Assignment 1 (3)(4)

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Problem 3

1.

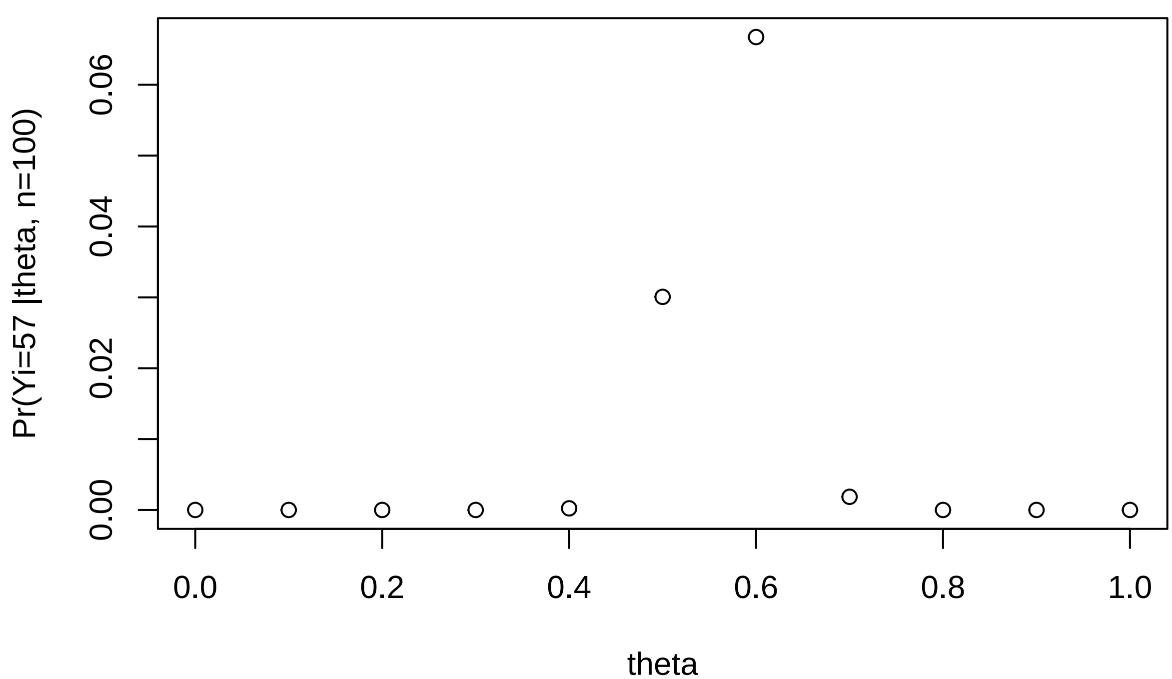
$$P(Y_1, Y_2, ..., Y_{100}|\theta) = \theta^{\sum_{i=1}^{100} y_i}(1-\theta)^{100-\sum_{i=1}^{100} y_i}$$

$$P(\sum Y_i = y|\theta) = \binom{100}{y}\theta^y(1-\theta)^{100-y}$$

2.

$$P(\sum Y_i = 57|\theta) = \binom{100}{57}\theta^{57}(1-\theta)^{43}$$

```
theta <- seq(0,1,by=0.1)
prob <- choose(100,57)*(theta^57)*(1-theta)^(100-57)
plot(theta, prob,ylab="Pr(Yi=57 |theta, n=100)",xlab="theta")
```

