Dynamic Programming

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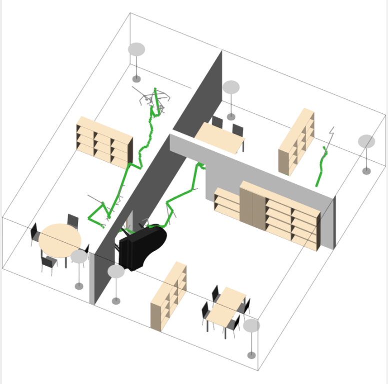
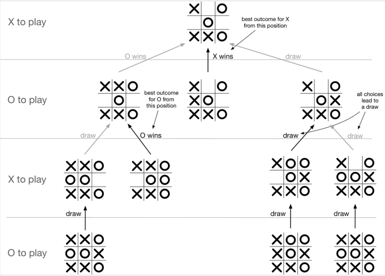
Outline

- Introduction
- Policy Evaluation
- Policy Iteration
- Value Iteration
- Extensions of DP

Introduction

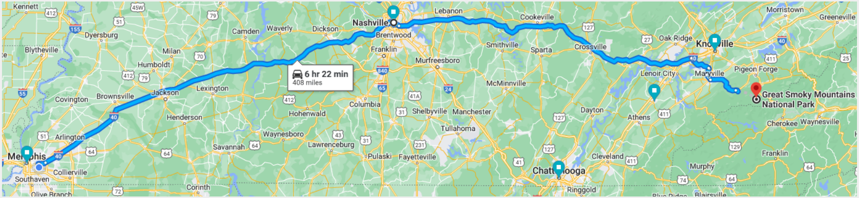
- Reinforcement Learning
  - Agent improves its policy by *interacting* with an *initially unknown* environment (model-free and model-based).
- Planning
  - Agent improves its policy by *computing* with its *internal model* of the environment (model-based).



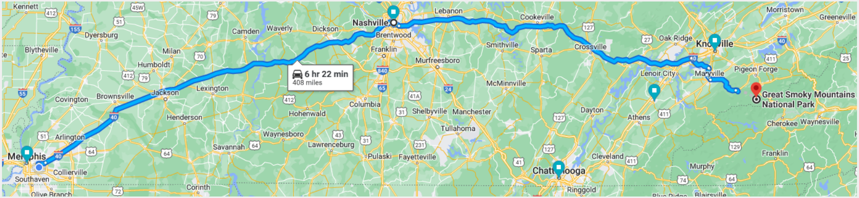


- DP assumes full knowledge (including transition function and reward function) of the MDP and is used for *planning* in the MDP.
- DP's focus is to solve MDP using the least computational effort.

- DP is a general method for problems which have two properties:
  - Optimal substructure: optimal solution can be decomposed for subproblems
  - Overlapping subproblems: subproblems recur many times and their solutions can be reused
  - E.g., finding the shortest path between A and B



- DP solves a complex problem, which involves sequential or temporal components, by breaking it down into subproblems
  - solve the subproblems
  - combine solutions to subproblems



- Scheduling algorithms
- String algorithms (e.g., sequence alignment)
- Graph algorithms (e.g., shortest path algorithms)
- Graphical models (e.g., Viterbi algorithm)
- Bioinformatics (e.g., lattice models)

- MDPs have both properties required by DP:
  - Bellman equation gives recursive decomposition
  - Value function stores solutions for reuse
- An optimal policy can be subdivided into two components:
  - An optimal first action A^*
  - Followed by an optimal policy from successor state S'

- Prediction: evaluate a given policy (evaluate the future)
  - E.g., what is the value function for the uniform random policy?
  - Input: MDP and policy π
  - Output: value function V^π
- Control: find the best policy (optimize the future)
  - What is the optimal value function over all possible policies, hence the optimal policy?
  - Input: MDP
  - Output: optimal value function V^* and optimal policy π^*

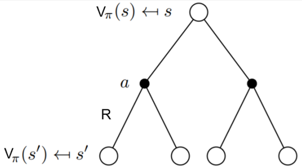
Policy Evaluation

Policy Evaluation

- Policy evaluation solves the *prediction* problem: compute the state-value function V^π for an arbitrary policy π.
- One approach is to solve the Bellman expectation equation directly using the normal equation (only works for small MDPs).
- The DP approach is to iteratively applying the Bellman expectation equation until converge to V^π.

