1. Form the PDE by eliminating the arbitrary constants in the following.

$z = (x+a)(y+b)$ — (1)

diff partially w.r.t 'x'   w.r.t 'y'

$\frac{\partial z}{\partial x} = y+b$ $\frac{\partial z}{\partial y} = x+a$

$p = y+b$ — (2) $q = x+a$ — (3)

sub (2) & (3) in (1)

Thus, $z = pq$ is the required PDE

(2) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

diff partially w.r.t 'x'   w.r.t 'y'

$\cancel{2}\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2}$ $\cancel{2}\frac{\partial z}{\partial y} = \frac{\cancel{2}y}{b^2}$

$\frac{p}{x} = \frac{1}{a^2}$ — (1) $\frac{q}{y} = \frac{1}{b^2}$ — (2)

sub (1) & (2) in Q^n

$2z = x^2 \times \frac{p}{x} + y^2 \times \frac{q}{y} = px + qy$

3. $z = xy + y\sqrt{x^2-a^2} + b$

$\frac{\partial z}{\partial x} = y + \frac{y}{2\sqrt{x^2-a^2}} \times 2x$ $\quad$ $\frac{\partial z}{\partial y} = x + \sqrt{x^2-a^2}$

$q - x = \sqrt{x^2-a^2}$ — ②

$p - y = \frac{xy}{\sqrt{x^2-a^2}}$ — ①

② in ①

$p - y = \frac{xy}{q - x}$

$(p-y)(q-x) = xy$

$pq - qy - xp + \cancel{xy} = \cancel{xy}$

$pq - xp - yq = 0$

Form the PDE by eliminating arbitrary function

3. $z = f(x^2+y^2)$

$\frac{\partial z}{\partial x} = f'(x^2+y^2)\, 2x$ $\quad$ $\frac{\partial z}{\partial y} = f'(x^2+y^2)\, 2y$

$\frac{p}{2x} = f'(x^2+y^2)$ $\quad$ $\frac{q}{2y} = f'(x^2+y^2)$

$\frac{p}{2x} = \frac{q}{2y}$

$py - qx = 0$

4. $z = x + y + f(xy)$

$\frac{\partial z}{\partial x} = 1 + f'(xy)y$ $\quad$ $\frac{\partial z}{\partial y} = 1 + f'(xy)x$

$\frac{p-1}{y} = f'(xy)$ $\quad$ $\frac{q-1}{x} = f'(xy)$

$\frac{p-1}{y} = \frac{q-1}{x}$

$xp - x = yq - y$

$x - y + yq - xp = 0$

5. Find the PDE of the family of all spheres whose centres lie on the plane $z=0$ and have a constant value 'r'.

The coordinates of the centre of the sphere can be taken as $(a, b, 0)$ where a and b are arbitrary, r is constant radius.

The eq^n of the sphere is,

$$(x-a)^2+(y-b)^2+z^2=r^2$$

diff w.r.t 'x'

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$

$$zp=-(x-a)$$

sq on b.s

$$z^2p^2=(x-a)^2$$

diff w.r.t 'y'

$$2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$$zq=-(y-b)$$

sq on b.s

$$z^2q^2=(y-b)^2$$

$$\therefore \text{Q}^n : z^2p^2+z^2q^2+z^2=r^2$$

6. $z=e^y f(x+y)$

$$\frac{\partial z}{\partial x}=e^y f'(x+y)=p$$

$$\frac{\partial z}{\partial y}=e^y f'(x+y)+e^y f(x+y)$$

$$q=e^y f'(x+y)+z$$

$$q-z=e^y f'(x+y)$$

$$\boxed{q-z=p}$$

7. $lx+my+nz=\phi(x^2+y^2+z^2)$

diff w.r.t 'x'

$$l+np=\phi'(x^2+y^2+z^2)\left[2x+2z\frac{\partial z}{\partial x}\right]$$

$$\frac{l+np}{2(x+zp)}=\phi'(x^2+y^2+z^2)$$

diff w.r.t 'y'

