Reinforcement Learning for Business, Economics, and Social Sciences

Unit 2-1: Markov Processes

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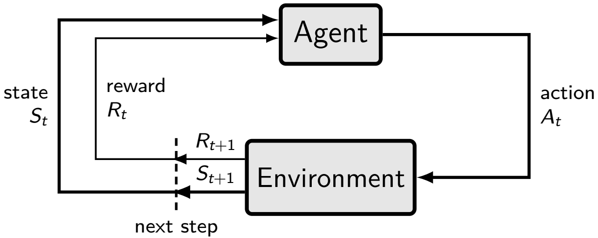
How to predict transitions?

Markov Chains

Unrolling the Problem



Goal: Learn to choose actions that maximize rewards

Unrolling the Problem

- Modeling environment dynamics
- Unrolling the control loop leads to a sequence of states, actions and rewards:

$$s_0, a_0, r_0, s_1, a_1, r_1, s_2, a_2, r_2, \ldots$$

- This sequence forms a stochastic process (due to some uncertainty in the dynamics of the process)

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  - Laws of the process do not change
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- **Example:** weather prediction
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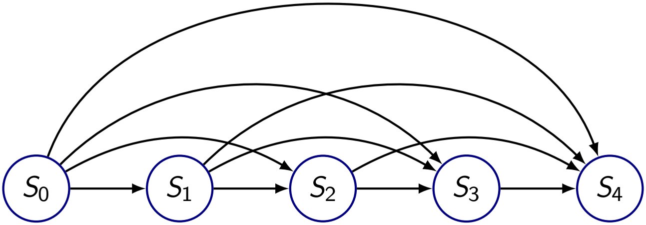
  - Weather measurements of past few days sufficient to predict weather
- **Example:** text prediction
  - Same model can be used in every conversation to predict next utterance
  - letter sequences of past texts sufficient to predict new sentences

Markovian and Stationary Processes

- Consider the sequence of states only
- Definition
  - Set of States: S
  - Stochastic dynamics: $\mathbb{P}(s_t | s_{t-1}, \dots, s_0)$



- Problem:
  - Infinitely large conditional distributions
- Solutions:
  - **Stationary process:**
    Dynamics do not change over time
  - **Markov assumption:**
    Current state depends only on a finite history of past states
  - Russell and Norvig (2016, Section 15.1)

K-Order Markov Process

- Assumption: last k states sufficient

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- First-order Markov Process
  - $\mathbb{P}(s_t | s_{t-1}, \ldots, s_0) = \mathbb{P}(s_t | s_{t-1})$

$$S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow S_3 \longrightarrow S_4$$

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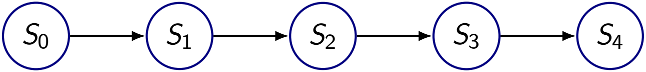
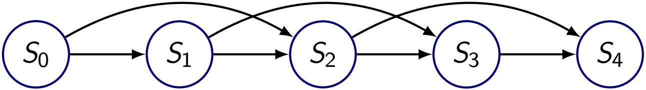


- Second-order Markov Process
  - $\mathbb{P}(s_t|s_{t-1},\ldots,s_0) = \mathbb{P}(s_t|s_{t-1},s_{t-2})$



Markov Process

- Commonly, a Markov Process refers to a
  - First-order process

$$\mathbb{P}(s_t \mid s_{t-1}, s_{t-2}, \ldots, s_0) = \mathbb{P}(s_t \mid s_{t-1}) \forall t$$

  - Stationary process

$$\mathbb{P}(s_t \mid s_{t-1}) = \mathbb{P}(s_{t'} \mid s_{t'-1}) \forall t'$$

- **Advantage:**
  can specify the entire process with a single concise conditional distribution

$$\mathbb{P}(s' \mid s)$$

Examples

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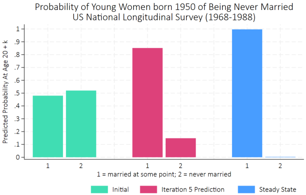
- Marrying decision of young women
  - **States:** relationship history
  - **Dynamics:** age



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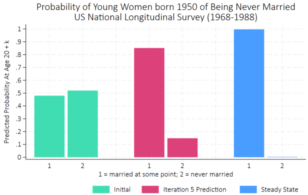
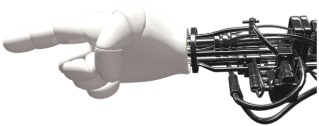


- Robotic control
  - **States:** $\langle x, y, z, \theta \rangle$ coordinates of joints
  - **Dynamics:** constant motion



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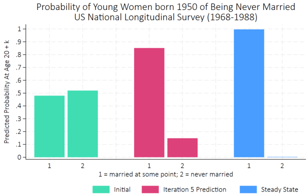

