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Lista 2.5

① Seja $X_1, \dots, X_n$ iid com pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad 0 < \theta < \infty.$$

ⓐ Encontre o MLE de θ, e mostre que a variancia $\to 0$ quando $n \to \infty$.

ⓑ Encontre o estimador de metodo dos momentos para θ.

$$f(x|\theta) = \prod \theta x_i^{\theta-1} = \theta^n \left(\prod x_i\right)^{\theta-1} = L(\theta|x)$$

$$\Rightarrow \quad \frac{\partial}{\partial \theta} \log L = \frac{\partial}{\partial \theta}\left[n \log \theta + (\theta-1)\log \prod x_i\right] = \frac{n}{\theta} + \sum \log x_i$$

$$\Rightarrow \quad \frac{\partial}{\partial \theta} \log L = \frac{n}{\theta} + \sum \log x_i = 0 \quad \Rightarrow \theta = -\frac{n}{\sum \ln x_i}$$

E a segunda derivada é $-\frac{n}{\theta^2} < 0$, então este é o MLE. Para calcular a variancia de $\hat{\theta}$, note que, $Y_i = -\log X_i \sim Exp(1/\theta)$, então $-\sum \log X_i \sim Gamma(n, 1/\theta)$. Seja $\hat{\theta} = n/T$, onde $T \sim Gamma(n, 1/\theta)$. Temos

$$E\frac{1}{T} = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1}$$

$$E\frac{1}{T^2} = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t^2} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)}$$

e daí $\quad E\hat{\theta} = \frac{n}{n-1}\theta \quad$ e $\quad Var\,\hat{\theta} = \frac{n^2}{(n-1)^2(n-2)}\theta^2 \to 0 \quad (n \to \infty)$

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(b)

$$EX_i = \int_0^1 x\,\theta x^{\theta-1}\,dx$$

$$= \theta \int_0^1 x^{\theta}\,dx = \theta \left[\frac{x^{\theta+1}}{\theta+1} \right]_0^1 = \frac{\theta}{\theta+1}$$

Daí $\frac{1}{n}\sum X_i = \frac{\hat{\theta}}{\hat{\theta}+1} \Rightarrow \hat{\theta} = \frac{\sum X_i}{n-\sum X_i}$

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② Seja $X_1, \ldots, X_n$ uma amostra aleatoria com pmf

$$P_\theta(X=x) = \theta^x (1-\theta)^{1-x}, \quad x = 0 \text{ ou } 1, \quad 0 \leq \theta \leq 1/2.$$

ⓐ Encontre o estimador de metodo de momentos e MLE de θ.

ⓑ Encontre o MSE de cada estimador.

ⓒ Qual você prefere? justifique.

ⓐ MM:

$$EX = \theta = \frac{1}{n}\sum X_i = \bar{X} \Rightarrow \tilde{\theta} = \bar{X}.$$

MLE:

$$f(x|\theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i} = L(\theta|x)$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left(\ln L(\theta|x)\right) = \frac{\partial}{\partial\theta}\left(\sum x_i \ln\theta + \sum(1-x_i)\ln(1-\theta)\right)$$

$$= \frac{\partial}{\partial\theta}\left(\ln\theta \sum x_i + \ln(1-\theta)n - \ln(1-\theta)\sum x_i\right)$$

$$= \frac{\sum x_i}{\theta} - \frac{n}{1-\theta} + \frac{\sum x_i}{1-\theta} = 0$$

$$\Rightarrow (1-\theta)\sum x_i - n\theta + \theta\sum x_i = 0 = \sum x_i - n\theta = 0$$

$$\Rightarrow \hat{\theta} = \frac{\sum x_i}{n}$$

quando $\bar{x} \leq 1/2$, pois $L(\theta|x)$ esta considerando o intervalo do θ, logo $\hat{\theta} = \min\{\bar{X}, 1/2\}$

(b) O MSE de $\tilde{\theta} = Var(\tilde{\theta}) + bias(\tilde{\theta})^2 = \left(\frac{(1-\theta)\theta}{n}\right) + \theta^2$.
O MSE de $\hat{\theta}$:

$$MSE(\hat{\theta}) = E(\hat{\theta}-\theta)^2 = \sum_{y=0}^{n} (\hat{\theta}-\theta)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y}$$
$$= \sum_{y=0}^{[n/2]} \left(\frac{y}{n}-\theta\right)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y}$$
$$+ \sum_{y=[n/2]+1} \left(\frac{1}{2}-\theta\right)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y},$$

Onde $Y = \sum X_i \sim binomial(n,\theta)$.

(c) Considerando o MSE, temos que

$$MSE(\tilde{\theta}) = E(\bar{X}-\theta)^2 = \sum_{y=0}^{n} \left(\frac{y}{n}-\theta\right)^2 \binom{n}{y} \theta^y (1-\theta)^{n-y}.$$

Portanto, $$MSE[\tilde{\theta}] - MSE(\hat{\theta}) = \sum_{y=\lfloor n/2 \rfloor+1}^{n} \left[\left(\frac{y}{n}-\theta\right)^2 - \left(\frac{1}{2}-\theta\right)^2\right] \binom{n}{y} \theta^y (1-\theta)^{n-y}$$
$$= \sum_{y=\lfloor n/2 \rfloor+1} \left(\frac{y}{n}+\frac{1}{2}-2\theta\right)\left(\frac{y}{n}-\frac{1}{2}\right) \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

Se $y/n > 1/2$ e $\theta \leq 1/2$ então os termos são positivos.
Daí, $MSE(\hat{\theta}) < MSE(\tilde{\theta})\ \forall\ 0 < \theta < 1/2$.

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⑥ Seja $X_1, \dots, X_n \overset{iid}{\sim}$ Bernoulli(p). Mostre que a variância de $\bar{X}$ satisfaz o Limite Inferior de Cramér-Rao, e por isso $\bar{X}$ é o melhor estimador não viesado de p.

Assunto não coberto.