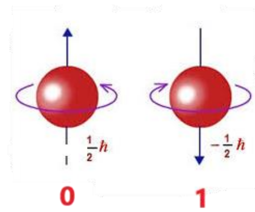
Module - 2
Introduction to Quantum Computing

Syllabus: Introduction to Quantum Computing- Moore's law and its end- quantum superposition-concept of qubit and its properties- conjugate of a matrix- transpose of a matrix- unitary matrix-orthogonality- ortho-normality. Single Qubit Gates: Quantum Not Gate- Pauli – X, Y and Z gates, Hadamard gate, phase gate (or S gate), T gate, Multiple Qubit Gates: CNOT gate-representation of swap gate, controlled-Z gate, Toffoli gate. Numerical Problems.

Introduction to Quantum computing

Quantum Computing is a new kind of computing based on quantum mechanics. Classical computers use binary bits, 0 and 1, to store and manipulate data.

In quantum computers, quantum bits, or Qubits, are used to store and manipulate information. Qubits are fundamentally a two-state quantum-mechanical system.



These Qubits uses the properties such as spins states of subatomic particles (atoms, electrons, and ions) or single photon polarization states, such horizontal polarization or vertical polarization, using them as the basis for computers.

Quantum computers are extremely powerful, and their applications range from data encryption, artificial intelligence and machine learning.

A quantum computer is defined as ***"a computer that utilizes the quantum states of subatomic particles, called Qubits, to store and manipulate information."***

Quantum superposition

Quantum superposition is a fundamental principle of quantum mechanics. A quantum system can exist in multiple states at the same time until measured. A particle like an electron can be in more than one position simultaneously.

In quantum mechanics, the state of a system is described by a wave function, which is a solution to the Schrödinger equation. Schrodinger equation can have more than one solution.

For example, if ψ_1 and ψ_2 are solutions to the Schrödinger equation, then any linear combination of these solutions, such as $c_1\psi_1+c_2\psi_2$ is also a solution, where c_1 and c_2 are constants with condition $|C_1|^2 + |C_2|^2 = 1$. This property is known as the superposition principle.

The superposition principle lays the groundwork for advanced concepts like quantum computing and quantum cryptography.

Superposition in quantum computing allows qubits to represent multiple states simultaneously. This enables quantum computers to perform many calculations at once, solving problems much faster.

Moore's law and its end

Introduction to Moore's Law

Moore's Law is a foundational principle in the field of electronics and computing, formulated by Gordon Moore, the co-founder of Intel, in 1965. It describes a long-term trend in Integrated circuit (IC) technology.

Moore states that "***The number of transistors on a microchip doubles approximately every two years (18-24 months).***"

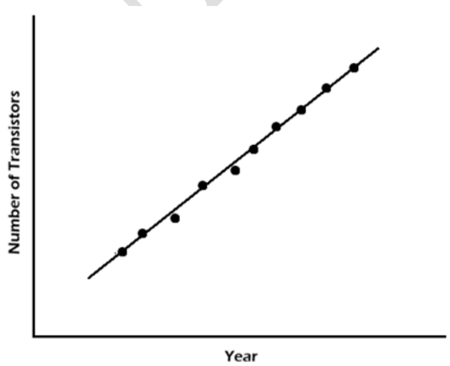
As the number of transistors increases, operations become faster, computers more powerful, consumes less power and production costs lower.

Failure of Moore`s law

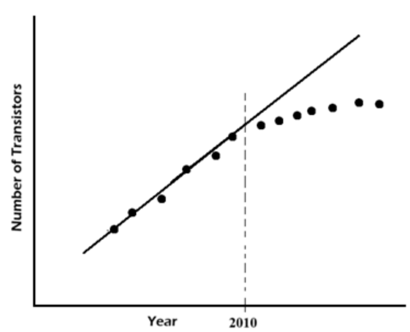
Up until 2010, the IC industry was able to follow Moore's Law. The pace of transistor scaling has slowed, and several factors are contributing to the end of Moore's Law. The factors are quantum effects like tunnelling and heat dissipation.

