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Assignment #1
0/7/25

1) Birthday Problems

a) - 10 people in room
- 4 people share a birthday
- 3 people share a birthday
- 2 people share a birthday
- 1 person has a unique birthday

$n = 365$ → days in a year

$k = 4$ → selected dates for birthdays

$\binom{365}{4}$ → ways to choose 4 unique birthdays

$4!$ → permutation of 4 selected birthdays

$\frac{10!}{4! \cdot 3! \cdot 2! \cdot 1!}$ → multinomial, ways to assign people to groups

$\left(\frac{1}{365}\right)^{10}$ → probability of a specific birthday for each person

$$P = \binom{365}{4}(4!)\left(\frac{10!}{4!3!2!1!}\right)\left(\frac{1}{365}\right)^{10}$$

$P = 2.18 \times 10^{-13}$

b) Let

A = no one is born on Memorial Day

B = no one is born on Independence Day

C = no one is born on Labor Day

$P\begin{pmatrix}\text{at least 1 person}\\ \text{born on each holiday}\end{pmatrix} = 1 - P(A \cup B \cup C) \rightarrow$ apply Inclusion-exclusion

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$P(A) = P(B) = P(C) = \left(\frac{364}{365}\right)^{10}$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \left(\frac{363}{365}\right)^{10}$$

$$P(A \cap B \cap C) = \left(\frac{362}{365}\right)^{10}$$

$P = 1.4 \times 10^{-5}$

2) Combinatorial Identity

a) $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

$\sum_{k=1}^{n} k\binom{n}{k}$ → $\binom{n}{k}$: number of ways to choose a committee of size k

→ k: number of ways to choose chairperson from the committee members

→ for each k, $k\binom{n}{k}$ counts the number of committees of size k with a chairperson, and summation adds over all possible committee sizes

↳ Total number of non-empty committees with a chairperson

$n2^{n-1}$ → n: ways to choose the chairperson

→ 2^{n-1}: number of subnets; for the rest of the committee, each of the remaining n-1 people can either be in or not in the committee

↳ Total number of non-empty committees with a chairperson

b) $\sum_{k=1}^{n} k^2\binom{n}{k}$

Combinatorial → $\binom{n}{k}$: number of ways to choose a committee of size k

→ k^2: choose 2 roles from within the k committee members, with repitition of roles allowed

Binary → for each of n people, they can be in or out of the committee, so 2^n subnets

→ for each subnet of size k, there are k^2 ways to pick 2 roles from within the subnet

$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$

↳ k=0 term → 0

$\sum_{k=1}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$

3) Union Bounds

a) $P(A \cup B \cup C \cup D) \leq P(A) + P(B) + P(C) + P(D)$

Boole's inequality $\rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

$A_1 = A$

$A_2 = B$

$A_3 = C$

$A_4 = D$

$\hookrightarrow P(A \cup B \cup C \cup D) \leq P(A) + P(B) + P(C) + P(D)$ ✓

b) $P(A \cup B \cup C \cup D) \geq P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(A \cap D)$
$- P(B \cap C) - P(B \cap D) - P(C \cap D)$

$\hookrightarrow$ Inclusion-exclusion

$$\begin{aligned} P(A \cup B \cup C \cup D) = {} & P(A) + P(B) + P(C) + P(D) \\ & - [P(A \cap B) + P(A \cap C) + P(A \cap D) + P(B \cap C) + P(B \cap D) \\ & \quad + P(C \cap D)] \\ & + [P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D)] \\ & - P(A \cap B \cap C \cap D) \end{aligned}$$

$\hookrightarrow$ Singles and doubles cancel out

$$\underbrace{P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) - P(A \cap B \cap C \cap D) \geq 0}_{\text{lower bound}}$$

$\hookrightarrow P(A \cup B \cup C \cup D) \geq P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(A \cap D)$
$- P(B \cap C) - P(B \cap D) - P(C \cap D)$ ✓

4) Rolling a die

a) $P \rightarrow$ probability of rolling two consecutive 6's before a 1

states:

S_0 = start, no history

S_1 = last roll was a 6

W = two 6's in a row

L = 1 is rolled

P_0 = probability of winning from state S_0

P_1 = probability of winning from state S_1

S_0: $\frac{1}{6} \rightarrow$ roll a $1 \rightarrow$ go to $L = 0$

$\frac{1}{6} \rightarrow$ roll a $6 \rightarrow$ go to S_1

$\frac{4}{6} \rightarrow$ roll 2-5 $\rightarrow$ stay in S_0

S_1: $\frac{1}{6} \rightarrow$ roll a $1 \rightarrow$ go to $L = 0$

$\frac{1}{6} \rightarrow$ roll a $6 \rightarrow$ go to $W = 1$

$\frac{4}{6} \rightarrow$ roll a 2-5 $\rightarrow$ back to S_0

$$p_0 = \frac{1}{6}(0) + \frac{1}{6}(p_1) + \frac{4}{6}(p_0)$$

$$p_1 = \frac{1}{6}(0) + \frac{1}{6}(1) + \frac{4}{6}(p_0)$$

$$p_0 = \frac{1}{6}\left(\frac{1}{6} + \frac{4}{6}p_0\right) + \frac{4}{6}p_0$$

$$6p_0 = \frac{1}{6} + \frac{4}{6}p_0 + 4p_0$$

$$2p_0 = \frac{1}{6} + \frac{4}{6}p_0$$

$$\frac{8}{6}p_0 = \frac{1}{6}$$

$$p_0 = \frac{1}{8}$$

$$P = \frac{1}{8}$$

b) See file Problem4b.py for code

Wins = 126

Emperical probability = 0.126

5) Independence

Intuitive: When flipping 3 fair coins, its always true that at least 2 are alike. The statement "the probability that all coins show the same side is equal to 1/2" assumes conditional probabilities without justifying the conditionals properly. Therefore, I do not agree with that statement.

Computed: Possible outcomes $= 2^3 = 8$

HHH →
HHT
HTH
HTT
THH
THT
TTH
TTT →

Only 2 of 8 possible outcomes that all 3 coins show same side

↳ $p = \frac{2}{8} = \frac{1}{4}$ → Statement is not true