Differential Equations:-

An Equation which contains 1 (or) more terms and the derivative of the dependent variable wrt Independent variable is called a differential Equation.

Eg:- ① $\frac{dy}{dx} = x + \sin x$

② $\frac{d^3y}{dx^3} + \frac{dy}{dx} + y = e^x$

Degree of diff. Eqn:- The degree of a diff. Eqn is the power of the highest order derivative.

Eg:- $\frac{d^3y}{dx^3} = \sqrt{\frac{d^2y}{dx^2} + 1}$ (Sq. on both the sides).

$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^2 = \frac{d^2y}{dx^2} + 1$ ∴ the degree is 2.

Order of a diff. Eqn:- The Order of the diff. Eqn is the highest Order derivative in a diff. Eqn.

Eg:- ① $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = e^x$

the Order of the diff Eqn is 3.

Linear differential Equation :- A differential Equation of the form $\frac{dy}{dx}+py=Q$, where p and Q are the func's of 'x'.

A diff. Eqn is said to be linear if every dependent variable and the derivative occurs in first degree only.

Solving method for a linear differential Equation.

$\frac{dy}{dx}+py=Q$., where p and Q are func'ns of x,

$\therefore$ the Soln is $y\cdot(I.F)=\int Q\,(I.F)\,dx+c$

Integrating factor $= e^{\int p\,dx}$

(1). Solve $\frac{dy}{dx}-\frac{2y}{x}=x+x^2$.

Soln:- The given diff. Eqn $\frac{dy}{dx}-\frac{2y}{x}=x+x^2$, is of the form $\frac{dy}{dx}+py=Q$, where $p=-2/x$ and $Q=x+x^2$

The Sol^n of the given diff. Eq^n is

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$$

$$\text{Integrating Factor} = e^{\int P dx} = e^{\int -2/x \, dx} = e^{-2\int 1/x \, dx} = e^{-2\log x}$$

$$= e^{\log x^{-2}} = x^{-2} = 1/x^2$$

$$\int Q \cdot e^{\int P dx} = \int (x + x^2) \cdot \frac{1}{x^2} dx + c$$

$$= \int \left(\frac{1}{x} + 1\right) dx + c$$

$$\therefore \; y \cdot \frac{1}{x^2} = \log x + x + c$$ is the required Sol^n

(2). Solve $\frac{dy}{dx} + y \cot x = \cos x$

Sol^n:- The given Eq^n $\frac{dy}{dx} + y \cot x = \cos x$ is of the form

$\frac{dy}{dx} + py = Q$, where $p = \cot x$ and $Q = \cos x$.

The Sol^n for the given Eq^n is given by

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$$

To find $e^{\int P dx} = e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x$

$$\Rightarrow \int Q \cdot e^{\int P dx} = \int \cos x \cdot \sin x \, dx = \int \frac{\sin 2x}{2} dx$$

$$\boxed{\begin{aligned} \sin 2x &= 2\sin x \cdot \cos x \\ \therefore \sin x \cdot \cos x &= \frac{\sin 2x}{2} \end{aligned}}$$

$$= \frac{1}{2}\int \sin 2x \, dx = \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= -\frac{1}{4}\cos 2x.$$

$\therefore \; y \cdot \sin x = -\frac{1}{4}\cos 2x + C$ is the required sol^n.

Bernoulli's differential Equation:-

The differential Eq^n of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are func^n's of x is called as Bernoulli's differential Equation.

$$\frac{dy}{dx} + Py = Qy^n \longrightarrow (1).$$

divide Equation (1) by y^n,

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + \frac{Py}{y^n} = Q$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P \cdot y^{1-n} = Q \longrightarrow (2).$$

Let $y^{1-n} = t$ (differentiate wrt x).

$$\therefore (1-n) y^{-n} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dt}{dx}$$

$\therefore$ Eqⁿ (2) becomes, $\frac{1}{(1-n)}\frac{dt}{dx} + pt = Q$.

$$\Rightarrow \frac{dt}{dx} + (1-n)\, pt = (1-n)\, Q \rightarrow (3).$$

Eqⁿ (3) is a linear diff. Eqⁿ in t.

Ly, $\frac{dx}{dy} + px = Q x^n$, where p and Q are fcns of y is called Bernoulli's Eqⁿ in x.