When transistors get as small as a few atoms, they start to behave differently, following the rules of quantum mechanics. This opens up new and unique possibilities for design.

Computing at the atomic scale using quantum mechanical laws gives birth to a new technology called **quantum computer**.



Moore's Law



Moore's law and its end

Concept of Qubit

Quantum Bit or **Qubit** is the fundamental unit of quantum information that represents the physical properties of subatomic particles such as atoms, electrons, etc. as a computer's memory and works based on quantum concept.

Properties of Qubits:

- Superposition, Entanglement, decoherence no-cloning and Tunnelling are all special properties that define a qubit.
- Qubits can be expressed in quantum mechanical states with mathematical notation called as Dirac or "bra–ket" notation. Bra is a row matrix and Ket is a column matrix.
- A qubit, $|\psi\rangle$, could be in a $|0\rangle$ or $|1\rangle$ state or superposition of both $|0\rangle$ and $|1\rangle$, (linear combination of states) which can be written as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

Where $|\alpha|^2$ and $|\beta|^2$ are probabilities of finding that state. (α and β also called as amplitude of the states)

$$\text{Here, } |\alpha|^2 + |\beta|^2 = 1$$

- Measurements on Superposition state results in $|0\rangle$ state with probability $|\alpha|^2 = 1$ and $|\beta|^2 = 0$, or $|1\rangle$ state with probability $|\beta|^2 = 1$ and $|\alpha|^2 = 0$. Measurement destroys superposition.
- A Qubit can be physically implemented by the two states of any quantum physical system.

Representation of Qubits

Matrix Representation of 0 and 1 states:

The state of a qubit (quantum binary digits) in quantum computation is represented as by $|0\rangle$ and $|1\rangle$ or $\langle 0|$ and $\langle 1|$. The first representation $|0\rangle$ and $|1\rangle$ read as ket zero or ket one and $\langle 0|$ and $\langle 1|$ | , read as bra zero or bra one. The 0 and 1 state of qubit can also be represented in matrix form as

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\langle 0| = [1\ 0]\ and\ \langle 1| = [0\ \ 1]$$

For a qubit state, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ we can write matrix form as

$$|\psi\rangle = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}$$

Multiple qubits

Consider two qubits. If these were two classical bits, then there would be four possible states, 00, 01, 10, and 11. Correspondingly, a two-qubit system has four computational basis states denoted |00⟩, |01⟩, |10⟩, |11⟩. A pair of qubits can also exist in superpositions of these four states.

$$|\psi\rangle = \alpha_{00}\ |00\rangle + \alpha_{01}\ |01\rangle + \alpha_{10}\ |10\rangle + \alpha_{11}\ |11\rangle$$

Where

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

Conjugate of a matrix:

It is possible to find the conjugate for a given matrix by replacing each element of the matrix with its complex conjugate. Conjugate of a matrix 'A' is represented by 'A*'

Example:

$$A = \begin{bmatrix} 1 & x + iy \\ x - iy & i \end{bmatrix}$$

Conjugate of the matrix A is given by

$$A^* = \begin{bmatrix} 1 & x - iy \\ x + iy & -i \end{bmatrix}$$

Transpose of a Matrix:

This matrix is found by interchanging its rows into columns or columns into rows. It is denoted by using the letter "T" in the superscript of the given matrix. For example, if A is the given matrix, then the transpose of the matrix is represented by A' or A^T.

Ex: $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The transpose of A is $$A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Unitary matrix 'U':

A matrix U is unitary, if the matrix product of U and its conjugate transpose $U^\dagger$ (called U-dagger) produces the identity matrix. i.e.

$$U\,U^\dagger = U^\dagger\,U = I = 1$$

Eg:

$$U = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

Then $U^* = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$ and $(U^*)^T = U^\dagger = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$

$$UU^\dagger = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} * \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Probability(Normalization and orthogonality) :

Let us consider a Quantum state

$$|\psi\rangle = \alpha\, |0\rangle + \beta\, |1\rangle$$

The above equation represents the quantum Superposition of states **|0⟩** and **|1⟩**

