A Radius–Bisector Relation and the Geometry of a Quadri-Arc-Lateral

Tadesse Weldegebreal
Independent Researcher
Newport News, Virginia, USA

January 9, 2026

Author Affirmation

The author affirms that this work is original, conducted independently, and has not been published elsewhere nor submitted concurrently to another journal. However, AI tools were used for refinement purposes to bridge language barriers and align with academic English prose. No institutional affiliation influenced the research. All results and interpretations presented are the sole responsibility of the author.

Abstract

We examine a geometric configuration consisting of a circle inscribed in a square and a right-angle bisector drawn from a tangent vertex to the circle. A fixed relation between the radius of the circle and the length of the bisector segment is obtained and used as the basis for a nonstandard geometric parameterization. Using this framework, a curvilinear planar figure bounded by four congruent quarter-circle arcs—termed the quadri-arc-lateral—is defined and analyzed. The figure admits two intrinsic perpendicular bisectors of unequal length, and closed-form expressions for its area are derived in terms of the radius, each bisector individually, and their product. A decomposition into congruent curved components yields a direct geometric derivation of these formulas and leads naturally to a corresponding volume expression under vertical extrusion. The novelty of the work lies in the formulation and intrinsic parameterization of the geometry rather than in the introduction of new numerical constants.

1 Introduction

Relations between circles, squares, and right triangles are classical topics in Euclidean geometry. While many results involving inscribed circles and right triangles are well known, alternative parameterizations of such configurations can lead to new perspectives and compact formulations. In this paper, we study a tangent–bisector construction that produces a fixed relation between the radius of an inscribed circle and the length of a right-angle bisector. This relation serves as the foundation for defining and analyzing a new curvilinear planar figure.

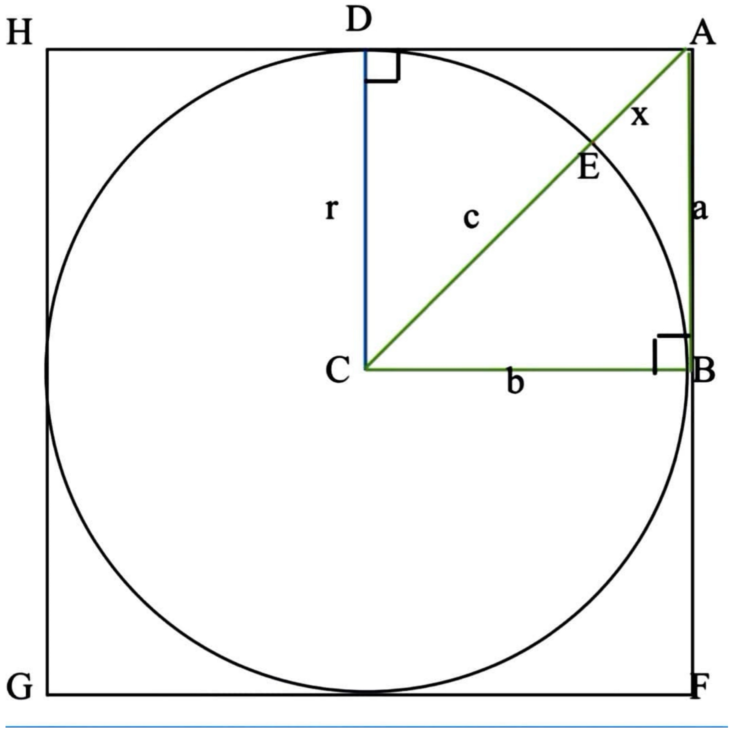

Figure 1: Geometric configuration showing radius r and bisector segment x.

2 Geometric Configuration

A circle of radius r is inscribed in a square. Two perpendicular tangent sides meet at a vertex A, forming a right angle. The bisector of this right angle intersects the circle at point E, and the length of this bisector segment is denoted by

$$x = AE.$$

Let C denote the center of the circle.

The radii drawn from C to the two tangent sides form an isosceles right triangle whose hypotenuse connects C to A.

3 Radius–Bisector Relation

Theorem 3.1. *Let r be the radius of the circle and x the length of the right-angle bisector from the tangent vertex to the circle. Then*

$$x = (\sqrt{2} - 1)r, \qquad r = (\sqrt{2} + 1)x.$$

Proof. The triangle formed by the two radii and the line segment joining the center C to the vertex A is an isosceles right triangle with legs of length r. Its hypotenuse therefore has length $\sqrt{2}\,r$. This hypotenuse is composed of a radius segment of length r and the bisector segment of length x, giving

$$\sqrt{2}\,r = r + x.$$

Solving for x yields $x = (\sqrt{2} - 1)r$, and rearranging gives $r = (\sqrt{2} + 1)x$. □

4 The Tad Constant

Definition 4.1. The ratio

$$t = \frac{r}{x} = \sqrt{2} + 1$$

is invariant for the configuration described above. We refer to this quantity as the Tad constant.

