MODULE-03.

PARTIAL DIFFERENTIATION- I

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→ **Partial Derivatives.**

Definition, simple problems on direct and indirect partial derivatives.

Euler's theorem (without proof)

Total Derivative, partial differentiation of composite functions - Problems.

→ **Jacobians.**

Definition and Problems.

Partial Differential Equations.

An equation which involves one dependent variable with respect to two (or) more independent variables is called partial differential equation.

Notations:

Let $u = f(x, y)$ be a real valued continous function defined by (or) an any other interval, then the first and second order partial derivatives of the function is notated as,

(1) $\dfrac{\partial u}{\partial x} = u_x = p$

(2) $\dfrac{\partial u}{\partial y} = u_y = q$

(3) $\dfrac{\partial^2 u}{\partial x^2} = u_{xx} = r \qquad \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$

(4) $\dfrac{\partial^2 u}{\partial x \partial y} = u_{xy} \,|\, u_{yx} = s. \qquad \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) \,\Big|\, \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

(5) $\dfrac{\partial^2 u}{\partial y^2} = u_{yy} = t. \qquad \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)$

Important Points.

1) Order of a PDE is the order of the highest derivative.

2) Degree of a PDE is the degree of the highest order derivative

3) PDE is said to be linear, if it is of first degree in dependent variable and its partial derivatives.

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4) If each term of the PDE contains either the dependent variable (or) one of its partial derivatives, the PDE is said to be homogeneous. Otherwise it is said to be non-homogeneous PDE.

Examples:

1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$: order = 1, degree = 1, homogeneous.

2) $\frac{\partial^2 z}{\partial x \partial y} = xy$: order = 2, degree = 1, non-homogeneous.

3) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$: order = 2, degree = 1, homogeneous.

) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$: order = 2, degree = 1, homogeneous.

) $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$: order: 3, degree = 1, non-homogeneous.

I) DIRECT PARTIAL DERIVATIVES.

1) If $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Given: $u = x^3 - 3xy^2 + x + e^x \cos y + 1$

Diff. u wrt x partially keeping y constant

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x \cos y. \longrightarrow ①$$

Again diff. ① wrt x. partially keeping y constant

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \text{ (or) } \frac{\partial^2 u}{\partial x^2} = 6x + e^x \cos y.$$

Diff u wrt y partially keeping x constant.

$$\frac{\partial u}{\partial y} = -6xy - e^x \sin y. \longrightarrow ②$$

Again diff ② wrt y partially keeping x constant.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \text{ (or) } \frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y.$$

Consider, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y$

$$= 0.$$

Hence proved.

2) If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, show that $x u_x + y u_y = 1$.

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Given: $u = \log\left(\frac{x^2+y^2}{x+y}\right) = \log(x^2+y^2) - \log(x+y)$

Diff. u. wrt x partially keeping y constant.

$$u_x = \frac{1}{(x^2+y^2)} \cdot 2x - \frac{1}{(x+y)}$$

Diff. u wrt y partially keeping x constant.

$$u_y = \frac{1}{(x^2+y^2)} \cdot 2y - \frac{1}{(x+y)}$$

Consider, $x u_x + y u_y$

$$= x \cdot \left[\frac{2x}{x^2+y^2}\right] - \frac{x}{x+y} + y \cdot \left[\frac{2y}{x^2+y^2}\right] - \frac{y}{x+y}$$

$$= \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} + \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}$$

$$= \frac{2(x^2+y^2)}{x^2+y^2} - \frac{(x+y)}{(x+y)}$$

$$= 2 - 1$$

$$= 1$$

Hence proved.

3) If $u = e^{ax-by} \cdot \sin(ax+by)$.

Show that $b \cdot \frac{\partial u}{\partial x} - a \cdot \frac{\partial u}{\partial y} = 2abu$.

Given: $u = e^{ax-by} \cdot \sin(ax+by) \longrightarrow ①$

Diff. u wrt. x keeping y constant.

$$\frac{\partial u}{\partial x} = e^{ax-by} \cos(ax+by) \cdot a + a \cdot e^{ax-by} \sin(ax+by)$$

$$\frac{\partial u}{\partial x} = ae^{ax-by} [\cos(ax+by) + \sin(ax+by)]$$

Diff. u wrt. y keeping x constant.

$$\frac{\partial u}{\partial y} = e^{ax-by} \cos(ax+by) \cdot b - b \cdot e^{ax-by} \sin(ax+by)$$

$$\frac{\partial u}{\partial y} = b \cdot e^{ax-by} [\cos(ax+by) - \sin(ax+by)]$$

Consider, $b \cdot \frac{\partial u}{\partial x} - a \cdot \frac{\partial u}{\partial y}$

$$= b \cdot ae^{ax-by} [\cos(ax+by) + \sin(ax+by)] - a \cdot be^{ax-by} [\cos(ax+by) - \sin(ax+by)]$$

$$= abe^{ax-by} [\cancel{\cos(ax+by)} + \sin(ax+by) - \cancel{\cos(ax+by)} + \sin(ax+by)]$$

$$= ab \cdot e^{ax-by} \cdot 2\sin(ax+by)$$

$= 2abu$ [from ①] $\therefore$ proved.

4) If $u = \tan^{-1}\left(\frac{y}{x}\right)$

show that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

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Given: $u = \tan^{-1}\left(\frac{y}{x}\right)$

Diff u wrt x keeping y constant.

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \quad \left\{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right\} \longrightarrow (1)$$

Diff (1) wrt y keeping x constant.

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(x^2+y^2)(-1) + y \cdot 2y}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \longrightarrow (2)$$

Diff u wrt y keeping x constant.

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} \longrightarrow (3)$$

Diff (3) wrt x keeping y constant.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \longrightarrow (4)$$

from (2) and (4), $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ $\therefore$ proved.

Symmetric Functions:

A function $f(x,y)$ is said to be symmetric function if $f(x,y) = f(y,x)$.

Similarly, $f(x,y,z) = f(y,z,x) = f(z,x,y)$.

Generally we can say that a function of several variables is symmetric if the function remains unchanged when the variable are cyclicoully rotated.

Ex: $x+y$, x^2+y^2, $xy+yz+zx$, $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, $\log(x^2+y^2)$ etc.

1) If $u = \log\sqrt{x^2+y^2+z^2}$, show that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = 1$$

Given, $u = \log\sqrt{x^2+y^2+z^2}$ is a symmetric function.