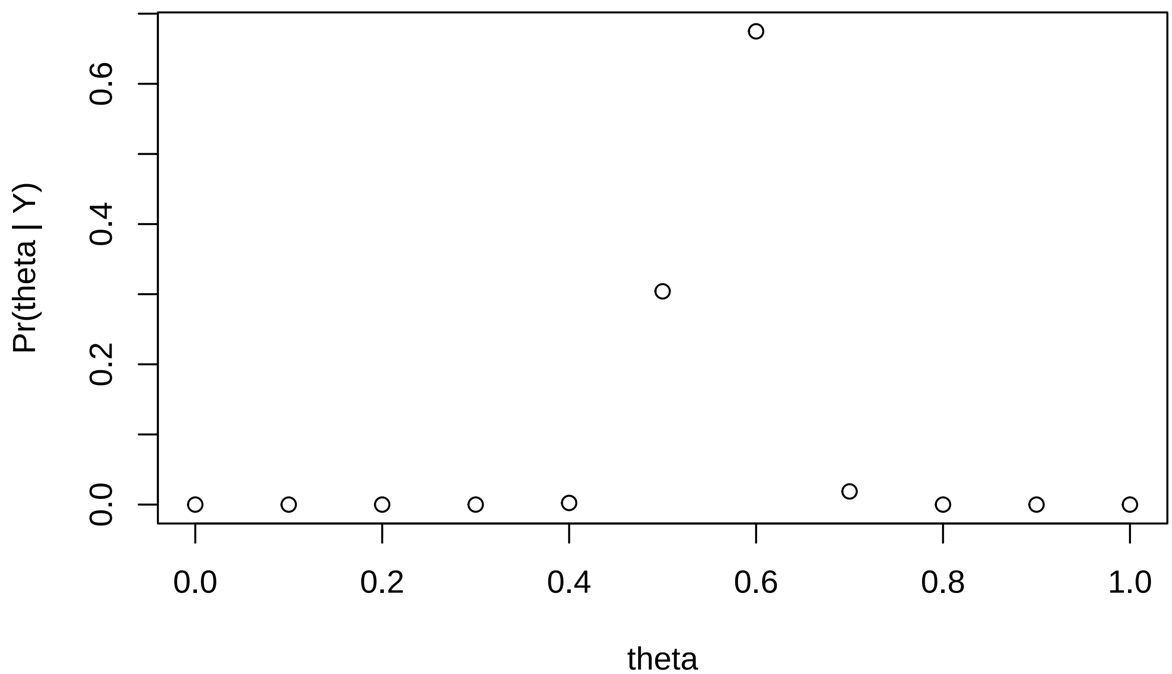


From the plot, we can see that the maximum likelihood estimate is for theta = 0.6.

3.

$$P(\theta) = 1/11$$
$$P(Y_i = y|\theta) = \binom{100}{57}\theta^{57}(1-\theta)^{43}$$
$$P(y) = \int P(y|\theta)\pi(\theta)d\theta = \sum P(\theta)P(Y_i = y|\theta)$$
$$P(\theta|Y_i = y) = \frac{P(Y_i = y|\theta)P(\theta)}{P(y)}$$

```
prior <- 1/11
prob <- (choose(100,57)*(theta^57)*(1-theta)^43)
p_y <- sum(prob*1/11)
posterior <- (prob*1/11)/p_y
plot(theta,posterior,xlab="theta",ylab="Pr(theta | Y)")
```

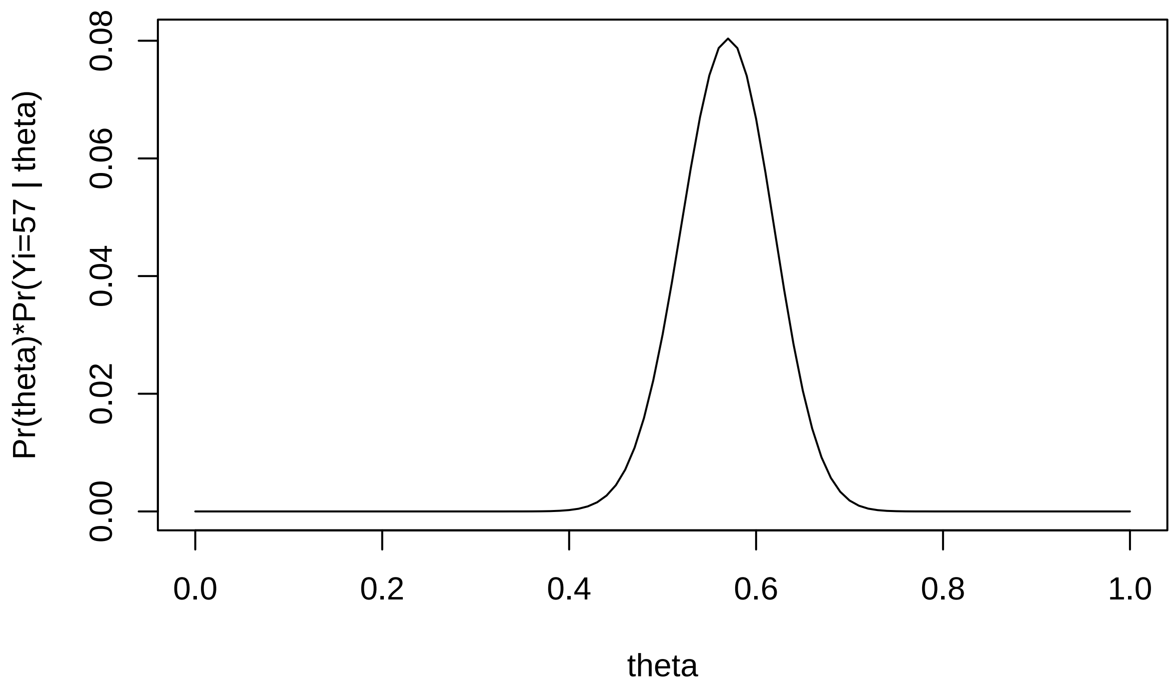


From the plot, we can see that the posterior mode is when theta = 0.6.

