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The function

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \quad -(1)$$

where n is a positive integer and $a_0, a_1, a_2 \cdots a_n$ are constants with $a_0 \neq 0$ is known as a polynomial of degree n.

The values of x making $f(x)$ zero are known as zeroes or roots of the polynomial $f(x)$ and every polynomial of degree 'n' has 'n' roots.

The equation of the form $f(x) = 0$ is called Algebraic or transcendental.

(1) If $f(x)$ is purely polynomial, then the equation is in algebraic form

(2) If $f(x)$ contains some other functions such as logarithmic, exponential and trigonometric functions etc.,

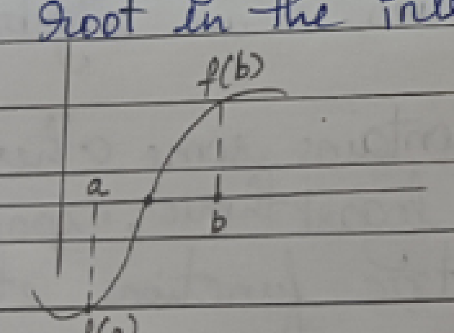
Iteration Process.

Numerical methods of finding approximate roots of the given equation is a repetitive type of process known as Iteration process.

Fundamental property.

If $f(x)=0$ is a real valued continuous function of the real variable x. We have the following property.

If there exist two values a, b such that $f(x)$ has opposite signs, say $f(a)<0$, $f(b)>0$. Equivalently $f(a) \cdot f(b)<0$, then there exist atleast one real root in the interval (a,b).



For $f(x) = x^2+4x+4$

$f(0)=4$ $f(1)=9$ $f(2)=16$

$f(-1)=1$ $f(-2)=0$

since $f(-2)=0$

one of the root is -2

... $f(b)$ are of opposite signs then there is a root between a and b

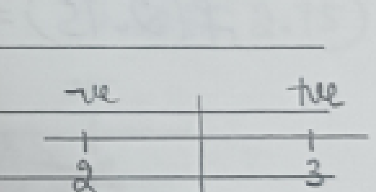
formula: $x_1 = \frac{a+b}{2}$

Problems

① Find a root of the equation $x^3-4x-9=0$, using bisection method correction to two decimal places.

Solution

Let $f(x) = x^3-4x-9$



$f(1) = 1-4-9 = -12 = -ve$

$f(2) = 8-8-9 = -9 = -ve$

$f(3) = 27-12-9 = 27-21 = 6 = +ve$

Iteration Process.

Numerical methods of finding approximate roots of the given equation is a repetative type of process known as iteration process.

Fundamental property.

If $f(x)=0$ is a real valued continuous function of the real variable x. We have the following property.

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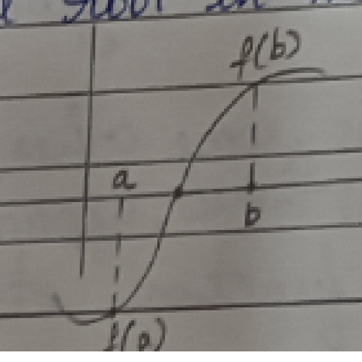


Bisection method

This method is used to find the root of the equation $f(x)=0$ in the interval (a,b).

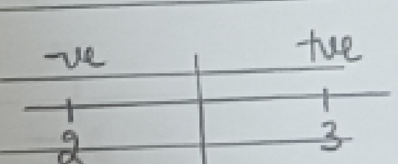
If $f(x)$ is continuous $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b

formula: $x_1 = \frac{a+b}{2}$

Problems

① Find a root of the equation $x^3-4x-9=0$, using bisection method correction to two decimal places.

Solution



Let $f(x) = x^3-4x-9$

$f(1) = 1-4-9 = -12 = -ve$

$f(2) = 8-8-9 = -9 = -ve$

$f(3) = 27-12-9 = 27-21 = 6 = +ve$

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Since $f(2)$ is -ve and $f(3)$ is +ve,
a root lies between 2 and 3.

∴ First approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

then $f(2.5) = -3.375$

-ve	-ve	+ve
2	2.5	3

∴ Root lies between $(2.5, 3)$

II Approximation:

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$f(2.75) = 0.7969 = +ve$

-ve	+ve	+ve
2.5	2.75	3

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∴ Root lies between $(2.5, 2.75)$

III Approximation:

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$f(2.625) = -1.4121 = -ve$

-ve	-ve	+ve
2.5	2.625	2.75

∴ Root lies between $(2.625, 2.75)$

IV Approximation:

-ve	-ve	+ve
2.625		2.75

$x_4 = 2.6875$

$f(x_4) = -0.3391 = -ve$

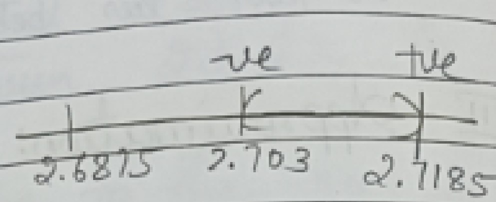
∴ root lies b/w $(2.6815, 2.75)$

V Approximation: $x_5 = 2.7185$

$f(x_5) = 0.2163 = +ve$

∴ root lie b/w $(2.6815, 2.7185)$

VI Approximation

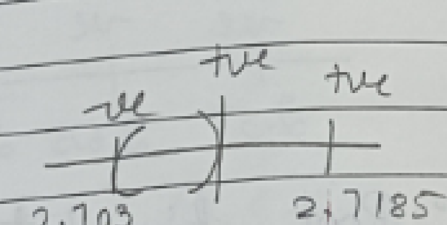


$x_6 = 2.703$

$f(x_6) = -0.0633 = -ve$

$\therefore$ roots lies b/w $(2.703, 2.7185)$

VII Approximation

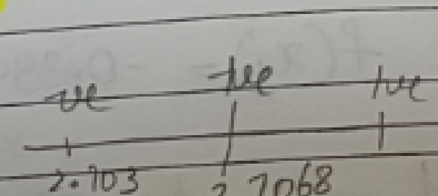


$x_7 = 2.71075$

$f(x_7) = 0.0760 = +ve$

$\therefore$ roots lies b/w $(2.703, 2.71075)$

VIII Approximation



$x_8 = 2.7068$

$f(x_8) = 0.0048 = +ve$

$\therefore$ root lies between $(2.703, 2.7068)$

IX Approximation

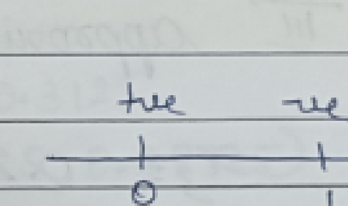
$$x_9 = \frac{2.703 + 2.7068}{2} = 2.7049$$

Hence the root is 2.7049 approximately.

② Find a root of the eq^n $x^3 - 4x + 1 = 0$, using bisection method.

