Discrete Fourier Transform
(DFT)

Recap

- 1-D Sampling
  - Sampling Revisited
  - Sampling Theorem
  - Signal Recovery

- 2-D Sampling

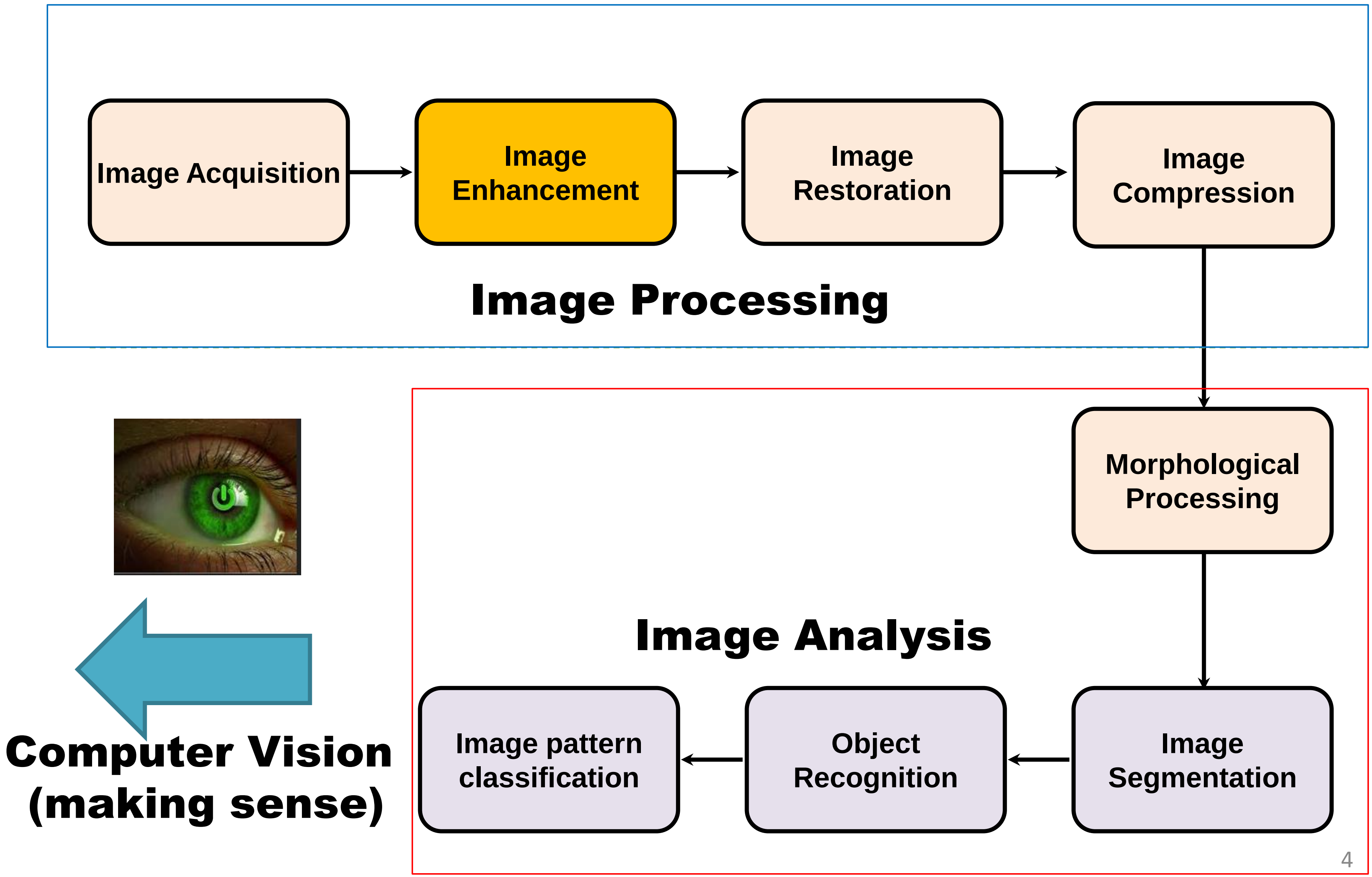
- Aliasing

- Aliasing in Images
  - How to reduce the effects of spatial aliasing?
  - Moiré Patterns
  - Halftoning

Lecture Objectives

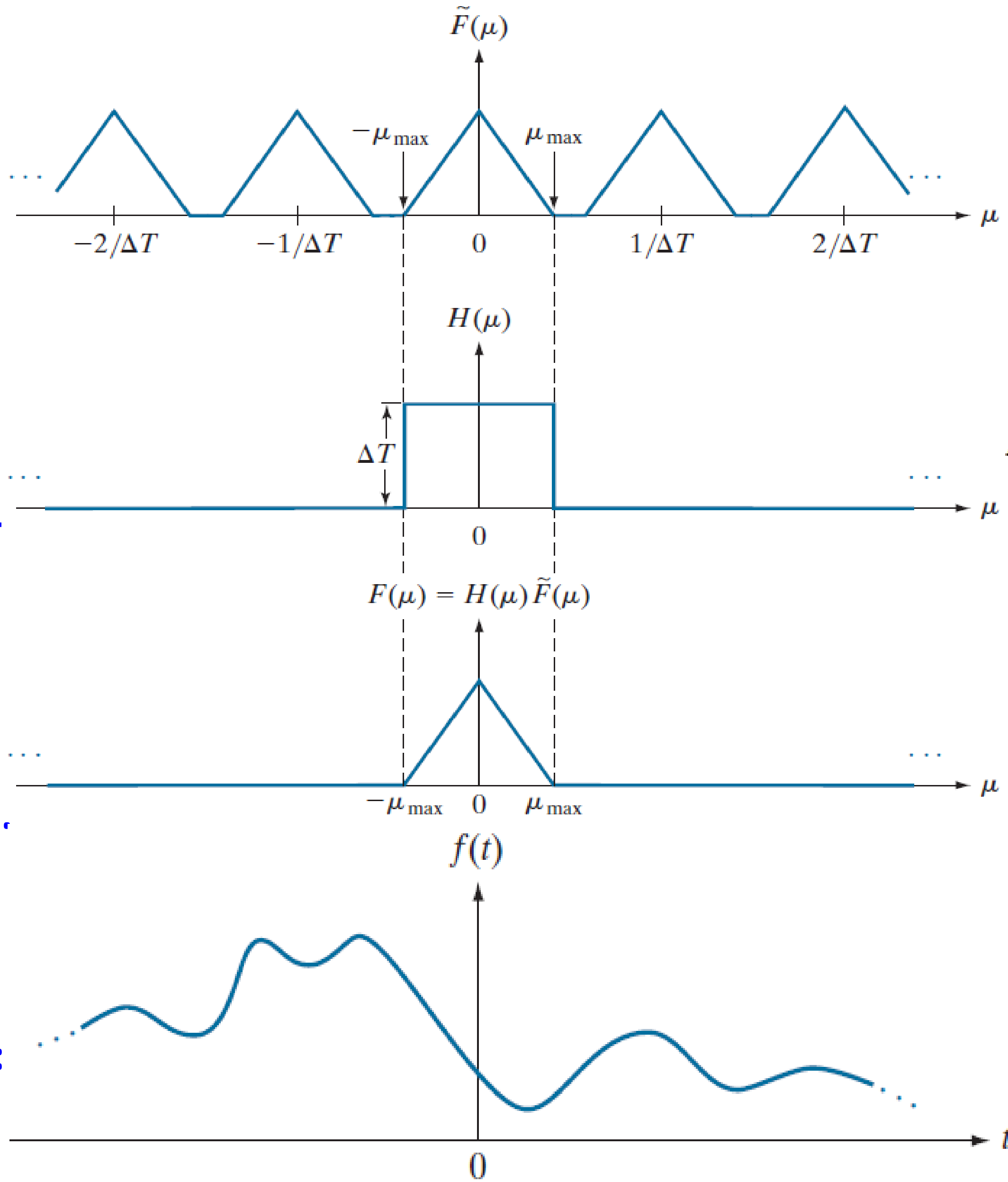
- DFT of one variable
- DFT of two variables
- How to overcome Wraparound Error?
- Properties of the 2-D DFT and IDFT

Key Stages in DIP



DFT of One Variable

Original Function *f(t)* Recovery



Fourier transform of a sampled, band-limited function

Step 1

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

(Ideal lowpass filter)

Step 2.

$$F(\mu) = H(\mu)\tilde{F}(\mu)$$

Step 3:

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Inverse Fourier transform

Why do we need DFT ?

- The **Fourier Transform** of a sampled, band-limited function extending from $-\infty$ to ∞ is a *continuous*, *periodic* function that also extends from $-\infty$ to ∞.

Fourier Transform of sampled function

$$\tilde{F}(\mu) = \Im\{\tilde{f}(t)\} = \Im\{f(t)s_{\Delta T}(t)\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

$$= (F \star S)(\mu)$$

where,

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

Points on the Fourier Transform of the sampled function at interval $1/\Delta T$

- In practice, we work with **finite number of samples**.

- **How to derive the DFT for such a signal?**

Sampled Function in Discrete Interval

- Let's assume that a function $\boldsymbol{f(t)}$ is nonzero over the **finite spatial interval** $\mathbf{[0, T]}$ and is bandlimited.

Then, the *sampled function* $\tilde{\boldsymbol{f}}(\boldsymbol{t})$ in the interval **[0, T]** is given as:

$$\tilde{f}(t) \approx \sum_{n=0}^{N-1} f(t)\delta(t - n\Delta T)$$

- where $\mathbf{\Delta T = T/N}$, with $\mathbf{N}$ being the total number of samples taken in $\mathbf{[0, T]}$.

Continuous Vs. Discrete Sampled Function

Continuous function

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

Vs.

$$\tilde{f}(t) \approx \sum_{n=0}^{N-1} f(t)\delta(t - n\Delta T)$$

Discrete function

Representing FT in terms of a Sampled Function

- An expression for Fourier Transform of sampled signal is given as:
  - In terms of continuous signal

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt$$

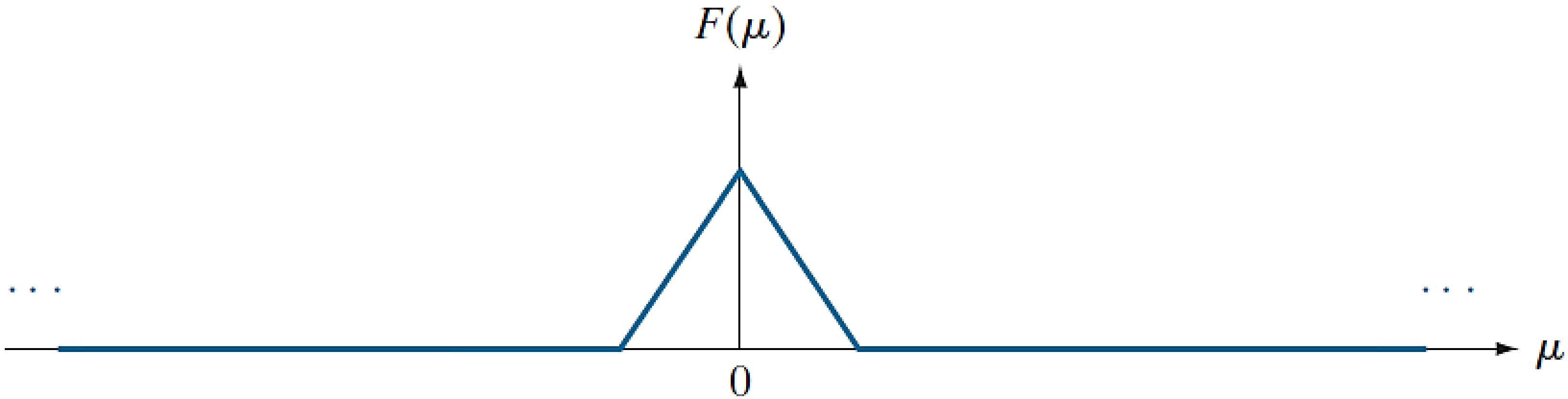
By substituting $\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{-\infty}^{\infty} f(t)\delta(t - n\Delta T)$

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

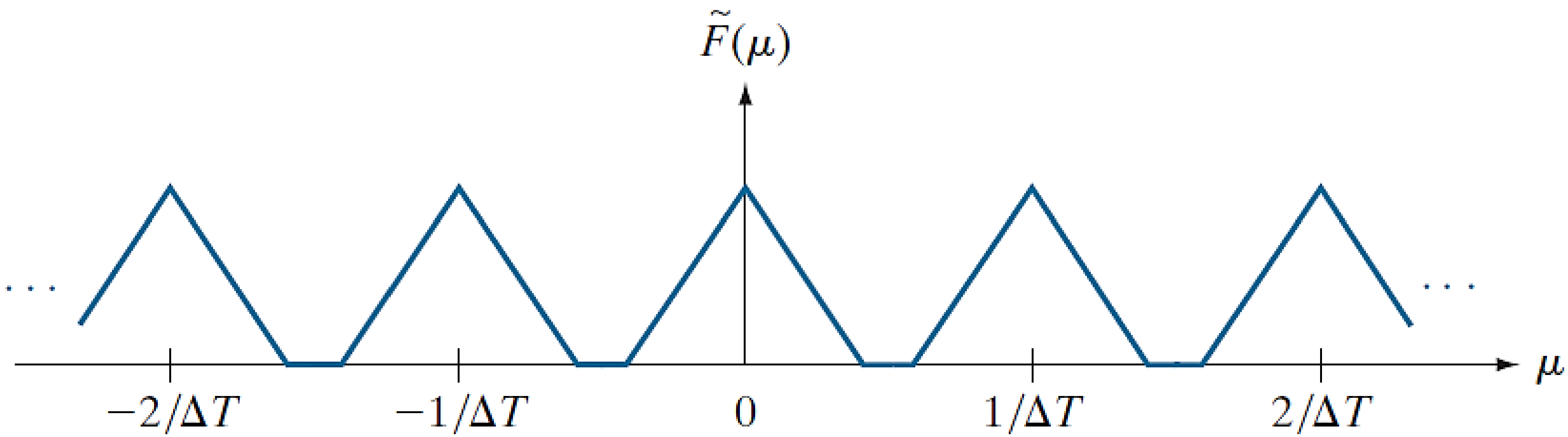
$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} f(t) e^{-j2\pi\mu n\Delta T}$$

Fourier Transform of the Sampled Function



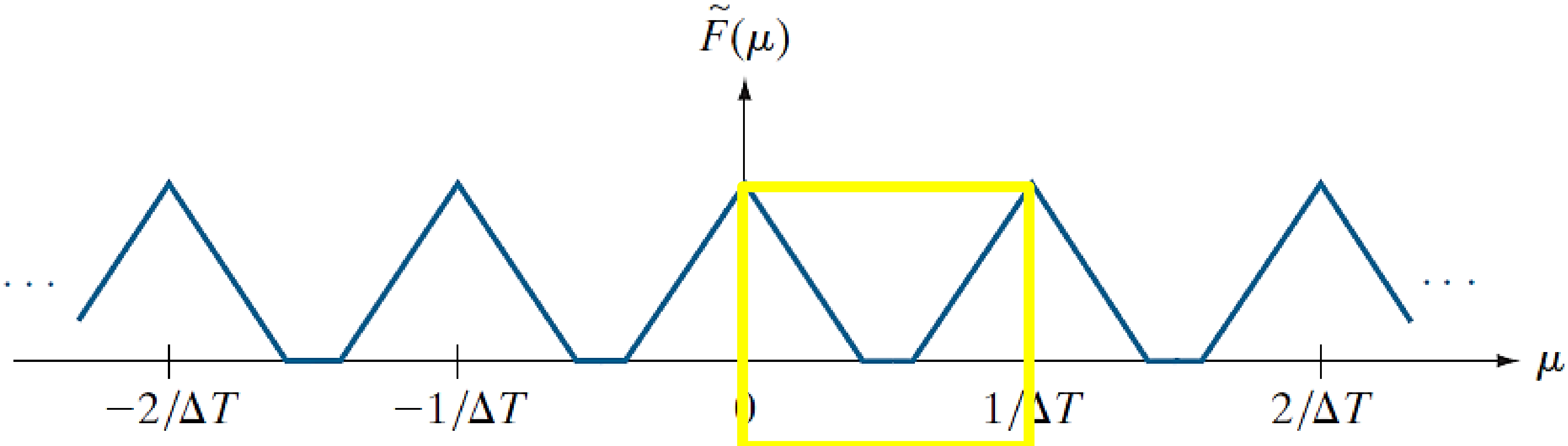
Fourier Transform $F(\mu)$ of the continuous function $f(t)$



Fourier Transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$

Building the 1-DFT

- Suppose we want to obtain **M** equally spaced samples of $\tilde{F}(\mu)$ taken over the **one period interval** from period: $\mu \in [0, 1/\Delta T]$



Fourier Transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$

- This is accomplished by taking the samples at the following frequencies:

$$\mu = \frac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$$

Building the 1-DFT

- Substituting $\mu = \dfrac{m}{M\Delta T} \qquad m = 0, 1, 2, \ldots, M-1$ in the following FT (*from slide-9*), and letting $\boldsymbol{F_m}$ denote the result gives us:

$$\tilde{F}(\mu) = \sum_{n=-\infty}^{\infty} f(t)\, e^{-j2\pi\mu n\Delta T}$$

$$\boxed{F_m = \sum_{n=0}^{M-1} f(t) e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1}$$

Discrete Fourier Transform

Building the 1-DFT

- Given a set $\{f_m\}$ consisting of $\mathbf{M}$ samples of a continuous function $f(n)$, the below equation yields a sample set $\{F_m\}$ of $\mathbf{M}$ complex, discrete values corresponding to the **DFT** of the *discrete input sample set*.

