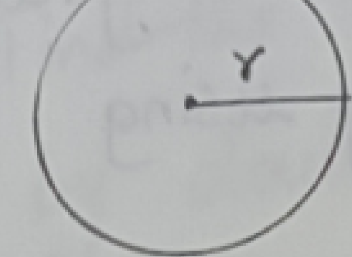
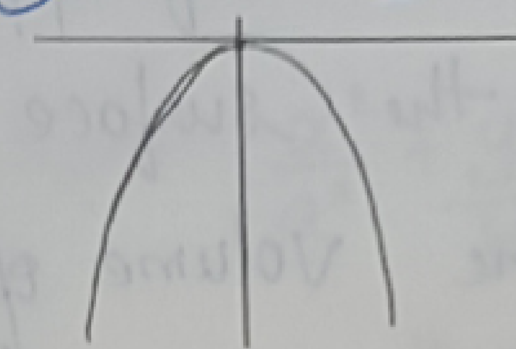
INTEGRAL CALCULUS - II

Some important curves and theirs shape

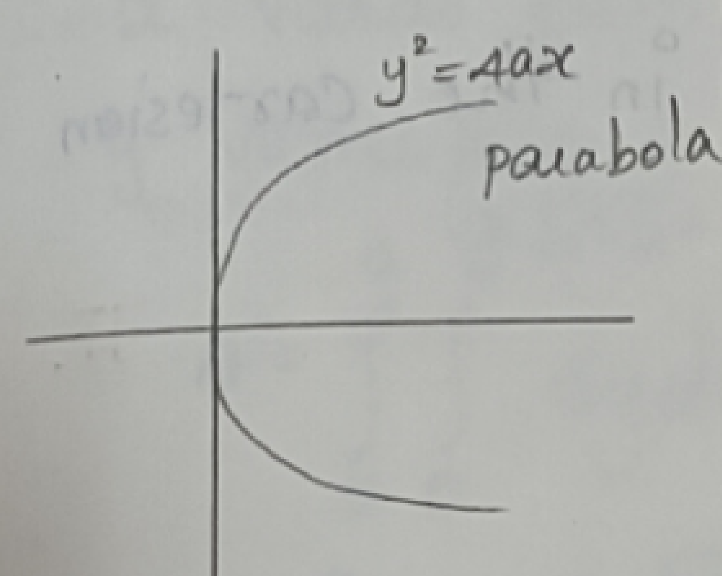
① $x^2+y^2=r^2$



④ $x^2=-4ay$



② $y^2=4ax$



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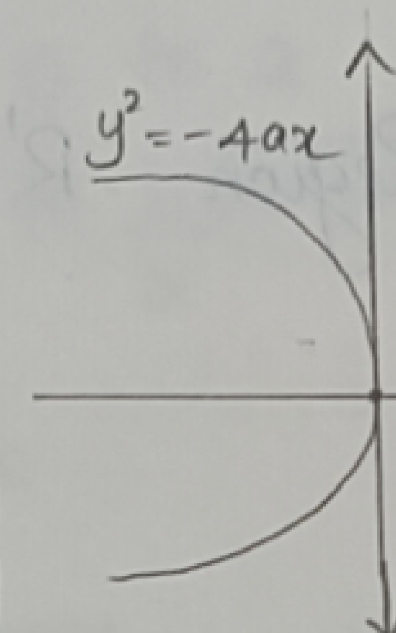
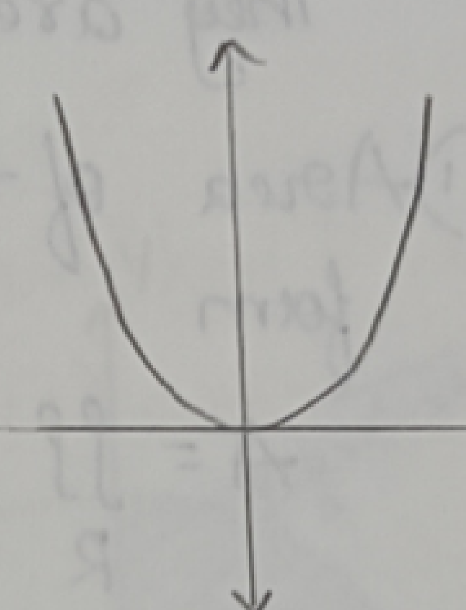
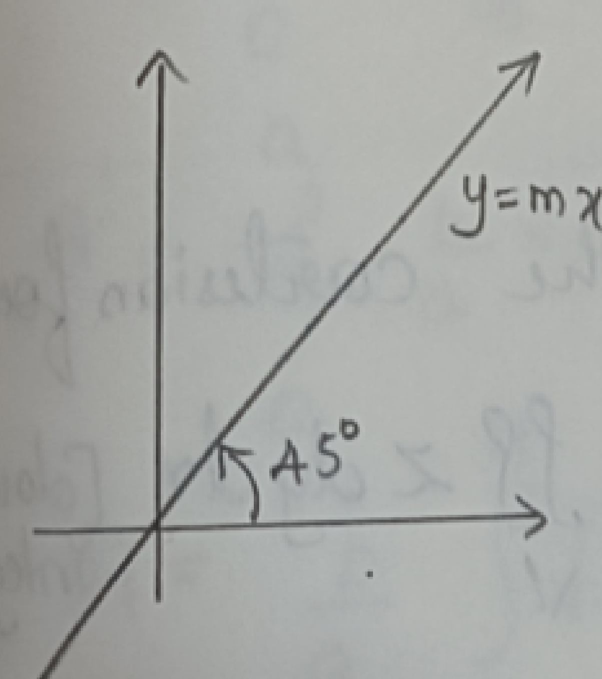