<u>solution</u>

$f(x) = x^3 - 4x + 1$



$f(0) = 1$

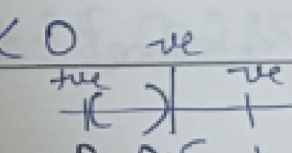
$f(1) = 1 - 4 + 1 = -2$

$f(2) = 8 - \cancel{8} + 1 = 1$

$\therefore$ root lies between $(0, 1)$

I approximation : $x_1 = \frac{0+1}{2} = 0.5$

$f(0.5) = -0.875 < 0$ -ve



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$\therefore$ root lies b/w $(0, 0.5)$

II approximation

$$x_2 = \frac{0+0.5}{2} = 0.25$$

$$f(0.25) = 0.0156 > 0$$

+ve   +ve   -ve

0   0.25   0.5

$\therefore$ root lies between $(0.25, 0.5)$

III approximation

$$x_3 = \frac{0.25+0.5}{2} = 0.375$$

$$f(0.375) = -0.4472 < 0$$

+ve   -ve   -ve

0.25   0.375   0.5

$\therefore$ root lies b/w $(0.25, 0.375)$

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IV approximation : $x_3 = 0.3125$

$f(0.3125) = 0.$ [illegible] $=$ -ve

+ve   -ve   -ve

0.25   0.3125   0.375

$\therefore$ root lies between $(0.25, 0.3125)$

V approximation : $x_4 = 0.28125$

$$f(0.28125) = -0.1027 < 0$$

+ve   -ve   -ve

0.25   0.28125   0.3125

$\therefore$ root lies b/w $(0.25, 0.28125)$

VI app : $x_5 = 0.2656$

$$f(x_5) = -0.0436 < 0$$

+ve   -ve

0.25   0.2656   0.28125

$\therefore$ root lies b/w $(0.25, 0.2656)$

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VII Approximation: $x_7 = 0.2578$

$f(x_7) = -0.0140 < 0$

+ve -ve -ve

0.25 0.2578 0.2650

$\therefore$ root lies b/w $(0.25, 0.2578)$

VIII iteration: $x_8 = 0.2539$

$f(x_8) = 0.00076 > 0$

$\therefore$ Root lies b/w $(0.2539, 0.2578)$

IX iteration: $x_9 = 0.25585$

$\therefore$ the root is 0.255

RF

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Regula Falsi Method (or) Method of False Position.

× To find the numerical solution of equation $f(x)=0$ by Regula falsi method (or) method of false position×

Let the root of the equation $f(x)=0$ lies between a and b then $f(a)$ and $f(b)$ are of opposite sign.

$\therefore$ The approximation of the root is given by

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

The procedure is repeated upto two consecutive approximate root becomes equal.

RF

1. Use Regula Falsi method to find a real root of $x^3-4x-9=0$ correct to three decimal places.

solution

Let $f(x)=x^3-4x-9$.

$f(0)=-9$

$f(1)=1-4-9=-12$

$f(2)=-9$

$f(3)=6$

$\therefore$ root lies b/w $(2,3)$ — -ve at 2, +ve at 3

Here, $a=2$, $b=3$; $f(a)=f(2)=-9$, $f(b)=f(3)=6$

I approximation:

$$x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

$$x_1=\frac{2\times 6+3\times 9}{6+9}$$

$$=\frac{12+27}{15}=\frac{39}{15}$$

$$x_1=\frac{13}{5}=2.6$$

$x_1=2.6$

$f(x_1)=-1.824<0$ — -ve at 2.6, +ve at 3

II approximation: $a=2.6$, $b=3$; $f(a)=-1.824$, $f(b)=6$

$$x_2=\frac{af(b)-bf(a)}{f(b)-f(a)}=\frac{15.6+5.472}{7.824}$$

$$=\frac{21.072}{7.824}$$

$x_2=2.6932$ — -ve at 2.6932, +ve at 3

$f(2.6932)=-0.2381<0$

x^3-4x-9

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III Approximation:

$a = 2.6932 \qquad f(a) = -0.2381$
$b = 3 \qquad f(b) = 6$

$$x_3 = \frac{16.1592 + 0.7143}{6.2381}$$

$x_3 = 2.7049$

-ve: 2.7049 | +ve: 3

$f(x_3) = -0.0292 < 0$

IV Approximation: $a = 2.7049 \quad f(a) = -0.0292$
$b = 3 \quad f(b) = 6$

$$x_4 = \frac{2.7049 \times 6 + 3 \times 0.0292}{6 + 0.0292}$$

$$= \frac{16.2294 + 0.0876}{6.0292} = \frac{16.317}{6.0292}$$

$x_4 = 2.7063$

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$f(x_4) = -0.00409$

-ve: 2.7063 | +ve: 3

V Approximation: $a = 2.7063 \quad f(a) = -0.00409$
$b = 3 \quad f(b) = 6$

$$x_5 = \frac{2.7063 \times 6 + 3 \times 0.00409}{6 + 0.00409}$$

$$= \frac{16.25007}{6.00409} = 2.7065$$

$\therefore$ 2.706 is the required approximate root correct to three decimal places.

2. Find the real root of the equation $x \log x - 1.2 = 0$ that lies b/w 2 and 3 correct to 3 decimal places by Regula Falsi method.

Solution: Let $f(x) = x \log x - 1.2$

Given the root lies b/w $(2, 3)$

$x\log x - 1.2$

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I Approximation: $a = 2$ $f(a) = -0.5979$
$b = 3$ $f(b) = 0.2313$

$\frac{2.2563}{0.8292}$

-ve +ve
2.7210 3

$x_1 = 2.7210$

$f(x_1) = -0.0171$

II Approximation: $a = 2.7210$ $f(a) = -0.0171$
$b = 3$ $f(b) = 0.2313$

0.62936

$x_2 = \frac{0.62936 + 0.0513}{0.9523}$

-ve +ve
2.7402 3

$x_2 = 2.7402$ $f(x_2) = -0.000389$

IV Approximation: $a = 2.7402$ $f(a) = -0.000389$
$b = 3$ $f(b) = 0.2313$

$x_3 = \frac{0.63497}{0.231689} = 2.7406$

$\therefore$ 2.7406 is the required approximate root correct to three decimal places.

RF

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3. Find a real root of $x^3 - 5x - 7 = 0$ by Regula Falsi method.

$f(2) = -9$ $f(3) = 5$

solution: $x_1 = 2.6428$ $f(x_1) = -1.755$
$\therefore$ root lies b/w $(2.6428, 3)$

II Approximation: $x_2 = 2.7356$ $f(x_2) = -0.2061$
$\therefore$ root $(2.7356, 3)$

III : $x_3 = 2.7461$ $f(x_3) = -0.02198$
$\therefore$ root lies $(2.7461, 3)$

IV : $x_4 = 2.7472$ $f(x_4) = -0.0026$
$\therefore$ root lies b/w $(2.7472, 3)$

V : $x_5 = 2.7473$

4. Use Regula Falsi method, to find
⑥ a real root of the equation
$2x - \log_{10} x = 7$ which lies between
3 and 4.

Solution

$a = 3$ $f(3) = -1.4771$
$b = 4$ $f(4) = 0.3979$

I App: $x_1 = 3.7877$ $f(x) = -0.0029$
root lies in $(3.7877, 4)$

II App: $x_2 = 3.7892$ $f(x_2) = -0.0001$
root lies in $(3.7892, 4)$

III App: $x_3 = 3.7892$

Thus the required approximate Value is 3.7892

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5. Find the real root of the equation
10 $x^3 - 2x - 5 = 0$ which lies between $(2,3)$
by using Regula Falsi Method.

Solution:

Check

$a = 2$ $f(2) = -1$
$b = 3$ $f(3) = 16$

VI $x_6 = 2.0943$
$f() = -0.0027$
VII $x_7 = 2.0945$
$f() = -0.0010$

I App: $x_1 = 2.0588$
$f(x_1) = -0.39105$ -0.3908
$\therefore$ root lies in $(2.0588, 3)$

II App: $x_2 = 2.0812$ 2.0813
$f(x_2) = -0.14792$
$\therefore$ root lies in $(2.0812, 3)$

III App: $x_3 = 2.0896$
$f(x_3) = -0.0551$
-0.0547
$\therefore$ root lies in $(2.0896, 3)$

IV App: $x_4 = 2.0927$
$f(x_4) = -0.0206$
0.0202
$\therefore$ root lies in $(2.0927, 3)$

V App: $x_5 = 2.093$, ~ 2.0939
$f(x_5) = -0.0015$
$\therefore$ root lies in $(2.0932, 3)$

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6. Find the real root of the equation
11 $x^3-3x+4=0$ which lies between $(-3,-2)$ by using Regula-Falsi method.