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

- Conversely, given $\{F_m\}$, we can *recover the sample set* $\{f_m\}$ using the **Inverse DFT**:

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

Building the 1-DFT

- Typically we use **x** as the discrete variable in **1D**, and **(x,y)** in **2D** and **u** as the frequency variable in **1D**, and **(u,v)** in **2D.** So, we represent **DFT** and **IDFT** equations respectively as follows:

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

Always use RADIAN convention while computing DFT & IDFT using Euler notation

$$\text{Euler's Relation: } e^{j\theta} = \cos\theta + j\sin\theta$$

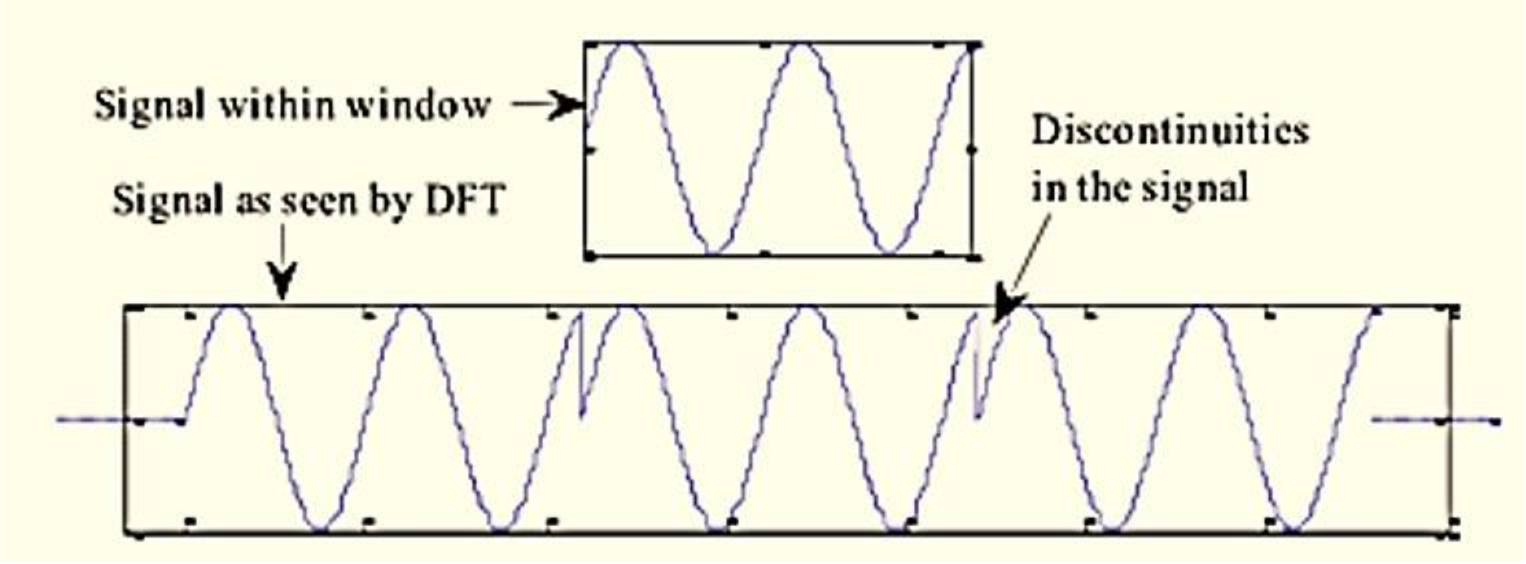
Periodic Properties of DFT Pair

- Both **DFT** and **IDFT** are infinitely periodic with period $\mathbf{M}$, i.e.

$$F(u)=F(u+kM)$$

and

$$f(x)=f(x+kM), \text{ where } k \text{ is an integer}$$



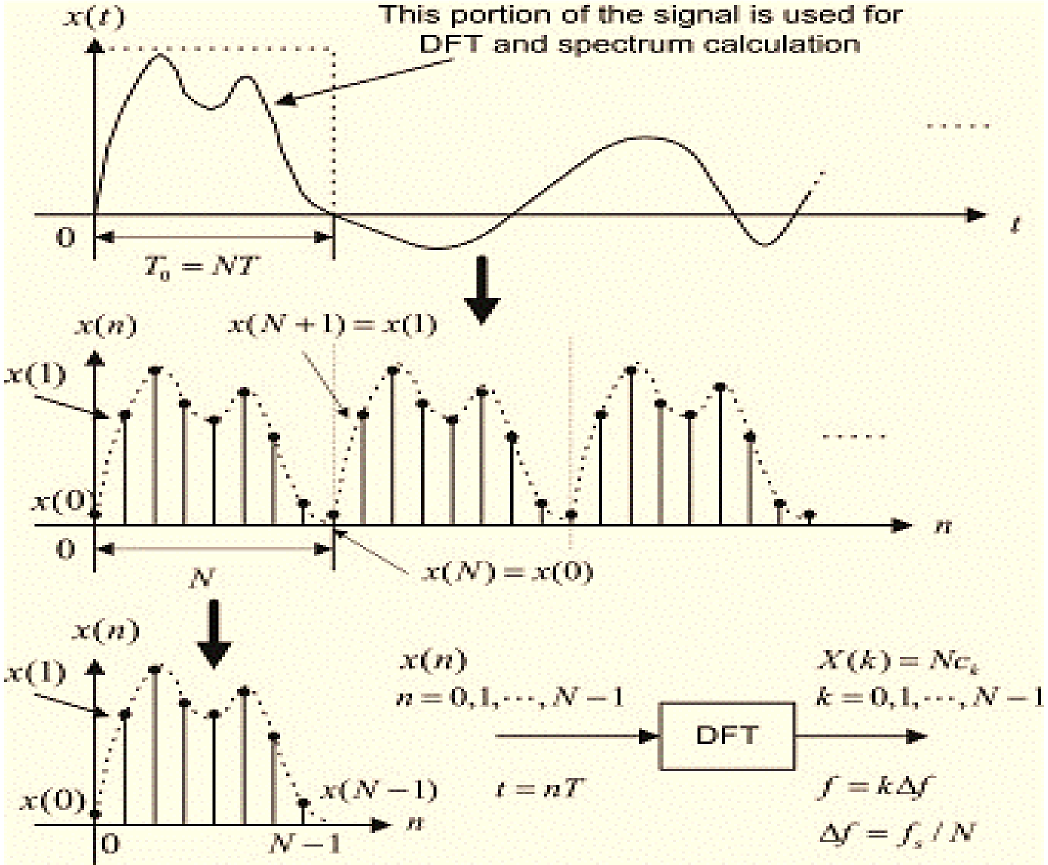
Periodic Properties of DFT Pair

- Both **DFT** and **IDFT** are infinitely periodic with period **M**, i.e.

$$F(u)=F(u+kM)$$

and

$$f(x)=f(x+kM),$$ where $\mathbf{k}$ is an integer



Discrete Equivalent of 1-D Convolution

- The basic convolution theorem for 1-D continuous variables is also applicable to 1-D discrete variables, with the following modification:

The Convolution Theorem for 1-D Continuous Variable :

$$(f \star h)(x) \Leftrightarrow (H \bullet F)(u)$$

$$(f \bullet h)(x) \Leftrightarrow (H \star F)(u)$$

The Convolution Theorem for 1-D Discrete Variable :

$$(f \star h)(x) \Leftrightarrow (H \bullet F)(u)$$

$$(f \bullet h)(x) \Leftrightarrow \frac{\mathbf{1}}{\mathbf{M}}(H \star F)(u)$$

Discrete Equivalent of 2-D Convolution

- The basic convolution theorem for 2-D continuous variables is also applicable to 2-D discrete variables, with the following modification:

The Convolution Theorem for 2-D Continuous Variable :

$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$$

$$(f \bullet h)(x,y) \Leftrightarrow (F \star H)(u,v)$$

The Convolution Theorem for 2-D Discrete Variables:

$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v) \longrightarrow$$

Foundation of linear filtering in the frequency domain

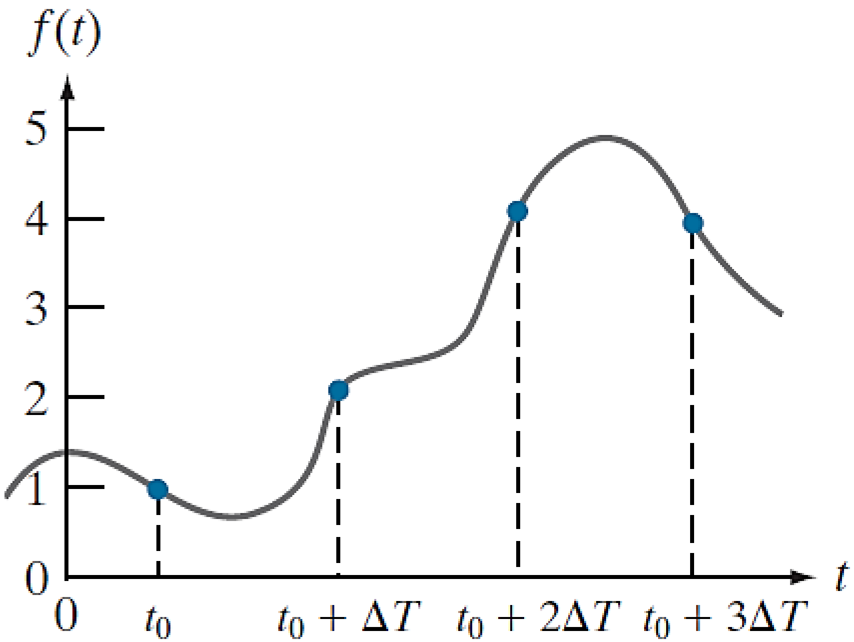
$$(f \bullet h)(x,y) \Leftrightarrow \frac{1}{MN}(F \star H)(u,v)$$

Facts of 1-DFT

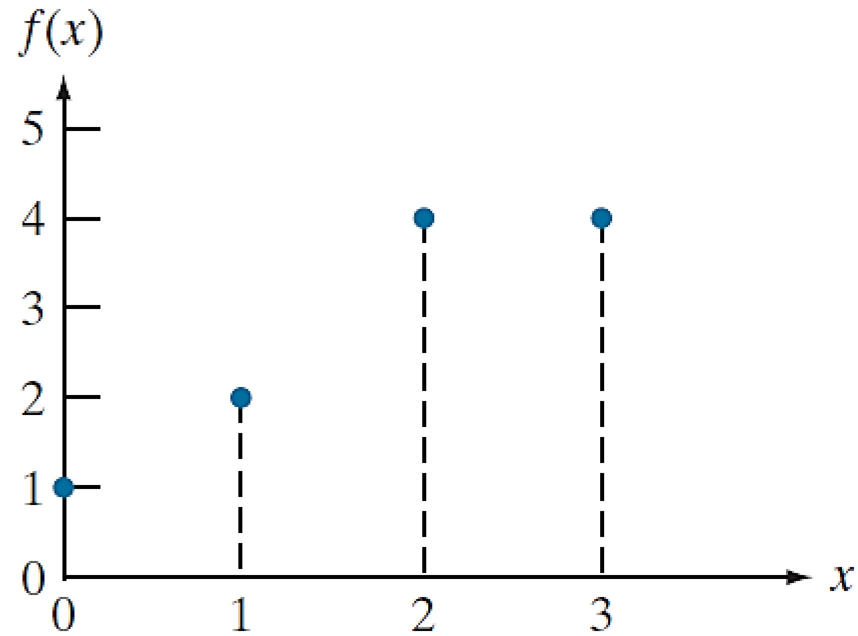
$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

- Neither expression depends on ΔT.
- DFT Pair is applicable to any finite signal that has been sampled uniformly.

Computing 1-DFT - Example



A continuous function sampled ΔT units apart

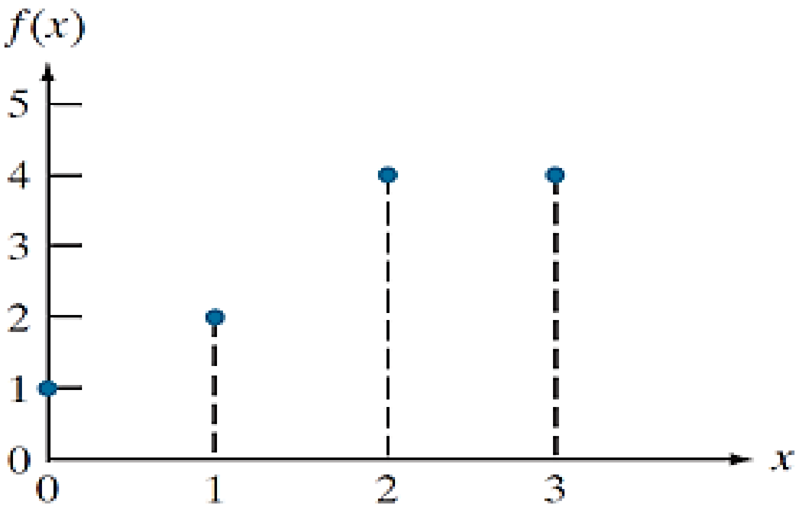


Samples in the *x*-domain (discrete)

Variable **t** is continuous, while **x** is discrete

- Four samples of a continuous function *f* (*t*) are taken ***T*** units apart.
- The values of ***x* [0, 1, 2, 3]** refer to the number of the samples in sequence

Computing 1-DFT - Example



$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

DFT

$$F(0) = \sum_{x=0}^{3} f(x) = \left[f(0) + f(1) + f(2) + f(3)\right] = 1 + 2 + 4 + 4 = 11$$

$$F(1) = \sum_{x=0}^{3} f(x) e^{-j2\pi(1)x/4} = 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

$$F(2) = -(1 + 0j)$$

Euler's Relation: $e^{j\theta} = \cos\theta + j\sin\theta$

$$F(3) = -(3 + 2j)$$

Computing 1-DFT - Example

$$F(0)=\sum_{x=0}^{3} f(x)=[f(0)+f(1)+f(2)+f(3)]=1+2+4+4=11$$

$$F(1)=\sum_{x=0}^{3} f(x)e^{-j2\pi(1)x/4}=1e^{0}+2e^{-j\pi/2}+4e^{-j\pi}+4e^{-j3\pi/2}=-3+2j$$

$$F(2)=-(1+0j)$$

$$F(3)=-(3+2j)$$

$$\boxed{f(x)=\frac{1}{M}\sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \quad x=0,1,2,\ldots,M-1}$$

IDFT

If we were given ***F(u)*** instead, we can compute its **inverse** as follows:

$$f(0)=\frac{1}{4}\sum_{u=0}^{3} F(u)e^{j2\pi u(0)}=\frac{1}{4}\sum_{u=0}^{3} F(u)=\frac{1}{4}[11-3+2j-1-3-2j]=\frac{1}{4}[4]=1$$

1-DFT Matrix

Frequency Spectrum | Multiplication Matrix | Time-Domain samples

$$
\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(N\text{-}2) \\ F(N\text{-}1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{4\pi}{N}} & \cdots & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{2(N-1)\pi}{N}} \\
1 & e^{-j\frac{4\pi}{N}} & e^{-j\frac{8\pi}{N}} & \cdots & e^{-j\frac{4(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} & \cdots & e^{-j\frac{2(N-2)^2\pi}{N}} & e^{-j\frac{2(N-2)(N-1)\pi}{N}} \\
1 & e^{-j\frac{2(N-1)\pi}{N}} & e^{-j\frac{4(N-1)\pi}{N}} & \cdots & e^{-j\frac{2(N-1)(N-2)\pi}{N}} & e^{-j\frac{(N-1)^2\pi}{N}}
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix}
$$

1-DFT - Practice

- Compute DFT and IDFT of the following two signals:
  - $f(x)=\cos(\pi x/2)$, $x=0,1,2,3$
  - $f(x)=2^x$, $x=0,1,2,3$

DFT of Two Variables

The 2-D Continuous Impulse and its Sifting Property

- The impulse, $\delta(t,z)$, of two continuous variables, ***t*** and ***z***, is defined as:

$$\delta(t,z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

and the area under curve is given as:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t,z)\; dt\; dz = 1$$

The 2-D Continuous Impulse and its Sifting Property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t,z)\;dt\;dz = f(0,0)$$

Sifting Property: Value of the function at the location of the impulse

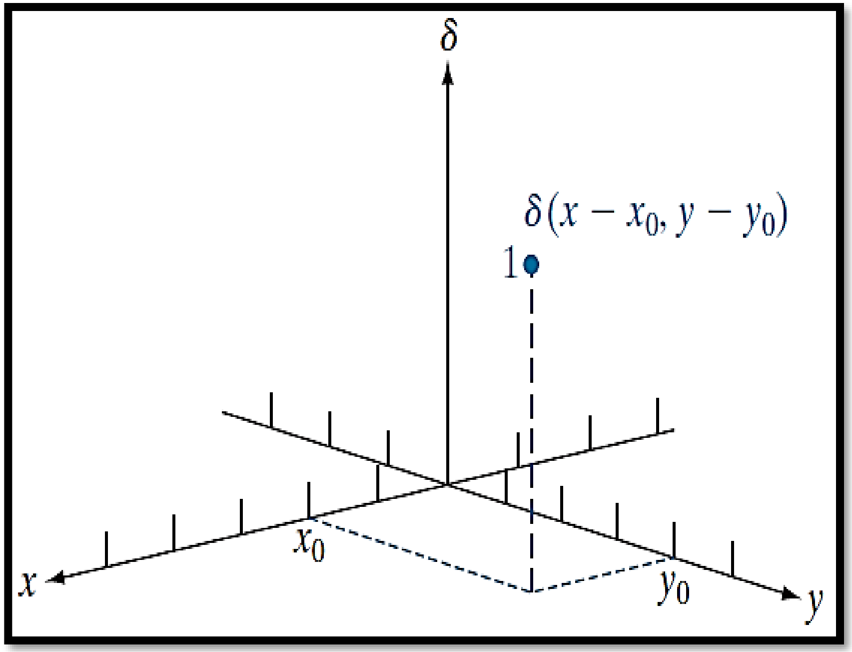
In the general case:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\,\delta(t-t_0,z-z_0)\;dt\;dz = f(t_0,z_0)$$

The 2-D Discrete Impulse and its Sifting Property

- The impulse $\delta(x,y)$ of two discrete variables, ***x*** and ***y***, is defined as:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$



and the sifting property is:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\ \delta(x,y) = f(0,0)$$

In the general case:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\ \delta(x-x_0, y-y_0) = f(x_0, y_0)$$

The 2-D Continuous Fourier Transform Pair

- Let *f*(*t*,*z*) be a continuous function of two continuous variables, ***t*** and ***z***.

- The two dimensional, continuous **Fourier transform pair** is given by:

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t + \nu z)}\,dt\,dz$$

and

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t + \nu z)}\,d\mu\,d\nu$$

where

— ***t*** and ***z*** the continuous *spatial* variables.

— ***μ*** and ***v*** are the *frequency* variables.

$$F(\mu) = \int_{-\infty}^{\infty} f(t)\,e^{-j2\pi\mu t}\,dt$$

1-D Fourier Transform

The 2-D Discrete Fourier Transform Pair

- Let ***f(x,y)*** be a **digital image** of size $M \times N$.

- The two dimensional **DFT pair** is given by:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

for **u** = 0, 1, 2,…,M–1 and **v** = 0, 1, 2,…, N-1

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

for **x** = 0, 1, 2,…,M–1 and **y** = 0, 1, 2,…, N-1

How to Overcome Wraparound Error?

(zero-padding)

2D Convolution Theorem

2D circular convolution

$$(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$

$$x = 0, 1, 2, \dots, M-1 \text{ and } y = 0, 1, 2, \dots, N-1$$

2D convolution theorem

$$(f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v) \longrightarrow$$

Foundation of linear filtering in the frequency domain

$$(f \bullet h)(x,y) \Leftrightarrow \frac{1}{MN}(F \star H)(u,v)$$

Convolution using Discrete Functions

- Consider two 1D functions ***f*** and ***h*** of same size
- Later, we will extend it to 2D….