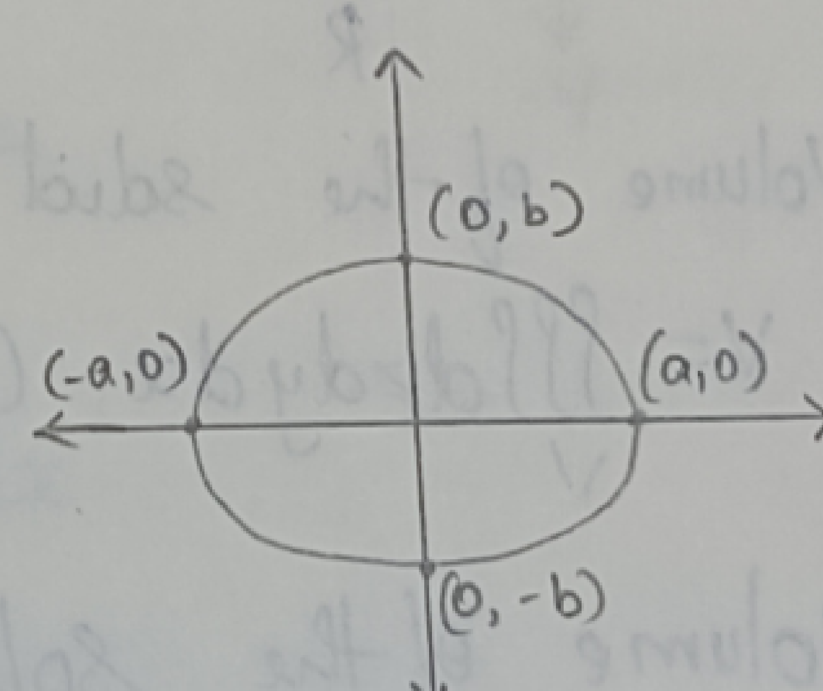
$x^2=4ay$



③ $y=mx$



⑤ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Applications to find the Area and Volume.

The main aim of applications are to find the area of the surface using double integration and the volume of the space by using triple (or) double integral.

They are,

1) Area of the Region 'R' in the cartesian form

$$A = \iint_R dx\,dy$$

2) Area of the Region 'R' in the polar form

$$A = \iint_R r\,dr\,d\theta$$

3) Volume of the solid V in the cartesian form

$$V = \iiint_V dz\,dy\,dx \quad (or) \quad V = \iint_V z\,dy\,dx \text{ [double integral]}$$

4) Volume of the solid obtained by the revolution of a curve enclosing an area 'A' about the initial ~~value~~ (polar form)

$$V = \iint_V 2\pi r^2 \sin\theta\,dr\,d\theta.$$

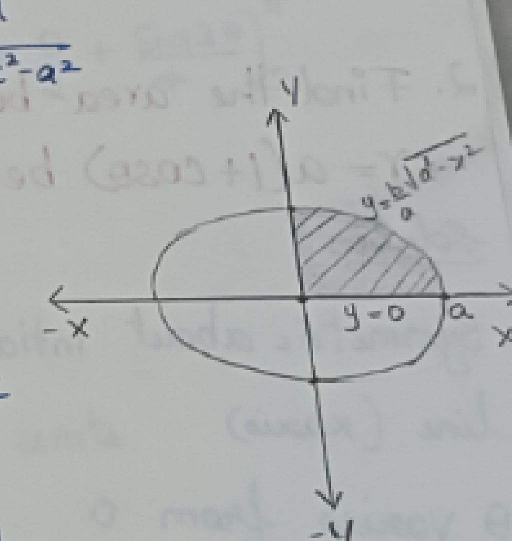
1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}$$

The area in first Quadrant

x varies from 0 to a

y varies from 0 to $\frac{b}{a}\sqrt{x^2 - a^2}$



$$A = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$A = \int_0^a y \Big|_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$A = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\Big|_0^a$$

$$= \frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a}$$

$$= 0 + \frac{a^2}{2} \times \frac{\pi}{2} \times \frac{b}{a}$$

$$= \frac{\pi ab}{4}$$

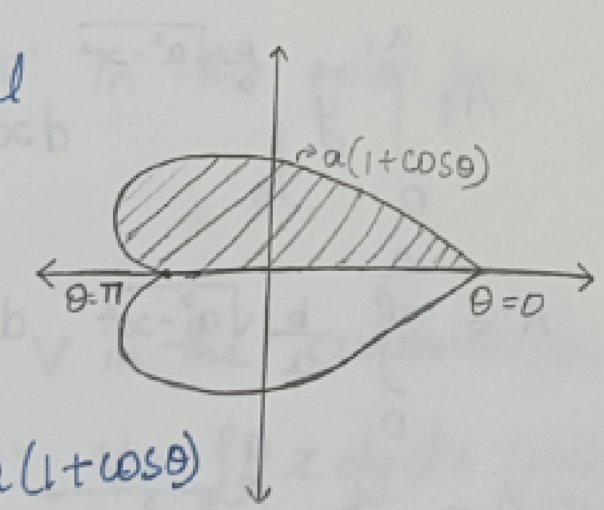
$\therefore$ Total area of the circle $= 4\,\frac{\pi a^2}{4}$

$= \pi a^2$ sq. units

2. Find the area bounded by the cardiod $r = a(1+\cos\theta)$ between $\theta = 0$ and $\theta = \pi$.

Solution

Symmetric about intial line (x axis)