Solution:

$f(-3)=-14$ $\quad$ $f(-2)=2$

$a=-3$ $\quad$ $b=-2$

I App: $x_1=-2.125$ $\quad f(x_1)=0.7792$

$\therefore$ root lies b/w $(-3,-2.125)$

II App: $x_2=-2.1711$ $\quad \therefore$ root lies b/w

$f(x_2)=0.2794$ $\quad (-3,-2.1711)$

III App: $x_3=-2.1873$ $\quad f(x_3)=0.0972$

root lies b/w $(-3,-2.1873)$.

IV App: $x_4=-2.1929$ $\quad f(x_4)=0.0334$

root lies b/w $(-3,-2.1929)$

V App: $x_5=-2.1948$ $\quad f(x_5)=0.01172$

root lies b/w $(-3,-2.1948)$

VI App: $x_6=-2.1954$

RF

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7. Show that a real root of the equation $\tan x+\tanh x=0$ lies between
4 2 and 3. Then apply the Regula Falsi method to find the third approximation.

Solution

$f(2)=-1.221$

$f(3)=0.8525$

I Approximation: $a=2$ $\quad f(2)=-1.2210$

$b=3$ $\quad f(3)=0.8525$

$x_1=\dfrac{5.368}{2.0735}=2.5888$

$f(x_1)=0.3718$

-ve at 2, +ve at 2.5888

$\therefore$ root lies b/w $(2, 2.5888)$

II Approximation: $a=2$ $\quad f(a)=-1.2216$

$b=2.5888$ $\quad f(b)=0.3178$

$x_2=2.4512$

$f(x_2)=0.1592$

-ve at 2, +ve at 2.4512

$\therefore$ root lies between $(2, 2.4512)$

III App: $a = 2$ $\quad f(a) = -0.221$
$b = 2.4512$ $\quad f(b) = 0.1592$

$x_3 = 2.3991$

Thus the required approximate root after

thus the required approxim is 2.399.

8. Using Regula Falsi method find the root of the equation $xe^x = \cos x$ that lies between 0.4 and 0.6 carry out four iterations. (round off)

Solution

Let $f(x) = xe^x - \cos x$

$a = 0.4$ -ve $\quad b = 0.6$ +ve

$f(0.4) = -0.3243$

$f(0.6) = 0.2679$

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I App: $a = 0.4$ $\quad f(a) = -0.3243$
$b = 0.6$ $\quad f(b) = 0.2679$

$x_1 = 0.5095$

$f(x_1) = -0.0249$

$\therefore$ root lies b/w $(0.5095, 0.6)$

II App: $a = 0.5095$ $\quad f(a) = -0.0249$
$b = 0.6$ $\quad f(b) = 0.2679$

$\therefore x_2 = 0.5171$

$f(x_2) = -0.00199$

$\therefore$ root lies b/w $(0.5171, 0.6)$

III App: $a = 0.5171$ $\quad f(a) = -0.00199$
$b = 0.6$ $\quad f(b) = 0.2679$

$x_3 = 0.5177$

$f(x_3) = -0.00017$

$\therefore$ root lies b/w $(0.5177, 0.6)$

IV App: $a = 0.5177$ $\quad f(a) = -0.00017$
$b = 0.6$ $\quad f(b) = 0.2679$

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$x_4 = 0.5177$

Thus the required root after four iteration is 0.5177

H.W. Find the real root of the equation $\cos x = xe^x$ to four decimal places using Regula falsi method.

(7)

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9. Find a real root of the equation $\cos x - 3x + 1 = 0$ correct to 3 decimal places using Regula Falsi method taking $x_0 = 0.5$

Date: / /

10. Find the real root of the equation $xe^x = 2$ by using Regula falsi method.

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Newton Raphson Method

Let $f(x)=0$ be the given equation and let x_0 be an approximate root of the equation. The first approximation to the root x_0 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The second approximation is obtained by replacing x_0 by x_1 in the RHS of the above ~~expansion~~

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is called the Newton Raphson ~~method.~~ formula

This procedure is repeated upto two consecutive approximated roots becomes equal.

Base change Rule $\log_a b = \frac{\log_c b}{\log_c a}$ $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a} = \frac{1}{x}\log_a e$

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① Using NRM find the root of $x\log x = 1.2$ near 2.5 carry out 3 iterations

Solution Let $f(x) = x\log_{10} x - 1.2$.

$\log_b a = \frac{\log_c a}{\log_c b}$ $f'(x) = \frac{1}{\log_e 10}(1+\log_e x)$ $f'(x)=x$

$$f'(x) = 0.4343(1+\log x)$$

Given $x_0 = 2.5$

$\Rightarrow$ ~~$f(x) = x\log.$~~

$x - ((x\log x - 1.2) \div (0.4343(1+\ln(x)))$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_1 = 2.7465$

$x_2 = 2.74065$

$x_3 = 2.7406$

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Thus the required approximate root after 3 iterations is 2.7406

2. Find a real root of the equation $x\sin x + \cos x = 0$ nearer $x=\pi$ correct to 4 decimal places using NRM

$f(x) = x\sin x + \cos x$

$f'(x) = x\cos x + \sin x - \sin x$

X-((xsin(x)+cos(x))÷(xcos(x))

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 2.8233$

$x_2 = 2.7986$

$x_3 = 2.7984$

$x_4 = 2.7984$

Thus required root is 2.7984

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4. Find a root of the equation $\tan x = x$ which is near to $x = 4.5$ using NRM

Solution:

Let $f(x) = \tan x - x$ $1+\tan^2 x = \sec^2 x$

$f'(x) = \sec^2 x - 1$

$f'(x) = \tan^2 x$

1 Approximation:

$x_1 = x_0 - \frac{\tan x_0 - x_0}{\tan^2 x_0}$

$x_1 = 4.4936$

$x_2 = 4.4934$

$x_3 = 4.4934$

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4. Find the real root of the equation $xe^x = 2$ near 0.5 correct to 3 decimal places using NRM.

Solution:

$f(x) = xe^x - 2$
$f'(x) = xe^x + e^x$

$$x_1 = x_0 - \frac{xe^x - 2}{xe^x + e^x}$$

$x_1 = 0.9153$

$x_2 = 0.8633$
$x_3 = 0.8526$
$x_4 = 0.8526$

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5. Find the root of the equation $x^3+5x-11=0$ nearer to 1.5, carry out iterations upto four decimal places using NRM

$f(x) = x^3+5x-11$
$f'(x) = 3x^2+5$

$$x_1 = x_0 - \frac{(x^3-5x-11)}{(3x^2-5)}$$

$= 1.5106$

$x_2 = 1.5105$
$x_3 = 1.5105$

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Interpolation

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable. ~~X~~ It is the art of reading x. For example we are given the following data

x	1941	1951	1961	1971	1981
f(x)	12	15	20	27	39

The process of finding $f(1956), f(1964)$, etc..., is known as interpolation.

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* It is used to fill gaps in the statistical data for the sake of continuity of information.

* For example if we know the records for the past five years except the third year which is not available due to unforeseen conditions, the interpolation technique helps to estimate the record for that year too under the assumption that the changes in the records over these five years have been uniform.

Extrapolation is the process of computing the value of the function outside the given range.
For example, in the above data, the process of finding $f(1931), f(1985), f(1991)$ is called extrapolation.

Forward difference operator : The forward difference operator, denoted by Δ, is a finite difference operator that calculates the difference b/w a function's value at a point and its value at the next point.
defined as $\Delta f(x) = f(x+h) - f(x)$

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Newton's Forward interpolation formula

It is a formula to be used to find the value of y, when the value of x is near to x_0.