400 samples

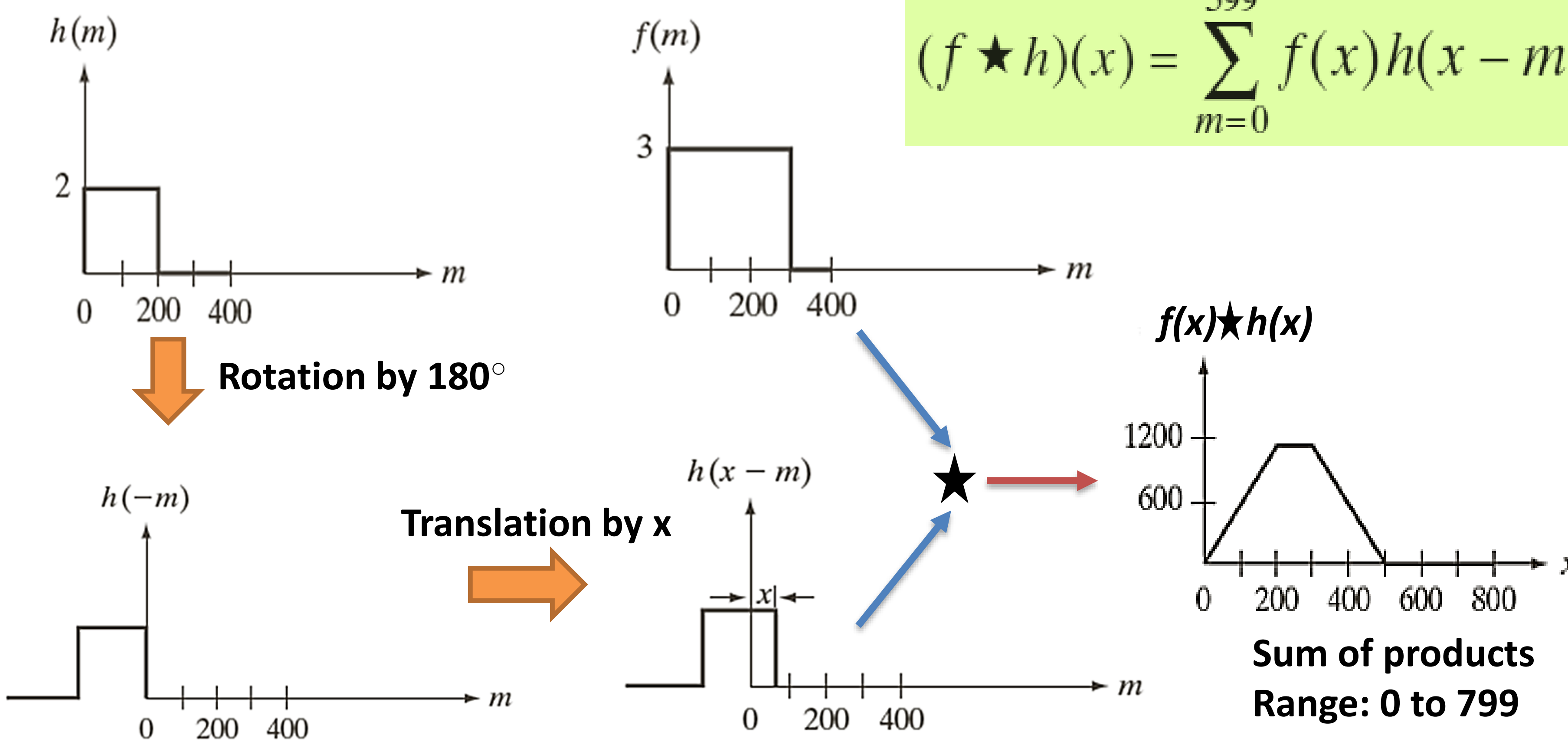
$$(f \star h)(x) = \sum_{m=0}^{399} f(x)h(x-m)$$

Multiplication for all x-values

Rotation of h(x) by 180^0

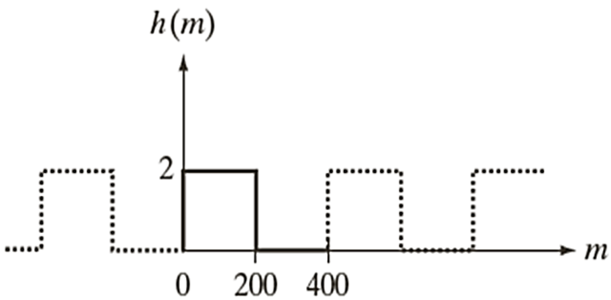
Example - Spatial Domain 1-D Convolution (Non-periodic functions)

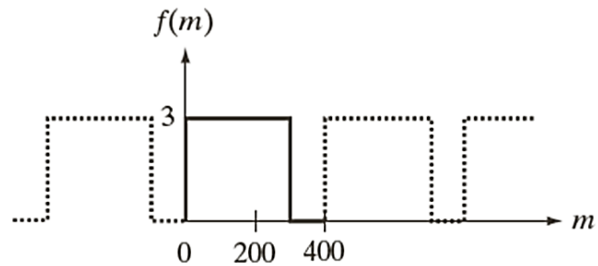
$$(f \star h)(x) = \sum_{m=0}^{399} f(x)h(x-m)$$

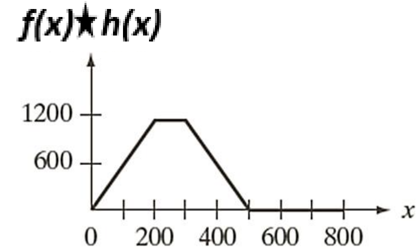


Sum of products
Range: 0 to 799

Example - DFT 1-D Convolution (Periodic functions)

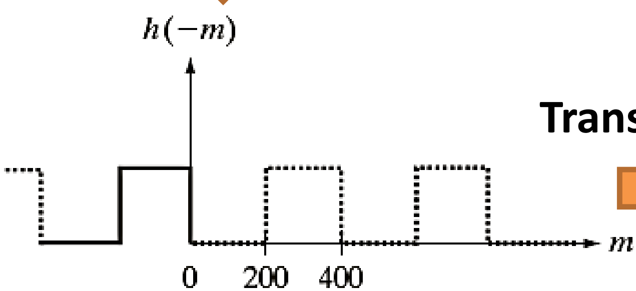




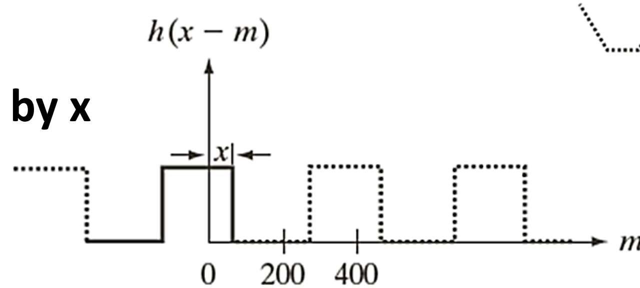


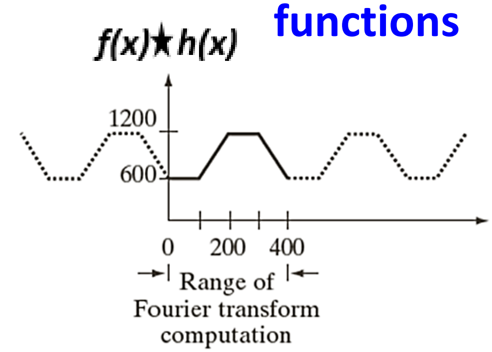
Convolution with **non-periodic functions**

Rotation by 180°



Translation by x





Convolution with **two periodic functions**

Recall:

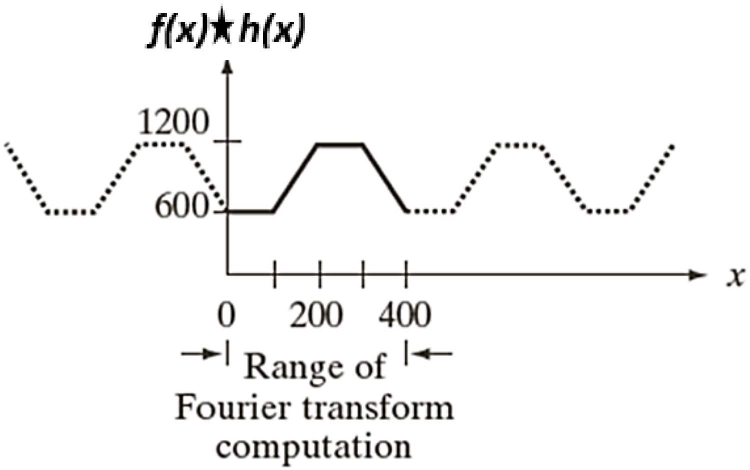
Both DFT and IDFT are infinitely periodic with period $\mathbf{M}$, i.e.

$$\mathbf{F(u)=F(u+kN)}$$

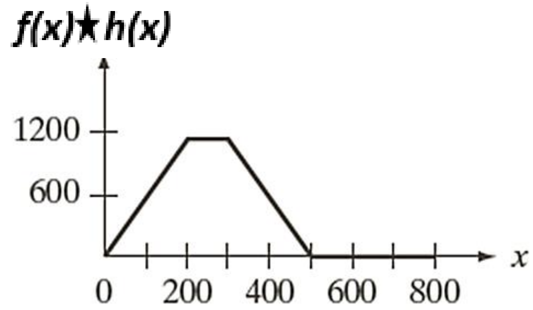
and

$\mathbf{f(x)=f(x+kN)}$, where $\mathbf{k}$ is an integer

Getting around the Wraparound Error in 1-D Convolution



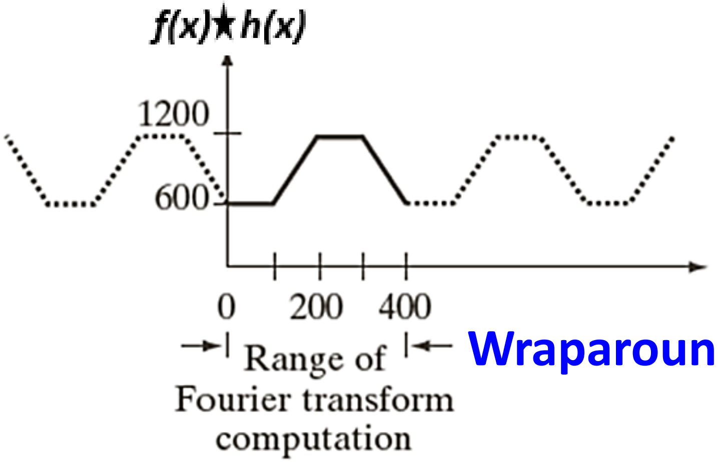
Wraparound Error



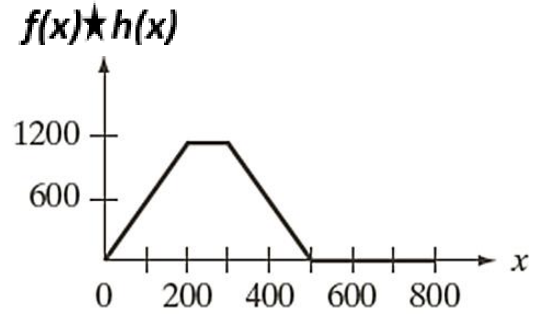
No wraparound Error

The closeness of the periods is such that *they interfere with each other* to cause what is commonly referred to as *wraparound error*.

Getting around the Wraparound Error in 1-D Convolution



Wraparound Error



No wraparound Error

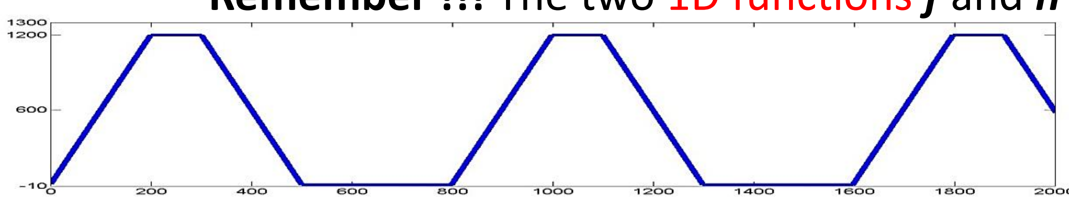
To fix:

$$f_p(x) = \begin{cases} f(x) & \text{if } 0 \le x \le M-1 \\ 0 & \text{if } M \le x \le P \end{cases}$$

$$h_p(x) = \begin{cases} h(x) & \text{if } 0 \le x \le M-1 \\ 0 & \text{if } M \le x \le P \end{cases}$$

$P \geq 2M - 1$

Remember !!! The two 1D functions ***f*** and ***h*** are of same size



wraparound Error Eliminated

Getting around the Wraparound Error in 2-D Convolution

$$\boxed{f_p(x,y) = \begin{cases} f(x,y) & \text{if } 0 \le x \le M-1 \text{ and } 0 \le y \le N-1 \\ 0 & \text{if } M \le x \le P \text{ or } N \le y \le Q \end{cases}}$$

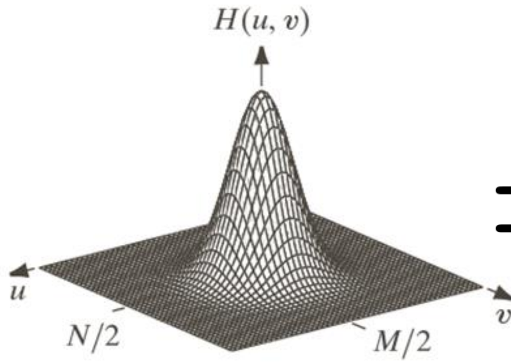
$$\boxed{h_p(x,y) = \begin{cases} h(x,y) & \text{if } 0 \le x \le M-1 \text{ and } 0 \le y \le N-1 \\ 0 & \text{if } M \le x \le P \text{ or } N \le y \le Q \end{cases}}$$

$P \geq 2M - 1$, and $Q \geq 2N - 1$

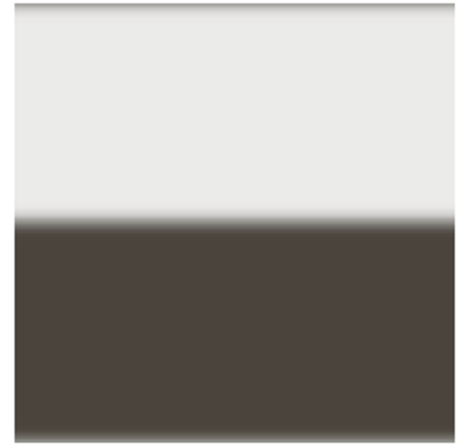
Note !!! The two 2D functions ***f*** and ***h*** are of same size

Getting around the Wraparound Error in 2-D Convolution

$*$



$=$

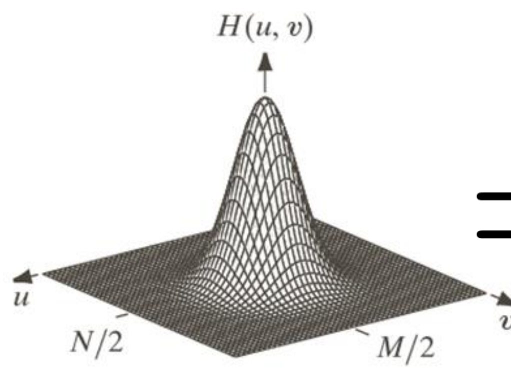


vertical white edges are not darkened

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

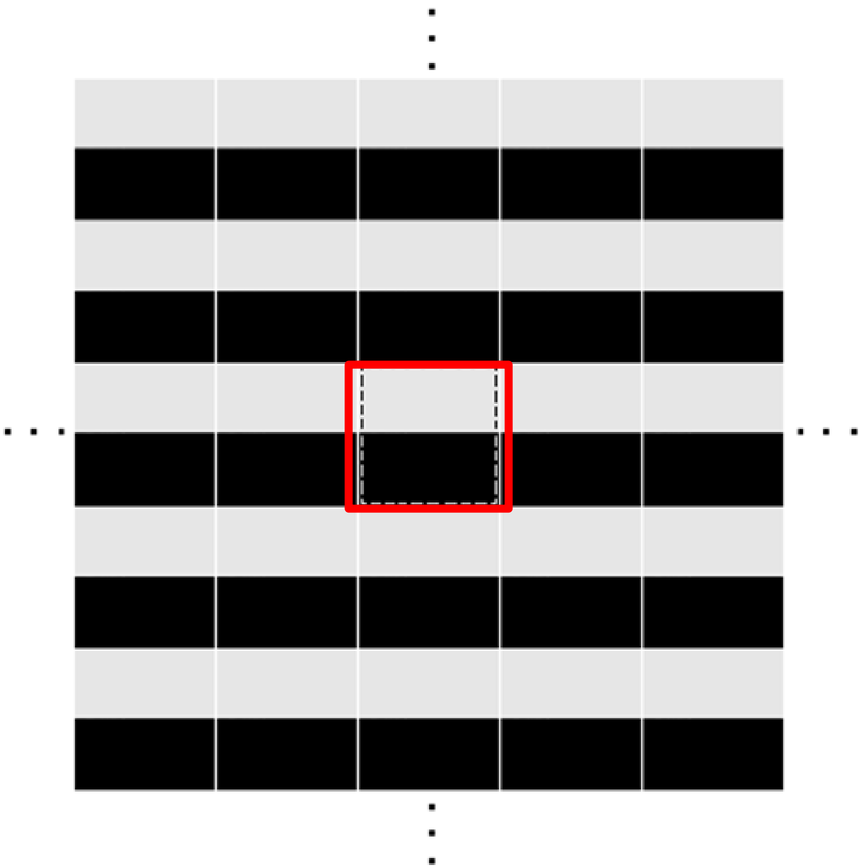
$*$



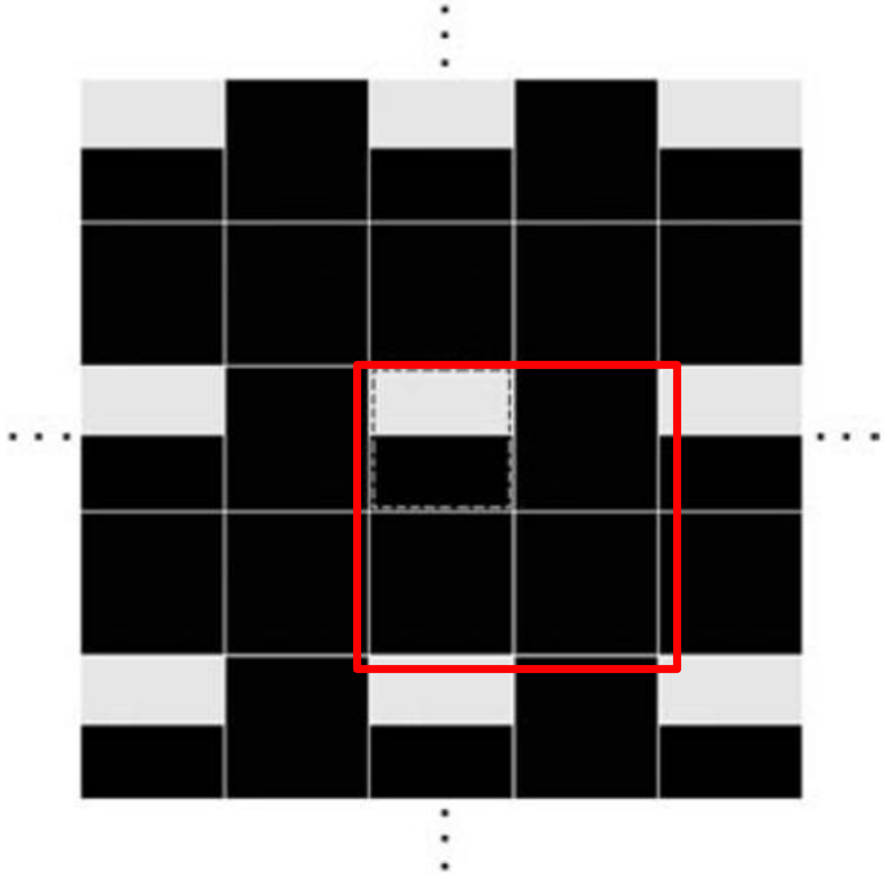
$=$

Uniform dark border resulting from zero padding

Getting around the Wraparound Error in 2-D Convolution



No Padding



Zero Padding

Properties of the 2-D DFT and IDFT

Odd and Even Functions

- **f(x)** is an **even function** when **f(-x) = f(x)**
  - $f(x) = x^2$
  - $f(x)=|x|$

- **f(x)** is an **odd function** when **f(-x) = -f(x)**
  - $f(x)=x$
  - $f(x)=x^3$

Symmetry Property

- Any *real or complex function*, $\mathbf{w(x,y)}$ can be represented as the sum of odd and even part, each of which can be real or complex.