Q. Solve $\frac{dy}{dx} + y/x = y^2 x \rightarrow (1)$

Solⁿ This is Bernoulli's Eqⁿ, dividing the given Eqⁿ by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x \rightarrow (2).$$

Substitute $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

(diff. wrt x) $\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dt}{dx}$

$\therefore$ Eq^n ② becomes.

$$-\frac{dt}{dx} + \frac{t}{x} = x$$

$$\Rightarrow \frac{dt}{dx} - \frac{t}{x} = -x \rightarrow ③$$

$\therefore$ Eq^n ③ is a lineal diff. Eq^n of the form

$\frac{dt}{dx} + pt = Q$, where $p = -1/x$ and $Q = -x$.

the general sol^n is $t \cdot (I.F) = \int Q \, (I.F) \, dx + c$

$$\Rightarrow t \cdot e^{\int p dx} = \int Q \cdot e^{\int p dx} \, dx + c$$

$$\Rightarrow \frac{1}{y} \cdot \frac{1}{x} = \int \left(-x \cdot \frac{1}{x}\right) dx + c$$

$$\Rightarrow 1/xy = -\int 1 \cdot dx + c$$

$$\left[I.F = e^{\int p dx} = e^{\int -1/x \, dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x \right]$$

$\Rightarrow \frac{1}{xy} = -x + c$ is the required sol^n

③. Solve $\frac{dy}{dx} + y/x = x^2 y^6 \rightarrow ①$.

Sol^n:- The given Equation is Bernoulli's diff. Eq^n.

divide Eq^n ① by 'y^6', we get.

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x \cdot y^5} = x^2 \rightarrow ②.$$

let $1/y^5 = t$ (diff. wrt x)

$$\Rightarrow -5\frac{1}{y^6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$\therefore$ Eq^n ② becomes,

$$-\frac{1}{5}\frac{dt}{dx} + \frac{5 \cdot t}{x} = x^2$$

$$\Rightarrow \frac{dt}{dx} - \frac{5t}{x} = -5x^2 \rightarrow ③.$$

Eq^n ③ is of linear diff. Eq^n form, where $P = -5/x$, $Q = -5x^2$.

$\therefore$ the Sol^n of the Eq^n is given by,

$$t \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c.$$

$$e^{\int P dx} = e^{\int -5/x \, dx} = e^{-5\int 1/x \, dx} = e^{-5 \log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$$

$$\int Q \cdot e^{\int P dx} dx = \int -5x^2 \cdot \frac{1}{x^5} dx = -5 \int \frac{1}{x^3} dx$$

$$= -5 \cdot \frac{x^{-2}}{-2} = \frac{5}{2} x^{-2}$$

$$\therefore \quad t \frac{1}{x^5} = \frac{5}{2} \cdot \frac{1}{x^2} + c$$

$$\Rightarrow \frac{1}{y^5 \cdot x^5} = \frac{5}{2x^2} + c \quad \text{is the required sol}^n.$$

(5). Solve $xy(1+xy^2)\frac{dy}{dx} = 1. \rightarrow (1).$

Sol^n:- The given Eq^n can be written as.

$$\frac{dy}{dx} = \frac{1}{xy(1+xy^2)} = \frac{1}{xy + x^2y^3}$$

$$\Rightarrow \frac{dx}{dy} = xy + x^2y^3$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \rightarrow (2).$$

$\therefore$ Eq^n (2) is in Bernoulli's form in x.

divide Eq^n (2) by x^2; $\Rightarrow \frac{1}{x^2}\frac{dx}{dy} - y/x = y^3 \rightarrow (3).$

let $\frac{1}{x} = t$ (diff. w.r.t y).

$$-\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy} \Rightarrow \frac{1}{x^2}\frac{dx}{dy} = -\frac{dt}{dy}$$

$\therefore$ Eq^n ③ becomes.

$$-\frac{dt}{dy} - ty = y^3$$

$$\Rightarrow \frac{dt}{dy} + ty = -y^3 \rightarrow ④.$$

$\therefore$ Eq^n ④ is linear diff. Eq^n form, where $P = y$ and $Q = -y^3$.

I.F $e^{\int P dy} = e^{\int y dy}$

$= e^{y^2/2}$

$\therefore$ the sol^n is.

$$\Rightarrow t \cdot I.F = \int Q \cdot (I.F) dy + C$$

$$t \cdot e^{y^2/2} = \int -y^3 \cdot e^{y^2/2} dy + C$$

Let $\frac{y^2}{2} = u$

$y^2 = 2u$

$2y dy = 2du$

$\Rightarrow y\,dy = du$

$$\frac{1}{x} \cdot e^{y^2/2} = -\int y^2 \cdot y\, e^{y^2/2} dy + C$$

$$= -\int 2u \cdot e^u \cdot du + C$$

$$= -2\int u \cdot e^u du + C$$

$$\frac{1}{x} \cdot e^u = -2[u \cdot e^u - e^u] + C$$

$\int u \cdot e^u du =$

$u \cdot \int e^u - \int \int e^u \cdot \frac{du}{dy}$

$= [u \cdot e^u - e^u]$

$$\Rightarrow \frac{1}{x} \cdot e^{y^2/2} = 2e^{y^2/2}[1 - y^2/2] + C$$

④. Solve $r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$

Sol?:- we have $r\sin\theta - \cos\theta \frac{dr}{d\theta} = r^2$

$$\Rightarrow -\cos\theta \frac{dr}{d\theta} + r\sin\theta = r^2$$

$$\Rightarrow \cos\theta \frac{dr}{d\theta} - r\sin\theta = -r^2 \rightarrow ①.$$