Diff. u wrt x partially

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{(x^2+y^2+z^2)} \longrightarrow (1)$$

Diff (1) wrt x partially

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2) - 2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2} \longrightarrow (2)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2} \longrightarrow (3)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \longrightarrow (4)$$

Consider, $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$

$$= (x^2+y^2+z^2)\cdot\frac{1}{(x^2+y^2+z^2)^2}\cdot\left[\cancel{-x^2}+y^2+\cancel{z^2}+\cancel{x^2}-\cancel{y^2}+z^2+x^2+y^2-\cancel{z^2}\right]$$

$$= (x^2+y^2+z^2)\cdot\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= 1$$

Hence proved.

2) If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0.$$

Given: $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a symmetric function.

Diff. u wrt x partially

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x = -x(x^2+y^2+z^2)^{-3/2} \longrightarrow (1)$$

Diff ① wrt x partially.

$$\frac{\partial^2 u}{\partial x^2} = -x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot 2x - (x^2+y^2+z^2)^{-3/2}$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

Consider, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$$
$$- (x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= (x^2+y^2+z^2)^{-5/2}[3x^2+3y^2+3z^2] - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3[(x^2+y^2+z^2)^{-3/2} - (x^2+y^2+z^2)^{-3/2}]$$

$$= 3[0]$$

$$= 0$$

Hence proved.

3) If $U = \log(\tan x + \tan y + \tan z)$,
show that $\sin 2x\, U_x + \sin 2y\, U_y + \sin 2z\, U_z = 2$.

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Given: $U = \log(\tan x + \tan y + \tan z)$ is a symmetric function.

Diff U partially w.r.t. x.

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot (\sec^2 x)$$

Diff u partially w.r.t. y.

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} (\sec^2 y)$$

Diff u partially w.r.t. z.

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \cdot (\sec^2 z)$$

Consider, $\sin 2x\, U_x + \sin 2y\, U_y + \sin 2z\, U_z$

$$= \sin 2x \cdot \frac{\sec^2 x}{\tan x + \tan y + \tan z} + \sin 2y \cdot \frac{\sec^2 y}{\tan x + \tan y + \tan z} + \sin 2z \cdot \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{1}{\tan x + \tan y + \tan z}\left[\cancel{2\sin x \cos x} \cdot \frac{1}{\cos^2 x} + 2\sin y \cos y \cdot \frac{1}{\cos^2 y} + 2\sin z\, \cancel{\cos z} \cdot \frac{1}{\cos^2 z}\right]$$

$$= \frac{2}{\cancel{\tan x+\tan y+\tan z}}\left[\cancel{\tan x+\tan y+\tan z}\right]$$

$$= 2$$

Hence proved.

4) If $u = \log(x^3+y^3+z^3-3xyz)$, then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and hence show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

→ Given: $u = \log(x^3+y^3+z^3-3xyz)$ is a symmetric function.

Diff. u wrt x partially

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\cdot(3x^2-3yz)$$

Diff. u wrt y partially

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

Diff. u wrt z partially

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

Consider,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}+\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}+\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$=\frac{1}{x^3+y^3+z^3-3xyz}\left[3x^2-3yz+3y^2-3xz+3z^2-3xy\right]$$

$$=\frac{3[x^2+y^2+z^2-yz-xz-xy]}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3\,\cancel{[x^2+y^2+z^2-yz-xz-xy]}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-xz)}}$$

$$=\frac{3}{x+y+z}\quad\left[a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\right]$$

Hence proved.

Again Consider,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2\cdot u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot u$$

$$=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right]$$

$$=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\left[\frac{3}{x+y+z}\right]$$ (from above result)

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Hence proved.

Euler's theorem on Homogeneous Functions

Statement: If $u = f(x, y)$ is a homogenous function of degree 'n' then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial z} = nu$.

1) If $u = \frac{x^3+y^3}{\sqrt{x+y}}$, show that $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{5}{2}u$

Given: $u = \frac{x^3+y^3}{\sqrt{x+y}}$

$$x^{3-1/2} = x^{\frac{6-1}{2}} = x^{5/2}$$

$$u = \frac{x^3+y^3}{(x+y)^{1/2}}$$

$$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x^{1/2}\left[1+\frac{y}{x}\right]^{1/2}} = \frac{x^{5/2}\left[1+\left(\frac{y}{x}\right)^3\right]}{\left[1+\frac{y}{x}\right]^{1/2}}$$

$$u = x^{5/2} f\left(\frac{y}{x}\right)$$

$\therefore$ u is homogenous of degree $\frac{5}{2}$.

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$\therefore n = \frac{5}{2}$

By Euler's theorem, $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$

Putting $n = \frac{5}{2}$, we get, $x\frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{5}{2}u$.

2) If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$

Show that $x\frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$.

Given: $u = \log\left(\frac{x^4+y^4}{x+y}\right)$

$$e^u = \frac{x^4+y^4}{x+y}$$

$$= \frac{x^4\left(1+\left(\frac{y}{x}\right)^4\right)}{x\left(1+\frac{y}{x}\right)}$$

$$= \frac{x^3\left[1+\left(\frac{y}{x}\right)^4\right]}{1+\frac{y}{x}}$$

$$e^u = x^3 f\left(\frac{y}{x}\right)$$

$\therefore e^u$ is homogenous function of degree 3.

$\therefore n = 3$.

By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

$$\Rightarrow x\frac{\partial e^u}{\partial x}+y\cdot\frac{\partial e^u}{\partial y}=ne^u$$

$$\Rightarrow x\,e^u\frac{\partial u}{\partial x}+y\cdot e^u\frac{\partial u}{\partial y}=3\cdot e^u \quad (\text{By } n=3)$$

Dividing both sides by e^u

We get, $x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=3.$

3) If $u=e^{\frac{x^3y^3}{x^2+y^2}}$, show that $x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=4u\log u.$

Given: $u=e^{\frac{x^3y^3}{x^2+y^2}}$

$$\log u=\frac{x^3y^3}{x^2+y^2}$$

$$\log u=\frac{x^3\cdot y^3\left(\frac{x^3}{x^3}\right)}{x^2+y^2}=\frac{x^6\cdot\left(\frac{y}{x}\right)^3}{x^2\left[1+\frac{y}{x}\right]^2}$$

Let $\log u=z.$

$$\therefore\quad z=\frac{x^4\left(\frac{y}{x}\right)^3}{\left(1+\frac{y}{x}\right)^2}=x^4f\left(\frac{y}{x}\right)$$

$\therefore$ z is a homogeneous function of degree 4.

ie, $n=4.$

By Euler's theorem.