- $\mathbf{w(x,y)=w_e(x,y)+w_o(x,y)}$

where,

$$w_e(x,y)=½[w(x,y)+w(-x,-y)]$$

and

$$w_o(x,y)= ½[w(x,y)-w(-x,-y)]$$

- $\mathbf{w_e(x,y) = w_e(-x,-y)}$ and $\mathbf{w_o(x,y) = -w_o(-x,-y)}$

Symmetry Property

- Suppose $\mathbf{w(x,y)=x^2+y^2+9xy+3x+6y+81}$
- Find $\mathbf{w_e(x,y)}$ and $\mathbf{w_o(x,y)}$

Symmetry Property

- Even functions are said to be ***symmetric***.
- Odd functions are ***antisymmetric***.
- In a **DFT** and **IDFT**, all the indices are **non-negative**: $[0, M-1]$ and $[0, N-1]$
  - The *point of symmetry* is then the **center of the interval** (not the origin)
  - $w_e(x,y) = w_e(M-x, N-y)$
  - $w_o(x,y) = -w_o(M-x, N-y)$

Symmetry Property

- **Product of functions**
  - even * even = even
  - odd * odd = even
  - even * odd = odd

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} w_e(x,y) w_o(x,y) = 0$$

Discrete Even and Odd 1-D Functions

Discrete Even Function :

- Let **f = { *f*(0), *f*(1), *f*(2), *f*(3) } = {2,1,1,1}**, where **M=4.**
  - To test for evenness, the condition ***f* (*x*) = *f* (4 − *x*)** must be satisfied for ***x* = 0,1, 2,3**. ▪ $w_e(x,y)=w_e(M-x,N-y)$
  - So, we require that: $f(0) = f(4)$, $f(1) = f(3)$, $f(2) = f(2)$, $f(3) = f(1)$
  - Because $f(4)$ is outside the range being examined and can be any value, the value of $f(0)$ is immaterial in the test for evenness.

- Any 4-point even sequences must have the form:

{a,b,c,b}

- **In general**:
  - when ***M*** is an **even number**, a **1-D even sequence** has *arbitrary values* for the points at locations *0* and *M/2*.
  - when ***M*** is an **odd number**, a **1-D even sequence** has *arbitrary value* for the point at location *0* , but the remaining terms form pairs with equal values.

Discrete Even and Odd 1-D Functions

Discrete Odd Function :

- Let $\boldsymbol{g} = \{g(0), g(1), g(2), g(3)\} = \{0, -1, 0, 1\}$, where $\mathbf{M=4}$.
  - Odd sequences have the property that their *first term*, $\boldsymbol{w_o(0,0)}$, is always **0**.
  - To test for oddness, the condition $\boldsymbol{g(x) = -g(4-x)}$ must be satisfied for $\boldsymbol{x = 1,2,3}$. ▪ $w_o(x,y) = -w_o(M-x, N-y)$
  - So, we require that: $g(1) = -g(3)$, $g(2) = -g(2)$, $g(3) = -g(1)$
  - All we have to do for $x = 0$ is to check that $g(0) = 0$.

- Any 4-point odd sequences must have the form:

$$\{0,-b,0,b\}$$

- **In general:**
  - when $\boldsymbol{M}$ is an **even number**, a **1-D odd sequence** has always **zero** values for the points at locations 0 and $M/2$.
  - when $\boldsymbol{M}$ is an **odd number**, a **1-D odd sequence** has its 1st term as 0 , but the remaining terms form pairs with equal value but opposite signs.

Discrete Even and Odd 1-D Functions

Some Observations :

- Oddness and evenness depends not only on the signal contents, but also on the length of the sequences:

  - {0, 1, 0, 1} is even
  - {0, -4, 0, -4} is even
  - {5, -4, 8, 7, 8, -4} is even
  - {5, -4, 8, 8, -4} is even

  - {0, -1, 0, 1} is odd
  - {0, 4, 0, -4} is odd
  - {0, -4, 8, 0, -8, 4} is odd
  - {0, -4, 8, -8, 4} is odd

  - {0, 1, 0, 2} is neither even nor odd
  - {0, -1, 0, 1, 0} is neither even nor odd

Even sequences {a,b,c,b}

- when ***M*** is an **even number**, a **1-D even sequence** has arbitrary values for the points at locations *0* and *M/2*.
- when ***M*** is an **odd number**, a **1-D even sequence** has arbitrary value for the point at location *0* , but the remaining terms form pairs with equal values.

Odd sequences {0,-b,0,b}

- when ***M*** is an **even number**, a **1-D odd sequence** has always **zero** values for the points at locations *0* and *M/2*.
- when ***M*** is an **odd number**, a **1-D odd sequence** has its 1st term as *0* , but the remaining terms form pairs with equal value but opposite signs.

Discrete Even and Odd 2-D Functions

$$f(x,y) = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Center location at (**M/2, N/2**)

For example:

{5, -4, 8, 7, 8, -4} is even

{0, -4, 8, 0, -8, 4} is odd

- Is f(x,y) even or odd? **It is an odd function**
- Adding a *single row and a column of zeros* would give a result that is **neither even or odd**.

Discrete Even and Odd 2-D Functions

- **In general:**

  – Inserting a 2-D array of *even dimensions* into a larger array of *zeros of even dimensions*, **preserves the symmetry** of the smaller array, provided that the centers coincide.

  – Similarly, inserting a 2-D array of *odd dimensions* into a larger array of *zeros of odd dimensions*, **preserves the symmetry** of the smaller array, provided that the centers coincide.

Conjugate Symmetry

- Suppose $\boldsymbol{z = x + iy}$ is a complex number, then the **complex conjugate** of this complex number $\boldsymbol{z}$ is denoted by:

$$\boldsymbol{z^{*} = x - iy}$$

Complex number	Conjugate complex number
-2 + 9i	-2 – 9i or -(2 + 9i)
2 – i√5	2 + i√5
2 – ib	2 + ib
2√2 + i(3√3)	2√2 – i(3√3)

Conjugate Symmetry

- A function f(x,y) is conjugate symmetric, if and only if **f*(x,y)=f(-x,-y)**, where f*(x,y) is the complex conjugate of f(x,y).
  - $f(x,y) = x^2 + y^2 + 10$
  - $f(x,y) = 3x^2 + iy$

$f(x,y) = x^2 + y^2 + 10 +$ i0 becomes a complex function

- A function f(x,y) is conjugate antisymmetric, if and only if **f*(-x,-y)=-f(x,y)**, where f*(x,y) is the complex conjugate of f(x,y).
  - $f(x,y) = 3x + 2y$
  - $f(x,y) = 10x + i5y^2$

$f(x,y) = 3x + 2y +$ i0 becomes a complex function

Conjugate Symmetry

- Let $r(x,y)$, $i(x,y)$ be the real and imaginary part of $\mathbf{f(x,y)}$, then $f(x,y)$ is conjugate symmetric if and only if $r(x,y)$ is even and $i(x,y)$ is odd.

- On the other hand, $\mathbf{f(x,y)}$ is conjugate antisymmetric if and only if $r(x,y)$ is odd and $i(x,y)$ is even.

Complex conjugate symmetry in DFT

- DFT of a real function $f(x,y)$ is conjugate symmetric, that is, $F^*(u,v)=F(-u,-v)$
  - Proof available in book
  - This property arises because the **image is real-valued function whereas the DFT operates on complex numbers**

- DFT of a imaginary function $f(x,y)$ is conjugate antisymmetric, that is, $F^*(-u,-v)=-F(u,v)$
  - Proof available in book

- *Complex conjugate symmetry* means that there exist negative frequencies which are **mirror images** of the corresponding positive frequencies as regarding the *Fourier spectrum*.

Properties of Discrete Fourier Transform

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

Properties of Discrete Fourier Transform - examples

Property	$f(x)$		$F(u)$
3	$\{1, 2, 3, 4\}$	$\Leftrightarrow$	$\{(10+0j),(-2+2j),(-2+0j),(-2-2j)\}$
4	$\{1j, 2j, 3j, 4j\}$	$\Leftrightarrow$	$\{(0+2.5j),(.5-.5j),(0-.5j),(-.5-.5j)\}$
8	$\{2, 1, 1, 1\}$	$\Leftrightarrow$	$\{5, 1, 1, 1\}$
9	$\{0, -1, 0, 1\}$	$\Leftrightarrow$	$\{(0+0j),(0+2j),(0+0j),(0-2j)\}$
10	$\{2j, 1j, 1j, 1j\}$	$\Leftrightarrow$	$\{5j, j, j, j\}$
11	$\{0j, -1j, 0j, 1j\}$	$\Leftrightarrow$	$\{0, -2, 0, 2\}$
12	$\{(4+4j), (3+2j), (0+2j), (3+2j)\}$	$\Leftrightarrow$	$\{(10+10j),(4+2j),(-2+2j),(4+2j)\}$
13	$\{(0+0j),(1+1j),(0+0j),(-1-j)\}$	$\Leftrightarrow$	$\{(0+0j),(2-2j),(0+0j),(-2+2j)\}$

{a,b,c,b}

Even sequence

{0,-b,0,b}

Odd sequence

1D and 2D DFT – IDFT pair

- **1D DFT**

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0, 1, 2, \ldots, M-1$$

- **2D DFT**

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

Fourier Spectrum and Phase Angle (important)

- 2D DFT in **polar form** is:
$$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

Where the **Magnitude** is: $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$

Fourier/Frequency spectrum

The **Phase** is: $\phi(u,v) = \arctan\left[\dfrac{I(u,v)}{R(u,v)}\right]$

Phase angle/phase spectrum

The **Power Spectrum** is:
$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

- All computations are carried out for the discrete variables **u** = 0, 1, 2,…,*M*–1 and **v** = 0, 1, 2,…, *N*–1. Therefore, magnitude, Phase and Power Spectrum are **arrays** of size ***M* × *N***.

Fourier Spectrum and Phase Angle

- The Fourier transform of a **real function** is conjugate symmetric**.** So, the **spectrum** has *even symmetry* *about the origin*:

$$|F(u,v)| = |F(-u,-v)|$$

- The **phase angle** exhibits *odd symmetry* about the origin:

$$\phi(u,v) = -\phi(-u,-v)$$

D C Component F(0,0) of Fourier Transform

- The 2-D DFT of a function is given by:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

Therefore,

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

Which means that the zero-frequency term of the DFT is proportional to the average of ***f*(*x*, *y*)**:

$$F(0,0) = MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$
$$= MN\bar{f}$$

Then,

$$|F(0,0)| = MN \left| \bar{f} \right|$$

|*F*(0,0)| typically is the largest component of the spectrum. Because frequency components ***u*** and ***v*** are zero at the origin, *F*(0,0) sometimes is called the ***dc component*** of the transform.

D C Component F(0,0) of Fourier Transform

Properties of F(0,0):

- **MN** is typically quite large.
- **Magnitude of the zero frequency** is the largest component of the Fourier Spectrum
  - From electrical engineering, "direct current" = current with zero frequency.

Translation Properties

- Multiplying **f(x,y)** with the exponential **shifts** (translates) the DFT to the point **(u_0, v_0)**:

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

- Multiplying **F(u,v)** with the negative of that exponential **shifts** (translates) the origin of **f(x,y)** to the location **(x_0, y_0):**

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

- Translation has **no effect** on the magnitude but has **effect** on the phase of ***F*(u,v)**.

- Rotating **f(x,y)** results in the **rotation** of its corresponding magnitude.

Periodicity

- The Fourier Transform and its Inverse of a **N × M** image repeat itself infinitely in both directions with periods **N** *and* **M**, i.e:

  $F(u,v) = F(u + k_1M, v + k_2N)$

  and

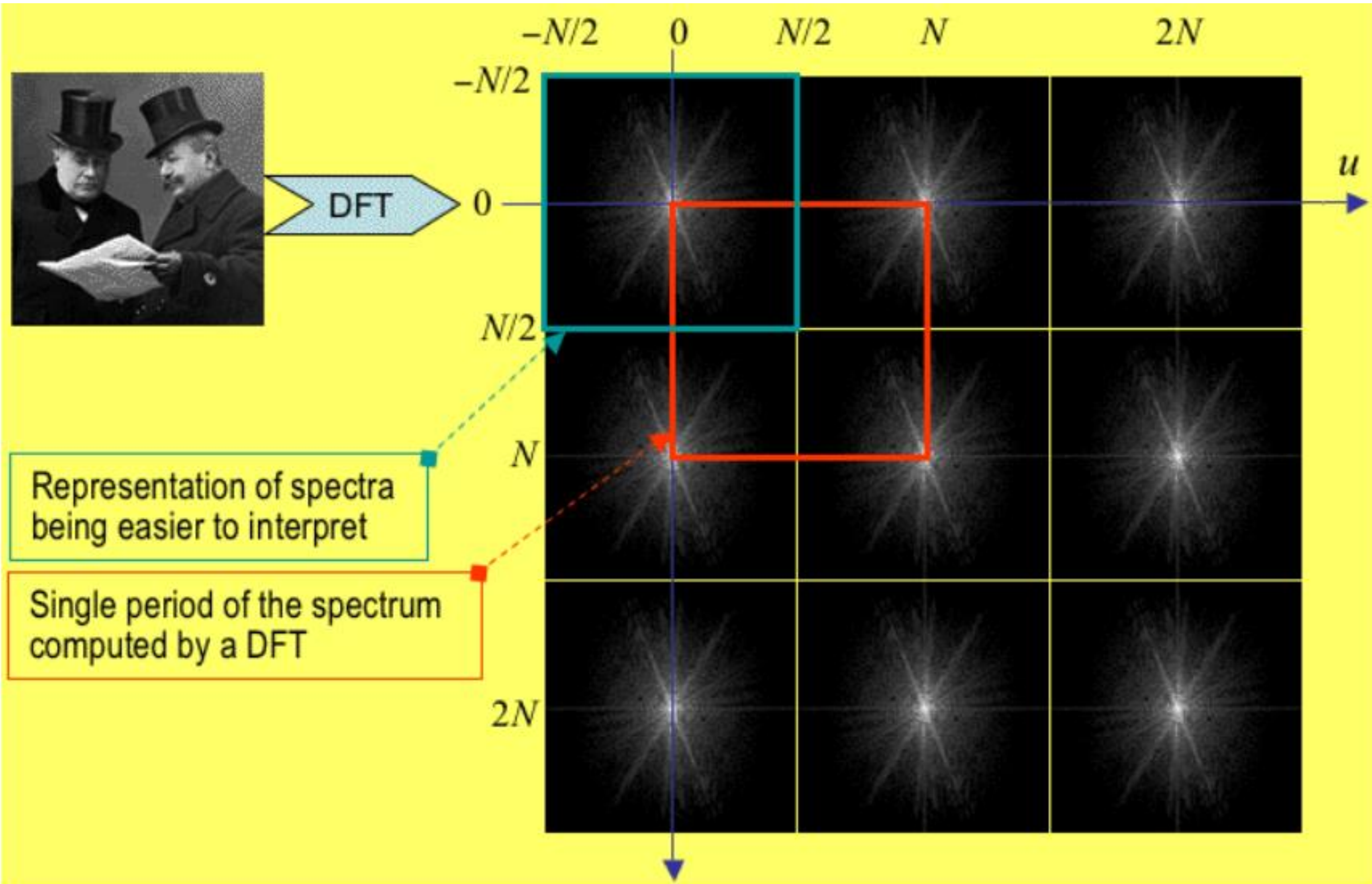
  $F(x,y) = F(x + k_1M, y + k_2N)$

  where $k_1, k_2 \in [-\infty, \ldots, -1, 0, 1, 2, \ldots, \infty]$.

The M×N block of the Fourier coefficients F(u,v) computed from an M×N image with the *2D DFT* is a single period from this *infinite sequence*.

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

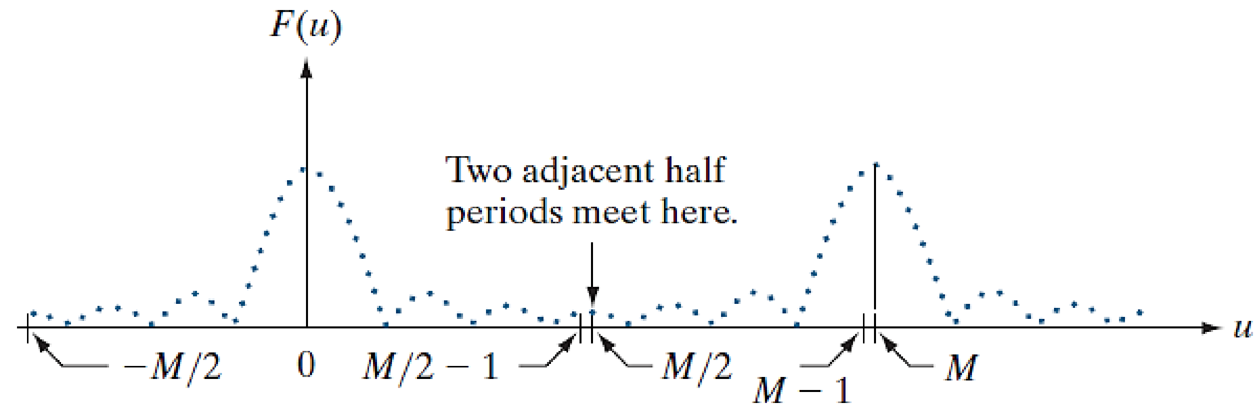
Periodicity - Extension to 2D



A portion of an infinite, periodic spectrum exhibiting complex conjugate symmetry, and the sample of the spectrum being computed by the DFT.

