

Lecture Notes of Multivariate Statistics

Lecture 01

Weizhong Zhang

School of Data Science, Fudan University

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1 Review of Linear Algebra

Theorem 1.1 (QR Factorization). *Prove the following results for Gram-Schmidt orthogonalization*

1. $r_{jj} \neq 0$ for all $i = 1, \dots, n$
2. $\|\mathbf{q}_i\|_2 = 1$ for all $i = 1, \dots, n$
3. $\mathbf{q}_i^\top \mathbf{q}_j = 0$ for all $i = 1, \dots, n$ and $j < i$.

Proof. Part 1: Since each \mathbf{q}_i is a linear combination of $\{\mathbf{a}_1, \dots, \mathbf{a}_i\}$, the entry r_{jj} is zero means

$$r_{jj} = \left\| \mathbf{a}_j - \sum_{i=1}^{j-1} r_{ij} \mathbf{q}_i \right\|_2 = 0,$$

then \mathbf{a}_j must be a linear combination of $\{\mathbf{a}_1, \dots, \mathbf{a}_{j-1}\}$, which validates the full rank assumption on \mathbf{A} .

Part 2: Just use the expression of r_{jj} .

Part 3: Recall that $r_{ij} = \mathbf{q}_i^\top \mathbf{a}_j$ for any $i \neq j$. We can verify

$$\mathbf{q}_1^\top \mathbf{q}_2 = \frac{\mathbf{q}_1^\top (\mathbf{a}_2 - r_{12} \mathbf{q}_1)}{r_{22}} = \frac{\mathbf{q}_1^\top (\mathbf{a}_2 - (\mathbf{q}_1^\top \mathbf{a}_2) \mathbf{q}_1)}{r_{22}} = \frac{\mathbf{q}_1^\top \mathbf{a}_2 - (\mathbf{q}_1^\top \mathbf{a}_2) \mathbf{q}_1^\top \mathbf{q}_1}{r_{22}} = 0$$

Suppose for $\mathbf{q}_i^\top \mathbf{q}_j = 0$ for all $\mathbf{q}_i^\top \mathbf{q}_j = 0$ for all $i = 1, \dots, n' - 1$ and $j < i$. Then for all $k = 1, 2, \dots, n' - 1$, we have

$$\mathbf{q}_k^\top \mathbf{q}_{n'} = \frac{\mathbf{q}_k^\top \mathbf{a}_{n'} - \sum_{i=1}^{n'-1} r_{in'} \mathbf{q}_k^\top \mathbf{q}_i}{r_{n'n'}} = \frac{\mathbf{q}_k^\top \mathbf{a}_{n'} - r_{kn'} \mathbf{q}_k^\top \mathbf{q}_k}{r_{n'n'}} = \frac{\mathbf{q}_k^\top \mathbf{a}_{n'} - r_{kn'}}{r_{n'n'}} = 0$$

Then we prove the result by induction. □

Theorem 1.2. *Prove $\|\mathbf{A}\|_2 = \sigma_1$.*

Proof. Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ be full SVD of \mathbf{A} . Then

$$\|\mathbf{A}\|_2 = \sup_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \sup_{\|\mathbf{x}\|_2=1} \|\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \mathbf{x}\|_2 = \sup_{\|\mathbf{x}\|_2=1} \|\mathbf{\Sigma}\mathbf{V}^\top \mathbf{x}\|_2$$

Then let $\mathbf{y} = \mathbf{V}^\top \mathbf{x}$. Since \mathbf{V} is orthogonal matrix, we have $\|\mathbf{y}\|_2 = \|\mathbf{V}^\top \mathbf{x}\|_2 = \|\mathbf{x}\|_2 = 1$. Hence,

$$\sup_{\|\mathbf{x}\|_2=1} \|\mathbf{\Sigma}\mathbf{V}^\top \mathbf{x}\|_2 = \sup_{\|\mathbf{y}\|_2=1} \|\mathbf{\Sigma}\mathbf{y}\|_2 = \sup_{\|\mathbf{y}\|_2=1} \sqrt{\sum_{i=1}^r (\sigma_i y_i)^2} \leq \sigma_1.$$

We attain the maximum by taking $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and the corresponding \mathbf{x} is $\mathbf{V} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ □

Theorem 1.3 (Cholesky Factorization). *The symmetric positive-definite matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ has the decomposition of the form*

$$\mathbf{A} = \mathbf{L}\mathbf{L}^\top$$

where $\mathbf{L} \in \mathbb{R}^{n \times n}$ is a lower triangular matrix with real and positive diagonal entries.

Proof. For $n = 1$, it is trivial. Suppose it holds for $n - 1$, then any $\tilde{\mathbf{A}} \in \mathbb{R}^{(n-1) \times (n-1)}$ can be written as

$$\tilde{\mathbf{A}} = \tilde{\mathbf{L}}\tilde{\mathbf{L}}^\top$$

where $\tilde{\mathbf{L}} \in \mathbb{R}^{(n-1) \times (n-1)}$ is a lower triangular matrix with real and positive diagonal entries. Consider the case of n such that

$$\mathbf{A} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{a} \\ \mathbf{a}^\top & \alpha \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{L}}\tilde{\mathbf{L}}^\top & \mathbf{a} \\ \mathbf{a}^\top & \alpha \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad \text{where } \mathbf{a} \in \mathbb{R}^{n-1}, \quad \alpha \in \mathbb{R}.$$

Let

$$\mathbf{L}_1 = \begin{bmatrix} \tilde{\mathbf{L}}^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

We have

$$\mathbf{L}_1^{-1}\mathbf{A}\mathbf{L}_1^{-\top} = \begin{bmatrix} \tilde{\mathbf{L}}^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{L}}\tilde{\mathbf{L}}^\top & \mathbf{a} \\ \mathbf{a}^\top & \alpha \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{L}}^{-\top} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{b} \\ \mathbf{b}^\top & \alpha \end{bmatrix} \triangleq \mathbf{B} \in \mathbb{R}^{n \times n} \quad \text{where } \mathbf{b} \in \tilde{\mathbf{L}}^{-1}\mathbf{a} \in \mathbb{R}^{n-1}.$$

Let

$$\mathbf{L}_2 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{b}^\top & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Then

$$\mathbf{L}_2^{-1}\mathbf{B}\mathbf{L}_2^{-\top} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{b}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{b} \\ \mathbf{b}^\top & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \alpha - \mathbf{b}^\top \mathbf{b} \end{bmatrix}.$$

Since \mathbf{A} is positive-definite, we have

$$\alpha - \mathbf{b}^\top \mathbf{b} = \alpha - \mathbf{a}^\top \tilde{\mathbf{L}}^{-\top} \tilde{\mathbf{L}}^{-1} \mathbf{a} = \alpha - \mathbf{a}^\top \tilde{\mathbf{L}}^{-\top} \tilde{\mathbf{L}}^{-1} \mathbf{a} = \alpha - \mathbf{a}^\top \tilde{\mathbf{A}}^{-1} \mathbf{a} > 0.$$

Let $\alpha - \mathbf{b}^\top \mathbf{b} = \lambda^2$, where $\lambda > 0$. Hence, we have

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \alpha - \mathbf{b}^\top \mathbf{b} \end{bmatrix} = \mathbf{L}_3 \mathbf{L}_3^\top, \quad \text{where } \mathbf{L}_3 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda \end{bmatrix}$$

which means $\mathbf{A} = \mathbf{L}\mathbf{L}^\top \in \mathbb{R}^{n \times n}$ where $\mathbf{L} = \mathbf{L}_1 \mathbf{L}_2 \mathbf{L}_3 \in \mathbb{R}^{n \times n}$ is a lower triangular matrix with real and positive diagonal entries. \square

Theorem 1.4. *Suppose $\nabla^2 f(\mathbf{x})$ is continuous in an open neighborhood of \mathbf{x}^* and that $\nabla f(\mathbf{x}^*) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}^*) \succ \mathbf{0}$. Then \mathbf{x}^* is a strict local minimizer of f .*

Proof. Because the Hessian is continuous and positive definite at \mathbf{x}^* , we can choose a radius $r > 0$ so that $\nabla^2 f(\mathbf{x})$ remains positive definite for all \mathbf{x} in the open ball $\mathcal{D} = \{\mathbf{z} : \|\mathbf{z} - \mathbf{x}^*\|_2 < r\}$. Taking any nonzero vector \mathbf{p} with $\|\mathbf{p}\|_2 < r$, we have $\mathbf{x}^* + \mathbf{p} \in \mathcal{D}$ and so

$$f(\mathbf{x}^* + \mathbf{p}) = f(\mathbf{x}^*) + \mathbf{p}^\top \nabla f(\mathbf{x}^*) + \frac{1}{2} \mathbf{p}^\top \nabla^2 f(\mathbf{z}) \mathbf{p} = f(\mathbf{x}^*) + \frac{1}{2} \mathbf{p}^\top \nabla^2 f(\mathbf{z}) \mathbf{p},$$

where $\mathbf{z} = \mathbf{x}^* + t\mathbf{p}$ for some $t \in (0, 1)$. Since $\mathbf{z} \in \mathcal{D}$, we have $\mathbf{p}^\top \nabla^2 f(\mathbf{z}) \mathbf{p} > 0$, and therefore $f(\mathbf{x}^* + \mathbf{p}) > f(\mathbf{x}^*)$, giving the result. \square

Theorem 1.5. Suppose \mathbf{x}^* is a local minimizer of twice differentiable $f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$ is continuous in an open neighborhood of \mathbf{x}^* , then $\nabla^2 f(\mathbf{x}^*) = \mathbf{0}$ and $\nabla^2 f(\mathbf{x}^*) \succeq \mathbf{0}$.

Proof. Suppose for contradiction that $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$. Define the vector $p = -\nabla f(\mathbf{x}^*)$, which leads to that $\mathbf{p}^\top \nabla f(\mathbf{x}^*) < 0$. Because ∇f is continuous near \mathbf{x}^* , there is a scalar $T > 0$ such that

$$\mathbf{p}^\top \nabla f(\mathbf{x}^* + t\mathbf{p}) < 0,$$

for all for any $t \in [0, T]$. We have by Taylor's theorem that

$$f(\mathbf{x}^* + \bar{t}\mathbf{p}) = f(\mathbf{x}^*) + \bar{t}\mathbf{p}^\top \nabla f(\mathbf{x}^* + t\mathbf{p}),$$

for some $t \in (0, \bar{t})$. Therefore, $f(\mathbf{x}^* + \bar{t}\mathbf{p}) < f(\mathbf{x}^*)$ for all $\bar{t} \in (0, T]$. We have found a direction leading away from \mathbf{x}^* along which f decreases, so \mathbf{x}^* is not a local minimizer, and we have $\nabla^2 f(\mathbf{x}^*) = \mathbf{0}$.

For contradiction, assume that $\nabla^2 f(\mathbf{x}^*)$ is not positive semidefinite. Then we can choose a vector \mathbf{p} such that $\mathbf{p}^\top \nabla^2 f(\mathbf{x}^*) \mathbf{p} < 0$. Because $\nabla^2 f(\mathbf{x})$ is continuous near \mathbf{x}^* , there is a scalar $T > 0$ such that

$$\mathbf{p}^\top \nabla^2 f(\mathbf{x}^* + t\mathbf{p}) \mathbf{p} < 0$$

for all $t \in [0, T]$. By doing a Taylor series expansion around \mathbf{x}^* , we have for all $\bar{t} \in (0, T]$ and some $t \in (0, \bar{t})$ that

$$f(\mathbf{x}^* + \bar{t}\mathbf{p}) = f(\mathbf{x}^*) + \bar{t}\mathbf{p}^\top \nabla f(\mathbf{x}^*) + \frac{1}{2}\bar{t}^2 \mathbf{p}^\top \nabla^2 f(\mathbf{x}^* + t\mathbf{p}) \mathbf{p} < f(\mathbf{x}^*).$$

We have found a direction from \mathbf{x}^* along which f is decreasing, and so again, \mathbf{x}^* is not a local minimizer. \square

Theorem 1.6. Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, the solution of minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \triangleq \frac{1}{2} \|\mathbf{Ax} - \mathbf{b}\|_2^2.$$

is $\hat{\mathbf{x}} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{y}$, where $\mathbf{y} \in \mathbb{R}^n$

Proof. The Hessian of $f(\mathbf{x})$ is $\mathbf{A}^\top \mathbf{A} \succeq \mathbf{0}$, which means $f(\mathbf{x})$ is convex. Let $\mathbf{A} = \mathbf{U}_r \Sigma_r \mathbf{V}_r^\top$ be the condense SVD, where r is the rank of \mathbf{A} . Since $\nabla f(\mathbf{x}) = \mathbf{A}^\top \mathbf{Ax} - \mathbf{A}^\top \mathbf{b}$, we only needs to solve the linear system

$$\mathbf{A}^\top \mathbf{Ax} - \mathbf{A}^\top \mathbf{b} = \mathbf{0}.$$

We denote the solution of $\mathbf{A}^\top \mathbf{Ax} - \mathbf{A}^\top \mathbf{b} = \mathbf{0}$ be

$$\mathcal{X} = \{\mathbf{x} : \mathbf{A}^\top \mathbf{Ax} - \mathbf{A}^\top \mathbf{b} = \mathbf{0}\}.$$

We can verify that $\hat{\mathbf{x}} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{y}$ is the solution of the linear system because

$$\begin{aligned} & \mathbf{A}^\top \mathbf{A} \hat{\mathbf{x}} - \mathbf{A}^\top \mathbf{b} \\ &= \mathbf{A}^\top \mathbf{A} (\mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{y}) - \mathbf{A}^\top \mathbf{b} \\ &= \mathbf{A}^\top (\mathbf{A} \mathbf{A}^\dagger - \mathbf{I}) \mathbf{b} + \mathbf{A}^\top \mathbf{A} (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A}) \mathbf{y} \\ &= \mathbf{V}_r \Sigma_r \mathbf{U}_r^\top (\mathbf{U}_r \Sigma_r \mathbf{V}_r^\top \mathbf{V}_r \Sigma_r^{-1} \mathbf{U}_r^\top - \mathbf{I}) \mathbf{b} + \mathbf{V}_r \Sigma_r \mathbf{U}_r^\top \mathbf{U}_r \Sigma_r \mathbf{V}_r^\top (\mathbf{I} - \mathbf{V}_r \Sigma_r^{-1} \mathbf{U}_r^\top \mathbf{U}_r \Sigma_r \mathbf{V}_r^\top) \mathbf{y} \\ &= \mathbf{V}_r \Sigma_r \mathbf{U}_r^\top (\mathbf{U}_r \mathbf{U}_r^\top - \mathbf{I}) \mathbf{b} + \mathbf{V}_r \Sigma_r^2 \mathbf{V}_r^\top (\mathbf{I} - \mathbf{V}_r \mathbf{V}_r^\top) \mathbf{y} \\ &= \mathbf{V}_r \Sigma_r (\mathbf{U}_r^\top - \mathbf{U}_r^\top) \mathbf{b} + \mathbf{V}_r \Sigma_r^2 (\mathbf{V}_r^\top - \mathbf{V}_r^\top) \mathbf{y} \\ &= \mathbf{0}. \end{aligned}$$

Hence, we have $\mathcal{X}_1 \subseteq \mathcal{X}$, where $\mathcal{X}_1 = \{\mathbf{x} : \mathbf{x} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{A}^\dagger \mathbf{A})\mathbf{y}, \mathbf{y} \in \mathbb{R}^n\}$.

We also have

$$\mathbf{A}^\top \mathbf{Ax} - \mathbf{A}^\top \mathbf{b} = \mathbf{0}$$

$$\begin{aligned}
&\Longleftrightarrow \mathbf{V}_r \boldsymbol{\Sigma}_r^2 \mathbf{V}_r^\top \mathbf{x} - \mathbf{V}_r \boldsymbol{\Sigma}_r \mathbf{U}_r^\top \mathbf{b} = \mathbf{0} \\
&\Longleftrightarrow \boldsymbol{\Sigma}_r^2 \mathbf{V}_r^\top \mathbf{x} - \boldsymbol{\Sigma}_r \mathbf{U}_r^\top \mathbf{b} = \mathbf{0} \\
&\Longleftrightarrow \mathbf{V}_r^\top \mathbf{x} = \boldsymbol{\Sigma}_r^{-1} \mathbf{U}_r^\top \mathbf{b} \\
&\Longleftrightarrow \mathbf{V}_r \mathbf{V}_r^\top \mathbf{x} = \mathbf{V}_r \boldsymbol{\Sigma}_r^{-1} \mathbf{U}_r^\top \mathbf{b} \\
&\Longleftrightarrow \mathbf{x} - (\mathbf{I} - \mathbf{V}_r \mathbf{V}_r^\top) \mathbf{x} = \mathbf{A}^\dagger \mathbf{b} \\
&\Longleftrightarrow \mathbf{x} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{V}_r \mathbf{V}_r^\top) \mathbf{x}
\end{aligned}$$

Hence, we have $\mathcal{X} = \{\mathbf{x} : \mathbf{x} = \mathbf{A}^\dagger \mathbf{b} + (\mathbf{I} - \mathbf{V}_r \mathbf{V}_r^\top) \mathbf{x}\} \subseteq \mathcal{X}_1$. In conclusion, we have $\mathcal{X} = \mathcal{X}_1$. \square

2 The Multivariate Normal Distributions

Statistical Independence If $F(x, y) = F(x)G(y)$, we have

$$\begin{aligned}
f(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial^2 F(x)G(y)}{\partial x \partial y} \\
&= \frac{dF(x)}{dx} \frac{dG(y)}{dy} \\
&= f(x)g(y).
\end{aligned}$$

If $f(x, y) = f(x)g(y)$, we have

$$\begin{aligned}
F(x, y) &= \int_{-\infty}^y \int_{-\infty}^x f(u, v) du dv = \int_{-\infty}^y \int_{-\infty}^x f(u)g(v) du dv \\
&= \int_{-\infty}^y \int_{-\infty}^x f(u, v) du dv = \int_{-\infty}^x f(u) du \int_{-\infty}^y g(v) dv \\
&= F(x)G(y).
\end{aligned}$$

Uncorrelated does not means independent Let $X \sim U(-1, 1)$ and

$$Y = \begin{cases} X, & X > 0 \\ -X, & X \leq 0 \end{cases}$$

Show X and Y are uncorrelated but they are NOT independent.

Conditional Distributions Let $y_1 = y$, $y_2 = y + \Delta$. Then for a continuous density, the mean value theorem implies

$$\int_y^{y+\Delta y} g(v) dv = g(y^*) \Delta y,$$

where $y \leq y^* \leq y + \Delta y$. We also have

$$\int_y^{y+\Delta y} f(u, v) dv = f(u, y^*(u)) \Delta y,$$

where $y \leq y^*(u) \leq y + \Delta y$. Connecting above results to

$$\Pr\{x_1 \leq X \leq x_2 \mid y_1 \leq Y \leq y_2\} = \frac{\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(u, v) dv du}{\int_{y_1}^{y_2} g(v) dv}$$

with $y_1 = y$ and $y_2 = y + \Delta y$, we have

$$\begin{aligned}
& \Pr\{x_1 \leq X \leq x_2 \mid y \leq Y \leq y + \Delta y\} \\
&= \frac{\int_{x_1}^{x_2} \int_y^{y+\Delta y} f(u, v) \, dv \, du}{\int_y^{y+\Delta y} g(v) \, dv} \\
&= \frac{\int_{x_1}^{x_2} f(u, y^*(u)) \Delta y \, du}{g(y^*) \Delta y} \\
&= \int_{x_1}^{x_2} \frac{f(u, y^*(u))}{g(y^*)} \, du.
\end{aligned} \tag{1}$$

For y such that $g(y) > 0$, we define $\Pr\{x_1 \leq X \leq x_2 \mid Y = y\}$, the probability that X lies between x_1 and x_2 , given that Y is y , as the limit of (1) as $\Delta y \rightarrow 0$. Thus

$$\Pr\{x_1 \leq X \leq x_2 \mid Y = y\} = \int_{x_1}^{x_2} \frac{f(u, y)}{g(y)} \, du = \int_{x_1}^{x_2} f(u \mid y) \, du. \tag{2}$$

Transform of Variables Let the density of X_1, \dots, X_p be $f(x_1, \dots, x_p)$. Consider the p real-valued functions $\mathbf{u} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ such that

$$y_i = u_i(x_1, \dots, x_p), \quad i = 1, \dots, p.$$

Assume the transformation \mathbf{u} from the x -space to the y -space is one-to-one, then the inverse transformation is \mathbf{u}^{-1} such that

$$x_i = u_i^{-1}(y_1, \dots, y_p), \quad i = 1, \dots, p.$$

Let the random variables Y_1, \dots, Y_p be defined by

$$Y_i = u_i(X_1, \dots, X_p), \quad i = 1, \dots, p,$$

and the density of Y_1, \dots, Y_p be $g(\mathbf{y})$. Then we have

$$\int_{\mathbf{u}(\Omega)} g(\mathbf{y}) \, d\mathbf{y} = \int_{\Omega} g(\mathbf{u}(\mathbf{x})) \, \text{abs}(|\mathbf{J}(\mathbf{x})|) \, d\mathbf{x}, \tag{3}$$

and

$$f(\mathbf{x}) = g(\mathbf{u}(\mathbf{x})) \, \text{abs}(|\mathbf{J}(\mathbf{x})|), \tag{4}$$

where the Jacobin matrix is

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_p} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_p} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_p}{\partial x_1} & \frac{\partial u_p}{\partial x_2} & \cdots & \frac{\partial u_p}{\partial x_p} \end{bmatrix}.$$

A roughly proof for above results:

- If $\mathbf{A} \in \mathbb{R}^{p \times p}$ and $\mathcal{S} \subset \mathbb{R}^p$ is a measurable set, then $m(\mathbf{A}\mathcal{S}) = |\det(\mathbf{A})|m(\mathcal{S})$. Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ where \mathbf{U} and \mathbf{V} are orthogonal and $\mathbf{\Sigma}$ is diagonal with nonnegative entries. Multiplying by \mathbf{V}^\top doesn't change the measure of \mathcal{S} . Multiplying by $\mathbf{\Sigma}$ scales along each axis, so the measure gets multiplied by $|\det(\mathbf{\Sigma})| = |\det(\mathbf{A})|$. Multiplying by \mathbf{U} doesn't change the measure.

- We consider the probability of \mathbf{x} in Ω and \mathbf{y} in $\mathbf{u}(\Omega)$; and partition Ω into $\{\Omega_i\}_i$. Then

$$\begin{aligned}
& \int_{\mathbf{u}(\Omega)} g(\mathbf{y}) d\mathbf{y} \\
&= \sum_i g(\mathbf{u}(\mathbf{x}_i)) m(\mathbf{u}(\Omega_i)) \\
&\approx \sum_i g(\mathbf{u}(\mathbf{x}_i)) m(\mathbf{u}(\mathbf{x}_i) + \mathbf{J}(\mathbf{x}_i)(\Omega_i - \mathbf{x}_i)) \\
&= \sum_i g(\mathbf{u}(\mathbf{x}_i)) m(\mathbf{J}(\mathbf{x}_i)\Omega_i) \\
&= \sum_i g(\mathbf{u}(\mathbf{x}_i)) \text{abs}(|\mathbf{J}(\mathbf{x}_i)|) m(\Omega_i) \\
&\approx \int_{\Omega} g(\mathbf{u}(\mathbf{x})) \text{abs}(|\mathbf{J}(\mathbf{x})|) d\mathbf{x}.
\end{aligned}$$

- Consider notation Ω such that

$$\int_{\Omega} = \int_{x_1}^{x'_1} \cdots \int_{x_p}^{x'_p}$$

where $x_1 \leq x'_1, x_2 \leq x'_2, \dots, x_p \leq x'_p$. Then the notation $\mathbf{u}(\Omega)$ in the integral should consider the order

$$\int_{\mathbf{u}(\Omega)} = \int_{\min\{u_1(x_1), u_1(x'_1)\}}^{\max\{u_1(x_1), u_1(x'_1)\}} \cdots \int_{\min\{u_p(x_p), u_p(x'_p)\}}^{\max\{u_p(x_p), u_p(x'_p)\}}$$

By using even tinier subsets Ω_i , the approximation would be even better so we see by a limiting argument that we actually obtain (3). On the other hand, we have (f is density functions of \mathbf{x} on Ω ; g is density function of \mathbf{y} on $\mathbf{u}(\Omega)$; $\mathbf{y} = \mathbf{u}(\mathbf{x})$ means \mathbf{x} and $\mathbf{y} = \mathbf{u}(\mathbf{x})$ are one-to-one mapping).

$$\int_{\Omega} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{u}(\Omega)} g(\mathbf{y}) d\mathbf{y} = \int_{\Omega} g(\mathbf{u}(\mathbf{x})) \text{abs}(|\mathbf{J}(\mathbf{x})|) d\mathbf{x}.$$

Since it holds for any Ω , then

$$f(\mathbf{x}) = g(\mathbf{u}(\mathbf{x})) \text{abs}(|\mathbf{J}(\mathbf{x})|).$$

Lemma 2.1. If \mathbf{Z} is an $m \times n$ random matrix, \mathbf{D} is an $l \times m$ real matrix, \mathbf{E} is an $n \times q$ real matrix, and \mathbf{F} is an $l \times q$ real matrix, then

$$\mathbb{E}[\mathbf{D}\mathbf{Z}\mathbf{E} + \mathbf{F}] = \mathbf{D}\mathbb{E}[\mathbf{Z}]\mathbf{E} + \mathbf{F}.$$

Proof. The element in the i -th row and j -th column of $\mathbb{E}[\mathbf{D}\mathbf{Z}\mathbf{E} + \mathbf{F}]$ is

$$\mathbb{E} \left[\sum_{h,g} d_{ih} z_{hg} e_{gj} + f_{ij} \right] = \sum_{h,g} d_{ih} \mathbb{E}[z_{hg}] e_{gj} + f_{ij}$$

which is the element in the i -th row and j -th column of $\mathbf{D}\mathbb{E}[\mathbf{Z}]\mathbf{E} + \mathbf{F}$. □

Lemma 2.2. If $\mathbf{y} = \mathbf{D}\mathbf{x} + \mathbf{f} \in \mathbb{R}^l$, where \mathbf{D} is an $l \times m$ real matrix, $\mathbf{x} \in \mathbb{R}^m$ is a random vector, then

$$\mathbb{E}[\mathbf{y}] = \mathbf{D}\mathbb{E}[\mathbf{x}] + \mathbf{f} \quad \text{and} \quad \text{Cov}[\mathbf{y}] = \mathbf{D}\text{Cov}[\mathbf{x}]\mathbf{D}^\top.$$

Proof. We have

$$\begin{aligned}
& \text{Cov}(\mathbf{y}) \\
&= \mathbb{E} [(\mathbf{y} - \mathbb{E}[\mathbf{y}])(\mathbf{y} - \mathbb{E}[\mathbf{y}])^\top] \\
&= \mathbb{E} [(\mathbf{D}\mathbf{x} + \mathbf{f} - \mathbb{E}[\mathbf{D}\mathbb{E}[\mathbf{x}] + \mathbf{f}])(\mathbf{D}\mathbf{x} + \mathbf{f} - \mathbb{E}[\mathbf{D}\mathbb{E}[\mathbf{x}] + \mathbf{f}])^\top] \\
&= \mathbb{E}[(\mathbf{D}\mathbf{x} - \mathbf{D}\mathbb{E}[\mathbf{x}])(\mathbf{D}\mathbf{x} - \mathbf{D}\mathbb{E}[\mathbf{x}])^\top] \\
&= \mathbb{E}[\mathbf{D}(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^\top \mathbf{D}^\top] \\
&= \mathbf{D}\mathbb{E}[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^\top] \mathbf{D}^\top \\
&= \mathbf{D}\text{Cov}[\mathbf{x}]\mathbf{D}^\top.
\end{aligned}$$

□

The Density Function of Multivariate Normal Distribution Let the spectral decomposition of \mathbf{A} be $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, then we take $\mathbf{C} = \mathbf{U}\mathbf{\Lambda}^{-1/2}$ and it satisfies $\mathbf{C}^\top \mathbf{A} \mathbf{C} = \mathbf{I}$ and \mathbf{C} is non-singular. Define $\mathbf{y} = \mathbf{C}^{-1}(\mathbf{x} - \mathbf{b})$, then

$$\begin{aligned}
K^{-1} &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{b})^\top \mathbf{A}(\mathbf{x} - \mathbf{b})\right) dx_1 \dots dx_p \\
&= \frac{1}{\det(\mathbf{C}^{-1})} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\mathbf{y}^\top \mathbf{y}\right) dy_1 \dots dy_p \\
&= \det(\mathbf{C}) \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2} \sum_{i=1}^n y_i^2\right) dy_1 \dots dy_p \\
&= \det(\mathbf{A}^{-\frac{1}{2}}) \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}y_p^2\right) \dots \exp\left(-\frac{1}{2}y_1^2\right) dy_1 \dots dy_p \\
&= \det(\mathbf{A}^{-\frac{1}{2}})(2\pi)^{\frac{p}{2}}.
\end{aligned}$$

Directly consider the expectation and variance of \mathbf{x} is not easy, so we first consider the ones of \mathbf{y} . The relation $\mathbf{y} = \mathbf{C}^{-1}(\mathbf{x} - \mathbf{b})$ means $\mathbf{x} = \mathbf{C}\mathbf{y} + \mathbf{b}$ and $\mathbb{E}[\mathbf{x}] = \mathbf{C}\mathbb{E}[\mathbf{y}] + \mathbf{b}$. The transformation implies the density function of \mathbf{y} is

$$\begin{aligned}
g(\mathbf{y}) &= \det(\mathbf{C})K \exp\left(-\frac{1}{2}(\mathbf{C}\mathbf{y} + \mathbf{b} - \mathbf{b})^\top \mathbf{A}(\mathbf{C}\mathbf{y} + \mathbf{b} - \mathbf{b})\right) dy_1 \dots dy_p \\
&= \det(\mathbf{C})K \exp\left(-\frac{1}{2}\mathbf{y}^\top \mathbf{C}^\top \mathbf{A} \mathbf{C} \mathbf{y}\right) dy_1 \dots dy_p \\
&= K \det(\mathbf{C}) \exp\left(-\frac{1}{2}\mathbf{y}^\top \mathbf{y}\right) dy_1 \dots dy_p \\
&= \frac{\det(\mathbf{C})}{\sqrt{(2\pi)^p \det(\mathbf{A})}} \exp\left(-\frac{1}{2} \sum_{i=1}^p y_i^2\right) dy_1 \dots dy_p \\
&= \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^p y_i^2\right) dy_1 \dots dy_p.
\end{aligned}$$

Then for each $i = 1, \dots, p$, we have

$$\begin{aligned}
\mathbb{E}[y_i] &= \frac{1}{(2\pi)^{p/2}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} y_i \exp\left(-\frac{1}{2} \sum_{j=1}^p y_j^2\right) dy_1 \dots dy_p \\
&= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y_i \exp\left(-\frac{1}{2}y_i^2\right) dy_i\right) \prod_{j=1, j \neq i}^p \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}y_j^2\right) dy_j
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y_i \exp\left(-\frac{1}{2}y_i^2\right) dy_i = 0.$$

Thus $\mathbb{E}[\mathbf{y}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{x}] = \mathbf{C}\mathbb{E}[\mathbf{y}] + \mathbf{b} = \boldsymbol{\mu}$ implies $\mathbf{b} = \boldsymbol{\mu}$.