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$$x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial}{\partial x}(\log u) + y\cdot\frac{\partial}{\partial y}(\log u) = n\log u.$$

$$x\cdot\frac{1}{u}\cdot\frac{\partial u}{\partial x} + y\cdot\frac{1}{u}\cdot\frac{\partial u}{\partial y} = 4\log u \quad (\text{By } n=4)$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 4u\log u.$$

$\Rightarrow$ If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

Given: $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\sin u = \frac{x^2+y^2}{x+y}.$$

$$= \frac{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}{x\left[1+\frac{y}{x}\right]}$$

$$= \frac{x\left[1+\left(\frac{y}{x}\right)^2\right]}{1+\frac{y}{x}}$$

$$\sin u = x f\left(\frac{y}{x}\right)$$

Let $\sin u = z$.

$$\therefore \quad z = x f\left(\frac{y}{x}\right)$$

$\therefore$ z is a homogeneous function of degree 1.

$\therefore$ ie, $n=1$

By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = nz$$

$$x\frac{\partial}{\partial x}(\sin u) + y\cdot\frac{\partial}{\partial y}(\sin u) = \sin u \quad (\text{By } n=1)$$

$$x\cdot\cos u\frac{\partial u}{\partial x} + y\cdot\cos u\frac{\partial u}{\partial y} = \sin u$$

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \tan u$$

5) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \sin 2u.$$

Given: $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\tan u = \frac{x^3+y^3}{x+y}$$

$$\tan u = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1+\frac{y}{x}\right]} = \frac{x^2\left[1+\left(\frac{y}{x}\right)^3\right]}{1+\frac{y}{x}}$$

Let $\tan u = z$

$$\therefore z = \frac{x^2\left[1+\left(\frac{y}{x}\right)^3\right]}{1+\frac{y}{x}} = x^2 f\left(\frac{y}{x}\right)$$

$\therefore$ z is a homogeneous function of degree 2.

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$\therefore\ n = 2.$

By Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = n\cdot z.$$

$$x\frac{\partial}{\partial x}(\tan u) + y\cdot\frac{\partial}{\partial y}(\tan u) = n\cdot\tan u.$$

$$x\sec^2 u\cdot\frac{\partial u}{\partial x} + y\cdot\sec^2 u\cdot\frac{\partial u}{\partial y} = n\cdot\tan u.$$

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = n\cdot\frac{\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 2\cdot\frac{\sin u}{\cos u}\cdot\cos^2 u \quad (\text{By } n=2)$$

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \sin 2u.$$

Hence proved.

TOTAL DERIVATIVES.

Total Differentiation / Exact Differentiation.

If $u = f(x,y)$ then the total differential (or) exact differential of 'u' is defined as,

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

Partial Derivatives involves two types:

1) Total Derivative Rule:

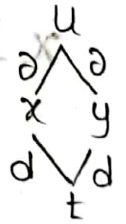
If $u = f(x,y)$ where $x = x(t)$ and $y = y(t)$, then 'u' is a composite function of a single variable 't', differentiating 'u' wrt 't' we have

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

$u \to (x,y) \to t$

$u \Rightarrow t$

$\frac{du}{dt}$



This is called total Derivative of u

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

2) Chain Rule:

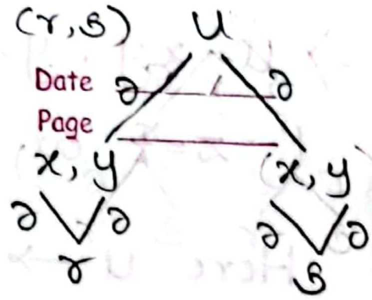
If $u = f(x,y)$ where $x = x(r,s)$ and $y = y(r,s)$ then 'u' is a composite function of two independent variable r, s. Differentiating 'u' wrt 'r' and 's' and using chain rule we have,

$$\boxed{\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}.\end{aligned}}$$

$u \to (x,y) \to (r,s)$

$u \Rightarrow (r,s)$

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

Problems on total derivative rule:

1) $z = xy^2 + x^2y$ where $x = at$, $y = 2at$.

Here $z \to (x,y) \to t \Longrightarrow z \to t$

and $\frac{dz}{dt}$ is the total derivative.

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Given: $z = xy^2 + x^2y$.

$$\Rightarrow \frac{dz}{dt} = (y^2+2xy)\cdot\frac{dx}{dt} + (2xy+x^2)\cdot\frac{dy}{dt}$$

Since $x = at \Longrightarrow \frac{dx}{dt} = a$.

$y = 2at \Longrightarrow \frac{dy}{dt} = 2a$

$$\begin{aligned}\therefore \frac{dz}{dt} &= (y^2+2xy)\cdot a + (2xy+x^2)\cdot 2a\\ &= (4a^2t^2 + 2at\cdot 2at)\cdot a + (2\cdot at\cdot 2at + a^2t^2)\cdot 2a\\ &= 4a^3t^2 + 4a^3t^2 + 8a^3t^2 + 2a^3t^2\\ &= 18a^3t^2\end{aligned}$$

2) $u = xy + yz + zx$ where $x = t\cos t$, $y = t\sin t$, $z = t$ at $t = \frac{\pi}{4}$.

Here $u \to (x, y, z) \to t \Rightarrow u \to t$ and $\frac{du}{dt}$ is the total derivative.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Given: $u = xy + yz + zx$.

$$\Rightarrow \frac{du}{dt} = (y+z) \cdot \frac{dx}{dt} + (x+z) \cdot \frac{dy}{dt} + (y+x) \cdot \frac{dz}{dt}$$

Since $x = t\cos t \Rightarrow \frac{dx}{dt} = -t\sin t + \cos t$

$y = t\sin t \Rightarrow \frac{dy}{dt} = t\cos t + \sin t$

$z = t \Rightarrow \frac{dz}{dt} = 1$

$$\therefore \frac{du}{dt} = (t\sin t + t)(-t\sin t + \cos t) + (t\cos t + t)(t\cos t + \sin t) + (t\sin t + t\cos t) \cdot 1$$

$$\frac{du}{dt} = -t^2\sin^2 t - t^2\sin t + t\sin t\cos t + t\cos t + t^2\cos^2 t + t^2\cos t + t\sin t\cos t + t\sin t + t\sin t + t\cos t.$$

Given,

$t = \frac{\pi}{4} \Rightarrow \sin\frac{\pi}{4}$ (or) $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\Rightarrow \frac{du}{dt} = -\frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right)^2 - \frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)$$

$$\frac{du}{dt} = 4\cdot\left[\frac{\pi}{4\sqrt{2}}\right] + 2\cdot\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^2 + 2\cdot\frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right)^2 - 2\cdot\frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right)^2$$

