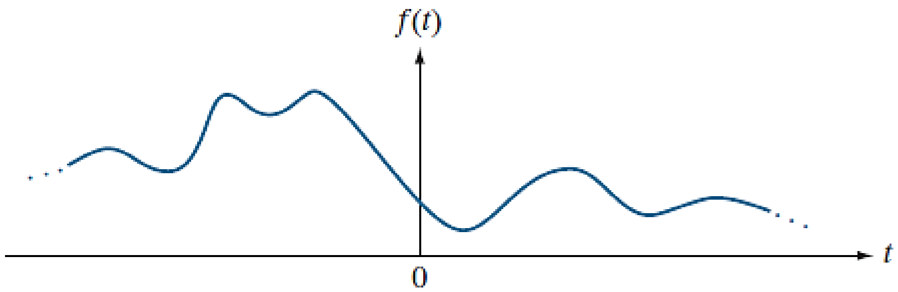
Sampling Revisited

- **More** the number of sampling & quantization levels, **better the quality** of image, but requires **more storage**.
  - Spatial sampling → Number of pixels
  - Intensity sampling → Number of grey levels



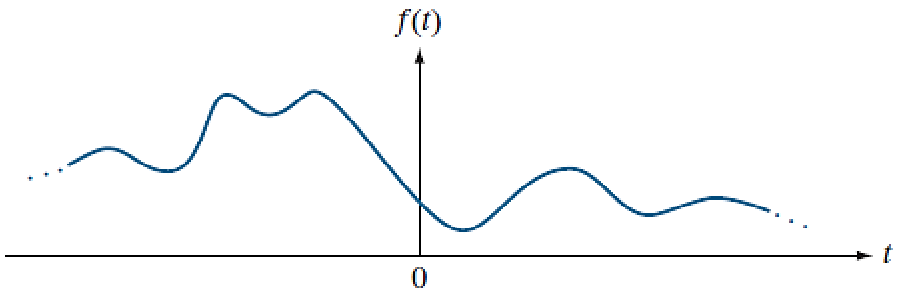
- 24 pixels
- 8 intensity levels

Sampling Revisited



A **continuous function**

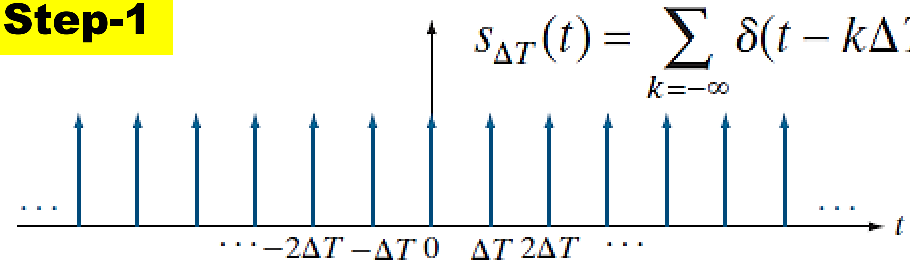
Sampling Revisited



A **continuous function**

Step-1

$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

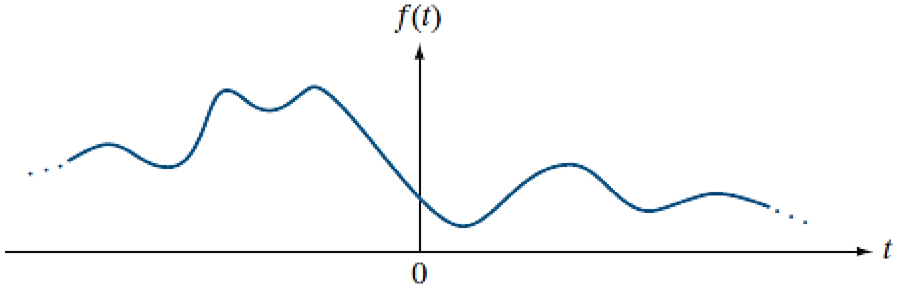


Train of impulses used to model the sampling process

ΔT=Sampling Period

$F_s=1/\Delta T$=Sampling Frequency

Sampling Revisited



A **continuous function**

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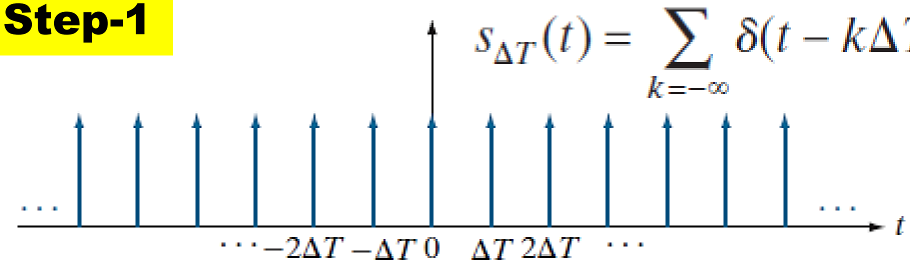


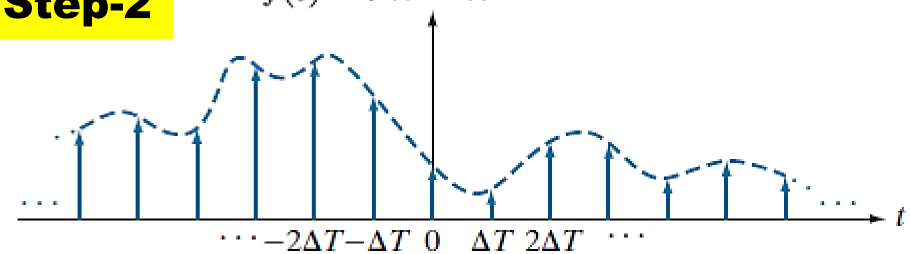
Train of impulses used to model the sampling process

ΔT=Sampling Period

$F_s = 1/\Delta T$=Sampling Frequency

Step-2

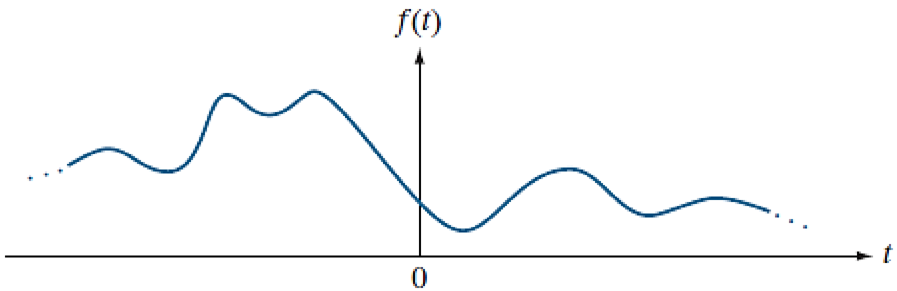
$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$



A **discrete sampled function** formed as a product

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

Sampling Revisited



A **continuous function**

Step-1

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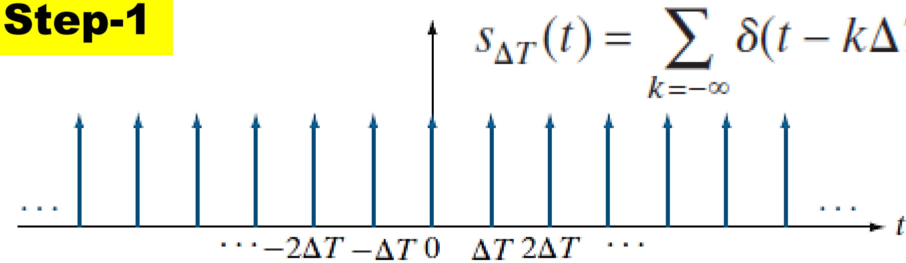

Train of impulses used to model the sampling process

ΔT=Sampling Period

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Step-2

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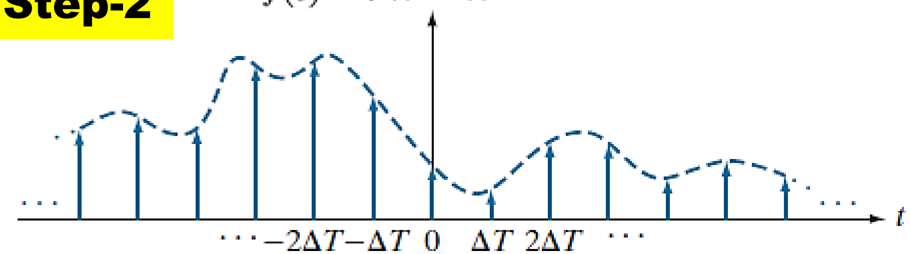


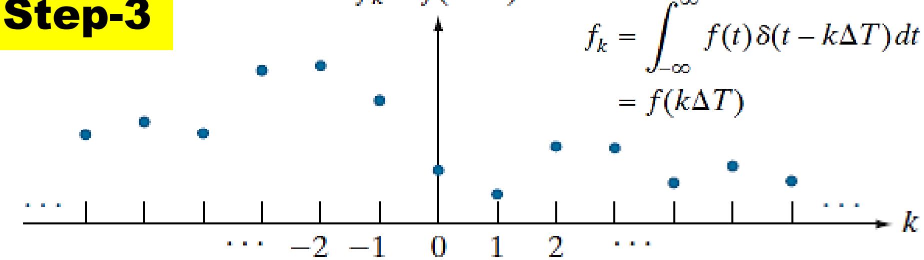
A **discrete sampled function** formed as a product

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

Step-3

$$f_k = f(k\Delta T)$$

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt = f(k\Delta T)$$



Sampled values obtained by *integration* and using the *sifting* properties of the impulse

Sampling - Summary

- Consider a continuous function, **f(t)**. We want to sample the function at equal intervals **(ΔT)** of the independent variable ***t***.

Steps in sampling the function:

- Compute the "**product**" of f(t) with a sampling function equal to a train of impulses unit ΔT apart:

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)$$

- Each component of this summation is an *impulse weighted by the value of* f(t) at the location of the impulse.

- The *value* of each sample is given by the **"strength"** of the weighted impulse obtained by *integration* and *sifting property*:

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$
$$= f(k\Delta T)$$

Fourier Transform of the Sampled Function

- The sampled function $\tilde{f}(t)$ is the product of continuous function $f(t)$ and **an impulse train**.

- From the convolution theorem, we know that the Fourier transform of the *product of two functions in the spatial domain is the convolution of the transforms of the two functions in the frequency domain*.

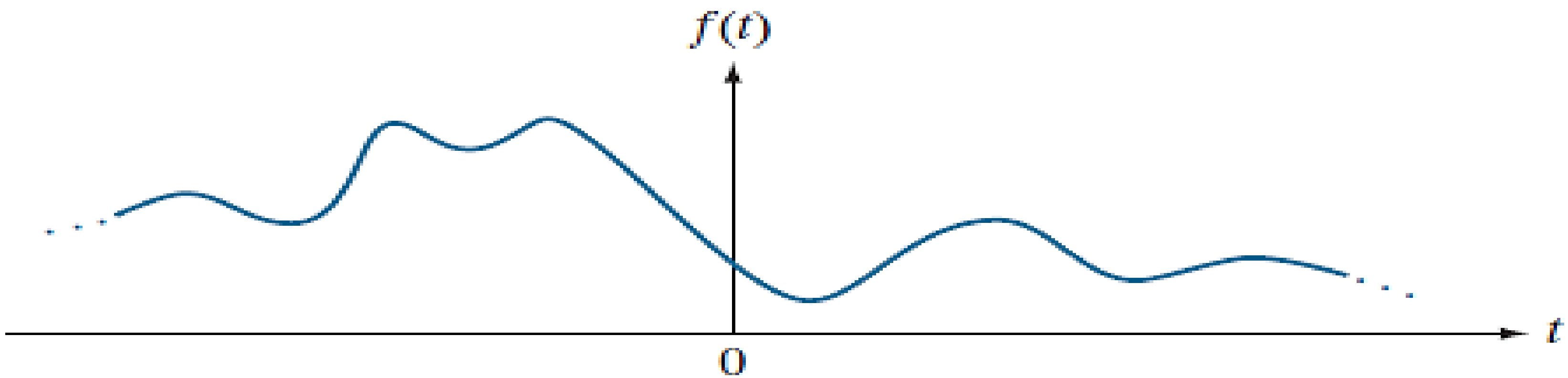
- So, the Fourier transform of sampled function $\tilde{f}(t)$ is:

$$\tilde{F}(\mu) = \Im\left\{\tilde{f}(t)\right\} = \Im\left\{f(t)s_{\Delta T}(t)\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right) = (F \star S)(\mu)$$

