Taylor Series :

Taylor series expansion about the point $x = a$ is given by.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Maclaurin's Series

If $a = 0$, then we have the following expansion for $f(x)$ around the origin $(0,0)$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

(5M)

1. Expand $f(x) = \tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$ upto third degree powers.

Solution : The general form of Taylor's series expansion of $f(x)$ about the point 'a' is

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

upto third degree terms —①

Here $a = \frac{\pi}{4}$

$$f(x) = f\left(\frac{\pi}{4}\right) + \frac{(x-a)}{1!}f'(\pi/4) + \frac{(x-\pi/4)^2}{2!}f''(\pi/4) + \frac{(x-\pi/4)^3}{3!}f'''(\pi/4) \quad \text{—②}$$

$f(x) = \tan x$

$f'(x) = \sec^2 x = 1 + \tan^2 x = 1 + (f(x))^2$

$f''(x) = 2f(x)\,f'(x)$

$f'''(x) = 2(f'(x))^2 + 2f(x)f''(x)$

$f(\pi/4) = \tan\frac{\pi}{4} = 1$

$f'(\pi/4) = 1 + (f(\pi/4))^2 = 1 + 1^2 = 2$

$f''(\pi/4) = 2 \times 1 \times 2 = 4$

$f'''(\pi/4) = 2 \times 4 + 2 \times 1 \times 4 = 16$

$$② \Rightarrow f(x) = 1 + \frac{(x-\pi/4)}{1!} \times 2 + \frac{(x-\pi/4)^2}{2!} \times 4 + \frac{(x-\pi/4)^3}{3!} \times 16$$

$$\tan x = 1 + 2(x-\pi/4) + 2(x-\pi/4)^2 + \frac{8}{3}(x-\pi/4)^3$$

(5M)

2. Using Maclaurin's series Expand $f(x)=\log(\sec x)$ upto term containing x^4.

<u>Solution</u>: $f(x)=\log(\sec x)$, $f(0)=\log 1=0$

$f'(x)=\frac{1}{\sec x}\sec x\tan x=\tan x$ $\quad f'(0)=\tan 0=0$

$f''(x)=\sec^2 x=1+\tan^2 x$ $\quad f''(0)=1+0=1$

$=1+(f'(x))^2$

$f'''(x)=2f'(x)f''(x)$ $\quad f'''(0)=2f'(0)f''(0)$

$=2\times 0=0$

$f^{iv}(x)=2(f''(x))^2+2f'(x)f'''(x)$ $\quad f^{iv}(0)=2\times 1+2\times 0$

$=2$

$$\log(\sec x)=0+x\times 0+\frac{x^2}{2!}\times 1+0+\frac{x^4}{4!}\times 2$$

$$=\frac{x^2}{2}+\frac{x^4}{12}$$

3. Expand $f(x)=\sin(e^x-1)$ in powers of x upto the terms containing x^4.

<u>Solution</u>

$f(x)=\sin(e^x-1)$ $\quad f(0)=\sin(1-1)=0$

$f'(x)=\cos(e^x-1)e^x$ $\quad f'(0)=\cos 0 . e^0=1$

$f''(x)=-\sin(e^x-1)e^x+\cos(e^x-1)e^x$ $\quad f''(0)=1$

$f''(x)=-e^{2x}f(x)+f'(x)$

$f'''(0)=-0-1+1$

$f'''(x)=-2e^{2x}f(x)-e^{2x}f'(x)+f''(x)$ $\quad =0$

$f^{iv}(x)=-4e^{2x}f(x)-2e^{2x}f'(x)-e^{2x}f''(x)-2e^{2x}f'(x)+f'''(x)$

$=-0-2-1-2+0=-3-2=-5$

$$\sin(e^x-1)=0+x\times 1+\frac{x^2}{2!}\times 1+0-5\frac{x^4}{4!}$$

$$=x+\frac{x^2}{2}-\frac{5x^4}{24}$$

4. Obtain Maclaurin series expansion $y = \sin^{-1}x$ upto the term containing x^3

Solution: $y = \sin^{-1}x$

$$y' = \frac{1}{\sqrt{1-x^2}}$$

method 1:

$y' = (1-x^2)^{-1/2}$

$y'' = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)$

$= x(1-x^2)^{-3/2}$

$y''' = (1-x^2)^{-3/2} + x\left(-\frac{3}{2}\right)(1-x^2)^{-5/2}(-2x)$

$y''' = (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$

Method 2: $\sqrt{1-x^2}\, y' = 1$

sq on b.s

$(1-x^2)y'^2 = 1$

$(1-x^2)2y'y'' - 2xy'^2 = 0$

$2y'[(1-x^2)y'' - xy'] = 0$ — ①

diff wrt 'x'

$-2xy'' + (1-x^2)y''' - y' - xy'' = 0$ — ②

$y(0) = \sin^{-1}(\sin 0) = 0$

$y'(0) = 1$

$y''(0) = 0$

$y'''(0) = 1$

$\therefore$ Maclaurin's series expansion is,

$$\sin^{-1}x = 0 + x + 0 + \frac{x^3}{3!} \times 1$$

$$\sin^{-1}x = x + \frac{x^3}{6}$$

5. Expand $e^{\sin x}$ using Maclaurin's theorem up to the term containing x^4.

solution: $f(x) = e^{\sin x}$

$f'(x) = e^{\sin x}\cos x = f(x)\cos x$

$f''(x) = f'(x)\cos x - f(x)\sin x$

$f'''(x) = f''(x)\cos x - f'(x)\sin x$
$\quad - f(x)\cos x - f'(x)\sin x$
$\quad = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x$

$f^{IV}(x) = f'''(x)\cos x - f''(x)\sin x$
$\quad - 2f''(x)\sin x - 2f'(x)\cos x$
$\quad - f'(x)\cos x + f(x)\sin x$

$f^{IV}(x) = f'''(x)\cos x - 3f''(x)\sin x$
$\quad - 3f'(x)\cos x + f(x)\sin x$

$f(0) = e^0 = 1$

$f'(0) = 1$

$f''(0) = 1$

$f'''(0) = 1 - 1 = 0$

$f^{IV}(0) = 0 - 3 = -3$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} + 0 - 3\frac{x^4}{4!}$$

$$= 1 + x + \frac{x^2}{2!} - \frac{x^4}{8}$$

⑥ Expand $\log(1+e^x)$ using Maclaurin's series up to the term containing x^4.