The relation $\mathbf{x} = \mathbf{C}\mathbf{y} + \mathbf{b}$ means $\text{Cov}[\mathbf{x}] = \mathbf{C}\text{Cov}[\mathbf{y}]\mathbf{C}^\top = \mathbf{C}\mathbb{E}[\mathbf{y}\mathbf{y}^\top]\mathbf{C}^\top$. For each $i \neq j$, we have

$$\begin{aligned} & \mathbb{E}[y_i y_j] \\ &= \frac{1}{(2\pi)^{p/2}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} y_i y_j \exp\left(-\frac{1}{2} \sum_{h=1}^p y_h^2\right) dy_1 \dots dy_p \\ &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y_i \exp\left(-\frac{1}{2}y_i^2\right) dy_i \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y_j \exp\left(-\frac{1}{2}y_j^2\right) dy_j \right) \prod_{j=1, h \neq i, j}^p \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}y_h^2\right) dy_h \\ &= 0 \end{aligned}$$

We also have

$$\begin{aligned} & \mathbb{E}[y_i^2] \\ &= \frac{1}{(2\pi)^{p/2}} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} y_i^2 \exp\left(-\frac{1}{2} \sum_{h=1}^p y_h^2\right) dy_1 \dots dy_p \\ &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y_i^2 \exp\left(-\frac{1}{2}y_i^2\right) dy_i \right) \prod_{j=1, h \neq i}^p \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}y_h^2\right) dy_h = 1, \end{aligned}$$

where the last step is due to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}y_h^2\right) dy_h$$

corresponds to the pdf of $y_h \sim \mathcal{N}(0, 1)$ and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y_i^2 \exp\left(-\frac{1}{2}y_i^2\right) dy_i$$

corresponds to the variance of $y_i \sim \mathcal{N}(0, 1)$. Hence, it holds that

$$\mathbb{E}[(y_i - \mathbb{E}[y_i])(y_j - \mathbb{E}[y_j])] = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

which implies $\boldsymbol{\Sigma} = \text{Cov}[\mathbf{x}] = \mathbf{C}\mathbb{E}[\mathbf{y}\mathbf{y}^\top]\mathbf{C}^\top = \mathbf{C}\mathbf{C}^\top$. Since $\mathbf{C}^\top \mathbf{A} \mathbf{C} = \mathbf{I}$, we obtain $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}^\top$ and $\boldsymbol{\Sigma} = \mathbf{A}^{-1} \succ \mathbf{0}$.

Theorem 2.1. Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma} \in \mathbb{R}^{p \times p}$ and $\boldsymbol{\Sigma} \succ \mathbf{0}$. Then

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

is distributed according to $\mathcal{N}_p(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)$ for non-singular $\mathbf{C} \in \mathbb{R}^{p \times p}$.

Proof. Let $f(\mathbf{x})$ be the density of \mathbf{x} such that

$$f(\mathbf{x}) = n(\boldsymbol{\mu} \mid \boldsymbol{\Sigma}) = \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right)$$

and $g(\mathbf{y})$ be the density function of \mathbf{y} . The relation $\mathbf{x} = \mathbf{C}^{-1}\mathbf{y}$ implies $g(\mathbf{y}) = f(\mathbf{u}^{-1}(\mathbf{y}))|\det(\mathbf{J}^{-1}(\mathbf{y}))|$ with $\mathbf{u}(\mathbf{x}) = \mathbf{C}\mathbf{x}$, $\mathbf{u}^{-1}(\mathbf{y}) = \mathbf{C}^{-1}\mathbf{y}$ and $\mathbf{J}^{-1}(\mathbf{y}) = \mathbf{C}^{-1}$. Hence, we have

$$g(\mathbf{y})$$

$$\begin{aligned}
&= f(\mathbf{C}^{-1}\mathbf{y}) |\det(\mathbf{C}^{-1})| \\
&= \frac{1}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{C}^{-1}\mathbf{y} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{C}^{-1}\mathbf{y} - \boldsymbol{\mu})\right) |\det(\mathbf{C}^{-1})| \\
&= \frac{|\det(\mathbf{C}^{-1})|}{\sqrt{(2\pi)^p \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{C}\boldsymbol{\mu})^\top \mathbf{C}^{-\top} \boldsymbol{\Sigma}^{-1} \mathbf{C}^{-1}(\mathbf{y} - \mathbf{C}\boldsymbol{\mu})\right) \\
&= \frac{1}{\sqrt{(2\pi)^p \det(\mathbf{C}\boldsymbol{\Sigma}^{-1}\mathbf{C}^\top)}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{C}\boldsymbol{\mu})^\top (\mathbf{C}\boldsymbol{\Sigma}^{-1}\mathbf{C}^\top)^{-1}(\mathbf{y} - \mathbf{C}\boldsymbol{\mu})\right) \\
&= n(\mathbf{C}\boldsymbol{\mu} \mid \mathbf{C}\boldsymbol{\Sigma}^{-1}\mathbf{C}^\top),
\end{aligned}$$

where we use the fact

$$\frac{|\det(\mathbf{C}^{-1})|}{\sqrt{\det(\boldsymbol{\Sigma})}} = \frac{1}{\sqrt{|\det(\mathbf{C})|^2 \det(\boldsymbol{\Sigma})}} = \frac{1}{\sqrt{|\det(\mathbf{C})| \det(\boldsymbol{\Sigma}) |\det(\mathbf{C}^\top)|}} = \frac{1}{\sqrt{|\det(\mathbf{C}\boldsymbol{\Sigma}\mathbf{C}^\top)|}}.$$

□

Theorem 2.2. If $\mathbf{x} = [x_1, \dots, x_p]^\top$ have a joint normal distribution. Let

1. $\mathbf{x}^{(1)} = [x_1, \dots, x_q]^\top$,
2. $\mathbf{x}^{(2)} = [x_{q+1}, \dots, x_p]^\top$.

for $q < p$. A necessary and sufficient condition for $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ to be independent is that each covariance of a variable from $\mathbf{x}^{(1)}$ and a variable from $\mathbf{x}^{(2)}$ is 0.

Proof. Let

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad \text{where } \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

such that

- $\boldsymbol{\mu}^{(1)} = \mathbb{E}[\mathbf{x}^{(1)}]$,
- $\boldsymbol{\mu}^{(2)} = \mathbb{E}[\mathbf{x}^{(2)}]$,
- $\boldsymbol{\Sigma}_{11} = \mathbb{E}[(\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)}) (\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)})^\top]$,
- $\boldsymbol{\Sigma}_{22} = \mathbb{E}[(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)}) (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})^\top]$,
- $\boldsymbol{\Sigma}_{12} = \boldsymbol{\Sigma}_{21}^\top = \mathbb{E}[(\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)}) (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})^\top]$.

Sufficiency (uncorrelated \implies independent): The random vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are uncorrelated means

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix}.$$

The quadratic form of $n(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is

$$\begin{aligned}
&(\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}) \\
&= [(\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)})^\top \quad (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})^\top] \begin{bmatrix} \boldsymbol{\Sigma}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)} \\ \mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)} \end{bmatrix} \\
&= (\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)})^\top \boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)}) + (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})^\top \boldsymbol{\Sigma}_{22}^{-1}(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})
\end{aligned}$$

and we have $\det(\mathbf{\Sigma}) = \det(\mathbf{\Sigma}_{11}) \det(\mathbf{\Sigma}_{22})$. Then

$$\begin{aligned}
& n(\boldsymbol{\mu} \mid \mathbf{\Sigma}) \\
&= \frac{1}{\sqrt{(2\pi)^p \det(\mathbf{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^\top \mathbf{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right) \\
&= \frac{1}{\sqrt{(2\pi)^q \det(\mathbf{\Sigma}_{11})}} \exp\left(-\frac{1}{2}(\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)})^\top \mathbf{\Sigma}^{-1}(\mathbf{x}^{(1)} - \boldsymbol{\mu}^{(1)})\right) \\
&\quad \cdot \frac{1}{\sqrt{(2\pi)^{p-q} \det(\mathbf{\Sigma}_{22})}} \exp\left(-\frac{1}{2}(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})^\top \mathbf{\Sigma}_{22}^{-1}(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})\right) \\
&= n(\boldsymbol{\mu}^{(1)} \mid \mathbf{\Sigma}^{(1)}) n(\boldsymbol{\mu}^{(2)} \mid \mathbf{\Sigma}^{(2)}).
\end{aligned}$$

Thus the marginal distribution of $\mathbf{x}^{(1)}$ is $\mathcal{N}(\boldsymbol{\mu}^{(1)}, \mathbf{\Sigma}_{11})$ and the marginal distribution of $\mathbf{x}^{(2)}$ is $\mathcal{N}(\boldsymbol{\mu}^{(2)}, \mathbf{\Sigma}_{22})$. We have prove two variables are independent.

Necessity (independent \implies uncorrelated): Let $1 \leq i \leq q$ and $q+1 \leq j \leq p$. The Independence means

$$\begin{aligned}
\sigma_{ij} &= \mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)] \\
&= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (x_i - \mu_i)(x_j - \mu_j) f(x_1, \dots, x_p) dx_1 \dots dx_p \\
&= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (x_i - \mu_i)(x_j - \mu_j) f(x_1, \dots, x_q) f(x_{q+1}, \dots, x_p) dx_1 \dots dx_p \\
&= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (x_i - \mu_i) f(x_1, \dots, x_q) dx_1 \dots dx_q \cdot \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} (x_j - \mu_j) f(x_{q+1}, \dots, x_p) dx_{q+1} \dots dx_p \\
&= 0.
\end{aligned}$$

□

Theorem 2.3. If $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} \succ \mathbf{0}$, the marginal distribution of any set of components of \mathbf{x} is multivariate normal with means, variances, and covariances obtained by taking the corresponding components of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$, respectively.

Proof. We shall make a non-singular linear transformation \mathbf{B} to subvectors

$$\begin{aligned}
\mathbf{y}^{(1)} &= \mathbf{x}^{(1)} + \mathbf{B}\mathbf{x}^{(2)} \\
\mathbf{y}^{(2)} &= \mathbf{x}^{(2)}
\end{aligned}$$

leading to the components of $\mathbf{y}^{(1)}$ are uncorrelated with the ones of $\mathbf{y}^{(2)}$. The matrix \mathbf{B} should satisfy

$$\begin{aligned}
\mathbf{0} &= \mathbb{E}\left[(\mathbf{y}^{(1)} - \mathbb{E}[\mathbf{y}^{(1)}])(\mathbf{y}^{(2)} - \mathbb{E}[\mathbf{y}^{(2)}])^\top\right] \\
&= \mathbb{E}\left[(\mathbf{x}^{(1)} + \mathbf{B}\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(1)} + \mathbf{B}\mathbf{x}^{(2)}])(\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(2)}])^\top\right] \\
&= \mathbb{E}\left[(\mathbf{x}^{(1)} - \mathbb{E}[\mathbf{x}^{(1)}] + \mathbf{B}(\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(2)}]))(\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(2)}])^\top\right] \\
&= \mathbb{E}\left[(\mathbf{x}^{(1)} - \mathbb{E}[\mathbf{x}^{(1)}])(\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(2)}])^\top\right] + \mathbf{B} \cdot \mathbb{E}\left[(\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(2)}])(\mathbf{x}^{(2)} - \mathbb{E}[\mathbf{x}^{(2)}])^\top\right] \\
&= \mathbf{\Sigma}_{12} + \mathbf{B}\mathbf{\Sigma}_{22}.
\end{aligned}$$

Thus $\mathbf{B} = -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$ and $\mathbf{y}^{(1)} = \mathbf{x}^{(1)} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{x}^{(2)}$. The vector

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{x}$$

is a non-singular transform of \mathbf{x} , and therefore has a normal distribution with

$$\mathbb{E} \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\Sigma_{12}\Sigma_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbb{E}[\mathbf{x}] = \begin{bmatrix} \mathbf{I} & -\Sigma_{12}\Sigma_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}^{(2)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nu}^{(1)} \\ \boldsymbol{\nu}^{(2)} \end{bmatrix}.$$

Since the transform is non-singular, we have

$$\begin{aligned} \text{Cov} \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & -\Sigma_{12}\Sigma_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\Sigma_{22}^{-1}\Sigma_{21} & \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \mathbf{0} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\Sigma_{22}^{-1}\Sigma_{21} & \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \mathbf{0} \\ \mathbf{0} & \Sigma_{22} \end{bmatrix} \end{aligned}$$

Thus $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are independent, which implies the marginal distribution of $\mathbf{x}^{(2)}$ is $\mathcal{N}(\boldsymbol{\mu}^{(2)}, \Sigma_{22})$. Because the numbering of the components of \mathbf{x} is arbitrary, we have proved this theorem. \square

Singular Normal Distribution The mass is concentrated on a linear set \mathcal{S} . For any $x \notin \mathcal{S}$, there exists $\mathcal{B}(x, r)$ such that $r > 0$ and $\mathcal{B} \cap \mathcal{S} = \emptyset$. If the distribution of x has density function f , then $f(x) = 0$ holds for any $x \notin \mathcal{S}$. Since the measure of \mathcal{S} is zero, we have $f(x) = 0$ almost everywhere, which means the integration of $f(x)$ on the whole space is 0.

Conditional Distribution by Schur Complement Recall that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}^{-1}\mathbf{C} & \mathbf{I} \end{bmatrix},$$

which directly means the inverse of covariance of Normal distribution.

Theorem 2.4. Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then

$$\mathbf{z} = \mathbf{D}\mathbf{x}$$

is distributed according to $\mathcal{N}_q(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top)$ for any $\mathbf{D} \in \mathbb{R}^{q \times p}$.

Proof. It is easy to verify $\mathbb{E}[\mathbf{z}] = \mathbf{D}\boldsymbol{\mu}$ and $\text{Cov}[\mathbf{z}] = \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top$. Hence, we only need to show \mathbf{z} follows normal distribution.

Since $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, it can be presented as

$$\mathbf{x} = \mathbf{A}\mathbf{y} + \boldsymbol{\lambda}$$

where $\mathbf{A} \in \mathbb{R}^{p \times r}$, r is the rank of $\boldsymbol{\Sigma}$ and $\mathbf{y} \sim \mathcal{N}_r(\boldsymbol{\nu}, \mathbf{T})$ with non-singular $\mathbf{T} \succ \mathbf{0}$. We can write

$$\mathbf{z} = \mathbf{D}\mathbf{A}\mathbf{y} + \mathbf{D}\boldsymbol{\lambda},$$

where $\mathbf{D}\mathbf{A} \in \mathbb{R}^{q \times r}$. If the rank of $\mathbf{D}\mathbf{A}$ is r , the formal definition of a normal distribution that includes the singular distribution implies \mathbf{z} follows normal distribution.

If the rank of $\mathbf{D}\mathbf{A}$ is less than r , say s , then

$$\mathbf{E} = \text{Cov}[\mathbf{z}] = \mathbf{D}\mathbf{A}\text{Cov}[\mathbf{y}]\mathbf{A}^\top\mathbf{D}^\top = \mathbf{D}\mathbf{A}\mathbf{T}\mathbf{A}^\top\mathbf{D}^\top \in \mathbb{R}^{q \times q}$$

is rank of s . There is a non-singular matrix

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} \in \mathbb{R}^{q \times q}$$

with $\mathbf{F}_1 \in \mathbb{R}^{s \times q}$ and $\mathbf{F}_2 \in \mathbb{R}^{(q-s) \times r}$ such that

$$\mathbf{F}\mathbf{E}\mathbf{F}^\top = \begin{bmatrix} \mathbf{F}_1\mathbf{E}\mathbf{F}_1^\top & \mathbf{F}_1\mathbf{E}\mathbf{F}_2^\top \\ \mathbf{F}_2\mathbf{E}\mathbf{F}_1^\top & \mathbf{F}_2\mathbf{E}\mathbf{F}_2^\top \end{bmatrix} \begin{bmatrix} (\mathbf{F}_1\mathbf{D}\mathbf{A})\mathbf{T}(\mathbf{F}_1\mathbf{D}\mathbf{A})^\top & (\mathbf{F}_1\mathbf{D}\mathbf{A})\mathbf{T}(\mathbf{F}_2\mathbf{D}\mathbf{A})^\top \\ (\mathbf{F}_2\mathbf{D}\mathbf{A})\mathbf{T}(\mathbf{F}_1\mathbf{D}\mathbf{A})^\top & (\mathbf{F}_2\mathbf{D}\mathbf{A})\mathbf{T}(\mathbf{F}_2\mathbf{D}\mathbf{A})^\top \end{bmatrix} = \begin{bmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Thus $(\mathbf{F}_1\mathbf{D}\mathbf{A})\mathbf{T}(\mathbf{F}_1\mathbf{D}\mathbf{A})^\top = \mathbf{I}_s$ means $\mathbf{F}_1\mathbf{D}\mathbf{A}$ is of rank s and the non-singularity of \mathbf{T} means $\mathbf{F}_2\mathbf{D}\mathbf{A} = \mathbf{0}$. Hence, we have

$$\mathbf{F}\mathbf{z}' = \mathbf{F}(\mathbf{D}\mathbf{A}\mathbf{y} + \mathbf{D}\boldsymbol{\lambda}) = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} \mathbf{D}\mathbf{A}\mathbf{y} + \mathbf{F}\mathbf{D}\boldsymbol{\lambda} = \begin{bmatrix} \mathbf{F}_1\mathbf{D}\mathbf{A}\mathbf{y} \\ \mathbf{F}_2\mathbf{D}\mathbf{A}\mathbf{y} \end{bmatrix} + \mathbf{F}\mathbf{D}\boldsymbol{\lambda} = \begin{bmatrix} \mathbf{F}_1\mathbf{D}\mathbf{A}\mathbf{y} \\ \mathbf{0} \end{bmatrix} + \mathbf{F}\mathbf{D}\boldsymbol{\lambda}.$$

Let $\mathbf{u}_1 = \mathbf{F}_1\mathbf{D}\mathbf{A}\mathbf{y} \in \mathbb{R}^s$. Since $\mathbf{F}_1\mathbf{D}\mathbf{A} \in \mathbb{R}^{s \times r}$ is of rank $s \leq r$, we conclude \mathbf{u}_1 has a non-singular normal distribution. Let $\mathbf{F}^{-1} = [\mathbf{G}_1, \mathbf{G}_2]$, where $\mathbf{G}_1 \in \mathbb{R}^{q \times s}$ and $\mathbf{G}_2 \in \mathbb{R}^{q \times (q-s)}$. Then

$$\mathbf{z} = \mathbf{F}^{-1} \left(\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{0} \end{bmatrix} + \mathbf{F}\mathbf{D}\boldsymbol{\lambda} \right) = [\mathbf{G}_1, \mathbf{G}_2] \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{0} \end{bmatrix} + \mathbf{D}\boldsymbol{\lambda} = \mathbf{G}_1\mathbf{u}_1 + \mathbf{D}\boldsymbol{\lambda}$$

which is of the form of the formal definition of normal distribution. \square

Theorem 2.5. For $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and every vector $\boldsymbol{\alpha} \in \mathbb{R}^{(p-q)}$, we have

$$\text{Var}[x_i^{(11.2)}] \leq \text{Var}[x_i - \boldsymbol{\alpha}^\top \mathbf{x}^{(2)}],$$

for $i = 1, \dots, q$, where $x_i^{(11.2)}$ and x_i are the i -th entry of $\mathbf{x}^{(11.2)}$ and the i -th entry of \mathbf{x} respectively.

Proof. We denote

$$\mathbf{B} = \begin{bmatrix} \boldsymbol{\beta}_{(1)}^\top \\ \vdots \\ \boldsymbol{\beta}_{(q)}^\top \end{bmatrix}.$$

Since $\mathbf{x}^{(11.2)}$ is uncorrelated with $\mathbf{x}^{(2)}$ and

$$\mathbb{E}[\mathbf{x}^{(11.2)}] = \mathbb{E}[\mathbf{x}^{(1)} - (\boldsymbol{\mu}^{(1)} + \mathbf{B}(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)}))] = \mathbb{E}[\mathbf{x}^{(1)}] - \boldsymbol{\mu}^{(1)} + \mathbf{B}(\mathbb{E}[\mathbf{x}^{(2)}] - \boldsymbol{\mu}^{(2)}) = \mathbf{0},$$

we have

$$\begin{aligned} & \text{Var}[x_i - \boldsymbol{\alpha}^\top \mathbf{x}^{(2)}] \\ &= \mathbb{E}[x_i - \boldsymbol{\alpha}^\top \mathbf{x}^{(2)} - \mathbb{E}[x_i - \boldsymbol{\alpha}^\top \mathbf{x}^{(2)}]]^2 \\ &= \mathbb{E}[x_i - \mu_i - \boldsymbol{\alpha}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]^2 \\ &= \mathbb{E}[x_i^{(11.2)} + \boldsymbol{\beta}_{(i)}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)}) - \boldsymbol{\alpha}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]^2 \\ &= \mathbb{E}[x_i^{(11.2)} - \mathbb{E}[x_i^{(11.2)}] + (\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha})^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]^2 \\ &= \text{Var}[x_i^{(11.2)}] + \mathbb{E}[(x_i^{(11.2)} - \mathbb{E}[x_i^{(11.2)}])(\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha})^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})] + \mathbb{E}[(\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha})^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]^2 \\ &= \text{Var}[x_i^{(11.2)}] + (\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha})^\top \mathbb{E}[(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})(\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})^\top] (\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha}) \\ &= \text{Var}[x_i^{(11.2)}] + (\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha})^\top \text{Cov}(\mathbf{x}^{(2)}) (\boldsymbol{\beta}_{(i)} - \boldsymbol{\alpha}) \\ &\geq \text{Var}[x_i^{(11.2)}], \end{aligned}$$

where the quadratic form attains its minimum of 0 at $\boldsymbol{\beta}_{(i)} = \boldsymbol{\alpha}$. \square

Remark 2.1. Observe that

$$\mathbb{E}[x_i] = \mu_i + \boldsymbol{\alpha}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})$$

Hence, the second equality in the proof means $\mu_i + \boldsymbol{\beta}_{(i)}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})$ is the best linear predictor of x_i in the sense that of all functions of $\mathbf{x}^{(2)}$ of the form $\boldsymbol{\alpha}^\top \mathbf{x}^{(2)} + c$, the mean squared error of the above is a minimum.

Theorem 2.6. Under the setting of Theorem 2.5, we have

$$\text{Corr}\left(x_i, \beta_{(i)}^\top \mathbf{x}^{(2)}\right) \geq \text{Corr}\left(x_i, \alpha^\top \mathbf{x}^{(2)}\right).$$

Proof. Since the correlation between two variables is unchanged when either or both is multiplied by a positive constant, we can assume that

$$\mathbb{E}\left[\alpha^\top \mathbf{x}^{(2)}\right]^2 = \mathbb{E}\left[\beta_{(i)}^\top \mathbf{x}^{(2)}\right]^2.$$

Using Theorem 2.5, we have

$$\begin{aligned} \text{Var}[x_i^{(11.2)}] &\leq \text{Var}[x_i - \alpha^\top \mathbf{x}^{(2)}] \\ \iff \mathbb{E}[x_i - \mu_i - \beta_{(i)}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]^2 &\leq \mathbb{E}[x_i - \mu_i - \alpha^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]^2 \\ \iff \text{Var}[x_i] - \mathbb{E}[(x_i - \mu_i)\beta_{(i)}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})] + \text{Var}[\beta_{(i)}^\top \mathbf{x}^{(2)}] \\ &\leq \text{Var}[x_i] - \mathbb{E}[(x_i - \mu_i)\alpha^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})] + \text{Var}[\alpha^\top \mathbf{x}^{(2)}] \\ \iff \frac{\mathbb{E}[(x_i - \mu_i)\alpha^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]}{\sqrt{\text{Var}[x_i]}\sqrt{\text{Var}[\alpha^\top \mathbf{x}^{(2)}]}} &\leq \frac{\mathbb{E}[(x_i - \mu_i)\beta_{(i)}^\top (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)})]}{\sqrt{\text{Var}[x_i]}\sqrt{\text{Var}[\beta_{(i)}^\top \mathbf{x}^{(2)}]}} \\ \iff \frac{\text{Cov}[x_i, \alpha^\top \mathbf{x}^{(2)}]}{\sqrt{\text{Var}[x_i]}\sqrt{\text{Var}[\alpha^\top \mathbf{x}^{(2)}]}} &\leq \frac{\mathbb{E}[x_i, \beta_{(i)}^\top \mathbf{x}^{(2)}]}{\sqrt{\text{Var}[x_i]}\sqrt{\text{Var}[\beta_{(i)}^\top \mathbf{x}^{(2)}]}} \end{aligned}$$

□

Theorem 2.7. Let $\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix}$. If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are independent and $g(\mathbf{x}) = g^{(1)}(\mathbf{x}^{(1)})g^{(2)}(\mathbf{x}^{(2)})$, its characteristic function is

$$\mathbb{E}[g(\mathbf{x})] = \mathbb{E}[g^{(1)}(\mathbf{x}^{(1)})]\mathbb{E}[g^{(2)}(\mathbf{x}^{(2)})].$$

Proof. Let $f(\mathbf{x}) = f^{(1)}(\mathbf{x}^{(1)})f^{(2)}(\mathbf{x}^{(2)})$ be the density of \mathbf{x} . If $g(x)$ is real-valued, we have

$$\begin{aligned} &\mathbb{E}[g(\mathbf{x})] \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} g(\mathbf{x})f(\mathbf{x}) \, dx_1 \dots dx_p \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} g^{(1)}(\mathbf{x}^{(1)})g^{(2)}(\mathbf{x}^{(2)})f^{(1)}(\mathbf{x}^{(1)})f^{(2)}(\mathbf{x}^{(2)}) \, dx_1 \dots dx_p \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} g^{(1)}(\mathbf{x}^{(1)})f^{(1)}(\mathbf{x}^{(1)}) \, dx_1 \dots dx_q \cdot \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} g^{(2)}(\mathbf{x}^{(2)})f^{(2)}(\mathbf{x}^{(2)}) \, dx_{q+1} \dots dx_p \\ &= \mathbb{E}[g^{(1)}(\mathbf{x}^{(1)})]\mathbb{E}[g^{(2)}(\mathbf{x}^{(2)})]. \end{aligned}$$

If $g(x)$ is complex-valued, then we have

$$\begin{aligned} &g(\mathbf{x}) \\ &= [g_1^{(1)}(\mathbf{x}^{(1)}) + i g_2^{(1)}(\mathbf{x}^{(1)})][g_1^{(2)}(\mathbf{x}^{(2)}) + i g_2^{(2)}(\mathbf{x}^{(2)})] \\ &= g_1^{(1)}(\mathbf{x}^{(1)})g_1^{(2)}(\mathbf{x}^{(2)}) - g_2^{(1)}(\mathbf{x}^{(1)})g_2^{(2)}(\mathbf{x}^{(2)}) + i [g_1^{(1)}(\mathbf{x}^{(1)})g_2^{(2)}(\mathbf{x}^{(2)}) + g_2^{(1)}(\mathbf{x}^{(1)})g_1^{(2)}(\mathbf{x}^{(2)})] \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}[g(\mathbf{x})] \\ &= \mathbb{E}[g_1^{(1)}(\mathbf{x}^{(1)})g_1^{(2)}(\mathbf{x}^{(2)})] - \mathbb{E}[g_2^{(1)}(\mathbf{x}^{(1)})g_2^{(2)}(\mathbf{x}^{(2)})] + i \mathbb{E}[g_1^{(1)}(\mathbf{x}^{(1)})g_2^{(2)}(\mathbf{x}^{(2)}) + g_2^{(1)}(\mathbf{x}^{(1)})g_1^{(2)}(\mathbf{x}^{(2)})] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}[g_1^{(1)}(\mathbf{x}^{(1)})] \mathbb{E}[g_1^{(2)}(\mathbf{x}^{(2)})] - \mathbb{E}[g_2^{(1)}(\mathbf{x}^{(1)})] \mathbb{E}[g_2^{(2)}(\mathbf{x}^{(2)})] \\
&\quad + i \mathbb{E}[g_1^{(1)}(\mathbf{x}^{(1)})] \mathbb{E}[g_2^{(2)}(\mathbf{x}^{(2)})] + i \mathbb{E}[g_2^{(1)}(\mathbf{x}^{(1)})] \mathbb{E}[g_1^{(2)}(\mathbf{x}^{(2)})] \\
&= \left[\mathbb{E}[g_1^{(1)}(\mathbf{x}^{(1)})] + i \mathbb{E}[g_2^{(1)}(\mathbf{x}^{(1)})] \right] \left[\mathbb{E}[g_1^{(2)}(\mathbf{x}^{(2)})] + i \mathbb{E}[g_2^{(2)}(\mathbf{x}^{(2)})] \right] \\
&= \mathbb{E}[g^{(1)}(\mathbf{x}^{(1)})] \mathbb{E}[g^{(2)}(\mathbf{x}^{(2)})].
\end{aligned}$$

□

Theorem 2.8. *The characteristic function of \mathbf{x} distributed according to $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is*

$$\phi(\mathbf{t}) = \exp \left(i \mathbf{t}^\top \boldsymbol{\mu} - \frac{1}{2} \mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t} \right).$$

for every $\mathbf{t} \in \mathbb{R}^p$.