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$$= \frac{\pi}{\sqrt{2}} + \frac{\pi}{4} + \cancel{\frac{\pi^2}{8}} - \cancel{\frac{\pi^2}{8}}$$

$$\frac{du}{dt} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

3) $u = x^2 + y^2 - z^2$ where $x = e^t$, $y = e^t\cosh t$, $z = e^t\sinh$

Here $u \to (x, y, z) \to t \Rightarrow u \to t$ and

$\frac{du}{dt}$ is the total derivative.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$\frac{d}{dx}(\sinh x) = \cosh$

$\frac{d}{dx}(\cosh x) = \sinh$

Given: $x^2 + y^2 - z^2$

$$\frac{du}{dt} = 2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} - 2z\cdot\frac{dz}{dt}$$

$$= 2\left[x\frac{dx}{dt} + y\cdot\frac{dy}{dt} - z\cdot\frac{dz}{dt}\right]$$

Since, $x = e^t \Rightarrow \frac{dx}{dt} = e^t$

$y = e^t\cosh t \Rightarrow \frac{dy}{dt} = e^t\sinh t + e^t\cosh t$

$z = e^t\sinh t \Rightarrow \frac{dz}{dt} = e^t\cosh t + e^t\sinh t$

$$\therefore \frac{du}{dt} = 2\left[e^t\cdot e^t + e^t\cosh t\,(e^t\sinh t + e^t\cosh t) - e^t\sinh t\,(e^t\cosh t + e^t\sinh t)\right]$$

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$$\frac{du}{dt} = 2\left[e^{2t} + e^{2t}\sinh t\cosh t + e^{2t}\cosh^2 t - e^{2t}\sinh t\cosh t - e^{2t}\sinh^2 t\right]$$

$$= 2e^{2t}\left[1 + \cancel{\sinh t\cosh t} + \cosh^2 t - \cancel{\sinh t\cosh t} - \sinh^2 t\right]$$

$$= 2e^{2t}\left[1 + \cosh^2 t - \sinh^2 t\right]$$

$$= 2e^{2t}\left[1+1\right]$$

$$\frac{du}{dt} = 4e^{2t}$$

Problems on Chain Rule:

1) If $z = x^2 + y^2$ where $x = e^u \sin v$, $y = e^u \cos v$ find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a composite function and verify the results by direct substitution.

Here $z \rightarrow (x,y) \rightarrow (u,v) \Rightarrow z \rightarrow (u,v)$ and

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{\partial z}{\partial u} = 2x\cdot\frac{\partial x}{\partial u} + 2y\cdot\frac{\partial y}{\partial u}$$

$$= 2(e^u\sin v)\cdot e^u\sin v + 2(e^u\cos v)\cdot e^u\cos v$$

$$= 2e^{2u}\left[\sin^2 v + \cos^2 v\right]$$

$$\frac{\partial z}{\partial u} = 2e^{2u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\rightarrow \frac{\partial z}{\partial v} = 2x\cdot\frac{\partial x}{\partial v} + 2y\cdot\frac{\partial y}{\partial v}$$

$$= 2(e^u\sin v)\cdot e^u\cos v + 2(e^u\cos v)\cdot e^u(-\sin v)$$

$$= 2e^{2u}[\sin v\cos v - \sin v\cos v]$$

$$\frac{\partial z}{\partial v} = 0.$$

from ① & ②, we have composite functions.

$$\frac{\partial z}{\partial u} = 2e^{2u} \text{ and } \frac{\partial z}{\partial v} = 0. \longrightarrow ①$$

Direct substitution:

Given: $z = x^2 + y^2$

$$\Rightarrow z = e^{2u}\sin^2 v + e^{2u}\cos^2 v.$$

$$z = e^{2u}[\sin^2 v + \cos^2 v]$$

$$z = e^{2u}.$$

Diff. z wrt u & v we get partially

$$\frac{\partial z}{\partial u} = 2e^{2u} \text{ and } \frac{\partial z}{\partial v} = 0. \longrightarrow ②$$

Thus from ① and ②

we conclude that, the result is verified.

2) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, Prove that,

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Here, $u = f(p, q, r)$ where $p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

ie, $u \to (p, q, r) \to (x, y, z) \Rightarrow u \to (x, y, z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot\left(\frac{1}{y}\right) + \cancel{\frac{\partial u}{\partial q}\left(0\right)} + \frac{\partial u}{\partial r}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}\cdot\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \cancel{\frac{\partial u}{\partial r}\left(0\right)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \cancel{\frac{\partial u}{\partial p}\left(0\right)} + \frac{\partial u}{\partial q}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial r}\left(\frac{1}{x}\right)$$

Consider, $x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= \cancel{\frac{x}{y}\cdot\frac{\partial u}{\partial p}} - \cancel{\frac{z}{x}\cdot\frac{\partial u}{\partial r}} + \cancel{\frac{y}{z}\cdot\frac{\partial u}{\partial q}} - \cancel{\frac{x}{y}\cdot\frac{\partial u}{\partial p}} - \cancel{\frac{y}{z}\cdot\frac{\partial u}{\partial q}} + \cancel{\frac{z}{x}\frac{\partial u}{\partial r}}$$

$= 0.$

Hence proved.

3) If $u = f(x-y,\, y-z,\, z-x)$

show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

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Here, $u = f(p, q, r)$ where $p = x-y$, $q = y-z$, $r = z-x$.

ie, $u \to (p, q, r) \to (x, y, z) \Rightarrow u \to (x, y, z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}(1) + \cancel{\frac{\partial u}{\partial q}}\cdot(0)^{\,0} + \frac{\partial u}{\partial r}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}\cdot(-1) + \frac{\partial u}{\partial q}(1) + \cancel{\frac{\partial u}{\partial r}}(0)^{\,0}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \cancel{\frac{\partial u}{\partial p}}(0)^{\,0} + \frac{\partial u}{\partial q}\cdot(-1) + \frac{\partial u}{\partial r}(1)$$

Consider, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

$$= \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial q}} - \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial r}}$$

$= 0$

Hence proved.

4) If $u = f(2x-3y, 3y-4z, 4z-2x)$.

Show that $6 \cdot \frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$.

