# 8.4 - DeGroot

- Derivar a densidade da Distribuição t-student

Temos que $T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$, $\bar{X} = \frac{\sum X_i}{n}$, $S = \frac{\sum(X_i - \bar{X})^2}{n-1}$

$X_i \sim$ IID
$\sim N(\mu, \sigma^2)$, $i = 1, \dots, n$

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \cdot \frac{\sigma}{\sigma} = \underbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}_{U \sim N(0,1)} \cdot \frac{\sigma}{S}$$

$$\left(\frac{1}{S^2/\sigma^2}\right)^{1/2} = \left[\frac{n-1}{(n-1)S^2/\sigma^2}\right]^{1/2}$$

$V \sim \chi^2(n-1)$

$$T = U\left(\frac{n-1}{V}\right)^{1/2}$$

Note que, $U = g_1(\bar{X})$ e $g_2(S^2)$, como $\bar{X}$ e $S^2$ são independentes então $U$ e $V$ são independentes

$$f_{U,V}(u,v) = f_U(u) f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{(n-1)/2}} v^{\frac{n-1}{2}-1} e^{-v/2}$$

$$= c \cdot e^{-u^2/2} v^{\frac{n-1}{2}-1} e^{-v/2}$$

$= c$

Jacobiano

$$f_{T,V}(t,w) = f_{U,V}(h_1(t,w), h_2(t,w))\,|J|$$

$$t = u\left(\frac{n-1}{v}\right)^{1/2};\quad u = t\left(\frac{v}{n-1}\right)^{1/2} = t\left(\frac{w}{n-1}\right)^{1/2}$$

$$w = v \quad ; \quad v = w$$

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial t} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial t} & \frac{\partial h_2}{\partial w} \end{vmatrix} = \left(\frac{w}{n-1}\right)^{1/2} \cdot 1 - \frac{t}{(n-1)^{1/2}} \frac{1}{2} w^{-1/2} \, 0 = \left(\frac{w}{n-1}\right)^{1/2}$$

Então,

$$f_{T,W}(t,w) = c\, e^{-t^2\left(\frac{w}{n-1}\right)\cdot\frac{1}{2}}\, w^{\frac{n-1}{2}-1}\, e^{-w/2} \cdot \frac{w^{1/2}}{(n-1)^{1/2}}$$

$$= \frac{c}{(n-1)^{1/2}}\, w^{\frac{n}{2}-1}\, e^{-\frac{w}{2}\left(\frac{t^2}{n-1}+1\right)}$$

$$f_T(t) = \frac{c}{(n-1)^{1/2}} \int_0^\infty w^{\frac{n}{2}-1} e^{-\frac{b}{2}w}\, dw, \quad b = \frac{t^2}{n-1}+1$$

Note que, conhecemos essa integral como

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy, \text{ com } y = \frac{b}{2}w \text{ e } \alpha = n/2$$

Assim, $\Gamma(n/2) = \int_0^\infty \left(\frac{b}{2}\right)^{\frac{n}{2}-1} w^{\frac{n}{2}-1} e^{-\frac{b}{2}w} \left(\frac{b}{2}\right) dw, \quad n \geq 2$

$$f_T(t) = \frac{c}{(n-1)^{1/2}}\, \Gamma(n/2) \left(\frac{2}{b}\right)^{n/2}$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right) \cancel{2^{(n-1)/2}}} \cdot \frac{1}{(n-1)^{1/2}} \cdot \Gamma(n/2) \cdot \frac{\cancel{2^{n/2}}}{\left(\frac{t^2}{n-1}+1\right)^{n/2}}$$

$$= \frac{\Gamma(n/2)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{\sqrt{\pi}(n-1)^{1/2}} \cdot \frac{1}{\left(\frac{t^2}{n-1}+1\right)^{n/2}}$$, $n-1$ graus de liberdade

$$f_T(t|p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \cdot \frac{1}{\sqrt{\pi p}} \cdot \frac{1}{\left(\frac{t^2}{p}+1\right)^{\left(\frac{p+1}{2}\right)}}$$, com $p$ graus de liberdade

$p \geq 1$