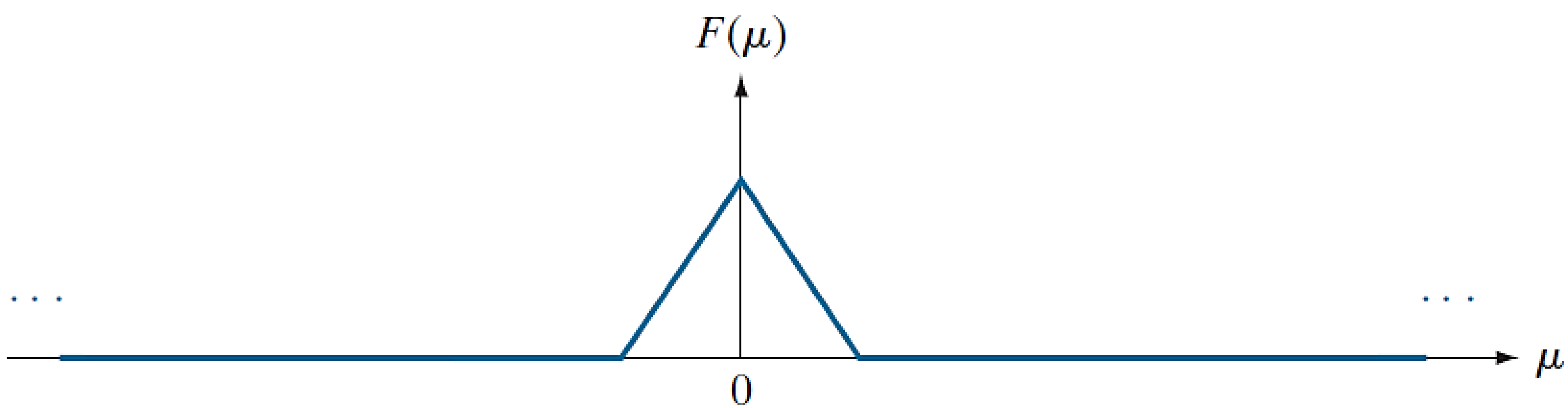
where,

$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$ is the Fourier transform of impulse train

Fourier Transform of the Sampled Function

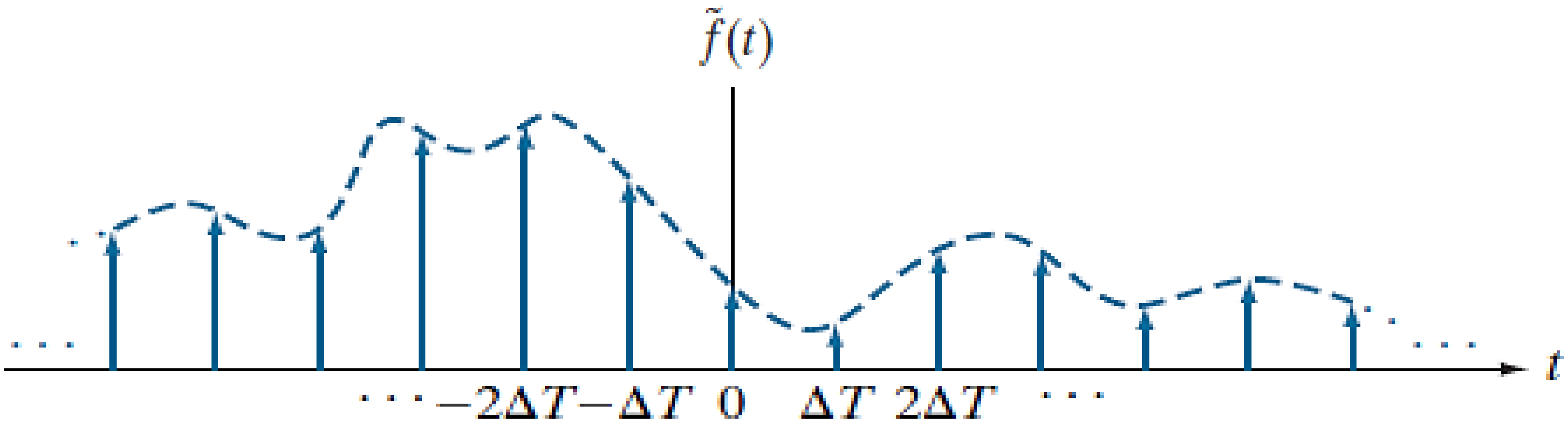


continuous function *f(t)*

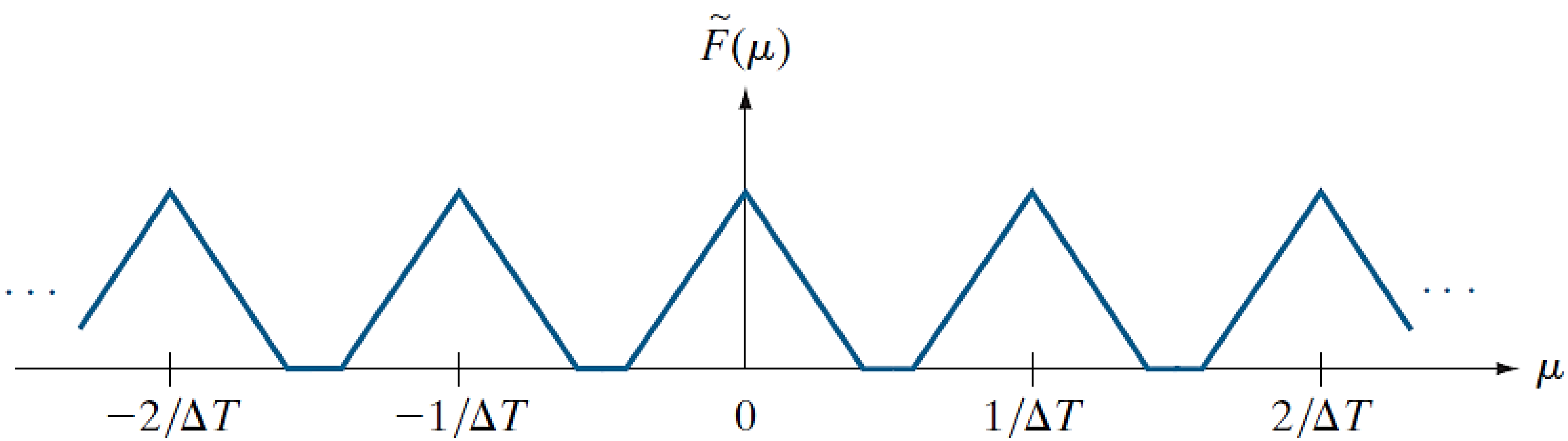


Fourier Transform $F(\mu)$ of the continuous function *f(t)*

Fourier Transform of the Sampled Function

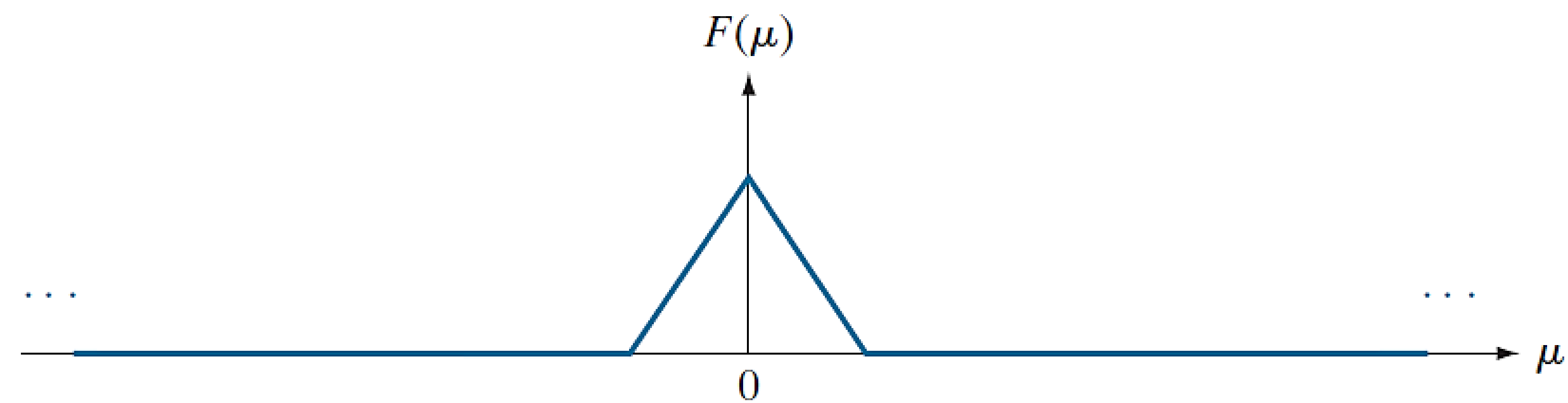


sampled function $\tilde{f}(t)$

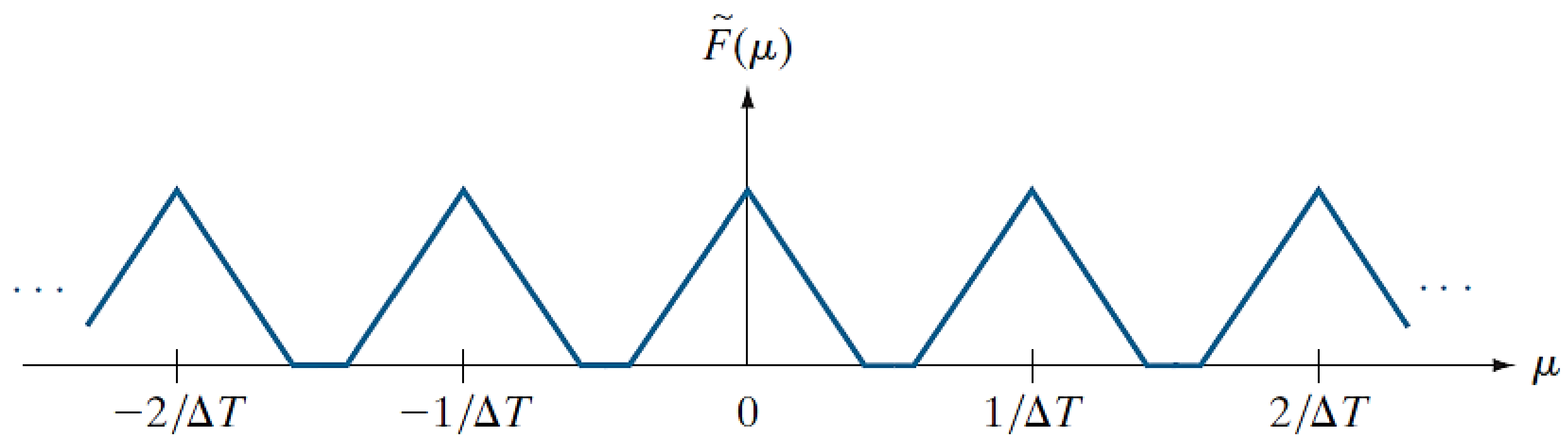


Fourier Transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$

Fourier Transform of the Sampled Function



Fourier Transform $F(\mu)$ of the function $f(t)$



Fourier Transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$

Fourier Transform of the Sampled Function

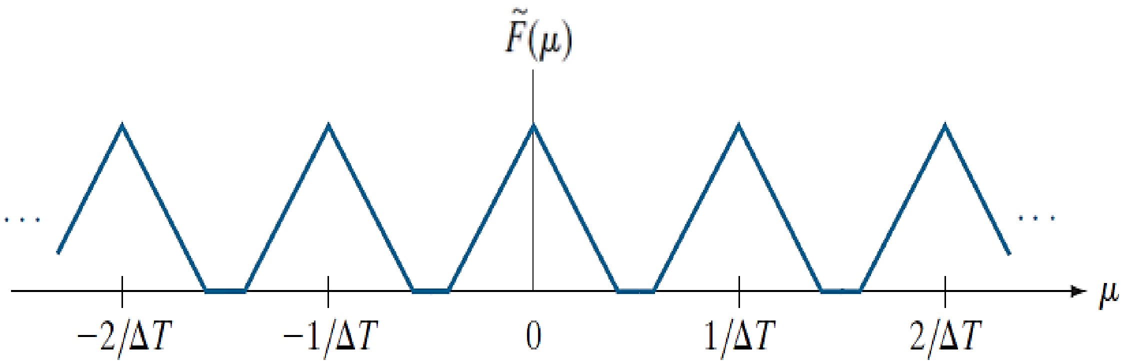
Properties of $\tilde{F}(\mu)$:

- It is an *infinite*, *periodic* sequence of *copies* of the **transform** $F(\mu)$ of the original, continuous function $f(t)$.

- The separation between the copies is $1/\Delta T$ which is known as **sampling rate / sampling frequency**

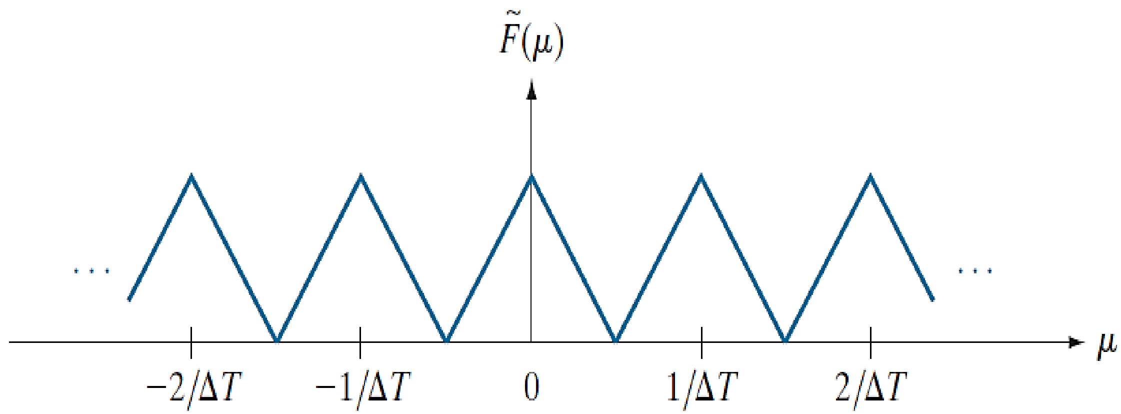
- Although the sampled function $\tilde{f}(t)$ is not continuous, its transform $\tilde{F}(\mu)$ is continuous because it contains copies of $F(\mu)$, which is a continuous function.