divide Eq^n ① by r^2, we get.

$$\Rightarrow \frac{1}{r^2}\cos\theta \frac{dr}{d\theta} - \frac{1}{r}\sin\theta = -1$$

let $\frac{+1}{r} = y$ (diff. wrt 'θ', we get)

$$\frac{-1}{r^2}\frac{dr}{d\theta} = \frac{dy}{d\theta}$$

$\therefore$ ① becomes.

$$\Rightarrow -\cos\theta \frac{dy}{d\theta} - y\sin\theta = -1$$

$$\Rightarrow \cos\theta \frac{dy}{d\theta} + y\sin\theta = 1 \rightarrow ②.$$

divide Eq^n ② by $\cos\theta$, we get.

$$\Rightarrow \frac{dy}{d\theta} + y\tan\theta = \sec\theta \rightarrow ③.$$

$\therefore$ Eq^n ③ is of the form $\frac{dy}{d\theta} + Py = Q$,

where $P = \tan\theta$, $Q = \sec\theta$.

where $P = \tan\theta$, $Q = \sec\theta$.

$\therefore$ the solⁿ is given by $y \cdot e^{\int P d\theta} = \int Q \cdot e^{\int P d\theta} d\theta + c$

$$I.F = e^{\int P d\theta} = e^{\int \tan\theta d\theta} = e^{\log \sec\theta} = \sec\theta.$$

$\Rightarrow y \cdot \sec\theta = \int (\sec\theta \cdot \sec\theta) d\theta + c$

$\Rightarrow y \cdot \sec\theta = \int \sec^2\theta \, d\theta + c$

$\Rightarrow \underline{y \cdot \sec\theta = \tan\theta + c}$ is the required solⁿ

(5). $\frac{dy}{dx} - y\tan x = \frac{\sin x \cdot \cos^2 x}{y^2}$

Solⁿ:- The given Eqⁿ can be written as

$$y^2 \frac{dy}{dx} - y^3 \tan x = \sin x \cdot \cos^2 x \rightarrow (1).$$

let $y^3 = t$ (diff. Eqⁿ w.r.t x).

$3y^2 \frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow y^2 \frac{dy}{dx} = \frac{1}{3}\frac{dt}{dx}$

$\therefore$ Eqⁿ (1) becomes

$\Rightarrow \frac{1}{3}\frac{dt}{dx} - t \cdot \tan x = \sin x \cdot \cos^2 x$

$\Rightarrow \frac{dt}{dx} - 3t\tan x = 3\sin x \cdot \cos^2 x \rightarrow (2)$

Eqⁿ (2) is of the form Linear diff. Eqⁿ, where $P = +3\tan x$
$Q = 3\sin x \cdot \cos^2 x$

$\Rightarrow$ The Solⁿ is of the form,

$$t \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + c$$

$I.F = e^{\int P dx} = e^{\int -3\tan x\, dx} = e^{-3\int \tan x\, dx} = e^{-3\log \sec x}$

$= e^{\log \sec^{-3} x} = \sec^{-3} x = \frac{1}{\sec^3 x}$

$= \cos^3 x.$

$\Rightarrow t \cdot \cos^3 x = \int 3\sin x \cdot \cos^2 x \cdot \cos^3 x\, dx + c$

$= 3\int \sin x \cdot \cos^5 x\, dx + c$ Let $\cos x = u$

$\therefore -\sin x\, dx = du$

$= -3\int u^5 du + c$ $\sin x\, dx = -du$

$= -3 \cdot \frac{u^6}{6} + c$

$y^3 \cdot \cos^3 x = -\frac{1}{2}\cos^6 x + c$ is the required Sol^n.

⑥. Solve $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$

Sol:- The given Eq^n is $x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x. \rightarrow$ ①.

but the general Bernoulli's Eq^n is $\frac{dy}{dx} + Py = Qy^n$.

