SAPTHAGIRI NPS UNIVERSITY

Department of Mathematics

Question Bank for 1st Sem B.Tech

Course Name: **Linear Algebra and Calculus** Course Code: **24BTPHY/ELY102**

MODULE-I: LINEAR ALGEBRA

Find the rank of the matrix by row elementary transformation:

1. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 91 & 92 & 93 & 94 & 95 \\ 92 & 93 & 94 & 95 & 96 \\ 93 & 94 & 95 & 96 & 97 \\ 94 & 95 & 96 & 97 & 98 \\ 95 & 96 & 97 & 98 & 99 \end{bmatrix}$

7. $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 12 & 13 & 14 & 15 \\ 13 & 14 & 15 & 16 \\ 14 & 15 & 16 & 17 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solve the following system of equations by Gauss Elimination Method and also by Gauss-Jordon Method:

1. $x + y + z = 9; x - 2y + 3z = 8; 2x + y - y = 3.$
2. $2x + y + 4z = 12; 4x + 11y - z = 33; 8x - 3y + 2z = 20.$
3. $2x + 5y + 7z = 52;\ 2x + y - z = 0;\ x + y + z = 9.$
4. $x - 2y + 3z = 2;\ 3x - y + 4z = 4;\ 2x + y - 2z = 5.$
5. $2x + 3y - z = 5; 4x + 4y - 3z = 3; 2x - 3y + 2z = 2.$
6. $3x + 4y + 5z = 18; 2x - y + 8z = 13; 5x - 2y + 7z = 20.$

Solve the following system of equations by Gauss-Seidel Method:

1. $10x + y + z = 12;\ x + 10y + z = 12;\ x + y + 10z = 12.$
2. $x + y + 54z = 110;\ 27x + 6y - z = 85;\ 6x + 15y + 2z = 72.$
3. $20x + y - 2z = 17;\ 3x + 20y - 2 = -18;\ 2x - 3y + 20z = 25.$
4. $28x + 4y - z = 32;\ 2x + 17y + 4z = 35;\ x + 3y + 10z = 24$ carryout 3 iterations correct to 3 decimal places.
5. $5x + 2y + z = 12;\ x + 4y + 2z = 15;\ x + 2y + 5z = 20$ by taking initial approximation as $(1, 0, 3)$.

Find the largest eigenvalues and the corresponding eigenvectors for the following matrices by Rayleigh's Power Method:

1. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by taking the initial eigen vector as $[1, 0.8, -0.8]^T$ perform 5 iterations

4. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by taking the initial eigen vector as $[1,\ 1,\ 1]^T$

5. $\begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$ by taking the initial eigen vector as $[0,\ 0,\ 1]^T$

MODULE-2: DIFFERENTIAL CALCULUS

Standard type of problems on Leibnitz Theorem:

1. If $y = \text{a} \cos(\log x) + b \sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

2. If $\tan y = x$, prove that $(1+x^2)y_2 + 2xy_1 = 0$ and hence show that

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

3. If $y = \frac{sinh^{-1}x}{\sqrt{1+x^2}}$, prove that

$$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$$

4. If $y = \log\left(x + \sqrt{1+x^2}\right)$, prove that

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$$

5. If $y^{1/m} + y^{-1/m} = 2x$, show that

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Find the angle between the radius vector and tangent to the curve:

i. $r = a(1-\cos\theta)$

ii. $r^2 \cos 2\theta = a^2$

iii. $r^m = a^m(\cos m\theta + \sin m\theta)$

iv. $\frac{2a}{r} = 1 - \cos\theta$ at $\theta = \frac{2\pi}{3}$

v. $rcos^2\left(\frac{\theta}{2}\right) = a$ at $\theta = \frac{2\pi}{3}$

Show that the following pairs of curves intersect each other orthogonally:

i. $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$

ii. $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$

iii. $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$

iv. $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$

v. $r = ae^\theta$ and $re^\theta = b$

Find the angle of intersection of the following pairs of curves:

i. $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

ii. $r = alog\theta$ and $r = \frac{a}{log\theta}$

iii. $r = a(1 - \cos\theta)$ and $r = 2a\cos\theta$

iv. $r = a(1 + \cos\theta)$ and $r^2 = a^2 \cos 2\theta$

v. $r = a\theta$ and $r = \frac{a}{\theta}$

Find the pedal equation of the following curves:

i. $\frac{2a}{r} = (1 + \cos\theta)$

ii. $r(1 - \cos\theta) = 2a$

iii. $r^n = a^n \cos n\theta$

iv. $r^m = a^m(\cos m\theta + \sin m\theta)$

v. $\frac{l}{r} = 1 + e\cos\theta$

MODULE 3: PARTIAL DIFFERENTIATION

Direct Partial Derivatives:

1. If $u = x^3 - 3xy^2 + x + e^x cosy + 1$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2. If $u = log\left(\frac{x^2+y^2}{x+y}\right)$, show that $xu_x + yu_y = 1$

3. If $u = e^{ax-by}\sin(ax + by)$, show that $b\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} = 2abu$

4. If $u = tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Symmetric functions:

A function $f(x,y)$ is said to be symmetric if $f(x,y) = f(y,x)$ and a function $f(x,y,z)$ is said to be symmetric if $f(x,y,z) = f(y,z,x) = f(z,x,y)$.

1. If $u = log\sqrt{x^2+y^2+z^2}$, show that $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 1$
2. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$
3. If $u = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2$
4. If $u = \log(x^3+y^3+z^3-3xyz)$, then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and hence

   show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Euler's theorem on homogeneous functions:

Statement: If $u = f(x,y)$ is a homogeneous function of degree n then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$

1. If $u = \frac{x^3+y^3}{\sqrt{x+y}}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{5}{2}u$
2. If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3$
3. If $u = e^{(x^3y^3/x^2+y^2)}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 4\,u\log u$
4. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u$
5. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u$

Total Differentiation:

If $u = f(x,y)$ then the total differential or the exact differential of u is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Total derivative rule:

If $u = f(x,y)$ where $x = x(t)$ and $y = y(t)$ then u is a composite function of the single variable t. Therefore in principle, we should be able to differentiate u with respect to t.

1. $z = x\,y^2 + x^2\,y$ where $x = at,\ y = 2at$
2. $u = xy + yz + zx$ where $x = t\,cos\,t,\ y = t\sin t,\ z = t\,,\ at\ t = \frac{\pi}{4}$
3. $u = x^2 + y^2 - z^2$ where $x = e^t, y = e^t\cosh t,\ z = e^t\sinh t$

Chain rule:

If $u = f(x,y)$ where $x = x(r,s)$ and $y = y(r,s)$ then u is a composite function of two independent variable $r,s.$ therefore in principle, we should be able to differentiate $u\ w.r.t\ r$ and also $w.r.t\ s$

$$u \to (x,y) \to (r,s) \Longrightarrow u \to (r,s) \begin{cases} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \end{cases}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

1. If $z = x^2 + y^2$ where $x = e^u \sin v,\ \ y = e^u \cos v$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a composite function and verify the results by direct substitution.
2. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$
3. If $u = f(x-y, y-z, z-x)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$
4. If $u = f(2x-3y, 3y-4z, 4z-2x)$, show that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$
5. If $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$ show that
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$
6. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Jacobian:

1. If $u = x + y + z,\ \ v = y + z\ \ and\ z = uvw$, find the value of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$
2. If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, find the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$
3. If $u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}$, find the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$
4. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $u = x^2 + y^2 + z^2, v = xy + yz + zx,\ \ w = x + y + z$
5. If $x = r\sin\theta\cos\emptyset,\ \ y = r\sin\theta\sin\emptyset,\ \ z = r\cos\theta$ show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = r^2\sin\theta$
6. If $u = x + 3y^2 - z^3, v = 4x^2yz,\ \ w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1,0)

MODULE 4: INTEGRAL CALCULUS

- Reduction formula for $\int sin^n x\, dx$ and $\int_0^{\pi/2} sin^n x\, dx, n$ is a positive integer
  - $\int sin^n x\, dx = -\frac{sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}$
  - $\int_0^{\pi/2} sin^n x\, dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\frac{2}{3}\ if\ n\ is\ odd \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\frac{1}{2}.\frac{\pi}{2}\ if\ n\ is\ even \end{cases}$

- Reduction formula for $\int cos^n x\, dx$ and $\int_0^{\pi/2} cos^n x\, dx,\quad n$ is a positive integer
  - $\int cos^n x\, dx = \frac{cos^{n-1}x\ \sin x}{n} + \frac{n-1}{n} I_{n-2}$
  - $\int_0^{\pi/2} cos^n x\, dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\frac{2}{3}\ if\ n\ is\ odd \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\frac{1}{2}.\frac{\pi}{2}\ if\ n\ is\ even \end{cases}$

- Reduction formula for $\int sin^m x\, cos^n x\, dx$ and $\int_0^{\pi/2} sin^m x\, cos^n x\, dx$ where m and n are positive integers.
  - $I_{m,n} = \int sin^m x\, cos^n x\, dx = -\frac{sin^{m-1}x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$
  - $\int_0^{\pi/2} sin^m x\, cos^n x\, dx = \frac{[(m-1)\,(m-3)\ldots]\,[(n-1)\,(n-3)\ldots]}{(m+n)(m+n-2)\,(m+n-4)\ldots} \times k$

    Where $k = \frac{\pi}{2}$ when m and n are even and $k = 1$ otherwise.