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots$$

where $r = \frac{x - x_0}{h}$

Newton's Backward interpolation formula:

It is a formula to be used to find the value of y, when x is near to x_n

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots$$

where $r = \frac{x - x_n}{h}$

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① Find $y(1.4)$ using Newton's forward interpolating formula.

x	1	2	3	4	5
$y = f(x)$	10	26	58	112	194

Solution

To find y at $x = 1.4$

Since, the value 1.4 is at beginning of table, so we have to use Newton's forward interpolation formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	10				
		16			
2	26		16		
		32		6	
3	58		22		0
		54		6	
4	112		28		
		86			
5	194				

we have Newton's forward interpolation formula.

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$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots$$

where $r = \frac{x - x_0}{h}$

$x = 1.4$ $\quad r = \frac{1.4-1}{1}$

$x_0 = 1$

$h = 1$ $\quad \boxed{r = 0.4}$

from the table,

$\Delta y_0 = 16 \quad \Delta^2 y_0 = 16$

$\Delta^3 y_0 = 6 \quad \Delta^4 y_0 = 6$

$$y(1.4) = 10 + 0.4 \times 16 + \frac{0.4(0.4-1)}{2!} \times 16 + \frac{0.4(0.4-1)(0.4-2)}{3!} \times 6 + 0$$

$= 10 + 6.4 - 1.92 + 0.384$

$y(1.4) = 14.864$

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2. Use Newton's Backward interpolation formula to find $f(410)$.

x	100	150	200	250	300	350	400
$f(x)$	10.63	13.03	15.03	16.81	18.42	19.9	21.27

Solution: To find $f(x)$ at $x = 410$

we find $f(410)$ using Newton's backward interpolation formula,

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
100	10.63	2.4					
150	13.03	2	-0.4	0.18	-0.13		
200	15.03	1.78	-0.22	0.05	-0.01	0.12	-0.13
250	16.81	1.61	-0.17	0.04	-0.02	-0.01	
300	18.42	1.48	-0.13	0.02			
350	19.9	1.37	-0.11				
400	21.27						

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$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots$$

$y_n = 21.27$ $\qquad \nabla^3 y_n = 0.02$

$\nabla y_n = 1.37$ $\qquad \nabla^4 y_n = -0.02$

$\nabla^2 y_n = -0.11$ $\qquad \nabla^5 y_n = -0.01$

$\nabla^6 y_n = -0.13$

$$r = \frac{x - x_n}{h} = \frac{410 - 400}{50} = 0.2$$

$$y(410) = 21.27 + 0.2 \times 1.37 + \frac{0.2(0.2+1)(-0.11)}{2!}$$

$$+ \frac{0.2 \times 1.2 \times 2.2 \times 0.02}{3!} + \frac{0.2 \times 1.2 \times 2.2 \times 3.2(-0.02)}{4!}$$

$$+ \frac{0.2 \times 1.2 \times 2.2 \times 3.2 \times 4.2(-0.01)}{5!} +$$

$$\frac{0.2 \times 1.2 \times 2.2 \times 3.2 \times 4.2 \times 5.2(-0.13)}{6!}$$

$$= 21.27 + 0.274 - 0.0132 + 0.00176 - 0.00141 - 0.0006 - 0.0066$$

$$= 21.5239$$

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3. The population of a town is given by table.

Year	1951	1961	1971	1981	1991
Population in thousand	19.19	39.65	58.81	77.21	94.64

Using Newton's forward and backward interpolation, Calculate increase population in 1955 to 1985.

Solution

Using Newton's forward interpolation formula

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} + \dots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1951	19.19	20.46			
961	39.65		-1.3	0.54	
1971	58.81	19.16	-0.76		-0.75
1981	77.21	18.4	-0.97	-0.21	
1991	94.64	17.43			

$$r = \frac{x - x_0}{h} = \frac{x - 1951}{10}$$

Newton forward interpolation.

$$r = \frac{1955 - 1951}{10} = \frac{4}{10} = 0.4$$

$$f(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \cdots$$

$$= 19.19 + 0.4 \times 20.46 + \frac{0.4(0.4-1)}{2!}(-1.3) +$$

$$+ \frac{0.4(0.4-1)(0.4-2)}{3!} \times 0.54 +$$

$$\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} \times (-0.75)$$

$$= 19.19 + 8.184 + 0.156 + 0.0346 + 0.0312$$

$$f(1955) = 27.5958$$

Newton Backward interpolation

$$r = \frac{1985 - 1991}{10} = \frac{-6}{10} = -0.6$$

$$(1985) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 94.64 + (-0.6)(17.43) + \frac{(-0.6)(-0.6+1)}{2}(-0.97)$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-0.21) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+3)}{24}(-0.75)$$

$$= 94.64 - 10.458 + 0.1164 + 0.0118 + 0.0252$$

$$= 84.3354$$

Thus the increase in population from year 1955 to 1985 is $84.3354 - 27.5958$
$= 56.7396$ thousand.

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4. From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Interval 40-45 lies in beginning of table so we have to use Newtons forward interpolation formula.

students who score 40 and below 40

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
below 40	31				
below 50	73	42	~~+09~~	~~-25~~	37
below 60	124	51	-16	12	
below 70	159	35	-4		
below 80	190	31			

$r = \frac{45-40}{10} = 0.5$

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$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$

$= 31 + 0.5 \times 42 + 0.5\frac{(0.5-1)}{2} \times 9 +$

$\frac{0.5(0.5-1)(0.5-2)}{6}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} 37$

$= 31 + 21 + (-1.125) - 1.5625 - 1.4453$

$= 47.8672$

≈ 48 (number of students having marks below 45 marks is 48)

But, we need to find $f(45) - f(40)$ $\{\because f(40) = 31$ by data$\}$

$= 48 - 31$

No. of students scoring b/w 40 and 45 $= 17$

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5. The area A of circle corresponding to the diameter (D) is given below:

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find area corresponding to the diameter 105 by using appropriate interpolation formula.

$x=D$	$y=A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026	648	40		
85	5674	688	38	-2	4
90	6362	726	40	2	
95	7088	766			
100	7854				

$$r = \frac{105-100}{5} = 1$$

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots$$

$$= 7854 + 766 + \frac{2}{2}\times 40 + \frac{1\times2\times3}{6}\times 2 + \frac{1(2)(3)(4)}{24}\times 4$$

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$$= 7854 + 766 + 40 + 2 + 4$$

$$= 8666$$

6. Estimate the probable number of persons in the age group 20 to 25 from the following data.

Income per day (Rs)	under 10	10-20	20-30	30-40	40-50
	20	45	115	210	115

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
under 10	20	45			
under 20	65	115	70	25	-215
under 30	180	210	95	-190	
under 40	390	115	-95		
under 50	505				

$$r = \frac{x - x_0}{h} = \frac{25-10}{10} = \frac{15}{10} = 1.5$$

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$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots$$

$$= 20 + 1.5\times 45 + \frac{1.5(1.5-1)}{2!}\times 70 +$$

$$\frac{1.5(1.5-1)(1.5-2)}{3!}\times 25 +$$

$$\frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}\times(-215)$$

$$= 20 + 67.5 + 26.25 - 1.5625 - 5.0391$$

$$= 107.1484$$

Number of person below 25 = 107.1484

person in age 20-25 = below 25 − below 20

$= 107.1484 - 65$

$= 42.1484$

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7. Given $f(0)=1$, $f(1)=3$, $f(2)=7$, $f(3)=13$. Find $f(0.1)$ and $f(2.9)$ using NIF