$\therefore$ to get Eq^n ① in Standard eq^n,

we need to divide Eq^n ① by x^3, we get

$\Rightarrow \frac{dy}{dx} - y/x = -\frac{y^4}{x^3}\cos x. \rightarrow$ ②.

Now Eq^n ② is in Bernoulli's Eq^n form. in y.

$\Rightarrow$ Now divide Eqn ② by y^4, we get,

$$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{xy^3} = -\frac{1}{x^3}\cos x \longrightarrow ③$$

$\Rightarrow$ let $1/y^3 = t \Rightarrow -3\frac{1}{y^4}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{1}{y^4}\frac{dy}{dx} = \frac{-1}{3}\frac{dt}{dx}$

$$\therefore ③ \Rightarrow \frac{-1}{3}\frac{dt}{dx} - \frac{t}{x} = -\frac{\cos x}{x^3}$$

$$\Rightarrow \frac{dt}{dx} + \frac{3t}{x} = \frac{3\cos x}{x^3} \longrightarrow ④.$$

$\therefore$ Equation ④ is of the form $\frac{dt}{dx} + pt = Q$, where $p = 3/x$, $Q = \frac{3\cos x}{x^3}$

Solⁿ is of the form $t \cdot I.F = \int Q \cdot I.F + c$

$$I.F = e^{\int P dx} = e^{\int 3/x\, dx} = e^{3\int 1/x\, dx} = e^{3\log x} = e^{\log x^3} = x^3$$

$$\Rightarrow \frac{1}{y^3} \cdot x^3 = 3\int\left(\frac{\cos x}{x^3} \cdot x^3\right)dx + c$$

$$= 3\int \cos x\, dx + c$$

$\Rightarrow \frac{x^3}{y^3} = 3\sin x + c$ is the required Solⁿ.

Exact differential Equations

The necessary and sufficient condition for the differential Equation $M(x,y)dx + N(x,y)dy = 0$ to be an Exact Equation is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

further the solution of the Exact Equation is given by

$$\int M dx + \int N(y) dy = C.$$

where $N(y)$ denotes, the terms in N not containing x.

① Solve $(2x+y+1)dx + (x+2y+1)dy = 0$

Sol?- Given, $(2x+y+1)dx + (x+2y+1)dy = 0$.

Let $M = 2x+y+1$ and $N = x+2y+1$

$\therefore \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ the given Equation is Exact.

The Solution of the is $\int M dx + \int N(y) dy = c$, where $N(y)$ denotes the terms in N not containing x.

i.e., $\int (2x+y+1)dx + \int (2y+1)dy = c$

$$\Rightarrow \left(\frac{2x^2}{2} + xy + x\right) + \frac{2y^2}{2} + y = c$$

$\Rightarrow x^2 + yx + x + y^2 + y = c$ is the required solution.

②. Solve $(y^3 - 3x^2y)dx - (x^3 - 3xy^2)dy = 0$.

Sol^n:- Given, $(y^3 - 3x^2y)dx - (x^3 - 3xy^2)dy = 0$.

Let $M = y^3 - 3x^2y$ and $N = -x^3 + 3xy^2$

$\therefore \frac{\partial M}{\partial y} = 3y^2 - 3x^2$ and $\frac{\partial N}{\partial x} = -3x^2 + 3y^2 = 3y^2 - 3x^2$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ the given Equation is Exact.

The Sol^n is $\int M dx + \int N(y)\, dy = c$.

$\int (y^3 - 3x^2y)dx + \int 0 = c$ ($\because$ there is no term in N which do not contain x).

$\Rightarrow \therefore y^3x - \frac{3x^3y}{3} = c$

$\Rightarrow y^3x - x^3y = c$ is the required Solution.

③. Solve $(y \cdot e^{xy})dx + (x \cdot e^{xy} + 2y)dy = 0$

Sol^n:- Given $(y \cdot e^{xy})dx + (x \cdot e^{xy} + 2y)dy = 0$.

Let $M = y \cdot e^{xy}$ and $N = x \cdot e^{xy} + 2y$

$\frac{\partial M}{\partial y} = y \cdot e^{xy} \cdot x + e^{xy} = e^{xy}(yx + 1)$

$\frac{\partial N}{\partial x} = x \cdot e^{xy} \cdot (y) + e^{xy}(1) = e^{xy}(xy + 1)$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore$ the given Equation is Exact Equation.