7. Form the PDE by eliminating the arbitrary functions $lx+my+nz=\phi(x^2+y^2+z^2)$

Solution

diff w.r.t 'x':

$$l+n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left[2x+2z\frac{\partial z}{\partial x}\right]$$

$$l+np = \phi'(x^2+y^2+z^2)(2x+2zp)$$

$$\frac{l+np}{2(x+zp)} = \phi'(x^2+y^2+z^2) \quad -①$$

diff w.r.t y partially,

$$m+n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left(2y+2z\frac{\partial z}{\partial y}\right)$$

$$m+nq = \phi'(x^2+y^2+z^2)\, 2(y+zq)$$

$$\frac{m+nq}{2(y+zq)} = \phi'(x^2+y^2+z^2) \quad -②$$

from ① & ②,

$$\frac{l+np}{\cancel{2}(x+zp)} = \frac{m+nq}{\cancel{2}(y+zq)}$$

$$(l+np)(y+zq) = (m+nq)(x+zp)$$

$$ly + lzq + npy + npzq = mx + mzp + nqx + npqz$$

$$mzp - npy + nqx - lzq = ly - mx$$

$(mz-ny)p + (nx-lz)q = ly - mx$ is the required PDE.

(1M)
8. Form the PDE by eliminating arbitrary functions in the following.

$$\phi(x+y+z,\ x^2+y^2-z^2) = 0$$

Let $\phi(u,v) = 0$ — ①

where $u = x+y+z$

$v = x^2+y^2-z^2$

diff u, w.r.t 'x' & 'y'

$$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1+p$$

$$\frac{\partial u}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1+q$$

*

diff v w.r.t x & y

$$\frac{\partial v}{\partial x} = 2x - 2z\frac{\partial z}{\partial x} = 2x - 2pz$$

$$\frac{\partial v}{\partial y} = 2y - 2z\frac{\partial z}{\partial y} = 2y - 2zq$$

*

differentiating ① w.r.t x & y by applying chain rule,

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} \text{ — ②}$$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \qquad \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} \text{ — ③}$$

dividing ② & ③, we get,

$$\frac{\cancel{\frac{\partial \phi}{\partial u}}\frac{\partial u}{\partial x}}{\cancel{\frac{\partial \phi}{\partial u}}\frac{\partial u}{\partial y}} = \frac{-\cancel{\frac{\partial \phi}{\partial v}}\frac{\partial v}{\partial x}}{-\cancel{\frac{\partial \phi}{\partial v}}\frac{\partial v}{\partial y}}$$

$$\frac{1+p}{1+q} = \frac{\cancel{2}(x-pz)}{\cancel{2}(y-zq)} \quad [\text{from } *]$$

$$(1+p)(y-zq) = (x-pz)(1+q)$$

$$y + py - zq - \cancel{pqz} = x + xq - pz - \cancel{pqz}$$

$$x - y + (x+z)q - (y+z)p = 0$$

9. Form the PDE by eliminating the arbitrary functions in

$\phi(xy+z^2, x+y+z)=0$

solution

By data $\phi(u,v)=0$ — (1)

where $u = xy+z^2$

$v = x+y+z$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= y+2z\frac{\partial z}{\partial x} = y+2zp\\ \frac{\partial u}{\partial y} &= x+2z\frac{\partial z}{\partial y} = x+2qz\\ \frac{\partial v}{\partial x} &= 1+\frac{\partial z}{\partial x} = 1+p\\ \frac{\partial v}{\partial y} &= 1+\frac{\partial z}{\partial y} = 1+q\end{aligned}\right\} - *$$

diff $\phi(u,v)$ w.r.t 'x' & 'y'

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}=0 \Rightarrow \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}=+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}=0 \qquad \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}=+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{y+2zp}{x+2zq}=\frac{1+p}{1+q}$$

$$y+yq+2pz+2pqz=x+px+2zq+2pqz$$

$$x-y+p(x-2z)+q(2z-y)=0$$

10. Form PDE by Eliminat arb func in the following

$\phi(x^2+y^2+z^2, z^2-2xy)=0$

Let, $\phi(u,v)=0$

$u = x^2+y^2+z^2$

$v = z^2-2xy$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x+2z\frac{\partial z}{\partial x} = 2(x+zp)\\ \frac{\partial u}{\partial y} &= 2y+2z\frac{\partial z}{\partial y} = 2(y+zq)\end{aligned}\right\}* \qquad \left.\begin{aligned}\frac{\partial v}{\partial x} &= 2z\frac{\partial z}{\partial x}-2y = 2(zp-y)\\ \frac{\partial v}{\partial y} &= 2z\frac{\partial z}{\partial y}-2x = 2(zq-x)\end{aligned}\right\}*$$

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}=0$$

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$\frac{\cancel{2}(x+zp)}{\cancel{2}(y+zq)}=\frac{\cancel{2}(zp-y)}{\cancel{2}(zq-x)}$$

$$(x+zp)(zq-x) = (zp-y)(y+zq)$$

$$xzq - x^2 + \cancel{z^2pq} - \underline{xzp} = \underline{yzp} - y^2 + \cancel{z^2pq} - yzq$$