Importance of Periodicity of the DFT

- 1D Fourier spectrum:



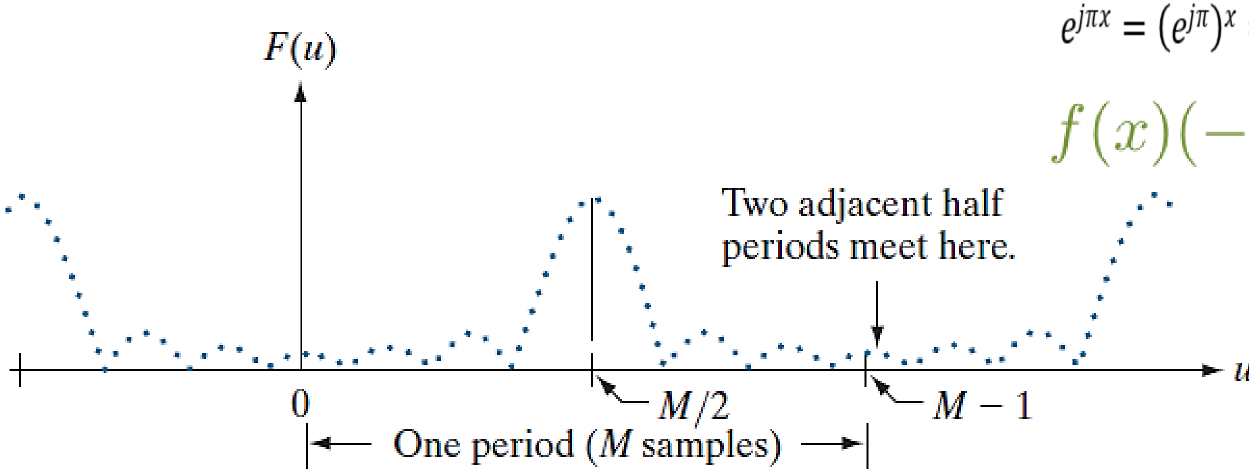
$$f(x)e^{j2\pi(u_0 x/M)} \Leftrightarrow F(u-u_0)$$

$$u_0 = M/2$$

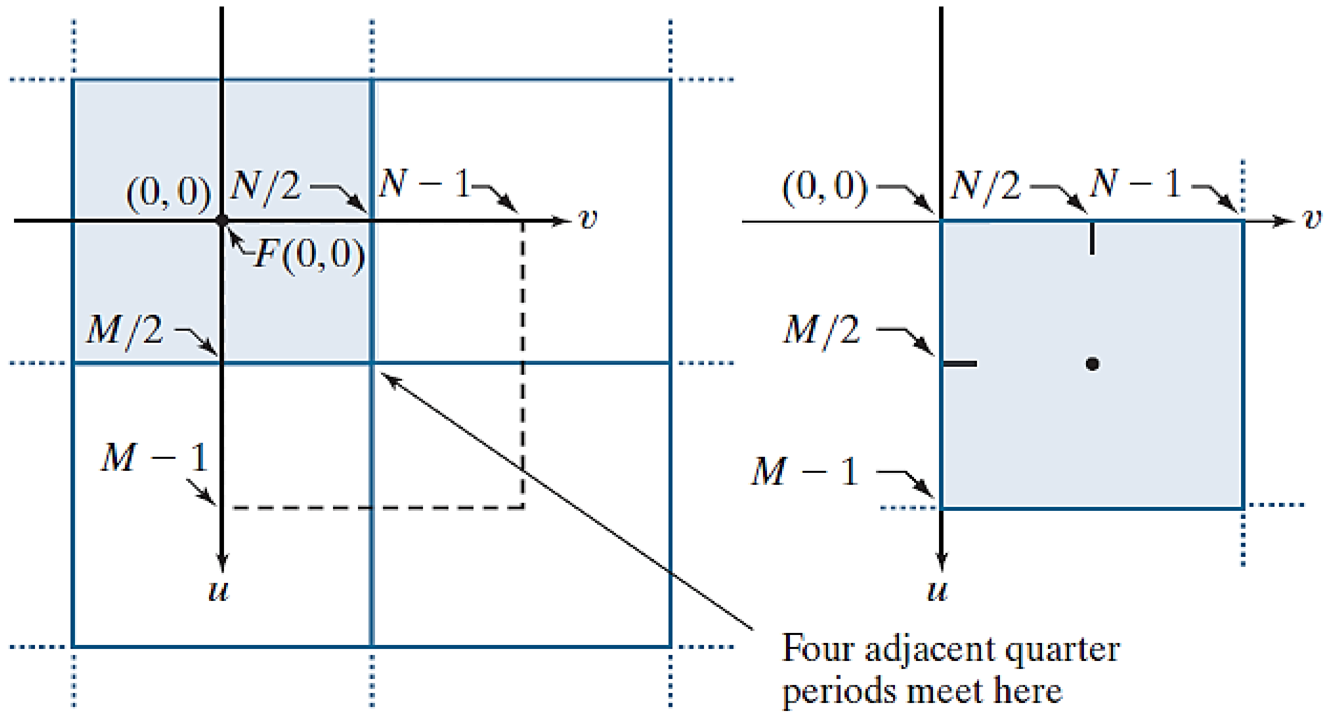
$$e^{j\pi x} = (e^{j\pi})^x = (\cos\pi + j\sin\pi)^x = (-1+0)^x = (-1)^x$$

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

- **What we want:**



Periodicity - Extension to 2D



= $M \times N$ data array computed by the DFT with $f(x, y)$ as input

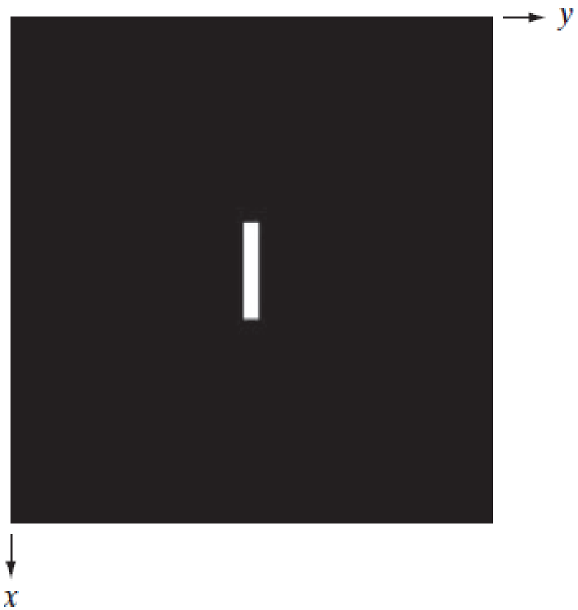
= $M \times N$ data array computed by the DFT with $f(x, y)(-1)^{x+y}$ as input

= Periods of the DFT

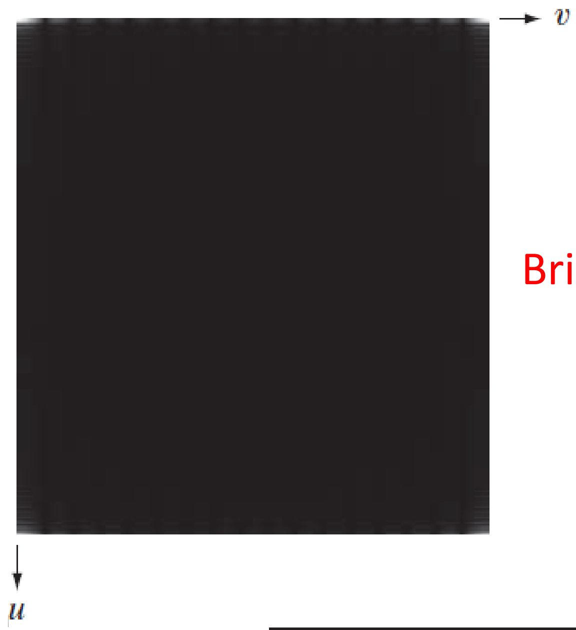
$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$

Example of Fourier Spectrum



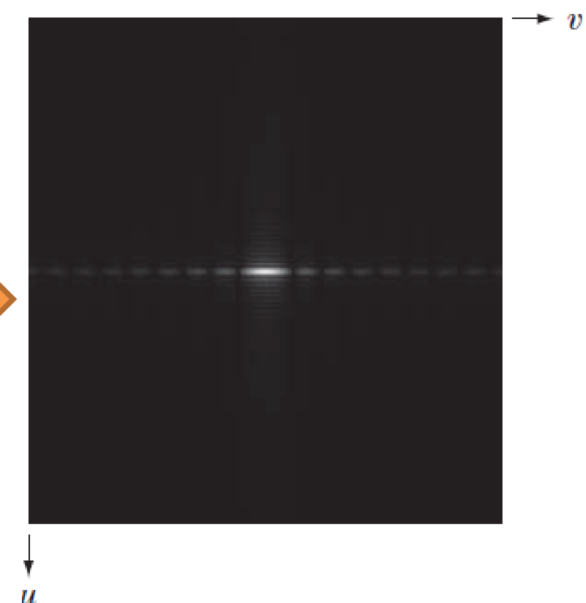
DFT



Fourier Spectrum -
Bright spots in the corners

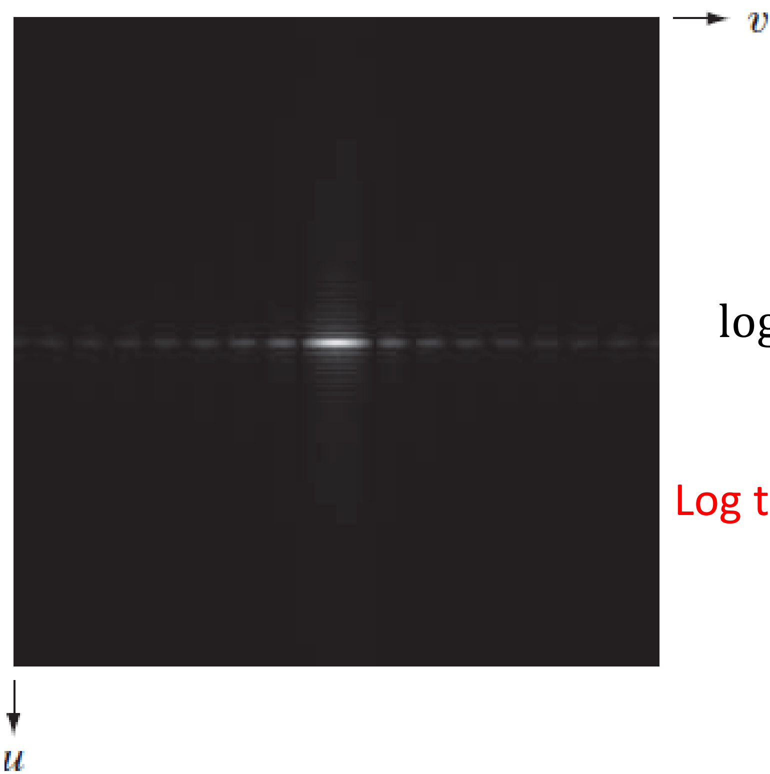
$\times (-1)^{x+y}$

DFT



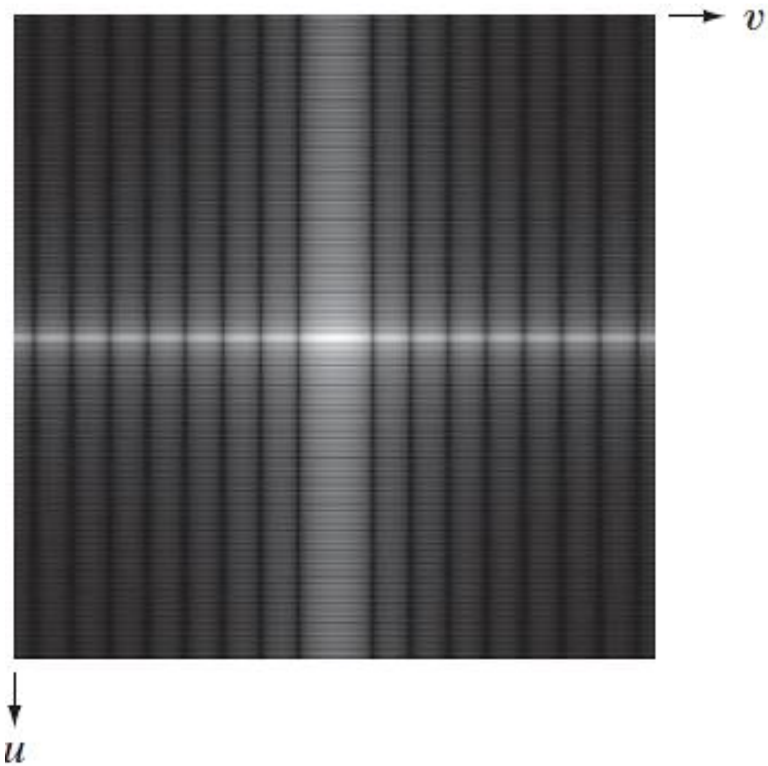
Centered
Fourier
Spectrum

Example of Fourier Spectrum

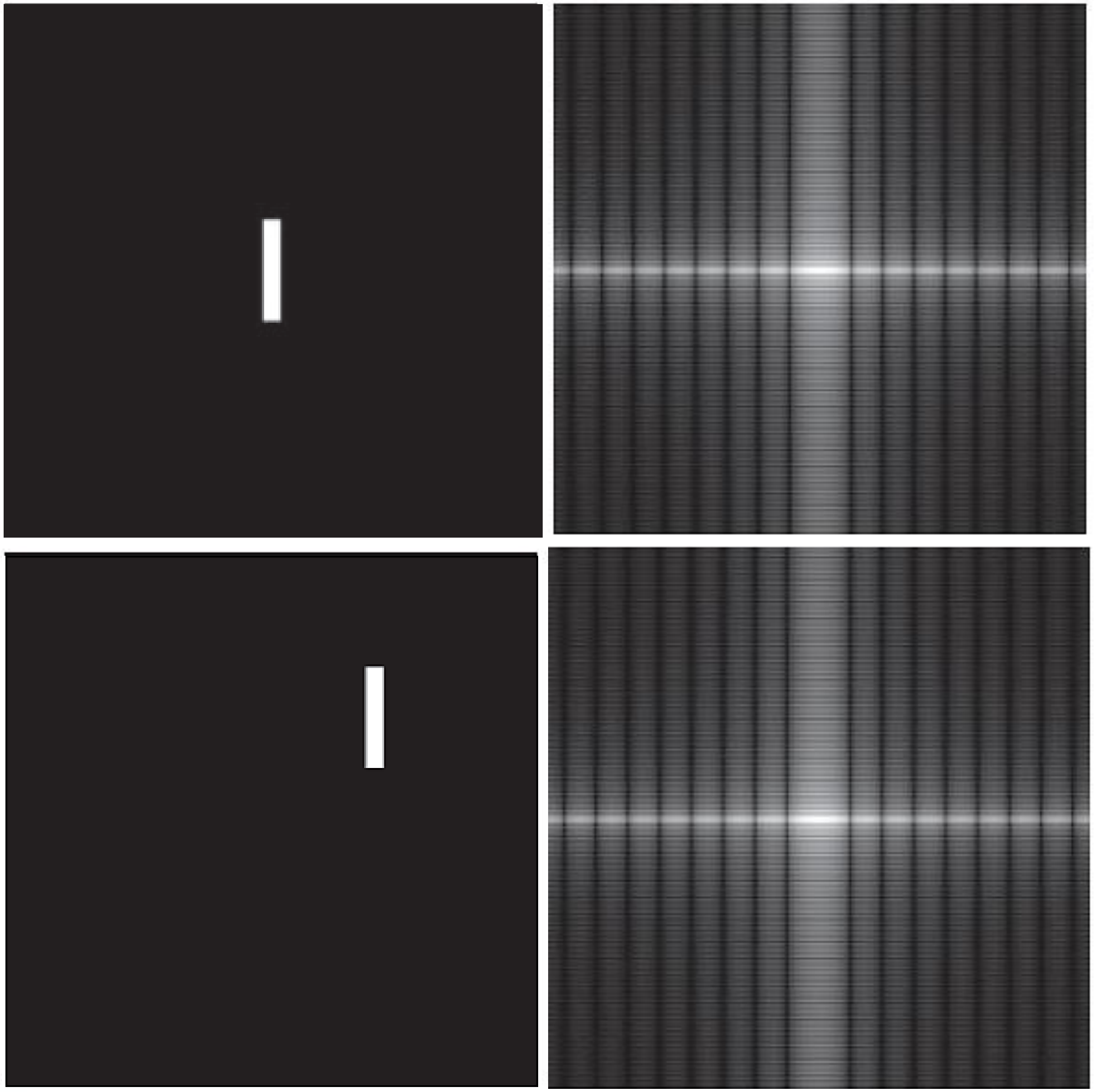


$\log|F(u,v)+1|$

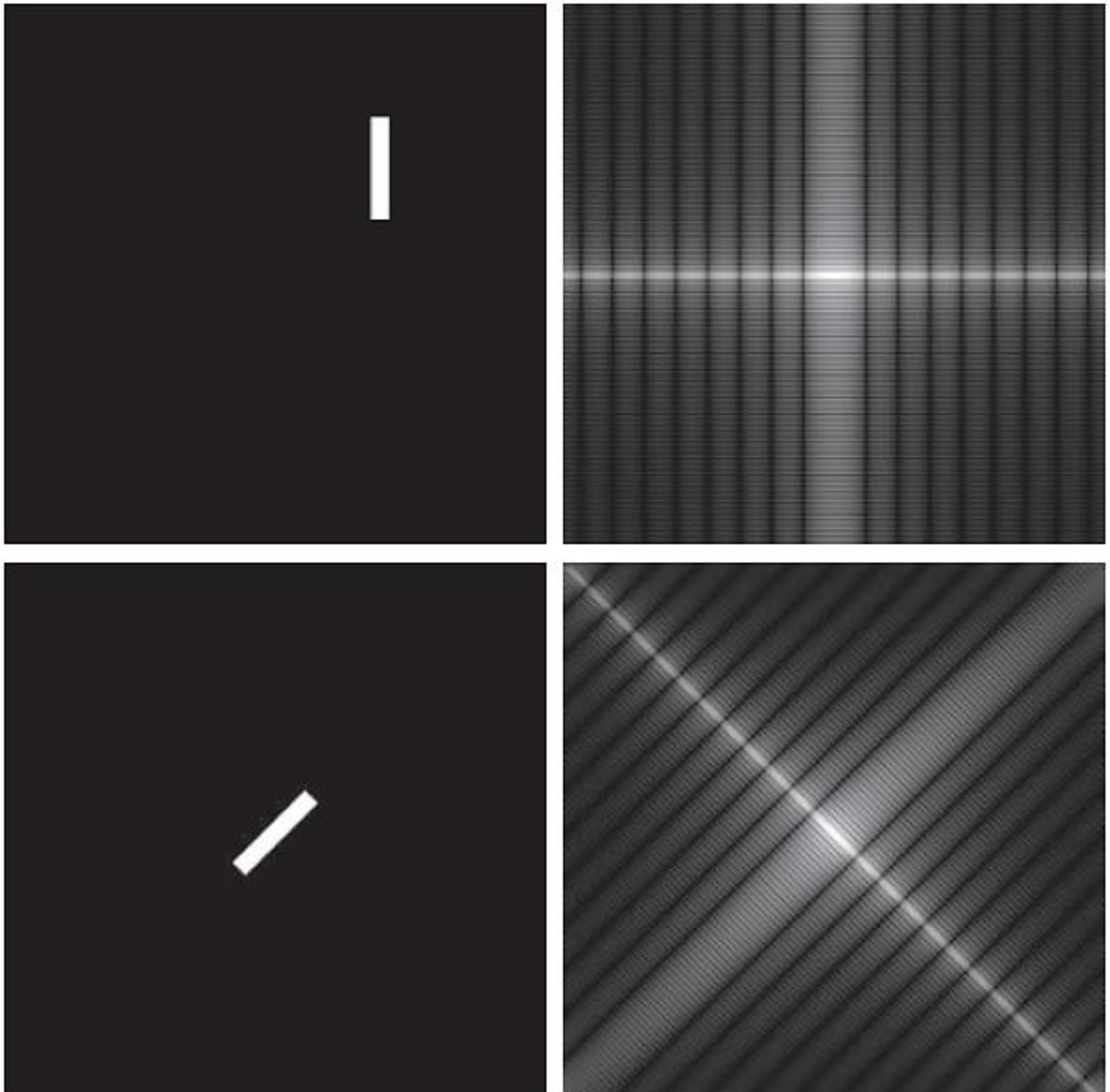
Log transformation



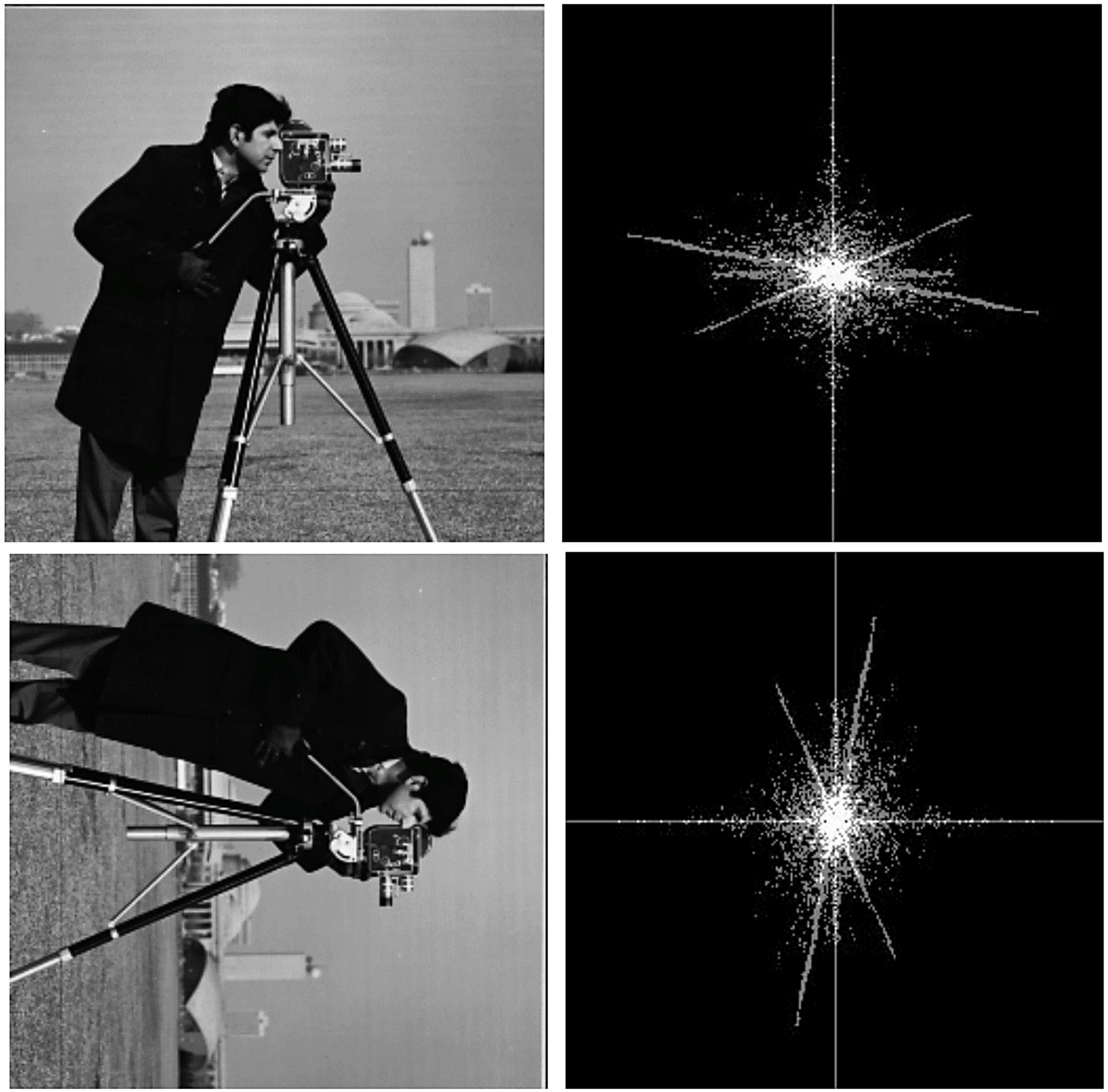
Effect of Translation on the Fourier Spectrum