Effect of the Sampling Period ΔT



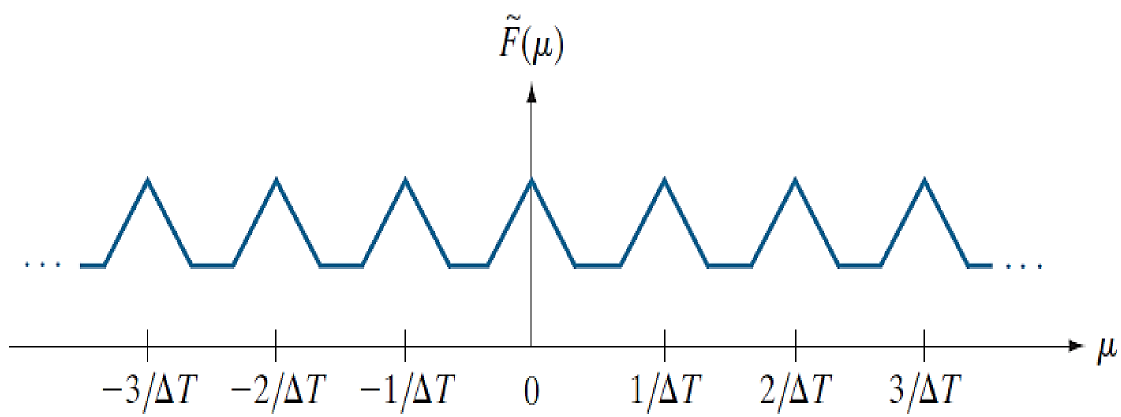
Over - Sampled

The value of ΔT is **very low**



Critically Sampled

The value of ΔT is **perfect**



Under - Sampled

The value of ΔT is **very high**

Sampling Theorem

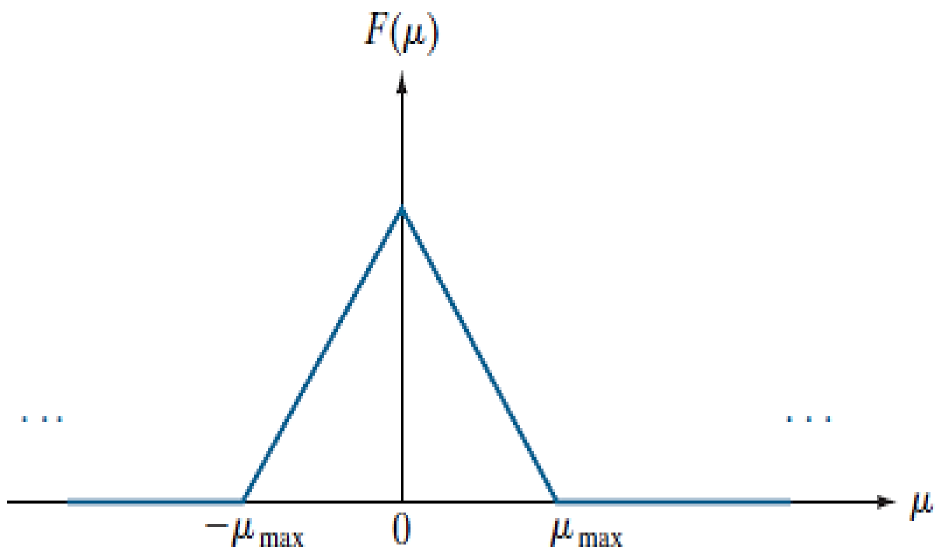
Band Limited Functions

- **Band ≈ range.**

- A band limited function has a limited range of frequencies:

$$[\,-\mu_{max}\,,\,\mu_{max}]$$

- A function $f(t)$ whose *Fourier transform* is **zero** for values of frequencies *outside a finite interval* (band) $[\,-\mu_{max}\,,\,\mu_{max}]$ *about the origin* is called a ***band-limited*** function.



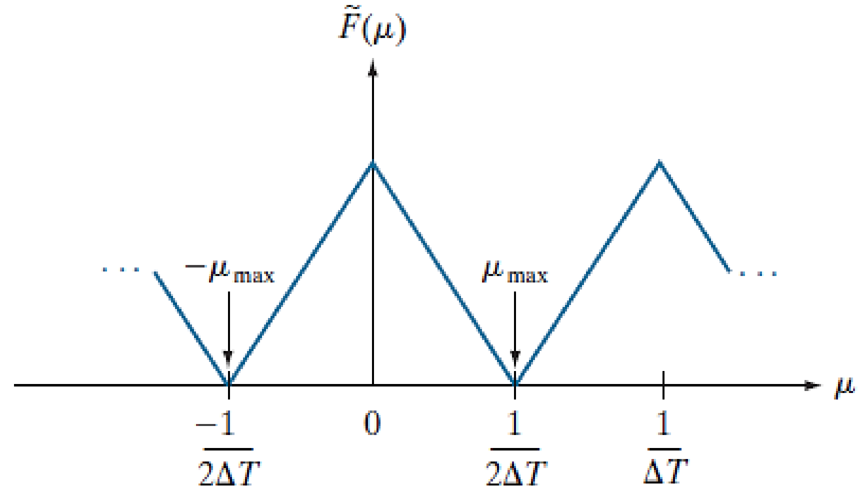
Fourier transform of a band-limited function

Critical Sampling of Band Limited Functions

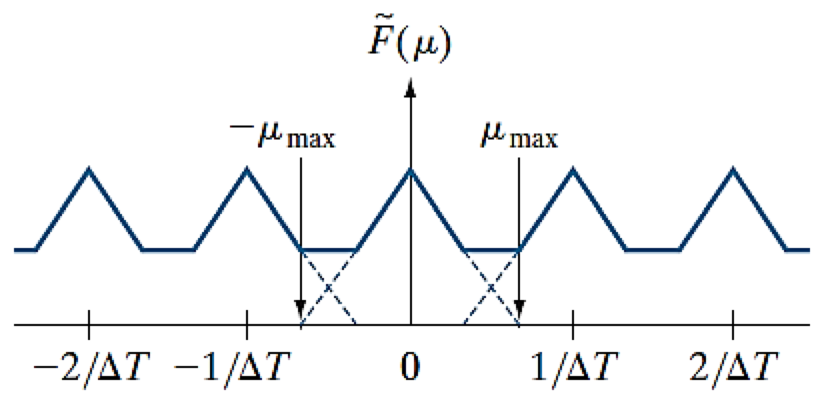
- If the **sampling rate** ($1/\Delta T$) is *reduced further* (by increasing value of ΔT), it would cause distinct bands to merge.
  - Loss of information

- So, a higher value of ΔT would cause the periods in $\tilde{F}(\mu)$ to **merge**; a lower value would provide a **clean separation** between the periods.

- There are multiple copies of $F(\mu)$ in $\tilde{F}(\mu)$
  - all we need is **one complete period** *to characterize the entire transform*.



Fourier transform of a critically sampled band-limited function



Fourier transform of a under sampled band-limited function

Sampling Theorem

Harry Nyquist
(1889–1976)

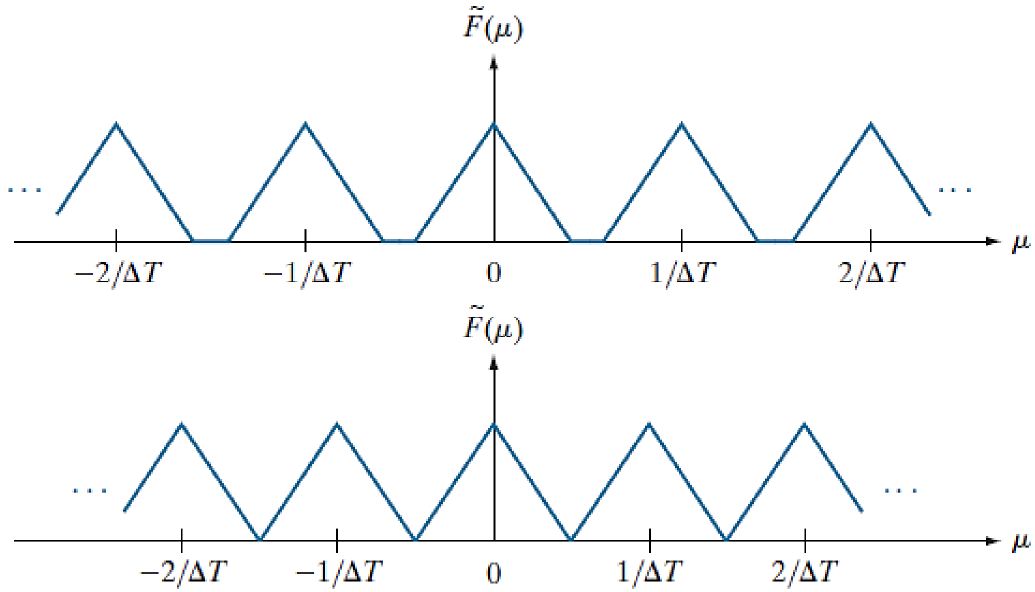
Formulated the sampling theorem in 1928

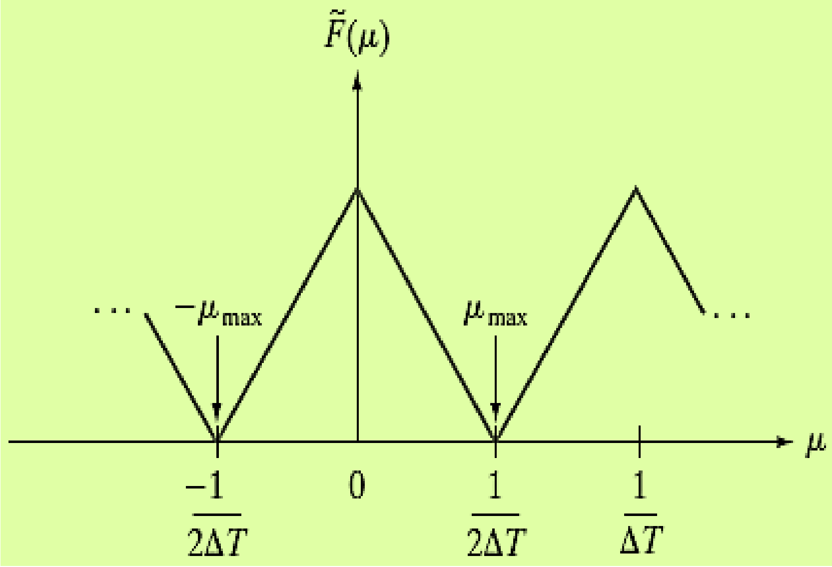
Claude Elwood Shannon
(1916–2001)

Formally proved the sampling theorem in 1949

Sampling Theorem

- Extracting from $\tilde{F}(\mu)$ a **single period** that is equal to $F(\mu)$ is possible if the separation between copies is sufficient:





- A sufficient separation is guaranteed if:

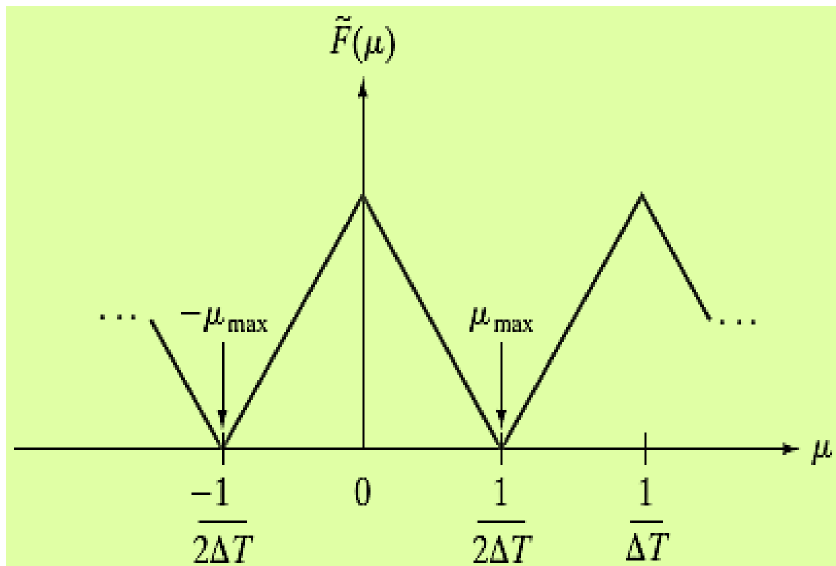
$$1/2\Delta T > \mu_{max}$$

OR

$$\frac{1}{\Delta T} > 2\mu_{max}$$

Sampling Theorem - Defined

- A continuous, band-limited function <u>*can be recovered completely*</u> from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.



$$\frac{1}{\Delta T} > 2\mu_{\max}$$

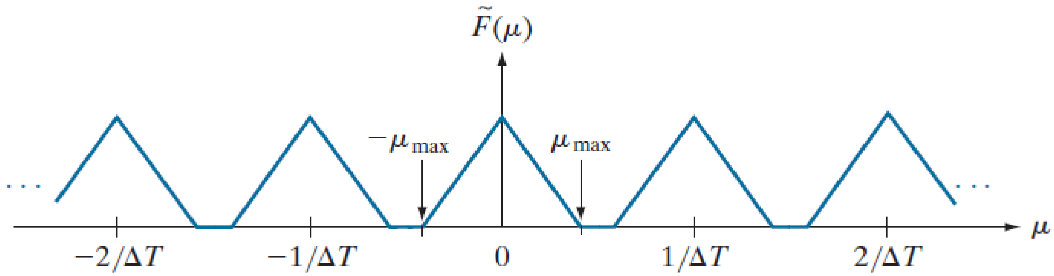
A sampling rate *exactly* equal to twice the highest frequency is called the ***Nyquist rate***.

