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Assignment #2
10/14/25

1) Game Problem

$E[x]$: expected number of rolls until two consecutive 6's or two consecutive 1's are observed

Fair die: $P(\text{roll } i) = \frac{1}{6}$ for $i \in \{1,2,3,4,5,6\}$

states:

S: Start or previow roll was 2-5

A: Previous roll was a 1

B: Previous roll was a 6

Let:

E_S: Expected number of rolls from start

E_A: Expected number of rolls after one 1

E_B: Expected number of rolls after one 6

From state S:

- Roll a 1 → go to A, $P = \frac{1}{6}$
- Roll a 6 → go to B, $P = \frac{1}{6}$
- Roll a 2-5 → stay in S, $P = \frac{4}{6}$

$$E_S = 1 + \frac{1}{6}E_A + \frac{1}{6}E_B + \frac{4}{6}E_S$$

$$\frac{1}{3}E_S = 1 + \frac{1}{6}E_A + \frac{1}{6}E_B$$

From state A:

- Roll a 1 → win, $P = \frac{1}{6}$ (stop rolling)
- Roll a 6 → go to B, $P = \frac{1}{6}$
- Roll a 2-5 → go to S, $P = \frac{4}{6}$

$$E_A = 1 + \frac{1}{6}(0) + \frac{1}{6}E_B + \frac{4}{6}E_S$$

$$E_A = 1 + \frac{1}{6}E_B + \frac{2}{3}E_S$$

From state B:

- Roll a 1 → go to A, $P = \frac{1}{6}$
- Roll a 6 → win, $P = \frac{1}{6}$ (stop rolling)
- Roll a 2-5 → go to S, $P = \frac{4}{6}$

$$E_B = 1 + \frac{1}{6}E_A + \frac{1}{6}(0) + \frac{4}{6}E_S$$

$$E_B = 1 + \frac{1}{6}E_A + \frac{2}{3}E_S$$

Solve system of equations:

E_A and E_B are symmetric in form by same linear combination

↳ $E_A = E_B$

$$E_B = 1 + \frac{1}{6}E_A + \frac{2}{3}E_S$$

$$E_B = 1 + \frac{1}{6}E_B + \frac{2}{3}E_S$$

$$\frac{5}{6}E_B = 1 + \frac{2}{3}E_S$$

$$\frac{1}{3}E_S = 1 + \frac{1}{6}E_A + \frac{1}{6}E_B$$

$$\frac{1}{3}E_S = 1 + \frac{1}{6}E_B + \frac{1}{6}E_B$$

$$\frac{1}{3}E_S = 1 + \frac{1}{3}E_B$$

$$\frac{1}{3}E_B = \frac{1}{3}E_S - 1$$

$$E_B = E_S - 3$$

$$\frac{2}{3}E_S = \frac{5}{6}E_B - 1$$

$$\frac{2}{3}E_S = \frac{5}{6}(E_S - 3) - 1$$

$$\frac{2}{3}E_S = \frac{5}{6}E_S - \frac{5}{2} - 1$$

$$\frac{7}{2} = \frac{1}{6}E_S$$

$$E_S = 21$$

↳ $E[X] = 21$ rolls

2) Elevator Problem

Number of floors above basement $\rightarrow N=40$

Number of people $\rightarrow k=21$

Probability a single person chooses any floor $\rightarrow p_0 = \frac{1}{40}$

a) Let X_i be a random variable of floor i, where $i \in \{1, 2, \ldots, 40\}$

$$X_i = \begin{cases} 1 & \text{if exactly 3 people exit at floor } i \\ 0 & \text{otherwise} \end{cases}$$

$X = \sum_{i=1}^{40} X_i \rightarrow$ total number of stops with exactly 3 people exit

Expected value:

$$E[X] = E\left[\sum_{i=1}^{40} X_i\right] = \sum_{i=1}^{40} E[X_i]$$

Number of people who choose floor i follows Binomial distribution,

let N_i be number of people who exit at floor i

$$\hookrightarrow P(X_i = 1) = P(N_i = 3) = \binom{K}{n} p_0^{\,n} (1-p_0)^{K-n}$$

$$P(N_i = 3) = \binom{21}{3}\left(\frac{1}{40}\right)^3 \left(1 - \frac{1}{40}\right)^{21-3}$$

$$= \frac{21!}{18! \cdot 3!}\left(\frac{1}{40}\right)^3 \left(\frac{39}{40}\right)^{18}$$

$$P(N_i = 3) = 0.01318$$

Since $E[X_i]$ is the same for all i:

$$E[X] = \sum_{i=1}^{40} E[X_i]$$

$$= 40(0.01318)$$

$$E[X] = 0.527$$

$$E[X^2] = E\left[\left(\sum_{i=1}^{40} X_i\right)^2\right] = E\left[\sum_{i=1}^{40} X_i^2 + \sum_{i \neq j} X_i X_j\right]$$