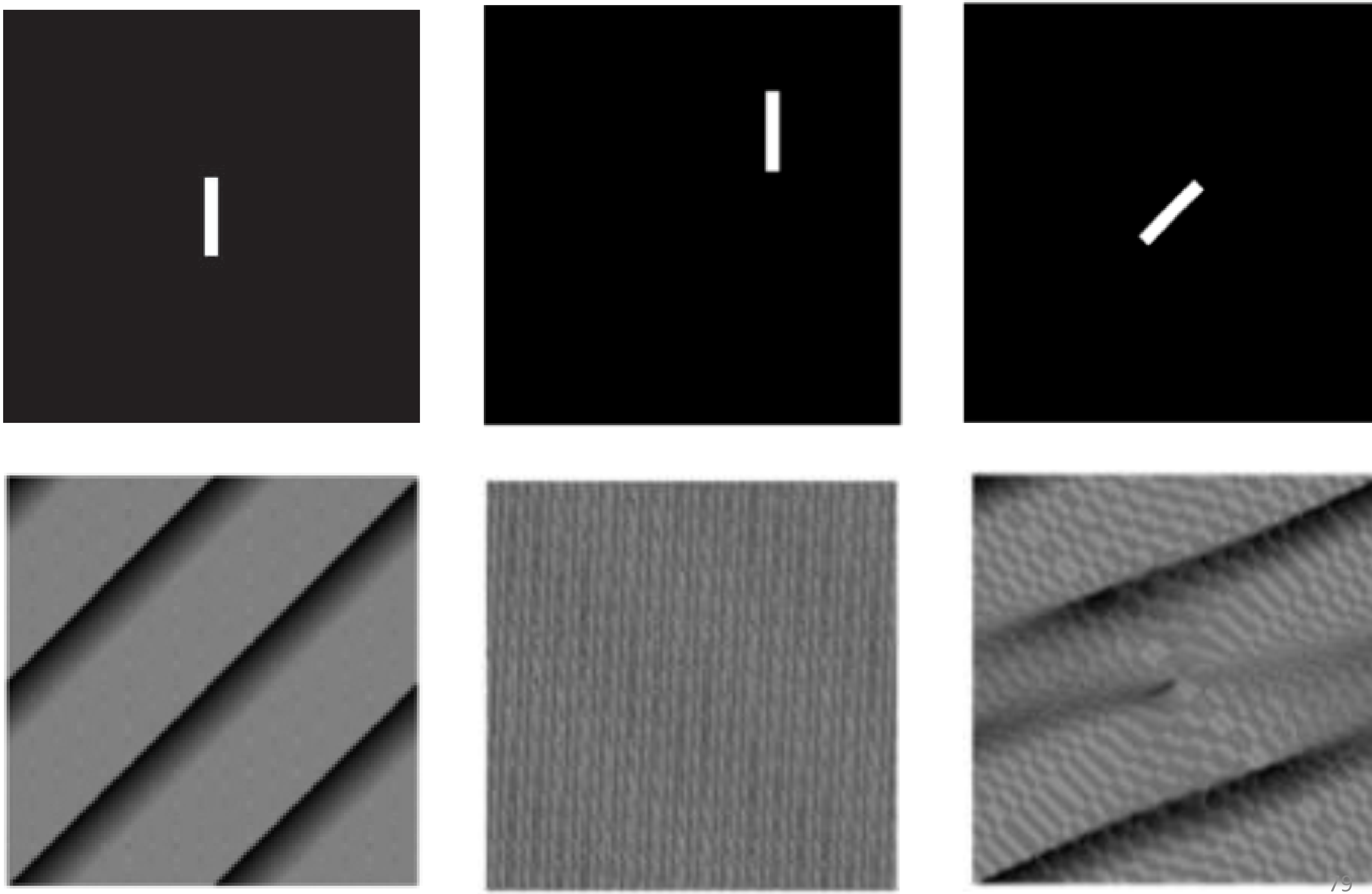
Effect of Rotation on the Fourier Spectrum



Effect of Rotation on the Fourier Spectrum



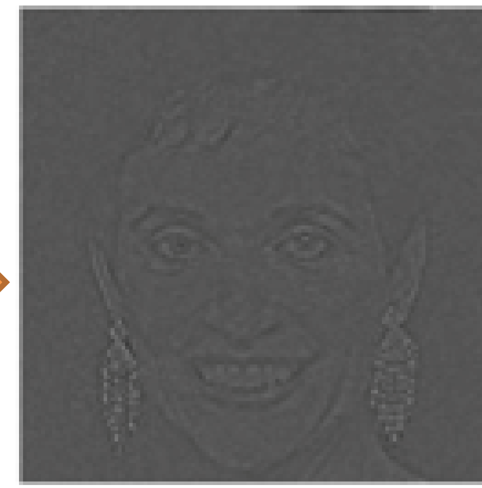
Effect of Translation and Rotation on Phase Angle



Importance of Phase Angle
example-1

DFT

IDFT



f(x,y) $\times$ $(-1)^{x+y}$

ϕ(u,v)

set |F(u,v)|=1

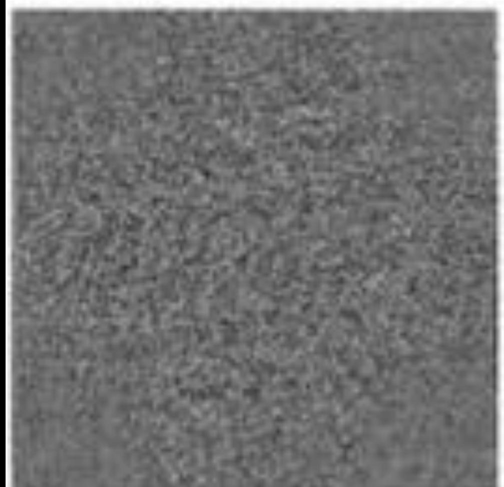
f'(x,y)
Reconstructed
using
phase angle only

$$
\begin{aligned}
F(u,v) &= R(u,v) + jI(u,v) \\
&= \left|F(u,v)\right| e^{j\phi(u,v)}
\end{aligned}
$$

Importance of Phase Angle example-1

$f(x,y) \times (-1)^{x+y}$

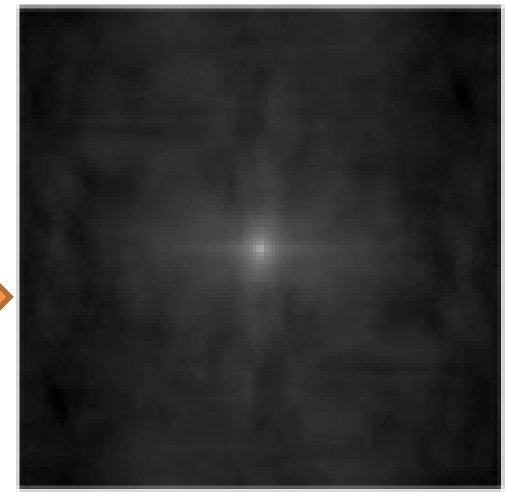
DFT



$|F(u,v)|$

set $\phi(u,v)=0$

IDFT

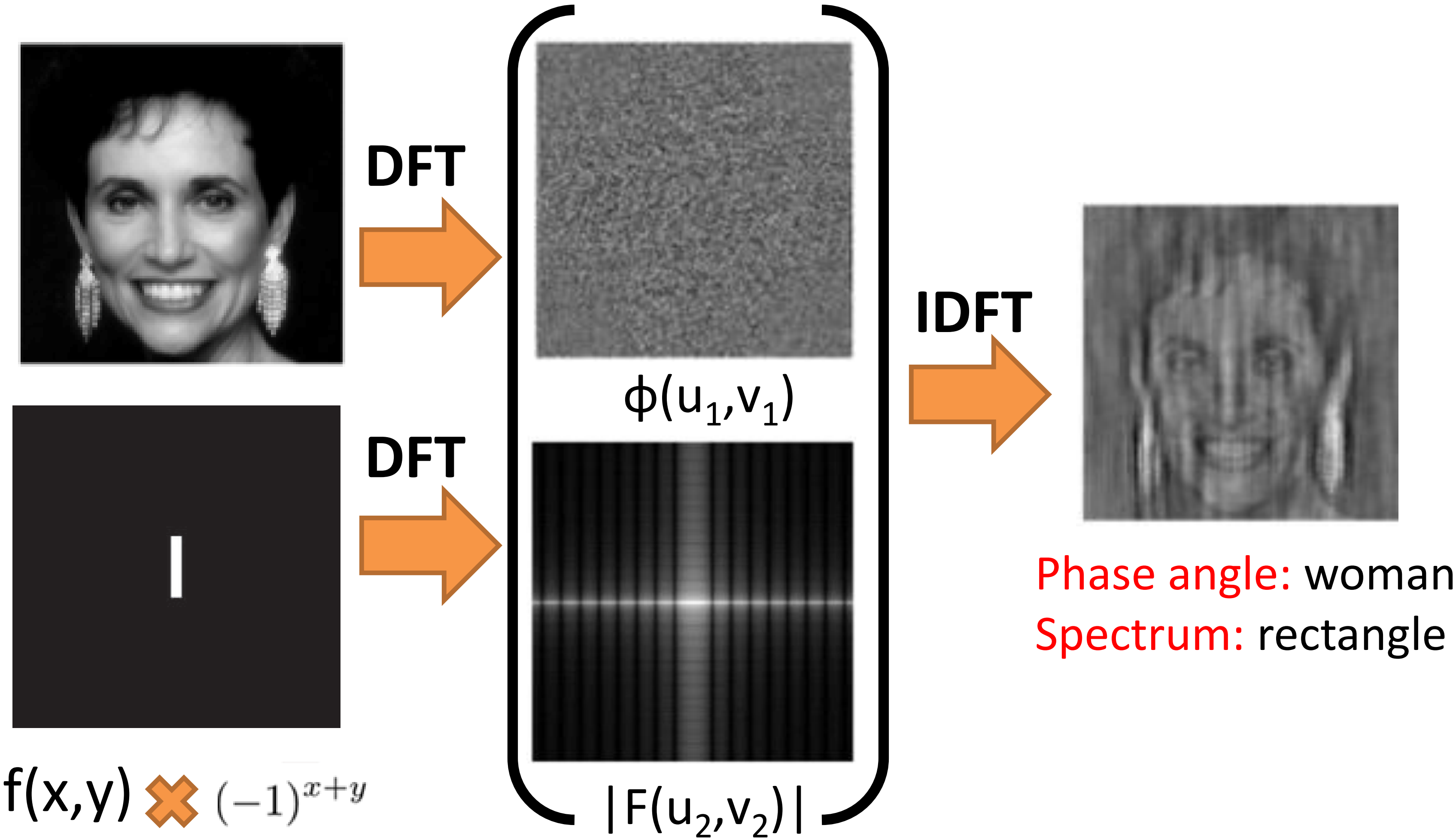


$f'(x,y)$

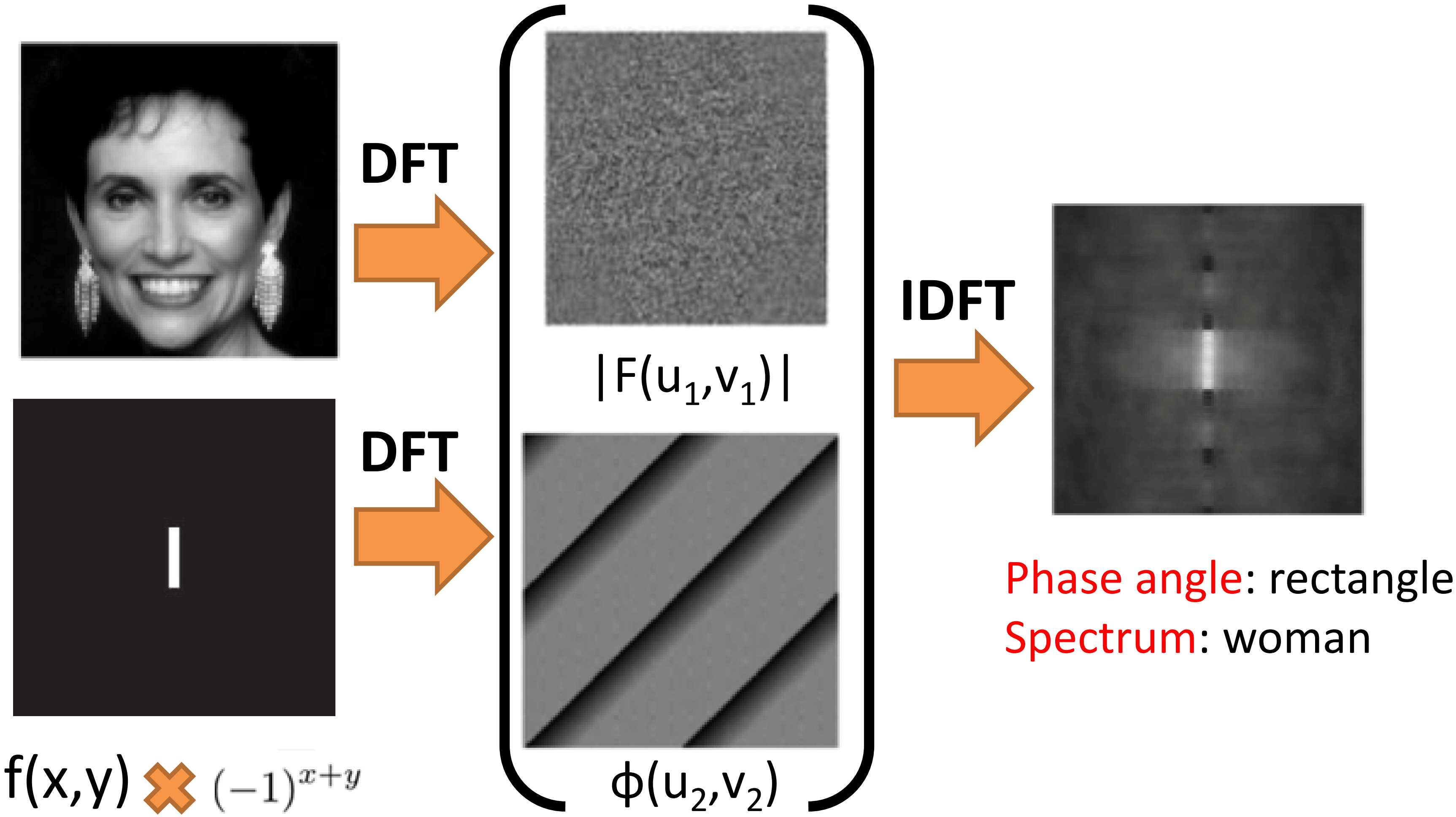
Reconstructed using spectrum only

$$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

Importance of Phase Angle
example-1



Importance of Phase Angle
example-1

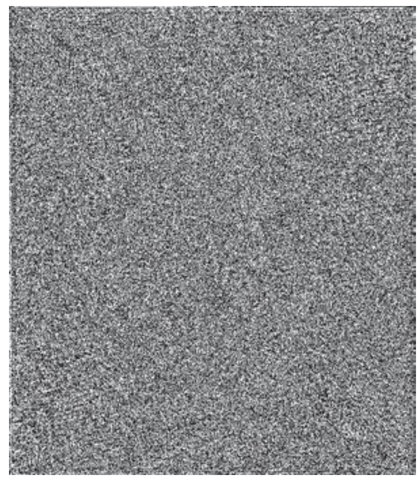


Importance of Phase Angle example-2

f(x,y) ✖ $(-1)^{x+y}$

$$F(u,v) = R(u,v) + jI(u,v)$$
$$= \left|F(u,v)\right| e^{j\phi(u,v)}$$

DFT



ϕ(u,v)

set |F(u,v)|=1

IDFT

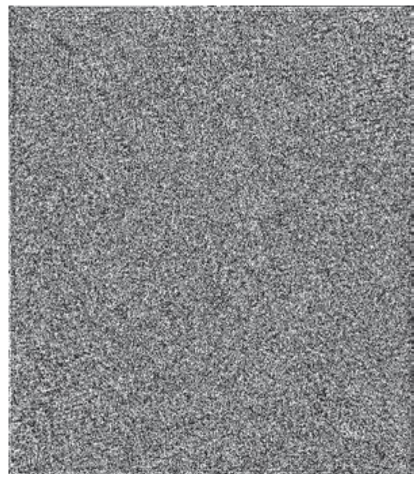
f’(x,y)
Reconstructed
using
phase angle only