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

① To find $f(0.1)$

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \ldots$$

$$r = \frac{x - x_0}{h} = \frac{0.1-0}{1} = 0.1$$

$$y = 1 + 0.1\times 2 + \frac{0.1(0.1-1)}{2!}\times 2 + 0$$

$$= 1 + 0.2 - 0.09 = 1.11$$

$$r = \frac{x - x_n}{h} = \frac{2.9 - 3}{1} = -0.1$$

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(11) To find $f(2.9)$, we use N.B.I.F

$$y = y_n + r\nabla y_n + r(r+1)\Delta^2 y_n + \cdots$$

$$= 13 + (-0.1)\times 6 + \frac{(-0.1)(1-0.1)}{2!} \times 2!$$

$$= 12.31$$

(8)

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9. Use Newton's Backward interpolation formula and find interpolating polynomial Hence find $f(12.5)$

x	10	11	12	13
$f(x)$	22	24	28	34

Solution: Using NBIF

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
10	22			
		2		
11	24		2	
		4		0
12	28		2	
		6		
13	34			

$r = \frac{x - x_n}{h}$

$= \frac{12.5-13}{1}$

$= -0.5$

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 34 + (-0.5)\times 6 + \frac{(-0.5)(-0.5+1)}{2}\times 2 + \frac{(-0.5)(1-0.5)(2-0.5)\times 0}{6}$$

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$$= 34 - 3 - 0.25$$

$$y(12.5) = 30.75$$

10. Given that $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$ $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$. Find $\sin 5\,^\circ$ Using Newton's forward interpolation formula

Solution Using NFIF

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071	0.0589	-0.0057	-0.0007
50°	0.7660	~~0.0532~~	-0.0064	
55°	0.8192	0.0468		
60°	0.8660			

$r = \frac{x - x_0}{h} = \frac{52-45}{5} = 1.4$

$$y(5\cdot\cdot) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 \cdots$$

$$= 0.7071 + 1.4\times 0.0589 + \frac{1.4(1.4-1)}{2}(-0.0057)$$

$$+ \frac{(1.4)(1.4-1)(1.4-2)}{3!}(-0.0007)$$

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$= 0.7071 + 0.08246 - 0.001596 + 0.0000392$

$= 0.7880$

11 Find $y(8)$ from $y(1)=24$, $y(3)=120$, $y(5)=336$, $y(7)=720$ by using Newton's Backward difference formula.

(11)

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Divide Difference

Let $f(x_0), f(x_1), f(x_2) \cdots f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of $x = x_0, x_1, x_2 \cdots x_n$ at unequal intervals.

The first order divided differences are defined as follows.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \text{ etc}$$

The second order divided differences are defined as follows.

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \text{ etc}$$

$f(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + \cdots$
$(x - x_0)(x - x_1) \cdots (x - x_{n-1})\Delta^n y_0$ This is called

Newton's divided difference interpolation formula for unequal intervals.

Date __/__/__

1. Fit an interpolating formula for the data $u_{10} = 355$, $u_0 = -5$, $u_8 = -21$, $u_4 = -125$, $u_1 = -14$

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$f(x_0)$ -5	$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{-14+5}{1-0} = -9$	$f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0} = \frac{-37+9}{4-0} = -7$	$f(x_0,x_1,x_2,x_3) = \frac{9-(-7)}{8-0} = 2$
1	$f(x_1)$ -14	$f(x_1,x_2) = \frac{-125+14}{4-1} = -37$	$f(x_1,x_2,x_3) = \frac{26+37}{8-1} = 9$	$f(x_1,x_2,x_3,x_4) = \frac{27-9}{10-1} = 2$
4	$f(x_2)$ -125	$f(x_2,x_3) = \frac{-21+125}{8-4} = 26$	$f(x_2,x_3,x_4) = \frac{188-26}{10-4} = 27$	
8	$f(x_3)$ -21	$f(x_3,x_4) = \frac{355+21}{10-8} = 188$		
10	$f(x_4)$ 355			

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Substituting in Newton's general interpolation formula.

$f(x) = f(x_0) + (x-x_0)\,\Delta y_0 + (x-x_0)(x-x_1)\,\Delta^2 y_0 \ldots \Delta^3 y_0 \ldots$

$(x-1)(-7) + x(x-1)(x-4)\times 2$

$+ 2(x^2-x)(x-4)$

$2x^3 - 8x^2 - 2x^2 + 8x$

5

Newton's divided difference interpolation formula for unequal intervals.

Date __/__/__

1. Fit an interpolating formula for the data $u_{10} = 355$, $u_0 = -5$, $u_8 = -21$, $u_1 = -14$, $u_4 = -125$

<u>solution</u>

x	y
0	$f(x_0)$ 5
1	$f(x_1)$ -14
4	$f(x_2)$ -125
8	$f(x_3)$ -21
10	$f(x_0)$ 355

Date __/__/__

Substituting in Newton's general interpolation formula.

$$f(x) = f(x_0) + (x - x_0) \cdot \Delta y_0 + (x - x_0)(x - x_1) \Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2) \Delta^3 y_0 - - -$$

$$= 5 + (x-0)(-9) + (x-0)(x-1)(-7) + x(x-1)(x-4) \times 2$$

$$= -5 - 9x - 7x(x-1) + 2(x^2 - x)(x-4)$$

$$= -5 - 9x - 7x^2 + 7x + 2x^3 - 8x^2 - 2x^2 + 8x$$

$$= 2x^3 - 17x^2 + 6x - 5$$

Date __/__/__

2. Find the equation of the polynomial which pass through the point (4, −43), (7, 83) (9, 327) (12, 1053). By using Newtons divided difference interpolation

x	y	1st D.D	2nd D.D	3rd D.D
4	−43	$f(x_0,x_1) = \frac{83+43}{7-4} = \frac{126}{3} = 42$	$f(x_0,x_1,x_2) = \frac{122-42}{9-4} = 16$	$f(x_0,x_1,x_2,x_3) = \frac{24-16}{12-4}$
7	83	$f(x_1,x_2) = \frac{327-83}{9-7} = 122$	$f(x_1,x_2,x_3) = \frac{242-122}{12-7} = 24$	$= 1$
9	327	$f(x_2,x_3) = \frac{1053-327}{12-9} = 242$		
12	1053			

Substituting in Newton's general interpolation formula.

$$f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + \dots$$

$$= -43 + (x-4)\,42 + (x-4)(x-7)\,16 + (x-4)(x-7)(x-9)\,1$$

$$= -43 + 42x - 168 + (x^2-11x+28)\,16 + (x^2-11x+28)(x-9)$$

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$$= -43 + 42x - 168 + 16x^2 - 176x + 448 + x^3 - 11x^2 + 28x - 9x^2 + 99x - 252$$

$$-7$$

$$x^3 - 4x^2 - 7x - 15 \quad 15$$

3. If $f(1) = 4$, $f(3) = 32$, $f(4) = 55$, $f(6) = 119$ find interpolating polynomial by Newton's divide difference formula.

x	y	1st D.D	2nd D.D
1	4	$f(x_0,x_1) = \frac{32-4}{3-1} = 14$	$f(x_0,x_1,x_2) = \frac{23-14}{4-1} = 3$
3	32	$f(x_1,x_2) = \frac{55-32}{4-3} = 23$	$f(x_1,x_2,x_3) = \frac{32-23}{6-3} = 3$
4	55	$f(x_2,x_3) = \frac{119-55}{6-4} = 32$	
6	119		

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Substituting in newton's general Interpolation formula.

