Multiple Integrals

Double and Triple Integrals

①. Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Sol^n: Given, $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx \longrightarrow ①$.

$$= \int_0^1 \left[x^2 \cdot y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^2 \cdot \sqrt{x} + \frac{1}{3} \cdot (\sqrt{x})^3\right) - \left(x^3 - \frac{x^3}{3}\right) dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right) dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{1}{3} \cdot \frac{x^{5/2}}{5/2} - \frac{x^4}{4} - \frac{x^4}{12}\right]_0^1$$

$$= \left[\frac{2}{7} + \frac{2}{15} - \frac{1}{4} - \frac{1}{12}\right]$$

$$= \frac{3}{35}$$

②. Evaluate $\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$

Sol:- Let $I=\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$

$$=\int_0^1 x\left[\frac{y^2}{2}\right]_x^{\sqrt{x}}dx$$

$$=\int_0^1 x\left[\frac{x}{2}-\frac{x^2}{2}\right]dx$$

$$=\int_0^1\left(\frac{x^2}{2}-\frac{x^3}{2}\right)dx$$

$$=\left[\frac{x^3}{6}-\frac{x^4}{8}\right]_0^1$$

$$=\left[\frac{1}{6}-\frac{1}{8}\right]$$

$$=\frac{1}{24}$$

③. Evaluate $\int_0^1\int_0^x (x^2+y^2)\,dy\,dx$

Sol:- Let $I=\int_0^1\int_0^x (x^2+y^2)\,dy\,dx$

$$=\int_0^1\left[x^2 y+\frac{y^3}{3}\right]_0^x dx$$

$$=\int_0^1\left(x^3+\frac{x^3}{3}\right)dx$$

$$=\left.\frac{x^4}{4}+\frac{x^4}{12}\right]_0^1$$

$$=\left[\frac{1}{4}+\frac{1}{12}\right]$$

$$=\frac{1}{6}$$

④. Evaluate $\int_0^1\int_0^{\sqrt{1-y^2}} x^3 y\,dy\,dx$

Sol:- Let $I=\int_0^1\int_0^{\sqrt{1-y^2}} x^3 y\,dy\,dx$

$$=\int_0^1\left[\frac{x^4}{4}\right]_0^{\sqrt{1-y^2}} y\,dy$$

$$=\int_0^1\frac{1}{4}(1-y^2)^2 y\,dy$$

$$=\int_0^1\frac{1}{4}(y^5-2y^3+y)\,dy$$

$$=\frac{1}{4}\left[\frac{y^6}{6}+\frac{y^2}{2}-\frac{2y^4}{4}\right]_0^1=\frac{1}{24}$$

⑤. Evaluate $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx$

Sol:- Let $I = \int_1^2\int_3^4 (xy+e^y)\,dy\,dx$

$= \int_1^2 \left[\frac{xy^2}{2}+e^y\right]_3^4 dx$

$= \int_1^2 \left[\frac{x}{2}(16-9)+e^4-e^3\right] dx$

$= \int_1^2 \left(\frac{7}{2}x+e^4-e^3\right) dx$

$= \left.\frac{7}{2}\left(\frac{x^2}{2}\right)+xe^4-xe^3\right]_1^2$

$= \frac{7}{4}(4-1)+(2-1)e^4-e^3$

$= \frac{21}{4}+e^4-e^4$

⑥. Evaluate $\int_1^2\int_0^{2-y} xy\,dx\,dy$

Sol:- Let $I = \int_1^2\int_0^{2-y} xy\,dx\,dy$

$= \int_1^2 \left[\frac{x^2}{2}y\right]_0^{2-y} dy$

$= \int_1^2 \frac{1}{2}\left((2-y)^2\cdot y\right) dy$

$= \int_1^2 \frac{1}{2}\left(y^3-4y^2+4y\right) dy$

$= \frac{1}{2}\left[\frac{y^4}{4}-\frac{4y^3}{3}+2y\right]_1^2$

$= \frac{1}{2}\left[\frac{(16-1)}{4}-\frac{4}{3}(8-1)+2(3)\right]$

$= \frac{1}{2}\left[\frac{5-112+72}{12}\right]$

$= \frac{5}{24}$

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(7). Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