4.

$$P(\theta_2) = 1$$
$$P(Y_i = y|\theta_2) = \binom{100}{57}\theta_2^{57}(1-\theta_2)^{43}$$
$$P(\theta_2)P(Y_i = y|\theta_2)$$

```r
theta_2 <- seq(0,1,by=0.01)
prior_2 <- 1
prob <- (choose(100,57)*(theta_2^57)*(1-theta_2)^43)
posterior_2 <- prob*prior_2
plot(theta_2,posterior_2,type="l",xlab="theta",ylab="Pr(theta)*Pr(Yi=57 | theta)")
```

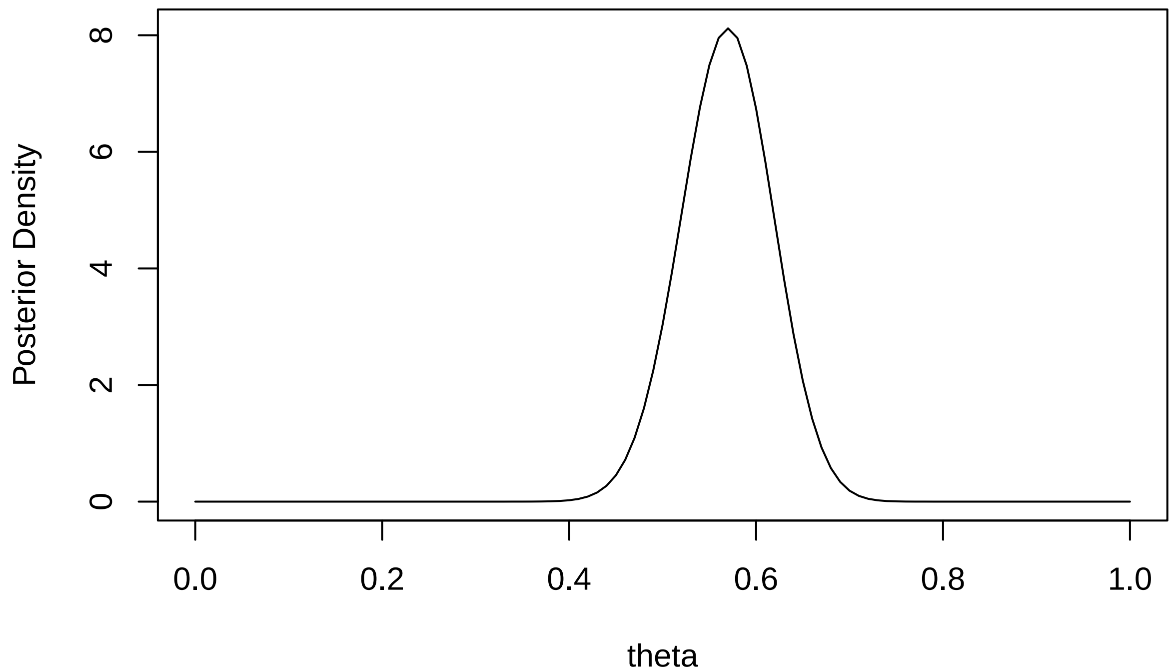


In this plot, we let the theta be approximately continuous as any value(on a 0.01 scale level) in the interval [0,1], and we can see the posterior mode now is around theta = 0.6 but not strictly equals 0.6.