θ varies from 0 to π

r varies from 0 to $a(1+\cos\theta)$

$$A = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$= \int \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(1+2\cos\theta+\frac{1}{2}+\frac{\cos2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin2\theta}{4}\right]_0^{\pi}$$

$$= \frac{a^2}{2}\left[\pi+\frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}\cdot\frac{3\pi}{2}$$

$$= \frac{3\pi a^2}{4} \text{ sq. units.}$$

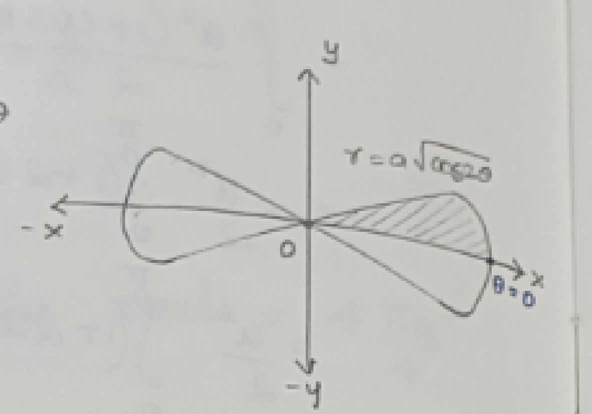
3. Find the area of the lemniscate

$r^2 = a^2 \cos 2\theta$.

Solution

Replace $r \Rightarrow -r$ & $\theta \rightarrow -\theta$ we get back eq^n

$\therefore$ It is symmetric along $\pi/2$



$0 = a^2 \cos 2\theta$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$

$\boxed{\theta = \frac{\pi}{4}}$

$$A = \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r \, dr \, d\theta$$

$$= \int \frac{r^2}{2} \Big|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int \frac{a^2 \cos 2\theta}{2} d\theta$$

$$A = \frac{a^2}{2} \frac{\sin 2\theta}{2} \Big|_0^{\pi/2}$$

$$= \frac{a^2}{4} \left[\sin \frac{\pi}{2} - \sin 0\right]$$

$$= \frac{a^2}{4}$$

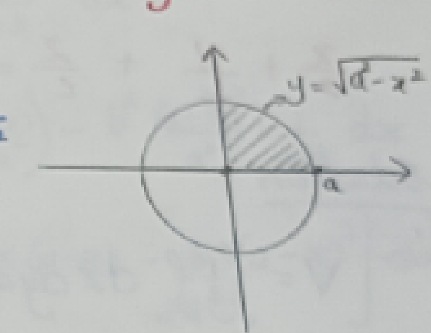
$\therefore$ Required Area $= 4 \times \frac{a^2}{4}$

$= a^2$ sq units.

4. Find the area of a circle $x^2 + y^2 = a^2$

x varies from 0 to a

y varies from 0 to $\sqrt{a^2 - x^2}$



$$A = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy \, dx$$

$$= \int_0^a y \Big|_0^{\sqrt{a^2-x^2}} dx$$

$$= \int \sqrt{a^2 - x^2} \, dx$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \Big|_0^a$$

$$= \frac{a^2}{2} \times \frac{\pi}{2} = \frac{\pi a^2}{4}$$

$\therefore$ Total area of the circle $= 4 \frac{\pi a^2}{4}$

$= \pi a^2$ sq units.

5. Find the volume of the tetrahedron

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Solution

$$V = \iiint_V dz\,dy\,dx = \iint_A z\,dy\,dx$$

consider $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\Rightarrow z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

if $z = 0$, $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow y = b\left(1 - \frac{x}{a}\right)$

if $z = 0, y = 0$, $\frac{x}{a} = 1 \Rightarrow x = a$

$$V = \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\,dx$$

$$= c\int_0^a y - \frac{yx}{a} - \frac{y^2}{2b}\Big|_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a y - \frac{yx}{a} - \frac{y^2}{2b}\,dx$$

$$= c\int_0^a b\left(1 - \frac{x}{a}\right) - \frac{bx}{a}\left(\frac{a-x}{a}\right) - b^2\frac{(a-x)^2}{a^2} \times \frac{1}{2\cancel{b}}\,dx$$

multiply

$$= c\int_0^a \left[\frac{b}{a}(a-x) - \frac{b}{a^2}x(a-x) - \frac{b}{2a^2}(a-x)^2\right] dx$$

$$= \frac{cb}{a}\int_0^a a - x - x + \frac{x^2}{a} - \frac{(a^2 + x^2 - 2ax)}{2a}\Big]\,dx$$

$$= \frac{cb}{a}\int_0^a a - 2x + \frac{x^2}{a} - \frac{a}{2} - \frac{x^2}{2a} + x\,dx$$

$$= \frac{cb}{a}\left[ax - x^2 + \frac{x^3}{3a} - \frac{a}{2}x - \frac{x^3}{6a} + \frac{x^2}{2}\right]_0^a$$

$$= \frac{cb}{a}\left[\cancel{a^2} - \cancel{a^2} + \frac{a^{3-1}}{3\cancel{a}} - \frac{\cancel{a^2}}{\cancel{2}} - \frac{a^{3-1}}{6\cancel{a}} + \frac{\cancel{a^2}}{\cancel{2}}\right]$$

$$= \frac{cb}{a}\left[+ \frac{a^2}{3} - \frac{a^2}{6}\right]$$

$$\frac{3a^2 + 2a^2}{6} = \frac{5a^2}{6}$$

$$= \frac{cb}{a}\left[\frac{a^2}{6}\right.$$

$$\frac{5a^2 - a^2}{6} = \frac{4a^2}{6}$$

$$= \frac{abc}{6}$$

6. Find the Volume generated by the revolution of the cardiod $r = a(1+\cos\theta)$ about the initial line.

Note: In polar form this formula becomes

$$V = \iint_A 2\pi r^2 \sin\theta \, d\theta \, dr.$$

θ varies from 0 to π

r varies from 0 to $a(1+\cos\theta)$

$$V = \int_0^{\pi} \int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= 2\pi \int_0^{\pi} \frac{a^3}{3} (1+\cos\theta)^3 \sin\theta \, d\theta$$

put, $1+\cos\theta = t$ if $\theta = 0$, $t = 2$

$-\sin\theta \, d\theta = dt$ $\theta = \pi$, $t = 0$

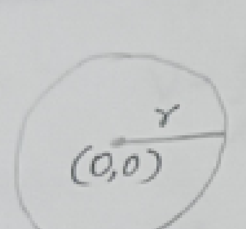
$$= \frac{2\pi a^3}{3} \int_2^0 t^3 (-dt)$$

