Reinforcement Learning for Business, Economics, and Social Sciences

Unit 1-4: Thompson Sampling

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How to update your priors about rewards?

Thompson Sampling

- Notation:
  - $r_t^a = r_t | A_t = a$ random variable for a's rewards
  - $R(a) = q(a) = \mathbb{E}\left[r_t^a\right]$ unknown average reward
- Idea:
  - Sample several potential average rewards:
    $R_1(a), \ldots, R_d(a) \sim \mathbb{P}\left(R(a) \mid r_1^a, \ldots, r_t^a\right)$ for each a
  - Sample empirical average

$$\hat{R}(a) = \frac{1}{d} \sum_{i=1}^{d} R_i(a)$$

  - Execute $\underset{\text{argmax}}{a} \hat{R}(a)$
  - Coin example
  - $\mathbb{P}\left(R(a) \mid r_1^a, \ldots, r_t^a\right) = \text{Beta}\left(\theta_a; \alpha_a, \beta_a\right)$
    where $\alpha_a - 1 = \#$heads and $\beta_a - 1 = \#$ tails

Bayesian Learning

Bayesian Learning

- Notation:
  - $\mathbb{P}\left(r^a;\theta\right)$: unknown distribution (parameterized by θ)
- Idea:
  - Express uncertainty about θ by a prior $\mathbb{P}(\theta)$
  - Compute posterior $\mathbb{P}\left(\theta \mid r_1^a, r_2^a, \ldots, r_t^a\right)$ based on
  - Samples $r_1^a, r_2^a, \ldots, r_t^a$ observed for a so far
- Bayes theorem:

$$\mathbb{P}\left(\theta \mid r_1^a, r_2^a, \ldots, r_t^a\right) \propto \mathbb{P}\left(\theta\right)\mathbb{P}\left(r_1^a, r_2^a, \ldots, r_t^a \mid \theta\right)$$

Distributional Information

- Posterior over θ allows us to estimate
  - Distribution over next reward r^a

$$\mathbb{P}\left(r^a \mid r_1^a, r_2^a, \ldots, r_t^a\right) = \int_\theta \mathbb{P}\left(r^a; \theta\right) \mathbb{P}\left(\theta \mid r_1^a, r_2^a, \ldots, r_t^a\right) d\theta$$

  - Distribution over $R(a)$ when θ includes the mean

$$\mathbb{P}\left(R(a) \mid r_1^a, r_2^a, \ldots, r_t^a\right) = \mathbb{P}\left(\theta \mid r_1^a, r_2^a, \ldots, r_t^a\right) \text{ if } \theta = R(a)$$

- To guide exploration:
  - UCB: $\mathbb{P}\left(R(a) > \text{bound}\left(r_1^a, r_2^a, \ldots, r_t^a\right)\right) \geq p$
  - Bayesian techniques: $\mathbb{P}\left(R(a) \mid r_1^a, r_2^a, \ldots, r_t^a\right)$

Coin Example

- Consider two biased coins C_1 and C_2

$$R(C_1) = \mathbb{P}(C_1 = \text{ head})$$
$$R(C_2) = \mathbb{P}(C_2 = \text{ head})$$

- Problem:
  - Maximize # of heads in d flips
  - Which coin should we choose for each flip?

Bernoulli Variables

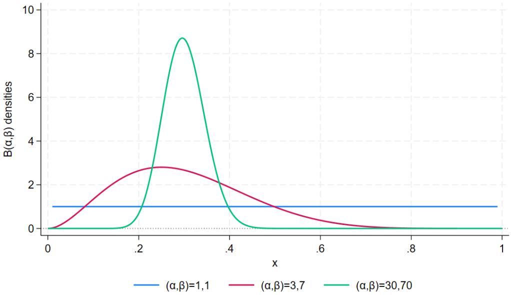
- r^{C_1}, r^{C_2} are Bernoulli variables with domain $\{0, 1\}$
- Bernoulli dist. are parameterized by their mean

$$\text{i.e. } \mathbb{P}\left(r^{C_1}; \theta_1\right) = \theta_1 = R(C_1)$$
$$\mathbb{P}\left(r^{C_2}; \theta_2\right) = \theta_2 = R(C_2)$$

Beta Distribution

- Let the prior $\mathbb{P}(\theta)$ be a Beta distribution $\text{Beta}(\theta; \alpha, \beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$

- $\alpha - 1$: # of heads
- $\beta - 1$: # of tails
- $\mathbb{E}[\theta] = \alpha/(\alpha+\beta)$