$x^2 - y^2 + pz(y+x) - qz(y+x) = 0$ is the required PDE

11. $z = f(y+x) + g(y+2x)$

$$p = \frac{\partial z}{\partial x} = f'(y+x) + g'(y+2x) \times 2$$

$$q = \frac{\partial z}{\partial y} = f'(y+x) + g'(y+2x)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(y+x) + 4g''(y+2x) \quad -(1)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''(y+x) + 2g''(y+2x) \quad -(2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(y+x) + g''(y+2x) \quad -(3)$$

subtract, (1) - (2)

$\Rightarrow r - s = 2g''(y+2x)$

$\Rightarrow s - t = g''(y+2x)$

dividing (4) & (5),

$$\frac{r-s}{s-t} = 2$$

$r - s = 2(s-t)$

$r - s = 2s - 2t$

$r - 3s + 2t = 0$

Thus $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$ is the required PDE.

3. $y^2p - xyq = x(z-2y)$

The given equation is of the form

$Pp - Qq = R$

The A.E is $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}$

equating 1st & 2nd term

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = -\frac{dy}{x}$$

$$-x\,dx = y\,dy$$

$$\frac{-x^2}{2} = \frac{y^2}{2} + c_1$$

$$c_1 = \frac{-x^2+y^2}{2}$$

$$2c_1 = -x^2+y^2$$

2nd & 3rd term

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$-\frac{dy}{y} = \frac{dz}{z-2y}$$

$$-(z-2y) = y\,dz$$

$$\cancel{-\log y = \log(z-2y)}$$

Solution of Lagrange's Linear PDE

1. Solve $P\cot x + q\cot y = \cot z$

Solution

The given eq^n is of the form

$$Pp + Qq = R$$

The A.E are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\tan x\, dx = \tan y\, dy = \tan z\, dz$$

taking 1st & 2nd term,

$$\tan x\, dx = \tan y\, dy$$

Integrating on b.s

$$\int \tan x\, dx = \int \tan y\, dy$$

$$\log(\sec x) = \log(\sec y) + \log c_1$$

$$\log \frac{\sec x}{\sec y} = \log c_1$$

$$\frac{\sec x}{\sec y} = c_1$$

taking 2nd and 3rd term,

$$\tan y\, dy = \tan z\, dz$$

Integrating on b.s

$$\log(\sec y) = \log(\sec z) + \log c_2$$

$$\log \frac{\sec y}{\sec z} = \log c_2$$

$$c_2 = \frac{\sec y}{\sec z}$$

Thus general solution is,

$$\phi(c_1, c_2) = 0$$

$$\phi\left(\frac{\sec x}{\sec y}, \frac{\sec y}{\sec z}\right) = 0$$

2. Solve $x^2 p + y^2 q = z$

Solution

$$Pp + Qq = R.$$

The A.E are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z}$$

taking 1st & 2nd terms

$$\int \frac{dx}{x^2} = \int y^{-2}\,dy$$

$$\frac{x^{-2+1}}{-1} = \frac{y^{-1}}{-1} + C_1$$

$$-x^{-1} = -y^{-1} + C_1$$

$$C_1 = \frac{1}{y} - \frac{1}{x}$$

Taking 2nd and 3rd terms

$$\int \frac{dy}{y^2} = \int \frac{dz}{z}$$

$$-y^{-1} = \log z + C_2$$

$$C_2 = -\frac{1}{y} - \log z$$

Thus $\phi\left(\frac{1}{y} - \frac{1}{x}, -\left(\frac{1}{y} + \log z\right)\right) = 0$

