Find the angle of intersection of the following pair of curves:

①. $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

Sol: ① $\Rightarrow r = \sin\theta + \cos\theta \rightarrow$ ①.

diff. Eqⁿ ① wrt 'θ'

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$\frac{d\theta}{dr} = \frac{1}{\cos\theta - \sin\theta}$$ (multiply by r on both the sides).

$$r \cdot \frac{d\theta}{dr} = \frac{r}{\cos\theta - \sin\theta} \rightarrow ②.$$

from ①, we have, $r = \sin\theta + \cos\theta$
Substituting ① r value in ②, we get.

$$r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} \rightarrow ③.$$

W.K.T,

$$r\frac{d\theta}{dr} = \tan\phi.$$

$$\therefore ③ \Rightarrow \tan\phi_1 = \frac{\cos\theta[\tan\theta + 1]}{\cos\theta[1 - \tan\theta]}$$

$$\therefore \tan\phi_1 = \left[\frac{\tan\theta + \tan\pi/4}{1 - \tan\pi/4 \cdot \tan\theta}\right]$$

$\therefore \tan\phi_1 = \tan\left[\frac{\pi}{4}+\theta\right]$

$\therefore \phi_1 = \frac{\pi}{4}+\theta \rightarrow$ (a).

Now, Consider Curve (2),

i.e., $r = 2\sin\theta \rightarrow$ (4).

diff. Eq^n (4) w.r.t 'θ', we get.

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$\frac{d\theta}{dr} = \frac{1}{2\cos\theta}$$ (multiplying by r on both the sides).

$$r\frac{d\theta}{dr} = \frac{r}{2\cos\theta} \left[\because r = 2\sin\theta\right]$$

$$\therefore \tan\phi_2 = \frac{\cancel{2}\sin\theta}{\cancel{2}\cos\theta} = \tan\theta$$

$\therefore \phi_2 = \theta \rightarrow$ (b)

To find the angle of intersection, we have

$|\phi_1 - \phi_2|$

$\therefore$ from (a) and (b), we have,

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{4}+\cancel{\theta}-\cancel{\theta}\right| = \pi/4$$

$$\therefore \phi_1 - \phi_2 = \pi/4$$

(2). $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$.

sol:- 1^{st} curve, $\Rightarrow r = a(1-\cos\theta) \rightarrow$ (1).

diff. Eq^n (1) w.r.t 'θ'!

$$\frac{dr}{d\theta} = a\sin\theta$$

$$\frac{d\theta}{dr} = \frac{1}{a\sin\theta}$$

$$r\cdot\frac{d\theta}{dr} = \frac{r}{a\sin\theta} = \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta} = \frac{\cancel{2}\sin^{\cancel{2}}\theta/2}{\cancel{2}\cancel{\sin\theta/2}\cdot\cos\theta/2}$$

$$\tan\phi_1 = \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2$$

$$\therefore \phi_1 = \theta/2 \rightarrow \text{(a)}.$$

2^{nd} curve $\Rightarrow r = 2a\cos\theta \rightarrow$ (2).

diff. Eq^n (2) w.r.t 'θ'!

$$\frac{dr}{d\theta} = -2a\sin\theta$$

$$\frac{d\theta}{dr} = -\frac{1}{2a\sin\theta}$$

$$r\cdot\frac{d\theta}{dr} = -\frac{r}{2a\sin\theta} = -\frac{\cancel{2a}\cos\theta}{\cancel{2a}\sin\theta} = -\cot\theta$$

$$\tan\phi_2 = -\cot\theta$$

$$\tan\phi_2 = \tan\left[\frac{\pi}{2}+\theta\right].$$

$$\therefore \phi_2 = \frac{\pi}{2}+\theta \rightarrow \text{(b)}$$

from ⓐ and ⓑ, we have

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \theta - \theta/2\right| = \left|\frac{\pi}{2} + \frac{\theta}{2}\right| \rightarrow ③.$$

from curves we have,

$r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$$\Rightarrow 2a\cos\theta = a(1-\cos\theta)$$

$$\Rightarrow 2a\cos\theta = a - a\cos\theta$$

$$\Rightarrow 2a\cos\theta + a\cos\theta = a$$

$$3a\cos\theta = a$$

$$\cos\theta = \frac{\cancel{a}}{\cancel{a}3}$$

$$\therefore \cos\theta = 1/3$$

$$\theta = \cos^{-1}(1/3)$$

Substituting θ value in ③, we get.

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{1}{2}\cos^{-1}(1/3)\right|$$

$$\therefore \phi_1 - \phi_2 = \frac{\pi}{2} + \frac{1}{2}\cos^{-1}(1/3)$$

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③. $r = a\theta$ and $r = \frac{a}{\theta}$

Sol^n:- 1^{st} curve, $r = a\theta \rightarrow ①$.

diff. ① w.r.t 'θ'.

$$\frac{dr}{d\theta} = a$$

$$\frac{d\theta}{dr} = \frac{1}{a}$$ (multiplying by r, both the sides)

$$r\frac{d\theta}{dr} = \frac{r}{a}$$

$$\tan\phi_1 = \frac{a\theta}{a}$$

$$\therefore \tan\phi_1 = \theta$$

$$\phi_1 = \tan^{-1}(\theta)$$

$$r = \frac{a}{\theta} \cancel{\rightarrow ②}. \Rightarrow r\theta = a \rightarrow ②$$

diff. Eq^n ②, w.r.t θ, we get

$$r \cdot 1 + \theta \cdot \frac{dr}{d\theta} = 0$$

$$\theta\frac{dr}{d\theta} = -r$$

$$\frac{dr}{d\theta} = -\frac{r}{\theta}$$

$$\frac{d\theta}{dr} = -\frac{\theta}{r}$$

$$\Rightarrow r\frac{d\theta}{dr} = -\theta \Rightarrow \tan\phi_2 = -\theta$$

$$\phi_2 = -\tan^{-1}(\theta)$$

$\phi_1 = \tan^{-1}(\theta)$ and $\phi_2 = -\tan^{-1}(\theta)$.

$\therefore |\phi_1 - \phi_2| = |\tan^{-1}(\theta) - (-\tan^{-1}(\theta))|$

$= |\tan^{-1}(\theta) + \tan^{-1}(\theta)|$

$= |2\tan^{-1}(\theta)|$

$\therefore |\phi_1 - \phi_2| = |2\tan^{-1}(\theta)| \rightarrow (3)$.

from curves, we have,

$r = a\theta$ and $r = \frac{a}{\theta}$

$\Rightarrow a\theta = \frac{a}{\theta} \Rightarrow a\theta^2 = a$

$\Rightarrow \frac{a}{a} = \theta^2 \Rightarrow \theta^2 = 1 \Rightarrow \theta = \pm 1$.

if $\theta = +1$, (3) $\Rightarrow |\phi_1 - \phi_2| = |2\tan^{-1}(1)|$

$= 2\,\pi/4 = \underline{\underline{\pi/2}}$

if $\theta = -1$, (3) $\Rightarrow |\phi_1 - \phi_2| = |2\tan^{-1}(-1)|$

$= |-2\tan^{-1}(1)|$

$= |-2 \cdot \pi/4|$

$= |-\pi/2|$

$\therefore \phi_1 - \phi_2 = \underline{\underline{\pi/2}}$