The inner product ⟨ψ| ψ ⟩ is = $|\psi|^2 = \psi\,\psi^* = |\alpha|^2 + |\beta|^2$,

Thus, the above equation represents **Probability Density.** Consider two states ψ and φ then,

If ⟨ψ| φ ⟩ = 1 the wavefunction is normalized, "Two states **|ψ⟩** and **|φ⟩** are said to be normalized if their **inner product is one".**

If ⟨ψ| φ ⟩ = 0 the wavefunction is orthogonal, "Two states **|ψ⟩** and **|φ⟩** are said to be orthogonal if their **inner product is zero".**

Introduction to Quantum Gates

Quantum gates are the fundamental building blocks of quantum circuits, analogous to classical logic gates such as OR, AND and NOT gates. All quantum gates are reversible and unitary. They manipulate qubits by changing their state through operations such as rotation, entanglement, and superposition. Common quantum gates include the Pauli-X (analogous to a classical NOT gate), Hadamard (which creates superposition), and CNOT (which entangles qubits). Quantum gates are essential for performing quantum algorithms, enabling powerful computations.

Single Qubit Gates

Single qubit gates are quantum operations that act on individual qubits and changing their state. These gates are the simplest type of quantum gates and are crucial for building quantum circuits. Common single qubit gates include:

Pauli-X Gate (X): Flips the state of a input qubit, turning |0⟩ into |1⟩ and |1⟩ into |0⟩. It's analogous to the classical NOT gate.

Matrix representation of Quantum Not gate is given by Pauli matrix σ_x or X,

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

X gate operating on qubits in superposition also flips. The schematic representation is given below

$$\alpha|0\rangle + \beta\,|1\rangle \quad \text{-----}X\text{------} \qquad \beta\,|0\rangle + \alpha|1\rangle$$

Truth table of NOT gate	
Input	output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle + \beta\,\|1\rangle$	$\alpha\|1\rangle + \beta\,\|0\rangle$

Pauli-Y Gate (Y): When applied on a bit it introduces a **complex phase shift** in addition to flipping the qubit.

Matrix representation of Pauli matrix σ_Y or Y,

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

The quantum Y gate representation is given below

$$\mathbf{|0\rangle + \beta\,|1\rangle\text{--------}Y\text{---------}i\alpha|1\rangle - i\beta|0\rangle}$$

Truth table of Y gate	
Input	output
$\|0\rangle$	$i\|1\rangle$
$\|1\rangle$	$-i\|0\rangle$
$\alpha\|0\rangle + \beta\,\|1\rangle$	$i\alpha\|1\rangle - i\beta\,\|0\rangle$

When applied on computational base set

$$Y|0\rangle = Y\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = i|1\rangle$$

$$Y|1\rangle = Y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix} = -i|0\rangle$$

Pauli-Z Gate (Z): Applies a phase shift to the qubit, flipping the phase of $|1\rangle$ while leaving $|0\rangle$ unchanged.

Matrix representation of Pauli matrix σ_Z or Z,

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The quantum Z gate representation is given below

α|0⟩ + β |1⟩ --------Z ------- α|0⟩ - β|1⟩

Truth table of Z gate	
Input	output
|0⟩	|0⟩
|1⟩	-|1⟩
α|0⟩ + β |1⟩	α|0⟩ - β |1⟩

Applying Z to computational basis states:

$$Z|0\rangle = Z\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$$

$$Z|1\rangle = Z\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = -|1\rangle$$

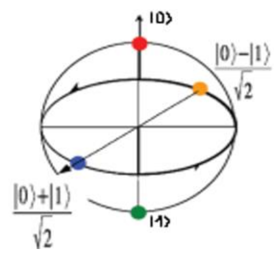
Hadamard Gate (H):

The Hadamard gate is a one-qubit quantum gate that plays a crucial role in quantum computing, particularly in quantum algorithms like quantum superposition and quantum parallelism. It transforms a single qubit into a superposition state, creating an equal probability of measuring either |0⟩ or |1⟩ when the qubit is in state |0⟩ or |1⟩.