Sol^n:- Let $I=\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

$$=\int_{-c}^{c}\int_{-b}^{b}\left[\frac{x^3}{3}+y^2x+z^2x\right]_{-a}^{a}dy\,dz$$

$$=\int_{-c}^{c}\int_{-b}^{b}\left[\left(\frac{a^3}{3}+ay^2+az^2\right)-\left(\frac{-a^3}{3}-ay^2-az^2\right)\right]dy\,dz$$

$$=\int_{-c}^{c}\int_{-b}^{b}\left[\frac{2a^3}{3}+2ay^2+2az^2\right]dy\,dz$$

$$=\int_{-c}^{c}\left[\frac{2a^3}{3}y+\frac{2ay^3}{3}+2az^2y\right]_{-b}^{b}dz$$

$$=\int_{-c}^{c}\left\{\left[\frac{2a^3b}{3}+\frac{2ab^3}{3}+2az^2b\right]-\left[\frac{-2a^3b}{3}-\frac{2ab^3}{3}-2az^2b\right]\right\}dz$$

$$=\int_{-c}^{c}\left\{\frac{4a^3b}{3}+\frac{4ab^3}{3}+4az^2b\right\}dz$$

$$=\left[\frac{4a^3b}{3}z+\frac{4ab^3}{3}z+\frac{4abz^3}{3}\right]_{-c}^{c}$$

$$=\left\{\left(\frac{4a^3bc}{3}+\frac{4ab^3c}{3}+\frac{4abc^3}{3}\right)-\left(\frac{-4a^3bc}{3}-\frac{4ab^3c}{3}-\frac{4abc^3}{3}\right)\right\}$$

$$=\frac{8a^3bc}{3}+\frac{8ab^3c}{3}+\frac{8abc^3}{3}$$

$$=\frac{8}{3}abc(a^2+b^2+c^2)$$

⑧. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

Sol^n: Let $I=\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$=\int_{-1}^{1}\int_{0}^{z}\left(xy+\frac{y^2}{2}+zy\right)_{x-z}^{x+z}dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right]-\left[x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}(4xz+2z^2)\,dx\,dz$$

$$=\int_{-1}^{1}\left(\frac{4zx^2}{2}+2z^2x\right)_{0}^{z}dz$$

$$=\int_{-1}^{1}(2z^3+2z^3)\,dz$$

$$=\int_{-1}^{1}(4z^3)\,dz$$

$$=\left.\frac{4z^4}{4}\right]_{-1}^{1}$$

$$=1-1$$

$$=\underline{\underline{0}}$$

(9). Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Solⁿ:- Let $I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

$$= \int_0^1\int_0^{\sqrt{1-x^2}} \left(\frac{xyz^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right) dx$$

$$= \frac{1}{4}\int_0^1 \left[x - x^3 - x^3 + x^5 - \frac{1}{2}x(1+x^4-2x^2)\right] dx$$

$$= \frac{1}{4}\int_0^1 \left[x - 2x^3 + x^5 - \frac{1}{2}(x + x^5 - 2x^3)\right] dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{x^6}{6} - \frac{2x^4}{4}\right)\right]_0^1$$

$$= \frac{1}{4}\left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6} - \frac{1}{2}\left(\cancel{\frac{1}{2}} + \frac{1}{6} - \cancel{\frac{1}{2}}\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{6} - \frac{1}{12}\right] = \frac{1}{48}$$

(10). Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Sol^n:- $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$$=\int_0^a e^x\left\{\int_0^x e^y\left[\int_0^{x+y} e^z\,dz\right]dy\right\}dx$$

$$=\int_0^a e^x\left\{\int_0^x e^y\left[e^z\right]_0^{x+y}dy\right\}dx$$

$$=\left\{\int_0^a e^x\left[\int_0^x e^y\left(e^{x+y}-e^0\right)dy\right]dx\right\}$$

$$=\int_0^a e^x\left[\int_0^x e^y\left(e^{x+y}-1\right)dy\right]dx$$

$$=\int_0^a e^x\left[\int_0^x\left(e^x\cdot e^{2y}-e^y\right)dy\right]dx$$

$$=\int_0^a e^x\left[e^x\cdot\frac{e^{2y}}{2}-e^y\right]_0^x dx$$

$$=\int_0^a e^x\left[\frac{1}{2}e^x\left(e^{2x}-e^0\right)-\left(e^x-e^0\right)\right]dx$$

