Reinforcement Learning for Business, Economics, and Social Sciences

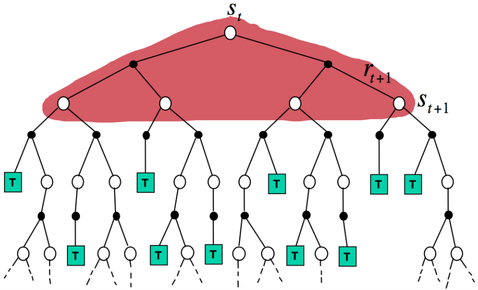
Unit 3-3: Temporal Difference Learning

Davud Rostam-Afschar (Uni Mannheim)

How can we learn by sampling from each step?

Dynamic Programming Backup

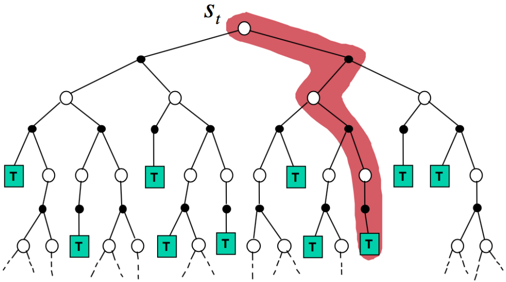
$$V(S_t) \leftarrow \mathbb{E}_\pi \left[R_{t+1} + \gamma V(S_{t+1})\right]$$



Source: David Silver

Monte Carlo Backup

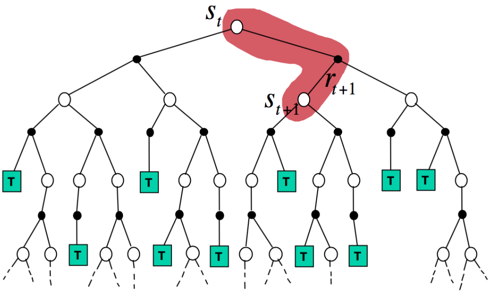
$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t - V(S_t)\right)$$



Source: David Silver

Temporal Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha\left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right)$$



Source: David Silver

- Given a policy π estimate $V^{\pi}(s)$ without any transition or reward model
- **Temporal difference** (TD) evaluation

$$V^{\pi}(s) = E[r \mid s, \pi(s)] + \gamma \sum_{s'} \mathbb{P}\left(s' \mid s, \pi(s)\right) V^{\pi}\left(s'\right)$$

$$\approx r + \gamma V^{\pi}\left(s'\right) \qquad \text{(one draw approximation)}$$

Toy Maze Example

3	**r**	**r**	**r**	$+1$
2	**u**		**u**	-1
1	**u**	**l**	**l**	**l**
	1	2	3	4

Start state: (1,1)
Terminal states: (4,2), (4,3)
No discount: $\gamma=1$

Reward is -0.04 for non-terminal states

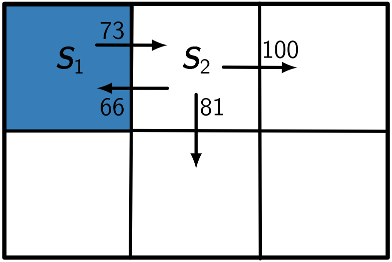
Four actions:

- up (**u**),
- left (**l**),
- right (**r**),
- down (**d**)

Do not know the transition probabilities

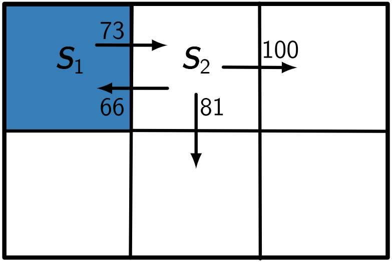
What is the value $V(s)$ of being in state s

Temporal Difference



$\gamma = 0.9, \alpha = 0.5, r = 0$ for non-terminal states

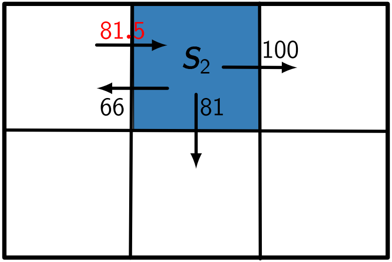
Temporal Difference



$\gamma = 0.9, \alpha = 0.5, r = 0$ for non-terminal states

$$
\begin{aligned}
Q(s_1, right) &= Q(s_1, right) + \alpha\left(r + \gamma \max_{a'} Q(s_2, a') - Q(s_1, right)\right) \\
&= 73 + 0.5(0 + 0.9 \max\{66, 81, 100\} - 73) \\
&= 73 + 0.5(17) \\
&= 81.5
\end{aligned}
$$

Temporal Difference



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Temporal Difference Evaluation

- Approximate value function: $V_n^{\pi}(s) \approx r + \gamma V^{\pi}(s')$
- **Incremental update** of sample (π, s', s)

$$V_n^{\pi}(s) \leftarrow V_{n-1}^{\pi}(s) + \alpha_n \left(r + \gamma V_{n-1}^{\pi}(s') - V_{n-1}^{\pi}(s)\right)$$

Stochastic approximation (Robbins-Monro algorithm)

- **Theorem**: If α_n is appropriately decreased with number of times a state is visited then $V_n^\pi(s)$ converges to correct value
- **Sufficient conditions** for α_n :

$$\sum_n \alpha_n \to \infty \tag{1}$$

$$\sum_n \alpha_n^2 < \infty \tag{2}$$

- Often $\alpha_n(s) = 1/n(s)$,
  where $n(s) = \#$ of times s is visited

Temporal Difference (TD) Evaluation

TDevaluation (π, V^{π})

Repeat

Execute $\pi(s)$

Observe s' and r

Update counts: $n(s) \leftarrow n(s) + 1$

Learning rate: $\alpha \leftarrow 1/n(s)$

Update value: $V^{\pi}(s) \leftarrow V^{\pi}(s) + \alpha \left(r + \gamma V^{\pi}(s') - V^{\pi}(s)\right)$

$s \leftarrow s'$

Until convergence of V^{π}

Return V^{π}

Temporal Difference Control

- Approximate Q-function:

$$\begin{aligned} Q^*(s,a) =& E[r \mid s,a] + \gamma \sum_{s'} \mathbb{P}\left(s' \mid s,a\right) \max_{a'} Q^*\left(s',a'\right) \\ &\approx r + \gamma \max_{a'} Q^*\left(s',a'\right) \end{aligned}$$

- **Incremental update**

  $Q_n^*(s,a) \leftarrow Q_{n-1}^*(s,a) + \alpha_n \left(r + \gamma \max_{a'} Q_{n-1}^*\left(s',a'\right) - Q_{n-1}^*(s,a)\right)$

Comparison

- **Monte Carlo evaluation**:
  - Unbiased estimate
  - High variance
  - Needs many trajectories
- **Temporal difference evaluation**:
  - Biased estimate
  - Lower variance
  - Needs less trajectories

References I

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Takeaways

How Does TD Learning Update Value Estimates Step-by-Step?

- No need to know transition probabilities or reward function
  $\rightarrow$ Model free
- Combine immediate reward with estimated value of next state
  $\rightarrow$ Biased value estimation from bootstrapped samples
- Revises estimates after each observed step
  $\rightarrow$ Needs few trajectories
- Lower variance than Monte Carlo at the cost of some bias
  $\rightarrow$ Bias-variance tradeoff
- Often used in algorithms such as Q-learning and SARSA