The Sol^n is $\int M dx + \int N(y)\, dy = c.$

$\Rightarrow \int y \cdot e^{xy} dx + \int 2y\, dy = c$

$\Rightarrow y \cdot \dfrac{e^{xy}}{y} + \dfrac{2y^2}{2} = c$

$\Rightarrow \underline{e^{xy} + y^2 = c}$, is the required Sol^n

(4). $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

Sol^n:- The given Equation can be written as.

$$\frac{dy}{dx} = -\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}$$

$\Rightarrow dy(\sin x + x\cos y + x) = -(y\cos x + \sin y + y)dx$

$\Rightarrow (y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0.$

Let $M = y\cos x + \sin y + y$, $\dfrac{\partial M}{\partial y} = \cos x + \cos y + 1$

$N = \sin x + x\cos y + x$, $\dfrac{\partial N}{\partial x} = \cos x + \cos y + 1.$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\therefore$ the given Eq^n is Exact.

The solution is

$$\int M\,dx + \int N(y)\,dy = c.$$

$$\Rightarrow \int (y\cos x + \sin y + y)\,dx + \int 0\,dy = c$$

$$\Rightarrow y\cdot \sin x + x\sin y + yx = c$$ is the required Solⁿ.

5). Solve $\cos x(e^y+1)dx + \sin x\, e^y dy = 0.$

Solⁿ: Given, $\cos x(e^y+1)\,dx + \sin x\, e^y\,dy = 0.$

Let $M = \cos x(e^y+1) \Rightarrow \frac{\partial M}{\partial y} = \cos x\cdot e^y$

$N = \sin x\, e^y \Rightarrow \frac{\partial N}{\partial x} = \cos x\cdot e^y$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ the given Eqⁿ is Exact differential Equation.

The Solution is $\int M\,dx + \int N(y)\,dy = c$

$$\Rightarrow \int \cos x(e^y+1)\,dx + \int 0\,dy = c$$

$$\Rightarrow \sin x(e^y+1) = c$$ is the required Solⁿ

Solution of a Homogeneous differential Equation:-

A differential Equation of the type

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n = 0.$$

where, $a_1, a_2, a_3, \ldots a_n$ are Constants, is a homogeneous differential Equation.

The Solution of the homogeneous differential Eqn is

$y =$ Complementary function. $\Rightarrow \{y = y_c\}$

Case (i): If the roots of the differential Equations are real and distinct i.e., $m_1 \neq m_2$

then the general soln of the Eqn is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} \Rightarrow \{\text{complementary fun}^n\}.$$

Case (ii). The roots are real and Equal i.e., $m_1 = m_2$,

then the general soln of Equation is

$$y_c = (c_1 + c_2 x) e^{mx}$$

Case (iii). The roots are complex, i.e., $a \pm ib$, then the general soln is

$$y_c = e^{ax} [c_1 \cos bx + c_2 \sin bx].$$

Q. Solve $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$.

Sol:- The given Equation can be written as

$(D^3 - 2D^2 + 4D - 8)y = 0$; where $D = \frac{dy}{dx}$

The auxillary Equation is

$m^3 - 2m^2 + 4m - 8 = 0$.

$m^2(m-2) + 4(m-2) = 0$

$(m^2+4)(m-2) = 0$.

$(m^2+4) = 0, (m-2) = 0$.

$m^2 = -4 \quad , m = 2$

$m = \sqrt{-4} \quad , m = 2$

$m = \pm 2i \quad , m = 2$

$\therefore m = 2, 2i, -2i$ {the roots are real and Complex}.

$\therefore$ the general sol^n is of the form

$y_c = c_1 e^{m_1 x} + e^{ax}\{c_2 \cos bx + c_3 \sin bx\}$

$= c_1 e^{2x} + e^{0x}\{c_2 \cos 2x + c_3 \sin 2x\}$

$y_c = c_1 e^{2x} + \{c_2 \cos 2x + c_3 \sin 2x\}$ is the required sol^n.

(2). Solve $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0.$

Sol:- The given Equation can be written as,

$$(D^3+6D^2+11D+6)y=0.$$

The auxillary Equation is.

$$m^3+6m^2+11m+6=0.$$

To find the roots, we use Synthetic division method.

for $m=-1$, the Equation $m^3+6m^2+11m+6=0.$

$\therefore$ $m=-1$ is one of the roots.

-1	1	6	11	6
		-1	-5	-6
	1	5	6	0

write the co-efficients of $m^3+6m^2+11m+6$

$$m^2+5m+6=0.$$

$$\Rightarrow m^2+2m+3m+6=0$$

$$\Rightarrow m(m+2)+3(m+2)=0$$

$$\Rightarrow (m+3)(m+2)=0$$

$$m=-3,-2$$

$\therefore$ $m=-1,-2,-3$ are the roots of the Equation. and the roots are real and distinct.

$$\therefore y_c=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}$$

$$=C_1e^{-x}+C_2e^{-2x}+C_3e^{-3x}$$ is the required sol^n.

③. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$.

Sol^n: The given Eq^n can be written as.

$(D^3 - 3D^2 + 2)y = 0$, where $D = \frac{dy}{dx}$.

∴ the auxiliary Eq^n is

$m^3 - 3m^2 + 2 = 0$. ⟹ for $m=1$, the given Eq^n is 0.

To find the roots using synthetic division method.

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$m^2 + m - 2 = 0$.

$\Rightarrow m^2 + 2m - m - 2 = 0$

$\Rightarrow m(m+2) - 1(m+2) = 0$

$\Rightarrow (m-1)(m+2) = 0$.

∴ $m = -2, 1$

∴ $m = 1, 1, -2$

∴ the general sol^n is of the form

$y_c = (C_1 + C_2 x)e^{x} + C_3 e^{mx}$

∴ $y = (C_1 + C_2 x)e^{1x} + C_3 e^{-2x}$ is the required sol^n.

(4). Solve $4\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 23\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0.$

Sol: The given diff. Eqn can be written as.

$(4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0.$

∴ the auxiliary Eqn is

$4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0.$

To find the roots using synthetic division method.

$m = 2$ is one of the roots (∵ for $m=2$, the given Eqn is 0).

2	4	−4	−23	12	36
		8	8	−30	−36
	4	4	−15	−18	0

$4m^3 + 4m^2 - 15m - 18 = 0.$

2	4	4	−15	−18
		8	24	18
	4	12	9	0

$4m^2 + 12m + 9 = 0.$

$4m^2 + 6m + 6m + 9 = 0.$

$2m(2m+3) + 3(2m+3) = 0$

$(2m+3)(2m+3) = 0$

∴ $m = -3/2, -3/2$

∴ $m = 2, 2, -3/2, -3/2$ are the roots of the given diff. Eqn.

∴ the Complementary funcn is

$y_c = (C_1 + C_2x)e^{2x} + (C_3 + C_4x)e^{-3/2x}$

⑤. Solve $4\frac{d^4y}{dx^4} - 8\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0.$

Sol^n:- The given Equation can be written as,

$$(4D^4 - 8D^3 - 7D^2 + 11D + 6)\,y = 0.$$

The auxiliary Eq^n is

$$4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0.$$

To find the roots using synthetic division.

$$\begin{array}{r|rrrrr}
2 & 4 & -8 & -7 & 11 & 6 \\
 & & 8 & 0 & -14 & -6 \\
\hline
-1 & 4 & 0 & -7 & -3 & 0 \\
 & & -4 & 4 & 3 & \\
\hline
 & 4 & -4 & -3 & 0 &
\end{array}
\Rightarrow 4m^3 - 7m - 3 = 0.$$

$$4m^2 - 4m - 3 = 0.$$

$$4m^2 - 6m + 2m - 3 = 0$$

$$(2m-3)(2m+1) = 0$$

$$m = 3/2,\ m = -1/2$$

$\therefore$ the roots are $m = 2, -1, 3/2, -1/2$

$\therefore$ Complementary func^n $\Rightarrow y_c = C_1e^{2x} + C_2e^{-x} + C_3e^{3/2\,x} + C_4e^{-1/2\,x}$

is the required Sol^n

Solution of Non-homogeneous differential Eq^n

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n = f(x)$$

where $a_1, a_2, a_3, \cdots a_n$ are constants and $f(x)$ is a function of x, is a non-homogeneous fun^n.

Type - I. if $f(x) = e^{ax}$,

then the general solution is y = Complementary fun^n + Particular Integral

$\Rightarrow y = y_c + y_p$

$$P.I = \frac{1}{f(D)} \cdot f(x) = \frac{1}{f(D)} \cdot e^{ax}$$

$$P.I = \begin{cases} \frac{1}{f(a)} \cdot e^{ax}, & \text{if } f(a) \neq 0 \\ \frac{x}{f'(a)} \cdot e^{ax}, & \text{if } f(a) = 0 \end{cases}$$

<u>Note</u>:- If the denominator $f(D) = 0$, then multiply the numerator by x, and differentiate the denominator $f(D)$, w.r.t D. Repeat the process until the denominator is not Equal to Zero.

(1). Solve $6\frac{d^2y}{dx^2}+17\frac{dy}{dx}+12y=e^{-x}$

Sol:- The given diff. Eq^n can be written as $(6D^2+17D+12)y=e^{-x}$

The auxillary Eq^n is $6m^2+17m+12=0$.

$m=-4/3$ and $m=-3/2$ are the roots.