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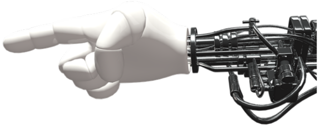

- Inventory management
  - **States:** inventory level
  - **Dynamics:** constant (stochastic) demand

- Common task is prediction: $\mathbb{P}\left(s_{t+k} \mid s_t\right)$
- Computation:

$$\mathbb{P}\left(s_{t+k} \mid s_t\right) = \sum s_{t+k} \ldots s_{t+k-1} \prod_{i=1}^{k} \mathbb{P}\left(s_{t+i} \mid s_{t+i-1}\right)$$

- Discrete states (matrix operations):
  - Let T be a $|S| \times |S|$ matrix representing $\mathbb{P}\left(s_{t+k} \mid s_t\right)$
  - Then $\mathbb{P}\left(s_{t+k} \mid s_t\right) = T^k$
  - Complexity: $\mathcal{O}\left(k|S|^3\right)$

Example: Marrying as 2-State Markov Process

Setup: Initial distribution $p_t = \left[\, 0.5_{\text{never married}} \;\; 0.5_{\text{married}} \,\right]$

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$T =$

	never married	married
never married	0.5	0.5
married	0	1

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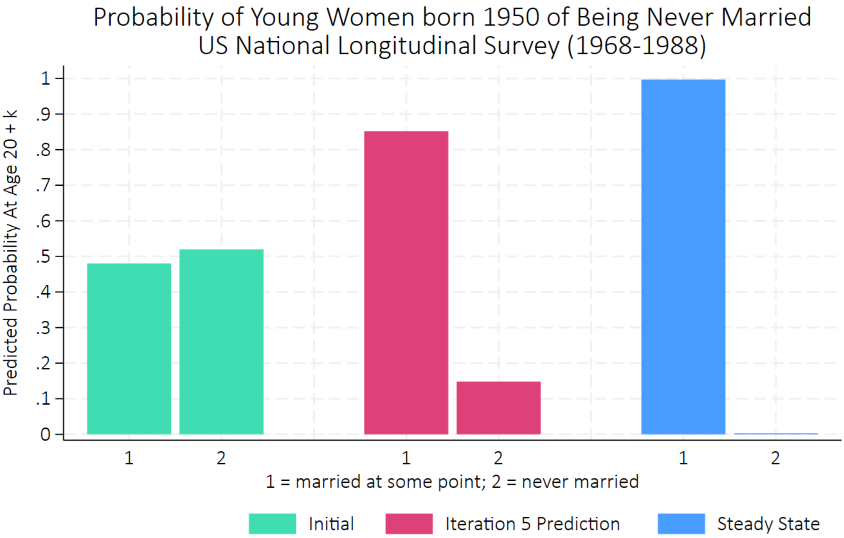
Long Run:

$$\boldsymbol{\pi} = \lim_{k \to \infty} p_{t+k} = [\,0 \quad 1\,] \quad \text{(everyone eventually marries)}$$

How Quickly Get Young Women Married?




```
xtsteadystate nev_mar if birth_yr ==50, tw 3dists ini ss pred twowayopt(.)
```

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  - Where do we stop?

Markovian Stationary Process

- **Problem:** adding components to the state description to force a process to be Markovian and stationary may significantly increase computational complexity
- **Solution:** try to find the smallest state description that is self-sufficient (i.e., Markovian and stationary)

- Predictions by themselves are useless
- They are only useful when they will influence future decisions
- Hence the ultimate task is decision making
- How can we influence the process to visit desirable states?
  - Model: Markov Decision Process

References I

Russell, S. J., and P. Norvig (2016): *Artificial intelligence: a modern approach*. Pearson.

Takeaways

How Can we Use Markov Processes to Predict Future States?

- Model sequences of states with probabilistic transitions
- First-order Markov and stationarity assumptions simplify prediction
- Adding state components can restore Markovian/stationary properties—at a computational cost
- Prediction relies on transition matrices
- Real goal: use predictions for decision-making

  $\rightarrow$ Markov Decision Processes

Appendix

Prediction and Steady State via Eigendecomposition

Objective: Predict future state distributions $\mathbb{P}(s_{t+k} \mid s_t)$ and compute the steady-state distribution using eigendecomposition

Inputs:

- Initial distribution: p_t
- Transition matrix: T where $T_{ij} = \mathbb{P}(s_{t+1} = j \mid s_t = i)$
- Horizon: k (number of steps ahead)

Procedure:

1. Eigendecompose: $T = U \Lambda U^{-1}$
2. Compute predicted distribution:

$$p_{t+k} = T^k p_t = U \Lambda^k U^{-1} p_t$$

3. Steady state distribution:

$$\boldsymbol{\pi} = \lim_{k \to \infty} p_{t+k}$$