Sampling Theorem - Application

- Note: human ear hears frequencies from **20Hz – 22 kHz**.
- That's why music CDs use sampling rate: **44.1 kHz**

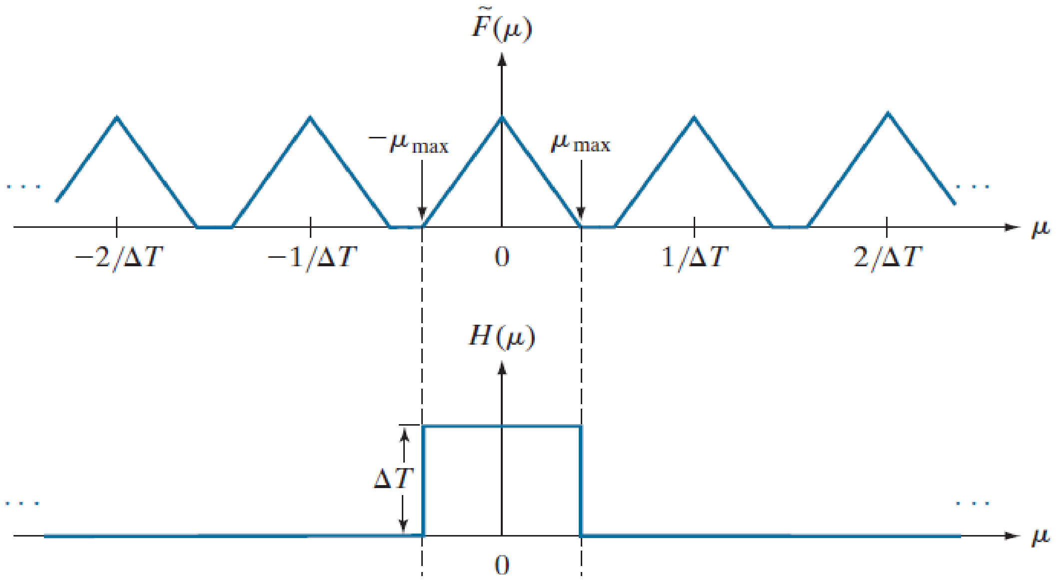
Signal Recovery

Original Function *f(t)* Recovery



Fourier transform of a sampled, band-limited function

Original Function *f(t)* Recovery



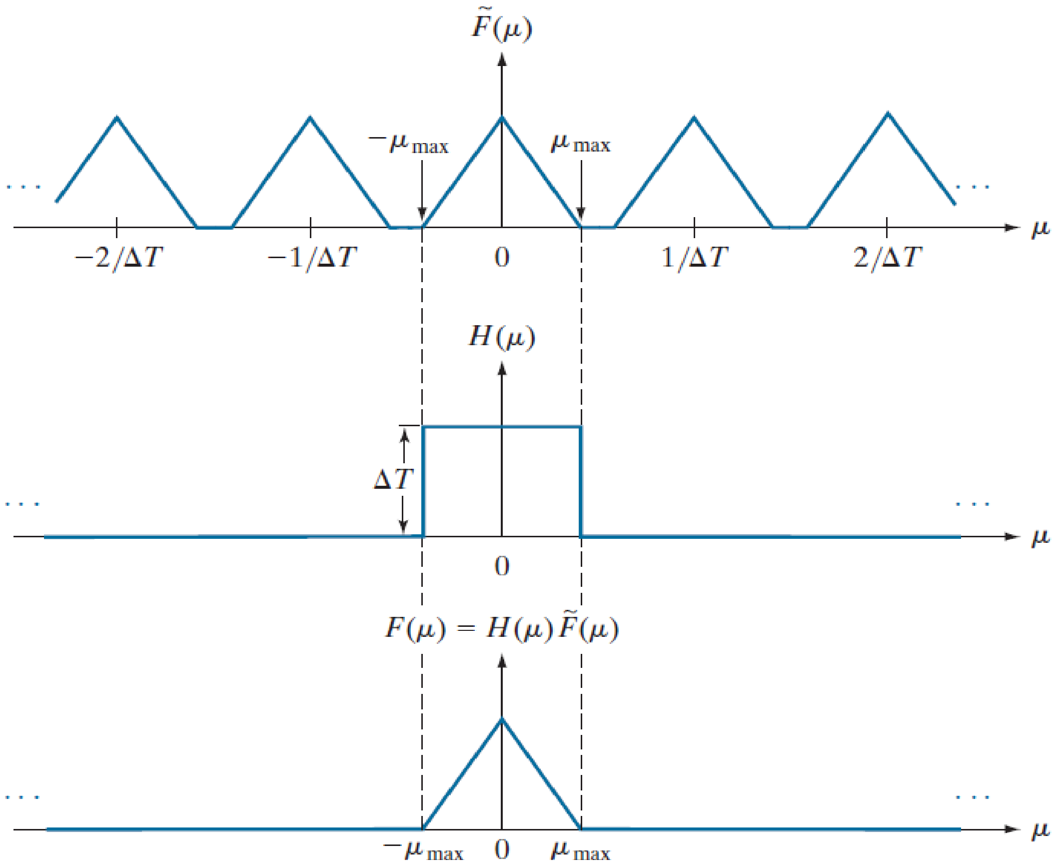
Fourier transform of a sampled, band-limited function

Step 1

$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$

(Ideal lowpass filter)

Original Function *f(t)* Recovery



Fourier transform of a sampled, band-limited function

Step 1

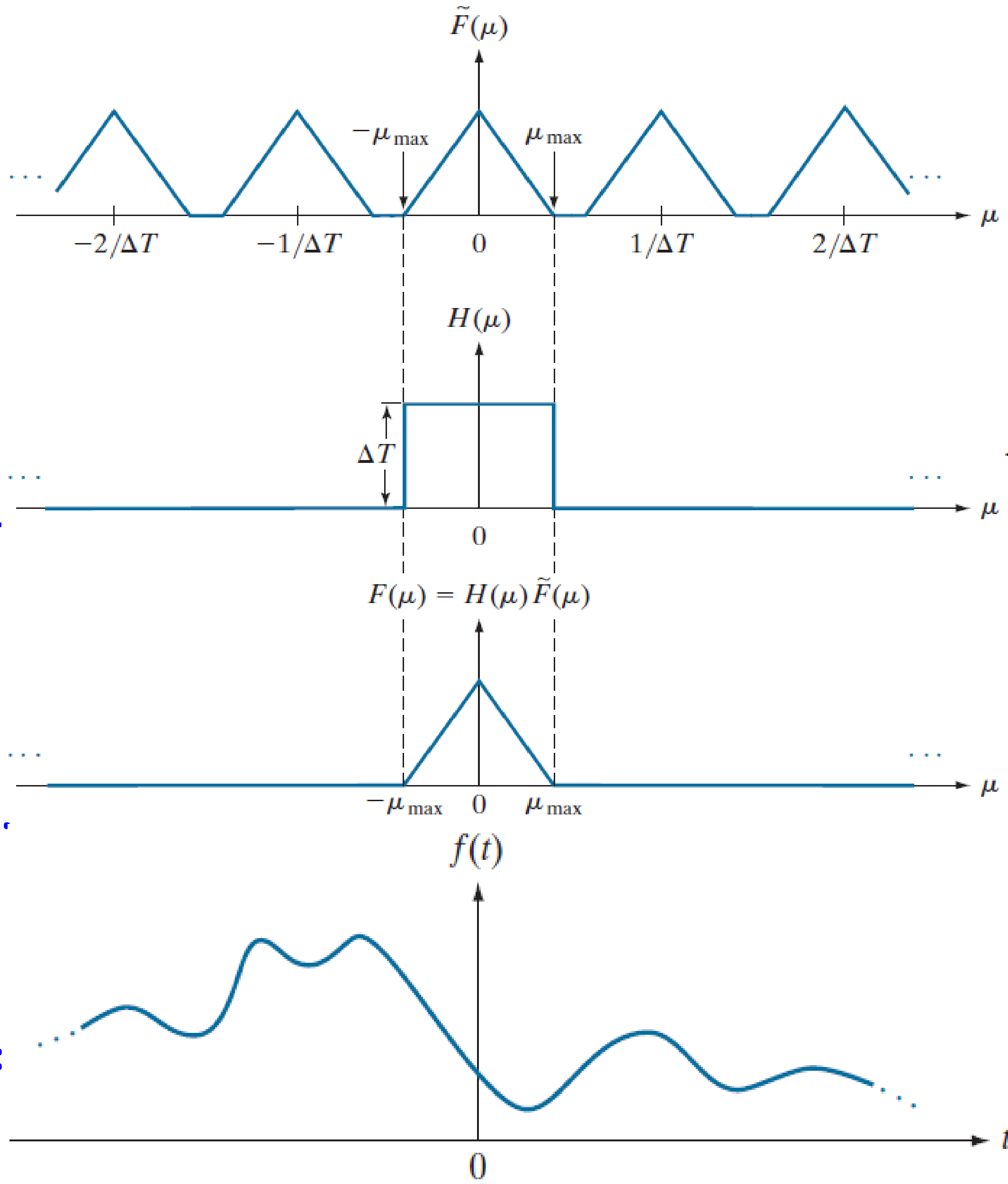
$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$

(Ideal lowpass filter)

Step 2.

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

Original Function *f(t)* Recovery



Fourier transform of a sampled, band-limited function

Step 1

$$H(\mu) = \begin{cases} \Delta T & -\mu_{max} \le \mu \le \mu_{max} \\ 0 & \text{otherwise} \end{cases}$$

(Ideal lowpass filter)

Step 2.

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

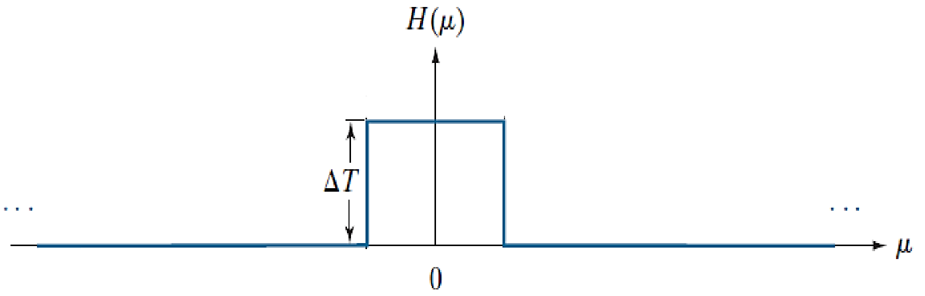
Step 3:

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Inverse Fourier transform

Original Function *f(t)* Recovery

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$



- *$H(\mu)$* is called a *lowpass filter.*
  - It allows frequencies at lower end to pass through, and eliminates the higher values of frequencies.

- *$H(\mu)$* is also an *ideal lowpass filter* because of its instantaneous transitions in amplitude (between 0 and ΔT at location $-\mu_{\max}$ and the reverse at $\mu_{\max}$).
  - **cannot be implemented physically in hardware**. We can simulate ideal filters in software, but even then there are limitations.

- *$H(\mu)$* is also known as *reconstruction filters*, since it is used to recover the original signal from the samples.