5.

$$(\theta|Y_i = y) \sim \mathsf{Beta}(58, 44)$$

```r
theta_2 <- seq(0,1,by=0.01)
posterior_3 <- dbeta(theta_2,1+57,1+100-57)
plot(theta_2,posterior_3,type="l",xlab = "theta",ylab = "Posterior Density")
```

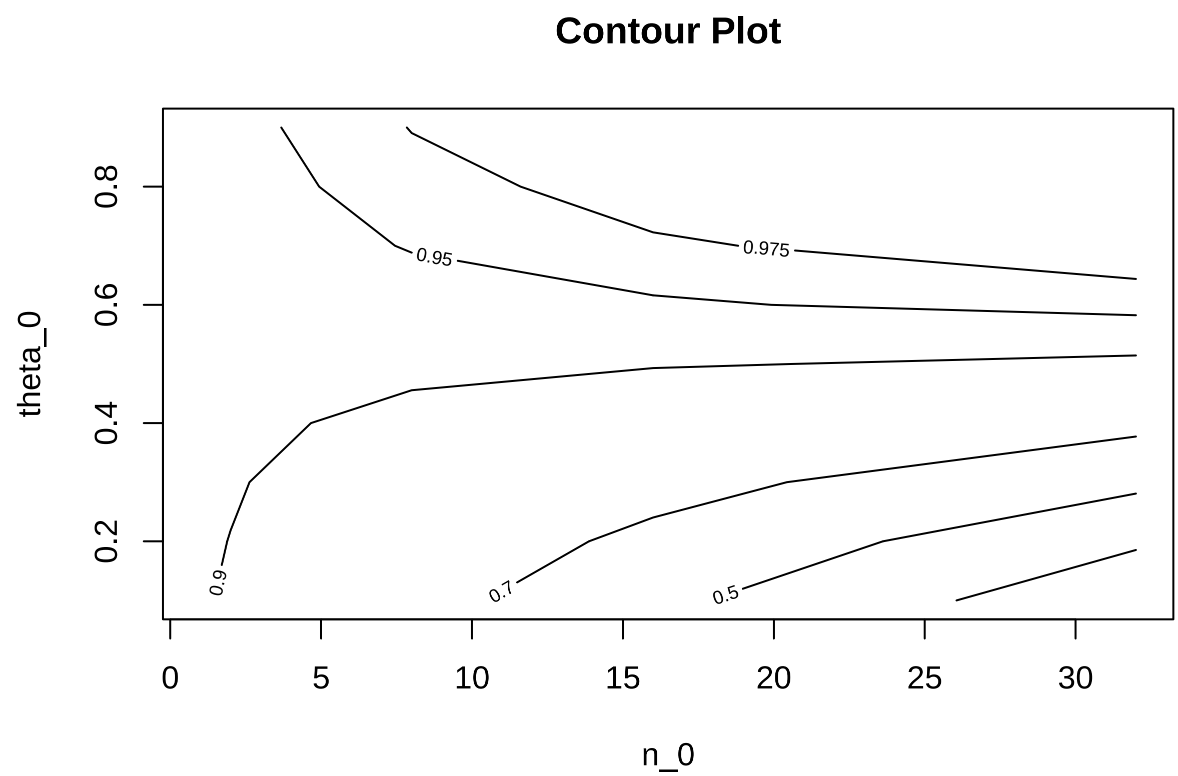


We plot the posterior with beta distribution. And we can see the estimate for theta is also around 0.6, but slightly lower than 0.6.

Problem 4

```
theta_0 <- seq(0.1,0.9,by=0.1)
n_0 <- c(1,2,8,16,32)
lst <- matrix(0,length(n_0),length(theta_0))
# create a matrix with length of n_0 and theta_0
for (i in 1:length(n_0)){
  for (j in 1:length(theta_0)){
    a = theta_0[j]*n_0[i]
    b = (1-theta_0[j])*n_0[i]
    # a=theta_0*n_0, b=(1-theta_0)*n_0
    n=100
    y=57
    lst[i,j] <- 1-pbeta(.5,a+y,b+n-y)
    # y follows binomial distribution with prior beta distribution of theta
    # The posterior follows Beta(a+y,b+n-y)
    # 1-pbeta(0.5,a+y, b+n-y) gives you Pr(  > 0.5|Y=57)
    # And we store each value into the matrix we created
  }
}
#lst
#contour plot
```

```
contour(n_0,theta_0,lst,main="Contour Plot",xlab="n_0", ylab="theta_0",
        level=c(0.975,0.95,0.9,0.7,0.5,0.3,0.1))
```



From the contour plot, we can see that for _0 less than 0.5, there are about 90% to have posterior that is larger than 0.5 when sample size(n_0) is small, while with larger sample size, the chance for such posterior decreases. For _0 larger than 0.5, the posterior has a high degree of certainty (above 0.95 or 0.975) that its is larger than 0.5 regardless of the sample size, although there is a small tendency that larger sample size will have a higher certainty.