Proof. For standard normal distribution $\mathbf{y} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$, we have

$$\begin{aligned}
\phi_0(\mathbf{t}) &= \mathbb{E} [\exp(i \mathbf{t}^\top \mathbf{y})] \\
&= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \frac{\exp(i \mathbf{t}^\top \mathbf{y})}{(2\pi)^{p/2}} \exp \left(-\frac{1}{2} \mathbf{y}^\top \mathbf{y} \right) dy_1 \dots dy_p \\
&= \prod_{j=1}^p \left(\int_{-\infty}^{+\infty} \frac{\exp(i t_j y_j)}{(2\pi)^{p/2}} \exp \left(-\frac{1}{2} y_j^2 \right) dy_j \right) \\
&= \prod_{j=1}^p \left(\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{p/2}} \exp \left(-\frac{1}{2} (y_j - i t_j)^2 - \frac{1}{2} t_j^2 \right) dy_j \right) \\
&= \prod_{j=1}^p \left(\exp \left(-\frac{1}{2} t_j^2 \right) \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{p/2}} \exp \left(-\frac{1}{2} z_j^2 \right) dz_j \right) \\
&= \prod_{j=1}^p \left(\exp \left(-\frac{1}{2} t_j^2 \right) \right) = \exp \left(-\frac{1}{2} \mathbf{t}^\top \mathbf{t} \right).
\end{aligned}$$

For the general case of $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we can write $\mathbf{x} = \mathbf{A}\mathbf{y} + \boldsymbol{\mu}$ such that $\mathbf{y} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$ and $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^\top$. Then we have

$$\begin{aligned}
\phi(\mathbf{t}) &= \mathbb{E} [\exp(i \mathbf{t}^\top \mathbf{x})] \\
&= \mathbb{E} [\exp(i \mathbf{t}^\top (\mathbf{A}\mathbf{y} + \boldsymbol{\mu}))] \\
&= \exp(i \mathbf{t}^\top \boldsymbol{\mu}) \mathbb{E} [\exp(i (\mathbf{A}^\top \mathbf{t})^\top \mathbf{y})] \\
&= \exp(i \mathbf{t}^\top \boldsymbol{\mu}) \phi_0(\mathbf{A}^\top \mathbf{t}) \\
&= \exp(i \mathbf{t}^\top \boldsymbol{\mu}) \exp \left(-\frac{1}{2} \mathbf{t}^\top \mathbf{A} \mathbf{A}^\top \mathbf{t} \right) \\
&= \exp \left(i \mathbf{t}^\top \boldsymbol{\mu} - \frac{1}{2} \mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t} \right).
\end{aligned}$$

□

Remark 2.2. *Denote the characteristic function of $\mathbf{x} \in \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ as $\phi_{\mathbf{x}}(\mathbf{t}) = \exp(i \mathbf{t}^\top \boldsymbol{\mu} - \frac{1}{2} \mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t})$. For $\mathbf{z} = \mathbf{D}\mathbf{x}$, the characteristic function of \mathbf{z} is*

$$\phi_{\mathbf{z}}(\mathbf{t}) = \mathbb{E} [\exp(i \mathbf{t}^\top \mathbf{z})] = \mathbb{E} [\exp(i \mathbf{t}^\top \mathbf{D}\mathbf{x})] = \mathbb{E} [\exp(i (\mathbf{D}^\top \mathbf{t})^\top \mathbf{x})] = \exp \left(i \mathbf{t}^\top (\mathbf{D}\boldsymbol{\mu}) - \frac{1}{2} \mathbf{t}^\top (\mathbf{D}^\top \boldsymbol{\Sigma} \mathbf{D}) \mathbf{t} \right)$$

which implies $\mathbf{z} \sim \mathcal{N}(\mathbf{D}\boldsymbol{\mu}, \mathbf{D}^\top \boldsymbol{\Sigma} \mathbf{D})$ and we prove Theorem 2.4.

Theorem 2.9. *If every linear combination of the components of a random vector \mathbf{y} is normally distributed, then \mathbf{y} is normally distributed.*

Proof. Let \mathbf{y} is a random vector with $\mathbb{E}[\mathbf{y}] = \boldsymbol{\mu}$ and $\text{Cov}[\mathbf{y}] = \boldsymbol{\Sigma}$. Suppose the univariate random variable $\mathbf{u}^\top \mathbf{y}$ (linear combination of \mathbf{y}) is normal distributed for any $\mathbf{u} \in \mathbb{R}^p$. The characteristic function of $\mathbf{u}^\top \mathbf{y}$ is

$$\begin{aligned}\phi_{\mathbf{u}^\top \mathbf{y}}(t) &= \mathbb{E} [\exp(i t \mathbf{u}^\top \mathbf{y})] \\ &= \exp \left(i t \mathbb{E}[\mathbf{u}^\top \mathbf{y}] - \frac{1}{2} t^2 \text{Cov}(\mathbf{u}^\top \mathbf{y}) \right) \\ &= \exp \left(i t \mathbf{u}^\top \boldsymbol{\mu} - \frac{1}{2} t^2 \mathbf{u}^\top \boldsymbol{\Sigma} \mathbf{u} \right).\end{aligned}$$

Set $t = 1$, then we have

$$\mathbb{E} [\exp(i \mathbf{u}^\top \mathbf{y})] = \exp \left(i \mathbf{u}^\top \boldsymbol{\mu} - \frac{1}{2} \mathbf{u}^\top \boldsymbol{\Sigma} \mathbf{u} \right).$$

which implies the characteristic function of \mathbf{y} is

$$\phi_{\mathbf{y}}(\mathbf{u}) = \exp \left(i \mathbf{u}^\top \boldsymbol{\mu} - \frac{1}{2} \mathbf{u}^\top \boldsymbol{\Sigma} \mathbf{u} \right),$$

that is, $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. □

Theorem 2.10. *Let $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$, $\mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$ and $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Suppose that \mathbf{x} and \mathbf{y} are independent. Prove $\mathbf{z} \sim \mathcal{N}_p(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2)$.*

Proof. Let $\phi_{\mathbf{x}}$, $\phi_{\mathbf{y}}$ and $\phi_{\mathbf{z}}$ be the characteristic functions of \mathbf{x} , \mathbf{y} and \mathbf{z} . Then we have

$$\begin{aligned}\phi_{\mathbf{z}}(\mathbf{t}) &= \mathbb{E} [\exp(i \mathbf{t}^\top (\mathbf{x} + \mathbf{y}))] \\ &= \mathbb{E} [\exp(i \mathbf{t}^\top \mathbf{x})] \mathbb{E} [\exp(i \mathbf{t}^\top \mathbf{y})] \\ &= \exp \left(-i \mathbf{t}^\top \boldsymbol{\mu}_1 + \frac{1}{2} \mathbf{t}^\top \boldsymbol{\Sigma}_1 \mathbf{t} \right) \exp \left(-i \mathbf{t}^\top \boldsymbol{\mu}_2 + \frac{1}{2} \mathbf{t}^\top \boldsymbol{\Sigma}_2 \mathbf{t} \right) \\ &= \exp \left(-i \mathbf{t}^\top (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) + \frac{1}{2} \mathbf{t}^\top (\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2) \mathbf{t} \right),\end{aligned}$$

which is the characteristic function of $\mathcal{N}_p(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2)$. □

3 Estimation of the Mean Vector and the Covariance

Theorem 3.1. *If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ constitute a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $p < N$, the maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are*

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

respectively.

Proof. The logarithm of the likelihood function is

$$\ln L = -\frac{PN}{2} \ln 2\pi - \frac{N}{2} \ln (\det(\boldsymbol{\Sigma})) - \frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \boldsymbol{\mu}).$$

We have

$$\begin{aligned}
& \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \boldsymbol{\mu}) \\
&= \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) + \sum_{\alpha=1}^N (\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) \\
&\quad + \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) + \sum_{\alpha=1}^N (\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) \\
&= \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) + \sum_{\alpha=1}^N (\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) \\
&\geq \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}),
\end{aligned}$$

where the equality holds when $\boldsymbol{\mu} = \bar{\mathbf{x}}$. Hence, the estimator of means should be $\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}}$.

Now, we only need to study how to maximize

$$-\frac{pN}{2} \ln 2\pi - \frac{N}{2} \ln (\det(\boldsymbol{\Sigma})) - \frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}).$$

We let $\boldsymbol{\Psi} = \boldsymbol{\Sigma}^{-1}$ and

$$\begin{aligned}
l(\boldsymbol{\Psi}) &= -\frac{pN}{2} \ln 2\pi - \frac{N}{2} \ln (\det(\boldsymbol{\Psi}^{-1})) - \frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Psi} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) \\
&= -\frac{pN}{2} \ln 2\pi + \frac{N}{2} \ln (\det(\boldsymbol{\Psi})) - \frac{1}{2} \sum_{\alpha=1}^N \text{tr}((\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Psi} (\mathbf{x}_\alpha - \bar{\mathbf{x}})) \\
&= -\frac{pN}{2} \ln 2\pi + \frac{N}{2} \ln (\det(\boldsymbol{\Psi})) - \frac{1}{2} \sum_{\alpha=1}^N \text{tr}((\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Psi}),
\end{aligned}$$

then

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\Psi})}{\partial \boldsymbol{\Psi}} &= \frac{\partial}{\partial \boldsymbol{\Psi}} \left(-\frac{pN}{2} \ln 2\pi + \frac{N}{2} \ln (\det(\boldsymbol{\Psi})) - \frac{1}{2} \sum_{\alpha=1}^N \text{tr}((\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Psi}) \right) \\
&= \frac{N}{2} \boldsymbol{\Psi}^{-1} - \frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.
\end{aligned}$$

We can verify $l(\boldsymbol{\Psi})$ is concave on the domain of symmetric positive definite matrices, which means the maximum is taken by $\frac{\partial f(\boldsymbol{\Psi})}{\partial \boldsymbol{\Psi}} = \mathbf{0}$, that is,

$$\boldsymbol{\Sigma} = \boldsymbol{\Psi}^{-1} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

□

Lemma 3.1. *If $\mathbf{D} \in \mathbb{R}^{p \times p}$ is positive definite, the maximum of*

$$f(\mathbf{G}) = -N \ln \det(\mathbf{G}) - \text{tr}(\mathbf{G}^{-1} \mathbf{D})$$

with respect to positive definite matrices \mathbf{G} exists, occurs at $\mathbf{G} = \frac{1}{N} \mathbf{D}$.

Proof. Let $\mathbf{D} = \mathbf{E}\mathbf{E}^\top$ and $\mathbf{E}^\top \mathbf{G}^{-1} \mathbf{E} = \mathbf{H}$. Then we have $\mathbf{G} = \mathbf{E}\mathbf{H}^{-1} \mathbf{E}^\top$,

$$\det(\mathbf{G}) = \det(\mathbf{E}) \det(\mathbf{H}^{-1}) \det(\mathbf{E}^\top) = \det(\mathbf{E}\mathbf{E}^\top) \det(\mathbf{H}^{-1}) = \frac{\det(\mathbf{D})}{\det(\mathbf{H})}$$

and

$$\text{tr}(\mathbf{G}^{-1} \mathbf{D}) = \text{tr}(\mathbf{G}^{-1} \mathbf{E}\mathbf{E}^\top) = \text{tr}(\mathbf{E}^\top \mathbf{G}^{-1} \mathbf{E}) = \text{tr}(\mathbf{H}).$$

Then the function to be maximized (with respect to positive definite \mathbf{H}) is

$$g(\mathbf{H}) = -N \ln \det(\mathbf{D}) + N \ln \det(\mathbf{H}) - \text{tr}(\mathbf{H}).$$

Let $\mathbf{H} = \mathbf{T}\mathbf{T}^\top$ here \mathbf{L} is lower triangular. Then the maximum of

$$\begin{aligned} g(\mathbf{H}) &= -N \ln \det(\mathbf{D}) + N \ln \det(\mathbf{H}) - \text{tr}(\mathbf{H}) \\ &= -N \ln \det(\mathbf{D}) + N \ln (\det(\mathbf{T}))^2 - \text{tr}(\mathbf{T}\mathbf{T}^\top) \\ &= -N \ln \det(\mathbf{D}) + N \ln \left(\prod_{i=1}^p t_{ii}^2 \right) - \sum_{i \geq j} t_{ij}^2 \\ &= -N \ln \det(\mathbf{D}) + \sum_{i=1}^p (N \ln(t_{ii}^2) - t_{ii}^2) - \sum_{i > j} t_{ij}^2 \end{aligned}$$

occurs at $t_{ii}^2 = N$ and $t_{ij} = 0$ for $i \neq j$; that is $\mathbf{H} = N\mathbf{I}$. Then

$$\mathbf{G} = \frac{1}{N} \mathbf{D}.$$

□

Theorem 3.2. Let $f(\theta)$ be a real-valued function defined on a set \mathcal{S} and let ϕ be a single-valued function, with a single-valued inverse, on \mathcal{S} to a set \mathcal{S}^* . Let

$$g(\theta^*) = f(\phi^{-1}(\theta^*)).$$

Then if $f(\theta)$ attains a maximum at $\theta = \theta_0$, then $g(\theta^*)$ attains a maximum at $\theta^* = \theta_0^* = \phi(\theta_0)$. If the maximum of $f(\theta)$ at θ_0 is unique, so is the maximum of $g(\theta^*)$ at θ_0^* .

Proof. By hypothesis $f(\theta_0) \geq f(\theta)$ for all $\theta \in \mathcal{S}$. Then for any $\theta^* \in \mathcal{S}^*$, we have

$$g(\theta^*) = f(\phi^{-1}(\theta^*)) = f(\theta) \leq f(\theta_0) = g(\phi(\theta_0)) = g(\theta_0^*).$$

Thus $g(\theta^*)$ attains a maximum at $\theta_0^* = \phi(\theta_0)$. If the maximum of $f(\theta)$ at θ_0 is unique, there is strict inequality above for $\theta \neq \theta_0$, and the maximum of $g(\theta^*)$ is unique. □

Theorem 3.3. If $\phi : \mathcal{S} \rightarrow \mathcal{S}^*$ is not one-to-one, we let

$$\phi^{-1}(\theta^*) = \{\theta : \theta^* = \phi(\theta)\}.$$

and the induced likelihood function

$$g(\theta^*) = \sup\{f(\theta) : \theta^* = \phi(\theta)\}.$$

If $\theta = \hat{\theta}$ maximize $f(\theta)$, then $\hat{\theta}^* = \phi(\hat{\theta})$ also maximize $g(\theta^*)$.

Proof. The definition means

$$\sup_{\boldsymbol{\theta}^* \in \mathcal{S}^*} g(\boldsymbol{\theta}^*) = \sup_{\boldsymbol{\theta}^* \in \mathcal{S}^*} \sup_{\boldsymbol{\theta}^* = \phi(\boldsymbol{\theta})} f(\boldsymbol{\theta}) = \sup_{\boldsymbol{\theta} \in \mathcal{S}} f(\boldsymbol{\theta}).$$

The definition of $\hat{\boldsymbol{\theta}}^* = \phi(\hat{\boldsymbol{\theta}})$ means

$$f(\hat{\boldsymbol{\theta}}) = \sup_{\hat{\boldsymbol{\theta}}^* = \phi(\boldsymbol{\theta})} f(\boldsymbol{\theta}) = g(\hat{\boldsymbol{\theta}}^*)$$

Since $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ maximize $f(\boldsymbol{\theta})$, we have

$$g(\hat{\boldsymbol{\theta}}^*) = f(\hat{\boldsymbol{\theta}}) = \sup_{\boldsymbol{\theta} \in \mathcal{S}} f(\boldsymbol{\theta}) = \sup_{\boldsymbol{\theta}^* \in \mathcal{S}^*} g(\boldsymbol{\theta}^*),$$

which implies $\hat{\boldsymbol{\theta}}^*$ maximize $g(\boldsymbol{\theta}^*)$. \square

Corollary 3.1. *If $\mathbf{x}_1, \dots, \mathbf{x}_N$ constitutes a sample from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, let $\rho_{ij} = \sigma_{ij}/(\sigma_i \sigma_j)$. Then the maximum likelihood estimator of ρ_{ij} is*

$$\hat{\rho}_{ij} = \frac{\sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}{\sqrt{\sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)^2} \sqrt{\sum_{\alpha=1}^N (x_{j\alpha} - \bar{x}_j)^2}}$$

Proof. The set of parameters $\mu_i = \mu_i$, $\sigma_i^2 = \sigma_{ii}$ and $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$ is a one-to-one transform of the set of parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Then the estimator of ρ is

$$\hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{\sqrt{\hat{\sigma}_{ii}\hat{\sigma}_{jj}}} = \frac{\sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)}{\sqrt{\sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)^2} \sqrt{\sum_{\alpha=1}^N (x_{j\alpha} - \bar{x}_j)^2}}.$$

\square

Theorem 3.4. *Suppose $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independent, where $\mathbf{x}_\alpha \sim \mathcal{N}_p(\boldsymbol{\mu}_\alpha, \boldsymbol{\Sigma})$. Let $\mathbf{C} \in \mathbb{R}^{N \times N}$ be an orthogonal matrix, then*

$$\mathbf{y}_\alpha = \sum_{\beta=1}^N c_{\alpha\beta} \mathbf{x}_\beta \sim \mathcal{N}_p(\boldsymbol{\nu}_\alpha, \boldsymbol{\Sigma}),$$

where $\boldsymbol{\nu}_\alpha = \sum_{\beta=1}^N c_{\alpha\beta} \boldsymbol{\mu}_\beta$ for $\alpha = 1, \dots, N$ and $\mathbf{y}_1, \dots, \mathbf{y}_N$ are independent.

Proof. The set of vectors $\mathbf{y}_1, \dots, \mathbf{y}_N$ have a joint normal distribution, because the entire set of components is a set of linear combinations of the components of $\mathbf{x}_1, \dots, \mathbf{x}_N$, which have a joint normal distribution. The expected value of \mathbf{y}_α is

$$\mathbb{E}[\mathbf{y}_\alpha] = \mathbb{E}\left[\sum_{\beta=1}^N c_{\alpha\beta} \mathbf{x}_\beta\right] = \sum_{\beta=1}^N c_{\alpha\beta} \mathbb{E}[\mathbf{x}_\beta] = \sum_{\beta=1}^N c_{\alpha\beta} \boldsymbol{\mu}_\beta.$$

The covariance matrix between \mathbf{y}_α and \mathbf{y}_γ is

$$\begin{aligned} & \text{Cov}[\mathbf{y}_\alpha, \mathbf{y}_\gamma] \\ &= \mathbb{E}[(\mathbf{y}_\alpha - \boldsymbol{\nu}_\alpha)(\mathbf{y}_\gamma - \boldsymbol{\nu}_\gamma)^\top] \\ &= \mathbb{E}\left[\left(\sum_{\beta=1}^N c_{\alpha\beta}(\mathbf{x}_\beta - \boldsymbol{\mu}_\beta)\right)\left(\sum_{\xi=1}^N c_{\gamma\xi}(\mathbf{x}_\xi - \boldsymbol{\mu}_\xi)^\top\right)\right] \\ &= \sum_{\beta=1}^N \sum_{\xi=1}^N c_{\alpha\beta} c_{\gamma\xi} \mathbb{E}[(\mathbf{x}_\beta - \boldsymbol{\mu}_\beta)(\mathbf{x}_\xi - \boldsymbol{\mu}_\xi)^\top] \end{aligned}$$

$$\begin{aligned}
&= \sum_{\beta=1}^N \sum_{\xi=1}^N c_{\alpha\beta} c_{\gamma\xi} \delta_{\beta\xi} \Sigma \\
&= \sum_{\beta=1}^N c_{\alpha\beta} c_{\gamma\beta} \Sigma,
\end{aligned}$$

where

$$\delta_{\beta\xi} = \begin{cases} 1, & \text{if } \beta = \xi, \\ 0, & \text{if } \beta \neq \xi. \end{cases}$$

If $\alpha = \gamma$, we have $\sum_{\beta=1}^N c_{\alpha\beta} c_{\gamma\beta} = \sum_{\beta=1}^N c_{\alpha\beta} c_{\alpha\beta} = 1$; otherwise, we have $\sum_{\beta=1}^N c_{\alpha\beta} c_{\gamma\beta} = 0$. Hence, we have

$$\text{Cov}[\mathbf{y}_\alpha, \mathbf{y}_\gamma] = \sum_{\beta=1}^N c_{\alpha\beta} c_{\gamma\beta} \Sigma = \delta_{\alpha\gamma} \Sigma.$$

The set of vectors $\mathbf{y}_1, \dots, \mathbf{y}_N$ have a joint normal distribution, we have proved $\text{Cov}[\mathbf{y}_\alpha] = \Sigma$ for $\alpha = 1, \dots, N$ and $\mathbf{y}_1, \dots, \mathbf{y}_N$ are independent. \square

Lemma 3.2. *If*

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \dots & c_{NN} \end{bmatrix} = \begin{bmatrix} c_1^\top \\ c_2^\top \\ \vdots \\ c_N^\top \end{bmatrix} \in \mathbb{R}^{N \times N}$$

is orthogonal, then $\sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top = \sum_{\beta=1}^N \mathbf{y}_\beta \mathbf{y}_\beta^\top$ where $\mathbf{y}_\alpha = \sum_{\beta=1}^N c_{\alpha\beta} \mathbf{x}_\beta$ for $\alpha = 1, \dots, N$.

Proof. Let

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^\top \\ \mathbf{x}_2^\top \\ \vdots \\ \mathbf{x}_N^\top \end{bmatrix} \in \mathbb{R}^{N \times p}.$$

We have

$$\sum_{\alpha=1}^N \mathbf{y}_\alpha \mathbf{y}_\alpha^\top = \sum_{\beta=1}^N \mathbf{X}^\top \mathbf{c}_\beta \mathbf{c}_\beta^\top \mathbf{X} = \mathbf{X}^\top \left(\sum_{\beta=1}^N \mathbf{c}_\beta \mathbf{c}_\beta^\top \right) \mathbf{X} = \mathbf{X}^\top (\mathbf{C}^\top \mathbf{C}) \mathbf{X} = \mathbf{X}^\top \mathbf{X} = \sum_{\beta=1}^N \mathbf{x}_\beta \mathbf{x}_\beta^\top.$$

\square

Remark 3.1. We can also write $\mathbf{y}_\alpha = \mathbf{X}^\top \mathbf{c}_\alpha$ and $\mathbf{Y} = \mathbf{C}\mathbf{X}$ by defining \mathbf{Y} like \mathbf{X} .

Theorem 3.5. Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be independent, each distributed according to $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$. Then the mean of the sample

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha$$

is distributed according to $\mathcal{N}(\boldsymbol{\mu}, \frac{1}{N}\boldsymbol{\Sigma})$ and independent of

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

Additionally, we have $N\hat{\boldsymbol{\Sigma}} = \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$, where $\mathbf{z}_\alpha \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ for $\alpha = 1, \dots, N$, and $\mathbf{z}_1, \dots, \mathbf{z}_{N-1}$ are independent.

Proof. There exists an orthogonal matrix $\mathbf{B} \in \mathbb{R}^{p \times p}$ such that

$$\mathbf{B} = \begin{bmatrix} \times & \times & \dots & \times \\ \times & \times & \dots & \times \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \dots & \frac{1}{\sqrt{N}} \end{bmatrix}$$

Let $\mathbf{A} = N\hat{\boldsymbol{\Sigma}}$ and let $\mathbf{z}_\alpha = \sum_{\beta=1}^N b_{\alpha\beta} \mathbf{x}_\beta$, then

$$\mathbf{z}_N = \sum_{\beta=1}^N b_{N\beta} \mathbf{x}_\beta = \sum_{\beta=1}^N \frac{\mathbf{x}_\beta}{\sqrt{N}} = \sqrt{N} \bar{\mathbf{x}}$$

By Lemma 3.2, we have

$$\begin{aligned} \mathbf{A} &= \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \\ &= \sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top - \sum_{\alpha=1}^N \mathbf{x}_\alpha \bar{\mathbf{x}}^\top - \sum_{\alpha=1}^N \bar{\mathbf{x}} \mathbf{x}_\alpha^\top + \sum_{\alpha=1}^N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top \\ &= \sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top + N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top \\ &= \sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top \\ &= \sum_{\alpha=1}^N \mathbf{z}_\alpha \mathbf{z}_\alpha^\top - \mathbf{z}_N \mathbf{z}_N^\top \\ &= \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top \end{aligned}$$

Lemma 3.2 also states \mathbf{z}_N is independent of $\mathbf{z}_1, \dots, \mathbf{z}_{N-1}$, then the mean vector $\bar{\mathbf{x}} = \frac{1}{\sqrt{N}} \mathbf{z}_N$ is independent of \mathbf{A} and $\hat{\boldsymbol{\Sigma}} = \frac{1}{N} \mathbf{A}$. Since $\bar{\mathbf{x}} = \frac{1}{\sqrt{N}} \mathbf{z}_n = \frac{1}{\sqrt{N}} \sum_{\beta=1}^N b_{N\beta} \mathbf{x}_\beta$, Theorem 3.4 implies

$$\mathbb{E}[\bar{\mathbf{x}}] = \mathbb{E} \left[\frac{1}{\sqrt{N}} \sum_{\beta=1}^N b_{N\beta} \mathbf{x}_\beta \right] = \frac{1}{\sqrt{N}} \sum_{\beta=1}^N \frac{1}{\sqrt{N}} \boldsymbol{\mu} = \boldsymbol{\mu}, \quad \text{and} \quad \text{Cov}[\bar{\mathbf{x}}] = \frac{1}{N} \text{Cov} \left[\sum_{\beta=1}^N b_{N\beta} \mathbf{x}_\beta \right] = \frac{1}{N} \boldsymbol{\Sigma}.$$

Hence, we have $\bar{\mathbf{x}} \sim \mathcal{N} \left(\boldsymbol{\mu}, \frac{1}{N} \boldsymbol{\Sigma} \right)$. For $\alpha = 1, \dots, N-1$, we also have

$$\mathbb{E}[\mathbf{z}_\alpha] = \mathbb{E} \left[\sum_{\beta=1}^N b_{\alpha\beta} \mathbf{x}_\beta \right] = \sum_{\beta=1}^N b_{\alpha\beta} \mathbb{E}[\mathbf{x}_\beta] = \sum_{\beta=1}^N b_{\alpha\beta} \boldsymbol{\mu} = \sum_{\beta=1}^N b_{\alpha\beta} b_{N\beta} \sqrt{N} \boldsymbol{\mu} = \mathbf{0}.$$

and Theorem 3.4 implies $\mathbf{z}_\alpha \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. □

Theorem 3.6. Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be p -dimensional random vector and they are independent. Denote

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha \quad \text{and} \quad \hat{\Sigma} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

If $\mathbb{E}[\mathbf{x}_1] = \dots = \mathbb{E}[\mathbf{x}_N] = \boldsymbol{\mu}$ and $\text{Cov}[\mathbf{x}_1] = \dots = \text{Cov}[\mathbf{x}_N] = \Sigma$, then we have

$$\mathbb{E}[\hat{\Sigma}] = \frac{N-1}{N} \Sigma.$$

Proof. We have

$$\Sigma = \text{Cov}[\mathbf{x}_\alpha] = \mathbb{E}[(\mathbf{x}_\alpha - \boldsymbol{\mu})(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top] = \mathbb{E}[\mathbf{x}_\alpha \mathbf{x}_\alpha^\top - \mathbf{x}_\alpha \boldsymbol{\mu}^\top - \boldsymbol{\mu} \mathbf{x}_\alpha^\top + \boldsymbol{\mu} \boldsymbol{\mu}^\top] = \mathbb{E}[\mathbf{x}_\alpha \mathbf{x}_\alpha^\top] - \boldsymbol{\mu} \boldsymbol{\mu}^\top$$

and

$$\frac{1}{n} \Sigma = \text{Cov}[\bar{\mathbf{x}}] = \mathbb{E}[(\bar{\mathbf{x}} - \mathbb{E}[\bar{\mathbf{x}}])(\bar{\mathbf{x}} - \mathbb{E}[\bar{\mathbf{x}}])^\top] = \mathbb{E}[\bar{\mathbf{x}} \bar{\mathbf{x}}^\top] - \boldsymbol{\mu} \boldsymbol{\mu}^\top.$$

Hence, we obtain

$$\begin{aligned} \mathbb{E}[\hat{\Sigma}] &= \mathbb{E}\left[\frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top\right] \\ &= \mathbb{E}\left[\frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha \mathbf{x}_\alpha^\top - \bar{\mathbf{x}} \mathbf{x}_\alpha^\top - \mathbf{x}_\alpha \bar{\mathbf{x}}^\top + \bar{\mathbf{x}} \bar{\mathbf{x}}^\top)\right] \\ &= \mathbb{E}\left[\frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha \mathbf{x}_\alpha^\top - \bar{\mathbf{x}} \bar{\mathbf{x}}^\top\right] \\ &= \mathbb{E}[\mathbf{x}_\alpha \mathbf{x}_\alpha^\top] - \mathbb{E}[\bar{\mathbf{x}} \bar{\mathbf{x}}^\top] \\ &= \Sigma + \boldsymbol{\mu} \boldsymbol{\mu}^\top - \left(\frac{1}{n} \Sigma + \boldsymbol{\mu} \boldsymbol{\mu}^\top\right) \\ &= \frac{n-1}{n} \Sigma. \end{aligned}$$

□

Theorem 3.7. Using the notation of Theorem 3.1, if $N > p$, the probability is 1 of drawing a sample so that

$$\hat{\Sigma} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

is positive definite.