5 Derived Lengths and Areas

5.1 Hypotenuse Length

The hypotenuse of the isosceles right triangle satisfies

$$AC = r + x = (2 + \sqrt{2})x.$$

5.2 Area Results

Circle

$$A_{\text{circle}} = \pi r^2 = (3 + 2\sqrt{2})\pi x^2.$$

Square

$$A_{\text{square}} = r^2 = (3 + 2\sqrt{2})x^2.$$

Right Triangle

$$A_{\triangle} = \frac{1}{2}r^2 = \frac{1}{2}(3 + 2\sqrt{2})x^2.$$

6 The Quadri-Arc-Lateral

A new curvilinear planar figure is defined, bounded by four congruent quarter-circle arcs arranged symmetrically. This figure is referred to as the quadri-arc-lateral.

The figure admits two intrinsic perpendicular bisectors:

$$D = 2r \quad \text{(long bisector)}, \qquad b = 2x \quad \text{(short bisector)}.$$

Consequently,

$$\frac{D}{b} = \frac{r}{x} = \sqrt{2} + 1.$$

7 Area of One Curved Component

The quadri-arc-lateral may be decomposed into four congruent curved components. Let ABD denote one such component.

Theorem 7.1. *The area of one curved component ABD is*

$$A_{ABD} = \left(1 - \frac{\pi}{4}\right) r^2.$$

Proof. The region ABD occupies a right-angle corner of a square of side length r. It is bounded by the two perpendicular sides of the square and a quarter-circle arc of radius r. Therefore, its area is equal to the area of the square corner minus the area of the quarter disk:

$$A_{ABD} = r^2 - \frac{\pi r^2}{4} = \left(1 - \frac{\pi}{4}\right) r^2.$$

Using $r = (\sqrt{2} + 1)x$, this may also be written as

$$A_{ABD} = (3 + 2\sqrt{2}) \left(1 - \frac{\pi}{4}\right) x^2.$$

□

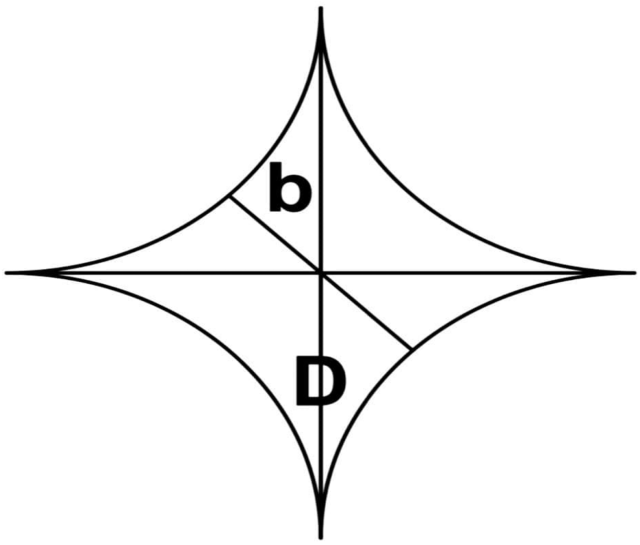

Figure 2: The quadri-arc-lateral showing intrinsic bisectors D and b.

8 Area of the Quadri-Arc-Lateral

Since the quadri-arc-lateral consists of four congruent curved components,

$$A_{\text{quad}} = 4A_{ABD} = (4 - \pi)r^2.$$

9 Bisector-Based Area Formulas

Using $D = 2r$ and $b = 2x$, the area of the quadri-arc-lateral can be expressed intrinsically as

$$A_{\text{quad}} = \frac{4 - \pi}{4} D^2 = \frac{(4 - \pi)(3 + 2\sqrt{2})}{4} b^2 = \frac{4 - \pi}{4} (\sqrt{2} + 1)\, Db.$$

10 Volume Extension

If the quadri-arc-lateral is extruded vertically by height h, the resulting solid has volume

$$V = A_{\text{quad}}\, h = \frac{4-\pi}{4} D^2 h = \frac{(4-\pi)(3+2\sqrt{2})}{4} b^2 h = \frac{4-\pi}{4} (\sqrt{2}+1)\, Dbh.$$

11 Novelty and Contribution

The constants appearing in this work are classical; however, the geometric framework and parameterization are nonstandard. The primary contribution lies in describing the configuration using a right-angle tangent bisector and two intrinsic perpendicular bisectors of a curvilinear figure. This approach yields closed-form area and volume expressions that are not typically presented in classical treatments, particularly those expressed in terms of the product of intrinsic bisectors. To the author's knowledge, this formulation and its associated results have not been explicitly documented in the literature.

12 Conclusion

A fixed radius–bisector relation provides a unifying framework for analyzing a new curvilinear planar figure bounded by four quarter-circle arcs. The resulting bisector-based parameterization leads naturally to compact expressions for area and volume and offers a geometric perspective that complements classical radius-based descriptions.

References

[1] Jon Rogawski, Colin Adams, and Robert Franzosa. *Calculus*, 2nd Edition. W. H. Freeman and Company, New York, 2012.

[2] Robin Hartshorne. *Geometry: Euclid and Beyond.* Springer-Verlag, New York, 2000.

[3] H. S. M. Coxeter. *Introduction to Geometry*, 2nd Edition. John Wiley & Sons, New York, 1969.

[4] Edwin E. Moise. *Elementary Geometry from an Advanced Viewpoint.* Addison-Wesley, Reading, MA, 1990.

[5] Sheldon Axler. *Linear Algebra Done Right*, 3rd Edition. Springer, New York, 2015.