Importance of Phase Angle example-2

$f(x,y) \times (-1)^{x+y}$

$$F(u,v) = R(u,v) + jI(u,v)$$
$$= |F(u,v)|e^{j\phi(u,v)}$$

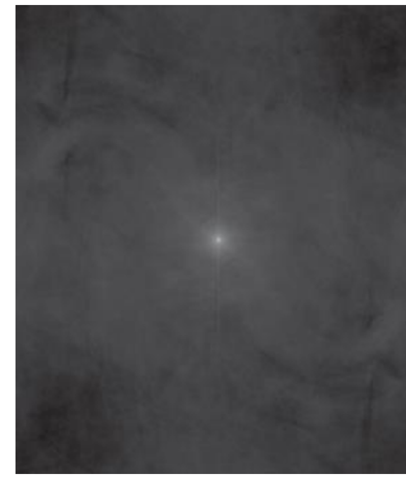
DFT



|F(u,v)|

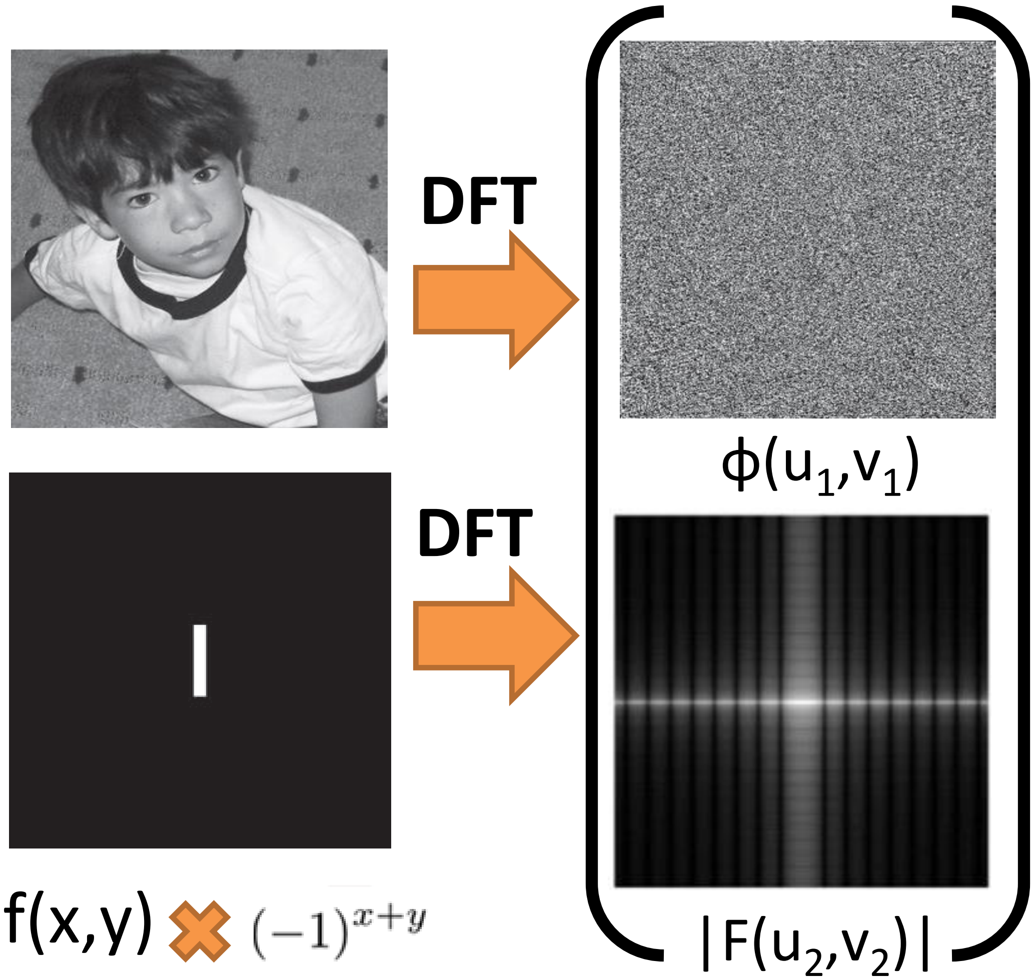
set ϕ(u,v)=0

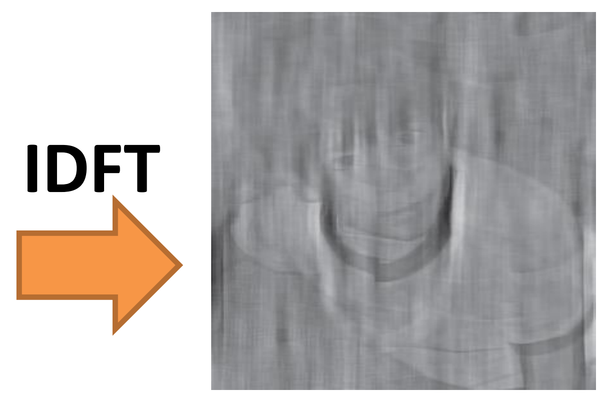
IDFT



f'(x,y)
Reconstructed using spectrum only

Importance of Phase Angle example-2

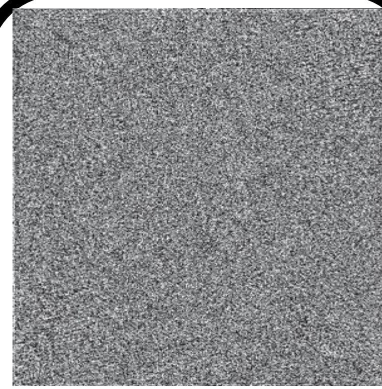




Phase angle: boy
Spectrum: rectangle

Importance of Phase Angle
example-2

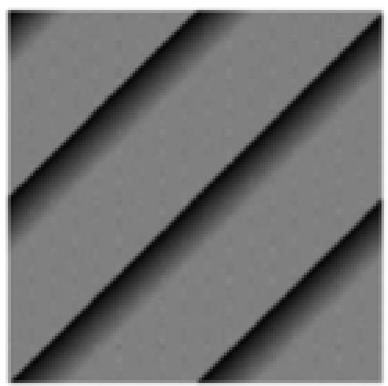
DFT



$|F(u_1,v_1)|$

$f(x,y) \times (-1)^{x+y}$

DFT



$\phi(u_2,v_2)$

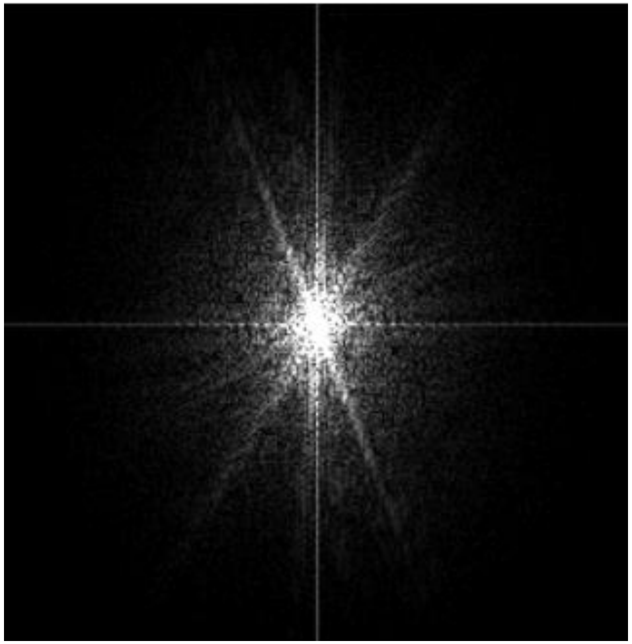
IDFT

Phase angle: rectangle

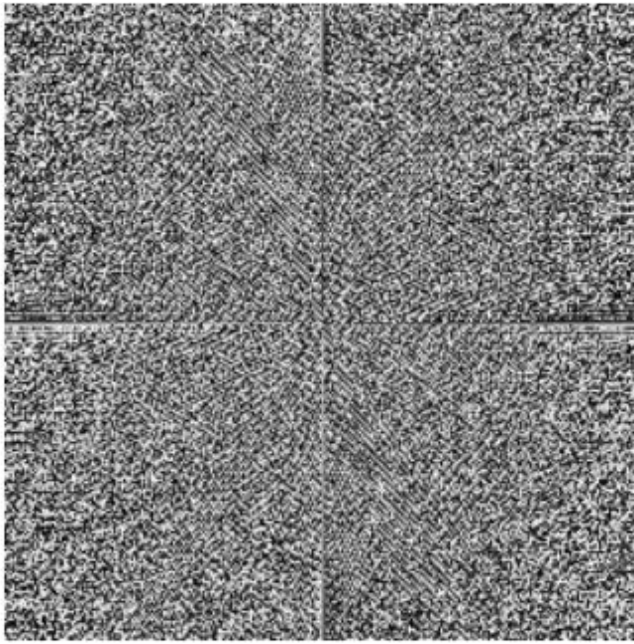
Spectrum: boy

Reading Fourier Spectrum

Image

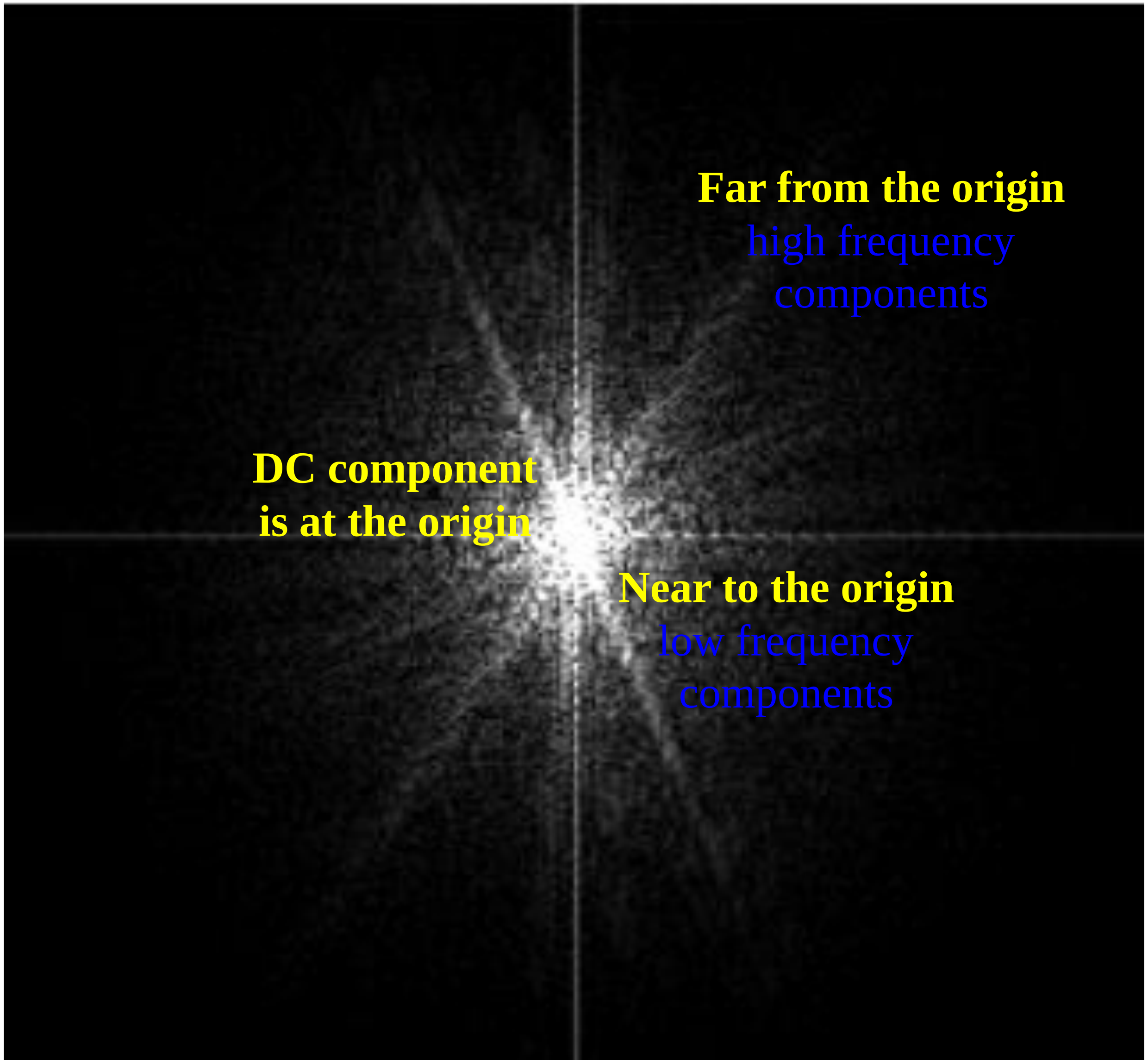


Its Fourier Spectrum



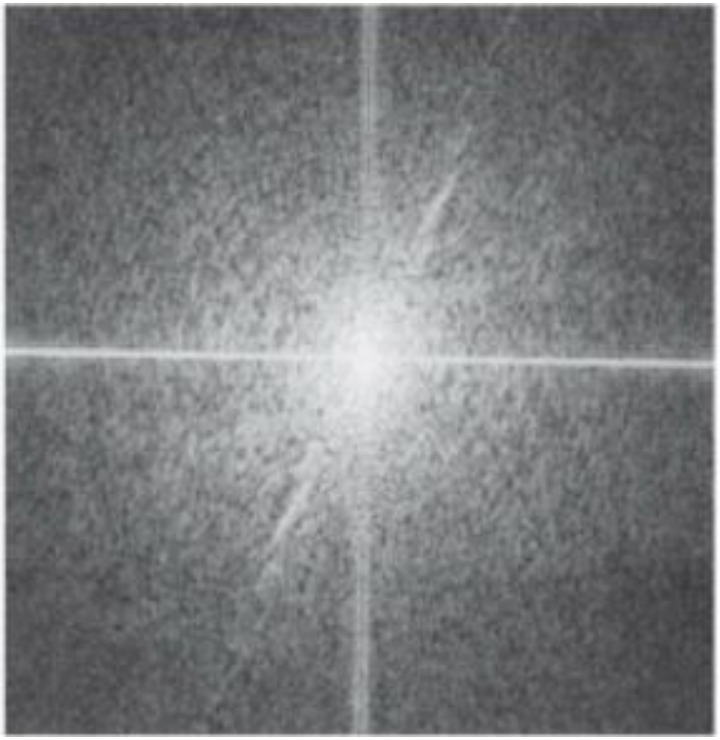
Its Phase Angle

Reading Fourier Spectrum



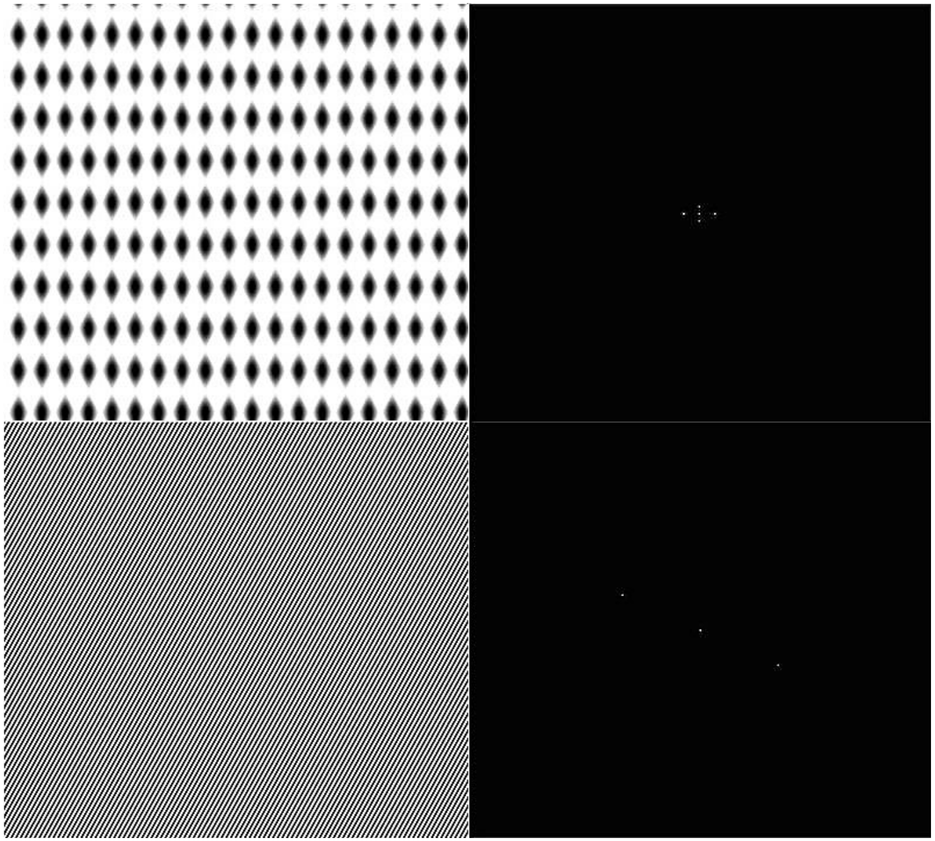
Fourier Spectrum

Reading Fourier Spectrum



- We can see that the **DC-value** is by far the **largest component** of the image.
- The result shows that the image contains components of most of the frequencies, but that their magnitude gets smaller for higher frequencies.
- Hence, low frequencies contain more image information than the higher ones.

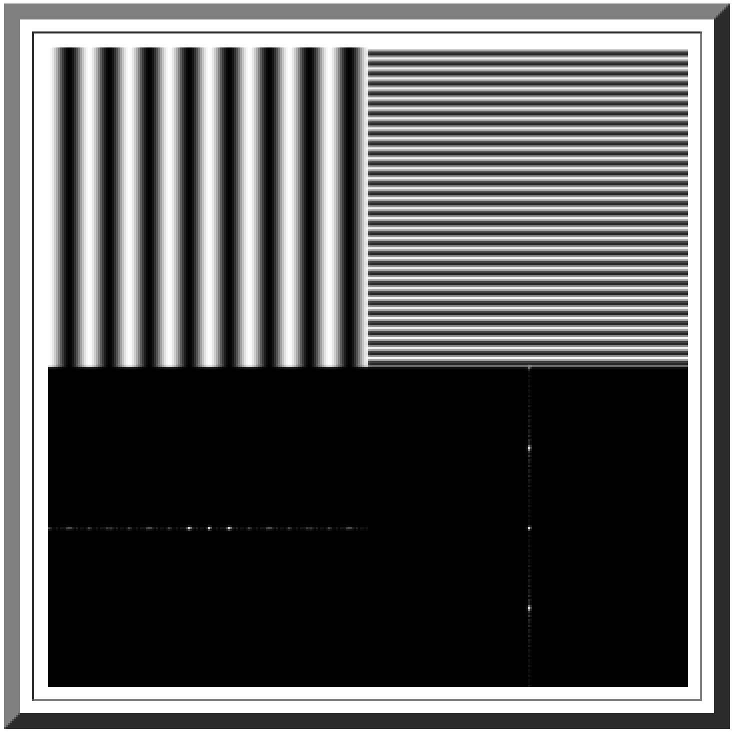
Reading Fourier Spectrum

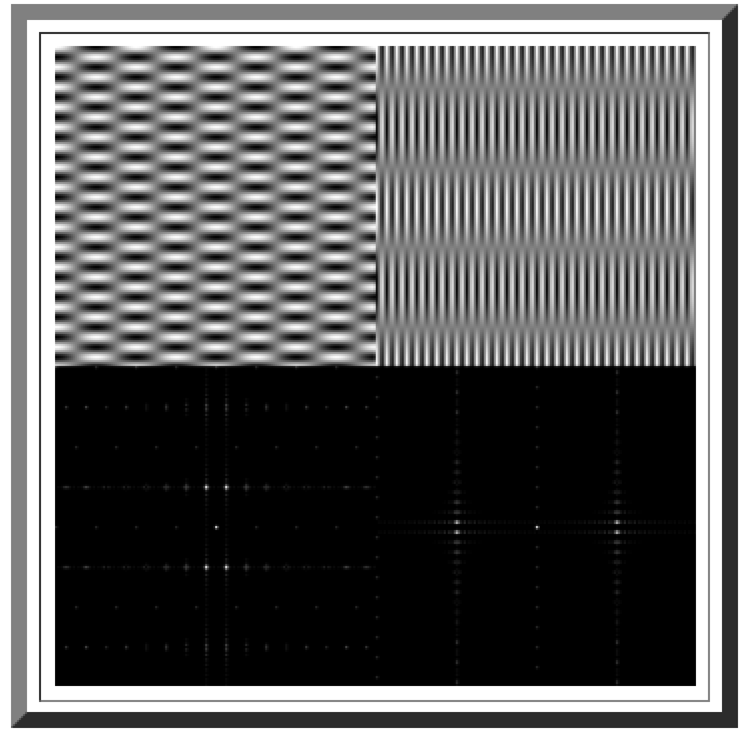


Top: The wave *sin(20x)+sin(10y)* and its Fourier transform, showing two bright pixels and their reflections representing the contributions of these *two waves*.

