(Chapter 38 P 316 - 319.)

Let $n \geqslant 0$ be an integer. We introduce the following notation:

$$n! = (1)(2)(3) \cdots (n-1)(n).$$

Read : n factorial

Example:

$$0! = 1$$
$$1! = 1 = 1$$
$$2! = (1)(2) = 2$$
$$3! = (1)(2)(3) = 6$$
$$4! = (1)(2)(3)(4) = 24.$$
$$5! = (1)(2)(3)(4)(5) = 120.$$

In this Lecture, we want to prove the following theorem.

<u>Theorem</u> (38.2 Binomial Theorem) The binomial coefficients in the expansion

$$(A+B)^n = \binom{n}{0} A^n + \binom{n}{1} A^{n-1} B + \cdots \binom{n}{k} A^{n-k} B^k + \cdots \binom{n}{n} B^n$$

are given by the formula:

$$\binom{n}{k} \overset{①}{=} \frac{n(n-1) \cdots (n-k+1)}{k!} \overset{②}{=} \frac{n!}{k!(n-k)!}$$

Example: (1) $\binom{4}{2} = \frac{4!}{(2!)(2!)} = \frac{24}{(2)(2)} = 6$

(2) $\binom{5}{2} = \frac{5!}{(2!)(3!)} = \frac{120}{(2)(6)} = 10$

(3) $\binom{n}{0} = \frac{n!}{(0!)(n!)} = \frac{n!}{(1)(n!)} = \frac{n!}{n!} = 1$

(4) $\binom{n}{n} = \frac{n!}{(n!)(0!)} = \frac{n!}{(n!)\cdot(1)} = \frac{n!}{n!} = 1.$

Proof of theorem:

We first prove ②: $\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$

$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots(2)(1)}{k!\cdot(n-k)(n-k-1)\cdots(2)(1)}$$

$$= \frac{n!}{k!(n-k)!} \quad ✓$$

Proof of ①: before the proof, we can look at an example:

we want to calculate $\binom{4}{2}$

$\binom{4}{2}$ = coefficient of A^2B^2 in $(A+B)^4$.

$(A+B)^4 = (A+B)(A+B)(A+B)(A+B)$.

We want to look at A^2B^2. This means:

we can consider each paranthesis as a box

(we have 4 boxes in total)

$\boxed{A, B}$ $\boxed{A, B}$ $\boxed{A, B}$ $\boxed{A, B}$

we want to pick up 2 "A" and 2 "B" in 4 boxes.

(Indeed, we only need to pick up 2 "A" since the left ones are "B")

To pick up one A, we have 4 choices

(since we have 4 boxes.)

After pick up one A, we want to pick one A again,

we have 3 choices.

(since we have 3 boxes left)

Therefore, we have $(4)(3) = 12$ choices.

However, in this process, we overcounted:

Step I Pick up A in 1st box

Step II Pick up A in 3rd box

Step I': Pick up A in 3rd box

Step II': Pick up A in 1st box.

They are different when we are picking up A but they give the same results for when we picked up A.

Indeed, in our process, the "order" of picking up A does <u>not</u> matter. That's why we over counted.

Therefore, we need to <u>divide</u> the total ways by the <u>order</u>.

Since we are picking up 2 A, the number of order is $2! = 2$.

This implies: $\binom{4}{2} = \dfrac{(4)(3)}{2!}$

The proof of general case is similar:

$\binom{n}{k}$ = coefficient of $A^{n-k}B^{k}$ in $(A+B)^{n}$

$$\underbrace{\boxed{A,B}\quad \boxed{A,B}\quad \boxed{A,B}\quad \cdots \quad \boxed{A,B}}_{n \text{ boxes.}}$$

Pick up k "A"

Total ~~numbe~~ number is $= n(n-1)(n-2)\cdots(n-k+1)$.

We need to divide by the order. Since we are picking up $\underline{k}$ "A", the number of order is $k!$

Therefore $\binom{n}{k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \dfrac{n!}{k!(n-k)!}$ $\square$.

Remark: The explicit formula also gives an proof for $\binom{n}{k} = \binom{n}{n-k}$.