$y = \log(1+e^x)$ $\qquad y(0) = \log 2$

$y' = \dfrac{e^x}{1+e^x}$ $\qquad y'(0) = \dfrac{1}{2}$

$(1+e^x)\,y' = e^x$

$(1+e^x)y'' + e^x y' = e^x$ — ②

② $\Rightarrow 2y'' + \frac{1}{2} = 1$ $\qquad \boxed{y'' = \frac{1}{4}}$

diff again

$(1+e^x)y''' + e^x y'' + e^x y' + e^x y'' = e^x$

$(1+e^x)y''' + 2e^x y'' + e^x y' = e^x$ — ③

③ $\Rightarrow 2y''' + 2\times\frac{1}{4} + \frac{1}{2} = 1$

$2y''' = 1 - 1$ $\qquad \boxed{y''' = 0}$

diff again

$(1+e^x)y^{iv} + e^x y''' + 2e^x y'' + 2e^x y''' + e^x y'' + e^x y' = e^x$

$(1+e^x)y^{iv} + 3e^x y''' + 3e^x y'' + e^x y' = e^x$ $\qquad 1 - \frac{5}{4} = \frac{-1}{4}$

$2y^{iv} + 0 + 3\times\frac{1}{4} + \frac{1}{2} = 1$ $\qquad \boxed{y^{iv} = -\frac{1}{8}}$

$\log(1+e^x) = \log 2 + \dfrac{x}{2} + \dfrac{x^2}{2}\times\dfrac{1}{4} + 0 - \dfrac{1}{8}\,\dfrac{x^4}{6\times 4}$

$= \log 2 + \dfrac{x}{2} + \dfrac{x^2}{8} - \dfrac{x^3}{48\times 4} = \log 2 + \dfrac{x}{2} + \dfrac{x^2}{8} - \dfrac{x^3}{192}$ //

7. Expand $f(x) = e^x$ by Maclaurin's series up to the term containing 4^{th} term

Solution: $f(x) = e^x$ $\qquad f(0) = 1$

$f'(x) = e^x$ $\qquad f'(0) = 1$

$f''(x) = e^x$ $\qquad f''(0) = 1$

$f'''(x) = e^x$ $\qquad f'''(0) = 1$

$f^{iv}(x) = e^x$ $\qquad f^{iv}(0) = 1$

$e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$

(7 or 8 marks)

1. Using Maclaurin's series expand $\log(\cos x)$ up to sixth degree terms.

Solution: $y = \log(\sec x)$ $\qquad y(0) = \log 1 = 0$ $\qquad \boxed{y(0) = 0}$

$y' = \dfrac{1}{\sec x}\sec x\tan x$ $\qquad y'(0) = \tan 0 = 0$ //

$= \tan x$

$y'' = \sec^2 x = 1 + \tan^2 x = 1 + (f'(x))^2$ $\qquad \boxed{y''(0) = 1}$

$y''' = 2f'(x)f''(x)$ $\qquad \boxed{y'''(0) = 0}$

$y^{iv} = 2(f''(x))^2 + 2f'(x)f'''(x)$ $\qquad y^{iv}(0) = 2$.

$y^{v} = 4f''(x)f'''(x) + 2f''(x)f'''(x) + 2f'(x)f^{iv}(x)$

$= 6f''(x)f'''(x) + 2f'(x)f^{iv}(x)$ $\qquad y^{v} = 0$

$$y^{vi} = 6(f'''(x))^2 + 6f''(x)f^{iv}(x) + 2f''(x)f^{iv}(x) + 2f'(x)f^{v}(x)$$

$$y^{vi} = 0 + 6\times2 + 2\times2 + 0$$

$$y^{vi} = 16$$

$$\log(\sec x) = 0 + 0 + \frac{x^2}{2!}\times1 + 0 + \frac{x^4}{4!}\times2 + 0 + \frac{x^6}{6!}\times16$$

$$= \frac{x^2}{2!} + \frac{x^4}{12} + \frac{x^6}{\cancel{6}\times5\times\cancel{4}\times3\times\cancel{2}}\times16$$

$$= \frac{x^2}{2!} + \frac{x^4}{12} + \frac{x^6}{45}$$

2. Expand $\log(1+\cos x)$ using Maclaurin's series up to the term containing x^4.

$y = \log(1+\cos x)$ — (1)

$y' = \dfrac{-\sin x}{1+\cos x}$

$y' = -\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$

$y' = -\tan\frac{x}{2}$ — (2)

$y'' = -\sec^2\frac{x}{2}\times\frac{1}{2}$

$y'' = -\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$= -\frac{1}{2}(1+y'^2)$ — (3)

$y''' = -\frac{1}{2}\,2y'y''$

$y''' = -y'y''$ — (4)

$y^{iv} = -y''^2 - y'y'''$ — (5)