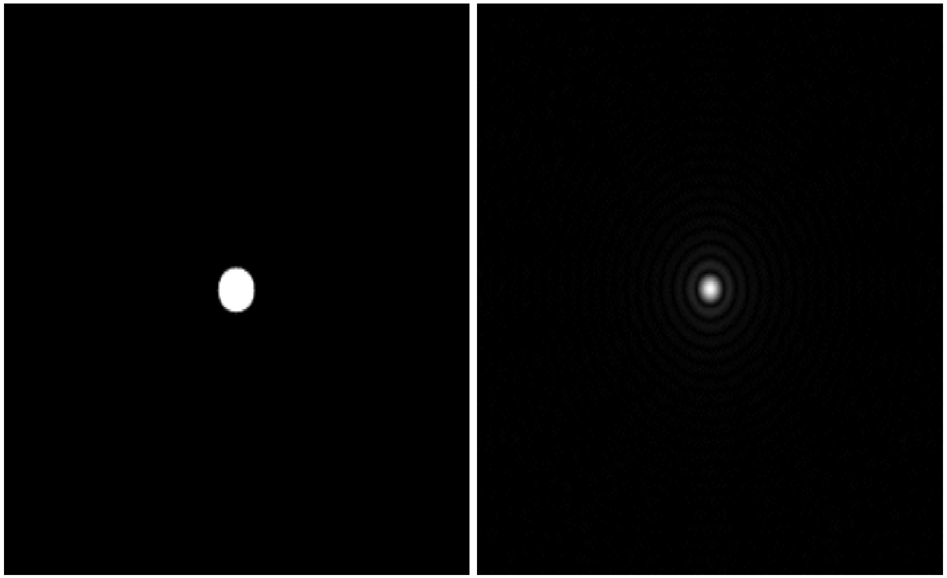
Bottom: The wave *sin(100x+50y)* and its Fourier transform, showing a bright pixel and its reflection representing the contribution of this *single wave*.

Reading Fourier Spectrum

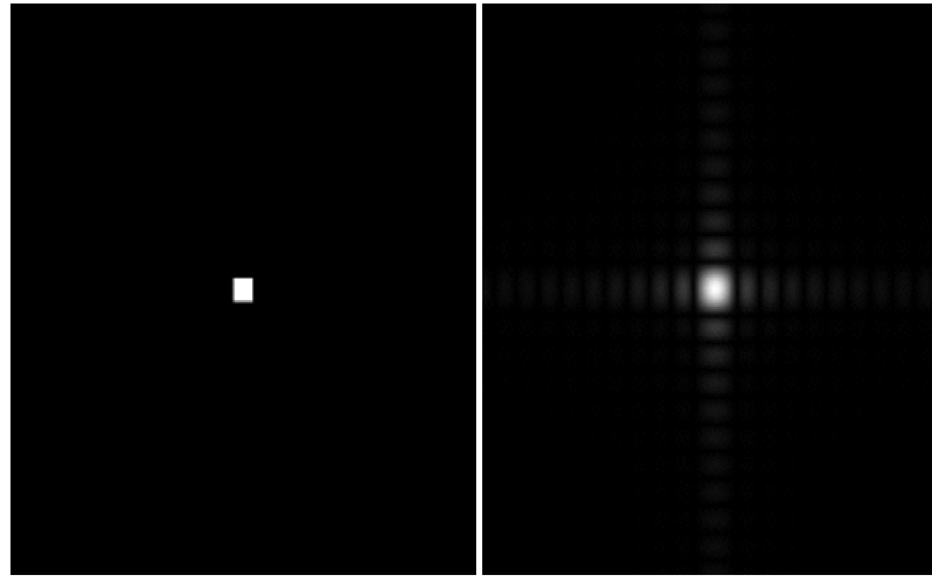




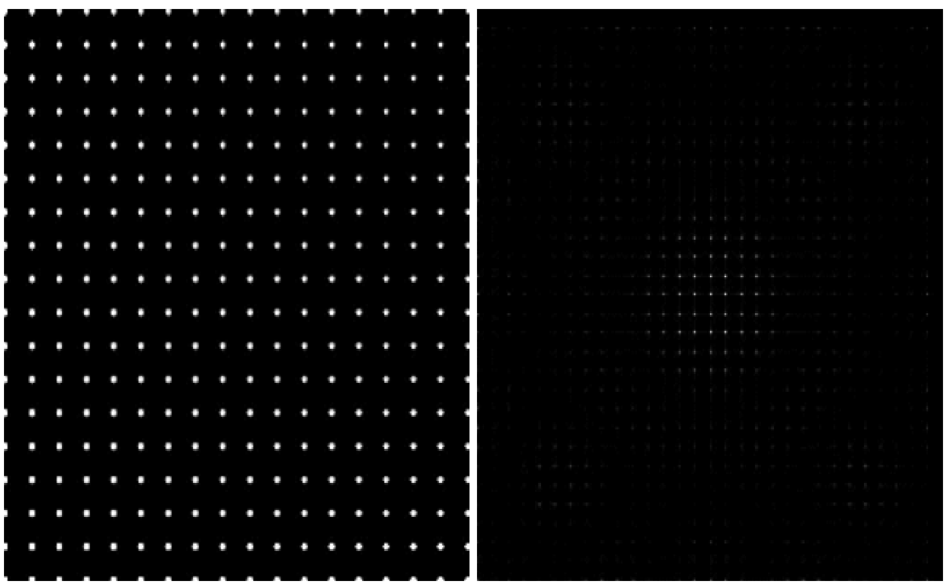
Reading Fourier Spectrum



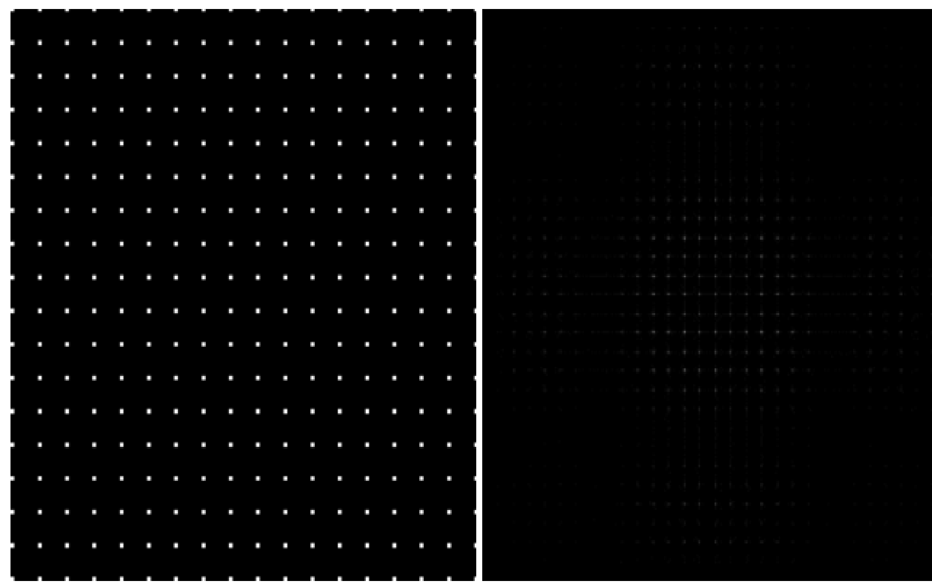
Circle



Square

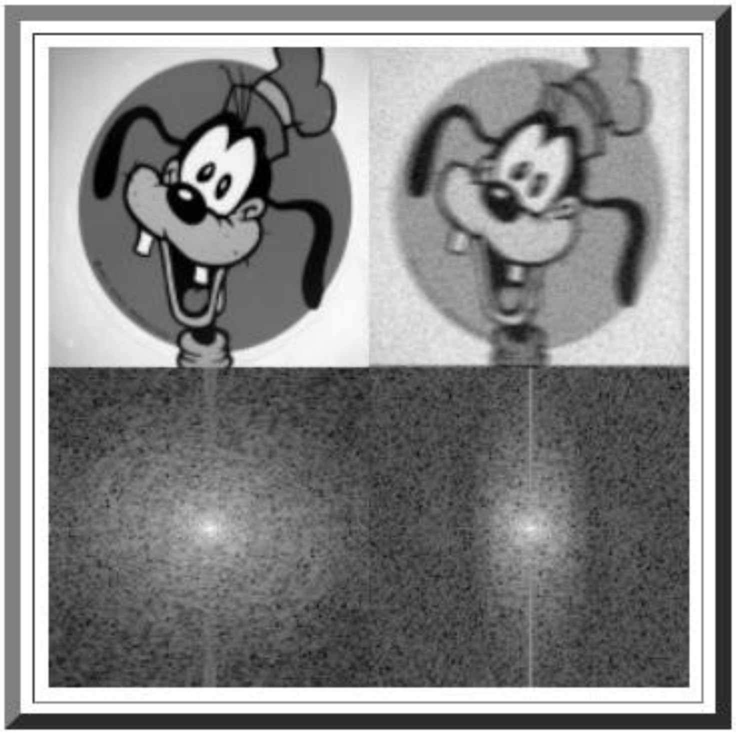


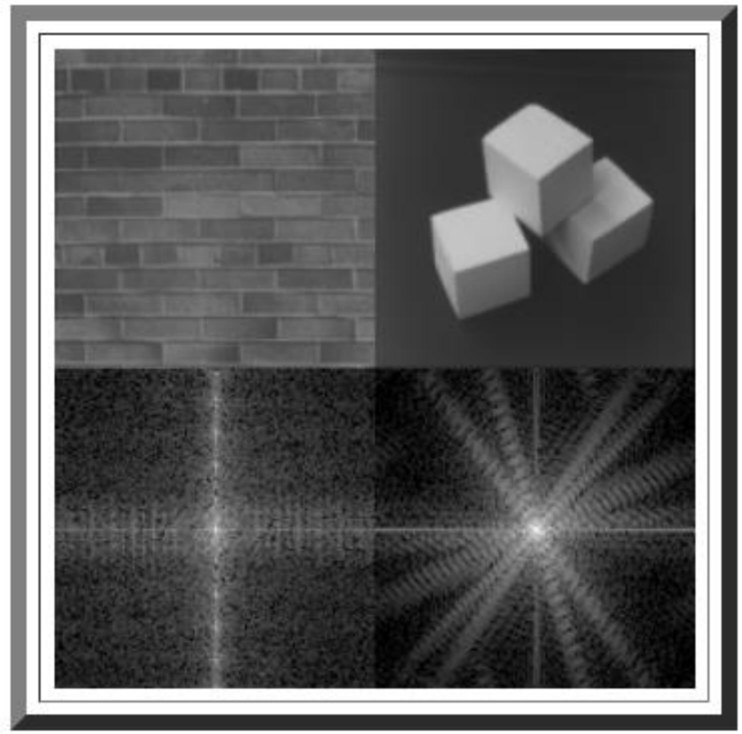
Periodic circles



Periodic squares

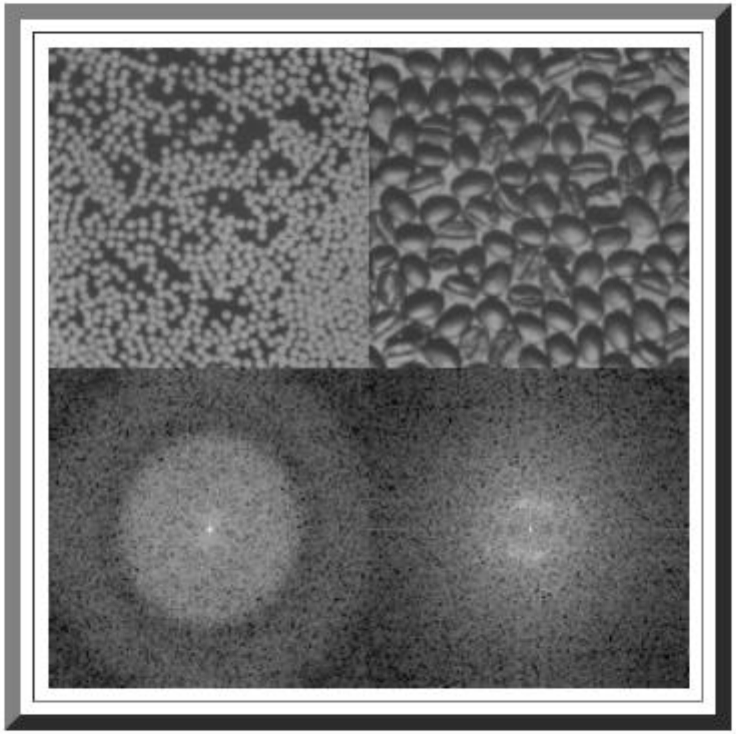
Reading Fourier Spectrum





Reading Fourier Spectrum

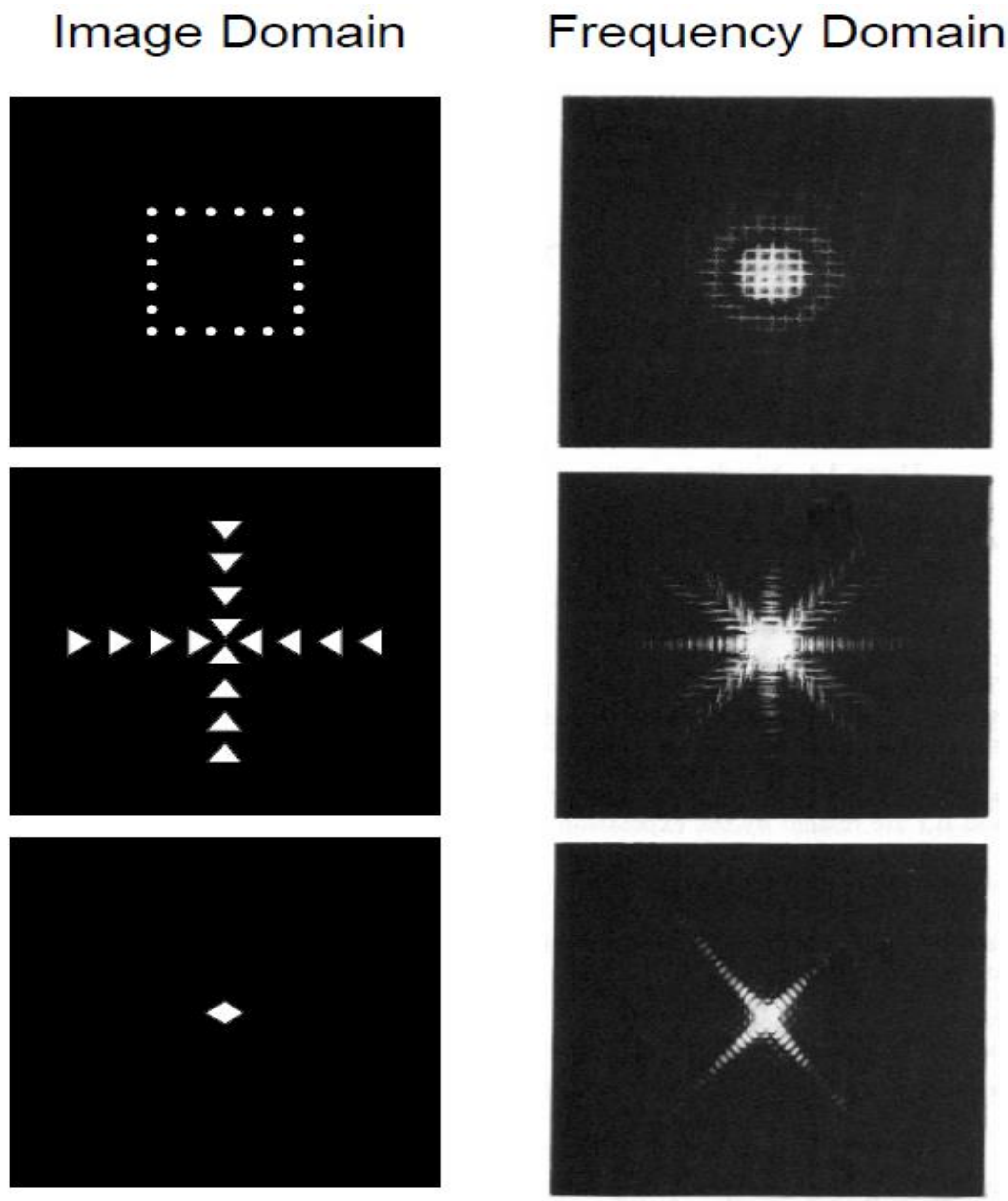
Reading Fourier Spectrum



- Notice the concentric ring structure in the FT of the white pellets image. It is due to each individual pellet. That is, if we took the FT of just one pellet, we would still get this pattern.

- The coffee beans have less symmetry and are more variably colored so they do not show the same ring structure.

Reading Fourier Spectrum



Summary

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x,y)$	$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u,v)$	$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3) Spectrum	$\lvert F(u,v)\rvert=\left[R^2(u,v)+I^2(u,v)\right]^{1/2}$ $\quad R=\text{Real}(F);\ I=\text{Imag}(F)$
4) Phase angle	$\phi(u,v)=\tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5) Polar representation	$F(u,v)=\lvert F(u,v)\rvert e^{j\phi(u,v)}$
6) Power spectrum	$P(u,v)=\lvert F(u,v)\rvert^2$
7) Average value	$\bar{f}=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)=\frac{1}{MN}F(0,0)$
8) Periodicity (k_1 and k_2 are integers)	$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)=F(u+k_1,v+k_2N)$ $f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)=f(x+k_1M,y+k_2N)$
9) Convolution	$(f\star h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$
10) Correlation	$(f \,☆\, h)(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m,y+n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.
12) Obtaining the IDFT using a DFT algorithm	$MNf^*(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting $F^*(u,v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x,y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.

Summary

	Name	DFT Pairs
1)	Symmetry properties	See Table 4.1
2)	Linearity	$af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$
3)	Translation (general)	$f(x,y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4)	Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5)	Rotation	$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$ $r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}(y/x) \quad \omega = \sqrt{u^2+v^2} \quad \varphi = \tan^{-1}(v/u)$
6)	Convolution theorem†	$f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$ $(f \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$
7)	Correlation theorem†	$(f \,☆\, h)(x,y) \Leftrightarrow (F^* \bullet H)(u,v)$ $(f^* \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \,☆\, H)(u,v)]$
8)	Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9)	Rectangle	$\text{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10)	Sine	$\sin(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{jMN}{2}[\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)]$
11)	Cosine	$\cos(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{1}{2}[\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)]$

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.

12)	Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m \boldsymbol{F}(\mu,\nu); \quad \dfrac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13)	Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

† Assumes that $f(x,y)$ and $h(x,y)$ have been properly padded. Convolution is associative, commutative, and distributive. Correlation is distributive (see Table 3.5). The products are elementwise products (see Section 2.6).

Next Lecture

- Filtering in Frequency Domain -Basic Observations
- Filtering in Frequency Domain – Requirements
- What About the Padding for Filters in Frequency Domain?
- Steps for Filtering in the Frequency Domain
- Correspondence Between Filtering in Spatial and Frequency Domain
- Constructing Spatial Filters from Frequency Domain Filters
- Constructing Frequency Domain Filters from Spatial Filters