is the general solution of PDE

5. $y^2p - xyq = x(z-2y)$

The given eq^n is of the form,

$Pp + Qq = R$

$$\therefore \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

taking 1st & 2nd term,

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$+x\,dx = -y\,dy$$

$$\frac{x^2}{2} + \frac{y^2}{2} = C_1$$

$$x^2 + y^2 = 2C_1$$

taking 2nd & 3rd term,

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$dy(z-2y) = -y\,dz$$

$$z\,dy + y\,dz = 2y\,dy$$

$$d(zy) = 2y\,dy$$

integrating,

$$zy = y^2 + C_2$$

$$C_2 = zy - y^2$$

Thus $\phi(x^2+y^2, zy-y^2) = 0$

4. Solve $y^2zp = x^2(zq + y)$

$y^2zp - x^2zq = yx^2$

The A.E is, $\frac{dx}{y^2z} = \frac{dy}{-x^2z} = \frac{dz}{yx^2}$

taking 1st & 2nd terms,

$\frac{dx}{y^2\cancel{z}} = \frac{dy}{-x^2\cancel{z}}$

$+x^2dx = y^2dy$

$\frac{x^3}{3} = -\frac{y^3}{3} + c_1$

$C_1 = \frac{x^3}{3} + \frac{y^3}{3}$

$C_1 = \frac{x^3+y^3}{3}$

taking 2nd & 3rd terms,

$\frac{dy}{-x^2z} = \frac{dz}{yx^2}$

$ydy = -zdz$

$\frac{y^2}{2} = -\frac{z^2}{2} + C_2$

$C_2 = \frac{y^2}{2} + \frac{z^2}{2}$

$C_2 = \frac{y^2+z^2}{2}$

∴ The general solution of the PDE is,

$\phi(x^3+y^3, y^2+z^2) = 0$

5. Solve $(y^2+z^2)p + x(yq - z) = 0$

$\frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz}$ —①

taking 3rd & 2nd terms,

$\frac{dz}{xz} = \frac{dy}{xy}$

integrating on B.S

$\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_1$

$\log\left(\frac{y}{z}\right) = \log c_1$

$\boxed{C_1 = \frac{y}{z}}$

Using multipliers $x, -y, -z$ in each ratio of ①

$\frac{xdx - ydy - zdz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{xdx - ydy - zdz}{0}$

$xdx - ydy - zdz = 0$

integrating,

$\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = C_2$

Thus,

$\phi\left(\frac{y}{z}, x^2 - y^2 - z^2\right) = 0$

<u>Note</u>: we have a property in ratio and proportion that a ratio

$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ is also equal to

$\frac{K_1 a_1 + K_2 a_2 + K_3 a_3}{K_1 b_1 + K_2 b_2 + K_3 b_3}$

6. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

The given eqn is of the form.

$Pp + Qq = R$

AE is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\therefore \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$

Using multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating,

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C_1$$

$$C_1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Using multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{yx} + \cancel{zx} - \cancel{zy}}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log C_2$$

$$\log xyz = \log C_2$$

$$\boxed{xyz = C_2}$$

Thus $\phi(C_1, C_2) = \phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right)$ is the required general solution.

7 Solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} + (mx - ly) = 0$

Solution

AE: $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$ — (1)

Using the multipliers l, m, n

$$\frac{l\,dx + m\,dy + n\,dz}{\cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{mlz} + \cancel{nly} - \cancel{mnx}} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$l\,dx + m\,dy + n\,dz = 0$$

On Integrating we get,

$$lx + my + nz = C_1$$

$\phi(lx + my + nz, x^2 + y^2 + z^2)$

Using multipliers x, y, z

$$\frac{x\,dx + y\,dy + z\,dz}{\cancel{mxz} - \cancel{nxy} + \cancel{nxy} - \cancel{lyz} + \cancel{lyz} - \cancel{mxz}}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$x^2 + y^2 + z^2 = 2C_2$$

8. Solve $(x^2-y^2-z^2)p+2xyq=2xz$

AE: $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

taking 2nd & 3rd terms,

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

$$\log y = \log z + \log c_1$$

$$\boxed{c_1 = \frac{y}{z}}$$

Using multipliers x, y, z

$$\frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{xdx+ydy+zdz}{x^3+xy^2+xz^2}$$

$$= \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

Let us consider

$$\frac{dy}{2xy} = \frac{2xdx+ydy+zdz}{x(x^2+y^2+z^2)} \qquad \int \frac{f'(x)}{f(x)}dx = \log f(x) + c$$

$$\frac{dy}{y} = \frac{2x\,dx}{x^2+y^2+z^2} + \frac{2y\,dy}{x^2+y^2+z^2} + \frac{2z\,dz}{x^2+y^2+z^2}$$

$$\log y = \log(x^2+y^2+z^2) + \log c_2$$

$$c_2 = \frac{y}{x^2+y^2+z^2}$$

Thus a general solution is

$$\phi\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0$$

Solution of homogeneous PDE

Case 1: If roots are real and equal.