Proof. The proof of Theorem 3.1 shows that $\mathbf{A} = \tilde{\mathbf{Z}}^\top \tilde{\mathbf{Z}}$ where

$$\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{z}_1^\top \\ \vdots \\ \mathbf{z}_{N-1}^\top \end{bmatrix} \in \mathbb{R}^{(N-1) \times p},$$

which means $\text{rank}(\hat{\Sigma}) = \text{rank}(\mathbf{A}) = \text{rank}(\tilde{\mathbf{Z}})$. Then the probability is 1 of $\hat{\Sigma} \succ \mathbf{0}$ is equivalent to

$$\Pr(\text{rank}(\tilde{\mathbf{Z}}) = p) = 1.$$

Since appending rows at the end of $\tilde{\mathbf{Z}}$ will not increase its rank, we only need to consider the case of $N = p + 1$ ($N - 1 = p$ and $\tilde{\mathbf{Z}} \in \mathbb{R}^{p \times p}$). We have

$$\begin{aligned}
& \Pr(\mathbf{z}_1, \dots, \mathbf{z}_p \text{ are linearly dependent}) \\
& \leq \sum_{i=1}^p \Pr(\mathbf{z}_i \in \text{span}\{\mathbf{z}_1, \dots, \mathbf{z}_{i-1}, \mathbf{z}_{i+1}, \dots, \mathbf{z}_p\}) \\
& = p \Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \dots, \mathbf{z}_p\}) \\
& = p \mathbb{E}[\Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_p\} \mid \mathbf{z}_2 = \boldsymbol{\alpha}_2, \dots, \mathbf{z}_p = \boldsymbol{\alpha}_p)] \\
& = p \mathbb{E}[0] = 0
\end{aligned}$$

The second equality is obtained as follows

$$\begin{aligned}
& \Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \dots, \mathbf{z}_p\}) \\
& = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \dots, \mathbf{z}_p\}, \mathbf{z}_2 = \boldsymbol{\alpha}_2, \dots, \mathbf{z}_p = \boldsymbol{\alpha}_p) d\boldsymbol{\alpha}_2 \dots d\boldsymbol{\alpha}_p \\
& = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \dots, \mathbf{z}_p\} \mid \mathbf{z}_2 = \boldsymbol{\alpha}_2, \dots, \mathbf{z}_p = \boldsymbol{\alpha}_p) \Pr(\mathbf{z}_2 = \boldsymbol{\alpha}_2, \dots, \mathbf{z}_p = \boldsymbol{\alpha}_p) d\boldsymbol{\alpha}_2 \dots d\boldsymbol{\alpha}_p \\
& = \mathbb{E}[\Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \dots, \mathbf{z}_p\} \mid \mathbf{z}_2 = \boldsymbol{\alpha}_2, \dots, \mathbf{z}_p = \boldsymbol{\alpha}_p)] \\
& = 0
\end{aligned}$$

The last equality holds since $\Pr(\mathbf{z}_1 \in \text{span}\{\mathbf{z}_2, \dots, \mathbf{z}_p\} \mid \mathbf{z}_2 = \boldsymbol{\alpha}_2, \dots, \mathbf{z}_p = \boldsymbol{\alpha}_p)$ is the probability of the event that \mathbf{z}_1 lies in a subspace with the dimension no higher than $p - 1$. \square

Theorem 3.8. *If $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independent observations from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then*

1. $\bar{\mathbf{x}}$ and \mathbf{S} are sufficient for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$;
2. if $\boldsymbol{\mu}$ is given, $\sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top$ is sufficient for $\boldsymbol{\Sigma}$;
3. if $\boldsymbol{\Sigma}$ is given, $\bar{\mathbf{x}}$ is sufficient for $\boldsymbol{\mu}$;

where

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha \quad \text{and} \quad \mathbf{S} = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

Proof. The density of $\mathbf{x}_1, \dots, \mathbf{x}_N$ is

$$\begin{aligned}
& \prod_{\alpha=1}^N n(\mathbf{x}_\alpha \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) \\
& = (2\pi)^{-\frac{pN}{2}} (\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} \exp\left(-\frac{1}{2} \text{tr}\left(\sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \boldsymbol{\mu})\right)\right) \\
& = (2\pi)^{-\frac{pN}{2}} (\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} \exp\left(-\frac{1}{2} \text{tr}\left(\boldsymbol{\Sigma}^{-1} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top (\mathbf{x}_\alpha - \boldsymbol{\mu})\right)\right) \\
& = (2\pi)^{-\frac{pN}{2}} (\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} \exp\left(-\frac{1}{2} (N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) + (N-1) \text{tr}(\boldsymbol{\Sigma}^{-1} \mathbf{S}))\right)
\end{aligned}$$

where the last step is due to

$$\sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \boldsymbol{\mu})$$

$$\begin{aligned}
&= \sum_{\alpha=1}^N (\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) + \sum_{\alpha=1}^N (\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) \\
&\quad + \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) + \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) \\
&= N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) + (N-1) \text{tr}(\boldsymbol{\Sigma}^{-1} \mathbf{S}).
\end{aligned}$$

Hence, the density is a function of $\mathbf{t}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \{\bar{\mathbf{x}}, \mathbf{S}\}$ and $\boldsymbol{\theta} = \{\boldsymbol{\mu}, \boldsymbol{\Sigma}\}$. If $\boldsymbol{\mu}$ is given, it is a function of $\mathbf{t}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top$ and $\boldsymbol{\theta} = \boldsymbol{\Sigma}$. If $\boldsymbol{\Sigma}$ is given, it is a function of $\mathbf{t}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \bar{\mathbf{x}}$ (since \mathbf{S} can be viewed a function of \mathbf{t} for given) and $\boldsymbol{\theta} = \boldsymbol{\mu}$. \square

Theorem 3.9 (Theorem 3.4.2, Page 84). *The sufficient set of statistics $\bar{\mathbf{x}}, \mathbf{S}$ is complete for $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ when the sample is drawn from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.*

Proof. We introduce $\mathbf{z}_1, \dots, \mathbf{z}_N$ by following the proof of Theorem 3.5. For any function $g(\bar{\mathbf{x}}, n\mathbf{S})$, we have

$$\begin{aligned}
0 &\equiv \mathbb{E}[g(\bar{\mathbf{x}}, n\mathbf{S})] \\
&= \int \cdots \int K(\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} g\left(\bar{\mathbf{x}}, \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top\right) \exp\left(-\frac{1}{2} \left(\sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha^\top \boldsymbol{\Sigma}^{-1} \mathbf{z}_\alpha + N(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu})\right)\right) d\mathbf{z}_1 \dots d\mathbf{z}_{N-1} d\bar{\mathbf{x}}.
\end{aligned}$$

for any $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, where $K = \sqrt{N}(2\pi)^{-\frac{1}{2}pN}$. Let $\boldsymbol{\Sigma}^{-1} = \mathbf{I} - 2\boldsymbol{\Omega}$ such that symmetric $\boldsymbol{\Omega}$ and $\mathbf{I} - 2\boldsymbol{\Omega} \succ 0$. Let $\boldsymbol{\mu} = (\mathbf{I} - 2\boldsymbol{\Omega})^{-1} \mathbf{t} = \boldsymbol{\Sigma} \mathbf{t}$. Then, we have

$$\begin{aligned}
&0 \\
&\equiv \int \cdots \int K(\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} g\left(\bar{\mathbf{x}}, \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top\right) \\
&\quad \exp\left(-\frac{1}{2} \left(\sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha^\top \boldsymbol{\Sigma}^{-1} \mathbf{z}_\alpha + N\bar{\mathbf{x}}^\top \boldsymbol{\Sigma}^{-1} \bar{\mathbf{x}} - 2N\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \bar{\mathbf{x}} + N\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right)\right) d\mathbf{z}_1 \dots d\mathbf{z}_{N-1} d\bar{\mathbf{x}} \\
&= \int \cdots \int K(\det(\boldsymbol{\Sigma}))^{-\frac{N}{2}} g\left(\bar{\mathbf{x}}, \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top\right) \\
&\quad \exp\left(-\frac{1}{2} \left(\sum_{\alpha=1}^{N-1} \text{tr}(\boldsymbol{\Sigma}^{-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top) + N\text{tr}(\boldsymbol{\Sigma}^{-1} \bar{\mathbf{x}} \bar{\mathbf{x}}^\top) - 2N\bar{\mathbf{t}}^\top \bar{\mathbf{x}} + N\mathbf{t}^\top \boldsymbol{\Sigma} \mathbf{t}\right)\right) d\mathbf{z}_1 \dots d\mathbf{z}_{N-1} d\bar{\mathbf{x}} \\
&= \int \cdots \int K(\det(\mathbf{I} - 2\boldsymbol{\Omega}))^{\frac{N}{2}} g\left(\bar{\mathbf{x}}, \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top\right) \\
&\quad \exp\left(-\frac{1}{2} \left(\text{tr}\left((\mathbf{I} - 2\boldsymbol{\Omega}) \left(\sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top + N\bar{\mathbf{x}} \bar{\mathbf{x}}^\top\right)\right) - 2N\bar{\mathbf{t}}^\top \bar{\mathbf{x}} + N\mathbf{t}^\top (\mathbf{I} - 2\boldsymbol{\Omega})^{-1} \mathbf{t}\right)\right) d\mathbf{z}_1 \dots d\mathbf{z}_{N-1} d\bar{\mathbf{x}} \\
&= (\det(\mathbf{I} - 2\boldsymbol{\Omega}))^{\frac{N}{2}} \exp\left(-\frac{1}{2} N\mathbf{t}^\top (\mathbf{I} - 2\boldsymbol{\Omega})^{-1} \mathbf{t}\right) \\
&\quad \int \cdots \int g(\bar{\mathbf{x}}, \mathbf{B} - N\bar{\mathbf{x}} \bar{\mathbf{x}}^\top) \exp(\text{tr}(\boldsymbol{\Omega} \mathbf{B}) + \mathbf{t}^\top (N\bar{\mathbf{x}})) n\left(\bar{\mathbf{x}} \mid \mathbf{0}, \frac{1}{N} \mathbf{I}\right) \prod_{\alpha=1}^{N-1} n(\mathbf{z}_\alpha \mid \mathbf{0}, \mathbf{I}) d\mathbf{z}_1 \dots d\mathbf{z}_{N-1} d\bar{\mathbf{x}} \\
&= (\det(\mathbf{I} - 2\boldsymbol{\Omega}))^{\frac{N}{2}} \exp\left(-\frac{1}{2} N\mathbf{t}^\top (\mathbf{I} - 2\boldsymbol{\Omega})^{-1} \mathbf{t}\right) \\
&\quad \int g(\bar{\mathbf{x}}, \mathbf{B} - N\bar{\mathbf{x}} \bar{\mathbf{x}}^\top) \exp(\text{tr}(\boldsymbol{\Omega} \mathbf{B}) + \mathbf{t}^\top (N\bar{\mathbf{x}})) n\left(\bar{\mathbf{x}} \mid \mathbf{0}, \frac{1}{N} \mathbf{I}\right) d\bar{\mathbf{x}} \\
&= (\det(\mathbf{I} - 2\boldsymbol{\Omega}))^{\frac{N}{2}} \exp\left(-\frac{1}{2} N\mathbf{t}^\top (\mathbf{I} - 2\boldsymbol{\Omega})^{-1} \mathbf{t}\right) \mathbb{E}[g(\bar{\mathbf{x}}, \mathbf{B} - N\bar{\mathbf{x}} \bar{\mathbf{x}}^\top) \exp(\text{tr}(\boldsymbol{\Omega} \mathbf{B}) + \mathbf{t}^\top (N\bar{\mathbf{x}}))].
\end{aligned}$$

where $\mathbf{B} = \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha \mathbf{z}_\alpha^\top + N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top$. Thus

$$\begin{aligned} 0 &\equiv \mathbb{E} \left[g(\bar{\mathbf{x}}, \mathbf{B} - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top) \exp(\text{tr}(\boldsymbol{\Omega} \mathbf{B}) + \mathbf{t}^\top (N \bar{\mathbf{x}})) \right] \\ &= \iint g(\bar{\mathbf{x}}, \mathbf{B} - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top) \exp(\text{tr}(\boldsymbol{\Omega} \mathbf{B}) + \mathbf{t}^\top (N \bar{\mathbf{x}})) h(\bar{\mathbf{x}}, \mathbf{B}) d\bar{\mathbf{x}} d\mathbf{B} \end{aligned}$$

where $h(\bar{\mathbf{x}}, \mathbf{B})$ is the joint density of $\bar{\mathbf{x}}$ and \mathbf{B} . Consider that

$$\iint g(\bar{\mathbf{x}}, \mathbf{B} - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top) \exp(\text{tr}(\boldsymbol{\Omega} \mathbf{B}) + \mathbf{t}^\top (N \bar{\mathbf{x}})) h(\bar{\mathbf{x}}, \mathbf{B}) d\bar{\mathbf{x}} d\mathbf{B}$$

is the Laplace transform of $g(\bar{\mathbf{x}}, \mathbf{B} - N \bar{\mathbf{x}} \bar{\mathbf{x}}^\top) h(\bar{\mathbf{x}}, \mathbf{B})$. Then we have $g(\bar{\mathbf{x}}, n\mathbf{S})h(\bar{\mathbf{x}}, \mathbf{B}) = 0$ almost everywhere. Hence, we have

$$\begin{aligned} 0 &= \iint |g(\bar{\mathbf{x}}, n\mathbf{S})h(\bar{\mathbf{x}}, \mathbf{B})| d\bar{\mathbf{x}} d\mathbf{B} \\ &= \iint |g(\bar{\mathbf{x}}, n\mathbf{S})| |h(\bar{\mathbf{x}}, \mathbf{B})| d\bar{\mathbf{x}} d\mathbf{B} \\ &= \iint |g(\bar{\mathbf{x}}, n\mathbf{S})| dm(\bar{\mathbf{x}}, \mathbf{B}). \end{aligned}$$

Hence, we have $g(\bar{\mathbf{x}}, n\mathbf{S}) = 0$ almost everywhere. \square

Cramer-Rao Inequality We first give some lemmas. We denote the density of observation with parameter $\boldsymbol{\theta}$ by $f(\mathbf{x}, \boldsymbol{\theta})$ and

$$\mathbf{s} = \frac{\partial \ln g(\mathbf{X}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}.$$

where g is the density on N samples and $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$.

Lemma 3.3. *We have $\mathbb{E}[\mathbf{s}] = \mathbf{0}$.*

Proof. We have

$$\begin{aligned} \mathbb{E}[s_j] &= \int g(\mathbf{X}, \boldsymbol{\theta}) \frac{\partial \ln g(\mathbf{X}, \boldsymbol{\theta})}{\partial \theta_j} d\mathbf{X} \\ &= \int g(\mathbf{X}, \boldsymbol{\theta}) \frac{1}{g(\mathbf{X}, \boldsymbol{\theta})} \frac{\partial g(\mathbf{X}, \boldsymbol{\theta})}{\partial \theta_j} d\mathbf{X} \\ &= \int \frac{\partial g(\mathbf{X}, \boldsymbol{\theta})}{\partial \theta_j} d\mathbf{X} \\ &= \frac{\partial}{\partial \theta_j} \int g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} \\ &= \frac{\partial}{\partial \theta_j} 1 = 0. \end{aligned}$$

\square

Remark 3.2. *Similarly, we also have*

$$\mathbb{E} \left[\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right] = \mathbf{0}.$$

Lemma 3.4. *For unbiased estimator \mathbf{t} of $\boldsymbol{\theta}$, we have $\mathcal{C}[\mathbf{t}, \mathbf{s}] = \mathbf{I}$.*

Proof. We have

$$\begin{aligned}
& \mathcal{C}[t_j, s_k] \\
&= \int (t_j - \theta_j) \frac{\partial \ln g(\mathbf{X}, \boldsymbol{\theta})}{\partial \theta_k} g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} \\
&= \int (t_j - \theta_j) \frac{\partial g(\mathbf{X}, \boldsymbol{\theta})}{\partial \theta_k} d\mathbf{X} \\
&= - \int g(\mathbf{X}, \boldsymbol{\theta}) \frac{\partial (t_j - \theta_j)}{\partial \theta_k} d\mathbf{X} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}
\end{aligned}$$

where the last line holds since

$$\begin{aligned}
& \int (t_j - \theta_j) g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} \\
&= \int t_j g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} - \theta_j \int g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} \\
&= \mathbb{E} t_j - \theta_j \\
&= 0
\end{aligned}$$

and therefore

$$\begin{aligned}
0 &= \frac{\partial}{\partial \theta_k} \int (t_j - \theta_j) g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} \\
&= \int \frac{\partial (t_j - \theta_j)}{\partial \theta_k} g(\mathbf{X}, \boldsymbol{\theta}) d\mathbf{X} + \int (t_j - \theta_j) \frac{\partial g(\mathbf{X}, \boldsymbol{\theta})}{\partial \theta_k}.
\end{aligned}$$

□

Theorem 3.10. *Under the regularity condition (everything is well-defined, integration and differentiation can be swapped), we have*

$$N \mathbb{E} [(\mathbf{t} - \boldsymbol{\theta})(\mathbf{t} - \boldsymbol{\theta})^\top] \succeq \left(\mathbb{E} \left[\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^\top \right] \right)^{-1},$$

where $\mathbb{E}[\mathbf{t}] = \boldsymbol{\theta}$ and $f(\mathbf{x}, \boldsymbol{\theta})$ is the density of the distribution with respect to the components of $\boldsymbol{\theta}$.

Proof. For any nonzero $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$, consider the correlation of $\mathbf{a}^\top \mathbf{t}$ and $\mathbf{b}^\top \mathbf{s}$, we have

$$1 \geq \frac{\mathcal{C}[\mathbf{a}^\top \mathbf{t}, \mathbf{b}^\top \mathbf{s}]}{\sqrt{\text{Var}[\mathbf{a}^\top \mathbf{t}] \text{Var}[\mathbf{b}^\top \mathbf{s}]}} = \frac{\mathbf{a}^\top \mathcal{C}[\mathbf{t}, \mathbf{s}] \mathbf{b}}{\sqrt{\mathbf{a}^\top \mathcal{C}[\mathbf{t}] \mathbf{a} \mathbf{b}^\top \mathcal{C}[\mathbf{s}] \mathbf{b}}} = \frac{\mathbf{a}^\top \mathbf{b}}{\sqrt{\mathbf{a}^\top \mathcal{C}[\mathbf{t}] \mathbf{a} \mathbf{b}^\top \mathcal{C}[\mathbf{s}] \mathbf{b}}}$$

Let $\mathbf{b} = (\mathcal{C}[\mathbf{s}])^{-1} \mathbf{a}$, we have

$$1 \geq \frac{\mathbf{a}^\top (\mathcal{C}[\mathbf{s}])^{-1} \mathbf{a}}{\sqrt{\mathbf{a}^\top \mathcal{C}[\mathbf{t}] \mathbf{a} \mathbf{a}^\top (\mathcal{C}[\mathbf{s}])^{-1} \mathbf{a}}}$$

which means

$$\mathbf{a}^\top \mathcal{C}[\mathbf{t}] \mathbf{a} \geq \mathbf{a}^\top (\mathcal{C}[\mathbf{s}])^{-1} \mathbf{a}$$

for any nonzero \mathbf{a} . Hence, we have

$$\begin{aligned}
& \mathbb{E} [(\mathbf{t} - \boldsymbol{\theta})(\mathbf{t} - \boldsymbol{\theta})^\top] = \mathcal{C}[\mathbf{t}] \succeq (\mathcal{C}[\mathbf{s}])^{-1} \\
&= \left(\mathcal{C} \left[\frac{\partial \ln g(\mathbf{X}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right] \right)^{-1} = \left(N \mathcal{C} \left[\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right] \right)^{-1} = \frac{1}{N} \left(\mathcal{C} \left[\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right] \right)^{-1} \\
&= \frac{1}{N} \left(\mathbb{E} \left[\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left(\frac{\partial \ln f(\mathbf{x}, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^\top \right] \right)^{-1}.
\end{aligned}$$

□

Theorem 3.11. Let p -component vectors $\mathbf{y}_1, \mathbf{y}_2, \dots$ be i.i.d with means $\mathbb{E}[\mathbf{y}_\alpha] = \boldsymbol{\nu}$ and covariance matrices $\mathbb{E}[(\mathbf{y}_\alpha - \boldsymbol{\nu})(\mathbf{y}_\alpha - \boldsymbol{\nu})^\top] = \mathbf{T}$. Then the limiting distribution of

$$\frac{1}{\sqrt{n}} \sum_{\alpha=1}^n (\mathbf{y}_\alpha - \boldsymbol{\nu})$$

as $n \rightarrow +\infty$ is $\mathcal{N}(\mathbf{0}, \mathbf{T})$.

Proof. Let

$$\phi_n(\mathbf{t}, u) = \mathbb{E} \left[\exp \left(i u \mathbf{t}^\top \frac{1}{\sqrt{n}} \sum_{\alpha=1}^n (\mathbf{y}_\alpha - \boldsymbol{\nu}) \right) \right],$$

where $u \in \mathbb{R}$ and $\mathbf{t} \in \mathbb{R}^p$. For fixed \mathbf{t} , the function $\phi_n(\mathbf{t}, u)$ can be viewed as the characteristic function of

$$\frac{1}{\sqrt{n}} \sum_{\alpha=1}^n (\mathbf{t}^\top \mathbf{y}_\alpha - \mathbf{t}^\top \mathbb{E}[\mathbf{y}_\alpha]).$$

By the univariate central limit theorem, the limiting distribution is $\mathcal{N}(0, \mathbf{t}^\top \mathbf{T} \mathbf{t})$. Therefore, we have

$$\lim_{n \rightarrow \infty} \phi_n(\mathbf{t}, u) = \exp \left(-\frac{1}{2} u^2 \mathbf{t}^\top \mathbf{T} \mathbf{t} \right),$$

for any $u \in \mathbb{R}$ and $\mathbf{t} \in \mathbb{R}^p$. Let $u = 1$, we obtain

$$\phi_n(\mathbf{t}, 1) = \mathbb{E} \left[\exp \left(i \mathbf{t}^\top \frac{1}{\sqrt{n}} \sum_{\alpha=1}^n (\mathbf{y}_\alpha - \boldsymbol{\nu}) \right) \right] \rightarrow \exp \left(-\frac{1}{2} \mathbf{t}^\top \mathbf{T} \mathbf{t} \right)$$

for any $\mathbf{t} \in \mathbb{R}^p$. Since $\exp \left(-\frac{1}{2} \mathbf{t}^\top \mathbf{T} \mathbf{t} \right)$ is continuous at $\mathbf{t} = \mathbf{0}$, the convergence is uniform in some neighborhood of $\mathbf{t} = \mathbf{0}$. The theorem follows. \square

Theorem 3.12. If $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independently distributed, each x_α according to $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and if $\boldsymbol{\mu}$ has an a prior distribution $\mathcal{N}(\boldsymbol{\nu}, \boldsymbol{\Phi})$, then the a posterior distribution of $\boldsymbol{\mu}$ given $\mathbf{x}_1, \dots, \mathbf{x}_N$ is normal with mean

$$\boldsymbol{\Phi} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \bar{\mathbf{x}} + \frac{1}{N} \boldsymbol{\Sigma} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \boldsymbol{\nu}$$

and covariance matrix

$$\boldsymbol{\Phi} - \boldsymbol{\Phi} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \boldsymbol{\Phi}.$$

Proof. Since $\bar{\mathbf{x}}$ is sufficient for $\boldsymbol{\mu}$, we need only consider $\bar{\mathbf{x}}$, which has the distribution of $\boldsymbol{\mu} + \mathbf{y}$, where

$$\mathbf{y} \sim \mathcal{N} \left(\mathbf{0}, \frac{1}{N} \boldsymbol{\Sigma} \right)$$

and is independent of $\boldsymbol{\mu}$. Then we have

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{y} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Phi} & \mathbf{0} \\ \mathbf{0} & \frac{1}{N} \boldsymbol{\Sigma} \end{bmatrix} \right)$$

which implies $\bar{\mathbf{x}} \sim \mathcal{N} \left(\boldsymbol{\nu}, \boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)$. Since we have

$$\begin{bmatrix} \boldsymbol{\mu} \\ \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{y} \end{bmatrix},$$

then

$$\begin{bmatrix} \boldsymbol{\mu} \\ \bar{\mathbf{x}} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\nu} \\ \boldsymbol{\nu} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Phi} \\ \boldsymbol{\Phi} & \boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \end{bmatrix} \right).$$

Consider the conditional distribution of $\boldsymbol{\mu}$ given $\bar{\mathbf{x}}$, we obtain the mean and covariance given $\bar{\mathbf{x}}$ is

$$\begin{aligned} & \boldsymbol{\nu} + \boldsymbol{\Phi} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} (\bar{\mathbf{x}} - \boldsymbol{\nu}) \\ &= \boldsymbol{\Phi} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \bar{\mathbf{x}} + \left(\mathbf{I} - \boldsymbol{\Phi} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \right) \boldsymbol{\nu} \\ &= \boldsymbol{\Phi} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \bar{\mathbf{x}} + \frac{1}{N} \boldsymbol{\Sigma} \left(\boldsymbol{\Phi} + \frac{1}{N} \boldsymbol{\Sigma} \right)^{-1} \boldsymbol{\nu}. \end{aligned}$$

□

Remark 3.3. Let

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right).$$

The conditional density of $\mathbf{x}^{(1)}$ given that $\mathbf{x}^{(2)}$ is

$$\mathbf{x}^{(1)} \mid \mathbf{x}^{(2)} \sim \mathcal{N} \left(\boldsymbol{\mu}^{(1)} + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (\mathbf{x}^{(2)} - \boldsymbol{\mu}^{(2)}), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{22} \right)$$

Lemma 3.5. If $f(x)$ is a function such that

$$f(b) - f(a) = \int_a^b f'(x) \, dx$$

for all $a < b$ and if

$$\int_{-\infty}^{+\infty} |f'(x)| \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx < +\infty,$$

then

$$\int_{-\infty}^{+\infty} f(x) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx = \int_{-\infty}^{+\infty} f'(x) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx. \quad (5)$$

Proof. Since $(x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right)$ is odd function, the LHS of (5) can be written as

$$\begin{aligned} & \int_{-\infty}^{+\infty} (f(x) - f(\theta)) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx \\ &= \int_{\theta}^{+\infty} (f(x) - f(\theta)) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx \\ & \quad + \int_{-\infty}^{\theta} (f(x) - f(\theta)) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx \\ &= \int_{\theta}^{+\infty} \int_{\theta}^x f'(y) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dy \, dx \\ & \quad - \int_{-\infty}^{\theta} \int_x^{\theta} f'(y) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dy \, dx \\ &= \int_{\theta}^{+\infty} \int_y^{+\infty} f'(y) (x - \theta) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} (x - \theta)^2 \right) \, dx \, dy \end{aligned}$$

$$\begin{aligned}
& - \int_{-\infty}^{\theta} \int_{-\infty}^y f'(y)(x - \theta) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x - \theta)^2\right) dx dy \\
&= \int_{\theta}^{+\infty} f'(y) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - \theta)^2\right) dy - \int_{-\infty}^{\theta} f'(y) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - \theta)^2\right) dy \\
&= \int_{-\infty}^{+\infty} f'(x) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x - \theta)^2\right) dx
\end{aligned}$$

where we use

$$\begin{aligned}
& \int (x - \theta) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x - \theta)^2\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int \exp\left(-\frac{1}{2}(x - \theta)^2\right) d\left(\frac{1}{2}(x - \theta)^2\right) \\
&= \frac{-1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x - \theta)^2\right)
\end{aligned}$$

and

$$\lim_{x \rightarrow +\infty} \frac{-1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x - \theta)^2\right) = \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x - \theta)^2\right) = 0.$$

□

Lemma 3.6. *Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independently distributed to $\mathcal{N}_p(\boldsymbol{\mu}, N\mathbf{I})$, we have*

$$\mathbb{E} \left[\|\bar{\mathbf{x}} - \boldsymbol{\mu}\|_2^2 \right] = \sum_{\alpha=1}^p \text{Var}(\bar{x}_\alpha) = p.$$

Proof. We have

$$\begin{aligned}
& \mathbb{E} \left[\|\bar{\mathbf{x}} - \boldsymbol{\mu}\|_2^2 \right] \\
&= \mathbb{E} \left[\text{tr}((\bar{\mathbf{x}} - \boldsymbol{\mu})^\top (\bar{\mathbf{x}} - \boldsymbol{\mu})) \right] \\
&= \mathbb{E} \left[\text{tr}((\bar{\mathbf{x}} - \boldsymbol{\mu})(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top) \right] \\
&= \text{tr}(\mathbb{E}[(\bar{\mathbf{x}} - \boldsymbol{\mu})(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top]) \\
&= \text{tr}(\mathbf{I}) = p.
\end{aligned}$$

□

Theorem 3.13. *Under the setting of Lemma 3.6, we let*

$$\mathbf{m}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right) (\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu}$$

and $p > 3$. Then $\mathbb{E} \left[\|\mathbf{m}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2 \right] < \mathbb{E} \left[\|\bar{\mathbf{x}} - \boldsymbol{\mu}\|_2^2 \right]$.