Here, $u = f(p, q, r)$

where $p = 2x-3y$, $q = 3y-4z$, $r = 4z-2x$.

ie, $u \rightarrow (p,q,r) \rightarrow (x,y,z) \Rightarrow u \rightarrow (x,y,z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p} \cdot (2) + \cancel{\frac{\partial u}{\partial q} \cdot (0)}^{0} + \frac{\partial u}{\partial r} \cdot (-2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p} \cdot (-3) + \frac{\partial u}{\partial q} (3) + \cancel{\frac{\partial u}{\partial r} (0)}^{0}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$= \cancel{\frac{\partial u}{\partial p} (0)}^{0} + \frac{\partial u}{\partial q} (-4) + \frac{\partial u}{\partial r} (4)$$

Consider, $6 \cdot \frac{\partial u}{\partial x} + 4 \cdot \frac{\partial u}{\partial y} + 3 \cdot \frac{\partial u}{\partial z}$

$$= 6 \cdot \left[\frac{\partial u}{\partial p} \cdot 2 - \frac{\partial u}{\partial r} 2\right] + 4\left[\frac{\partial u}{\partial q} 3 - \frac{\partial u}{\partial p} 3\right] + 3\left[4\frac{\partial u}{\partial r} - 4\frac{\partial u}{\partial q}\right]$$

$$= 12\cancel{\frac{\partial u}{\partial p}} - 12\cancel{\frac{\partial u}{\partial r}} + 12 \cdot \cancel{\frac{\partial u}{\partial q}} - 12\cancel{\frac{\partial u}{\partial p}} + 12\cancel{\frac{\partial u}{\partial r}} - 12\cancel{\frac{\partial u}{\partial q}}$$

$= 0$

Hence proved.

5) If $z=f(x,y)$ where $x=r\cos\theta$ and $y=r\sin\theta$.

Show that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

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Here, $z \rightarrow (x,y) \rightarrow (r,\theta) \Rightarrow z \rightarrow (r,\theta)$.

$$\therefore \frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$=\frac{\partial z}{\partial x}\cdot\cos\theta+\frac{\partial z}{\partial y}\cdot\sin\theta.$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}.$$

$$=\frac{\partial z}{\partial x}\cdot(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$$

Consider, $\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

$$=\left(\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial y}r\cos\theta-\frac{\partial z}{\partial x}r\sin\theta\right)^2$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+\cancel{2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta}+$$

$$\frac{1}{\cancel{r^2}}\left(\frac{\partial z}{\partial y}\right)^2\cancel{r^2}\cos^2\theta+\frac{1}{\cancel{r^2}}\left(\frac{\partial z}{\partial x}\right)^2\cancel{r^2}\sin^2\theta-\frac{2}{\cancel{r^2}}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}\cdot\cancel{r^2}\cancel{\cos\theta\sin\theta}$$

$$=\left(\frac{\partial z}{\partial x}\right)^2\left[\sin^2\theta+\cos^2\theta\right]+\left(\frac{\partial z}{\partial y}\right)^2\left[\sin^2\theta+\cos^2\theta\right]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

Hence proved.

6) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$,

Prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.

Here. $u = f(p,q)$ where $p = \frac{y-x}{xy}$, $q = \frac{z-x}{xz}$

ie, $u \rightarrow (p,q) \rightarrow (x,y,z) \Rightarrow u \rightarrow (x,y,z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot\left[\frac{xy(-1)-(y-x)\cdot y}{(xy)^2}\right] + \frac{\partial u}{\partial q}\left[\frac{xz(-1)-(z-x)z}{(xz)^2}\right]$$

$$= \frac{\partial u}{\partial p}\left[\frac{\cancel{-xy}-y^2+\cancel{xy}}{x^2y^2}\right] + \frac{\partial u}{\partial q}\left[\frac{\cancel{-xz}-z^2+\cancel{xz}}{x^2z^2}\right]$$

$$\frac{\partial u}{\partial x} = -\frac{1}{x^2}\frac{\partial u}{\partial p} - \frac{1}{x^2}\cdot\frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}$$

$$= \frac{\partial u}{\partial p}\cdot\left[\frac{xy-(y-x)\cdot x}{x^2y^2}\right] + \cancel{\frac{\partial u}{\partial q}\left[\frac{xz(0)-(z-x)\cdot 0}{x^2z^2}\right]}^{0}$$

$$= \frac{\partial u}{\partial p}\left[\frac{\cancel{xy}-\cancel{xy}+x^2}{x^2y^2}\right]$$

$$\frac{\partial u}{\partial y} = \frac{1}{y^2}\cdot\frac{\partial u}{\partial p}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z}$$

$$= \cancel{\frac{\partial u}{\partial p}[0]} + \frac{\partial u}{\partial q}\left[\frac{xz - (z-x)\cdot x}{x^2z^2}\right]$$

$$= \left[\frac{\cancel{xz} - \cancel{zx} + x^2}{x^2z^2}\right]\frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial z} = \frac{1}{z^2}\cdot\frac{\partial u}{\partial q}$$

Consider, $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$

$$= x^2\left[-\frac{1}{x^2}\cdot\frac{\partial u}{\partial p} - \frac{1}{x^2}\frac{\partial u}{\partial q}\right] + y^2\left[\frac{1}{y^2}\cdot\frac{\partial u}{\partial p}\right] + z^2\left[\frac{1}{z^2}\frac{\partial u}{\partial q}\right]$$

$$= -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$$

$$= 0$$

Hence proved.

JACOBIANS.

Suppose $u = u(x, y, z)$, $v = v(x, y, z)$, $w = (x, y, z)$ be the three functions in the independent variables of (x, y, z). Then the partial differentiation of u, v, w wrt x, y, z is called the Jacobian and which can be defined as,

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

[or]

The Jacobian of (x, y, z) wrt (u, v, w) can be defined as,

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = J\left(\frac{x, y, z}{u, v, w}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

similarly, the jacobian of (u,v) wrt (x,y) is defined as,

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$$J = J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

[or]

the Jacobian of (x,y) wrt (u,v) is defined as,

$$J = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Note: 1) If J is jacobian of (u,v) wrt (x,y) and J' be jacobian of (x,y) wrt (u,v) then we can prove that $JJ' = 1$.

2) If u and v are two independent variables of x and y, the u and v are said to be functionally dependent if $\frac{\partial(u,v)}{\partial(x,y)} = 0$. Also it is called null jacobian (or) zero jacobian.

1) If $u = x+y+z$, $v = y+z$, $z = uvw$ find the value $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$\rightarrow$ wkt,

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Given: $u = x+y+z$, $v = y+z$, $z = uvw$.