The Hadamard gate is defined as, the output may be considered midway between 0 and 1.

The gate transforms |0⟩ in to $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

The gate transforms |1⟩ in to $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$



The Matrix representation of Hadamard Gate is as follows $H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

The truth table,

Input	output
$\lvert 0\rangle$	$\frac{\lvert 0\rangle + \lvert 1\rangle}{\sqrt{2}}$
$\lvert 1\rangle$	$\frac{\lvert 0\rangle - \lvert 1\rangle}{\sqrt{2}}$
$\alpha\lvert 0\rangle + \beta\,\lvert 1\rangle$	$\alpha\frac{\lvert 0\rangle + \lvert 1\rangle}{\sqrt{2}} + \beta\frac{\lvert 0\rangle - \lvert 1\rangle}{\sqrt{2}}$

Hadamard Gate on Computational Base State

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Phase Gate or S-Gate:

The phase gate turns $|0\rangle$ into $|0\rangle$ and $|1\rangle$ into $i\,|1\rangle$. The Matrix representation of the S gate is given by
$$S = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$$

The quantum S gate representation is given below

$$\alpha|0\rangle + \beta\,|1\rangle \text{ ------ } \boxed{S} \text{ ---------- } \alpha|0\rangle + i\,\beta|1\rangle$$

The Truth table for S-Gate

Truth table of S gate	
Input	output
$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 1\rangle$	$i\lvert 1\rangle$
$\alpha\lvert 0\rangle + \beta\,\lvert 1\rangle$	$\alpha\lvert 0\rangle + i\beta\,\lvert 1\rangle$

T- Gate or π/8 Gate:

The T gate, also known as the π/8 gate, is a fundamental quantum gate in quantum computing. It's a single-qubit gate that applies a phase shift to the qubit, and it plays a crucial role in achieving universal quantum computation.

Mathematical Representation:

The T gate is represented by the following unitary matrix:

$$T = \begin{bmatrix}1 & 0\\ 0 & e^{\frac{i\pi}{4}}\end{bmatrix}$$

The gate transforms $|0\rangle$ in to $|0\rangle$

The gate transforms $|1\rangle$ in to $e^{i\pi/4}|1\rangle$

The quantum T gate representation is given below

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{T}—\; \alpha|0\rangle + \beta e^{i\pi/4}|1\rangle$$

The truth table of T-Gate is

Truth table of S gate	
Input	output
$|0\rangle$	$|0\rangle$
$|1\rangle$	$e^{i\pi/4}|1\rangle$
$\alpha|0\rangle + \beta|1\rangle$	$\alpha|0\rangle + \beta\, e^{i\pi/4}|1\rangle$

Multiple Qubit Gates

Controlled Gates

Multiple Qubit gates operate on two or more input qubits. Usually, one of them is a control qubit.

A Gate with operation of kind "**If 'A' is True then do 'B'**" is called Controlled Gate.

The '$|A\rangle$' Qubit is called Control qubit and '$|B\rangle$' is the Target qubit. The target qubit is altered only when the control qubit is $|1\rangle$. The control qubit remains unaltered during the transformations.

Controlled NOT Gate or CNOT Gate:

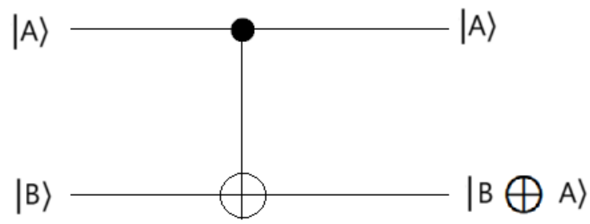
The CNOT gate is a typical multi-qubit logic gate

In CNOT, 1st qubit is control, and 2nd qubit is the target. When control qubit is $|0\rangle$ we do nothing, hence the Identity operator. When the control is a $|1\rangle$ we apply the X (classical NOT gate) operator.