$y(0) = \log 2$

$y'(0) = 0$

$y''(0) = -\frac{1}{2}$

$y'''(0) = 0$

$y^{iv}(0) = -\left(-\frac{1}{2}\right)^2 = -\frac{1}{4}$

$\therefore$ Maclaurin's Series

$$\log(1+\cos x) = \log 2 - \frac{1}{2}\frac{x^2}{2!} - \frac{1}{4}\frac{x^4}{4!}$$

$$= \log 2 - \frac{x^2}{4} - \frac{x^4}{96}$$

3. Using Maclaurin's theorem P.T $\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots$

Let $f(x) = \sqrt{1+\sin 2x} = \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}$

$f(x) = \sin x + \cos x$	$f(0) = 1$
$f'(x) = \cos x - \sin x$	$f'(0) = 1$
$f''(x) = -\sin x - \cos x$	$f''(0) = -1$
$f'''(x) = -\cos x + \sin x$	$f'''(0) = -1$
$f^{iv}(x) = \sin x + \cos x$	$f^{iv}(0) = 1$

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

4. Expand by using Maclaurin's series for the function $\log(1+\sin x)$ upto fourth degree terms.

Solution

Let, $f(x) = \log(1+\sin x)$ $\qquad f(0) = 0$

$f'(x) = \dfrac{\cos x}{1+\sin x}$ $\qquad f'(0) = 1$

$(1+\sin x) f'(x) = \cos x$

$\cos x f'(x) + (1+\sin x) f''(x) = -\sin x$

$1 + f''(0) = 0 \Rightarrow \boxed{f''(0) = -1}$

$-\sin x f'(x) + \cos x f''(x) + \cos x f''(x) + (1+\sin x) f'''(x) = -\cos x$

$-\sin x f'(x) + 2\cos x f''(x) + (1+\sin x) f'''(x) = -\cos x$

$0 - 2 + f'''(0) = -1 \Rightarrow \boxed{f'''(0) = 1}$

$-\cos x f'(x) - \sin x f''(x) - 2\sin x f''(x) + 2\cos x f'''(x)$

$\cos x f'''(x) + (1+\sin x) f^{IV}(x) = \sin x$

$-\cos x f'(x) - 3\sin x f''(x) + 3\cos x f'''(x) + (1+\sin x) f^{IV}(x) = \sin x$

$-1 + 3 + f^{IV}(0) = 0$

$\boxed{f^{IV}(0) = -2}$

$\log(1+\sin x) = 0 + x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{2x^4}{4\times3\times2}$

$= x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12}$

Indeterminate Forms

While evaluating certain limits, we come across expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0$ and 1^∞ which do not represent any value. Such expressions are called Indeterminate forms. There is no actual value for these expressions. We can evaluate such limits that lead to indeterminate forms by using L-Hospital's Rule.

L-Hospital's Rule: If $f(x)$ and $g(x)$ are two functions such that

(i) $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$

then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

(Differentiate both numerator & denominator seperately).

The above rule can be extended, i.e., if $f'(a)=0$ & $g'(a)=0$ then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to a} \frac{f''(x)}{g''(x)}$

Limits of the form $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$

The indeterminate forms of the given type can be evaluated using L Hospitals Rule by replacing the function by their derivative.

Limits of the for 0^0, ∞^0 and 1^∞

To evaluate such limits, where function to the power of function exist, we call such an expression as some constant, then take logarithm on both sides of the function whose limit is required and rewrite the expression to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply LHR.

Working procedure is as follows:

(i) To evaluate the limits of the form 0^0, ∞^0 and 1^∞ i.e., where function to the power of function exists, first identify the form.

(ii) Take such an expression as some constant A

(iii) Take logarithm on both sides and rewrite the expression to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply the LHR

(iv) See if any possible simplification can be done.

(v) Continue till we obtain the finite value.

Evaluate the following

Note: Revise trig reciprocal.

① Evaluate $\lim_{x \to 0} \frac{\sin x - x}{\tan^3 x}$

Let $K = \lim_{x \to 0} \frac{\sin x - x}{\tan^3 x} \quad \cdots\cdots \left(\frac{0}{0}\right)$

Applying LH rule,

$$K = \lim_{x \to 0} \frac{\cos x - 1}{3\tan^2 x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{3\frac{\sin^2 x}{\cos^2 x} \times \frac{1}{\cos^2 x}}$$

$$= \lim_{x \to 0} \frac{\cos^4 x(\cos x - 1)}{3\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\cos^5 x - \cos^4 x}{3\sin^2 x} = \frac{0}{0}$$

Apply LH rule,

$$= \lim_{x \to 0} \frac{-5\cos^4 x \cancel{\sin x} + 4\cos^3 x \cancel{\sin x}}{6\cancel{\sin x}\cos x}$$

$$= \lim_{x \to 0} \frac{-5\cos^3 x + 4\cos^2 x}{6} = \frac{-5+4}{6} = -\frac{1}{6}$$

(2) Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{\log(\cos x)}{\tan x}$

Solution

Let $K = \lim_{x \to \frac{\pi}{2}} \frac{\log(\cos x)}{\tan x} = -\frac{\infty}{\infty}$ $\{\because \log 0 = -\infty\}$

Applying LH Rule,

$K = \lim_{x \to \frac{\pi}{2}} \frac{-\tan x}{\sec^2 x} = -\lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} \times \cos^2 x$

$= \lim_{x \to \frac{\pi}{2}} -\sin x \cos x = 0$

(3) Evaluate $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Let $K = \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos} - \tan x = \infty - \infty$

$K = \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \frac{0}{0}$

Apply LH Rule,

$K = \lim_{x \to \frac{\pi}{2}} + \frac{\cos x}{\sin x} = \frac{\cos \pi/2}{\sin \pi/2} = \frac{0}{1} = 0$

1. (4) Evaluate $\lim_{x \to 1} x^{\frac{1}{1-x}}$

Let $K = \lim_{x \to 1} x^{\frac{1}{1-x}} = 1^{\frac{1}{1-1}} = 1^{\frac{1}{0}} = 1^{\infty}$