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) \ldots$$

$$= 4 + (x-1)\,14 + (x-1)(x-3)\,3$$

$$= 4 + 14x - 14 + (x^2 - 4x + 3)\,3$$

$$= 4 + 14x - 14 + 3x^2 - 12x + 9$$

$$= 3x^2 + 2x - 1$$

4. Use Newtons divide difference formula and find $f(4)$ given the data

x	0	2	3	6
$f(x)$	-4	2	14	158

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x	$f(x)$	1st D.D	2nd D.D	3rd D.D
0	-4	$\frac{2+4}{2-0} =$		
		$f(x_0,x_1) = 3$	$f(x_0,x_1,x_2) = \frac{12-3}{3-0}$	
2	2	$f(x_1,x_2) = \frac{14-2}{3-2}$	$= 3$	$f(x_0,x_1,x_2,x_3)$
		$= 12$	$f(x_1,x_2,x_3)$	$= \frac{9-3}{6-0}$
			$= \frac{48-12}{6-2}$	$= 1$
3	14	$f(x_2,x_3) = \frac{158-14}{6-3}$	$= \frac{36}{4}$	
		48	$= 9$	
6	158			

$$f(x) = y_0 + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$= -4 + (x)(3) + (x)(x-2)\,3 + \underbrace{x(x-2)}_{x^2-2x}(x-3)\,1$$

$$= -4 + 3x + \cancel{3x^2} - \cancel{6x} + x^3 - \cancel{3x^2} - 2x^2 + \cancel{6x}$$

$$f(x) = x^3 - 2x^2 + 3x - 4$$

$$f(4) = 40$$

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5. Using Newton's divide difference interpolation formula, find the interpolating polynomial

x	0	1	2	3	4	5
$f(x)$	3	2	7	24	59	118

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3	$f(x_0,x_1)=\frac{2-3}{1-0}$ $=-1$	$=\frac{5+1}{2-0}=3$	$\frac{6-3}{3-0}=1$	$\frac{1-2}{4}$ -0.25
1	2	$f(x_1,x_2)=\frac{7-2}{2-1}$ $=5$	$\frac{17-5}{3-1}=6$	$\frac{9-6}{4-1}=1$	$\frac{0.6666-1}{5-1}$ -0.083
2	7	$f(x_2,x_3)=\frac{24-7}{3-2}$ $=17$	$\frac{35-17}{4-2}=9$	$\frac{12-9}{5-2}=\cancel{0.6666}$ 1	
3	24	$f(x_3,x_4)=\frac{59-24}{4-3}$ $=35$	$\frac{59-35}{5-3}=12$		
4	59	$f(x_4,x_5)=\frac{118-59}{5-4}$ $=59$			
5	118				

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The fourth order differences are zero as third order differences are same.

We have Newton's general interpolation formula

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0\,x_1\,x_2\,x_3)+\cdots$$

$$=3+x(-1)+x(x-1)3+x(x-1)(x-2)1$$

$$=3-x+3x^2-3x+x(x^2-3x+2)$$

(6) $$=3-x+\cancel{3x^2}-3x+x^3-\cancel{3x^2}+2x$$
$$=x^3-2x+3$$

(6) Use Newton divided difference formula to find $f(8.2)$ given.

x	5	7	11	10	13
$f(x)$	150	392	1452	2366	5202

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	150				
		$=\frac{392-150}{7-5}$			
			$\frac{265-121}{11-5}$	$\frac{-393-24}{10-5}$	$\frac{220.443+83.4}{13-5}$
7	392	$=121$	$=24$	$=-83.4$	
		$\frac{1452-392}{11-7}$			
11	1452	$=265$	$\frac{-914-265}{10-7}$		$=37.9803$
			-393		
				$=\frac{929.66+393}{13-7}$	
10	2366	$\frac{2366-1452}{10-11}$			
		$=-914$	$\frac{945.333+914}{13-11}$	220.443	
13	5202	$\frac{5202-2366}{13-10}$	929.66		
		$=945.333$			

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4) \ldots$$

$x = 8.2$

$$= 150 + (x-5)121 + (x-5)(x-7)24 + (x-5)(x-7)(x-11)(-83.4) + (x-5)(x-7)(x-11)(x-10)37.9803$$

$$= 150 + 387.2 + 92.16 + 800.64 + 735.0555$$

$$= 2165.0555$$

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Module - 5

Numerical Methods - II

Lagrange's Interpolation formula.

If $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$ ... $y_n = f(x_n)$ be a set of values of an unknown function $y(x)$ corresponding values of x : $x_0, x_1, x_2 \ldots x_n$ not necessarily equal intervals then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 +$$

$$\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots + \frac{(x-x_0)\cdots(x-x_{n-1})}{(x_n-x_0)\cdots(x_n-x_{n-1})} y_n$$

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Problems

① Use Lagrange's interpolation formula to find f(4) given.

x	0	2	3	6
$f(x)$	-4	2	14	158

Solution : From the given data

$x_0 = 0 \quad y_0 = -4$

$x_1 = 2 \quad y_1 = 2 \quad \& \quad x = 4$

$x_3 = 3 \quad y_2 = 14$

$x_4 = 6 \quad y_3 = 158$

WKT formula

$$y = \frac{(4-2)(4-3)(4-6)}{(0-2)(0-3)(0-6)} -4 + \frac{(4-0)(4-3)(4-6)}{(2-0)(2-3)(2-6)} 2$$

$$+ \frac{(4-0)(4-2)(4-6)}{(3-0)(3-2)(3-6)} \times 14 + \frac{(4-0)(4-2)(4-3)}{(6-0)(6-2)(6-3)} \times 158$$

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$-0.4444 - 2 + 24.8889 + 17.5556$

$\boxed{y(4) = 40}$

② Use Lagrange's interpolation formula to find y at $x = 10$ given

x	5	6	9	11
y	12	13	14	16

Solution

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$= 2 - 4.3333 + 11.6667 + 5.3333$

$= 14.6667$

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③ The following table gives the normal weights of babies during first eight months of life.

Age (in months)	0	2	5	8
weight (in pounds)	6	10	12	16

Estimate the weight of the baby at the age of seven months using Lagrange's interpolation formula.

Solution.

$$y = \frac{(7-2)(7-5)(7-8)}{(0-2)(0-5)(0-8)} \times 6 + \frac{(7-0)(7-5)(7-8)}{(2-0)(2-5)(2-8)} \times 10$$

$$+ \frac{(7-0)(7-2)(7-8)}{(5-0)(5-2)(5-8)} \times 12 + \frac{(7-0)(7-2)(7-5)}{(8-0)(8-2)(8-5)} \times 16$$

$$= +0.75 - 3.8888 + 9.3333 + 7.777$$

$$= 13.9722$$

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The weight of the baby at the age of 7 months is 13.97 pounds.

4. Using Lagrange's formula find the interpolating polynomial that approximate to the function describe the following table.

x	0	1	2	5
$f(x)$	2	3	12	147

Solution: $$y = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$

$3x^2 - 6x$

$x^2 - x$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$= \frac{2(x^2-3x+2)(x-5)}{(+1)(-10)} + \frac{3x^3 - 15x^2 - 6x^2 + 30x}{-1 \times -4}$$

5

$2 \quad 6 \quad 49$

$$+ \frac{(x^3-5x^2-x^2+5x)12}{2 \times -3} + \frac{147x(x^2-3x+2)}{5 \times 4 \times 3}$$

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$$= \frac{x^3 - 3x^2 + 2x - 5x^2 + 15x - 10}{-5} +$$

$$\frac{3x^3 - 21x^2 + 30x - 2x^3 + 12x^2 - 10x}{4}$$

$$\frac{+49x^3 - 147x^2 + 98x}{20}$$

$$= \left(\frac{-1}{5} + \frac{3}{4} - 2 + \frac{49}{20}\right)x^3 + \left(\frac{3}{5} + 1 - \frac{21}{4} + 12 - \frac{147}{20}\right)x^2$$

$$+ \left(\frac{-2}{5} - 3 + \frac{30}{4} - 10 + \frac{98}{20}\right)x + 2$$

$$= x^3 + x^2 - x + 2$$

⑤ Use Lagrange's interpolation formula to fit a polynomial for the data.