Iterative Policy Evaluation

- At the $(k+1)$th iteration, update $V^{\pi}_{k+1}(s)$ from $V^{\pi}_{k}(s')$ (initialize $V^{\pi}_{0}(s) = 0$).
- DP uses *bootstrapping*: updates a guess $\left(V^{\pi}_{k+1}(s)\right)$ using a guess $\left(V^{\pi}_{k}(s')\right)$.

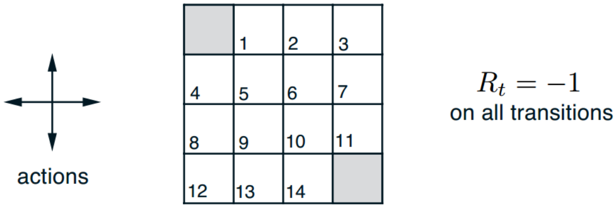


$$v_{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}^{a}_{s} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^{a}_{ss'} v_{\pi}(s') \right)$$

Iterative Policy Evaluation

- Example: evaluate the uniform random policy in an undiscounted, episodic task.

Example 4.1 Consider the 4×4 gridworld shown below.



Iterative Policy Evaluation

- E.g., from $k = 2$ to $k = 3$, $-1.7 \to -2.4$: -0.25(0+2+2+1.7)-1 = -2.425

v_k for the random policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

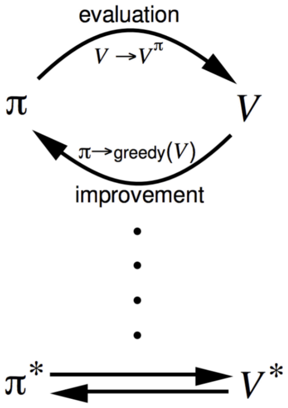
$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

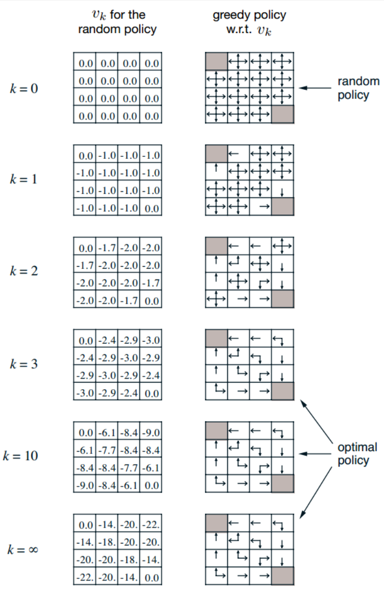
- In theory, iterative policy evaluation converges in the limit: $V_k^\pi \to V^\pi$ as $k \to \infty$
- In practice, we stop $\|V_{k+1}^\pi - V_k^\pi\| \leq \epsilon$.
- The existence and uniqueness of V^π are guaranteed providing either $\gamma < 1$ or eventual termination is guaranteed from all states under the policy π.

Policy Iteration

- Policy iteration solves the *control* problem, i.e., finding π^*
  - policy evaluation: compute V^π for a given π
  - policy improvement: produce $\pi' \geq \pi$ by acting greedily to V^π
- Policy iteration will converge to π^*



- For a simple task, we can get π^* with one iteration of policy evaluation and policy improvement.
- For the example in the figure, we can obtain π^* without converging to V^π in the first iteration of policy evaluation.



- One drawback of the vanilla policy iteration is that each iteration involves policy evaluation (if the number of states is small enough so that the direct method can be applied efficiently, this is no issue).