Since X_i is an indicator, $X_i^2 = X_i$

$$E[X^2] = \sum_{i=1}^{40} E[X_i] + \sum_{i \neq j} E[X_i X_j]$$

$E[X_i X_j] = P(X_i X_j = 1) = P(X_i = 1 \cap X_j = 1)$ for $i \neq j$

↳ Probability that exactly 3 people exit on floor i and exactly 3 people exit on floor j.

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floor i:

- Choose 3 people to exit: $\binom{21}{3}$
- Probability of choosing a random floor: $\left(\frac{1}{40}\right)^3$

floor j:

- Choose 3 people to exit from who's left: $\binom{18}{3}$
- Probability of choosing a random floor: $\left(\frac{1}{40}\right)^3$

remaining:

- 15 people left to choose other floors
- Probability of choosing a random floor: $\left(1 - \frac{2}{40}\right)^{15} = \left(\frac{38}{40}\right)^{15}$

$$P(X_i = 1 \cap X_j = 1) = \binom{21}{3}\binom{18}{3}\left(\frac{1}{40}\right)^3\left(\frac{1}{40}\right)^3\left(\frac{38}{40}\right)^{15}$$

$$= \left(\frac{21!}{18! \cdot 3!}\right)\left(\frac{18!}{15! \cdot 3!}\right)\left(\frac{1}{40}\right)^3\left(\frac{1}{40}\right)^3\left(\frac{38}{40}\right)^{15}$$

$$= 0.000123$$

Pairs: $N(N-1) = 40(40-1) = 40(39) = 1560$

$$\sum_{i \neq j} X_i X_j = N(N-1) P(X_i = 1 \cap X_j = 1) = 0.1915$$

$$E[X^2] = \sum_{i=1}^{40} E[X_i] + \sum_{i \neq j} X_i X_j$$

$$= 0.527 + 0.1915$$

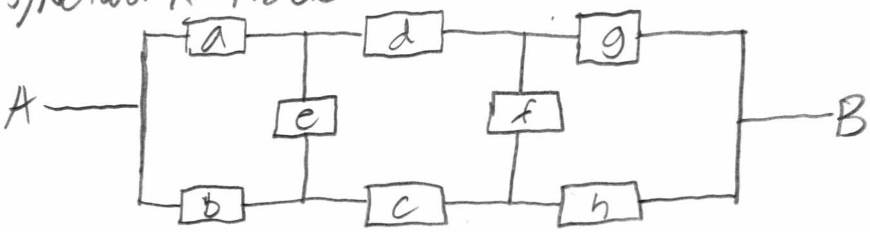
$$E[X^2] = 0.719$$

b) See file Problem2b.py for code

$E[X] = 0.5200$

$E[X^2] = 0.7060$

3) Network Problems



$S = \{a, b, c, d, e, f, g, h\} \rightarrow$ set of all links

$p \rightarrow$ probability of single link failing

$1-p \rightarrow$ probability of single link operating

a) $F \rightarrow$ event that exactly 5 links failed

$\binom{8}{5} = \frac{8!}{5!\cdot 3!} = 56 \rightarrow$ ways to choose 5 failing links

Since only 3 operational links, only way A can communicate with B is operational
is if one of the paths requiring 3 links
$\hookrightarrow$ two paths: $\{a, d, g\}$ and $\{b, c, h\}$

$C \rightarrow$ event that A can communicate with B

Number of ways event C and F happens: $N(C \cap F) = 2$

Number of possible ways event F happens: $N(F) = 56$

$$P(C|F) = \frac{N(C\cap F)}{N(F)} = \frac{2}{56} = \frac{1}{28}$$

$$P(C|F) = 0.0357$$

b) $F \rightarrow$ event that exactly 5 links failed

$G \rightarrow$ event that link g is operational

$H \rightarrow$ event that link h is operational

$J \rightarrow$ event that G or H is operational but not both

$\hookrightarrow J = (G \cap H^c) \cup (G^c \cap H)$

$G \cap H^c$:

- One operational link is g
- Need to choose 2 more operational links from remaining 6
- Number of ways for $G \cap H^c \cap F = \binom{6}{2} = \frac{6!}{4!\cdot 2!} = 15$

$G^c \cap H$:

- Same counting as previous step
- Number of ways $G^c \cap H \cap F = 15$

$(G \cap H^c \cap F)$ and $(G^c \cap H \cap F)$ are mutually exclusive

$N(J \cap F) = N(G \cap H^c \cap F) + N(G^c \cap H \cap F) = 15 + 15 = 30$

$P(J|F) = \dfrac{N(J \cap F)}{N(F)} = \dfrac{30}{56}$

$P(J|F) = 0.5357$

c) $K \rightarrow$ event that a, d, h have failed

$C \rightarrow$ event that A can communicate with B

$B_o, C_o, E_o, F_o, G_o \rightarrow$ events that links b, c, e, f, g are operational (respectively)

$P(L_o) = 1 - p$ for $L \in \{b, c, e, f, g\}$

Two paths left:

$P_1 : b \rightarrow e \rightarrow f \rightarrow g$

$P_2 : b \rightarrow c \rightarrow f \rightarrow g$

A can communicate with B only if P_1 or P_2 are operational

$A_1 \rightarrow$ event P_1 is operational $\rightarrow = B_o \cap E_o \cap F_o \cap G_o$

$A_2 \rightarrow$ event P_2 is operational $\rightarrow = B_o \cap C_o \cap F_o \cap G_o$

Event of each link operating is independent

$P(A_1) = P(B_o)P(E_o)P(F_o)P(G_o) = (1-p)^4$

$P(A_2) = P(B_o)P(C_o)P(F_o)P(G_o) = (1-p)^4$

$P(A_1 \cap A_2) = P(B_o)P(C_o)P(E_o)P(F_o)P(G_o) = (1-p)^5$

$P(C|K) = P(A_1 \cup A_2)$

Inclusion-exclusion $\rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$= (1-p)^4 + (1-p)^4 - (1-p)^5$

$= (1-p)^4(1 + 1 - (1-p))$

$P(C|K) = P(A_1 \cup A_2) = (1-p)^4(1+p)$

4) Subset Problems

$S = \{1, 2, \dots, n\}$ → set of integers

2^n → total number of subsets

$M = 2^n - 1$ → total number of non-empty subsets

$\frac{1}{M} = \frac{1}{2^n - 1}$ → probability of choosing a non-empty subset

A_{all} → collection of all non-empty subsets

$A \in A_{all}$ → chosen subset

X → size of the subset so $X = |A|$

$$E[X^2] = \sum_{A \in A_{all}} |A|^2 P(A)$$

$$= \sum_{A \in A_{all}} |A|^2 \frac{1}{2^n - 1}$$

$$= \frac{1}{2^n - 1} \sum_{A \in A_{all}} |A|^2$$

↳ includes all non-empty subsets, can rewrite sum over all subsets since $|\emptyset| = 0$

$$\sum_{A \in A_{all}} |A|^2 = \sum_{A \subseteq S, A \neq \emptyset} |A|^2 = \sum_{A \subseteq S} |A|^2$$

$|A|$ can range from $k=1$ to $k=n$

$\binom{n}{k}$ is number of subsets of size k

$$\sum_{A \subseteq S} |A|^2 = \sum_{k=0}^{n} k^2 \binom{n}{k} = \sum_{k=1}^{n} k^2 \binom{n}{k} = \sum_{k=1}^{n} k \left[n \binom{n-1}{k-1} \right] = n \sum_{k=1}^{n} k \binom{n-1}{k-1}$$

Let $j = k-1$

$$n \sum_{k=1}^{n} k \binom{n-1}{k-1} = n \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} = n \left[\sum_{j=0}^{n-1} j \binom{n-1}{j} + \sum_{j=0}^{n-1} \binom{n-1}{j} \right]$$