Proof. We have

$$\begin{aligned}
\Delta R(\boldsymbol{\mu}) &= \mathbb{E} \left[\|\bar{\mathbf{x}} - \boldsymbol{\mu}\|_2^2 - \|\mathbf{m}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2 \right] \\
&= \mathbb{E} \left[\|\bar{\mathbf{x}} - \boldsymbol{\mu}\|_2^2 - \left\| \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right) (\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu} - \boldsymbol{\mu} \right\|_2^2 \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[\sum_{i=1}^p (\bar{x}_i - \mu_i)^2 - \sum_{i=1}^p \left(\left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right) (\bar{x}_i - \nu_i) + \nu_i - \mu_i \right)^2 \right] \\
&= \mathbb{E} \left[\sum_{i=1}^p (\bar{x}_i - \mu_i)^2 - \sum_{i=1}^p \left(\bar{x}_i - \mu_i - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} (\bar{x}_i - \nu_i) \right)^2 \right] \\
&= \mathbb{E} \left[\frac{2(p-2)}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \sum_{i=1}^p (\bar{x}_i - \nu_i)(\bar{x}_i - \mu_i) - \sum_{i=1}^p \frac{(p-2)^2 (\bar{x}_i - \nu_i)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^4} \right] \\
&= \mathbb{E} \left[2(p-2) \sum_{i=1}^p \frac{\bar{x}_i - \nu_i}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \cdot (\bar{x}_i - \mu_i) - \frac{(p-2)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right].
\end{aligned}$$

Using Lemma 3.5 with $\theta = \mu_i$,

$$f(\bar{x}_i) = \frac{\bar{x}_i - \nu_i}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \quad \text{and} \quad f'(\bar{x}_i) = \frac{1}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} - \frac{2(\bar{x}_i - \nu_i)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^4}.$$

Hence, we obtain

$$\begin{aligned}
\Delta R(\boldsymbol{\mu}) &= \mathbb{E} \left[2(p-2) \sum_{i=1}^p \left(\frac{1}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} - \frac{2(\bar{x}_i - \nu_i)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^4} \right) - \frac{(p-2)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right] \\
&= \mathbb{E} \left[2(p-2) \sum_{i=1}^p \left(\frac{1}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} - \frac{2(\bar{x}_i - \nu_i)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^4} \right) - \frac{(p-2)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right] \\
&= \mathbb{E} \left[\frac{2p(p-2)}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} - \frac{4(p-2)}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} - \frac{(p-2)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right] \\
&= \mathbb{E} \left[\frac{(p-2)^2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2} \right] > 0
\end{aligned}$$

□

Remark 3.4. We consider the bias and variance decomposition

$$\begin{aligned}
&\mathbb{E} \|\mathbf{m}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2 \\
&= \mathbb{E} \|\mathbf{m}(\bar{\mathbf{x}}) - \mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})] + \mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})] - \boldsymbol{\mu}\|_2^2 \\
&= \mathbb{E} \|\mathbf{m}(\bar{\mathbf{x}}) - \mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})]\|_2^2 + 2\mathbb{E}[(\mathbf{m}(\bar{\mathbf{x}}) - \mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})])^\top (\mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})] - \boldsymbol{\mu})] + \mathbb{E} \|\mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})] - \boldsymbol{\mu}\|_2^2 \\
&= \mathbb{E} \|\mathbf{m}(\bar{\mathbf{x}}) - \mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})]\|_2^2 + \|\mathbb{E}[\mathbf{m}(\bar{\mathbf{x}})] - \boldsymbol{\mu}\|_2^2.
\end{aligned}$$

Unbiased estimator may leads to larger variance.

Lemma 3.7. Suppose that $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$, then

$$\mathbb{E} \|g^+(\|\mathbf{x}\|_2) \mathbf{x} - \boldsymbol{\mu}\|_2^2 \leq \mathbb{E} \|g(\|\mathbf{x}\|_2) \mathbf{x} - \boldsymbol{\mu}\|_2^2,$$

where

$$g^+(u) = \begin{cases} g(u), & \text{if } g(u) \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

for any function $g(u)$.

Proof. We have

$$\begin{aligned}
& \mathbb{E} \|g(\|\mathbf{x}\|_2)\mathbf{x} - \boldsymbol{\mu}\|_2^2 - \mathbb{E} \|g^+(\|\mathbf{x}\|_2)\mathbf{x} - \boldsymbol{\mu}\|_2^2 \\
&= \mathbb{E} \left[(g(\|\mathbf{x}\|_2))^2 \|\mathbf{x}\|_2^2 \right] - \mathbb{E} \left[(g^+(\|\mathbf{x}\|_2))^2 \|\mathbf{x}\|_2^2 \right] + 2\mathbb{E} [\boldsymbol{\mu}^\top \mathbf{x} (g^+(\|\mathbf{x}\|_2) - g(\|\mathbf{x}\|_2))] \\
&\geq 2\mathbb{E} [\boldsymbol{\mu}^\top \mathbf{x} (g^+(\|\mathbf{x}\|_2) - g(\|\mathbf{x}\|_2))] .
\end{aligned}$$

Let \mathbf{P} be the orthogonal matrix such that $\mathbf{P}\mathbf{P}^\top = \mathbf{I}$ and

$$\mathbf{P} = \left[\frac{\boldsymbol{\mu}}{\|\boldsymbol{\mu}\|_2}, \times, \dots, \times \right],$$

which means

$$\mathbf{P}^\top \boldsymbol{\mu} = [\|\boldsymbol{\mu}\|_2, 0, \dots, 0]^\top.$$

Let $\mathbf{y} = \mathbf{P}^\top \mathbf{x}$, then we have $\boldsymbol{\mu}^\top \mathbf{x} = \boldsymbol{\mu}^\top \mathbf{P}\mathbf{y} = (\mathbf{P}^\top \boldsymbol{\mu})^\top \mathbf{y} = \|\boldsymbol{\mu}\|_2 y_1$ and

$$\begin{aligned}
& \mathbb{E} [\boldsymbol{\mu}^\top \mathbf{x} (g^+(\|\mathbf{x}\|_2) - g(\|\mathbf{x}\|_2))] \\
&= \mathbb{E} [\|\boldsymbol{\mu}\|_2 y_1 (g^+(\|\mathbf{y}\|_2) - g(\|\mathbf{y}\|_2))] \\
&= \|\boldsymbol{\mu}\|_2 \int_{-\infty}^{+\infty} y_1 (g^+(\|\mathbf{y}\|_2) - g(\|\mathbf{y}\|_2)) \frac{1}{(2\pi)^{\frac{p}{2}}} \exp\left(-\frac{1}{2} \left(\sum_{i=1}^p y_i^2 - 2y_1 \|\boldsymbol{\mu}\|_2 + \|\boldsymbol{\mu}\|_2^2\right)\right) \mathrm{d}\mathbf{y} \\
&= \frac{\|\boldsymbol{\mu}\|_2 \exp\left(-\frac{1}{2} \|\boldsymbol{\mu}\|_2^2\right)}{(2\pi)^{\frac{p}{2}}} \int_{-\infty}^{+\infty} y_1 (g^+(\|\mathbf{y}\|_2) - g(\|\mathbf{y}\|_2)) \exp\left(-\frac{1}{2} \sum_{i=1}^p y_i^2\right) \exp(y_1 \|\boldsymbol{\mu}\|_2) \mathrm{d}\mathbf{y} \\
&= \frac{\|\boldsymbol{\mu}\|_2 \exp\left(-\frac{1}{2} \|\boldsymbol{\mu}\|_2^2\right)}{(2\pi)^{\frac{p}{2}}} \\
&\quad \cdot \int_{-\infty}^{+\infty} \dots \int_0^{+\infty} y_1 (g^+(\|\mathbf{y}\|_2) - g(\|\mathbf{y}\|_2)) \exp\left(-\frac{1}{2} \sum_{i=1}^p y_i^2\right) (\exp(y_1 \|\boldsymbol{\mu}\|_2) - \exp(-y_1 \|\boldsymbol{\mu}\|_2)) \mathrm{d}y_1 \dots \mathrm{d}y_p,
\end{aligned}$$

where the last step use $\exp(z) - \exp(-z) \geq 0$ for all $z \geq 0$. \square

Theorem 3.14. *Let*

$$\mathbf{m}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right) (\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu} \quad \text{and} \quad \tilde{\mathbf{m}}(\bar{\mathbf{x}}) = \left(1 - \frac{p-2}{\|\bar{\mathbf{x}} - \boldsymbol{\nu}\|_2^2}\right)^+ (\bar{\mathbf{x}} - \boldsymbol{\nu}) + \boldsymbol{\nu},$$

where $\bar{\mathbf{x}} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$. Then we have $\mathbb{E} \|\tilde{\mathbf{m}}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2 \leq \mathbb{E} \|\mathbf{m}(\bar{\mathbf{x}}) - \boldsymbol{\mu}\|_2^2$.

Proof. Use Lemma 3.7 with $g(u) = 1 - (p-2)/u$, $\mathbf{x} = \bar{\mathbf{x}} - \boldsymbol{\nu}$ and replace $\boldsymbol{\mu}$ by $\boldsymbol{\mu} - \boldsymbol{\nu}$. \square

4 T^2 -Statistic

Theorem 4.1. *For $y \sim \chi^2(n)$, we have $\mathbb{E}[y] = n$ and $\text{Var}[y] = 2n$.*

Proof. We can write

$$y = \sum_{i=1}^n x_i^2,$$

where x_1, \dots, x_n are independent standard normal variables. Then, we have

$$\mathbb{E}[y] = \mathbb{E} \left[\sum_{i=1}^n x_i^2 \right] = \sum_{i=1}^n \mathbb{E} [x_i^2] = \sum_{i=1}^n \text{Var} [x_i] = n$$

and

$$\text{Var}[y] = \text{Var} \left[\sum_{i=1}^n x_i^2 \right] = \sum_{i=1}^n \text{Var} [x_i^2] = \sum_{i=1}^n \mathbb{E} [x_i^4 - (\mathbb{E}[x_i^2])^2] = \sum_{i=1}^n \mathbb{E} [3 - 1] = 2n.$$

We use the fact $\mathbb{E}[x_i^4] = 3$ because of $\phi(t) = \exp(-\frac{1}{2}t^2)$ and

$$\mathbb{E}[x_i^4] = \frac{1}{i^4} \frac{d^4 \phi(t)}{dt^4} \Big|_{t=0} = (t^4 - 6t^2 + 3) \exp\left(-\frac{1}{2}t^2\right) \Big|_{t=0} = 3.$$

□

Theorem 4.2. *The density of $y \sim \chi^2(n)$ is*

$$f(y; n) = \begin{cases} \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} \exp\left(-\frac{y}{2}\right), & y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) dt.$$

Proof. We first provide the following results:

1. We have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, because

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2} \exp(-t) dt \\ &= \int_0^\infty \left(\frac{1}{2}x^2\right)^{-1/2} \exp\left(-\frac{1}{2}x^2\right) d\left(\frac{1}{2}x^2\right) \\ &= \int_0^\infty \frac{\sqrt{2}}{x} \exp\left(-\frac{1}{2}x^2\right) x dx \\ &= \sqrt{2} \int_0^\infty \exp\left(-\frac{1}{2}x^2\right) dx \\ &= 2\sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx \\ &= \sqrt{\pi}. \end{aligned}$$

2. For $y_1 = x^2$ with $x \sim \mathcal{N}(0, 1)$, the density function of y_1 is

$$\frac{1}{\sqrt{2\pi y_1}} \exp\left(-\frac{1}{2}y_1\right).$$

We define the positive random variable \hat{x} whose density function is

$$\frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\hat{x}^2\right).$$

Then the transform $\hat{x} = \sqrt{y_1}$ is one to one and the density of y_1 is

$$\frac{2}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y_1\right) \frac{d\sqrt{y_1}}{dy_1} = \frac{1}{\sqrt{2\pi y_1}} \exp\left(-\frac{1}{2}y_1\right).$$

3. For beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt,$$

we have

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Consider that

$$\begin{aligned} & \Gamma(\alpha)\Gamma(\beta) \\ &= \int_0^\infty x^{\alpha-1} \exp(-x) dx \int_0^\infty y^{\beta-1} \exp(-y) dy \\ &= \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} \exp(-(x+y)) dy dx. \end{aligned}$$

Using the substitution $x = uv$ and $y = u(1-v)$, then the Jacobian matrix of the transformation is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix}$$

and $\det(\mathbf{J}) = -u$. Since $u = x + y$ and $v = x/(x + y)$, we have that the limits of integration for u are 0 to ∞ and the limits of integration for v are 0 to 1. Thus

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} \exp(-(x+y)) dy dx \\ &= \int_0^1 \int_0^\infty (uv)^{\alpha-1} (u(1-v))^{\beta-1} \exp(-(uv + u(1-v))) | -u | du dv \\ &= \int_0^1 \int_0^\infty u^{\alpha+\beta-1} v^{\alpha-1} (1-v)^{\beta-1} \exp(-u) du dv \\ &= \int_0^1 v^{\alpha-1} (1-v)^{\beta-1} dv \int_0^\infty u^{\alpha+\beta-1} \exp(-u) du \\ &= B(\alpha, \beta) \Gamma(\alpha + \beta). \end{aligned}$$

4. If

$$F(z) = \int_{a(z)}^{b(z)} f(y, z) dy,$$

then

$$F'(z) = \int_{a(z)}^{b(z)} \frac{\partial f(y, z)}{\partial z} dx + f(b(z), z)b'(z) - f(a(z), z)a'(z).$$

We prove the density of Chi-square distribution by induction. For $n = 1$ and $y > 0$, we have

$$f(y; 1) = \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{1}{2}y\right) = \frac{1}{2^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)} y^{\frac{1}{2}-1} \exp\left(-\frac{y}{2}\right).$$

Suppose the statement holds for $n - 1$, that is

$$f(y; n - 1) = \begin{cases} \frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} y^{\frac{n-1}{2}-1} \exp\left(-\frac{y}{2}\right), & y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

We consider $y_n = y_{n-1} + x_n^2$ such that $y_{n-1} \sim \chi^2(n-1)$ and $x_n \sim \mathcal{N}(0, 1)$ are independent. Let F_1 be the corresponding cdf of $f(y; 1)$. Then the cdf of y_n is

$$\begin{aligned} & \Pr(y_n \leq z) \\ &= \int_0^z \int_0^{z-y} f_{n-1}(y) f_1(x) dx dy \\ &= \int_0^z (F_1(z-y) - F_1(0)) f_{n-1}(y) dy \\ &= \int_0^z F_1(z-y) f_{n-1}(y) dy \end{aligned}$$

and the pdf of y_n is (let $y = tz$)

$$\begin{aligned} & \int_0^z \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} (z-y)^{\frac{1}{2}-1} \exp\left(-\frac{z-y}{2}\right) \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} y^{\frac{n-1}{2}-1} \exp\left(-\frac{y}{2}\right) dy \\ &= \frac{1}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})} \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \int_0^z (z-y)^{\frac{1}{2}-1} y^{\frac{n-1}{2}-1} \exp\left(-\frac{z}{2}\right) dy \\ &= \frac{\exp\left(-\frac{z}{2}\right) z^{\frac{n-1}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2})} \int_0^1 (1-t)^{\frac{1}{2}-1} t^{\frac{n-1}{2}-1} dt \\ &= \frac{\exp\left(-\frac{z}{2}\right) z^{\frac{n}{2}-1}}{2^{\frac{n}{2}} \Gamma(\frac{1}{2}) \Gamma(\frac{n-1}{2})} B\left(\frac{n-1}{2}, \frac{1}{2}\right) \\ &= \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} z^{\frac{n}{2}-1} \exp\left(-\frac{z}{2}\right). \end{aligned}$$

□

Theorem 4.3. *If the n -component vector \mathbf{y} is distributed according to $\mathcal{N}(\boldsymbol{\nu}, \mathbf{T})$ with $\mathbf{T} \succ \mathbf{0}$, then*

$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y} \sim \chi_n^2(\boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}).$$

If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is the central χ^2 -distribution.

Proof. Let \mathbf{C} be a non-singular matrix such that $\mathbf{C} \mathbf{T} \mathbf{C}^\top = \mathbf{I}$. Define $\mathbf{z} = \mathbf{C} \mathbf{y}$, then \mathbf{z} is normally distributed with mean

$$\mathbf{C} \mathbb{E}[\mathbf{y}] = \mathbf{C} \boldsymbol{\nu} \triangleq \boldsymbol{\lambda}$$

and covariance matrix

$$\mathbb{E}[(\mathbf{z} - \boldsymbol{\lambda})(\mathbf{z} - \boldsymbol{\lambda})^\top] = \mathbf{C} \mathbb{E}[(\mathbf{y} - \boldsymbol{\nu})(\mathbf{y} - \boldsymbol{\nu})^\top] \mathbf{C}^\top = \mathbf{C} \mathbf{T} \mathbf{C}^\top = \mathbf{I}.$$

Then we have

$$\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y} = \mathbf{z}^\top \mathbf{C}^{-\top} \mathbf{T}^{-1} \mathbf{C}^{-1} \mathbf{z} = \mathbf{z}^\top (\mathbf{C} \mathbf{T} \mathbf{C}^\top)^{-1} \mathbf{z} = \mathbf{z}^\top \mathbf{z},$$

which is the sum of squares of the components of \mathbf{z} . Similarly, we have $\boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu} = \boldsymbol{\lambda}^\top \boldsymbol{\lambda}$. Thus, the random variable $\mathbf{y}^\top \mathbf{T}^{-1} \mathbf{y}$ is distributed as $\sum_{i=1}^n z_i^2$, where z_1, \dots, z_n are independently normally distributed with means $\lambda_1, \dots, \lambda_n$ respectively, and variances 1. By definition this is the noncentral χ^2 -distribution with noncentrality parameter $\sum_{i=1}^n \lambda_i^2 = \boldsymbol{\nu}^\top \mathbf{T}^{-1} \boldsymbol{\nu}$. □

Theorem 4.4. *The probability density function (pdf) for the noncentral χ^2 -distribution is*

$$f(v; p, \tau^2) = \begin{cases} \frac{\exp\left(-\frac{1}{2}(\tau^2 + v)\right) v^{\frac{p}{2}-1}}{2^{\frac{p}{2}} \sqrt{\pi}} \sum_{\beta=0}^{\infty} \frac{\tau^{2\beta} v^\beta \Gamma\left(\beta + \frac{1}{2}\right)}{(2\beta)! \Gamma\left(\frac{p}{2} + \beta\right)} & v > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. $\chi_p^2(\tau^2)$ with $\tau^2 = \sum_{i=1}^p \lambda_i^2$ can be constructed via $\mathbf{y}^\top \mathbf{y}$ with $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\lambda}, \mathbf{I})$.

Let \mathbf{Q} be $p \times p$ orthogonal matrix with elements of the first row being

$$q_{i1} = \frac{\lambda_i}{\sqrt{(\boldsymbol{\lambda})^\top \boldsymbol{\lambda}}}$$

for $i = 1, \dots, p$. Then $\mathbf{z} = \mathbf{Q}\mathbf{y}$ is distributed according to $\mathcal{N}(\boldsymbol{\tau}, \mathbf{I})$, where

$$\boldsymbol{\tau} = \begin{bmatrix} \tau \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\tau = \sqrt{\boldsymbol{\lambda}^\top \boldsymbol{\lambda}}$. Let $\mathbf{v} = \mathbf{y}^\top \mathbf{y} = \mathbf{z}^\top \mathbf{z} = \sum_{i=1}^p z_i^2$. Then $w = \sum_{i=2}^p z_i^2$ has a χ^2 -distribution with $p - 1$ degrees of freedom, and z_1 and w have as joint density

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z_1 - \tau)^2\right) \frac{1}{2^{\frac{p-1}{2}} \Gamma\left(\frac{p-1}{2}\right)} w^{\frac{p-1}{2}-1} \exp\left(-\frac{w}{2}\right) \\ &= C \exp\left(-\frac{1}{2}(\tau^2 + z_1^2 + w)\right) w^{\frac{p-3}{2}} \exp(\tau z_1) \\ &= C \exp\left(-\frac{1}{2}(\tau^2 + z_1^2 + w)\right) w^{\frac{p-3}{2}} \sum_{\alpha=0}^{\infty} \frac{\tau^\alpha z_1^\alpha}{\alpha!} \end{aligned}$$

where $C^{-1} = 2^{\frac{p}{2}} \sqrt{\pi} \Gamma\left(\frac{p-1}{2}\right)$. The joint density of $v = w + z_1^2$ and z_1 is obtained by substituting $w = v - z_1^2$ (the Jacobian being 1):

$$C \exp\left(-\frac{1}{2}(\tau^2 + v)\right) (v - z_1^2)^{\frac{p-3}{2}} \sum_{\alpha=0}^{\infty} \frac{\tau^\alpha z_1^\alpha}{\alpha!}.$$

The joint density of v and $u = z_1/\sqrt{v}$ is ($dz_1 = \sqrt{v} du$)

$$C \exp\left(-\frac{1}{2}(\tau^2 + v)\right) v^{\frac{p-2}{2}} (1 - u^2)^{\frac{p-3}{2}} \sum_{\alpha=0}^{\infty} \frac{\tau^\alpha v^{\frac{\alpha}{2}} u^\alpha}{\alpha!}.$$

The admissible range of z given v is $-\sqrt{v}$ to \sqrt{v} , and the admissible range of u is -1 to 1 . When we integrate above joint density with respect to u term by term, the terms for a odd integrate to 0, since such a term is an odd function of u . In the other integrations we substitute $u = \sqrt{s}$ ($du = \frac{\sqrt{s}}{2} ds$) to obtain

$$\begin{aligned} & \int_{-1}^1 (1 - u^2)^{\frac{p-3}{2}} u^{2\beta} du \\ &= 2 \int_0^1 (1 - u^2)^{\frac{p-3}{2}} u^{2\beta} du \\ &= \int_0^1 (1 - s)^{\frac{p-3}{2}} s^{\beta-\frac{1}{2}} ds \\ &= B\left(\frac{p-1}{2}, \beta + \frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{p-1}{2}\right) \Gamma\left(\beta + \frac{1}{2}\right)}{\Gamma\left(\frac{p}{2} + \beta\right)} \end{aligned}$$

by the usual properties of the beta and gamma functions. Thus the density of v is

$$\frac{1}{2^{\frac{p}{2}} \sqrt{\pi}} \exp\left(-\frac{1}{2}(\tau^2 + v)\right) v^{\frac{p}{2}-1} \sum_{\beta=0}^{\infty} \frac{\tau^{2\beta} v^\beta \Gamma\left(\beta + \frac{1}{2}\right)}{(2\beta)! \Gamma\left(\frac{p}{2} + \beta\right)}$$

for $v > 0$. □

Theorem 4.5. Define the likelihood ratio criterion as

$$\lambda = \frac{\max_{\mathbf{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}_0, \mathbf{\Sigma})}{\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \mathbf{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \mathbf{\Sigma})},$$

where

$$L(\boldsymbol{\mu}, \mathbf{\Sigma}) = (2\pi)^{-\frac{pN}{2}} (\det(\mathbf{\Sigma}))^{-\frac{N}{2}} \exp \left(-\frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu})^\top \mathbf{\Sigma}^{-1} (\mathbf{x}_\alpha - \boldsymbol{\mu}) \right).$$

then we have

$$\lambda^{\frac{2}{N}} = \frac{1}{1 + T^2/(N-1)},$$

where $T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$.