x	0	1	3	4
y	-12	0	6	12

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$$y = \frac{(x-1)(x-3)(x-4)}{(0+1)(0-3)(0-4)}(+12) + \frac{(x-0)(x-3)(x-4)}{(-12-0)(-12-3)(-12-4)} \times 0$$

$$\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \times 6 + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times 12$$

$$= \frac{(x^2-4x+3)(x-4)}{-1} + 0 + \frac{(x^2-x)(x-4)}{5 \times 2} +$$

$$(x^2-x)(x-3)$$

$$= +(x^3 - 4x^2 + 3x - 4x^2 + 16x - 12)$$
$$- (x^3 - x^2 - 4x^2 + 4x) + x^3 - x^2 - 3x^2 + 3x$$

$$= +x^3 - 8x^2 + 19x - 12 - x^3 + 5x^2 - 4x$$

$$\cancel{+x^3} - 4x^2 + 3x$$
$$(-8+5-4)x^2 + (19-4+3)$$

$$= x^3 - 7x^2 + 18x - 12 \,//$$

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⑥ If $y(1)=3,\ y(3)=9,\ y(4)=30,\ y(6)=132$, find Lagrange's interpolation polynomial that takes on these values.

Solution

x	1	3	4	6
y	3	9	30	132

$$y = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}\times 3 + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}\times 9$$

$$+\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}\times 30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}\times 132$$

$$(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+bc+ca)x - abc$$

$$= \frac{(x^3-13x^2+(12+24+18)x-72)\times \cancel{3}}{-2\times-\cancel{3}\times-5}$$

$$+\frac{(x^3-11x^2+34x-24)\times \cancel{9}^{3}}{2\times(-1)\times(-\cancel{3})}$$

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$$-\frac{(x^3-10x^2+27x-18)\,\cancel{30}^{\cancel{10}\,5}}{\cancel{3}\times 1\times(-\cancel{2})}$$

$$+\frac{x^3-8x^2+19x-12}{5\times\cancel{3}\times\cancel{2}}\times \cancel{132}^{\cancel{66}\,22}$$

$$= -\left\{\frac{x^3-13x^2+54x-72}{10}\right\} + \frac{3}{2}(x^3-11x^2+34x-24)$$

$$-5(x^3-10x^2+27x-18) + \frac{22}{5}(x^3-8x^2+19x-12)$$

$$= \frac{4}{5}x^3 - \frac{2}{5}x^2 - \frac{29}{5}x + \frac{42}{5}$$

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Lagrange's Inverse Interpolation formula

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\cdots(y_0-y_n)}x_0$$

$$+\frac{(y-y_0)(y-y_2)(y-y_3)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\cdots(y_1-y_n)}x_1+\cdots$$

$$+\frac{(y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\cdots(y_n-y_{n-1})}x_n$$

Problems

① Applying Lagrange's formula inversely find x when $y=6$ given the data

x	20	30	40
y	2	4.4	7.9

Solution: $x = \frac{(6-4.4)(6-7.9)}{(2-4.4)(2-7.9)}\times 20$

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$$+\frac{(6-2)(6-7.9)}{(4.4-2)(4.4-7.9)}\times 30+\frac{(6-2)(6-4.4)}{(7.9-2)(7.9-4.4)}\times 40$$

$x = -4.2938 + 20.6521 + 12.3971$

$x = 35.2461$

2. Apply Lagrange's formula inversely to find a root of the equation $f(x)=0$ given that $f(30)=-30$, $f(34)=-13$, $f(38)=3$, $f(42)=18$

Solution

Given $y=0$

x	30	34	38	42
y	-30	-13	3	18

$$x = \frac{(0-(-13))(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}\times 30 + \frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}\times 34$$

$$+\frac{(0+30)(0+13)(0-18)}{(+3+30)(3+13)(3-18)}\times 38+\frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}\times 42$$

$= -0.7821 + 6.5322 + 33.6818 - 2.2016$

$= \cancel{6.91}\ 37.2304$

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Numerical Integration

The process of obtaining approximately ~~the~~ value of the definite integeal $I = \int_a^b y\,dx$ without actually integrating the function but only using the values of y at some point of x equally spaced over $[a, b]$ is called ~~integra~~ numerical integration.

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Explain

The area bounded by the curve $f(x)$ and x-axis between limit a and b is denoted by $I = \int_a^b f(x)\,dx$ — (1)

divide the interval (a,b) into n equal interval with length h (step size)

i.e., $(a,b) = (a = x_0, x_1, x_2 \cdots x_{n-1}, x_n = b)$

$h = \underline{b-a}$

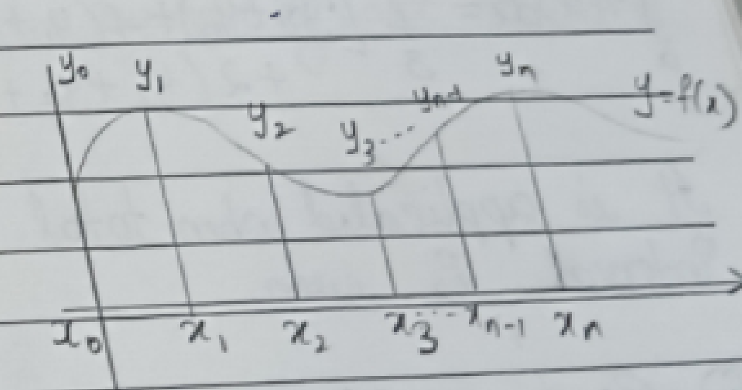
$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$\vdots$

$x_n = x_{n-1} + h$



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Eq^n ① can be evaluated by.

① Trapezoidal Rule (

$$\int_a^b f(x)dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+\dots+y_{n-1})+y_n\right]$$

It is applicable on any number of interval.

⑪ Simpson $1/3^{rd}$ Rule (even interval

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\right]$$

It is applicabd when total number of interval is even.

(iii) Simpson 3/8 Rule (3 multiple)

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+\dots+y_{n-3}) + 3(y_1+y_2+y_4+\dots)\right]$$

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Here * $h \rightarrow$ common difference $h = \frac{b-a}{n}$

* n is the total number of subintervals of the interval

* If there are n intervals then there are $(n+1)$ number of ordinates.

Problems

① Evaluate $\int_0^2 \frac{dx}{16+x^2}$ by applying Trapezoidal Rule by taking six equal parts.

Solution

Given $n=6$

we have to divide $[0,2]$ into 6 equal parts ($a = 0$, $b = 2$)

$$h = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$$

$y = \frac{1}{16+x^2}$ $h = \frac{1}{3}$

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
y	0.0625	0.0621	0.0608	0.0588	0.0562	0.0532	0.05

By trapezoidal rule

$$\int_0^2 y\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{3\times 2}\left[(0.0625+0.05)+2(0.0621+0.0608+0.0588+0.0562+0.0532)\right]$$

$$\boxed{\int_0^2 \frac{dx}{16+x^2} = 0.1158}$$

② Evaluate $\int_0^5 \frac{1}{4x+5}\,dx$ by using Trapezoidal rule by taking 11 ordinates.

Solution

$n = 10$ $y = \frac{1}{4x+5}$

$h = \frac{5-0}{10} = \frac{1}{2}$

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
y	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0435	0.04

$$\int_0^5 \frac{dx}{4x+5} = \frac{1}{2\times 2}\left[(0.2+0.04)+2(0.1429+0.1111+0.0909+0.0769+0.0667+0.0588+0.0526+0.0476+0.0435)\right]$$

$= 0.4055$

③ Evaluate $\int_0^{\pi/2} \cos x\,dx$ by applying trapezoidal rule, taking eleven ordinates.