Taking log on both sides,

$\log K = \lim_{x \to 1} \frac{\log x}{1-x} = \frac{0}{0}$

Apply LH Rule,

$\Rightarrow \log K = \lim_{x \to 1} \frac{\frac{1}{x}}{-1} = -1$

$\log K = -1$

$K = e^{-1}$

(8)

(5) Evaluate $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Let $K = \lim_{x \to \pi/2} (\sin x)^{\tan x} = 1^{\infty}$

take log on both sides,

$\log K = \lim_{x \to \frac{\pi}{2}} \tan x \log \sin x = \lim_{x \to \frac{\pi}{2}} \frac{\log \sin x}{\cot x} = \frac{0}{0}$

Apply L H Rule,

$$\log K = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x} = -\frac{\cos x}{\sin x} \times \sin^2 x$$

$$= -\lim_{x \to \frac{\pi}{2}} \sin x \cos x$$

$$= -\sin \pi/2 \cos \pi/2 = 0$$

$$\log K = 0$$

$$K = e^0 = 1$$

(6) Evaluate $\lim_{x \to 0} \left[\frac{a^x + b^x}{2}\right]^{1/x}$

4

Let $K = \lim_{x \to 0} \left[\frac{a^x + b^x}{2}\right]^{1/x} = 1^\infty$

Taking log on b.s

$$\log K = \lim_{x \to 0} \frac{1}{x} \{\log(a^x + b^x) - \log 2\} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cancel{1}}{\cancel{x}} \frac{\left[\frac{1}{a^x + b^x}[a^x \log a + b^x \log b]\right]}{1}$$

$$= \frac{\log a - \log b}{1+1} = \frac{\log a + \log b}{2}$$

$$= \frac{1}{2} \log ab$$

$$\log K = \log (ab)^{1/2}$$

$$K = (ab)^{1/2}$$

$$K = \sqrt{ab}$$

2.

(7) $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Let $K = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} = 1^\infty$

take log on b.s

$$\log K = \lim_{x \to a} \tan \frac{\pi x}{2a} \log\left(2 - \frac{x}{a}\right)$$

$$= \lim_{x \to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot \frac{\pi x}{2a}} = \frac{0}{0}$$

Apply L H rule,

$$\log K = \lim_{x \to a} \frac{\frac{1}{2-\frac{x}{a}}\left(-\frac{1}{a}\right)}{-\operatorname{cosec}^2 x\left(\frac{\pi x}{2a}\right) \times \frac{\pi}{2a}}$$

$$= \frac{\frac{1}{1}\left(+\frac{1}{a}\right)}{+\frac{\pi}{2a}} = \frac{1}{a} \times \frac{2a}{\pi}$$

$$\log K = \frac{2}{\pi}$$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

8. Evaluate $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$ --- 1

$$\log K = \lim_{x \to 0} \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \sin x - \log x$$

Apply LH Rule,

$$= \lim_{x \to 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^2 \sin x} \times \frac{x}{x}$$

$$= \cancel{\lim_{x \to 0} \frac{x}{\sin x}} \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3} = \frac{0}{0}$$

Apply L-Hospital Rule,

$$\log K = \lim_{x \to 0} \frac{-x\sin x + \cancel{\cos x} - \cancel{\cos x}}{6x^2}$$

$$= -\frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x}$$

$$\log K = -\frac{1}{6}$$

$$\boxed{K = e^{-1/6}}$$

9. Evaluate $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x + d^x}{4}\right)^{1/x}$

$$\log K = \frac{\log(a^x + b^x + c^x + d^x) - \log 4}{x}$$

$$\log K = \lim_{x \to 0} \frac{a^x \log a + b^x \log b + c^x \log c + d^x \log d}{a^x + b^x + c^x + d^x}$$

$$\log K = \frac{\log(abcd)}{4}$$

$$K = (abcd)^{1/4}$$

10. Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$ $\qquad \frac{\cos x}{\cos^2 x \sin x} - \frac{1}{x}$

$$\log K = \lim_{x\to 0} \frac{1}{x} \log\tan x - \log x$$

$$= \lim_{x\to 0} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{1}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x\tan x} \times \frac{x}{x}$$

$$= \lim_{x\to 0} \frac{x}{\tan x} \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} = \frac{0}{0}$$

Apply LH rule,

$$= \lim_{x\to 0} \frac{\cancel{2}x\sec^2 x\tan x + \cancel{\sec^2 x} - \cancel{\sec^2 x}}{\cancel{2}x}$$

$$= 1\times 0 = 0$$

11. Evaluate $\lim_{x\to 0} x^{\sin x} = 0^0$

$$\log K = \lim_{x\to 0} \sin x \log x = \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} = -\frac{\infty}{\infty}$$

LHR, $\lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cot x} = -\lim_{x\to 0} \frac{\sin x}{x\cot x} = -\lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \tan x$

$$\log K = -1\times 0 \Rightarrow \boxed{K = 1}$$

12. $\lim_{x\to 0} (\cot x)^{\tan x}$ --- ∞^0

$$\log K = \lim_{x\to 0} \tan x \log\cot x$$

$$\log K = \lim_{x\to 0} \frac{\log\cot x}{\cot x} = \frac{\infty}{\infty}$$

Apply LHR, $\log K = \lim_{x\to 0} \frac{\frac{-\operatorname{cosec}^2 x}{\cot x}}{-\operatorname{cosec}^2 x} = \lim_{x\to 0} \tan x = 0$

13. $\lim_{x\to 0}\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x} = (abc)^{1/3}$