Proof. The maximum likelihood estimators of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ are

$$\hat{\boldsymbol{\mu}}_\Omega = \bar{\mathbf{x}} \quad \text{and} \quad \hat{\mathbf{\Sigma}}_\Omega = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

If we restrict $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, the likelihood function is maximized at

$$\hat{\mathbf{\Sigma}}_\omega = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu}_0)(\mathbf{x}_\alpha - \boldsymbol{\mu}_0)^\top.$$

Furthermore, we have

$$\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \mathbf{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \mathbf{\Sigma}) = (2\pi)^{-\frac{pN}{2}} (\det(\mathbf{\Sigma}_\Omega))^{-\frac{N}{2}} \exp \left(-\frac{1}{2} pN \right)$$

because of

$$\begin{aligned} & \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\boldsymbol{\mu}})^\top \hat{\mathbf{\Sigma}}_\Omega^{-1} (\mathbf{x}_\alpha - \bar{\boldsymbol{\mu}}) \\ &= \text{tr} \left(\hat{\mathbf{\Sigma}}_\Omega^{-1} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\boldsymbol{\mu}})(\mathbf{x}_\alpha - \bar{\boldsymbol{\mu}})^\top \right) \\ &= \text{tr}(n\mathbf{I}_p) = np. \end{aligned}$$

Similarly, we also have

$$\max_{\mathbf{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}_0, \mathbf{\Sigma}) = (2\pi)^{-\frac{pN}{2}} (\det(\mathbf{\Sigma}_\omega))^{-\frac{N}{2}} \exp \left(-\frac{1}{2} pN \right).$$

Thus the likelihood ratio criterion is

$$\begin{aligned} \lambda &= \frac{(2\pi)^{-\frac{pN}{2}} (\det(\mathbf{\Sigma}_\Omega))^{-\frac{N}{2}} \exp(-\frac{1}{2} pN)}{(2\pi)^{-\frac{pN}{2}} (\det(\mathbf{\Sigma}_\omega))^{-\frac{N}{2}} \exp(-\frac{1}{2} pN)} = \frac{(\det(\mathbf{\Sigma}_\omega))^{\frac{N}{2}}}{(\det(\mathbf{\Sigma}_\Omega))^{\frac{N}{2}}} \\ &= \frac{\left(\det \left(\sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \right) \right)^{\frac{N}{2}}}{\left(\det \left(\sum_{\alpha=1}^N (\mathbf{x}_\alpha - \boldsymbol{\mu}_0)(\mathbf{x}_\alpha - \boldsymbol{\mu}_0)^\top \right) \right)^{\frac{N}{2}}} = \frac{(\det(\mathbf{A}))^{\frac{N}{2}}}{(\det(\mathbf{A} + N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top))^{\frac{N}{2}}} \end{aligned}$$

where $\mathbf{A} = \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top = (N-1)\mathbf{S}$. Hence, we obtain

$$\begin{aligned}\lambda^{\frac{2}{N}} &= \frac{\det(\mathbf{A})}{\det(\mathbf{A} + (\sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0))(\sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top))} \\ &= \frac{1}{1 + N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{A}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)} \\ &= \frac{1}{1 + T^2/(N-1)}\end{aligned}$$

where $T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0) = (N-1)N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top \mathbf{A}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$ and we use the property of Schur complement to obtain

$$\det\left(\begin{bmatrix} \mathbf{A} & \mathbf{u} \\ -\mathbf{u}^\top & 1 \end{bmatrix}\right) = \det(\mathbf{A} + \mathbf{u}\mathbf{u}^\top) = \det\left(\begin{bmatrix} 1 & -\mathbf{u}^\top \\ \mathbf{u} & \mathbf{A} \end{bmatrix}\right) = \det(\mathbf{A})(1 + \mathbf{u}^\top \mathbf{A}^{-1}\mathbf{u})$$

with $\mathbf{u} = \sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$. Recall that The decomposition

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}^{-1}\mathbf{C} & \mathbf{I} \end{bmatrix}$$

means we have $\det(\mathbf{M}) = \det(\mathbf{D})\det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})$. □

Lemma 4.1. *For any $p \times p$ non-singular matrices \mathbf{C} and \mathbf{H} and any vector \mathbf{k} , we have*

$$\mathbf{k}^\top \mathbf{H}^{-1} \mathbf{k} = (\mathbf{C}\mathbf{k})^\top (\mathbf{C}\mathbf{H}\mathbf{C}^\top)^{-1} (\mathbf{C}\mathbf{k}).$$

Proof. We have $(\mathbf{C}\mathbf{k})^\top (\mathbf{C}\mathbf{H}\mathbf{C}^\top)^{-1} (\mathbf{C}\mathbf{k}) = \mathbf{k}^\top \mathbf{C}^\top (\mathbf{C}^\top)^{-1} (\mathbf{H})^{-1} \mathbf{C}^{-1} (\mathbf{C}\mathbf{k}) = \mathbf{k}^\top \mathbf{H}^{-1} \mathbf{k}$. □

Remark 4.1. *This lemma means*

$$T^{*2} = N(\bar{\mathbf{x}}^* - \mathbf{0})^\top (\mathbf{S}^*)^{-1} (\bar{\mathbf{x}}^* - \mathbf{0}) = N(\mathbf{C}\bar{\mathbf{x}} - \mathbf{0})^\top (\mathbf{C}\mathbf{S}\mathbf{C})^{-1} (\mathbf{C}\bar{\mathbf{x}}^* - \mathbf{0}) = N(\bar{\mathbf{x}} - \mathbf{0})^\top \mathbf{S}^{-1} (\bar{\mathbf{x}}^* - \mathbf{0}) = T^2.$$

Theorem 4.6. *Suppose $\mathbf{y}_1, \dots, \mathbf{y}_m$ are independent with \mathbf{y}_α distributed according to $\mathcal{N}(\mathbf{\Gamma}\mathbf{w}_\alpha, \Phi)$, where \mathbf{w}_α is an r -component vector. Let $\mathbf{H} = \sum_{\alpha=1}^m \mathbf{w}_\alpha \mathbf{w}_\alpha^\top$ assumed non-singular, $\mathbf{G} = \sum_{\alpha=1}^m \mathbf{y}_\alpha \mathbf{w}_\alpha^\top \mathbf{H}^{-1}$ and*

$$\mathbf{C} = \sum_{\alpha=1}^m (\mathbf{y}_\alpha - \mathbf{G}\mathbf{w}_\alpha)(\mathbf{y}_\alpha - \mathbf{G}\mathbf{w}_\alpha)^\top = \sum_{\alpha=1}^m \mathbf{y}_\alpha \mathbf{y}_\alpha^\top - \mathbf{G}\mathbf{H}\mathbf{G}^\top.$$

Then \mathbf{C} is distributed as

$$\sum_{\alpha=1}^{m-r} \mathbf{u}_\alpha \mathbf{u}_\alpha^\top$$

where $\mathbf{u}_1, \dots, \mathbf{u}_{m-r}$ are independently distributed according to $\mathcal{N}(\mathbf{0}, \Phi)$ independently of \mathbf{G} .

Proof. Theorem 4.3.3 of “Theodore W. Anderson. An Introduction to Multivariate Statistical Analysis. John Wiley & Sons Inc; 3rd Edition.” □

Theorem 4.7. *Let $T^2 = \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y}$, where \mathbf{y} is distributed according to $\mathcal{N}_p(\boldsymbol{\nu}, \boldsymbol{\Sigma})$ and $n\mathbf{S}$ is independently distributed as $\sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$ with $\mathbf{z}_1, \dots, \mathbf{z}_n$ independent, each with distribution $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. Then the random variable*

$$\frac{T^2}{n} \cdot \frac{n-p+1}{p}$$

is distributed as a noncentral F -distribution with p and $n-p+1$ degrees of freedom and noncentrality parameter $\boldsymbol{\nu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is central F .

Theorem 4.8. Let $T^2 = \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y}$, where \mathbf{y} is distributed according to $\mathcal{N}_p(\boldsymbol{\nu}, \boldsymbol{\Sigma})$ and $n\mathbf{S}$ is independently distributed as $\sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$ with $\mathbf{z}_1, \dots, \mathbf{z}_n$ independent, each with distribution $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. Then the random variable

$$\frac{T^2}{n} \cdot \frac{n-p+1}{p}$$

is distributed as a noncentral F -distribution with p and $n-p+1$ degrees of freedom and noncentrality parameter $\boldsymbol{\nu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\nu}$. If $\boldsymbol{\nu} = \mathbf{0}$, the distribution is central F .

Proof. Let \mathbf{D} be a non-singular matrix such that $\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top = \mathbf{I}$, and define

$$\mathbf{y}^* = \mathbf{D}\mathbf{y}, \quad \mathbf{S}^* = \mathbf{D}\mathbf{S}\mathbf{D}^\top, \quad \boldsymbol{\nu}^* = \mathbf{D}\boldsymbol{\nu}.$$

Lemma 4.1 means

$$T^2 = (\mathbf{y}^*)^\top (\mathbf{S}^*)^{-1} \mathbf{y}^*,$$

where \mathbf{y}^* is distributed according to $\mathcal{N}(\boldsymbol{\nu}^*, \mathbf{I})$ and

$$n\mathbf{S}^* = \sum_{\alpha=1}^{N-1} \mathbf{z}_\alpha^* (\mathbf{z}_\alpha^*)^\top = \sum_{\alpha=1}^{N-1} \mathbf{D}\mathbf{z}_\alpha (\mathbf{D}\mathbf{z}_\alpha)^\top$$

with $\mathbf{z}_\alpha^* = \mathbf{D}\mathbf{z}_\alpha$ independent, each with distribution $\mathcal{N}(\mathbf{0}, \mathbf{I})$. We also have

$$\boldsymbol{\nu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\nu} = (\mathbf{D}\boldsymbol{\nu})^\top (\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}^\top)^{-1} (\mathbf{D}\boldsymbol{\nu}^*) = (\boldsymbol{\nu}^*)^\top \boldsymbol{\nu}^*.$$

Let the first row of a $p \times p$ orthogonal matrix \mathbf{Q} be defined by

$$q_{i1} = \frac{y_i^*}{\sqrt{(\mathbf{y}^*)^\top \mathbf{y}^*}}$$

for $i = 1, \dots, p$. Since \mathbf{Q} depends on \mathbf{y}^* , it is a random matrix. Now let

$$\mathbf{u} = \mathbf{Q}\mathbf{y}^* \quad \text{and} \quad \mathbf{B} = \mathbf{Q}(n\mathbf{S}^*)\mathbf{Q}^\top,$$

where $n = N - 1$. The definition of \mathbf{Q} means

$$u_1 = \sum_{i=1}^p q_{1i} y_i^* = \frac{\sum_{i=1}^p (y_i^*)^2}{\sqrt{(\mathbf{y}^*)^\top \mathbf{y}^*}} = \sqrt{(\mathbf{y}^*)^\top \mathbf{y}^*}$$

and

$$u_j = \sum_{i=1}^p q_{ji} y_i^* = \sqrt{(\mathbf{y}^*)^\top \mathbf{y}^*} \sum_{i=1}^p q_{ji} q_{1i} = 0$$

for $j = 2, \dots, p$. Then

$$\begin{aligned} \frac{T^2}{n} &= \frac{(\mathbf{y}^*)^\top (\mathbf{S}^*)^{-1} \mathbf{y}^*}{n} = (\mathbf{Q}\mathbf{u})^\top (\mathbf{Q}^\top \mathbf{B} \mathbf{Q})^{-1} \mathbf{Q}^\top \mathbf{u} = \mathbf{u}^\top \mathbf{Q}^\top (\mathbf{Q}^\top)^{-1} \mathbf{B}^{-1} \mathbf{Q}^{-1} \mathbf{Q}^\top \mathbf{u} = \mathbf{u}^\top \mathbf{B}^{-1} \mathbf{u} \\ &= \begin{bmatrix} u_1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b^{11} & b^{12} & \dots & b^{1p} \\ b^{21} & b^{22} & \dots & b^{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b^{p1} & b^{p2} & \dots & b^{pp} \end{bmatrix} \begin{bmatrix} u_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = u_1^2 b^{11} \end{aligned}$$

where b^{ij} is the (i, j) -th entry of \mathbf{B}^{-1} . Using Schur Complement, we have

$$\frac{1}{b^{11}} = b_{11} - \mathbf{b}_{(1)}^\top \mathbf{B}_{22}^{-1} \mathbf{b}_{(1)} \triangleq b_{11.2, \dots, p} \quad (6)$$

with

$$\mathbf{B} = \begin{bmatrix} b_{11} & \mathbf{b}_{(1)}^\top \\ \mathbf{b}_{(1)} & \mathbf{B}_{22} \end{bmatrix}$$

and

$$\frac{T^2}{n} = \frac{u_1^2}{b_{11.2, \dots, p}} = \frac{(\mathbf{y}^*)^\top \mathbf{y}^*}{b_{11.2, \dots, p}}.$$

The conditional distribution of \mathbf{B} given \mathbf{Q} is that of

$$\mathbf{B} = \sum_{\alpha=1}^n \mathbf{Q} \mathbf{z}_\alpha^* (\mathbf{Q} \mathbf{z}_\alpha^*)^\top = \sum_{\alpha=1}^n \mathbf{v}_\alpha^* (\mathbf{v}_\alpha^*)^\top = \begin{bmatrix} \sum_{\alpha=1}^n (\mathbf{v}_{\alpha 1}^*)^2 & \sum_{\alpha=1}^n \mathbf{v}_{\alpha,1}^* (\mathbf{v}_{\alpha,2-p}^*)^\top \\ \sum_{\alpha=1}^n \mathbf{v}_{\alpha,1}^* (\mathbf{v}_{\alpha,2-p}^*) & \sum_{\alpha=1}^n (\mathbf{v}_{\alpha,2-p}^*) (\mathbf{v}_{\alpha,2-p}^*)^\top \end{bmatrix} = \begin{bmatrix} b_{11} & \mathbf{b}_{(1)}^\top \\ \mathbf{b}_{(1)} & \mathbf{B}_{22} \end{bmatrix},$$

where $\mathbf{v}_\alpha = \mathbf{Q} \mathbf{z}_\alpha^*$ are independent, each with distribution $\mathcal{N}(\mathbf{0}, \mathbf{I})$ since $\mathbf{Q} \mathbf{D} \Sigma \mathbf{D}^\top \mathbf{Q}^\top = \mathbf{I}$. We denote

$$\mathbf{G} = b_{(1)}^\top \mathbf{B}_{22}^{-1} = \sum_{\alpha=1}^m \mathbf{v}_{\alpha,1}^* (\mathbf{v}_{\alpha,2-p}^*)^\top \mathbf{B}_{22}^{-1}$$

By Theorem 4.6, the random variable

$$\begin{aligned} b_{11.2, \dots, p} &= b_{11} - \left(b_{(1)}^\top \mathbf{B}_{22}^{-1} \right) \mathbf{B}_{22} \mathbf{B}_{22}^{-1} b_{(1)} \\ &= \sum_{\alpha=1}^n (\mathbf{v}_{\alpha 1}^*)^2 - \mathbf{G} \mathbf{B}_{22}^{-1} \mathbf{G}^\top \end{aligned}$$

is conditionally distributed as

$$\sum_{\alpha=1}^{n-(p-1)} w_\alpha^2$$

where conditionally the w_α^2 are independent, each with the distribution $\mathcal{N}(0, 1)$; that is, $b_{11.2, \dots, p}$ is conditionally distributed as χ^2 with $n - (p - 1)$ degrees of freedom. Since the conditional distribution of $b_{11.2, \dots, p}$ does not depend on \mathbf{Q} , it is unconditionally distributed as χ^2 . The quantity $(\mathbf{y}^*)^\top \mathbf{y}^*$ has a noncentral χ^2 -distribution with p degrees of freedom and noncentrality parameter $(\boldsymbol{\nu}^*)^\top \boldsymbol{\nu}^* = \boldsymbol{\nu}^\top \Sigma^{-1} \boldsymbol{\nu}$. Then T is distributed as the ratio of a noncentral χ^2 and an independent χ^2 . \square

Remark 4.2. The equation (6) is based on the fact

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B} \mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}^{-1} \mathbf{C} & \mathbf{I} \end{bmatrix} \quad (7)$$

and

$$\begin{aligned} \mathbf{M}^{-1} &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \left(\begin{bmatrix} \mathbf{I} & \mathbf{B} \mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}^{-1} \mathbf{C} & \mathbf{I} \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{D}^{-1} \mathbf{C} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{B} \mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} \mathbf{B} \mathbf{D}^{-1} \\ -\mathbf{D}^{-1} \mathbf{C} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{C} (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1} \mathbf{B} \mathbf{D}^{-1} \end{bmatrix}. \end{aligned}$$

Theorem 4.9. Let u be distributed according to the χ^2 -distribution with a degrees of freedom and w be distributed according to the χ^2 -distribution with b degrees of freedom. The density of $v = u/(u + w)$, when u and w are independent is

$$\frac{1}{B\left(\frac{a}{2}, \frac{b}{2}\right)} v^{\frac{a}{2}-1} (1-v)^{\frac{b}{2}-1}, \quad (8)$$

where $B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$.

Proof. Let

$$v = \frac{u}{u+w} \quad \text{and} \quad z = u + w.$$

Then $u = vz$, $w = (1-v)z$ and

$$\det(\mathbf{J}(v, z)) = \det \left(\begin{bmatrix} \frac{\partial u}{\partial v} & \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial v} & \frac{\partial w}{\partial z} \end{bmatrix} \right) = \det \left(\begin{bmatrix} z & v \\ -z & 1-v \end{bmatrix} \right) = z.$$

Since v and w are independent, the joint density of u and w is

$$f_{u,w}(u, w) = \frac{1}{2^{\frac{a}{2}} \Gamma(\frac{a}{2})} u^{\frac{a}{2}-1} \exp\left(-\frac{u}{2}\right) \cdot \frac{1}{2^{\frac{b}{2}} \Gamma(\frac{b}{2})} w^{\frac{b}{2}-1} \exp\left(-\frac{w}{2}\right)$$

and the joint density of v and z is

$$\begin{aligned} f_{v,z}(v, z) &= f_{u,w}(vz, (1-v)z) \det(\mathbf{J}(v, z)) \\ &= \frac{1}{2^{\frac{a}{2}} \Gamma(\frac{a}{2})} (vz)^{\frac{a}{2}-1} \exp\left(-\frac{vz}{2}\right) \cdot \frac{1}{2^{\frac{b}{2}} \Gamma(\frac{b}{2})} ((1-v)z)^{\frac{b}{2}-1} \exp\left(-\frac{(1-v)z}{2}\right) \cdot z \\ &= \frac{1}{2^{\frac{a+b}{2}} \Gamma(\frac{a}{2}) \Gamma(\frac{b}{2})} v^{\frac{a}{2}-1} \cdot (1-v)^{\frac{b}{2}-1} z^{\frac{a+b}{2}-1} \exp\left(-\frac{z}{2}\right). \end{aligned}$$

Consider that the density of χ^2 -distribution with $a+b$ degrees of freedom, we have

$$\int_{-\infty}^{\infty} \frac{1}{2^{\frac{a+b}{2}} \Gamma(\frac{a+b}{2})} z^{\frac{a+b}{2}-1} \exp\left(-\frac{z}{2}\right) dz = 1.$$

Hence,

$$\begin{aligned} f_z(z) &= \int_{-\infty}^{\infty} f_{v,z}(v, z) dv \\ &= \frac{1}{2^{\frac{a+b}{2}} \Gamma(\frac{a}{2}) \Gamma(\frac{b}{2})} v^{\frac{a}{2}-1} (1-v)^{\frac{b}{2}-1} \int_{-\infty}^{\infty} z^{\frac{a+b}{2}-1} \exp\left(-\frac{z}{2}\right) dz \\ &= \frac{2^{\frac{a+b}{2}} \Gamma(\frac{a+b}{2})}{2^{\frac{a+b}{2}} \Gamma(\frac{a}{2}) \Gamma(\frac{b}{2})} v^{\frac{a}{2}-1} (1-v)^{\frac{b}{2}-1} \\ &= \frac{1}{B(\frac{a}{2} + \frac{b}{2})} v^{\frac{a}{2}-1} (1-v)^{\frac{b}{2}-1}. \end{aligned}$$

□

Remark 4.3. Beta distribution is a conjugate prior the binomial random variable. The binomial random variable X with parameters n and θ has the probability mass function

$$f(X = k | n, \theta) = C_n^k \theta^k (1-\theta)^{n-k}.$$

Let θ follows Beta distribution (prior distribution) with parameters a and b whose density function is

$$g(\theta | a, b) = \frac{1}{B(a, b)} v^{a-1} (1-v)^{b-1}.$$

Then we can write the density for the posterior distribution of θ by Bayes rule

$$P(\theta | X = k) = \frac{P(X = k | \theta) P(\theta)}{P(X = k)}$$

$$\begin{aligned}
&= \frac{C_n^k \theta^k (1-\theta)^{n-k} \cdot \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}}{P(X=k)} \\
&= \frac{C_n^k}{P(X=k)B(a,b)} \theta^{k+a-1} (1-\theta)^{n-k+b-1}.
\end{aligned}$$

Since $C_n^k/(P(X=k)B(a,b))$ is independent on θ , it follows Beta distribution with parameters $k+a$ and $n-k+b$ is density.

Theorem 4.10. Let x_1, x_2, \dots be a sequence of independently identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let

$$\hat{\mathbf{x}}_N = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_\alpha, \quad \hat{\mathbf{S}}_N = \frac{1}{N-1} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

and

$$T_N^2 = N(\bar{\mathbf{x}}_N - \boldsymbol{\mu}_0)^\top \mathbf{S}_N^{-1} (\bar{\mathbf{x}}_N - \boldsymbol{\mu}_0).$$

Then the limiting distribution of T_N^2 as $N \rightarrow \infty$ is the χ^2 -distribution with p degrees of freedom if $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

Proof. By the central limit theorem, the limiting distribution of $\sqrt{N}(\bar{\mathbf{x}}_N - \boldsymbol{\mu})$ is $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. The sample covariance matrix converges stochastically to $\boldsymbol{\Sigma}$. Then the limiting distribution of T^2 is the distribution of

$$\mathbf{y}^\top \boldsymbol{\Sigma}^{-1} \mathbf{y}$$

where \mathbf{y} has the distribution $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. The theorem follows from Theorem 4.3. \square

Lemma 4.2. If \mathbf{v} is a vector of p components and if \mathbf{B} is a non-singular $p \times p$ matrix, then $\mathbf{v}^\top \mathbf{B}^{-1} \mathbf{v}$ is the nonzero root of

$$\det(\mathbf{v}\mathbf{v}^\top - \lambda \mathbf{B}) = 0.$$

Proof. The non-zero root λ_1 of $\det(\mathbf{v}\mathbf{v}^\top - \lambda \mathbf{B}) = 0$ associate with vector $\boldsymbol{\beta} \neq \mathbf{0}$ satisfying

$$(\mathbf{v}\mathbf{v}^\top - \lambda_1 \mathbf{B})\boldsymbol{\beta} = \mathbf{0} \implies \mathbf{v}\mathbf{v}^\top \boldsymbol{\beta} = \lambda_1 \mathbf{B}\boldsymbol{\beta} \implies (\mathbf{v}^\top \mathbf{B}^{-1} \mathbf{v}) \mathbf{v}^\top \boldsymbol{\beta} = \lambda_1 \mathbf{v}^\top \boldsymbol{\beta}.$$

We can obtain that $\mathbf{v}^\top \boldsymbol{\beta} \neq 0$, otherwise $(\mathbf{v}\mathbf{v}^\top - \lambda_1 \mathbf{B})\boldsymbol{\beta} = \mathbf{0}$ means $\mathbf{B}\boldsymbol{\beta} = \mathbf{0}$ which is impossible since \mathbf{B} is non-singular. Hence $\lambda_1 = \mathbf{v}^\top \mathbf{B}^{-1} \mathbf{v}$.

Remark 4.4. Using this lemma with $\mathbf{v} = \sqrt{N}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$ and $\mathbf{B} = \mathbf{A}$, we can prove $T^2/(N-1)$ is the non-zero root of $\det(N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)^\top - \lambda \mathbf{A}) = 0$. \square

Lemma 4.3. For any positive definite matrix $\mathbf{S} \in \mathbb{R}^{p \times p}$ and $\mathbf{y}, \boldsymbol{\gamma} \in \mathbb{R}^p$, we have

$$(\boldsymbol{\gamma}^\top \mathbf{y})^2 \leq (\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma})(\mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y}).$$

Proof. For $\boldsymbol{\gamma} = \mathbf{0}$, the result is trivial. Otherwise, let

$$b = \frac{\boldsymbol{\gamma}^\top \mathbf{y}}{\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma}}.$$

Then we have

$$\begin{aligned}
0 &\leq (\mathbf{y} - b\mathbf{S}\boldsymbol{\gamma})^\top \mathbf{S}^{-1} (\mathbf{y} - b\mathbf{S}\boldsymbol{\gamma}) \\
&= \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y} - b\mathbf{y}^\top \mathbf{S}^{-1} \mathbf{S} \boldsymbol{\gamma} - b\boldsymbol{\gamma}^\top \mathbf{S} \mathbf{S}^{-1} \mathbf{y} + b^2 \boldsymbol{\gamma}^\top \mathbf{S} \mathbf{S}^{-1} \mathbf{S} \boldsymbol{\gamma} \\
&= \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y} - 2b\mathbf{y}^\top \boldsymbol{\gamma} + b^2 \boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma} \\
&= \mathbf{y}^\top \mathbf{S}^{-1} \mathbf{y} - \frac{(\boldsymbol{\gamma}^\top \mathbf{y})^2}{\boldsymbol{\gamma}^\top \mathbf{S} \boldsymbol{\gamma}},
\end{aligned}$$

which implies the desired result. \square

Theorem 4.11. Let $\{\mathbf{x}_\alpha^{(i)}\}$ for $\alpha = 1, \dots, N_i$, $i = 1, \dots, q$ be samples from $\mathcal{N}(\boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma})$, $i = 1, \dots, q$, respectively and suppose

$$\sum_{i=1}^q \beta_i \boldsymbol{\mu}^{(i)} = \boldsymbol{\mu}.$$

where β_1, \dots, β_q are given scalars and $\boldsymbol{\mu}$ is a given vector. Define the criterion

$$T^2 = c \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right) \mathbf{S}^{-1} \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right)^\top$$

where

$$\bar{\mathbf{x}}^{(i)} = \frac{1}{N_i} \sum_{\alpha=1}^{N_i} \mathbf{x}_\alpha^{(i)}, \quad \frac{1}{c} = \sum_{i=1}^q \frac{\beta_i^2}{N_i}$$

and

$$\left(\sum_{i=1}^q N_i - q \right) S = \sum_{i=1}^q \sum_{\alpha=1}^{N_i} (\mathbf{x}_\alpha^{(i)} - \bar{\mathbf{x}}^{(i)}) (\mathbf{x}_\alpha^{(i)} - \bar{\mathbf{x}}^{(i)})^\top.$$

Then this T^2 has the T^2 -distribution with $\sum_{i=1}^q N_i - q$ degrees of freedom.

Proof. Since $\mathbf{x}_\alpha^{(i)} \sim \mathcal{N}(\boldsymbol{\mu}^{(i)}, \boldsymbol{\Sigma})$, we have

$$\bar{\mathbf{x}}^{(i)} \sim \mathcal{N}\left(\boldsymbol{\mu}^{(i)}, \frac{1}{N_i} \boldsymbol{\Sigma}\right) \implies \beta_i (\bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu}^{(i)}) \sim \mathcal{N}\left(0, \frac{\beta_i^2}{N_i} \boldsymbol{\Sigma}\right).$$

and

$$\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} = \sum_{i=1}^q \beta_i (\bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu}^{(i)}) \sim \mathcal{N}\left(\mathbf{0}, \sum_{i=1}^q \frac{\beta_i^2}{N_i} \boldsymbol{\Sigma}\right) \implies \sqrt{c} \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right) \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}).$$

On the other hand, we can write

$$\sum_{i=1}^q \sum_{\alpha=1}^{N_i} (\mathbf{x}_\alpha^{(i)} - \bar{\mathbf{x}}^{(i)}) (\mathbf{x}_\alpha^{(i)} - \bar{\mathbf{x}}^{(i)})^\top = \sum_{i=1}^q \sum_{\alpha=1}^{N_i-1} \mathbf{z}_\alpha^{(i)} (\mathbf{z}_\alpha^{(i)})^\top$$

where $\mathbf{z}_\alpha^{(i)}$ are independent and $\mathbf{z}_\alpha^{(i)} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. Hence,

$$T^2 = \sqrt{c} \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right) \mathbf{S}^{-1} \left(\sqrt{c} \left(\sum_{i=1}^q \beta_i \bar{\mathbf{x}}^{(i)} - \boldsymbol{\mu} \right) \right)^\top$$

has the T^2 -distribution with $\sum_{i=1}^q N_i - q$ degrees of freedom. □

Lemma 4.4. Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ be independent samples from $\mathcal{N}(\boldsymbol{\mu}_\alpha, \boldsymbol{\Sigma}_\alpha)$ for $i = 1, \dots, m$. Define

$$\mathbf{z}_1 = \sum_{\alpha=1}^N a_\alpha \mathbf{x}_\alpha \quad \text{and} \quad \mathbf{z}_2 = \sum_{\alpha=1}^N b_\alpha \mathbf{x}_\alpha,$$

then

$$\text{Cov}(\mathbf{z}_1, \mathbf{z}_2) = \sum_{\alpha=1}^N a_\alpha b_\alpha \boldsymbol{\Sigma}_\alpha.$$

Proof. The definitions mean

$$\mathbf{z}_1 = \begin{bmatrix} a_1 \mathbf{I} & a_2 \mathbf{I} & \dots & a_N \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \dots \\ \mathbf{x}_N \end{bmatrix} \quad \text{and} \quad \mathbf{z}_2 = \begin{bmatrix} b_1 \mathbf{I} & b_2 \mathbf{I} & \dots & b_N \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \dots \\ \mathbf{x}_N \end{bmatrix},$$

then

$$\begin{aligned} \text{Cov}(\mathbf{z}_1, \mathbf{z}_2) &= \begin{bmatrix} a_1 \mathbf{I} & a_2 \mathbf{I} & \dots & a_N \mathbf{I} \end{bmatrix} \text{Cov} \left(\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \dots \\ \mathbf{x}_N \end{bmatrix}, \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \dots \\ \mathbf{x}_N \end{bmatrix} \right) \begin{bmatrix} b_1 \mathbf{I} \\ b_2 \mathbf{I} \\ \vdots \\ b_N \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} a_1 \mathbf{I} & a_2 \mathbf{I} & \dots & a_N \mathbf{I} \end{bmatrix} \begin{bmatrix} \Sigma_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Sigma_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Sigma_N \end{bmatrix} \begin{bmatrix} b_1 \mathbf{I} \\ b_2 \mathbf{I} \\ \vdots \\ b_N \mathbf{I} \end{bmatrix} \\ &= \sum_{\alpha=1}^N a_\alpha b_\alpha \Sigma_\alpha. \end{aligned}$$

□

Lemma 4.5. Let $\{\mathbf{x}_\alpha^{(i)}\}$ for $\alpha = 1, \dots, N_i$ be independent samples from $\mathcal{N}(\boldsymbol{\mu}^{(i)}, \Sigma_i)$ for $i = 1, 2$, respectively. We suppose $N_1 < N_2$ and define

$$\mathbf{y}_\alpha = \mathbf{x}_\alpha^{(1)} - \sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} \mathbf{x}_\gamma^{(2)},$$

for $\alpha = 1, \dots, N_1$. Then we have

$$\bar{\mathbf{y}} = \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \mathbf{y}_\alpha = \bar{\mathbf{x}}_\alpha^{(1)} - \bar{\mathbf{x}}_\alpha^{(2)}$$

and

$$\text{Cov}(\mathbf{y}_\alpha, \mathbf{y}_{\alpha'}) = \begin{cases} \Sigma_1 + \frac{N_1}{N_2} \Sigma_2, & \alpha = \alpha', \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} \bar{\mathbf{y}} &= \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \mathbf{y}_\alpha \\ &= \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \left(\mathbf{x}_\alpha^{(1)} - \sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} \mathbf{x}_\gamma^{(2)} \right) \\ &= \bar{\mathbf{x}}^{(1)} - \bar{\mathbf{x}}^{(2)} - \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \left(\sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} \right) \\ &= \bar{\mathbf{x}}^{(1)} - \bar{\mathbf{x}}^{(2)} - \frac{1}{N_1} \sum_{\alpha=1}^{N_1} \sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} \\ &= \bar{\mathbf{x}}^{(1)} - \bar{\mathbf{x}}^{(2)}. \end{aligned}$$

We first consider the case of $\alpha = \alpha'$. The independence means the covariance matrix of $[\mathbf{x}_\alpha^{(1)}; \mathbf{z}_\alpha]^\top$ has the form of

$$\begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \times \end{bmatrix},$$

where

$$\mathbf{z}_\alpha = -\sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} \mathbf{x}_\gamma^{(2)}.$$

Hence, we only needs to focus on the covariance matrix of

$$\begin{aligned} \mathbf{z}_\alpha &= -\sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} \mathbf{x}_\gamma^{(2)} \\ &= \sum_{\gamma=1}^{\alpha-1} \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right) \mathbf{x}_\gamma^{(2)} + \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} - \sqrt{\frac{N_1}{N_2}} \right) \mathbf{x}_\alpha^{(2)} \\ &\quad + \sum_{\gamma=\alpha+1}^{N_1} \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right) \mathbf{x}_\gamma^{(2)} + \sum_{\gamma=N_1+1}^{N_2} \left(-\frac{1}{N_2} \right) \mathbf{x}_\gamma^{(2)} \end{aligned}$$

Lemma 4.4 means

$$\begin{aligned} \text{Cov}(\mathbf{z}_\alpha, \mathbf{z}_\alpha) &= \left((\alpha-1) \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right)^2 + \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} - \sqrt{\frac{N_1}{N_2}} \right)^2 \right. \\ &\quad \left. + (N_1 - \alpha) \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right)^2 + (N_2 - N_1) \sum_{\gamma=N_1+1}^{N_2} \left(-\frac{1}{N_2} \right)^2 \right) \mathbf{\Sigma}_2 \\ &= \left((N_1 - 1) \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right)^2 + \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} - \sqrt{\frac{N_1}{N_2}} \right)^2 + \frac{(N_2 - N_1)^2}{N_2^2} \right) \mathbf{\Sigma}_2 \\ &= \frac{N_1}{N_2} \mathbf{\Sigma}_2, \end{aligned}$$

which means $\text{Cov}(\mathbf{y}_\alpha, \mathbf{y}_\alpha) = \mathbf{\Sigma}_1 + (N_1/N_2) \mathbf{\Sigma}_2$.