Mode + 7 (table)

$f(x) = \cos(x)$

Start? 0

End? $\pi/2$

Step? $\pi/20$

you will get table

Solution: $n = 10$

$h = \frac{\frac{\pi}{2}-0}{10} = \frac{\pi}{20}$

$y = \cos x \quad h = \frac{\pi}{20}$

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x	0	$\pi/20$	$2\pi/20$	$3\pi/20$	$4\pi/20$	$5\pi/20$
y	1	0.9877	0.9511	0.8910	0.8090	0.7075

$6\pi/20$	$7\pi/20$	$8\pi/20$	$9\pi/20$	$10\pi/20$
0.5878	0.4540	0.3090	0.1564	0

$$\int_0^{\pi/2} \cos x\,dx = 0.998 \approx 1$$

Simpon 1/3 rule (even integral)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\right]$$

$y = \frac{1}{1+x^2}$

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(4) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using simpson's 1/3rd rule, dividing interval (0,1) into six equal parts and hence find approximate value of π

$h = \frac{1-0}{6} \qquad \because n = 6$

$x_n = x_0 + nh$

$1 = 0 + 6 \times 1/6$

x	0	1/6	1/3	1/2	2/3	5/6	1
y	1	0.9730	0.9	0.8	0.6923	0.5902	0.5

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{18}\left[(1+0.5) + 2(0.9 + 0.6923) + 4(0.9730 + 0.8 + 0.5902)\right]$$

$$\tan^{-1} x \Big|_0^1 = 0.7854$$

$$\tan^{-1} 1 - \tan^{-1} 0 = 0.7854$$

$$\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0) = 0.7854$$

$$\frac{\pi}{4} = 0.7854 \qquad \pi = 3.1416$$

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⑤ Evaluate $\int_0^5 \frac{dx}{4x+5}$ by using Simpson's $1/3^{rd}$ rule taking 10 equal strips and hence find log5.

Solution

$x_n = x_0 + nh$
$5 = 0 + 10 \times \frac{1}{2}$

$$h = \frac{b-a}{n} = \frac{5-0}{10} = \frac{1}{2}$$

x	0	$1/2$	$2/2$	$3/2$	2
y	0.2	0.1429	0.1111	0.909	0.0769
	y_0	y_1	y_2	y_3	y_4

x	$5/2$	3	$7/2$	4	$9/2$	5
y	0.0667	0.0588	0.0526	0.0476	0.0435	0.04
	y_5	y_6	y_7	y_8	y_9	y_{10}

By simpson's $1/3^{rd}$ rule,

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

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$$= \frac{1}{6}\left[(0.2+0.04) + 4(0.1429+0.0909+0.0667+0.0526+0.0435) + 2(0.1111+0.0769+0.0588+0.0476)\right]$$

$$= \frac{1}{6}\left[0.24 + 1.5864 + 0.5888\right]$$

$$\int_0^5 \frac{dx}{4x+5} = 0.4025$$

$$\frac{1}{4}\log(4x+5)\Big|_0^5 = 0.4025$$

$$\frac{1}{4}\left[\log(25) - \log(5)\right] = 0.4025$$

$$\log \frac{25}{5} = 1.6100$$

$$\log 5 = 1.61$$

(6) Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^{0.6} e^{-x^2}\,dx$ by dividing 7 ordinates.

Solution: $h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.99	0.9607	0.9139	0.8521	0.7788	0.6976
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6976)+4(0.99+0.9139+0.7788)+2(0.9607+0.8521)\right]$$

$1.6976 + 5.3654 + 7.2512$

$= 0.5351$

(7) Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\,dx$ using Simpsons $\frac{1}{3}^{rd}$ rule, taking 8 equal parts

Solution

$h = \frac{\pi/2 - 0}{8} = \frac{\pi}{16}$

x	0	$\pi/16$	$2\pi/16$	$3\pi/16$	$4\pi/16$	$5\pi/16$	$6\pi/16$	$7\pi/16$	$8\pi/16$
y	0	0.4416	0.6186	0.7453	0.8408	0.9118	0.9611	0.9903	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\int_0^{\pi/2} \sqrt{\sin x}\,dx = \frac{h}{3}\left[(0+1)+4(0.4416+0.7453+0.9118+0.9903)+2(0.6186+0.8408+0.9611)\right]$$

$$= \frac{\pi}{16\times 3}(1+12.3560+4.8410)$$

$= 1.1910$

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⑧ Evaluate $\int_4^{5.2} \log_e x \, dx$ using simpson's $(1/3)^{rd}$ rule, taking 6 equal parts

Solution

$h = \frac{5.2-4}{6} = 0.2$

x	4	4.2	4.4	4.6	4.8	5	5.2
y	1.3862	1.435	1.4816	1.526	1.5686	1.6094	1.6486
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$y = \frac{0.2}{3}\left[(1.3862+1.6486)+4(1.435+1.526+1.6094)+2(1.4816+1.5686)\right]$

$= 1.8278$

⑨ Use simpson's $1/3^{rd}$ rule with seven ordinates to evaluate $\int_2^8 \frac{dx}{\log_{10} x}$

$h = \frac{b-a}{n} = \frac{8-2}{6} = 1$

1.4307 1.2851

x	2	3	4	5	6	7	8
y	3.3219	2.0959	1.6609	1.4307	1.2851	1.1833	1.1073 1.1073
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$y = \frac{1}{3}\left[(3.3219+1.1073)+4(2.0959+1.4307+1.1833)+2(1.6609+1.2851)\right]$

9.7203

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Weddle's Rule

$n = 6$

$$I = \int_{x_0}^{x_0+6h} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

where $h = \frac{b-a}{n}$

Note: To apply weddle's Rule n must be a multiple of 6.

1. Evaluate $\int_4^{5.2} \log_e x\,dx$ using weddle's rule taking 7 ordinates

$h = \frac{5.2-4}{6}$

$= 0.2$

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x	4	4.2	4.4	4.6	4.8	5	5.2
y	1.3862	1.435	1.4816	1.526	1.5686	1.6094	1.6486
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_4^{5.2} \log_e x\,dx = \frac{3\times 0.2}{10}\left[1.3862 + 5\times 1.435 + 1.4816 + 6\times 1.526 + 1.5686 + 5\times 1.6094 + 1.6486\right]$$

$= 1.8278$

Date __/__/__

2) Evaluate $\int_0^1 \frac{x}{1+x^2}$ by using weddle's rule taking 7 ordinates and hence find $\log_e 2$.

$h = \frac{1-0}{6} = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6
y	0	0.1621	0.3	0.4	0.4615

5/6	1
0.4918	0.5

$$\int_0^1 \frac{x}{x^2+1} dx = \frac{3\times 1}{10\times 6}\left[0+5\times0.1621+0.3+6\times0.4+0.4615+5\times0.4918+0.5\right]$$

$= 0.3466$

Date __/__/__

$1+x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$\int_0^1 \frac{x}{1+x^2} dx$$

$$\int \frac{1}{t} \frac{dt}{2}$$

$$\frac{1}{2}\int \frac{1}{t} dt = \frac{1}{2}\log t = \frac{1}{2}\log(1+x^2)$$

$$\int_0^1 \frac{x}{1+x^2} dx = 0.3466$$

$$\frac{\log(1+x^2)}{2}\Big|_0^1 = 0.3466$$

$$\frac{1}{2}\left[\log 2 - \cancel{\log 1}^{\,0}\right] = 0.3466$$

$$\frac{1}{2}\log_e 2 = 0.3466$$

$$\log_e 2 = 2\times 0.3466$$
$$= 0.6932$$