∴ the Complementary fun^n of the given Eq^n is

$y_c=C_1e^{-4/3x}+C_2e^{-3/2x}$

also, to find the Particular Integral.

$y_p=\frac{1}{f(D)}\cdot f(x)=\frac{e^{-x}}{6D^2+17D+12}$ { Substitute $D=-1$ (Co-efficient of x).

$y_p=\frac{e^{-x}}{6(-1)^2+17(-1)+12}=e^{-x}$.

∴ the sol^n for the given Eq^n is

$\underline{y=C_1e^{-4/3x}+C_2e^{-3/2x}+e^{-x}}$

(2). Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\cosh(x/2)$

Sol^n:- The given diff. Eq^n is $(D^2+2D+1)y=\cosh(x/2)$, where $D=\frac{dy}{dx}$.

The auxillary Eq^n is $m^2+2m+1=0$. $\Rightarrow (m+1)^2=0$.

∴ $m=-1,-1$, are the roots.

∴ Complementary fun^n is $y_c=(C_1+C_2x)e^{-x}$.

$\cosh x=\frac{e^{x}+e^{-x}}{2}$

$\cosh x/2=\frac{e^{x/2}+e^{-x/2}}{2}$

The particular Integral is

$y_p=\frac{\cosh(x/2)}{(D+1)^2}=\frac{e^{x/2}+e^{-x/2}}{2(D+1)^2}$

$$= \frac{1}{2}\left\{\frac{e^{x/2}}{(D+1)^2} + \frac{e^{-x/2}}{(D+1)^2}\right\} = \frac{1}{2}\left\{\frac{e^{x/2}}{(\frac{1}{2}+1)^2} + \frac{e^{-x/2}}{(-\frac{1}{2}+1)^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{x/2}}{9/4} + \frac{e^{-x/2}}{1/4}\right\} = \frac{1}{2}\left\{\frac{4}{9}e^{x/2} + 4e^{-x/2}\right\}$$

$\therefore$ The Complete Solⁿ is.

$$\underline{y = (C_1 + C_2 x)e^{-x} + \frac{2}{9}e^{x/2} + 2e^{-x/2}}$$

(3). Solve $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

Solⁿ:- The given Eqⁿ Can be written as $(D^2-4)y = \cosh(2x-1) + 3^x$.

The auxillary Eqⁿ is $m^2 - 4 = 0$.

$$m = \pm 2.$$

$\therefore$ the Complementary funⁿ is

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

3^x can be written as $e^{\log 3^x} = e^{x\log 3}$ $\therefore$ the Co-eff of x is $\log 3$.

The Particular Integral is

$$y_p = \frac{\cosh(2x-1) + 3^x}{(D^2-4)} = \frac{1}{2}\left\{\frac{e^{2x-1} + e^{-(2x-1)}}{D^2-4}\right\} + \frac{3^x}{(D^2-4)}$$

$$= \frac{1}{2}\left\{\frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4}\right\} + \frac{3^x}{D^2-4}$$

for $D = 2$ and -2, the denominator is 0,

If $f(a) = 0$. then, $\frac{x \cdot e^{ax}}{f'(a)}$

$$\Rightarrow \frac{1}{2}\left\{\frac{x \cdot e^{2x-1}}{2D} + \frac{x \cdot e^{-2x+1}}{2D}\right\} + \frac{e^{\log 3x}}{D^2-4}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{x \cdot e^{2x-1}}{4} + \frac{x \cdot e^{-2x+1}}{2(-2)}\right\} + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$$

$$= \frac{x \cdot e^{2x-1}}{8} - \frac{x e^{-2x+1}}{8} + \frac{3^x}{(\log 3)^2 - 4}$$

If $f(x) = \cos ax$ (or) $\sin ax$. (Type-II).

then the $P.I = \frac{f(x)}{f(D)} = \frac{\cos ax}{f(D^2)}$ (or) $\frac{\sin ax}{f(D^2)}$

replace D^2 by $-a^2$.

$\therefore P.I = \frac{\sin ax \text{ (or) } \cos ax}{f(-a^2)}$.

Q. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = \cos 2x$.

Solⁿ:- The given Eqⁿ can be written as $(D^2-4D+13)y = \cos 2x$.
The auxiliary Eqⁿ is $m^2 - 4m + 13 = 0$.

$\therefore m = 2 \pm 3i$. are the roots.

and the roots are complex.

$\therefore$ the complementary funⁿ, $y_c = e^{2x}[C_1 \cos 3x + C_2 \sin 3x]$.

The particular integral is $P.I = \frac{\cos 2x}{(D^2-4D+13)}$, replace D^2 by -2^2

$\therefore P.I = \frac{\cos 2x}{(-2^2)-4D+13} = \frac{\cos 2x}{9-4D}$ {Rationalize the denominator.