2-D Sampling

2-D Sampling

- Sampling in 2-D can be modeled using a sampling function (2-D impulse train) as:

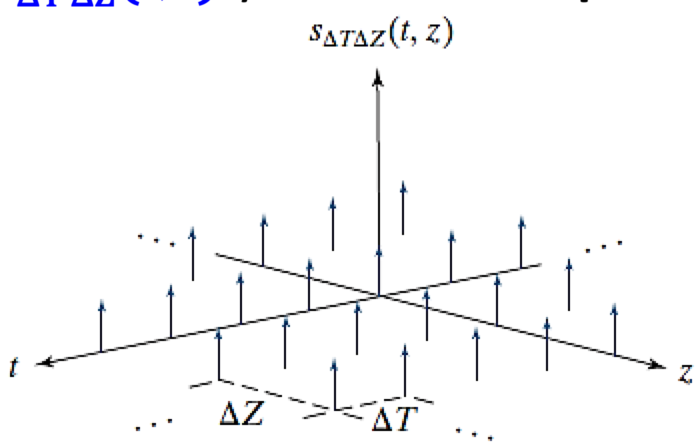
$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

1-D impulse train

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

where Δ**T** and Δ**Z** are the separations between samples along the ***t***-and ***z***-axis

- Multiplying ***f(t,z)*** by $S_{\Delta T\,\Delta Z}(t,z)$ yields the **sampled function**.



2-D Sampling

- The ***two-dimensional sampling theorem*** states that a continuous, band-limited function ***f*(*t*,*z*)** can be recovered with no error if the ***sampling rate*** is:

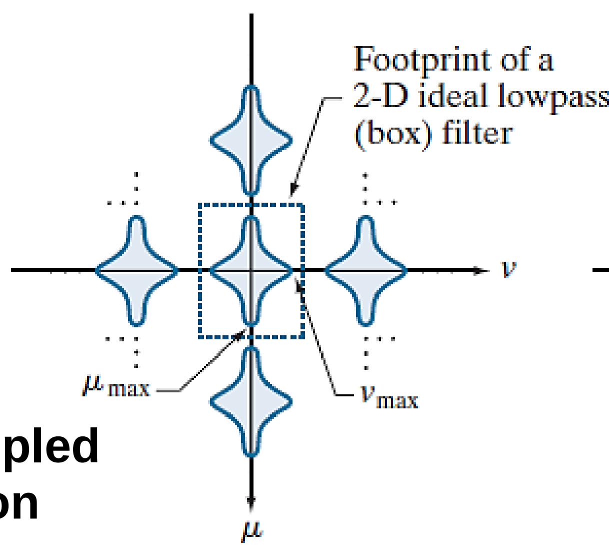
$$\frac{1}{\Delta T} > 2\mu_{\max}$$

and

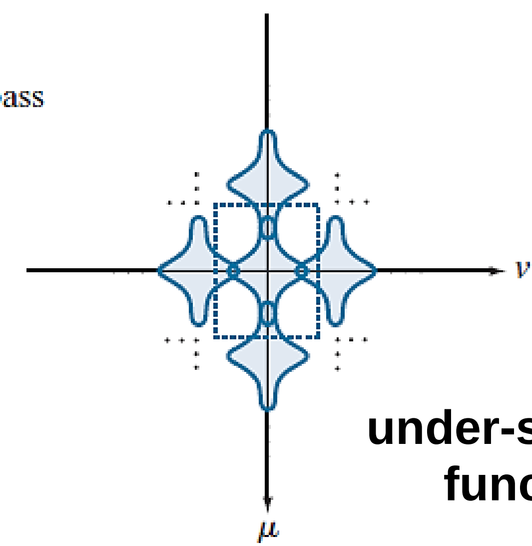
$$\frac{1}{\Delta Z} > 2\nu_{\max}$$

Function ***f*(*t*,*z*)** is said to be *band limited* if its Fourier transform is **0** outside the following frequency rectangle:

$$F(\mu,\nu) = 0 \quad \text{for} \ |\mu| \geq \mu_{\max} \ \text{and} \ |\nu| \geq \nu_{\max}$$



over-sampled function

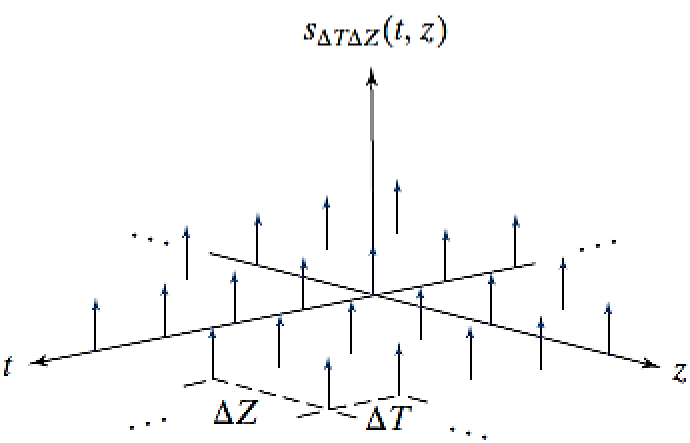


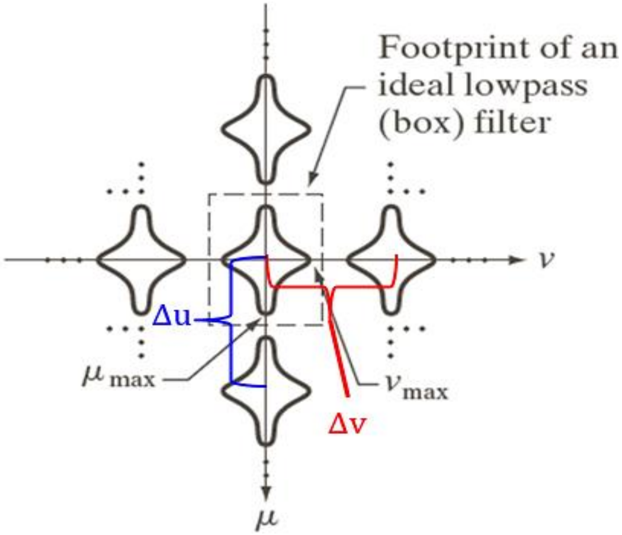
under-sampled function

Relationships Between Spatial and Frequency Intervals

- Suppose that a continuous function $f(t,z)$ **is sampled to form a** digital image $f(x,y)$ consisting of $\mathbf{M \times N}$ samples taken in the **t** and **z** directions, respectively.

- Let ΔT and ΔZ denote the separations between samples in Spatial domain. The separation between the corresponding discrete, frequency domain variables are given by: $\mathbf{\Delta u = 1/(M \times \Delta T)}$ and $\mathbf{\Delta v = 1/(N \times \Delta Z)}$



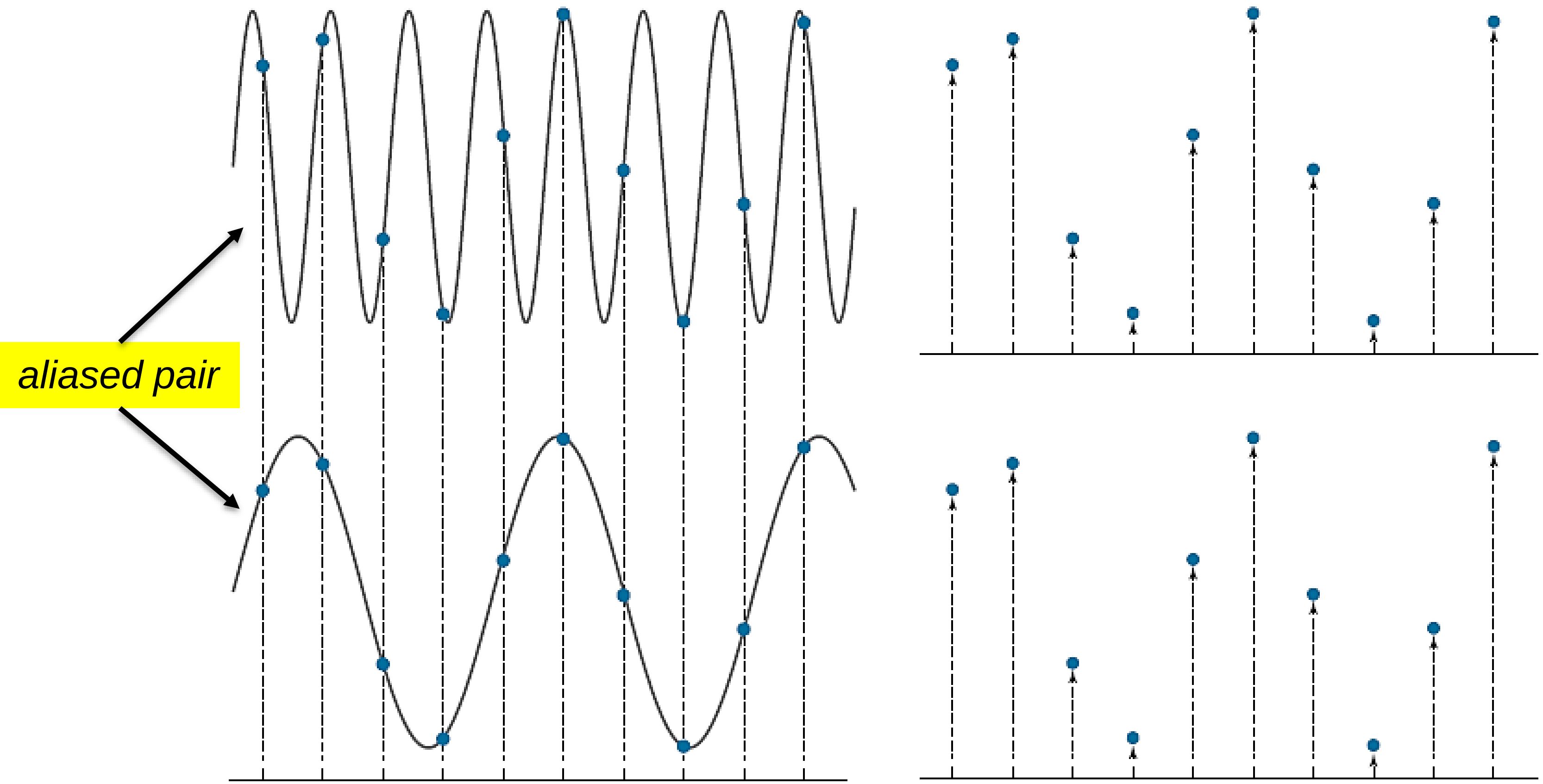


Aliasing

“A false identity”

What happens if a band-limited function is sampled at a sampling rate < Nyquist rate ($2\ \mu_{max}$) ??

Aliasing



The two functions are totally different, but their sampling rate is same

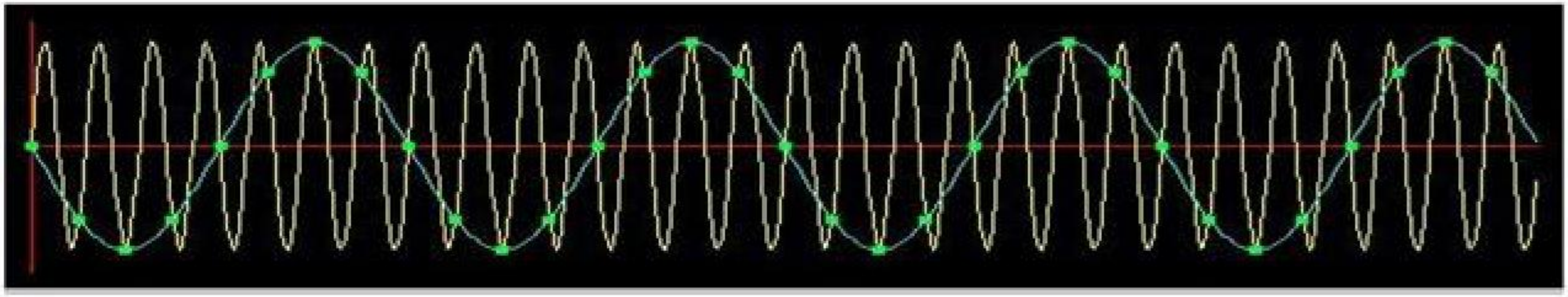
Digitized versions of the functions after sampling

Aliasing refers to sampling phenomena that cause different signals to become indistinguishable from one another after sampling