$\Rightarrow x+y+z = u$. $\quad v = y+z$

$x+v = u$ $\quad y = v-z$

$\boxed{x = u-v}$ $\quad \boxed{y = v-uvw}$ $\quad \boxed{z = uvw}$

$$\therefore J = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix} \end{array}$$

$$= 1\begin{vmatrix} 1-uw & -uv \\ uw & uv \end{vmatrix} + 1\begin{vmatrix} -vw & -uv \\ vw & uv \end{vmatrix} + 0\cancel{\begin{vmatrix} -vw & 1-uw \\ vw & uw \end{vmatrix}}$$

$$= [(1-uw)(uv) - (uw)(-uv)] + [(-vw)(uv) - (vw)(-uv)]$$

$$= uv - \cancel{u^2vw} + \cancel{u^2vw} - \cancel{uv^2w} + \cancel{uv^2w}.$$

$J = 0.$

2) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

find the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

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$\rightarrow$ wkt,

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Given: $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$\begin{array}{cccc} & + & - & + \\ \therefore J = & \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} \end{array}$$

$$= -\frac{yz}{x^2}\begin{vmatrix} -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} - \frac{z}{x}\begin{vmatrix} \frac{z}{y} & \frac{x}{y} \\ \frac{y}{z} & -\frac{xy}{z^2} \end{vmatrix} + \frac{y}{x}\begin{vmatrix} \frac{z}{y} & -\frac{zx}{y^2} \\ \frac{y}{z} & \frac{x}{z} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left(\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(-\frac{xyz}{yz^2} - \frac{xy}{zy}\right) + \frac{y}{x}\left(\frac{zx}{yz} + \frac{xyz}{y^2z}\right)$$

$$= -\frac{x^2y^2z}{x^2y^2z} + \frac{x^2yz}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} + \frac{xyz}{xyz} + \frac{xy^2z}{xy^2z}$$

$$= \cancel{-1} + \cancel{1} + 1 + 1 + 1 + 1$$

$$J = 4$$

3) If $u = \dfrac{2yz}{x}$, $v = \dfrac{3zx}{y}$, $w = \dfrac{4xy}{z}$

find the value of $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

$\rightarrow$ wkt,

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Given: $u = \dfrac{2yz}{x}$, $v = \dfrac{3zx}{y}$, $w = \dfrac{4xy}{z}$.

$$\Rightarrow J = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \frac{-2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & \frac{-3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & \frac{-4xy}{z^2} \end{vmatrix} \end{array}$$

$$= -\frac{2yz}{x^2}\left(\frac{12x^2yz}{y^2z^2} - \frac{12x^2}{yz}\right) - \frac{2z}{x}\left(\frac{-12xyz}{yz^2} - \frac{12xy}{yz}\right)$$

$$+\frac{2y}{x}\left(\frac{12xz}{yz} + \frac{12xyz}{y^2z}\right)$$

$$= \frac{-24x^2y^2z^2}{x^2y^2z^2} + \frac{24x^2yz}{x^2yz} + \frac{24xyz^2}{xyz^2} + \frac{24xyz}{xyz} + \frac{24xy^2}{xyz}$$

$$+ \frac{24xy^2z}{xy^2z} = \cancel{-24} + \cancel{24} + 24 + 24 + 24 + 24$$

$$= 96.$$

4) Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$, where $u = x^2+y^2+z^2$

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$v = xy+yz+zx$, $w = x+y+z$

$\rightarrow$ wkt.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Given: $u = x^2+y^2+z^2$, $v = xy+yz+zx$, $w = x+y+z$.

$$\Rightarrow J = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix} \end{array}$$

$$= 2x[(x+z)-(y+x)] - 2y[(y+z)-(y+x)] + 2z[(y+z)-(x+z)]$$

$$= 2x[\cancel{x}+z-y-\cancel{x}] - 2y[\cancel{y}+z-\cancel{y}-x] + 2z[y+\cancel{z}-x-\cancel{z}]$$

$$= \cancel{2xz} - \cancel{2xy} - \cancel{2yz} + \cancel{2yx} + \cancel{2zy} - \cancel{2zx}$$

$J = 0.$

5) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

$\longrightarrow$ wkt,

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

Given: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\begin{array}{cccc} & + & - & + \\ \Longrightarrow J = & \left|\begin{matrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{matrix}\right. & \begin{matrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{matrix} & \left.\begin{matrix} -r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{matrix}\right| \end{array}$$

$$= \sin\theta\cos\phi\left(0 + r^2\sin^2\theta\cos\phi\right) - r\cos\theta\cos\phi\,(r\sin\theta\cos\theta\cos\phi) - r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi).$$

$$= r^2\sin^3\theta\cos^2\phi - r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi.$$

$$= r^2\left[\sin^3\theta\,(\overset{1}{\cancel{\cos^2\phi + \sin^2\phi}}) + \sin\theta\cos^2\theta\,(\overset{1}{\cancel{\cos^2\phi + \sin^2\phi}})\right]$$

$$= r^2\sin\theta\,[\overset{1}{\cancel{\sin^2\theta + \cos^2\theta}}]$$

$$J = r^2\sin\theta.$$

6) If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

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$\rightarrow$ WKt,

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Given: $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$.

$$\Rightarrow J = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix} \end{array}$$

$$= 1(16x^2z + 4x^3y) - 6y(32xyz^2 + 4x^2y^2) - 3z^2(-8x^2yz + 4x^2yz)$$

$$= 16x^2z + 4x^3y - 192xy^2z^2 - 24x^2y^3 + 24x^2yz^3 - 12x^2yz^3$$

$$J_{(1,-1,0)} = 0 - 4 - 0 + 24 + 0 - 0.$$

$$J_{(1,-1,0)} = 20$$

Vidya (666)

Two marks Questions:

1) If $u = 3x^2y + 6xy^2 + 7$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Given: $u = 3x^2y + 6xy^2 + 7$

$$\frac{\partial u}{\partial x} = 6xy + 6y^2$$

$$\frac{\partial u}{\partial y} = 3x^2 + 12xy.$$

2) If $u = \sin(xy)$; find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Given: $u = \sin(xy)$

$$\frac{\partial u}{\partial x} = \cos(xy) \cdot y = y\cos(xy)$$

$$\frac{\partial u}{\partial y} = \cos(xy) \cdot x = x\cos(xy)$$

3) If $u = e^{4x+3y}$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Given: $u = e^{4x+3y}$

$$\frac{\partial u}{\partial x} = e^{4x+3y} \cdot 4 = 4e^{4x+3y}$$

$$\frac{\partial u}{\partial y} = e^{4x+3y} \cdot 3 = 3e^{4x+3y}.$$

4) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ for the function

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$u = x^3 - 3xy^2 + x + e^x \cos y + 1.$