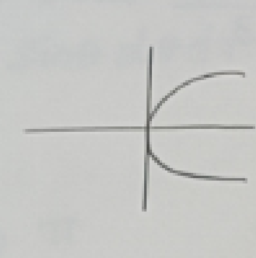
$$= -\frac{2\pi a^3}{3} \frac{t^4}{4}\Big|_2^0 = -2\pi\left(-\frac{2^4}{2^2}\right)$$

$$= \frac{8\pi a^3}{3} \text{ cubic units.}$$

Some important curves and their shapes.

1. $x^2+y^2=r^2$ ② $y^2=x$



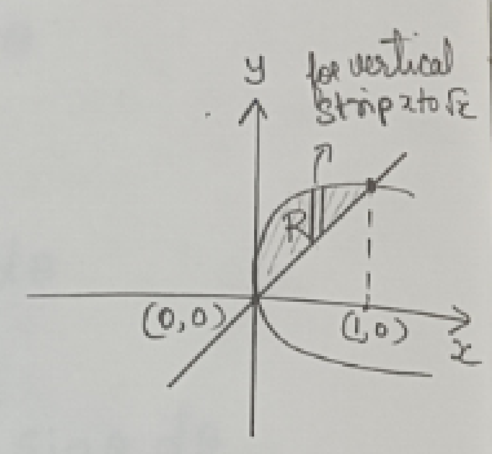


↑↑ limits of y
⇉ limits for x

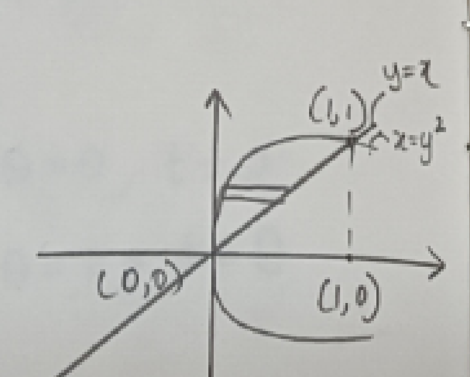
Evaluate the following by changing order of integration

1. $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

$I = \int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} xy \, dy \, dx$



R is the region bounded by the curve $y=x$, $y=\sqrt{x}$ between $x=0$ & $x=1$.



$y=x$ $\quad$ $y=\sqrt{x}$

$x=\sqrt{x}$ $\quad$ $x^2-x=0$

$x^2=x$ $\quad$ $x(x-1)=0$

$x=0, \; x=1$

$\therefore$ when $x=0, \; y=0$

$x=1, \; y=1$

$\therefore$ point of intersection are

$(0,0)$ & $(1,1)$

y varies from 0 to 1

x varies from y^2 to y

$I = \int_0^1 \int_{y^2}^{y} xy \, dx \, dy$

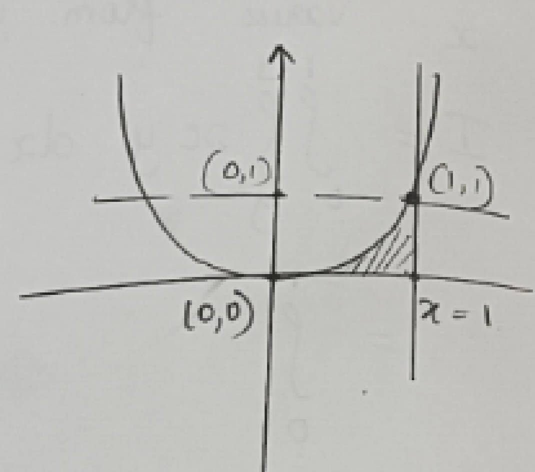
$= \int_0^1$

$\frac{1}{24}$

2. $\int_0^1 \int_{\sqrt{y}}^1 dx\, dy$

$x = \sqrt{y}$ $\quad x = 1$

$y = 1$



x varies from

y varies from

$I = \int_0^1 \int_0^{x^2} dy\, dx$

$\frac{1}{3}$

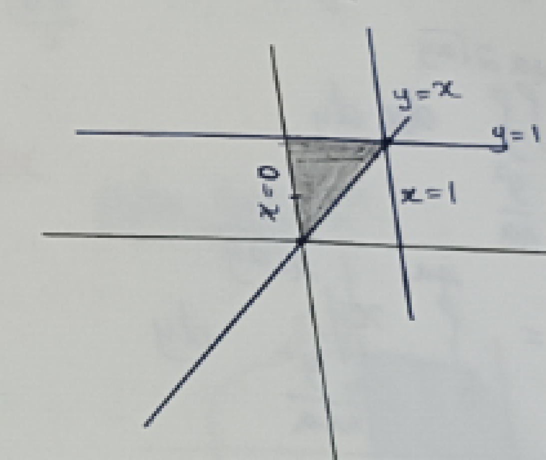
$= \int_0^1 y\Big|_0^{x^2} dx$

$= \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

3. $\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

out of QB

$y = x \quad x = 0$
$y = 1 \quad x = 1$
x



solution

x varies from 0 to y

y varies from 0 to 1

$I = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$

put $t = x^2+y^2$, $\quad 2x\,dx = dt$
$x\,dx = \frac{dt}{2}$

$x = 0 \quad t =$

$= \int_0^1 \int \frac{1}{\sqrt{t}} \frac{dt}{2}$

4. $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$

<u>solution</u>

x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$

y varies from 0 to $4a$

$$\int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$

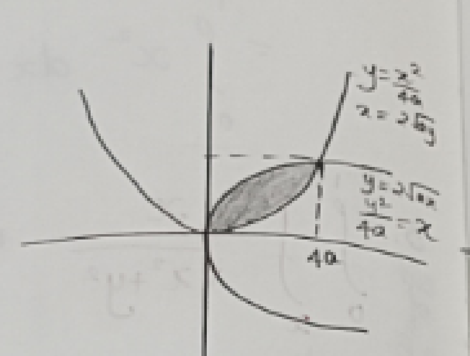
$$I = \int_0^{4a} x \Big|_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\Big|_0^{4a}$$