14. $\lim_{x\to 0}\left(\frac{1}{x}\right)^{2\sin x}$ --- ∞^0

$$\log K = 2\sin x \log 1/x$$

$$\log K = 2\lim_{x\to 0} \frac{\log 1/x}{\operatorname{cosec} x}$$

$$= 2\lim_{x\to 0} \frac{\log 1 - \log x}{\operatorname{cosec} x}$$

$$= 2\lim_{x\to 0} \frac{-\log x}{\operatorname{cosec} x}$$

Apply LHR

$$= +2\lim_{x\to 0} \frac{1}{x\operatorname{cosec} x \cot x}$$

$$= 2\lim_{x\to 0} \frac{\sin x}{x} \tan x$$

$$\log K = 2\lim_{x\to 0} \tan x$$

$$\boxed{K = 1}$$

15. $\lim_{x\to 0} \frac{a^x - b^x}{x}$ --- $\frac{0}{0}$

$K = \lim_{x\to 0} \frac{a^x \log a - b^x \log b}{1} = \log a - \log b$

$K = \log \frac{a}{b}$

16. $\lim_{x\to 0}\left(\frac{2^x + 3^x + 4^x}{3}\right)^{1/x}$ ---- 1^∞

$\log K = \lim_{x\to 0} \frac{1}{x}\left[\log(2^x + 3^x + 4^x) - \log 3\right]$

$= \lim_{x\to 0} \frac{2^x \log 2 + 3^x \log 3 + 4^x \log 4}{2^x + 3^x + 4^x}$

$\log K = \log \frac{2\times 3\times 4}{3}$

$\boxed{K = (24)^{1/3}}$

17. $\lim_{x\to 0} (\cos x)^{1/x^2} = 1^\infty$

__Solution__

$\log K = \frac{1}{x^2} \log \cos x$

$\log K = \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to 0} \frac{\tan x}{-2x}$

$= \lim_{x\to 0} -\frac{\sin x}{x} \lim_{x\to 0} \frac{1}{2\cos x}$

$\log K = -\frac{1}{2}$

$K = e^{-1/2}$

18. $\lim_{x\to \frac{\pi}{2}} (\sec x)^{\cot x}$

$\log K = \lim_{x\to \frac{\pi}{2}} \cot x \log \sec x$

$= \lim_{x\to \frac{\pi}{2}} \frac{\log \sec x}{\tan x}$

$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{\sec x \tan x}{\sec x}}{\sec^2 x}$

$= \lim_{x\to \frac{\pi}{2}} \frac{\sin x}{\cos x} \times \cos^2 x$

$= 0 \times 1$

$\log K = 0$

$\boxed{K = 1}$

Radius Of Curvature

1. Find the radius of curvature at any point $P(x,y)$ on the parabola $y^2 = 4ax$.

Solution

By data, we have

$$y^2 = 4ax \quad -(1)$$

diff eqⁿ (1) wrt x, we get

$$2y\frac{dy}{dx} = 4a \quad \therefore \; y' = \frac{dy}{dx} = \frac{4a}{2y} \quad -(2)$$

diff (2) wrt x,

$$y'' = \frac{d^2y}{dx^2} = \frac{-2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2} \times \frac{2a}{y} = \frac{-4a^2}{y^3} \quad (3)$$

we have $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2}}{\frac{-4a^2}{y^3}}$

$$= \frac{\left(1+\frac{\cancel{4}a^{\cancel{2}}}{\cancel{4a}x}\right)^{3/2}}{\frac{-\cancel{4}a^2}{\cancel{4a}x \cdot 2\sqrt{ax}}} = -\left(\frac{x+a}{x}\right)^{3/2} \times \frac{2\sqrt{ax}.x}{a}$$

$$= \frac{-(x+a)^{3/2} \times 2\sqrt{a}\,\cancel{x^{3/2}}}{\cancel{x^{3/2}}\sqrt{a}\sqrt{a}} = \frac{-2(x+a)^{3/2}}{\sqrt{a}}$$

$$\boxed{|\rho| = \frac{2(x+a)^{3/2}}{\sqrt{a}}}$$

* The amount of bending of a curve at given point on it is called curvature.

(1) Cartesian curve (or) form.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

2. Find the radius of curvature of the curve $y = a\log\sec\left(\frac{x}{a}\right)$ at any point (x,y)

Solution: $y = a\log\sec\left(\frac{x}{a}\right) \quad -(1)$

diff eqⁿ (1) wrt x, we get,

$$\frac{dy}{dx} = \cancel{a}\frac{1}{\cancel{\sec\left(\frac{x}{a}\right)}}\cancel{\sec\left(\frac{x}{a}\right)}\tan\left(\frac{x}{a}\right) \times \cancel{\frac{1}{a}}$$

$$\frac{dy}{dx} = \tan\left(\frac{x}{a}\right) \quad -(2)$$

diff (2) wrt 'x', we get,

$$\frac{d^2y}{dx^2} = \frac{\sec^2\left(\frac{x}{a}\right)}{a}$$

$$\rho = \frac{\left(1+\tan^2\frac{x}{a}\right)^{3/2}}{\sec^2\left(\frac{x}{a}\right)} \times a$$

$$\rho = \frac{\left(\sec^{\cancel{2}}\frac{x}{a}\right)^{3/\cancel{2}} \times a}{\sec^2\frac{x}{a}}$$

$$\boxed{\rho = a\sec\frac{x}{a}}$$

(3) Find the curvature of the curve $x^4+y^4=2$ at the point $(1,1)$

<u>solution</u> :

$$x^4+y^4=2 \quad \text{---} (1)$$

diff (1) w.r.t 'x', we get,

$$4x^3+4y^3\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^3}{y^3} \quad \text{---} (2)$$

$$\rho = \frac{(1+1)^{3/2}}{-6}$$

$$= -\frac{2^{3/2}}{6}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -1 \qquad \boxed{y_1=-1}$$

diff (2) wrt x, we get,

$$\frac{d^2y}{dx^2} = -\left\{\frac{3x^2y^3-3y^2y_1x^3}{y^6}\right\}$$

$$y_{2(1,1)} = -\left\{\frac{3+3}{1}\right\} = -6$$

$$= \frac{\frac{3a^2}{\cancel{4}}\times(-\cancel{4}a)-\frac{3a^2}{\cancel{4}}\times\cancel{4}a}{\frac{9a^4}{16}} = -\frac{\cancel{6}^{2}a^3}{\cancel{9}_{3}a^4}\times 16$$

$$= -\frac{32}{3a}.$$

(4) Find the radius of curvature of the curve $x^3+y^3=3axy$ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$ on it.