Problems:

1. Evaluate $\int_0^{\pi} x\, sin^8\, x\, dx$
2. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^8\, x\, dx$
3. Evaluate $\int_0^{\pi} x\, sin^2\, x\, cos^4\, x\, dx$
4. Evaluate $\int_0^{\pi} sin^6\, x\, cos^6\, x\, dx$
5. Evaluate $\int_0^{\frac{\pi}{6}} cos^4\, 3x\, sin^2\, 6x\, dx$
6. Evaluate $\int_0^{\pi} \frac{sin^4\theta}{(1+cos\theta)^2} d\theta$
7. Evaluate $\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a-x}}\, dx$
8. Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\, dx$

9. Evaluate $\int_0^1 x^2 sin^{-1} x\, dx$

10. Show that where n is a positive integer $\int_0^{2a} x^n\sqrt{2ax - x^2}\, dx = \pi a^2 \left(\frac{a}{2}\right)^n \cdot \frac{(2n+1)!}{(n+2)!\, n!}$

Double and Triple integrals:

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$
2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$
3. Evaluate $\int_0^1 \int_0^x (x^2 + y^2)\, dy\, dx$
4. Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3\, y\, dx\, dy$
5. Evaluate $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$
6. Evaluate $\int_1^2 \int_0^{2-y} xy\, dx\, dy$
7. Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$
8. Evaluate $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x + y + z)\, dy\, dx\, dz$
9. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$
10. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$
11. Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$
12. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dr\, d\theta\, dz$
13. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\, dy\, dx}{(1+x+y+z)^3}$

MODULE 5: DIFFERENTIAL EQUATIONS

Linear differential equation:

1. Solve $\frac{dy}{dx} - \frac{2y}{x} = x + x^2$
2. Solve $\frac{dy}{dx} + y\cot x = \cos x$

Bernoulli's differential equation:

1. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2\, x$
2. Solve $\frac{dy}{dx} + \frac{y}{x} = x^2\, y^6$

3. Solve $xy(1+xy^2)\frac{dy}{dx}=1$

4. Solve $r\sin\theta-\cos\theta\ \frac{dr}{d\theta}=r^2$

5. Solve $\frac{dy}{dx}-y\tan x=\frac{\sin x\,cos^2 x}{y^2}$

6. Solve $x^3\frac{dy}{dx}-x^2\,y=-y^4\cos x$

Exact differential equations:

1. Solve $(2x+y+1)\,dx+(x+2y+1)\,dy=0$

2. Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+y}=0$

3. Solve $y\,e^{xy}dx+(xe^{xy}+2y\,)\,dy=0$

4. Solve $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\,dy=0$

5. Solve $\cos x\,(e^y+1)dx+\sin x\,e^y\,dy=0$

Solution of Homogeneous differential equation:

1. Solve $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}+4\frac{dy}{dx}-8y=0$

2. Solve $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

3. Solve $\frac{d^3y}{dx^3}-3\frac{dy}{dx}+2y=0$

4. Solve $4\frac{d^4y}{dx^4}-4\frac{d^3y}{dx^3}-23\frac{d^2y}{dx^2}+12\frac{dy}{dx}+36y=0$

5. Solve $4\frac{d^4y}{dx^4}-8\frac{d^3y}{dx^3}-7\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

Solution of non-homogeneous differential equation:

Type-1:

1. Solve $6\frac{d^2y}{dx^2}+17\frac{dy}{dx}+12y=e^{-x}$

2. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cosh\left(\frac{x}{2}\right)$

3. Solve $\frac{d^2y}{dx^2} - 4y = \cosh(2x - 1) + 3^x$

Type-2:

4. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = \cos 2x$

5. Solve $\frac{d^2y}{dx^2} + 9y = \cos 2x\,.\cos x$

6. Solve $\frac{d^3y}{dx^3} - y = 3\cos 2x$

Type-3:

7. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 12\,x^2$

8. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x + \ x^2$

9. Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2 - 4x - 6$