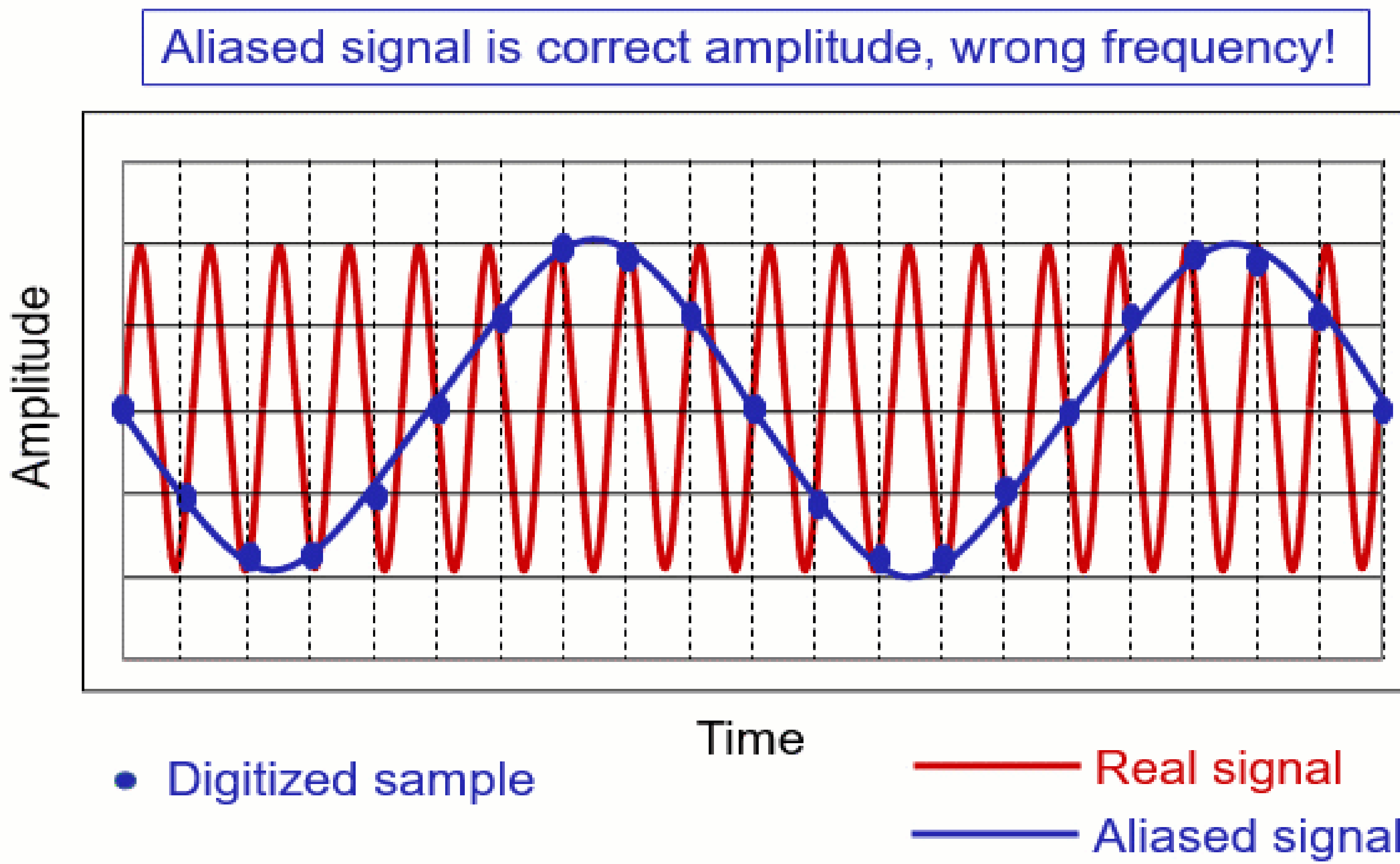
Sampling and Aliasing

- A high frequency signal is being sampled using a *<u>low sampling frequency</u>* (green dots).

- This makes it indistinguishable from a low frequency signal.

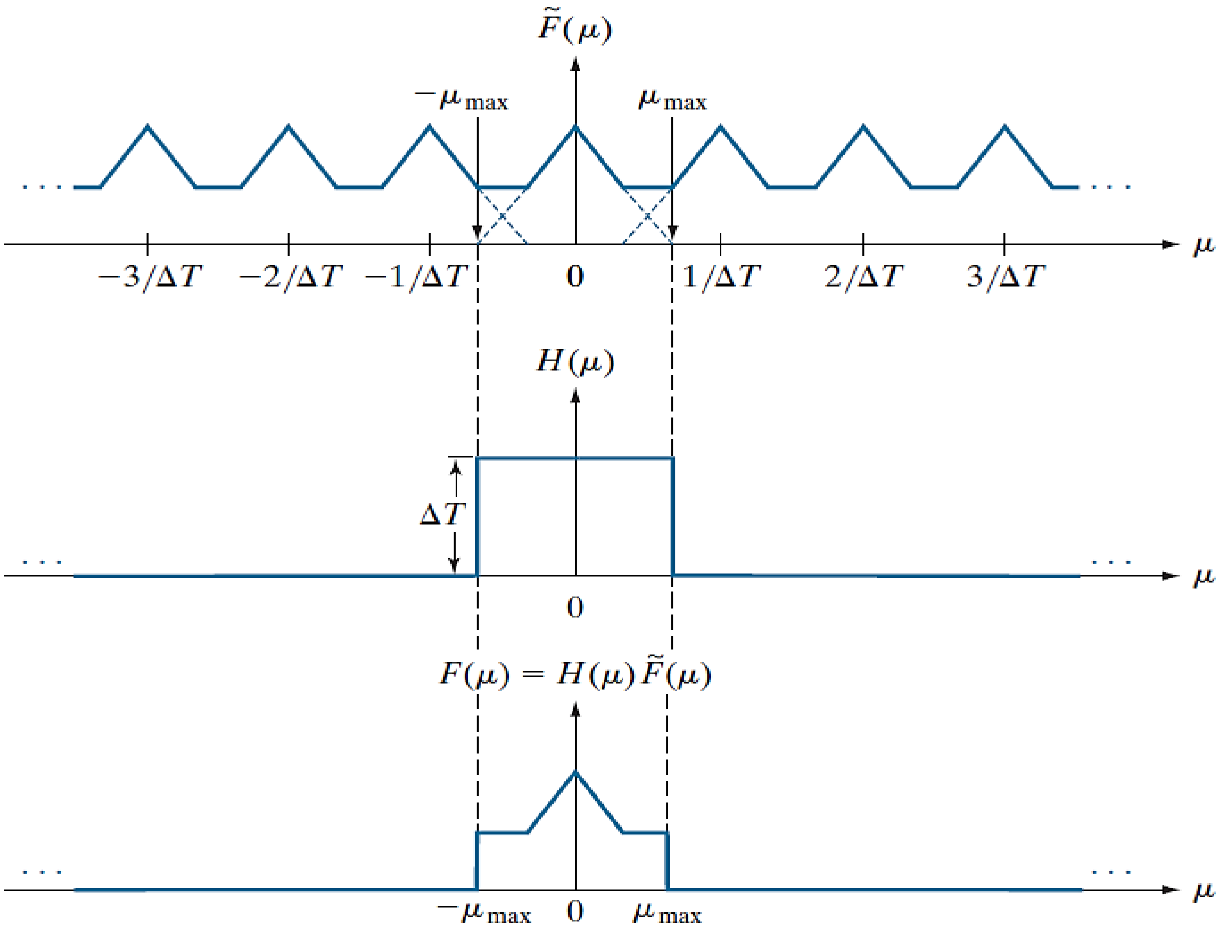


Sampling and Aliasing

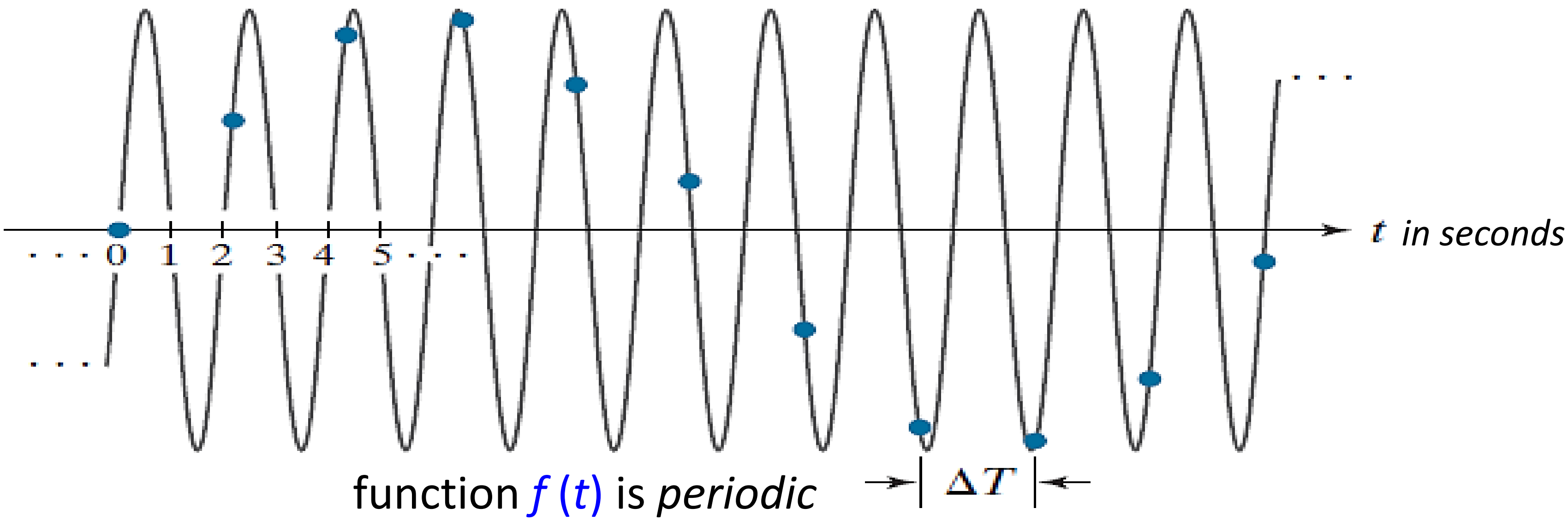


Sampling and Aliasing

- **A signal is sampled at a rate that is less than twice its highest frequency.**
- Interference from adjacent bands results in loss of information from the original signal.
- This is *Aliasing* or frequency aliasing.
- Unable to distinguish a high frequency signal from a low frequency signal.



Sampling and Aliasing - example



function *f* (*t*) is *periodic*

- The *period* is the time it takes to complete one cycle of the wave.
- The *frequency* of a *periodic* function is the number of periods (cycles) that the function completes in one unit of time.
- The *sampling rate* is the number of samples taken per one unit of time.

Period (P) = 2 sec

Frequency = $\frac{1}{\text{Period}}$ = $\frac{1}{2}$ cycles/sec

Sampling rate $\frac{1}{\Delta T}$ is < 1 samples/sec

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

The separation **ΔT** between samples has to be *less than* **0.5 sec**.

Sampling and Aliasing - Facts

- *Under sampling* is the reason for Aliasing.

- If *sampling rate is increased*, more and more of the differences between the continuous functions would be revealed in the sampled signals.

- In real-world applications, *sampling at higher frequencies* results in **better reconstructed signals**. However, higher sampling frequencies require faster converters and more storage.

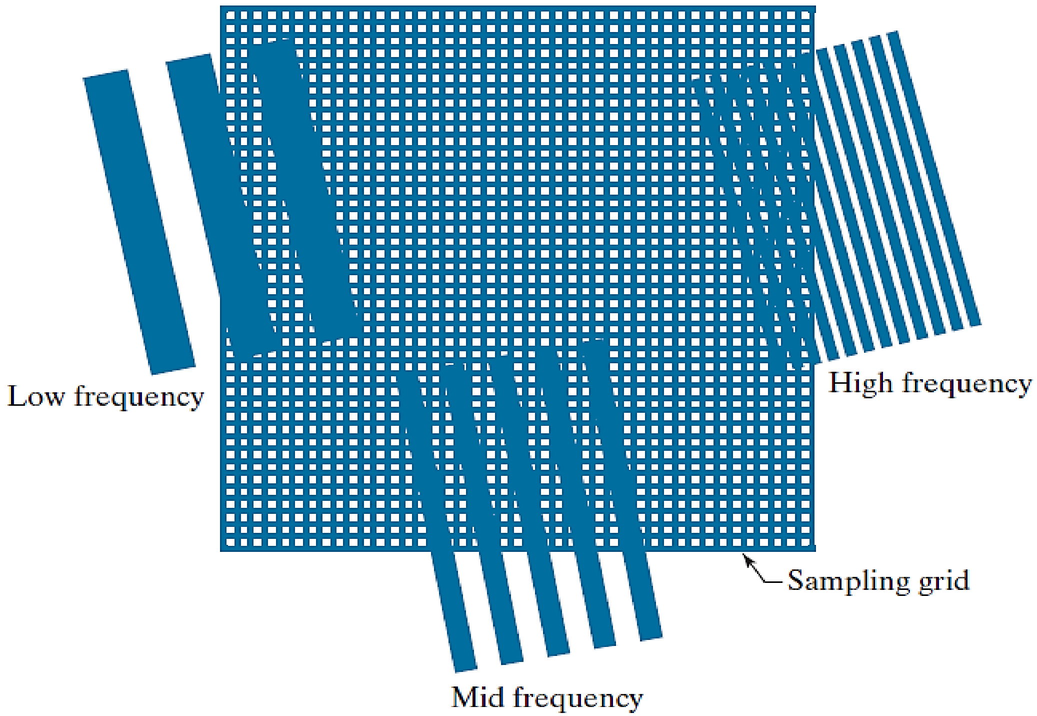
Aliasing in Images

Aliasing in Images

- Because we cannot sample a function ***f*(*t*,*z*)** of two continuous variables, ***t*** and ***z***, infinitely, aliasing is always present in digital images.

- There are two types of image-specific aliasing phenomena:
  - **Spatial aliasing** - due to *under-sampling* of the *image in spatial domain*.
  - **Temporal aliasing** - due to *under-sampling* of a *sequence of images in time* (“wagon wheel” effect).

Aliasing in Images



- The ***low frequency*** region is rendered reasonably well, with some mild jaggedness around the edges.
- The ***jaggedness increases*** as the frequency of the region increases from Low to High.