Then we consider the case of $\alpha \neq \alpha'$. We have

$$\begin{aligned} &\mathbf{y}_\alpha - \mathbb{E}[\mathbf{y}_\alpha] \\ &= \mathbf{x}_\alpha^{(1)} - \sqrt{\frac{N_1}{N_2}} \mathbf{x}_\alpha^{(2)} + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} \mathbf{x}_\beta^{(2)} - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} \mathbf{x}_\gamma^{(2)} - (\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}) \\ &= \mathbf{x}_\alpha^{(1)} - \boldsymbol{\mu}^{(1)} - \sqrt{\frac{N_1}{N_2}} (\mathbf{x}_\alpha^{(2)} - \boldsymbol{\mu}^{(2)}) + \frac{1}{\sqrt{N_1 N_2}} \sum_{\beta=1}^{N_1} (\mathbf{x}_\beta^{(2)} - \boldsymbol{\mu}^{(2)}) - \frac{1}{N_2} \sum_{\gamma=1}^{N_2} (\mathbf{x}_\gamma^{(2)} - \boldsymbol{\mu}^{(2)}) \\ &= \mathbf{x}_\alpha^{(1)} - \boldsymbol{\mu}^{(1)} - \sqrt{\frac{N_1}{N_2}} (\mathbf{x}_\alpha^{(2)} - \boldsymbol{\mu}^{(2)}) + \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right) \sum_{\beta=1}^{N_1} (\mathbf{x}_\beta^{(2)} - \boldsymbol{\mu}^{(2)}) - \frac{1}{N_2} \sum_{\gamma=N_1+1}^{N_2} (\mathbf{x}_\gamma^{(2)} - \boldsymbol{\mu}^{(2)}) \end{aligned}$$

and

$$\begin{aligned} &\mathbf{y}_{\alpha'} - \mathbb{E}[\mathbf{y}_{\alpha'}] \\ &= \mathbf{x}_{\alpha'}^{(1)} - \boldsymbol{\mu}^{(1)} - \sqrt{\frac{N_1}{N_2}} (\mathbf{x}_{\alpha'}^{(2)} - \boldsymbol{\mu}^{(2)}) + \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right) \sum_{\beta=1}^{N_1} (\mathbf{x}_\beta^{(2)} - \boldsymbol{\mu}^{(2)}) - \frac{1}{N_2} \sum_{\gamma=N_1+1}^{N_2} (\mathbf{x}_\gamma^{(2)} - \boldsymbol{\mu}^{(2)}). \end{aligned}$$

The independence means

$$\begin{aligned}
& \mathbb{E} \left[(\mathbf{y}_\alpha - \mathbb{E}[\mathbf{y}_\alpha]) (\mathbf{y}_{\alpha'} - \mathbb{E}[\mathbf{y}_{\alpha'}])^\top \right] \\
&= -2\sqrt{\frac{N_1}{N_2}} \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right) \mathbf{\Sigma}_2 + \left(\frac{1}{\sqrt{N_1 N_2}} - \frac{1}{N_2} \right)^2 N_1 \mathbf{\Sigma}_2 + \frac{N_2 - N_1}{N_2^2} \mathbf{\Sigma}_2 \\
&= \left(-2 \left(\frac{1}{N_2} - \frac{\sqrt{N_1}}{N_2 \sqrt{N_2}} \right) + \left(\frac{1}{N_1 N_2} - \frac{2}{N_2 \sqrt{N_1 N_2}} + \frac{1}{N_2^2} \right) N_1 + \frac{1}{N_2} - \frac{N_1}{N_2^2} \right) \mathbf{\Sigma}_2 \\
&= \mathbf{0}.
\end{aligned}$$

□

5 Sample Correlation Coefficients

Lemma 5.1. *If $\mathbf{y}_1, \dots, \mathbf{y}_N$ are independently distributed, if*

$$\mathbf{y}_\alpha = \begin{bmatrix} \mathbf{y}_\alpha^{(1)} \\ \mathbf{y}_\alpha^{(2)} \end{bmatrix}$$

has the density $f(\mathbf{y}_\alpha)$ and if the conditional density of $\mathbf{y}_\alpha^{(2)}$ given $\mathbf{y}_\alpha^{(1)}$ is $f(\mathbf{y}_\alpha^{(2)} | \mathbf{y}_\alpha^{(1)})$ for $\alpha = 1, \dots, n$. Then in the conditional distribution of $\mathbf{y}_1^{(2)}, \dots, \mathbf{y}_N^{(2)}$ given $\mathbf{y}_1^{(1)}, \dots, \mathbf{y}_N^{(1)}$, the random vectors $\mathbf{y}_1^{(2)}, \dots, \mathbf{y}_N^{(2)}$ are independent and the density of $\mathbf{y}_\alpha^{(2)}$ is $f(\mathbf{y}_\alpha^{(2)} | \mathbf{y}_\alpha^{(1)})$.

Proof. The marginal density of $\mathbf{y}_1^{(1)}, \dots, \mathbf{y}_N^{(1)}$ is

$$\prod_{\alpha=1}^N f_1(\mathbf{y}_\alpha^{(1)})$$

where $f_1(\mathbf{y}_\alpha^{(1)})$ is the marginal density of $\mathbf{y}_\alpha^{(1)}$, and the conditional density of $\mathbf{y}_1^{(2)}, \dots, \mathbf{y}_N^{(2)}$ given $\mathbf{y}_1^{(1)}, \dots, \mathbf{y}_N^{(1)}$ is

$$\frac{\prod_{\alpha=1}^N f(\mathbf{y}_\alpha)}{\prod_{\alpha=1}^N f_1(\mathbf{y}_\alpha^{(1)})} = \prod_{\alpha=1}^N \frac{f(\mathbf{y}_\alpha^{(1)}, \mathbf{y}_\alpha^{(2)})}{f_1(\mathbf{y}_\alpha^{(1)})} = \prod_{\alpha=1}^N f(\mathbf{y}_\alpha^{(2)} | \mathbf{y}_\alpha^{(1)}).$$

□

Theorem 5.1. *If the pairs $(z_{11}, z_{21}), \dots, (z_{1n}, z_{2n})$ are independent and each pair are distributed according to*

$$\begin{bmatrix} z_{1\alpha} \\ z_{2\alpha} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{bmatrix} \right), \quad \text{where } \alpha = 1, \dots, n,$$

then given $z_{11}, z_{12}, \dots, z_{1n}$, the conditional distributions of

$$b = \frac{\sum_{\alpha=1}^n z_{2\alpha} z_{1\alpha}}{\sum_{\alpha=1}^n z_{1\alpha}^2} \quad \text{and} \quad \frac{u}{\sigma^2} = \sum_{\alpha=1}^n \frac{(z_{2\alpha} - b z_{1\alpha})^2}{\sigma^2}$$

are $\mathcal{N}(\beta, \sigma^2/c^2)$ and χ^2 -distribution with $n-1$ degrees of freedom, respectively; and b and u are independent, where

$$\beta = \frac{\rho \sigma_2}{\sigma_1}, \quad \sigma^2 = \sigma_2^2(1 - \rho^2) \quad \text{and} \quad c^2 = \sum_{\alpha=1}^n z_{1\alpha}^2.$$

Proof. The conditional distribution of $z_{2\alpha}$ given $z_{1\alpha}$ is $\mathcal{N}(\beta z_{1\alpha}, \sigma^2)$. Let $\mathbf{v}_i = [z_{i1}, \dots, z_{in}]^\top$ for $i = 1, 2$. Lemma 5.1 means the density of \mathbf{v}_2 given \mathbf{v}_1 is $\mathcal{N}(\beta \mathbf{v}_1, \sigma^2 \mathbf{I})$ since z_{21}, \dots, z_{2n} are independent. We also have

$$\mathbf{v}_1^\top (\mathbf{v}_2 - b \mathbf{v}_1) = \mathbf{v}_1^\top \left(\mathbf{v}_2 - \frac{\mathbf{v}_1^\top \mathbf{v}_2}{\mathbf{v}_1^\top \mathbf{v}_1} \mathbf{v}_1 \right) = 0$$

and

$$u = (\mathbf{v}_2 - b \mathbf{v}_1)^\top (\mathbf{v}_2 - b \mathbf{v}_1) = \mathbf{v}_2^\top \mathbf{v}_2 - 2b \mathbf{v}_1^\top \mathbf{v}_2 + b^2 \mathbf{v}_1^\top \mathbf{v}_1 = \mathbf{v}_2^\top \mathbf{v}_2 - b^2 \mathbf{v}_1^\top \mathbf{v}_1.$$

Apply Theorem 3.4 with $x_\alpha = z_{2\alpha}$ and $y_\alpha = \sum_{\gamma=1}^n c_{\alpha\gamma} z_{2\gamma}$ for $\alpha = 1, \dots, n$, where the first row of orthogonal matrix \mathbf{C} is $(1/c) \mathbf{v}_1^\top$. Then y_1, \dots, y_n are independently normally distributed with variance σ^2 and means

$$\mathbb{E}[y_1] = \sum_{\gamma=1}^n c_{1\gamma} \mathbb{E}[z_{2\gamma}] = \sum_{\gamma=1}^n c_{1\gamma} \beta z_{1\gamma} = \beta c,$$

and

$$\mathbb{E}[y_\alpha] = \sum_{\gamma=1}^n c_{\alpha\gamma} \mathbb{E}[z_{2\gamma}] = \sum_{\gamma=1}^n c_{\alpha\gamma} \beta z_{1\gamma} = 0.$$

Thus, we have

$$b = \frac{\sum_{\alpha=1}^n z_{2\alpha} z_{1\alpha}}{\sum_{i=1}^n z_{1\alpha}^2} = \frac{\sum_{\alpha=1}^n c z_{2\alpha} c_{1\alpha}}{c^2} = \frac{y_1}{c} \sim \mathcal{N}\left(\beta, \frac{\sigma^2}{c^2}\right).$$

and

$$u = \sum_{\alpha=1}^n z_{2\alpha}^2 - b^2 \sum_{\alpha=1}^n z_{1\alpha}^2 = \sum_{\alpha=1}^n y_\alpha^2 - y_1^2 = \sum_{\alpha=2}^n y_\alpha^2,$$

which is independent of b . Since we have $y_\alpha \sim \mathcal{N}(0, \sigma^2)$ for $\alpha = 2, \dots, n$, the random variable u/σ^2 has a χ^2 -distribution with $n-1$ degrees of freedom. \square

Theorem 5.2. *If x and y are independently distributed, x having the distribution $\mathcal{N}(0, 1)$ and y having the χ^2 -distribution with m degrees of freedom, then $t = x/\sqrt{y/m}$ (has t -distribution with m degrees of freedom) has the density*

$$\frac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi} \Gamma(\frac{m}{2})} \left(1 + \frac{t^2}{m}\right)^{-\frac{m+1}{2}}.$$

Proof. The joint density of x and y is

$$f_{x,y}(x, y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \cdot \frac{1}{2^{\frac{m}{2}} \Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} \exp\left(-\frac{y}{2}\right).$$

The definition of t means $x = t\sqrt{y/m}$, then the joint density of t and y is

$$\begin{aligned} f_{t,y}(t, y) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2 y}{2m}\right) \cdot \frac{1}{2^{\frac{m}{2}} \Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} \exp\left(-\frac{y}{2}\right) \cdot \frac{dt \sqrt{y/m}}{dt} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2 y}{2m}\right) \cdot \frac{1}{2^{\frac{m}{2}} \Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} \exp\left(-\frac{y}{2}\right) \cdot \left(\frac{y}{m}\right)^{\frac{1}{2}} \\ &= \frac{1}{2^{\frac{m+1}{2}} \sqrt{m\pi} \Gamma(\frac{m}{2})} \exp\left(-\left(\frac{t^2}{2m} + \frac{1}{2}\right) y\right) \cdot y^{\frac{m-1}{2}}. \end{aligned} \tag{9}$$

The density of t can be obtained by integrating out y . Consider the expression of gamma function

$$\begin{aligned}\Gamma(\alpha) &= \int_0^{+\infty} \tilde{t}^{\alpha-1} \exp(-\tilde{t}) d\tilde{t} \\ &= \int_0^{+\infty} \left(\frac{t^2}{2m} + \frac{1}{2}\right)^{\alpha-1} y^{\alpha-1} \exp\left(-\left(\frac{t^2}{2m} + \frac{1}{2}\right)y\right) \left(\frac{t^2}{2m} + \frac{1}{2}\right) dy \\ &= \left(\frac{t^2}{2m} + \frac{1}{2}\right)^{\alpha} \int_0^{+\infty} y^{\alpha-1} \exp\left(-\left(\frac{t^2}{2m} + \frac{1}{2}\right)y\right) dy\end{aligned}\quad (10)$$

where we use the substitution

$$\tilde{t} = \left(\frac{t^2}{2m} + \frac{1}{2}\right)y.$$

Connecting (9) and (10) with $\alpha = \frac{m+1}{2}$, we have

$$\begin{aligned}f_t(t) &= \int_0^{+\infty} f_{t,y}(t, y) dy \\ &= \frac{1}{2^{\frac{m}{2}} \sqrt{m\pi} \Gamma\left(\frac{m+1}{2}\right)} \int_0^{+\infty} \exp\left(-\left(\frac{t^2}{2m} + \frac{1}{2}\right)y\right) \cdot y^{\frac{m-1}{2}} dy \\ &= \frac{1}{2^{\frac{m}{2}} \sqrt{m\pi} \Gamma\left(\frac{m+1}{2}\right)} \left(\frac{t^2}{2m} + \frac{1}{2}\right)^{-\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right) \\ &= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi} \Gamma\left(\frac{m}{2}\right)} \left(\frac{t^2}{m} + 1\right)^{-\frac{m+1}{2}}.\end{aligned}$$

□

Theorem 5.3. *Let us consider the likelihood ratio test of the hypothesis that $\rho = \rho_0$ based on a sample $\mathbf{x}_1, \dots, \mathbf{x}_N$ from the bivariate normal distribution*

$$\mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{bmatrix}\right).$$

The set Ω consists of $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ such that

$$\sigma_1 > 0, \quad \sigma_2 > 0 \quad \text{and} \quad -1 < \rho < 1$$

and the set ω is the subset for which $\rho = \rho_0$. The likelihood ratio criterion is

$$\frac{\sup_{\omega} L(\mathbf{x}, \boldsymbol{\theta})}{\sup_{\Omega} L(\mathbf{x}, \boldsymbol{\theta})} = \left(\frac{(1 - \rho_0^2)(1 - r^2)}{(1 - \rho_0 r)^2} \right)^{\frac{N}{2}},$$

where

$$r = \frac{a_{12}}{\sqrt{a_{11}} \sqrt{a_{22}}}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top} \quad \text{and} \quad \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha}.$$

Proof. We have shown in the section of T^2 -statistic that the likelihood maximized in Ω is

$$\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \boldsymbol{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{pN}{2}} (\det(\boldsymbol{\Sigma}_{\Omega}))^{-\frac{N}{2}} \exp\left(-\frac{1}{2}pN\right)$$

where

$$\boldsymbol{\Sigma}_{\Omega} = \frac{1}{N} \mathbf{A} \quad \text{with} \quad \bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha}, \quad \mathbf{A} = \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})^{\top} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad p = 2.$$

Then we have

$$\det(\mathbf{\Sigma}_\Omega) = \frac{a_{11}a_{22} - a_{12}a_{21}}{N^2},$$

which implies

$$\max_{\boldsymbol{\mu} \in \mathbb{R}^p, \mathbf{\Sigma} \in \mathbb{S}_p^{++}} L(\boldsymbol{\mu}, \mathbf{\Sigma}) = \frac{N^N \exp(-N)}{(2\pi)^N (a_{11}a_{22} - a_{12}a_{21})^{\frac{N}{2}}} = \frac{N^N \exp(-N)}{(2\pi)^N (1-r^2)^{\frac{N}{2}} a_{11}^{\frac{N}{2}} a_{22}^{\frac{N}{2}}}.$$

Let $\sigma^2 = \sigma_1\sigma_2$ and $\tau = \sigma_1/\sigma_2$. Under the null hypothesis ($\rho = \rho_0$), we have

$$\det(\mathbf{\Sigma}) = \sigma_1^2\sigma_2^2 - \sigma_1^2\sigma_1^2\rho_0^2 = \sigma^4(1 - \rho_0^2), \quad \mathbf{\Sigma}^{-1} = \frac{1}{1 - \rho_0^2} \begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho_0}{\sigma_1\sigma_2} \\ -\frac{\rho_0}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

and

$$\begin{aligned} & \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \mathbf{\Sigma}^{-1} (\mathbf{x}_\alpha - \bar{\mathbf{x}}) = \text{tr}(\mathbf{\Sigma}^{-1} \mathbf{A}) \\ &= \frac{1}{1 - \rho_0^2} \left(\frac{a_{11}}{\sigma_1^2} - \frac{2\rho_0 a_{12}}{\sigma_1\sigma_2} + \frac{a_{22}}{\sigma_2^2} \right) \\ &= \frac{1}{(1 - \rho_0^2)\sigma^2} \left(\frac{a_{11}}{\tau} - 2\rho_0 a_{12} + \tau a_{22} \right). \end{aligned}$$

Then the likelihood function under the null hypothesis ($\rho = \rho_0$) is

$$\frac{1}{(2\pi)^N (1 - \rho_0^2)^{\frac{N}{2}} (\sigma^2)^N} \exp \left(-\frac{a_{11}/\tau - 2\rho_0 a_{12} + \tau a_{22}}{2\sigma^2(1 - \rho_0^2)} \right) \quad (11)$$

The maximum of (11) with respect to τ occurs at

$$\hat{\tau} = \sqrt{a_{11}/a_{22}},$$

then the concentrated likelihood is

$$\frac{1}{(2\pi)^N (1 - \rho_0^2)^{\frac{N}{2}} (\sigma^2)^N} \exp \left(-\frac{\sqrt{a_{11}} \sqrt{a_{22}} (1 - \rho_0 r)}{\sigma^2(1 - \rho_0^2)} \right). \quad (12)$$

The maximum of (12) occurs at

$$\hat{\sigma}^2 = \frac{\sqrt{a_{11}} \sqrt{a_{22}} (1 - \rho_0 r)}{N(1 - \rho_0^2)},$$

which is because of $f(x) = \exp(-b/x)/x^N$ leads to

$$f'(x) = \frac{\exp(-\frac{b}{x}) \cdot \frac{b}{x^2} \cdot x^N - \exp(-\frac{b}{x}) \cdot Nx^{N-1}}{x^{2N}} = \frac{\exp(-\frac{b}{x}) x^{N-2} (b - Nx)}{x^{2N}}.$$

The likelihood ratio criterion is, therefore,

$$\frac{\sup_{\omega} L(\mathbf{x}, \boldsymbol{\theta})}{\sup_{\Omega} L(\mathbf{x}, \boldsymbol{\theta})} = \left(\frac{(1 - \rho_0^2)(1 - r^2)}{(1 - \rho_0 r)^2} \right)^{\frac{N}{2}}.$$

□

Lemma 5.2. For random vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}.$$

Then $\mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)] = 0$ and $\mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_l - \mu_l)] = \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$.

Theorem 5.4. Let

$$\mathbf{A}(n) = \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}}_N)(\mathbf{x}_\alpha - \bar{\mathbf{x}}_N)^\top,$$

where $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independently distributed according to $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $n = N - 1$. Then the limiting distribution of

$$\mathbf{B}(n) = \frac{1}{\sqrt{n}}(\mathbf{A}(n) - n\boldsymbol{\Sigma})$$

is normal with mean $\mathbf{0}$ and covariance $\mathbb{E}[b_{ij}(n)b_{kl}(n)] = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$.

Proof. We have

$$\mathbf{A}(n) = \sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top,$$

where $\mathbf{z}_1, \dots, \mathbf{z}_n$ are distributed according to $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. We arrange the elements of $\mathbf{z}_\alpha \mathbf{z}_\alpha^\top$ in a vector such as

$$\mathbf{y}_\alpha = \begin{bmatrix} z_{1\alpha}^2 \\ z_{1\alpha}z_{2\alpha} \\ \vdots \\ z_{2\alpha}^2 \\ \vdots \\ z_{p\alpha}^2 \end{bmatrix}.$$

The second moments of \mathbf{y}_α can be deduced from the forth moments of \mathbf{z}_α by using Lemma 5.2, that is,

$$\mathbb{E}[z_{i\alpha}z_{j\alpha}] = \sigma_{ij}, \quad \mathbb{E}[z_{i\alpha}z_{j\alpha}z_{k\alpha}z_{l\alpha}] = \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk},$$

and

$$\mathbb{E}[(z_{i\alpha}z_{j\alpha} - \sigma_{ij})(z_{k\alpha}z_{l\alpha} - \sigma_{kl})] = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}. \quad (13)$$

Arranging the elements of $\boldsymbol{\Sigma}$ and $\mathbf{A}(n)$ as

$$\boldsymbol{\nu} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \vdots \\ \sigma_{22} \\ \vdots \\ \sigma_{pp} \end{bmatrix} \quad \text{and} \quad \mathbf{w}(n) = \begin{bmatrix} a_{11}(n) \\ a_{12}(n) \\ \vdots \\ a_{22}(n) \\ \vdots \\ a_{pp}(n) \end{bmatrix}$$

we obtain

$$\frac{1}{\sqrt{n}}(\mathbf{w}(n) - n\boldsymbol{\nu}) = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^n (\mathbf{y}_\alpha - \boldsymbol{\nu}).$$

Since $\mathbb{E}[\mathbf{y}_\alpha] = \boldsymbol{\mu}$ and covariance of \mathbf{y}_α satisfies (13), the multivariate central limit theorem implies the desired result. \square

Remark 5.1. In the analysis for the asymptotic distribution of sample correlation, we apply this theorem with

$$\mathbf{A}(n) = \mathbf{C}(n) \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$

Then the covariance matrix of limiting distribution of the vector

$$\sqrt{n}(\mathbf{u}(n) - \mathbf{b}) = \frac{1}{\sqrt{n}} \left(\begin{bmatrix} c_{ii}(n) \\ c_{jj}(n) \\ c_{ij}(n) \end{bmatrix} - n\mathbf{b} \right)$$

is

$$\begin{bmatrix} \sigma_{11}\sigma_{11} + \sigma_{11}\sigma_{11} & \sigma_{12}\sigma_{12} + \sigma_{12}\sigma_{12} & \sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{11} \\ \sigma_{12}\sigma_{12} + \sigma_{12}\sigma_{12} & \sigma_{22}\sigma_{22} + \sigma_{22}\sigma_{22} & \sigma_{21}\sigma_{22} + \sigma_{22}\sigma_{21} \\ \sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{11} & \sigma_{21}\sigma_{22} + \sigma_{22}\sigma_{21} & \sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21} \end{bmatrix} = \begin{bmatrix} 2 & 2\rho^2 & 2\rho \\ 2\rho^2 & 2 & 2\rho \\ 2\rho & 2\rho & 1 + \rho^2 \end{bmatrix}.$$

Theorem 5.5. Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be a sample from $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and partition the variables as

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}.$$

Define $\mathbf{B} = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}$, $\boldsymbol{\Sigma}_{11.2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$,

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}^{(1)} \\ \bar{\mathbf{x}}^{(2)} \end{bmatrix} = \frac{1}{N} \sum_{\alpha=1}^N \begin{bmatrix} \mathbf{x}_\alpha^{(1)} \\ \mathbf{x}_\alpha^{(2)} \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \frac{1}{N} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top.$$

Then the maximum likelihood estimators of $\boldsymbol{\mu}^{(1)}$, $\boldsymbol{\mu}^{(2)}$, \mathbf{B} , $\boldsymbol{\Sigma}_{11.2}$ and $\boldsymbol{\Sigma}_{22}$ are

$$\hat{\boldsymbol{\mu}}^{(1)} = \bar{\mathbf{x}}^{(1)}, \quad \hat{\boldsymbol{\mu}}^{(2)} = \bar{\mathbf{x}}^{(2)}, \quad \hat{\mathbf{B}} = \mathbf{A}_{12}\mathbf{A}_{22}^{-1},$$

$$\hat{\boldsymbol{\Sigma}}_{11.2} = \frac{1}{N}(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}) \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_{22} = \frac{1}{N}\mathbf{A}_{22}.$$

Proof. The correspondence between $\boldsymbol{\Sigma}$ and $(\boldsymbol{\Sigma}_{11.2}, \mathbf{B}, \boldsymbol{\Sigma}_{22})$ is one-by-one since

$$\boldsymbol{\Sigma}_{12} = \mathbf{B}\boldsymbol{\Sigma}_{22} \quad \text{and} \quad \boldsymbol{\Sigma}_{11} = \boldsymbol{\Sigma}_{11.2} + \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}^\top,$$

which implies the desired result. \square

6 The Wishart Distribution

Theorem 6.1. Let $\mathbf{z}_1, \dots, \mathbf{z}_n$ be independently distributed, each according to $\mathcal{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$, where $n \geq p$; let

$$\mathbf{A} = \sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top = \mathbf{T}^* \mathbf{T}^{*\top},$$

where $t_{ij}^* = 0$ for $i < j$, and $t_{ii}^* > 0$ for $i = 1, \dots, p$. Then the density of \mathbf{T}^* is

$$\frac{\prod_{i=1}^p t_{ii}^{*n-i} \exp\left(-\frac{1}{2}\text{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{T}^*\mathbf{T}^{*\top})\right)}{2^{\frac{p(n-2)}{2}} \pi^{\frac{p(p-1)}{4}} (\det(\boldsymbol{\Sigma}))^{\frac{n}{2}} \prod_{i=1}^p \Gamma\left(\frac{1}{2}(n+1-i)\right)}.$$

Proof. Let \mathbf{C} be the lower triangular matrix ($c_{ij} = 0, i < j$) such that $\mathbf{\Sigma} = \mathbf{C}\mathbf{C}^\top$ and $c_{ii} > 0$. Define $\mathbf{y}_\alpha = \mathbf{C}^{-1}\mathbf{z}_\alpha$ for $\alpha = 1, \dots, n$, which are be independently distributed, each according to $\mathcal{N}_p(\mathbf{0}, \mathbf{I})$. We have $\mathbf{T}^*\mathbf{T}^{*\top} = \sum_{\alpha=1}^n \mathbf{C}\mathbf{y}_\alpha\mathbf{y}_\alpha^\top\mathbf{C}^\top = \mathbf{C}\mathbf{T}\mathbf{T}^\top\mathbf{C}^\top$. Let $\mathbf{T} = \mathbf{C}^{-1}\mathbf{T}^*$, then the matrix \mathbf{T} is the lower triangular with $t_{ii} > 0$ and we have

$$\mathbf{T}\mathbf{T}^\top = \mathbf{C}^{-1}\mathbf{T}^*\mathbf{T}^{*\top}\mathbf{C}^{-1} = \sum_{\alpha=1}^n \mathbf{C}^{-1}\mathbf{z}_\alpha\mathbf{z}_\alpha^\top\mathbf{C}^{-1} = \sum_{\alpha=1}^n \mathbf{y}_\alpha\mathbf{y}_\alpha^\top.$$