Given: $u = x^3 - 3xy^2 + x + e^x \cos y + 1.$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1 + e^x \cos y$$

$$\frac{\partial u}{\partial y} = -6xy - e^x \sin y$$

5) If $f(x,y) = x^2y - 3y^2$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Given: $f = x^2y - 3y^2$

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 - 6y.$$

6) If $u = x^y$ then find $\frac{\partial^2 u}{\partial x \partial y}$

Given: $u = x^y$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\therefore \frac{\partial u}{\partial y} = x^y \log x.$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^y \cdot \frac{1}{x} + \log x \cdot y x^{y-1}$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1}[1 + y \log x]$$

7) If $u = f(x+ay) + g(x-ay)$ then find $\frac{\partial^2 u}{\partial y^2}$

Given: $u = f(x+ay) + g(x-ay)$

$$\frac{\partial u}{\partial y} = f'(x+ay)\cdot a + g'(x-ay)(-a)$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x+ay)\cdot a^2 + g''(x-ay)\cdot a^2$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = a^2\left[f''(x+ay) + g''(x-ay)\right]$$

8) If $f(x,y) = x^2y + 3xy^2$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

Given: $f = x^2y + 3xy^2$.

$$\frac{\partial f}{\partial x} = 2xy + 3y^2$$

$$\frac{\partial f}{\partial y} = x^2 + 6xy.$$

9) If $f(x,y) = \cos(xy)$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

Given: $f = \cos(xy)$

$$\frac{\partial f}{\partial x} = -y\sin(xy)$$

$$\frac{\partial f}{\partial y} = -x\sin(xy).$$

10) Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = \frac{xy}{x^2+y^2}$

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Given: $f = \frac{xy}{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2).y - xy.(2x)}{[x^2+y^2]^2}$$

$$= \frac{x^2y + y^3 - 2x^2y}{[x^2+y^2]^2}$$

$$\frac{\partial f}{\partial x} = \frac{y^3 - x^2y}{[x^2+y^2]^2}$$

$$\frac{\partial f}{\partial y} = \frac{[x^2+y^2]\cancel{y}x - xy(2y)}{[x^2+y^2]^2}$$

$$= \frac{x^3 + y^2x - 2xy^2}{[x^2+y^2]^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^3 - xy^2}{[x^2+y^2]^2}$$

11) Determine $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial z}$ for $f(x,y,z) = xe^{yz}$

Given: $f = xe^{yz}$

$$\frac{\partial f}{\partial x} = e^{yz}, \quad \frac{\partial f}{\partial z} = xye^{yz}.$$

12) Compute $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x,y) = \log(x^2+y^2)$.

Given: $f = \log(x^2+y^2)$

$$\frac{\partial f}{\partial y} = \frac{1}{(x^2+y^2)} \cdot 2y.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2) \cdot (0) - 2y(2x)}{(x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-4xy}{(x^2+y^2)}.$$

13) Calculate $\frac{\partial^2 f}{\partial y^2}$ for $f(x,y) = x\cos(y)$.

Given: $f = x\cos(y)$

$$\frac{\partial f}{\partial y} = -x\sin y.$$

$$\frac{\partial^2 f}{\partial y^2} = -x\cos y.$$

14) If $u = e^x \cos y + 1$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Given: $u = e^x \cos y + 1$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y.$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0.$$

15) If $u = x^2+y^2$, find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

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Given: $u = x^2+y^2$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y$

$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = 2.$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2+2 = 4.$

16) Find total derivative of $z = xy^2 + x^2y$.
$x = at$ and $y = 2at$.

$\rightarrow$ WKT,
Here $z \rightarrow (x,y) \rightarrow t \Rightarrow z \rightarrow t$

$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= (y^2+2xy)\cdot\frac{dx}{dt} + (2xy+x^2)\frac{dy}{dt}$

Since $x = at$ and $y = 2at$

$\therefore \frac{dz}{dt} = (4a^2t^2 + 4a^2t^2)\cdot a + (4a^2t^2 + a^2t^2)\cdot 2a$

$= 4a^3t^2 + 4a^3t^2 + 8a^3t^2 + 2a^3t^2$

$\frac{dz}{dt} = 18a^3t^2$

17) If $z = u^2+v^2$ and $v = at^2$ find $\frac{dz}{dt}$.

$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$

$= 2u\cdot\frac{du}{dt} + 2v\cdot\frac{dv}{dt} = 2u\frac{du}{dt} + 4a^2t^3$

18) If (i) $x^3+xy^2+y^3$ (ii) $x=r\sin\theta\sin\phi$. then find the value of total derivative.

(i) Let $u=x^3+xy^2+y^3$

$$\therefore \frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$=(3x^2+y^2)\cdot\frac{dx}{dt}+(2xy+3y^2)\cdot\frac{dy}{dt}.$$

(ii) Given $x=r\sin\theta\sin\phi$

$$\therefore \frac{dx}{dt}=\frac{\partial x}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial x}{\partial \theta}\cdot\frac{d\theta}{dt}+\frac{\partial x}{\partial \phi}\cdot\frac{d\phi}{dt}$$

$$=(\sin\theta\sin\phi)\cdot\frac{dr}{dt}+(r\cos\theta\sin\phi)\cdot\frac{d\theta}{dt}+$$

$$(r\sin\theta\cos\phi)\cdot\frac{d\phi}{dt}.$$

19) If $u=f(x,y)$ where $x=x(t)$ and $y=y(t)$ find $\frac{du}{dt}$.

Given: $u=f(x,y)$, $x=x(t)$, $y=y(t)$

ie, $u\rightarrow(x,y)\rightarrow t \Rightarrow u\rightarrow t$

$$\therefore \frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

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20) If $z = xe^{y/x}$

show that $x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = z$

Given: $z = xe^{y/x}$

$$\frac{\partial z}{\partial x} = xe^{y/x}\left(\frac{y}{-x^2}\right) + e^{y/x} = e^{y/x}\left[1 - \frac{y}{x}\right]$$

$$\frac{\partial z}{\partial y} = xe^{y/x}\cdot\frac{1}{x} = e^{y/x}$$

Consider, $x\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y}$

$$= e^{y/x}\left[1 - \frac{y}{x}\right]\cdot x + e^{y/x}\cdot y$$

$$= xe^{y/x} - \cancel{ye^{y/x}} + \cancel{ye^{y/x}}$$

$$= xe^{y/x}$$

$$= z$$

21) If $u = \frac{x}{y}$, then show that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Given: $u = \frac{x}{y}$

$$\frac{\partial u}{\partial x} = \frac{1}{y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-x}{y^2}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{y} - \frac{xy}{y^2} = \frac{x}{y} - \frac{x}{y}$$

$$= 0$$

Hence proved

22) If $u = x^2 + 2xy + y^2$ then show that

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 2u$$

Given: $u = x^2 + 2xy + y^2$.