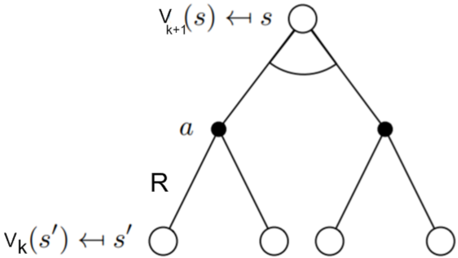
- Do we need to converge to V^π to obtain the optimal policy? No.
  - Use a stopping condition, e.g., $\|V^\pi_{k+1} - V^\pi_k\| \leq \epsilon$
  - Simply stop after k iterations (we may then have the optimal policy)
  - In fact, the policy evaluation step can be truncated in several ways without losing the convergence guarantees of policy iteration

- Do we need to update policy in every iteration to obtain the optimal policy? No, we can use value iteration.

Value Iteration

- Value iteration solves the *control* problem, i.e., finding π^*.
- Solution: iteratively applying the Bellman optimality equation until converge to V^* (no direct method exists due to the nonlinearity of the Bellman optimality equation).
- Unlike policy iteration, there is no explicit policy update during the process
- Intermediate value functions may not correspond to any effective policy

- At the $(k+1)$th iteration, update $V_{k+1}(s)$ from $V_k(s')$.



$$v_{k+1}(s) = \max_a \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

Value Iteration: Undiscounted Example

Example: Shortest Path

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Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

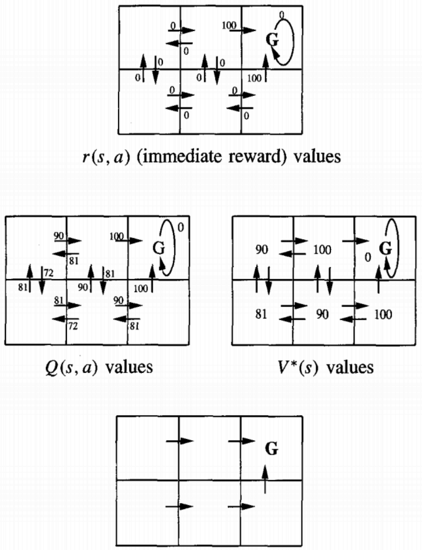
0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Value Iteration: Discounted Example ($\gamma = 0.9$)



$r(s,a)$ (immediate reward) values

$Q(s,a)$ values

$V^*(s)$ values

One optimal policy

- Policy iteration
  - policy evaluation (multiple iterations) + policy improvement (multiple iterations)
  - uses Bellman expectation equation
- Value iteration
  - finding optimal value function (multiple iterations) + optimal policy extraction (once)
  - uses Bellman optimality equation
- The performance is task-dependent.
  - policy converge rate (policy iteration) vs. value convergence rate (value iteration)
  - if optimal policy is unnecessary, we can use policy iteration

- Both are widely used and converge much faster than their theoretical worst-case run times.
- There is a spectrum of algorithms between "Policy Interation" and "Value Iteration" conducting policy evaluation and policy improvement at different granularity.
- In theory, initialization is independent of convergence rate. In practice, good initial value functions or policies can speed-up the convergence.

Extensions of DP

- Synchronous DP processes all states at once using previous value functions.

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

- Algorithms based on state-value function V, complexity $O(mn^2)$ per iteration, based on action-value function Q, complexity $O(m^2n^2)$ per iteration, for m actions and n states.
- DP is guaranteed to find an optimal policy in polynomial time, even though the total number of (deterministic) policies is m^n, which is better than direct search and linear programming.

- Asynchronous DP does not require processing all states at once, instead it can select any state to process, in any order.
- Asynchronous DP can significantly reduce the computation and guarantee to converge if all states continue to be selected.
- Example asynchronous DP algorithms:
  - In-place dynamic programming
  - Prioritized sweeping
  - Real-time dynamic programming

- DP uses full-width updates by considering every successor state and action.
- DP is effective and efficient for medium-sized problems (millions of states).
- For large problems, DP suffers Bellman's curse of dimensionality (the number of states grows exponentially with the number of state variables). Even one update can be too expensive.

- Update via sampling the environment transition function and reward function has two advantages
  - Model-free: do not need to know the full MDP.
  - Overcomes the curse of dimensionality: cost is constant, independent from the number of state variables.