The lemma in slides have shown that random variables t_{i1}, \dots, t_{ii-1} are independently distributed and t_{ij} is distributed according to $\mathcal{N}(0, 1)$ for $i > j$; and t_{ii} has the χ^2 -distribution with $n - i + 1$ degrees of freedom. Hence, the density of $w = t_{ii}^2$ is

$$\frac{1}{2^{\frac{1}{2}(n+1-i)}\Gamma\left(\frac{1}{2}(n+1-i)\right)} w^{\frac{1}{2}(n+1-i)-1} \exp\left(-\frac{w}{2}\right)$$

and the density of $t_{ii} = \sqrt{w}$ is (using $dw/dt_{ii} = 2t_{ii}$)

$$\frac{1}{2^{\frac{1}{2}(n+1-i)}\Gamma\left(\frac{1}{2}(n+1-i)\right)} (t_{ii}^2)^{\frac{1}{2}(n+1-i)-1} \exp\left(-\frac{t_{ii}^2}{2}\right) \cdot (2t_{ii}) = \frac{1}{2^{\frac{n-i-1}{2}}\Gamma\left(\frac{1}{2}(n+1-i)\right)} t_{ii}^{n-i} \exp\left(-\frac{t_{ii}^2}{2}\right)$$

Then the joint density of t_{ij} for $j = 1, \dots, i, i = 1, \dots, p$ is

$$\begin{aligned} & \prod_{i=1}^p \prod_{j=1}^{i-1} \frac{\exp\left(-\frac{1}{2}t_{ij}^2\right)}{\sqrt{2\pi}} \cdot \prod_{i=1}^p \frac{t_{ii}^{n-i} \exp\left(-\frac{1}{2}t_{ii}^2\right)}{2^{\frac{n-i-1}{2}}\Gamma\left(\frac{1}{2}(n+1-i)\right)} \\ &= \prod_{i=1}^p \frac{\exp\left(-\frac{1}{2}\sum_{j=1}^{i-1} t_{ij}^2\right)}{(2\pi)^{\frac{i-1}{2}}} \cdot \prod_{i=1}^p \frac{t_{ii}^{n-i} \exp\left(-\frac{t_{ii}^2}{2}\right)}{2^{\frac{n-i-1}{2}}\Gamma\left(\frac{1}{2}(n+1-i)\right)} \\ &= \prod_{i=1}^p \frac{\exp\left(-\frac{1}{2}\sum_{j=1}^i t_{ij}^2\right) t_{ii}^{n-i}}{2^{\frac{n}{2}-1}\pi^{\frac{i-1}{2}}\Gamma\left(\frac{1}{2}(n+1-i)\right)} \\ &= \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^p \sum_{j=1}^i t_{ij}^2\right) \prod_{i=1}^p t_{ii}^{n-i}}{2^{\frac{p(n-2)}{2}}\pi^{\frac{p(p-1)}{4}} \prod_{i=1}^p \Gamma\left(\frac{1}{2}(n+1-i)\right)}. \end{aligned}$$

The Jacobian of the transformation from \mathbf{T} to $\mathbf{T}^* = \mathbf{C}\mathbf{T}$ can be written as

$$\begin{bmatrix} t_{11}^* \\ t_{21}^* \\ t_{22}^* \\ \vdots \\ t_{p1}^* \\ \vdots \\ t_{pp}^* \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \times & c_{22} & 0 & \cdots & 0 & \cdots & 0 \\ \times & \times & c_{22} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \times & \times & \times & \cdots & c_{pp} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \times & \times & \times & \cdots & \times & \cdots & c_{pp} \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{21} \\ t_{22} \\ \vdots \\ t_{p1} \\ \vdots \\ t_{pp} \end{bmatrix}.$$

Since the matrix of the transformation is triangular, its determinant is the product of the diagonal elements, namely, $\prod_{i=1}^p c_{ii}$. The Jacobian of the transformation from \mathbf{T} to \mathbf{T}^* is the reciprocal of the determinant. We also have $t_{ii} = t_{ii}^*/c_{ii}$, $\prod_{i=1}^p c_{ii}^2 = \det(\mathbf{C})\det(\mathbf{C}^\top) = \det(\mathbf{\Sigma})$ and

$$\begin{aligned} & \sum_{i=1}^p \sum_{j=1}^i t_{ij}^2 = \text{tr}(\mathbf{T}\mathbf{T}^\top) = \text{tr}\left(\mathbf{C}^{-1}\mathbf{T}^*\mathbf{T}^{*\top}\mathbf{C}^{-\top}\right) \\ &= \text{tr}\left(\mathbf{T}^*\mathbf{T}^{*\top}\mathbf{C}^{-\top}\mathbf{C}^{-1}\right) = \text{tr}\left(\mathbf{T}^*\mathbf{T}^{*\top}\mathbf{\Sigma}^{-1}\right) \end{aligned}$$

Then the density of \mathbf{T}^* is

$$\begin{aligned} & \frac{\exp\left(-\frac{1}{2}\text{tr}\left(\mathbf{T}^*\mathbf{T}^{*\top}\boldsymbol{\Sigma}^{-1}\right)\right)\prod_{i=1}^p(t_{ii}^*/c_{ii})^{n-i}}{2^{\frac{p(n-2)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)}\cdot\left(\prod_{i=1}^p c_{ii}^i\right)^{-1} \\ &= \frac{\exp\left(-\frac{1}{2}\text{tr}\left(\mathbf{T}^*\mathbf{T}^{*\top}\boldsymbol{\Sigma}^{-1}\right)\right)\prod_{i=1}^p t_{ii}^{*n-i}}{2^{\frac{p(n-2)}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)}\cdot\left(\prod_{i=1}^p c_{ii}\right)^n \\ &= \frac{\exp\left(-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{T}^*\mathbf{T}^{*\top}\right)\right)\prod_{i=1}^p t_{ii}^{*n-i}}{2^{\frac{p(n-2)}{2}}\pi^{\frac{p(p-1)}{4}}(\det(\boldsymbol{\Sigma}))^{\frac{n}{2}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)}. \end{aligned}$$

□

Theorem 6.2. Let $\mathbf{z}_1, \dots, \mathbf{z}_n$ be independently distributed, each according to $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$, where $n \geq p$. Then the density of $\mathbf{A} = \sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$ is

$$\frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)\right)}{2^{\frac{np}{2}}\pi^{\frac{p(p-1)}{4}}(\det(\boldsymbol{\Sigma}))^{\frac{n}{2}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)}$$

for \mathbf{A} positive definite, and 0 otherwise.

Proof. Following the proof of Theorem 6.1, we only need to consider the transformation from \mathbf{T}^* to \mathbf{A} . The relation $\mathbf{A} = \mathbf{T}^*\mathbf{T}^{*\top}$ means we can write

$$a_{hi} = \sum_{j=1}^i t_{hj}^* t_{ij}^* \quad \text{for } h \geq i.$$

Then we have

$$\frac{\partial a_{hi}}{\partial t_{kl}^*} = 0 \quad \text{for } k > h; \text{ or } k = h, l > i.$$

that is, $\partial a_{hi}/\partial t_{kl}^* = 0$ if k, l is beyond h, i in the lexicographic ordering. The Jacobian matrix of the transformation from \mathbf{A} to \mathbf{T}^* is a lower triangular matrix with diagonal elements

$$\begin{aligned} \frac{\partial a_{hh}}{\partial t_{hh}^*} &= 2t_{hh}^* \quad \text{for } h = 1, \dots, p; \\ \frac{\partial a_{hi}}{\partial t_{hi}^*} &= t_{ii}^* \quad \text{for } h > i; \end{aligned}$$

The determinant of the Jacobian matrix is therefore

$$2^p \prod_{i=1}^p t_{ii}^{*p+1-i}$$

The Jacobian of the transformation from \mathbf{T}^* to \mathbf{A} is the reciprocal. Hence, the density of \mathbf{A} is

$$\begin{aligned} & \frac{\prod_{i=1}^p t_{ii}^{*n-i} \exp\left(-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)\right)}{2^{\frac{p(n-2)}{2}}\pi^{\frac{p(p-1)}{4}}(\det(\boldsymbol{\Sigma}))^{\frac{n}{2}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)}\cdot\left(2^p \prod_{i=1}^p t_{ii}^{*p+1-i}\right)^{-1} \\ &= \frac{\prod_{i=1}^p t_{ii}^{*n-p-1} \exp\left(-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)\right)}{2^{\frac{pn}{2}}\pi^{\frac{p(p-1)}{4}}(\det(\boldsymbol{\Sigma}))^{\frac{n}{2}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)} \\ &= \frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2}\text{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)\right)}{2^{\frac{pn}{2}}\pi^{\frac{p(p-1)}{4}}(\det(\boldsymbol{\Sigma}))^{\frac{n}{2}}\prod_{i=1}^p\Gamma\left(\frac{1}{2}(n+1-i)\right)}. \end{aligned}$$

□

Corollary 6.1. Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be independently distributed, each according to $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $N > p$. Then the distribution of $\mathbf{S} = \frac{1}{n} \sum_{\alpha=1}^N (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$ is $\mathcal{W}(\frac{1}{n}\boldsymbol{\Sigma}, n)$.

Proof. The matrix \mathbf{S} has the distribution of

$$\mathbf{S} = \sum_{\alpha=1}^n \frac{\mathbf{z}_\alpha}{\sqrt{n}} \left(\frac{\mathbf{z}_\alpha}{\sqrt{n}} \right)^\top,$$

where each $\frac{\mathbf{z}_1}{\sqrt{n}}, \dots, \frac{\mathbf{z}_n}{\sqrt{n}}$ are independently distributed, each according to $\mathcal{N}(\mathbf{0}, \frac{1}{n}\boldsymbol{\Sigma})$. Theorem 6.2 implies this corollary. \square

Lemma 6.1. Given \mathbf{B} positive semidefinite and \mathbf{A} positive definite, there exists a non-singular matrix \mathbf{F} such that $\mathbf{F}^\top \mathbf{B} \mathbf{F} = \mathbf{D}$ and $\mathbf{F}^\top \mathbf{A} \mathbf{F} = \mathbf{I}$, where \mathbf{D} is diagonal.

Proof. Let the spectral decomposition of \mathbf{A} be $\mathbf{A} = \mathbf{U}_\mathbf{A} \boldsymbol{\Sigma}_\mathbf{A} \mathbf{U}_\mathbf{A}^\top$ and $\mathbf{E} = \mathbf{U}_\mathbf{A} \boldsymbol{\Sigma}_\mathbf{A}^{-\frac{1}{2}}$, then $\mathbf{E}^\top \mathbf{A} \mathbf{E} = \mathbf{I}$. Let the spectral decomposition of $\mathbf{B}^* = \mathbf{E}^\top \mathbf{B} \mathbf{E}$ be $\mathbf{B}^* = \mathbf{U}_{\mathbf{B}^*} \boldsymbol{\Sigma}_{\mathbf{B}^*} \mathbf{U}_{\mathbf{B}^*}^\top$, then

$$\boldsymbol{\Sigma}_{\mathbf{B}^*} = \mathbf{U}_{\mathbf{B}^*}^\top \mathbf{B}^* \mathbf{U}_{\mathbf{B}^*} = \mathbf{U}_{\mathbf{B}^*}^\top \mathbf{E}^\top \mathbf{B} \mathbf{E} \mathbf{U}_{\mathbf{B}^*}.$$

Letting $\mathbf{F} = \mathbf{E} \mathbf{U}_{\mathbf{B}^*}$ and $\mathbf{D} = \boldsymbol{\Sigma}_{\mathbf{B}^*}$ proves this lemma. \square

Lemma 6.2. The characteristic function of chi-square distribution with the degree of freedom n is

$$\phi(t) = (1 - 2it)^{-\frac{n}{2}}.$$

Proof. Let x be distributed according to χ^2 -distribution with the degree of freedom n , then its density is

$$f(x) = \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right).$$

We have (using the density of χ^2 -distribution with the degree of freedom $2k + n$)

$$\begin{aligned} \phi(t) &= \mathbb{E}[\exp(itx)] \\ &= \int_0^{+\infty} \exp(itx) \cdot \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right) dx \\ &= \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_0^{+\infty} \left(\sum_{k=0}^{\infty} \frac{(itx)^k}{k!} \right) x^{\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right) dx \\ &= \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \int_0^{+\infty} x^{k+\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right) dx \\ &= \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \cdot 2^{k+\frac{n}{2}} \Gamma\left(k + \frac{n}{2}\right) \int_0^{+\infty} \frac{1}{2^{k+\frac{n}{2}} \Gamma\left(k + \frac{n}{2}\right)} x^{k+\frac{n}{2}-1} \exp\left(-\frac{x}{2}\right) dx \\ &= \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \cdot 2^{k+\frac{n}{2}} \Gamma\left(k + \frac{n}{2}\right) \\ &= 1 + \sum_{k=1}^{\infty} \frac{(2it)^k}{k!} \cdot \prod_{j=0}^{k-1} \left(j + \frac{n}{2}\right) \\ &= (1 - 2it)^{-\frac{n}{2}}. \end{aligned}$$

For the last step, we consider Taylor expansion on $f(x) = (1 - x)^{-\frac{n}{2}}$ at $x = 0$, that is

$$f(x) = \sum_{k=0}^{\infty} \frac{f'(0)x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \prod_{j=0}^{k-1} \left(j + \frac{n}{2}\right).$$

We take $x = 2it$. \square

Theorem 6.3. If $\mathbf{z}_1, \dots, \mathbf{z}_n$ are independent, each with distribution $\mathcal{N}_p(\mathbf{0}, \Sigma)$, then the characteristic function of $a_{11}, \dots, a_{pp}, 2a_{12}, \dots, 2a_{p-1,p}$, where a_{ij} is the (i, j) -th element of

$$\mathbf{A} = \sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$$

is given by $\mathbb{E}[\exp(i \operatorname{tr}(\mathbf{A}\Theta))] = (\det(\mathbf{I} - 2i\Theta\Sigma))^{-\frac{n}{2}}$, where $\Theta \in \mathbb{R}^{p \times p}$ is symmetric.

Proof. The characteristic function of $a_{11}, \dots, a_{pp}, 2a_{12}, \dots, 2a_{p-1,p}$ is

$$\begin{aligned} & \mathbb{E}[\exp(i \operatorname{tr}(\mathbf{A}\Theta))] \\ &= \mathbb{E}\left[\exp\left(i \operatorname{tr}\left(\sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top \Theta\right)\right)\right] \\ &= \mathbb{E}\left[\exp\left(i \operatorname{tr}\left(\sum_{\alpha=1}^n \mathbf{z}_\alpha^\top \Theta \mathbf{z}_\alpha\right)\right)\right] \\ &= \mathbb{E}\left[\exp\left(i \sum_{\alpha=1}^n \mathbf{z}_\alpha^\top \Theta \mathbf{z}_\alpha\right)\right] \\ &= \prod_{\alpha=1}^n \mathbb{E}[\exp(i \mathbf{z}_\alpha^\top \Theta \mathbf{z}_\alpha)] \\ &= (\mathbb{E}[\exp(i \mathbf{z}^\top \Theta \mathbf{z})])^n, \end{aligned}$$

where $\mathbf{z} \sim \mathcal{N}_p(\mathbf{0}, \Sigma)$. Lemma 6.1 means there exists non-singular matrix \mathbf{F} such that

$$\mathbf{F}^\top \Sigma^{-1} \mathbf{F} = \mathbf{I} \quad \text{and} \quad \mathbf{F}^\top \Theta \mathbf{F} = \mathbf{D},$$

where $\mathbf{D} \in \mathbb{R}^{p \times p}$ is diagonal. If we set $\mathbf{z} = \mathbf{F}\mathbf{y}$, then

$$\begin{aligned} & \mathbb{E}[\exp(i \mathbf{z}^\top \Theta \mathbf{z})] \\ &= \mathbb{E}[\exp(i \mathbf{y}^\top \mathbf{F}^\top \Theta \mathbf{F} \mathbf{y})] \\ &= \mathbb{E}[\exp(i \mathbf{y}^\top \mathbf{D} \mathbf{y})] \\ &= \mathbb{E}\left[\prod_{j=1}^p \exp(i d_{jj} y_j^2)\right] \\ &= \prod_{j=1}^p \mathbb{E}[\exp(i d_{jj} y_j^2)]. \end{aligned}$$

Note that the term of $\mathbb{E}[\exp(i d_{jj} y_j^2)]$ is the characteristic function of the χ^2 -distribution with one degree of freedom, namely $(1 - 2i d_{jj})^{-\frac{1}{2}}$. Thus, we have

$$\mathbb{E}[\exp(i \mathbf{z}^\top \Theta \mathbf{z})] = \prod_{j=1}^p (1 - 2i d_{jj})^{-\frac{1}{2}} = (\det(\mathbf{I} - 2i\mathbf{D}))^{-\frac{1}{2}}.$$

We also have

$$\begin{aligned} & \det(\mathbf{I} - 2i\mathbf{D}) \\ &= \det(\mathbf{F}^\top \Sigma^{-1} \mathbf{F} - 2i\mathbf{F}^\top \Theta \mathbf{F}) \\ &= \det(\mathbf{F}^\top (\Sigma^{-1} - 2i\Theta) \mathbf{F}) \\ &= (\det(\mathbf{F}))^2 \det(\Sigma^{-1} - 2i\Theta) \end{aligned}$$

and $\mathbf{F}^\top \boldsymbol{\Sigma}^{-1} \mathbf{F} = \mathbf{I}$ means $\det(\mathbf{F}) = (\det(\boldsymbol{\Sigma}))^{\frac{1}{2}}$. Combing the above results, we obtain

$$\det(\mathbf{I} - 2i\mathbf{D}) = \det(\boldsymbol{\Sigma}) \det(\boldsymbol{\Sigma}^{-1} - 2i\boldsymbol{\Theta}) = \det(\mathbf{I} - 2i\boldsymbol{\Theta}\boldsymbol{\Sigma})$$

and

$$\mathbb{E}[\exp(i \operatorname{tr}(\mathbf{A}\boldsymbol{\Theta}))] = (\det(\mathbf{I} - 2i\boldsymbol{\Theta}\boldsymbol{\Sigma}))^{-\frac{n}{2}}.$$

□

Theorem 6.4. *Let \mathbf{A} and $\boldsymbol{\Sigma}$ be partitioned into q and $p - q$ rows and columns,*

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

If \mathbf{A} is distributed according to $\mathcal{W}(\boldsymbol{\Sigma}, n)$, then \mathbf{A}_{11} is distributed according to $\mathcal{W}(\boldsymbol{\Sigma}_{11}, n)$.

Proof. The assumption means \mathbf{A} is distributed as $\mathbf{A} = \sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$, where the \mathbf{z}_α are independent, each with the distribution $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. Partition \mathbf{z}_α into subvectors of q and $p - q$ components such that

$$\mathbf{z}_\alpha = \begin{bmatrix} \mathbf{z}_\alpha^{(1)} \\ \mathbf{z}_\alpha^{(2)} \end{bmatrix}.$$

Then $\mathbf{z}_1^{(1)}, \dots, \mathbf{z}_\alpha^{(n)}$ are independent, each with the distribution $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{11})$, and \mathbf{A}_{11} is distributed as

$$\sum_{\alpha=1}^n \mathbf{z}_\alpha^{(1)} (\mathbf{z}_\alpha^{(1)})^\top,$$

which has the distribution $\mathcal{W}(\boldsymbol{\Sigma}_{11}, n)$. □

Theorem 6.5. *Let \mathbf{A} and $\boldsymbol{\Sigma}$ be partitioned into p_1, \dots, p_q rows and columns with $p = p_1, \dots, p_q$,*

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1q} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{q1} & \cdots & \mathbf{A}_{qq} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \cdots & \boldsymbol{\Sigma}_{1q} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_{q1} & \cdots & \boldsymbol{\Sigma}_{qq} \end{bmatrix}$$

If $\boldsymbol{\Sigma} = \mathbf{0}$ for $i \neq j$ and if $\mathbf{A} \sim \mathcal{W}(\boldsymbol{\Sigma}, n)$, then $\mathbf{A}_{11}, \dots, \mathbf{A}_{qq}$ are independently distributed and $\mathbf{A}_{jj} \sim \mathcal{W}(\boldsymbol{\Sigma}_{jj}, n)$ for $j = 1, \dots, q$.

Proof. The assumption means \mathbf{A} is distributed as $\mathbf{A} = \sum_{\alpha=1}^n \mathbf{z}_\alpha \mathbf{z}_\alpha^\top$, where the \mathbf{z}_α are independent, each with the distribution $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$. Partition \mathbf{z}_α into subvectors

$$\mathbf{z}_\alpha = \begin{bmatrix} \mathbf{z}_\alpha^{(1)} \\ \vdots \\ \mathbf{z}_\alpha^{(q)} \end{bmatrix}.$$

as \mathbf{A} and $\boldsymbol{\Sigma}$ be portioned. Since $\boldsymbol{\Sigma}_{ij} = \mathbf{0}$, the sets $\mathbf{z}_1^{(1)}, \dots, \mathbf{z}_n^{(1)}, \dots, \mathbf{z}_1^{(q)}, \dots, \mathbf{z}_n^{(q)}$ are independent. Then $\mathbf{A}_{11} = \sum_{\alpha=1}^n \mathbf{z}_\alpha^{(1)} (\mathbf{z}_\alpha^{(1)})^\top, \dots, \mathbf{A}_{qq} = \sum_{\alpha=1}^n \mathbf{z}_\alpha^{(q)} (\mathbf{z}_\alpha^{(q)})^\top$ are independent. The rest of the proof follows from Theorem 6.4. □

Theorem 6.6. *If $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independent, each with distribution $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where*

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & 0 & \cdots & 0 \\ 0 & \sigma_{22} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{pp} \end{bmatrix}$$

then the density of the sample correlation coefficients is given by

$$\frac{(\Gamma(\frac{n}{2}))^p (\det([r_{ij}]_{ij}))^{\frac{n-p-1}{2}}}{\Gamma_p(\frac{n}{2})}.$$

where $n = N - 1$.

Proof. The density of \mathbf{A} is

$$\frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^p \frac{a_{ii}}{\sigma_{ii}}\right)}{2^{\frac{np}{2}} \prod_{i=1}^p \sigma_{ii}^{\frac{n}{2}} \Gamma_p\left(\frac{n}{2}\right)}.$$

We consider the transformation

1. $a_{ij} = \sqrt{a_{ii}} \sqrt{a_{jj}} r_{ij}$ for $i < j$,
2. $a_{ii} = a_{ii}$ otherwise,

which is from

$$\{r_{ij} : i < j, \ i, j = 1, \dots, p\} \cup \{a_{ii} : i = 1, \dots, p\}$$

to

$$\{a_{ij} : i < j, \ i, j = 1, \dots, p\} \cup \{a_{ii} : i = 1, \dots, p\}.$$

The determinant of Jacobian for this transformation is

$$\prod_{i=1}^p \prod_{j=1}^{i-1} \sqrt{a_{ii}} \sqrt{a_{jj}} = \prod_{i=1}^p a_{ii}^{\frac{p-1}{2}}.$$

The joint density of $\{r_{ij} : i < j, \ i, j = 1, \dots, p\} \cup \{a_{ii} : i = 1, \dots, p\}$ is

$$\begin{aligned} & \frac{(\det([\sqrt{a_{ii}} \sqrt{a_{jj}} r_{ij}]_{ij}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^p \frac{a_{ii}}{\sigma_{ii}}\right)}{2^{\frac{np}{2}} \prod_{i=1}^p \sigma_{ii}^{\frac{n}{2}} \Gamma_p\left(\frac{n}{2}\right)} \cdot \prod_{i=1}^p a_{ii}^{\frac{p-1}{2}} \\ &= \frac{(\prod_{i=1}^p a_{ii})^{\frac{n-p-1}{2}} (\det([r_{ij}]_{ij}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^p \frac{a_{ii}}{\sigma_{ii}}\right)}{2^{\frac{np}{2}} \prod_{i=1}^p \sigma_{ii}^{\frac{n}{2}} \Gamma_p\left(\frac{n}{2}\right)} \cdot \prod_{i=1}^p a_{ii}^{\frac{p-1}{2}} \\ &= \frac{(\det([r_{ij}]_{ij}))^{\frac{n-p-1}{2}}}{\Gamma_p\left(\frac{n}{2}\right)} \cdot \prod_{i=1}^p \frac{a_{ii}^{\frac{n}{2}-1} \exp\left(-\frac{a_{ii}}{2\sigma_{ii}}\right)}{2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}}}, \end{aligned}$$

where $r_{ii} = 1$. Let $u_i = a_{ii}/(2\sigma_{ii})$, then

$$\int_0^\infty \frac{a_{ii}^{\frac{n}{2}-1} \exp\left(-\frac{a_{ii}}{2\sigma_{ii}}\right)}{2^{\frac{n}{2}} \sigma_{ii}^{\frac{n}{2}}} da_{ii} = \int_0^\infty u_i^{\frac{n}{2}-1} \exp(-u_i) du_i = \Gamma\left(\frac{n}{2}\right).$$

Combing all above results proves this theorem. \square

Theorem 6.7. If \mathbf{A} has the distribution $\mathcal{W}(\boldsymbol{\Sigma}, n)$ and $\boldsymbol{\Sigma}$ has the a prior distribution $\mathcal{W}^{-1}(\boldsymbol{\Psi}, m)$, then the conditional distribution of $\boldsymbol{\Sigma}$ given \mathbf{A} is the inverted Wishart distribution $\mathcal{W}^{-1}(\mathbf{A} + \boldsymbol{\Psi}, n + m)$.

Proof. The joint density of \mathbf{A} and Σ ,

$$\begin{aligned} f(\mathbf{A}, \Sigma) &= \frac{(\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2} \text{tr}(\Sigma^{-1} \mathbf{A})\right)}{2^{\frac{np}{2}} (\det(\Sigma))^{\frac{n}{2}} \Gamma_p\left(\frac{n}{2}\right)} \cdot \frac{(\det(\Psi))^{\frac{m}{2}} (\det(\Sigma))^{-\frac{m+p+1}{2}} \exp\left(-\frac{1}{2} \text{tr}(\Psi \Sigma^{-1})\right)}{2^{\frac{mp}{2}} \Gamma_p\left(\frac{m}{2}\right)} \\ &= \frac{(\det(\Psi))^{\frac{m}{2}} (\det(\Sigma))^{-\frac{n+m+p+1}{2}} (\det(\mathbf{A}))^{\frac{n-p-1}{2}} \exp\left(-\frac{1}{2} \text{tr}((\mathbf{A} + \Psi) \Sigma^{-1})\right)}{2^{\frac{(m+n)p}{2}} \Gamma_p\left(\frac{n}{2}\right) \Gamma_p\left(\frac{m}{2}\right)} \end{aligned} \quad (14)$$

for \mathbf{A} and Σ are positive definite. The marginal density of \mathbf{A} is the integral of (14) over the set of Σ positive definite. Since

$$\begin{aligned} 1 &= \int w^{-1}(\Sigma \mid \mathbf{A} + \Psi, n+m) d\Sigma \\ &= \frac{(\det(\mathbf{A} + \Psi))^{\frac{n+m}{2}} (\det(\Sigma))^{-\frac{n+m+p+1}{2}} \exp\left(-\frac{1}{2} \text{tr}((\mathbf{A} + \Psi) \Sigma^{-1})\right)}{2^{\frac{(m+n)p}{2}} \Gamma_p\left(\frac{n+m}{2}\right)}, \end{aligned}$$

we have

$$\begin{aligned} f(\mathbf{A}) &= \int f(\mathbf{A}, \Sigma) d\Sigma \\ &= \frac{(\det(\Psi))^{\frac{m}{2}} (\det(\mathbf{A}))^{\frac{n-p-1}{2}}}{\Gamma_p\left(\frac{n}{2}\right) \Gamma_p\left(\frac{m}{2}\right)} \int \frac{(\det(\Sigma))^{-\frac{n+m+p+1}{2}} \exp\left(-\frac{1}{2} \text{tr}((\mathbf{A} + \Psi) \Sigma^{-1})\right)}{2^{\frac{(m+n)p}{2}}} d\Sigma \\ &= \frac{(\det(\Psi))^{\frac{m}{2}} (\det(\mathbf{A}))^{\frac{n-p-1}{2}}}{\Gamma_p\left(\frac{n}{2}\right) \Gamma_p\left(\frac{m}{2}\right)} \cdot \Gamma_p\left(\frac{n+m}{2}\right) (\det(\mathbf{A} + \Psi))^{-\frac{n+m}{2}}. \end{aligned}$$

Then

$$\begin{aligned} f(\Sigma \mid \mathbf{A}) &= \frac{f(\Sigma, \mathbf{A})}{f(\mathbf{A})} \\ &= \frac{(\det(\mathbf{A} + \Psi))^{\frac{n+m}{2}} (\det(\Sigma))^{-\frac{n+m+p+1}{2}} \exp\left(-\frac{1}{2} \text{tr}((\mathbf{A} + \Psi) \Sigma^{-1})\right)}{2^{\frac{(m+n)p}{2}} \Gamma_p\left(\frac{n+m}{2}\right)} \\ &= w^{-1}(\Sigma \mid \mathbf{A} + \Psi, n+m). \end{aligned}$$

□