$= \frac{\cos 2x}{9-4D} \times \frac{9+4D}{9+4D} = \frac{9\cos 2x + 4 \cdot D \cdot \cos 2x}{81 - 16D^2}$

$= \frac{9\cos 2x}{81-16D^2} + \frac{4(-\sin 2x(2))}{81-16D^2}$

{ $D \cdot \cos 2x$
↓
diff. w.r.t x.
$\Rightarrow -\sin 2x \cdot 2$

$= \frac{9\cos 2x}{81-16(-2^2)} - \frac{8\sin 2x}{81-16(-2^2)}$

$= \frac{9\cos 2x}{145} - \frac{8\sin 2x}{145}$

$\therefore y = e^{2x}\left[C_1\cos 3x + C_2\sin 3x\right] + \frac{9\cos 2x - 8\sin 2x}{145}$ is the required sol^n

Q. Solve $\frac{d^2y}{dx^2} + 9y = \cos 2x \cdot \cos x$.

Sol^n:- The given diff. Eq^n can be written as.

$(D^2+9)y = \cos 2x \cdot \cos x$.

The auxiliary Eq^n is $m^2+9=0$. $\therefore m = \pm 3i$

$\therefore$ the complementary fun^n is $y_c = C_1\cos 3x + C_2\sin 3x$

The particular integral is.

$$P.I = y_p = \frac{\cos 2x \cdot \cos x}{D+9} = \frac{\frac{1}{2}(\cos 3x + \cos x)}{D^2+9}$$

$$= \frac{1}{2}\left\{\frac{\cos 3x}{D^2+9} + \frac{\cos x}{D^2+9}\right\}$$

for $D^2 = -3^2$, the denominator is zero.

$$\therefore P.I = \frac{1}{2}\left\{\frac{x \cdot \cos 3x}{2D} + \frac{\cos x}{D^2+9}\right\}$$

$\frac{1}{D}$ is integration

$$= \frac{1}{2}\left\{\frac{1}{2}\frac{x\cos 3x}{(D)}\right\} + \frac{1}{2}\left\{\frac{\cos x}{-1^2+9}\right\}$$

$$= \frac{x}{4}\left\{\int \cos 3x\right\} + \frac{1}{2}\frac{\cos x}{(8)}$$

$$= \frac{x}{4}\frac{\sin 3x}{3} + \frac{\cos x}{16}$$

$$= \frac{x\sin 3x}{12} + \frac{\cos x}{16}$$

$$\therefore y = C_1\cos 3x + C_2\sin 3x + \frac{x\sin 3x}{12} + \frac{\cos x}{16}$$

(3). Solve $\left(\frac{d^3y}{dx^3}-1\right)y = 3\cos 2x$

Sol^n:- The given diff. Eq^n can be written as

$(D^3-1)y = 3\cos 2x$

The auxiliary Eq^n is

$a^3-b^3 = (a-b)(a^2+b^2+ab)$

$m^3-1=0.$

$m^3-1^3=0$

$(a-b)(a^2+b^2+ab)=0.$

$(m-1)(m^2+1+m)=0.$

$m=-1, \left(-\frac{1}{2} \pm \frac{3i}{2}\right)$ are the roots.

$m^2+m+1=0.$

$a=1, b=1, c=1$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{3i}{2}$

$$P.I = \frac{3\cos 2x}{(D^3-1)} = \frac{3\cos 2x}{(D \cdot D^2-1)} = \frac{3\cos 2x}{D(-2^2)-1}$$

$$= \frac{3\cos 2x}{-4D-1}$$

$$= \frac{3\cos 2x}{-4D-1} \times \frac{-4D+1}{-4D+1}$$

$$= \frac{-12D\cos 2x + 3\cos 2x}{16D^2-1^2}$$

$$= \frac{-12D\cos 2x}{16D^2-1^2} + \frac{3\cos 2x}{16D^2-1}$$

$$= \frac{-12(-\sin 2x)(2)}{16(-2^2)-1} + \frac{3\cos 2x}{16(-2^2)-1}$$

$$= \frac{24\sin 2x}{-64-1} + \frac{3\cos 2x}{-64-1}$$

$$= \frac{24\sin 2x}{-65} - \frac{3\cos 2x}{65}$$

$$\therefore\ y = c_1 e^{-x} + e^{-1/2\, x}\left[c_2 \cos \tfrac{3}{2}x + c_3 \sin \tfrac{3}{2}x\right] - \frac{24\sin 2x}{65} - \frac{3\cos 2x}{65}$$