$$=\int_0^a e^x\left[\frac{1}{2}e^{3x}-\frac{e^x}{2}-e^x+1\right]dx$$

$$=\int_0^a\left(\frac{e^{4x}}{2}-\frac{3e^{2x}}{2}+e^x\right)dx$$

$$=\left[\frac{e^{4x}}{8}-\frac{3e^{2x}}{4}+e^x\right]_0^a=\left[\frac{e^{4a}}{8}-\frac{3}{4}e^{2a}+e^a\right]-\left[\frac{1}{8}-\frac{3}{4}+1\right]$$

$$=\frac{e^{4a}}{8}-\frac{3}{4}e^{2a}+e^a-\frac{3}{8}$$

(11). Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

Sol:- Let $I = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$

$$= \int_0^{\log 2}\int_0^{x} e^{x}.e^{y}\left[e^{z}\right]_0^{x+\log y} dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}.e^{y}\left[e^{x}.e^{\log y}-1\right] dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}.e^{y}\left[y.e^{x}-1\right] dy\,dx$$

$$= \int_0^{\log 2}\int_0^{x} e^{x}\left\{y.e^{y}.e^{x}-e^{y}\right\} dy\,dx$$

$$= \int_0^{\log 2} e^{x}\left\{(y.e^{y}-e^{y})e^{x}-e^{y}\right\}_0^{x} dx$$

$$= \int_0^{\log 2} e^{x}\left(\left\{x.e^{x}-e^{x}\right\}e^{x}-e^{x}\right)-\left(e^{x}\left\{(0-1)e^{x}-1\right\}\right)dx$$

$$= \int_0^{\log 2}\left[\left(x.e^{3x}-e^{3x}-e^{2x}\right)+e^{2x}+e^{x}\right]dx$$

$$= \int_0^{\log 2}\left(xe^{3x}-e^{3x}\cancel{-e^{2x}}\cancel{+e^{2x}}+e^{x}\right)dx$$

$$= \left[x.\frac{e^{3x}}{3}-\int\frac{e^{3x}}{3}dx\right]_0^{\log 2}-\left(\frac{e^{3x}}{3}\right)_0^{\log 2}+\left(e^{x}\right)_0^{\log 2}$$

$$= \left(x.\frac{e^{3x}}{3}-\frac{e^{3x}}{9}\right)_0^{\log 2}-\frac{1}{3}\left(e^{3\log 2}-1\right)+\left(e^{\log 2}-1\right)$$

$$= \left(\frac{\log 2 \cdot e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} - 0 + \frac{1}{9}\right) - \left(\frac{8}{3} - \frac{1}{3}\right) + (2-1)$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} + \frac{1}{9} - \frac{7}{3} + 1$$

$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

(12). Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r\, dr\, d\theta\, dz$

Sol:- Let $I = \displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r \cdot dr\, d\theta\, dz$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} r \cdot z\Big]_0^{\frac{a^2-r^2}{a}} dr\, d\theta$$

We have,

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

($\because$ n is Even)

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} r\left(\frac{a^2-r^2}{a}\right) dr\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\int_0^{a\sin\theta} (ra^2 - r^3)\, dr\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(\frac{r^2a^2}{2} - \frac{r^4}{4}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2}\left(\frac{a^4\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4}\right) d\theta$$

$$\therefore I = \frac{1}{a}\left[\frac{a^4}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{a^4}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right] = \frac{5\pi a^3}{64}$$

(13). Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$

Sol:- Let $I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$

$$= \int_0^1\int_0^{1-x}\left[\frac{-1}{2(1+x+y+z)^2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{-1}{2(1+\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y})^2} + \frac{1}{2(1+x+y)^2}\right] dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{-1}{2(4)} + \frac{1}{2(1+x+y)^2}\right] dy\,dx$$

$$= \int_0^1 \left[-\frac{1}{8}y - \frac{1}{2(1+x+y)}\right]_0^{1-x} dx$$

$$= \int_0^1\left[-\frac{1}{8}(1-x) - \frac{1}{2(1+\cancel{x}+1-\cancel{x})} + 0 + \frac{1}{2(1+x)}\right] dx$$

$$= \int_0^1\left(-\frac{1}{8}(1-x) - \frac{1}{4} + \frac{1}{2(x+1)}\right) dx$$

$$= \int_0^1\left[-\frac{3}{8}(1-x) + \frac{1}{2(x+1)}\right] dx$$

$$= \left[-\frac{3}{8}x + \frac{x^2}{16} + \frac{1}{2}\log(x+1)\right]_0^1$$

$$= \left(-\frac{3}{8} + \frac{1}{16} + \frac{1}{2}\log 2\right) - (0) = -\frac{5}{16} + \log\sqrt{2}$$