Sapthagiri NPS University

Department of Mathematics

<u>Question Bank</u>

MODULE 1: Differential calculus – II

Taylor Series:

Taylor series expansion about the point $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Maclaurin's series Expansion:

If $a = 0$, then we have the following expansion for $f(x)$ around the origin (0,0)

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

SOLVE THE FOLLOWING (5 Marks)

1. Expand $f(x) = tanx$ in powers of $\left(\mathrm{x} - \frac{\pi}{4}\right)$ upto third degree terms.
2. Using Maclaurin's series expand $f(x) = log(secx)$ up to the term containing x^4.
3. Expand $f(x) = sin(\mathrm{e}^{\mathrm{x}} - 1)$ in powers of x upto the terms containing x^4.
4. Obtain the Maclaurin's expansion of $sin^{-1}x$ upto the term containing x^3.
5. Expand e^{sinx} using Maclaurin's theorem up to the terms containing x^4.
6. Expand $f(x) = e^x$ by the Maclaurin's series up to the term containing of 4^{th} term.
7. Expand $\log(1 + \mathrm{e}^{\mathrm{x}})$ using Maclaurin's series up to the term containing x^4.

SOLVE THE FOLLOWING (7 or 8 marks)

1. Using Maclaurin's series expand $log(secx)$ up to sixth degree terms.
2. Expand $\log(1 + \mathrm{cosx})$ using Maclaurin's series up to the term containing x^4.
3. Using Maclaurin's theorem prove that $\sqrt{1 + \mathrm{sin2x}} = 1 + \mathrm{x} - \frac{\mathrm{x}^2}{2} - \frac{\mathrm{x}^3}{6} + \frac{\mathrm{x}^4}{24} \mp \cdots$
4. Expand by using Maclaurin's series for the function $\log(1 + \mathrm{sinx})$ up to fourth degree terms.

Indeterminate forms

While evaluating certain limits, we come across expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0 and\, 1^\infty$ which do not represent any value. Such expressions are called Indeterminate Forms. There is no actual value for these expressions.

We can evaluate such limits that lead to indeterminate forms by using L'Hospital's Rule.

L'Hospital's Rule

If f(x) and g(x) are two functions such that

(i) $\lim_{x\to a} f(x) = 0 \; and \; \lim_{x\to a} g(x) = 0$

(ii) $f'(x) \; and g'(x) \; exist \; and \; g'(a) \neq 0$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

(Differentiate both numerator and denominator separately)

The above rule can be extended, i.e, if

$f'(a) = 0 \; and \; g'(a) = 0 \; then \; \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = ...$

Limits of the form $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$: The indeterminate forms of this type can be evaluated using L'Hospital's Rule by replacing the functions by their derivatives.

Limits of the form: $0^0, \infty^0 \; and \; 1^\infty$

To evaluate such limits, where function to the power of function exists, we call such an expression as some constant, then take logarithm on both sides of the function whose limit is required and rewrite the expressions to get $\frac{0}{0} \; or \; \frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.

Working procedure is as follows:

- To evaluate the limits of the form $0^0, \infty^0 \; and \; 1^\infty$ i.e, where function to the power of function exists, first identify the form
- Take such an expression as some constant A.

- Take logarithm on both sides and rewrite the expressions to get $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ form and then apply the L'Hospital's Rule.
- See any possible simplification can be done
- Continue till we obtain the finite value

Evaluate the following (5 Marks)

1. $\lim_{x\to 1} x^{\left(\frac{1}{1-x}\right)}$
2. $\lim_{x\to a} \left(2-\frac{x}{a}\right)^{\tan\left(\pi x/2a\right)}$
3. $\lim_{x\to 0} x^{x}$
4. $\lim_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$
5. $\lim_{x\to 0} \left(\frac{sinx}{x}\right)^{\frac{1}{x^2}}$
6. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$
7. $\lim_{x\to 0} \left(\frac{tanx}{x}\right)^{\frac{1}{x}}$
8. $\lim_{x\to \frac{\pi}{2}} (sinx)^{tanx}$
9. $\lim_{x\to 0} (x)^{sinx}$
10. $\lim_{x\to 0} (cotx)^{tanx}$
11. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$
12. $\lim_{x\to 0} \left(\frac{1}{x}\right)^{2sinx}$
13. $\lim_{x\to 0} \frac{a^x-b^x}{x}$
14. $\lim_{x\to 0} \left(\frac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}}$
15. $\lim_{x\to 0} (cosx)^{\frac{1}{x^2}}$
16. $\lim_{x\to \frac{\pi}{4}} (tanx)^{tan2x}$
17. $\lim_{x\to \frac{\pi}{2}} secx^{cotx}$

Radius of Curvature

(5, 6 or 7 Marks)

1. Find the radius of curvature at any point P(x, y) on the parabola $y^2 = 4ax$
2. Find the radius of curvature of the curve $y = alogsec\left(\frac{x}{a}\right)$ at any point (x,y)
3. Find the radius of curvature of the curve $x^4 + y^4 = 2$ at the point $(1,1)$
4. Find the radius of curvature of $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on it.
5. Find the radius of curvature of the curve $x^2 y = a(x^2 + y^2)$ at the point $(-2a, 2a)$.
6. Find the radius of curvature of the curve $a^2 y = x^3 - a^3$ at the point where the curve cuts x-axis or at the point $(a, 0)$.
7. Find the radius of curvature for the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point where the curve cuts x-axis or at the point $(a, 0)$.
8. Find the radius of curvature of the curve $x = a(cost + tsint)\ and\ y = a(sint - tcost)$
9. Find the radius of curvature of the curve $x = a\left[cost + logtan\left(\frac{t}{2}\right)\right], y = asint.$
10. Find the radius of curvature of the curve $x = a(t + sint)\ and\ y = a(1 - cost)$
11. Find the radius of curvature of the curve $r^n = a^n sinn\theta$
12. Find the radius of curvature of the curve $r^n = a^n \cos n\theta$
13. S.T for the curve $r(1 - cos\theta) = 2a,\ \rho^2\ \ varies\ \ as\ \ r^3$
14. Prove that $\frac{\rho^2}{r}$ is a constant for the cardioid $r = a(1 + cos\theta)$ where ρ is the radius of curvature.
15. For the curve $r = ae^{\theta cot\propto}$, prove that $\frac{\rho}{r}\ is\ constant$
16. Find the radius of curvature of the curve $pa^2 = r^3$