$y = \frac{x^2}{4a}$ $\quad x = 0$

$y = 2\sqrt{ax}$ $\quad x = 4a$



$y = \frac{x^2}{4a}$ $\quad y^2 = 4ax$

$$\frac{x^4}{16a^2} = 4ax$$

$$x^4 = 64a^3x$$

$$x^3 - 64a^3 = 0$$

$$x^3 - (4a)^3 = 0$$

$$(x-4a)(x^2+16a^2+4ax)$$

$$x = 4a$$

$$y = \frac{\cancel{16}^{4}a^{\cancel{2}}}{\cancel{4a}}$$

$$\boxed{y = 4a}$$

$$= \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{(4a)^3}{12a}$$

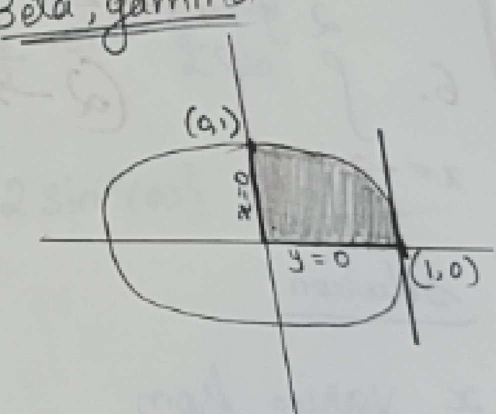
$$= \frac{4}{3} a^{1/2+3/2} \sqrt{4^3} - \frac{\cancel{64}^{16}a^3}{\cancel{12}_{3}a}$$

$$= \frac{4\times\sqrt{64}}{3} a^2 - \frac{16}{3}a^2$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16a^2}{3}$$

5. $\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} y^2\,dy\,dx$

<u>Beta, gamma</u>



x varies from 0 to $\sqrt{1-y^2}$

y varies from 0 to 1

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} y^2\,dx\,dy$$

$$= \int_0^1 xy^2 \Big|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \sqrt{(1-y^2)} \times y^2\,dy$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$

$y = \sin\theta$

$dy = \cos\theta \, d\theta$

$y = 0 \quad \theta = 0$

$y = 1 \quad \theta = \frac{\pi}{2}$

$$\frac{4}{4}\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta.$$

$$\frac{1}{4}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{\cancel{4}\cancel{2}}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi}$$

$$= \frac{1}{8}\frac{\pi}{2}$$

$$= \frac{\pi}{16}$$

6. $$\int_{x=-2}^{2}\int_{y=0}^{y=\sqrt{4-x^2}} (2-x)\, dy\, dx$$

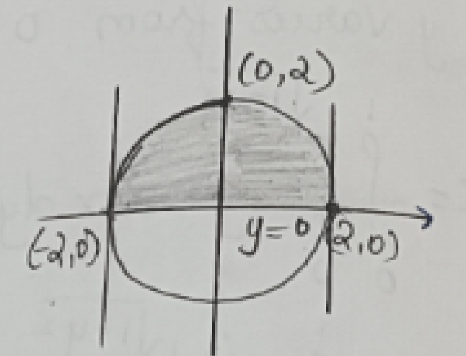
$x^2 + y^2 = 4$

$x = \sqrt{4-y^2}$

Solution

x varies from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$

y varies from 0 to 2



$$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2-x)\, d.\, d..$$

$$= \int_0^2 2x - \frac{x^2}{2}\Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \, dy.$$

$$= \int_0^2 2\sqrt{4-y^2} - \cancel{\frac{4-y^2}{2}} + 2\sqrt{4-y^2} + \cancel{\frac{(4-y^2)}{2}} \, dy$$

$$= \int_0^2 4\sqrt{4-y^2} \, dy.$$

$$= 4\left\{\cancel{\frac{y}{2}\sqrt{4-y^2}}^{\,0} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2$$

$$= 4\{+2\sin^{-1} 1 + 2\sin^{-1}(0)\}$$

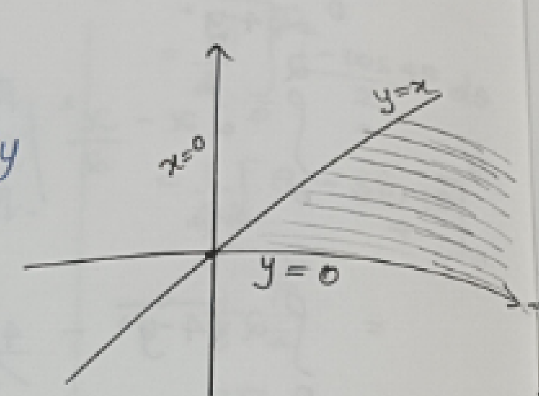
$$= 4\left\{+2\,\frac{\pi}{2}\right\}$$

$$= 4\pi$$

$$\int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

7. $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy \, dx$

$I = \int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} \, dx \, dy$



take $t = \frac{-x^2}{y}$

$dt = -\frac{2x}{y} dx$

$x \, dx = -\frac{y}{2} dt$

$x = y$ $\quad$ $x = \infty$

$t = -y$ $\quad$ $t = -\infty$

$I = \int_0^\infty \int_{-y}^{-\infty} e^t \, \frac{-y}{2} \, dt \, dy$

$= \int_0^\infty -\frac{y}{2} \; e^t \Big|_{-y}^{-\infty} dy$

$= \int_0^\infty -\frac{y}{2} \left(e^{-\infty} - e^{-y}\right) dy$

$= \frac{1}{2} \int_0^\infty y e^{-y} \, dy$ $\qquad$ ILATE (e^{-y}, y)