↳ Identities:

$$\left.\begin{aligned} &\sum_{j=0}^{m} j \binom{m}{j} = m 2^{m-1} \\ &\sum_{j=0}^{m} \binom{m}{j} = 2^m \end{aligned}\right\} m = n-1$$

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n\left[m2^{m-1} + 2^m\right]$$

$$= n\left[(n-1)2^{(n-1)-1} + 2^{n-1}\right]$$

$$= n\left[(n-1)2^{n-2} + 2^{n-1}\right]$$

$$= n2^{n-2}\left[(n-1)1 + 2^1\right]$$

$$= n2^{n-2}(n-1+2)$$

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n2^{n-2}(n+1) = \sum_{A \in A_{all}} |A|^2$$

$$E[X^2] = \frac{1}{2^n - 1}\left[n2^{n-2}(n+1)\right]$$

$$E[X^2] = \frac{n2^{n-2}(n+1)}{2^{n-1}}$$

5) Poisson Random Variables

$X \sim Poisson(\lambda)$

$n \geq 2$

PMF for $X \sim Poisson(\lambda)$:

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k=1,2,\ldots$$

$$E[X^n] = \sum_{k=0}^{\infty} k^n P(X=k)$$

$$= \sum_{k=0}^{\infty} k^n \frac{e^{-\lambda}\lambda^k}{k!}$$

$\hookrightarrow P(X=0)=0 \rightarrow$ can drop $k=0$ term

$\hookrightarrow$ property $k! = k \cdot (k-1)!$

$$E[X^n] = \sum_{k=1}^{\infty} k^n \frac{e^{-\lambda}\lambda^k}{k \cdot (k-1)!}$$

Let $j = k-1$

$\hookrightarrow$ when $k=1, j=0$

$\hookrightarrow$ when $k \to \infty, j \to \infty$

$$E[X^n] = \sum_{k=1}^{\infty} k^n \frac{e^{-\lambda}\lambda^k}{k \cdot (k-1)!}$$

$$= \sum_{k=1}^{\infty} k^{n-1} \frac{e^{-\lambda}\lambda\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \sum_{k=1}^{\infty} k^{n-1} \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}$$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{e^{-\lambda}\lambda^{j}}{j!}$$

$\hookrightarrow P(X=j) = \frac{e^{-\lambda}\lambda^j}{j!}$

$$E[X^n] = \lambda \underbrace{\sum_{j=0}^{\infty} (j+1)^{n-1} P(X=j)}$$

$$E[g(X)] = \sum_{k=0}^{\infty} g(k) P(X=k)$$

$\hookrightarrow$ function is $g(X) = (X+1)^{n-1}$

$\hookrightarrow$ summation term becomes $E[(X+1)^{n-1}]$ by definition of expectation

$$E[X^n] = \lambda E[(X+1)^{n-1}] \checkmark$$

6) Minimum of Geometric Random Variables

Let $X_1, X_2, X_3 \overset{iid}{\sim}$ geometric(p)

$E[X_i] = \frac{1}{p}$

Geometric $\rightarrow P(X=k) = (1-p)^{k-1}p \rightarrow P(X \geq k) = (1-p)^{k-1}$

Let $M_n = \min(X_1, \ldots, X_n)$

$$E[M_n] = \sum_{k=1}^{\infty} P(M_n \geq k)$$

$$= \sum_{k=1}^{\infty} P(X_1 \geq k, X_2 \geq k, \ldots, X_n \geq k)$$

$$= \sum_{k=1}^{\infty} (P(X \geq k))^n$$

$$P(M_n \geq k) = \left[(1-p)^{k-1}\right]^n = (1-p)^{n(k-1)}$$

$$E[M_n] = \sum_{k=1}^{\infty} (1-p)^{n(k-1)}$$

Let $j = k-1$

$$E[M_n] = \sum_{j=0}^{\infty} \left[(1-p)^n\right]^j$$

↳ Geometric series

$$E[M_n] = \frac{1}{1-(1-p)^n}$$

for $n=2$:

$$E[\min(X_1, X_2)] = \frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p-p^2}$$

for $n=3$:

$$E[\min(X_1, X_2, X_3)] = \frac{1}{1-(1-p)^3} = \frac{1}{1-(1-2p+p^2-p+2p^2-p^3)}$$

$$= \frac{1}{1-(1-3p+3p^2-p^3)}$$

$$= \frac{1}{3p-3p^2+p^3}$$