7.37 Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \frac{1}{2\theta}, \quad -\theta < x < \theta, \quad \theta > 0.$$

Find, if one exists, a best unbiased estimator of θ.

7.41 Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2.

(a) Show that the estimator $\sum_{i=1}^n a_i X_i$ is an unbiased estimator of μ if $\sum_{i=1}^n a_i = 1$.

(b) Among all unbiased estimators of this form (called *linear unbiased estimators*) find the one with minimum variance, and calculate the variance.

7.45 Let $X_1, X_2, \ldots, X_n$ be iid from a distribution with mean μ and variance σ^2, and let S^2 be the usual unbiased estimator of σ^2. In Example 7.3.4 we saw that, under normality, the MLE has smaller MSE than S^2. In this exercise will explore variance estimates some more.

(a) Show that, for any estimator of the form aS^2, where a is a constant,

$$\text{MSE}(aS^2) = \text{E}[aS^2 - \sigma^2]^2 = a^2 \text{Var}(S^2) + (a-1)^2\sigma^4.$$

(b) Show that

$$\text{Var}(S^2) = \frac{1}{n}\left(\kappa - \frac{n-3}{n-1}\sigma^4\right),$$

where $\kappa = \text{E}[X-\mu]^4$ is the *kurtosis*. (You may have already done this in Exercise 5.8(b).)

(c) Show that, under normality, the kurtosis is $3\sigma^4$ and establish that, in this case, the estimator of the form aS^2 with the minimum MSE is $\frac{n-1}{n+1}S^2$. (Lemma 3.6.5 may be helpful.)

(d) If normality is not assumed, show that $\text{MSE}(aS^2)$ is minimized at

$$a = \frac{n-1}{(n+1) + \frac{(\kappa-1)(n-1)}{n}},$$

which is useless as it depends on a parameter.

(e) Show that

(i) for distributions with $\kappa > 3$, the optimal a will satisfy $a < \frac{n-1}{n+1}$;

(ii) for distributions with $\kappa < 3$, the optimal a will satisfy $\frac{n-1}{n+1} < a < 1$.

See Searls and Intarapanich (1990) for more details.

7.47 Suppose that when the radius of a circle is measured, an error is made that has a $\text{n}(0, \sigma^2)$ distribution. If n independent measurements are made, find an unbiased estimator of the area of the circle. Is it best unbiased?

7.55 For each of the following pdfs, let $X_1, \ldots, X_n$ be a sample from that distribution. In each case, find the best unbiased estimator of θ^r. (See Guenther 1978 for a complete discussion of this problem.)

(a) $f(x|\theta) = \frac{1}{\theta}, \quad 0 < x < \theta,\ r < n$

(b) $f(x|\theta) = e^{-(x-\theta)}, \quad x > \theta$

(c) $f(x|\theta) = \frac{e^{-x}}{e^{-\theta} - e^{-b}}, \quad \theta < x < b, \quad b$ known

7.57 Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| p\right),$$

the probability that the first n observations exceed the $(n+1)$st.

(a) Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

is an unbiased estimator of $h(p)$.

(b) Find the best unbiased estimator of $h(p)$.

7.59 Let $X_1, \ldots, X_n$ be iid $\mathrm{n}(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a known positive constant, not necessarily an integer.