$C.F = A e^{m_1 x} + B e^{m_2 x} + \ldots$

Case 2: If roots are real and equal

$C.F = (A + Bx) e^{m_1 x}$

Case 3: If roots are imaginary,

$\alpha \pm i\beta$

$C.F = e^{\alpha x}[A \cos\beta x + B \sin\beta x]$

1. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

Let us suppose that z is a function of x only, The given PDE assumes the form of ODE,

$$\frac{d^2 z}{dx^2} + z = 0$$

$$D^2 z + z = 0$$

$$(D^2 + 1) z = 0$$

$AE: m^2 + 1 = 0$

$m = \pm i$

Solution of the ODE is given by,

$$z = C_1 \cos x + C_2 \sin x$$

we get Solution of the PDE by replacing C_1 and C_2 by function of y

Hence, solution of the PDE is given by

$$z = f(y) \cos x + g(y) \sin x \quad \text{— (1)}$$

Given $x = 0$, $z = e^y$ & $\frac{\partial z}{\partial x} = 1$

$e^y = f(y) + 0$

$\boxed{f(y) = e^y}$

$\frac{\partial z}{\partial x} = -f(y) \sin x + g(y) \cos x$

$1 = 0 + g(y)$

$\boxed{g(y) = 1}$

$(1) \Rightarrow z = e^y \cos x + \sin x$

2. Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given $\frac{\partial z}{\partial x} = a\sin y$ and $z=0$ when $x=0$

$D^2 z - a^2 z = 0$

$(D^2 - a^2) z = 0$

AE: $m^2 - a^2 = 0$

$m = \pm a$

$CF = C_1 e^{ax} + C_2 e^{-ax}$

$z = f(y) e^{ax} + g(y) e^{-ax}$ — ⓐ

$0 = f(y) + g(y)$ — ①

$\frac{\partial z}{\partial x} = a f(y) e^{ax} - a g(y) e^{-ax}$

$\cancel{a} \sin y = \cancel{a}[f(y) - g(y)]$

$\sin y = f(y) - g(y)$ — ②

Adding ① & ②

$f(y) + \cancel{g(y)} + f(y) - \cancel{g(y)} = \sin y$

$f(y) = \frac{\sin y}{2}$

① $\Rightarrow$ $g(y) = -\frac{\sin y}{2}$

ⓐ $\Rightarrow$ $z = \frac{\sin y}{2} e^{ax} - \frac{\sin y}{2} e^{-ax}$

$z = \frac{\sin y}{2}\left[e^{ax} - e^{-ax}\right]$

$z = \sin y \sinh ax$

3. Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$ given that $z = 1$ and $\frac{\partial z}{\partial x} = y$ when $x = 0$

Solution

Suppose z is a function of x only.

$\frac{d^2 z}{dx^2} + 3\frac{dz}{dx} - 4z = 0$

$(D^2 + 3D - 4) z = 0$

$(D+4)(D-1) = 0$

$z = C_1 e^{-4x} + C_2 e^{x}$

Replacing C_1 & C_2 with functions of y

$z = f(y) e^{-4x} + g(y) e^{x}$ — ①

$z=1 \quad x=0$

$1 = f(y) + g(y)$ — ①

$\frac{\partial z}{\partial x} = y \quad x=0$

$\frac{\partial z}{\partial x} = -4f(y)e^{-4x} + g(y)e^{x}$

$y = -4f(y) + g(y)$ — ②

① - ②

$1-y = f(y)+g(y) + 4f(y) - g(y)$

$1-y = 5f(y)$

$f(y) = \frac{1-y}{5}$

$1 = \frac{1-y}{5} + g(y)$

$1 - \frac{1-y}{5} = g(y)$

$\frac{5-1+y}{5} = g(y)$

$\boxed{g(y) = \frac{4+y}{5}}$

$z = \frac{1-y}{5} e^{-4x} + \frac{4+y}{5} e^{x}$

Solve $\frac{\partial^2 z}{\partial y^2} = z$ $\quad z=0$ & $\frac{\partial z}{\partial y} = \sin x$ when $y=0$

$(D^2-1)z=0$

$D = \pm 1$

$z = c_1 e^{y} + c_2 e^{-y} \quad = f(x)e^{y} + g(x)e^{-y}$

$\frac{\partial z}{\partial y} = c_1 e^{y} - c_2 e^{-y}$

$z=0 \quad y=0$

$0 = c_1 + c_2$

$y=0 \quad \frac{\partial z}{\partial y} = \sin x$

$\sin x = c_1 - c_2$

Adding

$c_1 + \cancel{c_2} + c_1 - \cancel{c_2} = \sin x \qquad g(\,)c_2 = -\frac{\sin x}{2}$

$2c_1 = \sin x$

$f(\,) = c_1 = \frac{\sin x}{2}$

$z = \sin x\left[\frac{e^{y} - e^{-y}}{2}\right] = \sin x \sinh y$