Belief Update

- Prior: $\mathbb{P}(\theta) = \text{Beta}(\theta; \alpha, \beta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$
- Posterior after coin flip:

$$
\begin{aligned}
\mathbb{P}(\theta \mid \text{head}) &\propto \quad \mathbb{P}(\theta) \qquad\qquad \mathbb{P}(\text{head} \mid \theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \qquad\qquad \theta \\
&= \theta^{(\alpha+1)-1}(1-\theta)^{\beta-1} \\
&\propto \text{Beta}(\theta; \alpha+1, \beta) \\
\mathbb{P}(\theta \mid \text{tail}) &\propto \quad \mathbb{P}(\theta) \qquad\qquad \mathbb{P}(\text{tail} \mid \theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \qquad\qquad (1-\theta) \\
&= \theta^{\alpha-1}(1-\theta)^{(\beta+1)-1} \\
&\propto \text{Beta}(\theta; \alpha, \beta+1)
\end{aligned}
$$

Thompson Sampling Algorithm: Bernoulli Rewards

ThompsonSampling (T)

$V \leftarrow 0$

For $t = 1$ to T

Sample $R_1(a), \ldots, R_d(a) \sim \mathbb{P}(R(a)) \; \forall a$

$\hat{R}(a) \leftarrow \frac{1}{d} \sum_{i=1}^{d} R_i(a) \; \forall a$

$a^* \leftarrow \underset{\text{argmax}}{a} \hat{R}(a)$

Execute a^* and receive r

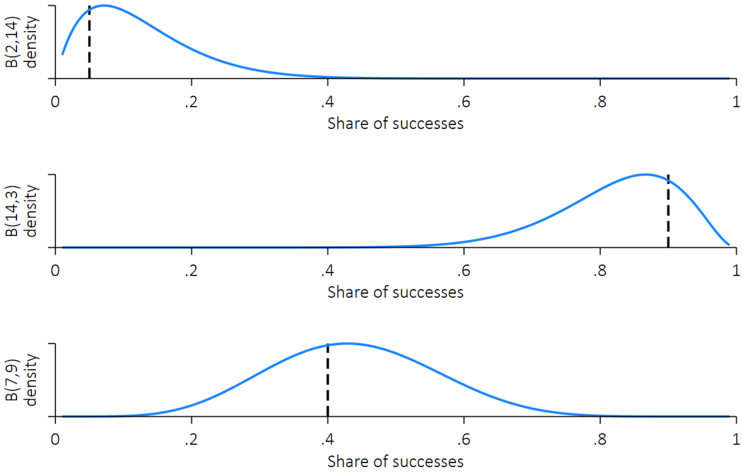
$V \leftarrow V + r$

Update $\mathbb{P}\,(R\,(a^*))$ based on r

Return V

Exploration vs Exploitation

Thompson (1933, 1935) Sampling



- Beta-Bernoulli Thompson sampling
- Models uncertainty about the shape of the distribution and the expected outcome R explicitly

Click to watch!

Comparison

Thompson Sampling

- Samples
  $r_i^a \sim \mathbb{P}\left(r^a;\theta\right)$
  $R_i(a) \sim \mathbb{P}\left(R_i(a) \mid r_1^a \dots r_t^a\right)$
- Empirical mean
  $\hat{R}(a) = \frac{1}{d}\sum_{i=1}^{d} R_i(a)$
- Action Selection
  $a^* = \underset{a}{\operatorname{argmax}}\ \hat{R}(a)$
- Some exploration

Greedy Strategy

- Samples
  $r_i^a \sim \mathbb{P}\left(r^a;\theta\right)$
- Empirical mean
  $\tilde{R}(a) = \frac{1}{t}\sum_{i=1}^{t} r_i^a$
- Action Selection
  $a^* = \underset{a}{\operatorname{argmax}}\ \tilde{R}(a)$
- No exploration

(Russo, Van Roy, Kazerouni, Osband, Wen, et al., 2018)

Sample Size

- In Thompson sampling, amount of data t and sample size d regulate amount of exploration
- As t and d increase, $\hat{R}(a)$ becomes less stochastic, which reduces exploration
  - As $t \uparrow, \mathbb{P}\left(R(a) \mid r_1^a, \ldots, r_t^a\right)$ becomes more peaked
  - As $d \uparrow, \hat{R}(a)$ approaches $\mathbb{E}\left[R(a) \mid r_1^a, \ldots, r_t^a\right]$
- The stochasticity of $\hat{R}(a)$ ensures that all actions are chosen with some probability

Analysis

- Thompson sampling converges to best arm
- Theory:
  - Expected cumulative regret: $\mathcal{O}(\log T)$
  - On par with UCB and ε-greedy
- Practice:
  - Sample size d often set to 1

References

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Takeaways

What is Thompson Sampling?

- Models uncertainty about expected rewards using probability distributions
- Samples from posterior of each arm's reward distribution
- Selects the arm with the highest sampled value
- Posterior is updated after each observation
- Achieves log regret
- Applied at, e.g., Google, Amazon, Facebook, Salesforce, and Netflix

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