<u>solution</u>

$$3x^2+3y^2\frac{dy}{dx}=3a\left\{y+x\frac{dy}{dx}\right\}$$

$$\cancel{3}\left\{x^2+y^2\frac{dy}{dx}\right\}=\cancel{3}a\left\{y+x\frac{dy}{dx}\right\}$$

$$x^2+y^2\frac{dy}{dx}=ay+ax\frac{dy}{dx}$$

$$(y^2-ax)\frac{dy}{dx}=ay-x^2$$

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}=y_1 \quad \text{---} (1)$$

$$y_1\left(\frac{3a}{2},\frac{3a}{2}\right)=\frac{a\times\frac{3a}{2}-\frac{9a^2}{4}}{\frac{9a^2}{4}-a\frac{3a}{2}}=\frac{\frac{3a^2}{2}-\frac{9a^2}{4}}{-\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)}=-1$$

diff (1) wrt x,

$$\frac{d^2y}{dx^2}=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

$$\text{at}\left(\frac{3a}{2},\frac{3a}{2}\right);\ y_2=\frac{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)\left(-a-\frac{6a}{2}\right)-\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)\left(2\times\frac{3a}{2}-a\right)}{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)^2}$$

$$\frac{9a^2-6a^2}{4}$$

(5) Find the ROC of the curve $x^2y = a(x^2+y^2)$ at the point $(-2a, 2a)$.

Solution

By data, we have

$$x^2y = a(x^2+y^2) \quad -(1)$$

diff eq^n (1) w.r.t x, we get,

$$2xy + x^2y_1 = a(2x + 2yy_1)$$

$$(x^2 - 2ya)y_1 = 2ax - 2xy$$

$$y_1 = \frac{2ax - 2xy}{x^2 - 2ya}$$

at $(-2, 2)$, y_1 is infinity

Hence, $\frac{dx}{dy} = \frac{1}{y_1} = \frac{x^2 - 2ay}{2ax - 2xy}$

$$\frac{dx}{dy} = x_1 = 0 \text{ at } (-2a, 2a)$$

Also, $(2ax - 2xy)x_1 = x^2 - 2ay$

$$(2ax_1 - 2yx_1 - 2x \quad)x_1 + (2ax - 2xy)x_2 = 2xx_1 - 2a$$

$$(2ax_1 - 2x_1y - 2x)x_1 + (2ax - 2xy)x_2 = 2xx_1 - 2a$$

$$0 - 0 - 2(-2a)\times 0 + [2a(-2a) - 2(-2a)2a]x_2 = 0 - 2a$$

$$[-4a^2 + 8a^2]x_2 = -2a$$

$$x_2 = \frac{-2a}{4a^2} = -\frac{1}{2a}$$

$$\therefore \rho = \frac{[1 + x_1^2]^{3/2}}{x_2} = \frac{1^{3/2}}{-1/2a} = -2a$$

(6) Find the radius of curvature of the curve $a^2y = x^3 - a^3$ at $(a, 0)$

$$a^2y_1 = 3x^2 - 0$$

$$y_1 = \frac{3x^2}{a^2} \quad -(1)$$

At $(a,0)$, $y_1 = \frac{3a^2}{a^2} = 3$

diff eq^n (1) wrt 'x', we get

$$y_2 = \frac{1}{a^2}(6x)$$

At $(a,0)$ $y_2 = \frac{6\times a}{a^2} = \frac{6}{a}$

we have, $\rho = \frac{(1 + 3^2)^{3/2}}{\frac{6}{a}} = \frac{a\ 10^{3/2}}{6}$

$$= \frac{a\sqrt{10^3}}{6} = \frac{a\sqrt{10^2\times 10}}{6} = \frac{10a\sqrt{10}}{6}$$

$$= \frac{5a\sqrt{10}}{3}$$

(7) Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at $(a,0)$

Solution

$$xy^2 = a^2(a-x)$$

$$x\,2yy_1 + y^2 = -a^2$$

$$\frac{dy}{dx} \cdot y_1 = \frac{-a^2-y^2}{2xy}$$

At $(a,0)$ $\quad y_1 = \infty$

$$\therefore \frac{dx}{dy} = \frac{2xy}{-a^2-y^2}$$

diff w.r.t 'y',

$$(-a^2-y^2)x_1 = 2xy$$

$$(-a^2-y^2)x_2 + (-2y\,x_1) = 2x_1y + 2x$$

$$-a^2x_2 = 2a$$

$$x_2 = \frac{-2}{a}$$

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

$$\rho = \frac{(1+0)^{3/2}}{-\frac{2}{a}} = \frac{-a}{2}$$

$$|\rho| = \frac{a}{2}$$

Note: Revise half angles //

8. Find the radius of curvature of the curve $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

Solution

Given $x = a(\cos t + t\sin t)$

$y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a(-\cancel{\sin t} + t\cos t + \cancel{\sin t}) = at\cos t \quad \text{--- (1)}$$

$$\frac{dy}{dt} = a(\cancel{\cos t} + t\sin t - \cancel{\cos t}) = at\sin t$$

$$\text{Now, } y_1 = \frac{dy}{dx} = \frac{at\sin t}{at\cos t} = \tan t$$

$$y_2 = \sec^2 t \times \frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} \quad [\text{from (1)}]$$

$$= \frac{\sec^3 t}{at}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+\tan^2 t)^{3/2}}{\sec^3 t} \times at$$

$$= \frac{(\sec^2 t)^{3/2}}{\sec^3 t} \times at$$

$$\boxed{\rho = at}$$

9. Find the ROC of the curve
$x = a[\cos t + \log\tan\frac{t}{2}]$, $y = a\sin t$

Solution:

Given, $x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right], y = a\sin t$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}} \times \frac{1}{2}\right] = a\left[-\sin t + \frac{\cancel{\cos^2\frac{t}{2}}}{2\sin\frac{t}{2}\cos^2\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$= a\,\frac{\cos^2 t}{\sin t} \quad \text{— (1)}$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{\cancel{a}\cos t \times \sin t}{\cancel{a}\cos^2 t} = \tan t$$

$$y_2 = \sec^2 t\,\frac{dt}{dx} = \frac{\sec^2 t \times \sin t}{a\cos^2 t} \quad [\text{from (1)}]$$

$$\rho = \frac{(1+\tan^2 t)^{3/2} \times a\cos^2 t}{\sec^2 t \times \sin t} = \frac{\sec^3 t \times a}{\sec^4 t \times \sin t} = \underline{a\cot t}$$

10. Find the roc of the curve
$x = a(t+\sin t)$ and $y = a(1-\cos t)$

Solution

$x = a(t+\sin t)$ and $y = a(1-\cos t)$

$\frac{dx}{dt} = a(1+\cos t)$ $\quad \frac{dy}{dt} = a\sin t$

$$\frac{dy}{dx} = \frac{a\sin t}{a(1+\cos t)}$$

$\cos 2t = 2\cos^2 t - 1$
$\cos t + 1 = 2\cos^2\frac{t}{2}$

$$= \frac{2a\sin\frac{t}{2}\cos\frac{t}{2}}{2a\cos^2\frac{t}{2}}$$

$$y_1 = \frac{dy}{dx} = \tan\frac{t}{2}$$

$$y_2 = \frac{1}{2}\sec^2\frac{t}{2}\,\frac{dt}{dx} = \frac{\sec^2 t}{2} \times \frac{1}{2a(1+\cos t)}$$

$$= \frac{\sec^2\frac{t}{2}}{2a\,2\cos^2\frac{t}{2}}$$

$$= \frac{\sec^4\frac{t}{2}}{4a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{(1+\tan^2 t/2)^{3/2}}{\sec^4 t/2} \times 4a$$

$$= \frac{\sec^3 t/2}{\sec^4 t/2} \times 4a$$

$$\rho = \frac{4a}{\sec t/2} = 4a \cos t/2$$

Polar Curve

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

Note: $1+\cot^2\theta = \operatorname{cosec}^2\theta$

Pedal Curve

$$r\frac{dr}{dp} = \rho$$

11. Find the roc of the curve $r^n = a^n \sin n\theta$

Solution

$$r^n = a^n \sin n\theta$$

$$n r^{n-1} \frac{dr}{d\theta} = n a^n \cos n\theta$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

$$r_1 = \frac{dr}{d\theta} = r \cot n\theta \quad \text{—(1)}$$

$$r_2 = -rn\operatorname{cosec}^2 n\theta + r_1 \cot n\theta$$

we have, $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\cot^2 n\theta)^{3/2}}{r^2+2\times r^2\cot^2 n\theta + r^2 n\operatorname{cosec}^2 n\theta - r_1 \cot n\theta}$$

$$= \frac{(r^2)^{3/2}(1+\cot^2 n\theta)^{3/2}}{r^2+2r^2\cot^2 n\theta + r^2 n\,\mathrm{cosec}^2 n\theta - \underline{r^2\cot^2 n\theta}}$$

from ①

$$= \frac{r^3 (\mathrm{cosec}^2 n\theta)^{3/2}}{r^2+r^2\cot^2 n\theta + r^2 n\,\mathrm{cosec}^2 n\theta}$$

$$= \frac{r^3\,\mathrm{cosec}^3 n\theta}{r^2[\mathrm{cosec}^2 n\theta + n\,\mathrm{cosec}^2 n\theta]}$$

$$= \frac{r\,\mathrm{cosec}^3 n\theta}{(1+n)\,\mathrm{cosec}^2 n\theta}$$

$$\boxed{\rho = \frac{r\,\mathrm{cosec}\,n\theta}{1+n}}$$

12. Find the roc of $r^n = a^n\cos n\theta$

Solution

$r^n = a^n \cos n\theta$

$$n r^{n-1}\frac{dr}{d\theta} = -a^n \sin n\theta \cdot n$$

$$\frac{n r^n}{r}\frac{dr}{d\theta} = -n\sin n\theta \cdot a^n$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin n\theta\, a^n}{a^n\cos n\theta} = -\tan n\theta$$

$$\frac{dr}{d\theta} = r_1 = -r\tan n\theta$$

$$r_2 = -r_1\tan n\theta - r n\sec^2 n\theta$$

we have $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$$= \frac{(r^2+r^2\tan^2 n\theta)^{3/2}}{r^2+2r^2\tan^2 n\theta + r(r_1\tan n\theta + rn\sec^2 n\theta)}$$

$$= \frac{r^3 \cdot \sec^3 n\theta}{r^2+2r^2\tan^2 n\theta - r^2\tan^2 n\theta + r^2 n\sec^2 n\theta}$$

$$\rho = \frac{r^3\sec^3 n\theta}{r^2+r^2\tan^2 n\theta + r^2 n\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta + r^2 n\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta(1+n)}$$