The Matrix representation of CNOT Gate is given by

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Circuit symbol of the CNOT gate is given as



where ⊕ is the logical exclusive OR operator .

Operation of CNOT Gate

Consider the operations of CNOT gate on the four inputs |00⟩, |01⟩, |10⟩ and |11⟩

1. CNOT(|00⟩)= |00⟩;

Here input of control qubit is 0 state, So there is no change in the state of target qubit.

2. CNOT(|01⟩)= |01⟩;

Here input of control qubit is 0 state, So there is no change in the state of target qubit.

3. CNOT(|10⟩)= |11⟩;

Here input of control qubit is 1 state, So target qubit flips from 0 state to 1 state.

4. CNOT(|11⟩)= |10⟩;

Here input of control qubit is 1 state, So target qubit flips from 1 state to 0 state.

The truth table of CNOT gate

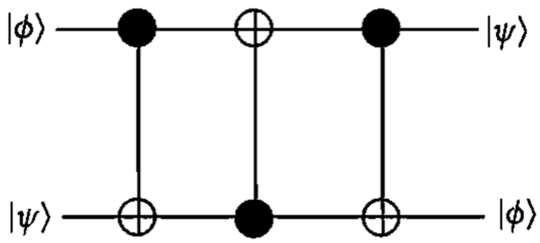
Truth table of CNOT gate	
Input	output
|00⟩	|00⟩
|01⟩	|01⟩
|10⟩	|11⟩
|11⟩	|10⟩

SWAP Gate:

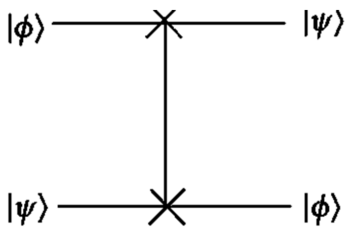
The SWAP gate is two qubit operation. The SWAP gate can swap (exchange) the state of the two qubits. The matrix representation of the SWAP gate is given below

$$U_{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The schematic symbol of SWAP gate circuit is



Equivalent circuit representation of SWAP gate:



The SWAP gate is a combined circuit of 3 CNOT gates and the overall effect is that two input qubits are swapped at the output

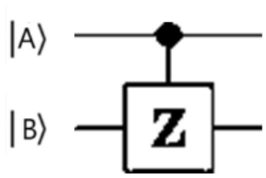
Truth table of SWAP gate	
Input	output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|10\rangle$
$\|10\rangle$	$\|01\rangle$
$\|11\rangle$	$\|11\rangle$

Controlled Z Gate:

In Controlled Z Gate, the operation of Z Gate is controlled by a Control Qubit. Let $|AB\rangle$ be the input to the gate, where $|A\rangle$ is the control qubit and $|B\rangle$ is the target qubit. If the control Qubit is $|A\rangle = |1\rangle$ then only the Z gate transforms the Target Qubit $|B\rangle$ as per the Pauli-Z operation. The action of Controlled Z-Gate could be specified by a matrix as follows.

$$U_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Circuit Symbol:



Truth table:

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|10\rangle$
$\|11\rangle$	$-\|11\rangle$

Toffoli Gate:

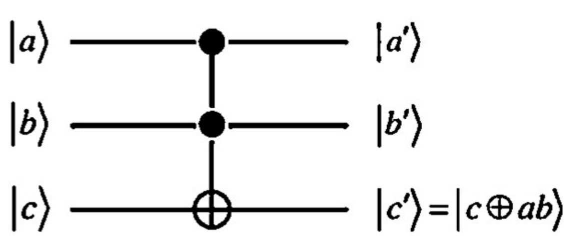
The Toffoli Gate is also known as CCNOT Gate (Controlled-Controlled-Not). It has three inputs out of which two are Control Qubits and one is the Target Qubit. The Target Qubit flips only when both the Control Qubits are $|1\rangle$. The two Control Qubits are not altered during the operation.