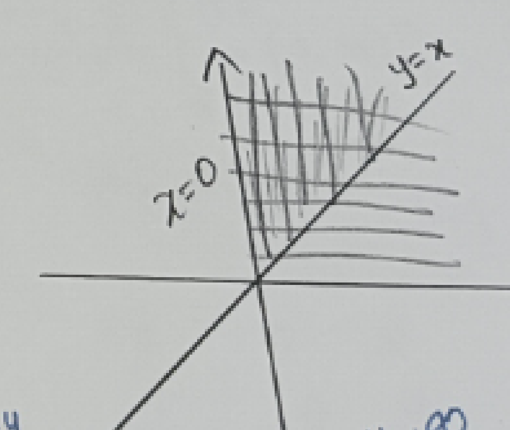
$= \frac{1}{2} \left[y \frac{e^{-y}}{-1} - \int \frac{e^{-y}}{-1} \cdot 1 \cdot dy \right]$

$= \frac{1}{2} \left[-ye^{-y} + \frac{e^{-y}}{-1} \right]_0^\infty$

$= \frac{1}{2} \left[\cancelto{0}{-\infty \times e^{-\infty}} + \cancelto{0}{\frac{e^{-\infty}}{-1}} - \left\{ -0 e^0 + \frac{e^{-0}}{-1} \right\} \right]$

$= \frac{1}{2} \times 1 = \frac{1}{2}$ //

8. $\int_{x=0}^{x=\infty} \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$



$\int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy$

$\int_0^y \frac{e^{-y}}{y} \times x \Big|_0^y dy = \int_0^y \frac{e^{-y}}{y} \times y \, dy = \frac{e^{-y}}{-1} \Big|_0^\infty = 1$

Beta and gamma functions

Gamma function is defined as,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx, \quad n>0.$$

Beta function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, \quad m>0, n>0$$

$$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

Properties of beta and gamma function

1. $\beta(m,n) = \beta(n,m)$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

W.K.T $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$= \int_0^1 (1-x)^{m-1}(1-1+x)^{n-1}\, dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\, dx$$

$$\beta(m,n) = \beta(n,m)$$

2. $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$$

$$= x^n \frac{e^{-x}}{-1}\Big|_0^\infty - \int_0^\infty \frac{e^{-x}}{-1} \cdot n x^{n-1}\, dx$$

$$= x^\infty \frac{e^{-\infty}}{-1} + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(6) = \Gamma(5+1)$$
$$= 5\Gamma(5) = 5\Gamma(4+1)$$
$$= 5\times 4\Gamma(4)$$
$$= 5\times 4\times 3\Gamma(3)$$
$$= 5\times 4\times 3\times 2\Gamma$$
$$= 5\times 4\times 3\times 2\times 1$$
$$\Gamma(n+1) = n!$$

3. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

put $x = \sin^2\theta$ $\quad dx = 2\sin\theta\cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

$$\beta(m,n) = \int_0^{\pi/2} (\sin\theta)^{2m-2} (1-\sin^2\theta)^{n-1} (2\sin\theta\cos\theta)\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-2} (\cos\theta)^{2n-2} \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$$

~~Transformation of Beta function~~

$$\cancel{\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{n+m}}\, dt}$$

$$\cancel{= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}\, dt}$$

1. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$x^{-1/2}$

$$\cancel{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx}$$

put $x = t^2$ $\quad dx = 2t\, dt$

$x=0$ $t=0$ $\quad x=\infty$ $t=\infty$

$$\cancel{\Gamma(n) = \int_0^\infty e^{-t^2} t^{2(n-1)}\, 2t\, dt}$$

$$\cancel{= 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt}$$

put $n = \frac{1}{2}$

$$\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-t^2} t^0 \, dt$$

$$= 2\int_0^\infty e^{-t^2} dt$$

$$\Gamma\tfrac{1}{2} \cdot \Gamma\tfrac{1}{2} = \left[2\int_0^\infty e^{-x^2} dx\right] \times \left[2\int_0^\infty e^{-y^2} dy\right]$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} \, dx\, dy$$

$x = 0 \quad x = \infty$
$\theta = \pi/2 \quad \theta =$

$x = r\cos\theta$ & $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

r varies from 0 to ∞, θ varies from 0 to $\pi/2$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} \, r\,dr\,d\theta$$

put $r^2 = t$
$2r\,dr = dt$

$$\left(\Gamma\tfrac{1}{r}\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-t} \frac{dt}{2} \, d\theta$$

$$= 2\int_0^{\pi/2} \left.\frac{e^{-t}}{-1}\right|_0^\infty d\theta$$

$$= 2\int_0^{\pi/2} -e^{-\infty} + e^0 \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\theta\Big|_0^{\pi/2}$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Evaluate the following definite integrals by converting into gamma functions.