$$\boxed{\rho = \frac{r\sec n\theta}{1+n}}$$

18. S.T for the curve $r(1-\cos\theta) = 2a$, ρ^2 varies as r^3.

Solution

$r(1-\cos\theta) = 2a$ — ①

$r - r\cos\theta = 2a$ — ⓐ

① differential
② simplify $[r_1]$
③ Half angle

diff ⓐ, w.r.t θ, we get

$$\frac{dr}{d\theta} + r\sin\theta - \frac{dr}{d\theta}\cos\theta = 0$$

$$\frac{dr}{d\theta}[1-\cos\theta] = -r\sin\theta$$

$$r_1 = \frac{dr}{d\theta} = \frac{-r\sin\theta}{1-\cos\theta} = \frac{-r\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$r_1 = -r\cot\theta/2 \quad — ②$$

diff eqⁿ ② w.r.t θ, we get

$$r_2 = -r_1\cot\frac{\theta}{2} + r\,\mathrm{cosec}^2\left(\frac{\theta}{2}\right)\times\frac{1}{2}$$

$$= +r\cot^2\frac{\theta}{2} + \frac{r}{2}\mathrm{cosec}^2\frac{\theta}{2}$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\cot^2\theta/2)^{3/2}}{r^2+2\times r^2\cot^2\frac{\theta}{2} - r^2\cot^2\frac{\theta}{2} - \frac{r^2}{2}\mathrm{cosec}^2\frac{\theta}{2}}$$

$$= \frac{r^3\,\mathrm{cosec}^3\theta/2}{r^2\left[1+\cot^2\frac{\theta}{2} - \frac{\mathrm{cosec}^2\theta/2}{2}\right]}$$

$$\rho = \frac{r\,\mathrm{cosec}^3\theta/2}{\frac{1}{2}\mathrm{cosec}^2\theta/2} = 2r\,\mathrm{cosec}\,\theta/2 \quad — ③$$

But, $r(1-\cos\theta) = 2a$

$$r\,2\sin^2\theta/2 = 2a$$

$$\sin^2\theta/2 = \frac{a}{r}$$

$$\sin\theta/2 = \sqrt{\frac{a}{r}}$$

$$\mathrm{cosec}\,\theta/2 = \sqrt{\frac{r}{a}}$$

③ becomes

$$\rho = 2r\sqrt{\frac{r}{a}}$$

$$\rho = \frac{2r^{3/2}}{\sqrt{a}}$$

$$\rho^2 = \frac{4r^3}{a}$$

$$\rho^2 \propto r^3$$

14. P.T $\frac{\rho^2}{r}$ is a constant for the cardiod $r = a(1+\cos\theta)$ where ρ is the radius of curvature.

Solution

$r = a(1+\cos\theta)$ Apply log on b.S

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{dr}{d\theta} = \frac{-r\sin\theta}{1+\cos\theta} = \frac{-r\cancel{2}\sin\theta/2\,\cos\theta/2}{\cancel{2}\cos^2\theta/2}$$

$$\therefore r_1 = -r\tan\theta/2$$

Hence $r_2 = -\frac{r}{2}\sec^2\frac{\theta}{2} - r_1\tan\frac{\theta}{2}$

$$= \frac{-r}{2}\sec^2\frac{\theta}{2} - (-r\tan\tfrac{\theta}{2})\tan\frac{\theta}{2}$$

$$r_2 = -\frac{r}{2}\sec^2\frac{\theta}{2} + r\tan^2\frac{\theta}{2}$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{(r^2+r^2\tan^2\theta/2)^{3/2}}{r^2+2(r^2\tan^2\frac{\theta}{2}) - r(-\frac{r}{2}\sec^2\frac{\theta}{2}+r\tan^2\frac{\theta}{2})}$$

$$= \frac{r^3\sec^3\theta/2}{r^2\left[1+\tan^2\frac{\theta}{2}+\frac{\sec^2\theta/2}{2}\right]}$$

$$= \frac{r\sec^3\theta/2}{\frac{3}{2}\sec^2\theta/2} = \frac{2r\sec^3\theta/2}{3\sec^2\theta/2}$$

$$= \frac{2r\sec\theta/2}{3}$$

$$\frac{\rho^2}{r} = \frac{8a}{9}$$

15. For the curve $r = ae^{\theta\cot\alpha}$, P.T $\frac{\rho}{r}$ is constant

Solution

Apply log on b.s

$\log r = \log a + \theta\cot\alpha\,\log e$

diff w.r.t θ we have

$$\frac{1}{r}\frac{dr}{d\theta} = \cancel{0}+\cot\alpha$$

$$r_1 = r\cot\alpha$$

$$r_2 = r_1\cot\alpha = r\cot^2\alpha$$

$$\rho = \frac{(r^2 + r^2\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha}$$

$$= \frac{r^3 \operatorname{cosec}^3\alpha}{r^2 + r^2\cot^2\alpha}$$

$$\rho = \frac{r \operatorname{cosec}^3\alpha}{\operatorname{cosec}^2\alpha} = r\operatorname{cosec}\alpha$$

$$\rho = r\operatorname{cosec}\alpha$$

Thus $\frac{\rho}{r} = \operatorname{cosec}\alpha =$ constant

16. Find the radius of curvature of the curve $pa^2 = r^3$

Solution

$pa^2 = r^3$, diff w.r.t p

$a^2 = 3r^2 \frac{dr}{dp}$ or $\frac{dr}{dp} = \frac{a^2}{3r^2} \Rightarrow r\frac{dr}{dp} = \frac{a^2}{3r}$

we have $\rho = r\frac{dr}{dp} = \frac{r a^2}{3r^2}$

$$\boxed{\rho = \frac{a^2}{3r}}$$