The matrix representation, Gate circuit and the truth table of Toffoli Gates as follows.

$$U_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The circuit symbol and truth table,

Inputs			Ouputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0



Numerical on Introduction to Quantum Computing

Solved Numerical:

1. Find the transpose of the given matrix. $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{bmatrix}$

Solution : Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Taking Transpose of matrix A (convert rows in to columns and columns in to rows)

We get $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

2. Verify the given matrix is unitary or not $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{i} \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & 1 \\ 1 & i \end{bmatrix}$

Complex conjugate $A^* = \begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix}$

The transpose of A^* = $A^\dagger = \begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix}$

Unitary matrix $AA^\dagger = \begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix}$

$$A\,A^\dagger = \begin{bmatrix} 1+1 & 1-i \\ 1-i & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 1-i \\ 1-i & 0 \end{bmatrix} \neq I$$

A is not a unitary matrix because the product is not an identity matrix.

3. Find the complex conjugate $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{x+iy} \\ \mathbf{x-iy} & \mathbf{i} \end{bmatrix}$

Solution: Given $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{x+iy} \\ \mathbf{x-iy} & \mathbf{i} \end{bmatrix}$

Taking complex conjugate of matrix A

We get $A^* = \begin{bmatrix} 1 & x-iy \\ x+iy & -i \end{bmatrix}$

4. Verify the given matrix is unitary or not $\mathbf{U} = \frac{1}{2}\begin{bmatrix} \mathbf{1+i} & \mathbf{1-i} \\ \mathbf{1-i} & \mathbf{1+i} \end{bmatrix}$

Solution: Given $\mathbf{U} = \frac{1}{2}\begin{bmatrix} \mathbf{1+i} & \mathbf{1-i} \\ \mathbf{1-i} & \mathbf{1+i} \end{bmatrix}$

Taking complex conjugate of matrix U

$$U^* = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \quad \text{and} \quad U^\dagger = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$$

$$UU^{\dagger} = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} * \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

5. **Verify the given vector bra 0 and ket 1 are normalized**

Solution: Take the inner product between(0) state and (1)state

$\langle 0|1\rangle = (1 \quad 0)\binom{0}{1}$ = (1 x 0) + (0 x1) = 0

The given vectors are not normalized

6. **Verify the given vector bra 0 and ket 1 are orthogonal**

Solution: Take the inner product between(0) state and (1)state

$\langle 0|1\rangle = (1 \quad 0)\binom{0}{1}$ = (1 x 0) + (0 x1) = 0

The given vectors are orthogonal

7. **Show that the Pauli X gate acts as classical NOT gate.**

Solution: The Pauli X gate is $\sigma_X = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Consider a state, $|0\rangle = \binom{1}{0}$ then $X\,|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\binom{1}{0} = \binom{0+0}{1+0} = \binom{0}{1} = |1\rangle$

The input $|0\rangle$ is converted in to $|1\rangle$, it shows that Pauli X gate is NOT gate.

Similarly

Consider a state, $|1\rangle = \binom{0}{1}$ then $X\,|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\binom{0}{1} = \binom{1+0}{0+0} = \binom{1}{0} = |0\rangle$

The input $|1\rangle$ is converted in to $|0\rangle$, it shows that Pauli X gate is NOT gate.

8. **Find how the Hadamard gate transforms $|\mathbf{0}\rangle$**

Solution: The Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

9. **Show the matrix formulation of Pauli Z gate**

Solution: $\sum |\text{inputs}\rangle\langle\text{output}| = \text{matrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \quad 0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0 \quad -1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise Problems:

1. A Linear Operator 'X' operates such that X $|0\rangle$ = $|1\rangle$ and X $|1\rangle$= $|0\rangle$. Find the matrix representation of 'X'.
2. Given $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ prove that $A = A^{\dagger}$
3. Show that the Matrix $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ is Unitary.
4. Using Matrix multiplication show that on applying Hadamard gate twice to a $|0\rangle$ results in its original state.

1. Using two X-gates in series show that two not gates in series are equivalent to a quantum wire.
2. Show that Hadamard Gate is Unitary.
3. Show that S gate can be formed by connecting two T gates in Series.