(1) $\displaystyle\int_0^\infty x^{3/2} e^{-x} \, dx.$ $\qquad \Gamma(n) = \displaystyle\int_0^\infty e^{-x} x^{n-1} \, dx$

taking $n - 1 = 3/2$
$n = 5/2.$

$\Gamma(n+1) = n\Gamma(n)$
$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$$I = \Gamma(5/2)$$

$$= 3/2 \times 1/2 \, \Gamma\tfrac{1}{2} = \frac{3}{4}\sqrt{\pi}$$

ii) $\int_0^1 (\log x)^3 \, dx$

out of AB put $\log x = -t$ $\quad x = e^{-t}$ $\qquad x=0 \; t=\infty$, $\; x=1 \; t=0$

$\therefore \; dx = -e^{-t} dt$

$$I = \int_\infty^0 (-t)^3 (-e^{-t}) \, dt$$

$$= + \int_\infty^0 t^3 e^{-t} \, dt$$

$$= - \int_0^\infty t^3 e^{-t} \, dt$$

$$= - \Gamma(4) \qquad \Gamma(n+1) = n!$$

$$= -3!$$

$$I = \underline{\underline{-6}}$$

iii) $\int_0^\infty x^{3/2} e^{-4x} \, dx \qquad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

put $t = 4x \qquad x = \frac{t}{4}$

$dx = \frac{dt}{4}$

$$I = \int_0^\infty e^{-t} \left(\frac{t}{4}\right)^{3/2} \frac{dt}{4}$$

$$= \frac{1}{32} \int_0^\infty e^{-t} t^{3/2} \, dt$$

$$= \frac{1}{32} \int_0^\infty e^{-t} t^{\frac{5}{2}-1} \, dt$$

$$= \frac{1}{32} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{32} \Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{3}{2 \times 32} \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{2 \times 2 \times 32} \sqrt{\pi}$$

$$= \frac{3}{128} \sqrt{\pi}$$

iv) $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$ $\quad \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$t = x^3$ $\quad x = \sqrt[3]{t}$

$dx = \frac{1}{3} t^{-2/3} dt$

$$I = \int_0^\infty (t^{1/3})^{1/2}\, e^{-t} \frac{dt}{3} \times t^{-2/3}$$

$$= \frac{1}{3} \int_0^\infty t^{1/6} t^{-2/3} e^{-t}\, dt$$

$$= \frac{1}{3} \int_0^\infty t^{-1/2} e^{-t}\, dt$$

$$= \frac{1}{3} \int_0^\infty t^{+1/2-1} \cdot e^{-t}\, dt$$

$$= \frac{1}{3} \Gamma \tfrac{1}{2}$$

$$= \frac{\sqrt{\pi}}{3}$$

$\frac{1}{6} - \frac{2}{3}$

$\frac{1-2\times 2}{6}$

$\frac{1-4}{6}$

$-\frac{3}{6}$

v) $\int_0^\infty x^6 e^{-2x} dx$ $\quad \Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx$

$2x = t$

$x = \frac{t}{2}$ $\quad dx = \frac{dt}{2}$

$x = 0$ $\quad t = 0$

$x = \infty$ $\quad t = \infty$

$$I = \int_0^\infty \frac{t^6}{2^6}\, e^{-t}\, \frac{dt}{2}$$

$$= \frac{1}{2^7} \int_0^\infty e^{-t}\, t^{7-1}\, dt$$

$$= \frac{1}{2^7} \Gamma 7 \qquad \Gamma(n+1) = n!$$

$$= \frac{6!}{2^7} = \frac{720}{128} = \frac{45}{8}$$

Express the following in terms of beta function and then evaluate using gamma function.

(1) $\int_0^1 x^{3/2} (1-x)^{1/2}\, dx$

we have $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

taking $m-1 = \frac{3}{2}$ $\quad n-1 = \frac{1}{2}$

$m = \frac{5}{2}$ $\quad n = \frac{3}{2}$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(5/2)\,\Gamma(3/2)}{\Gamma(4)}$$

$$= \frac{\frac{3}{2}\Gamma(\frac{3}{2})\,\frac{1}{2}\Gamma(\frac{1}{2})}{3!}$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\Gamma(\frac{1}{2}) \times \frac{1}{2}\sqrt{\pi}}{6}$$

$$= \frac{3\pi}{8\times 6} = \frac{\pi}{16}$$

(2) $\int_0^2 (4-x^2)^{3/2}\, dx$

out of OB

Solution

$x = 2\sin\theta \qquad dx = 2\cos\theta\, d\theta$

$$= \int_0^2 (4 - 4\sin^2\theta)^{3/2}\, 2\, \cancel{dx} \cos\theta\, d\theta$$

$$= \int_0^2 4^{3/2} (\cos^2\theta)^{3/2} \times 2\cos\theta\, d\theta$$

$$= 8\int_0^2 \cos^4\theta\, d\theta$$

ii) $\int_0^4 x^{3/2}(4-x)^{5/2}\,dx$

Solution

put $x = 4\sin^2\theta$

$dx = 4\cos\theta\,d\theta\,\sin\theta$

$$I = \int_0^{\pi/2} (4\sin^2\theta)^{3/2}(4-4\sin^2\theta)^{5/2}\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 8\sin^3\theta\; 4^{5/2}(1-\sin^2\theta)^{5/2}\,4\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 16\times 2^5\times \sin\theta\, 2^2\cos^6\theta\,d\theta$$

$$= 2^{11}\int_0^{\pi/2} \sin^{\overset{2\times2-1}{4}}\theta\,\cos^6\theta\,d\theta$$

$\beta(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

$2m-1 = 4$

$2m = 5$

$\boxed{m = \frac{5}{2}}$

$$= 2^{11}\int_0^{\pi/2}\sin^4\theta\cos^6\theta\,d\theta$$

$$= 2^{10}\,\beta\left(\frac{5}{2},\frac{7}{2}\right)$$

$2n-1 = 6$

$2n = 7$

$\boxed{n = \frac{7}{2}}$

$$= \frac{2^{10}\,\Gamma\frac{5}{2}\,\Gamma\frac{7}{2}}{\Gamma 6}$$

$$= \frac{2^{10}\,\frac{3}{2}\Gamma\frac{3}{2}\;\frac{5}{2}\Gamma\frac{5}{2}}{5!}$$

$$= \frac{2^{10-5}\,\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\cancel{5}}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{\cancel{5}\times4\times3\times2}$$

$$= \frac{2^{5-3}\times \overset{3}{9}\pi}{\cancel{24}_{8}} = 12\pi$$

iii) $\int_0^\infty \frac{dx}{1+x^4}$

put $x^4 = \tan^2\theta$

$x = (\tan\theta)^{1/2}$

$dx = \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta$.