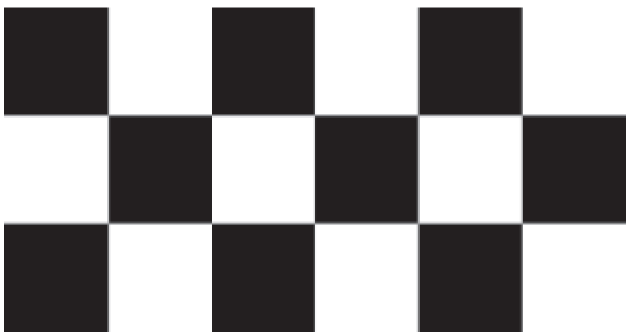
- *Sampling grid* in the center is a 2-D representation of the *impulse train*.
- In the grid, the little *white squares* correspond to the *location of the impulses* (where the image is sampled) and *black* represents the *separation between samples*.

Spatial Aliasing

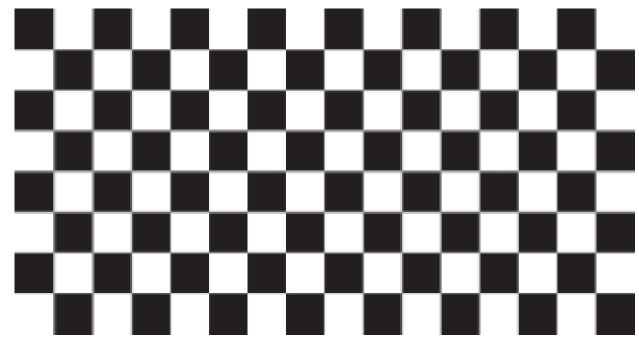
Consider a *perfect imaging system* (noiseless, produces an exact digital image) used to digitize checkerboard images.

- Fixed samples : 96 x 96 pixels = 9,216 pixels.
- Pixels are square shaped.
- What happens when the checkerboard square size is **less than 1 pixel** (*under sampling*) ?

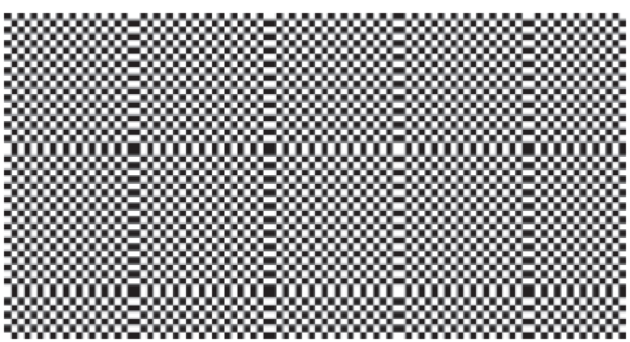
Square size is 16 x 16 pixels



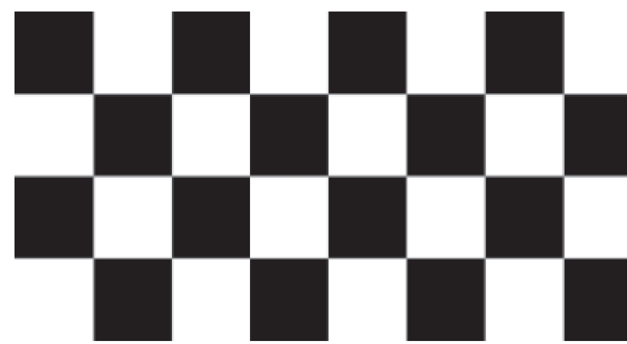
Square size is 6 x 6 pixels



Square size is 0.95 x 0.95 pixels



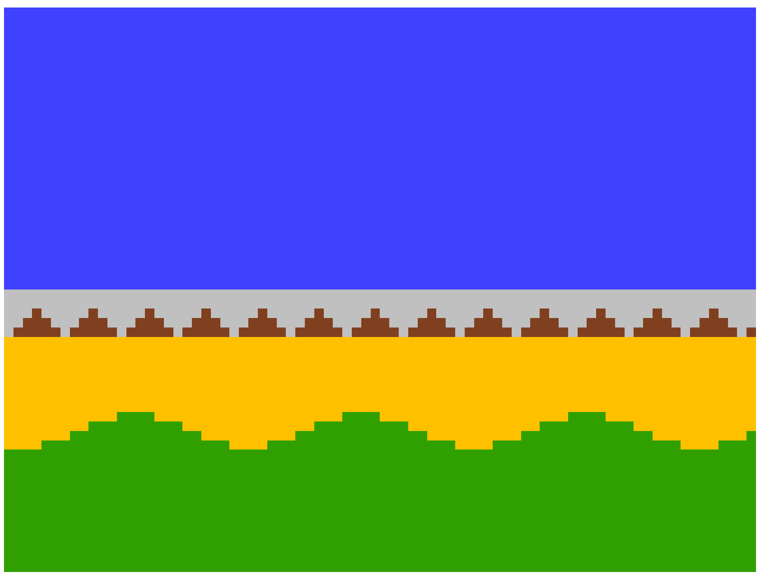
Square size is 0.48 x 0.48 pixels

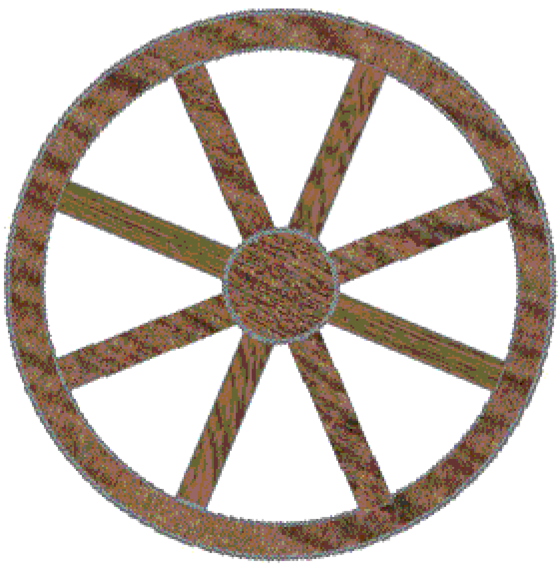


"masquerading" ≡ 16 x 16 pixels

Temporal Aliasing

- Wagon wheel effect - car tires seem to be rotating backwards
  - *Frame rate being too low* with respect to the speed of wheel rotation in the sequence





How to Reduce the Effects of Spatial Aliasing?

Anti-aliasing

- Aliasing can be reduced by applying a blur filter *before the signal is sampled*.
  - Attenuate the high frequency components

- There is no “after-the-fact” anti-aliasing filters available.

- A significant number of commercial digital cameras have true anti-aliasing filtering built in, either in the lens or on the *surface of the sensor* itself.

Anti-aliasing

The effects of aliasing generally are *worsened* when the **size** of a digital image is *reduced*.