$$\frac{\partial u}{\partial x} = 2x + 2y$$

$$\frac{\partial u}{\partial y} = 2x + 2y$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (2x+2y)x + (2x+2y)y$$

$$= 2x^2 + 4xy + 2y^2$$

$$= 2u$$

23) Verify Euler's theorem for

$f(x,y) = x^3 - 3x^2y + y^3$

Given: $f(x,y) = x^3 - 3x^2y + y^3$

By Euler's theorem: $f(x,y) = x^3\left[1 - 3y/x + y^3/x^3\right]$

$$= x^3 f(x,y)$$

ie, $n = 3$.

$$\therefore \quad x\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = 3u.$$

Verification: $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = x\cdot(3x^2 - 6xy) + y(-3x^2 + 3y^2)$

$$= 3x^3 - 6x^2y - 3x^2y + 3y^3$$

$$= 3[x^3 - 3x^2y + y^3]$$

$$= 3u \qquad \therefore \text{Verified.}$$

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24) If $z = e^{ax+by} f(ax-by)$ then show that $b\cdot\frac{\partial z}{\partial x} + a\cdot\frac{\partial z}{\partial y} = 2abz$.

Given: $z = e^{ax+by} f(ax-by)$.

$$\therefore\ b\cdot\frac{\partial z}{\partial x} + a\cdot\frac{\partial z}{\partial y}$$

$$= b\left[e^{ax+by} f'(ax-by)\cdot a + f(ax-by)\cdot e^{ax+by}\cdot a\right]$$

$$+ a\left[e^{ax+by} f'(ax-by)(-b) + f(ax-by)\cdot e^{ax+by}\cdot b\right]$$

$$= 2ab e^{ax+by} f(ax-by)$$

$$= 2abz \quad \text{(from given)}$$

25) If $u = \alpha e^{ax} + \beta e^{-ax}$ Prove thot $\frac{\partial^2 u}{\partial x^2} - a^2 u = 0$.

Given: $u = \alpha e^{ax} + \beta e^{-ax}$.

$$\frac{\partial u}{\partial x} = \alpha a e^{ax} - \beta a e^{-ax}$$

$$\frac{\partial^2 u}{\partial x^2} = \alpha a^2 e^{ax} + \beta a^2 e^{ax}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} - a^2 u = \alpha a^2 e^{ax} + \beta a^2 e^{ax} - \alpha a^2 e^{ax} - \beta a^2 e^{-ax}$$

$$= 0$$

Hence proved.

26) If $u = \frac{y}{z} + \frac{z}{x}$,

find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Given: $u = \frac{y}{z} + \frac{z}{x}$.

$\therefore \; x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$= x\cdot\left(\frac{-z}{x^2}\right) + y\cdot\left(\frac{1}{z}\right) + z\left(\frac{-y}{z^2}\right) + z\left(\frac{1}{x}\right)$

$= \cancel{\frac{-z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{y}{z}} + \cancel{\frac{z}{x}}$

$= 0.$

27) Find the Jacobian of u, v, w wrt x, y, z given $u = x+y+z$, $v = y+z$, $w = z$.

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Given: $u = x+y+z$, $v = y+z$, $w = z$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) - 1(0) + 1(0)$$

$= 1$

$J = 1$

28) Find $\frac{\partial u}{\partial x}$, if $x^2+xy+y^2=1$

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Given: $x^2+xy+y^2=1$

$\therefore \frac{\partial u}{\partial x} = 2x+y.$

29) If $u=x+3y^2-z^3$, $v=4x^2yz$, $w=2z^2-xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Given: $u=x+3y^2-z^3$, $v=4x^2yz$, $w=2z^2-xy$

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= 1(16x^2z^2+4x^3y) - 6y(32xyz^2+4x^2y^2) - 3z^2(-8x^2yz+4x^2yz)$$

$$= 16x^2z^2+4x^3y-192xy^2z^2-24x^2y^3 + 24x^2yz^3 - 12x^2yz^3$$

$$= 16x^2z^2+4x^3y-192xy^2z^2-24x^2y^3 + 12x^2yz^3$$

30) If $u = x + y$, $v = x - y$, find $J\left(\frac{u,v}{x,y}\right)$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -1 - 1$$

$$J = -2.$$

31) If $u = x + y$, $v = x^2 + y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Given: $u = x + y$, $v = x^2 + y^2$

$$J = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix}$$

$$= 2y - 2x$$

$$J = 2(y - x)$$

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32) If $x = u + v$ and $y = u - v$ find $J\left(\frac{x,y}{u,v}\right)$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Given $x = u + v$, $y = u - v$.

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -1 - 1$$

$$J = -2.$$

33) If $x = u^2 + v^2$ and $y = uv$ find $J\left(\frac{x,y}{u,v}\right)$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Given: $x = u^2 + v^2$, $y = uv$

$$J = \begin{vmatrix} 2u & 2v \\ v & u \end{vmatrix}$$

$$J = 2u^2 - 2v^2.$$

$$J = 2(u^2 - v^2)$$

34) If $x = r\cos\theta$ and $y = r\sin\theta$ find $J\left(\frac{x,y}{r,\theta}\right)$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Given: $x = r\cos\theta$, $y = r\sin\theta$.

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta.$$

$$= r\left[\cos^2\theta + \sin^2\theta\right]$$

$$J = r$$

35) If $u = xy$, $v = x^3$ find $\frac{\partial(u,v)}{\partial(x,y)}$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Given: $u = xy$, $v = x^3$

$$J = \begin{vmatrix} y & x \\ 3x^2 & 0 \end{vmatrix}$$

$$J = -3x^3.$$

36) If $x = uv$ and $y = \frac{u}{v}$ then find $J\left(\frac{x,y}{u,v}\right)$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Given: $x = uv$, $y = \frac{u}{v}$

$$J = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix}$$

$$J = -u/v - u/v$$

$$J = -\frac{2u}{v}.$$

37) Given $u = x^2 + y^2$ and $v = 2xy$ find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Given: $u = x^2 + y^2$, $v = 2xy$

$$J = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix}$$

$$J = 4x^2 - 4y^2$$

$$J = 4(x^2 - y^2).$$