θ varies from 0 to $\pi/2$

$$I = \int_0^{\pi/2}\frac{1}{1+\tan^2\theta}\,\frac{(\tan\theta)^{-1/2}\sec^2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{(\tan\theta)^{-1/2}}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{-1/2} \cos\theta^{1/2}\, d\theta$$

$$= \frac{1}{2}\, \beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$$

$$= \frac{1}{2\times 2}\, \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4} \frac{\Gamma\frac{1}{4}\, \Gamma\frac{3}{4}}{\Gamma 1} \qquad \Gamma n\, \Gamma 1-n = \frac{\pi}{\sin(\pi n)}$$

$$= \frac{1}{4} \frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{1}{4} \times \pi\sqrt{2}$$

$$= \frac{\pi\sqrt{2}}{4} = \frac{\pi}{2\sqrt{2}}$$

iv) $\int_0^{\infty} \frac{dx}{\sqrt{x}\,(1+x)}$

$x = \tan^2\theta$

$dx = 2\tan\theta \sec^2\theta\, d\theta$

θ varies from 0 to $\pi/2$

$$I = \int_0^{\pi/2} \frac{1}{\cancel{\sec^2\theta}} \frac{2\tan\theta\cancel{\sec^2\theta}\, d\theta}{\tan\theta}$$

$$= 2\int_0^{\pi/2} 1.\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta\, d\theta \qquad 2m-1=0$$

$$= 2\times\beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{\Gamma\frac{1}{2}\, \Gamma\frac{1}{2}}{\Gamma 1} = \pi$$

Show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \pi$

$$I = \int_0^{\frac{\pi}{2}} \sin\theta^{-1/2}\, d\theta \times \int_0^{\frac{\pi}{2}} (\sin\theta)^{1/2}\, d\theta$$

$2m-1 = -\frac{1}{2}$, $m = \frac{1}{4}$; $2n-1=0$, $n = \frac{1}{2}$; $2m-1 = \frac{1}{2}$, $m = \frac{3}{4}$, $n = \frac{1}{2}$

WKT $\int_0^{\frac{\pi}{2}} \sin^p\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}} \times \frac{\sqrt{\pi}}{2}$

$$I = \frac{\cancel{\Gamma\frac{3}{4}}}{\Gamma\frac{5}{4}} \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{1}{4}}{\cancel{\Gamma\frac{3}{4}}} \frac{\sqrt{\pi}}{2} =$$

$\frac{1}{4}+\frac{1}{2}$

$\frac{1+2}{4}$

$$= \frac{\pi}{4} \frac{\Gamma\frac{1}{4}}{\Gamma\frac{1}{4}+1}$$

$\frac{3}{4}+\frac{1}{2}$

$\frac{3+2}{4}$

$$= \frac{\pi}{4} \frac{\cancel{\Gamma\frac{1}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}} = \pi$$

$$= \frac{\beta\left(\frac{1}{4}, \frac{1}{2}\right)}{2} \times \frac{\beta\left(\frac{3}{4}, \frac{1}{2}\right)}{2}$$

$$= \frac{1}{4}\left[\frac{\Gamma\frac{1}{4}\, \Gamma\frac{1}{2}}{\cancel{\Gamma\frac{3}{4}}} \times \frac{\cancel{\Gamma\frac{3}{4}}\, \Gamma\frac{1}{2}}{\Gamma\frac{5}{4}}\right]$$

$$= \frac{\pi}{4}\left[\frac{\cancel{\Gamma\frac{1}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}}\right.$$

P.T $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^{\infty} \sqrt{x}\, e^{-x^2}\, dx = \frac{\pi}{2\sqrt{2}}$

$I_2 = \int_0^{\infty} e^{-x^2} x^{-1/2}\, dx$ | $I_1 = \int_0^{\infty} x^{1/2} e^{-x^2}\, dx$

we have $\frac{1}{2}\Gamma n = \int_0^{\infty} e^{-x^2} x^{2n-1}\, dx$

WKT $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx$ $\qquad x = t^2$, $dx = 2t\,dt$

$$\Gamma n = \int_0^{\infty} e^{-t^2} t^{2n-2}\, 2t\, dt$$

$$\frac{\Gamma n}{2} = \int_0^{\infty} e^{-t^2} t^{2n-1}\, dt \quad \text{— (1)}$$

compare I_1 & (1)

$2n-1 = \frac{1}{2}$

$n = \frac{3}{4}$

$I_1 = \frac{1}{2}\Gamma\frac{3}{4}$

compare I_2 & (1)

$2n-1 = -\frac{1}{2}$ $\quad n = \frac{1}{4}$

$I_2 = \frac{1}{2}\Gamma\frac{1}{4}$

Hence $I_1 \times I_2 = \frac{1}{2}\Gamma\frac{3}{4} \times \frac{1}{2}\Gamma\frac{1}{4}$

$$= \frac{1}{4}\,\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi\sqrt{2}}{4}$$

$$= \frac{\pi}{2\sqrt{2}}$$

Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$

solution

Let $m, n = \frac{1}{2}$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1}$$

$$\left(\Gamma\frac{1}{2}\right)^2 = 2\int_0^{\pi/2} \sin^{2(\frac{1}{2})-1}\theta \cos^{2(\frac{1}{2})-1} d\theta$$

$$= 2\int_0^{\pi/2} 1 \cdot d\theta = \cancel{2}\left(\frac{\pi}{\cancel{2}} - 0\right)$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$