Original image

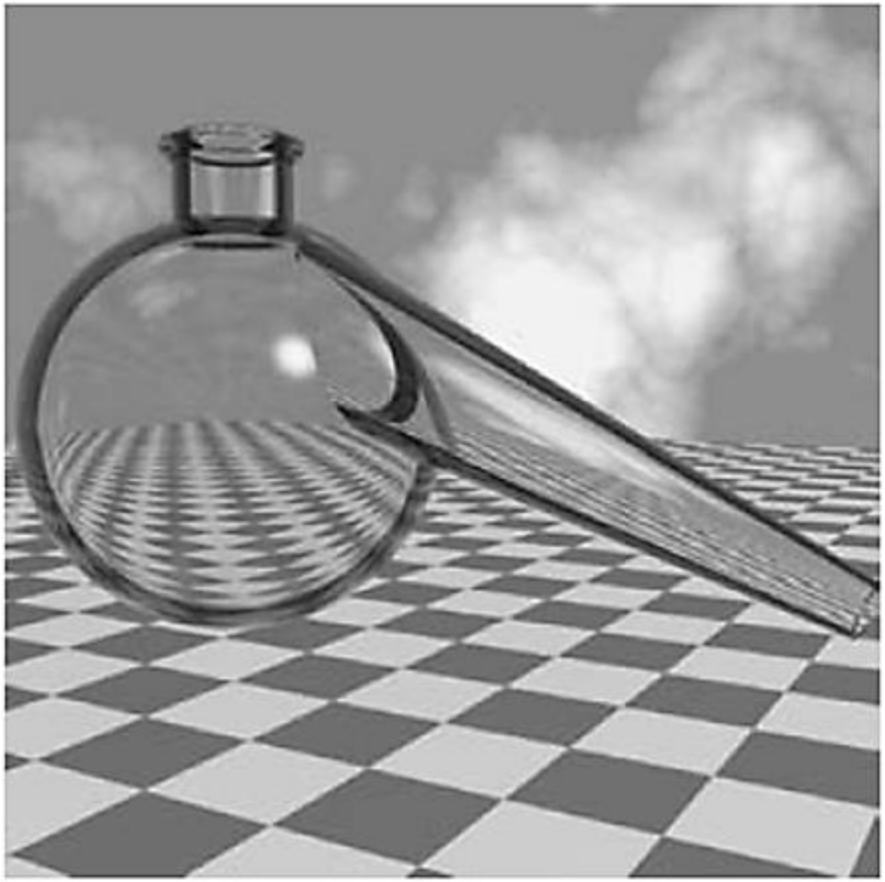
33% of the original image size

Anti-aliasing

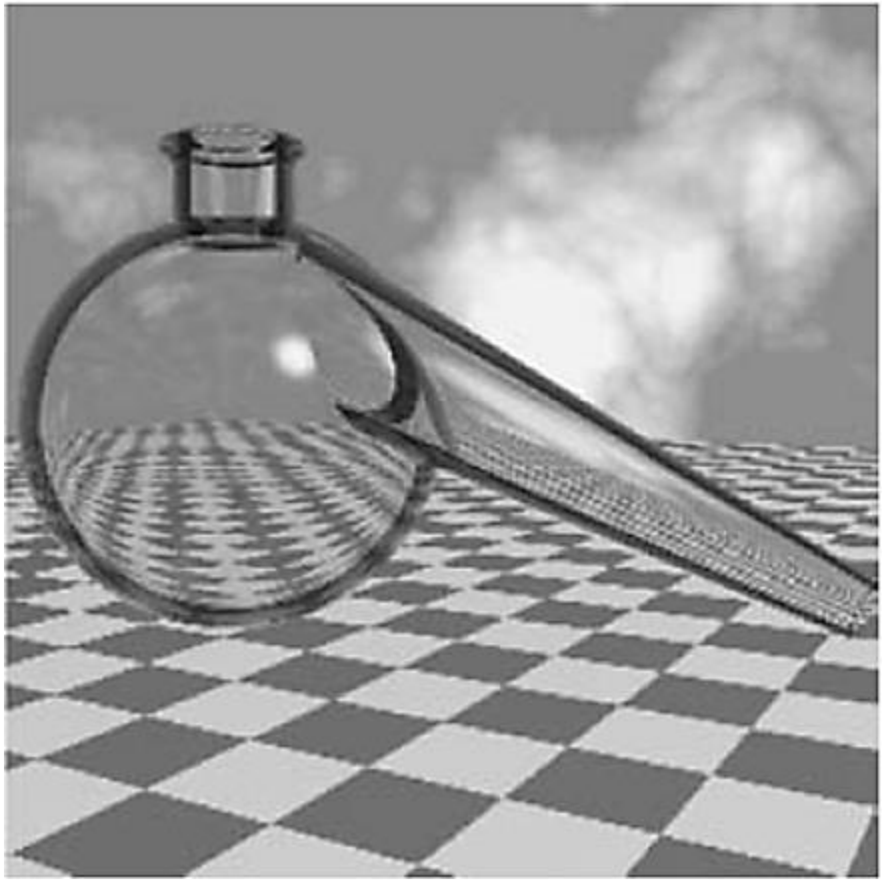
Original image

Preprocessed with a 3×3 averaging blur filter before resizing

Anti-aliasing: Jaggies

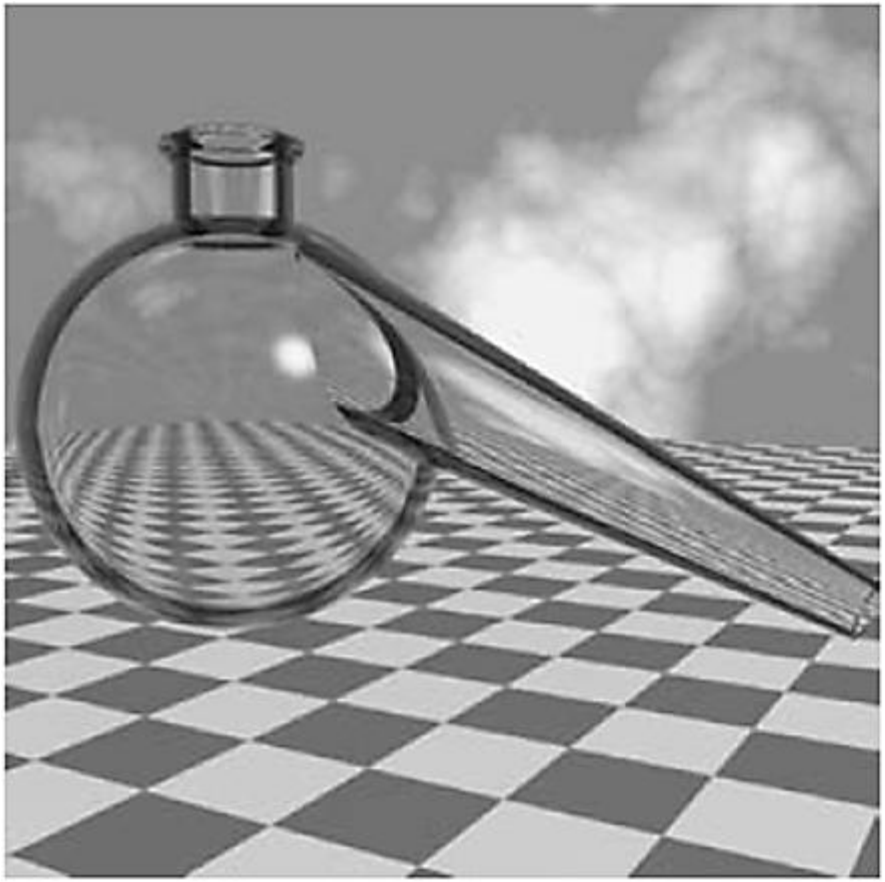


Original image

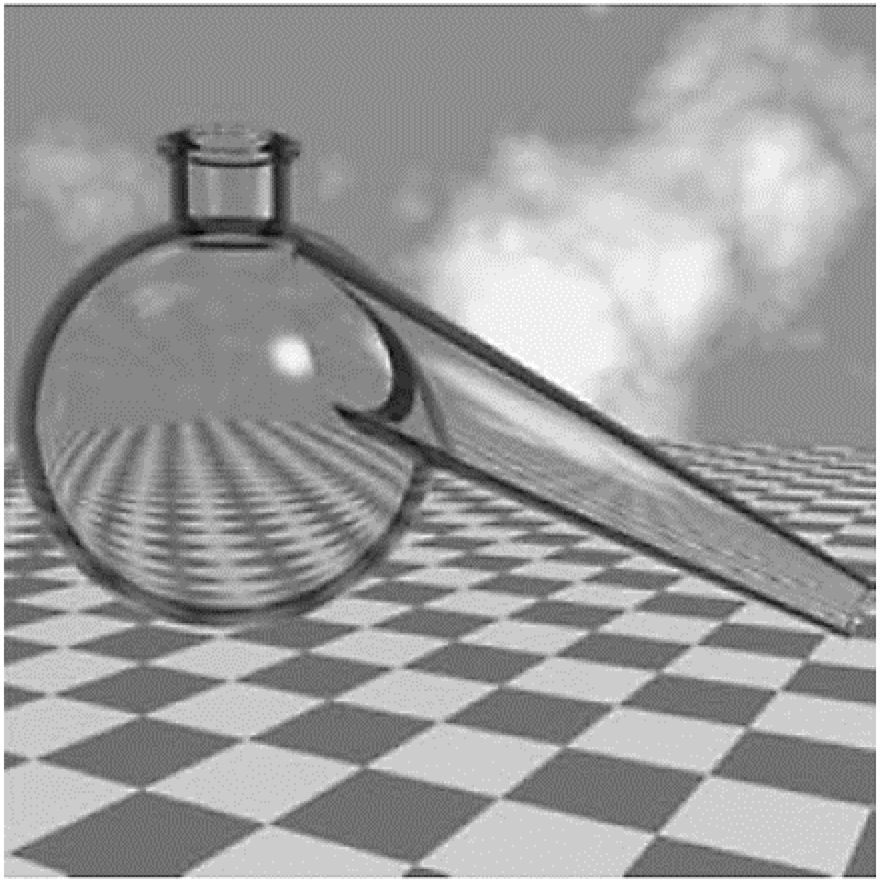


25% of the original image
Bilinear Interpolation

Anti-aliasing: Jaggies



Original image



Preprocessed with a 5×5 averaging blur filter

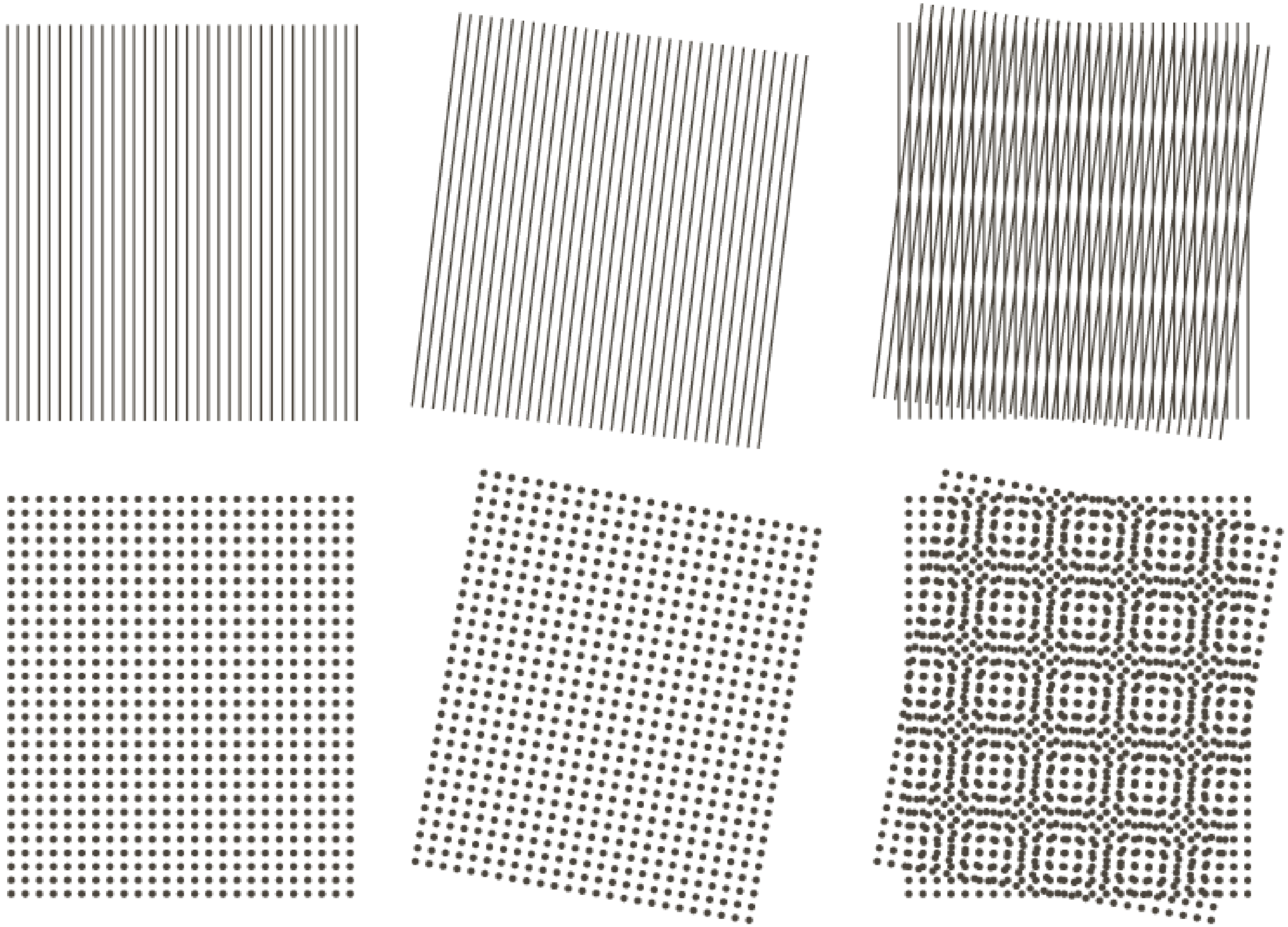
Moiré Patterns

Moiré Patterns

- In digital image processing, moiré-like patterns arise when sampling is done with **periodic components** (like pixels) whose *spacing* is comparable to the **spacing between samples**.

- Everyday occurrences:
  - Overlapping window screens
  - Newspaper printing
  - Interference between TV raster lines

Moiré Patterns



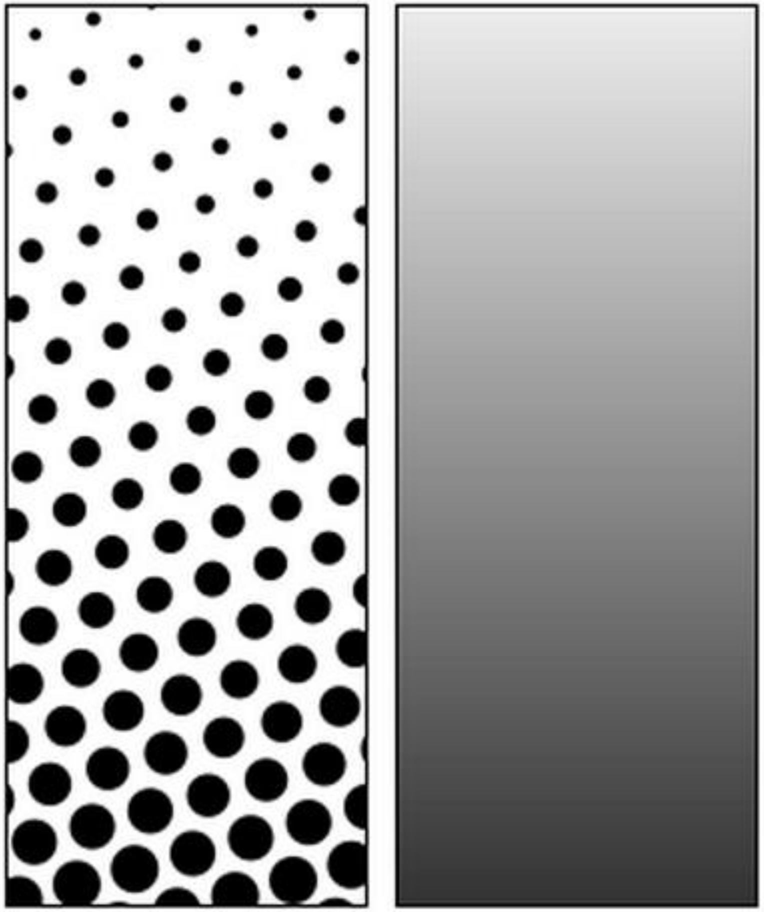
Individual patterns

Superimposition “multiplication”

Halftoning

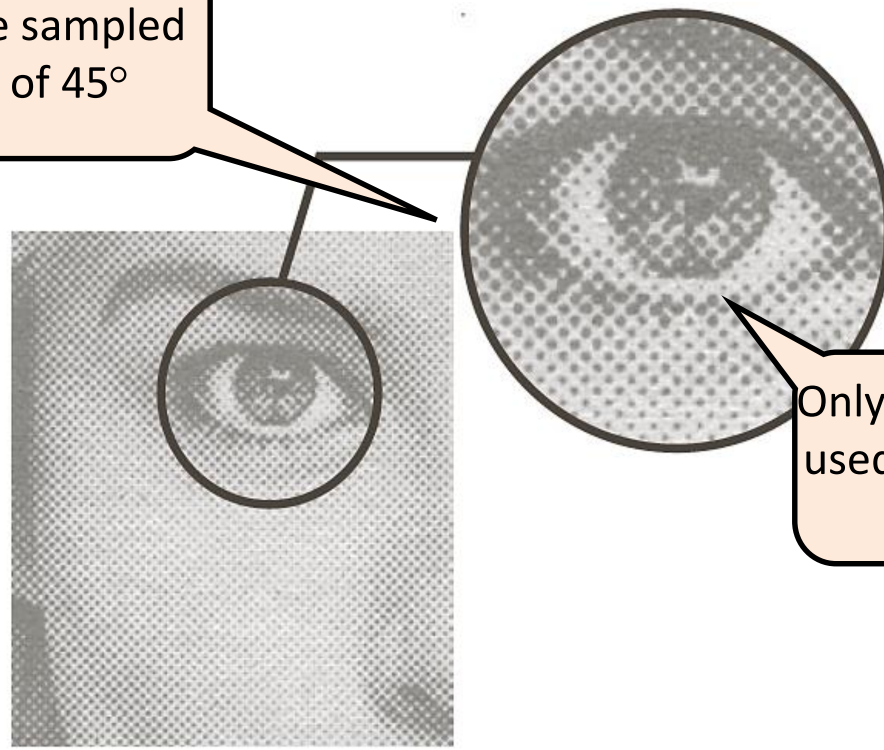
Halftoning

- Common printing practice for newspapers use *halftone dots*.
- Show gray levels with variable size dots of black ink.
- The *sampling grid* and *dot patterns* (oriented at $\pm 45°$) interact to create a uniform moiré-like pattern.



Halftoning

Gray levels are sampled at an angle of 45°



Only dots of one color are used to simulate the gray levels

400 dpi

Halftoning

246×168, 75 dpi

Next Week

- DFT of one variable
- DFT of two variables
- How to overcome Wraparound Error?
- Properties of the 2-D DFT and IDFT