*9. Suppose data X has density p_θ, $\theta \in \Omega \subset \mathbb{R}$, and that these densities are regular enough that the derivative of the power function of any test function φ can be evaluated differentiating under the integral sign,

$$\beta'_\varphi(\theta) = \int \varphi(x) \frac{\partial p_\theta(x)}{\partial \theta}\, d\mu(x).$$

A test φ^* is called *locally most powerful* testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ if it maximizes $\beta'_\varphi(\theta_0)$ among all tests φ with level α. Determine the form of the locally most powerful test.

*10. Suppose $X = (X_1, \dots, X_n)$ with the X_i i.i.d. with common density f_θ. The locally most powerful test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ from Problem 12.9 should reject H_0 if an appropriate statistic T exceeds a critical value c. Use the central limit theorem to describe how the critical level c can be chosen when n is large to achieve a level approximately α. The answer should involve Fisher information at $\theta = \theta_0$.

8.2 In a given city it is assumed that the number of automobile accidents in a given year follows a Poisson distribution. In past years the average number of accidents per year was 15, and this year it was 10. Is it justified to claim that the accident rate has dropped?

8.4 Prove the assertion made in the text after Definition 8.2.1. If $f(x|\theta)$ is the pmf of a discrete random variable, then the numerator of $\lambda(\mathbf{x})$, the LRT statistic, is the maximum probability of the observed sample when the maximum is computed over parameters in the null hypothesis. Furthermore, the denominator of $\lambda(\mathbf{x})$ is the maximum probability of the observed sample over all possible parameters.

8.6 Suppose that we have two independent random samples: $X_1, \dots, X_n$ are exponential(θ), and $Y_1, \dots, Y_m$ are exponential(μ).

(a) Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\Sigma X_i}{\Sigma X_i + \Sigma Y_i}.$$

(c) Find the distribution of T when H_0 is true.

8.8 A special case of a normal family is one in which the mean and the variance are related, the $n(\theta, a\theta)$ family. If we are interested in testing this relationship, regardless of the value of θ, we are again faced with a nuisance parameter problem.

(a) Find the LRT of $H_0: a = 1$ versus $H_1: a \neq 1$ based on a sample $X_1, \ldots, X_n$ from a $n(\theta, a\theta)$ family, where θ is unknown.

(b) A similar question can be asked about a related family, the $n(\theta, a\theta^2)$ family. Thus, if $X_1, \ldots, X_n$ are iid $n(\theta, a\theta^2)$, where θ is unknown, find the LRT of $H_0: a = 1$ versus $H_1: a \neq 1$.

8.10 Let $X_1, \ldots, X_n$ be iid Poisson(λ), and let λ have a gamma(α, β) distribution, the conjugate family for the Poisson. In Exercise 7.24 the posterior distribution of λ was found, including the posterior mean and variance. Now consider a Bayesian test of $H_0: \lambda \leq \lambda_0$ versus $H_1: \lambda > \lambda_0$.

(a) Calculate expressions for the posterior probabilities of H_0 and H_1.

(b) If $\alpha = \frac{5}{2}$ and $\beta = 2$, the prior distribution is a chi squared distribution with 5 degrees of freedom. Explain how a chi squared table could be used to perform a Bayesian test.

8.12 For samples of size $n = 1, 4, 16, 64, 100$ from a normal population with mean μ and known variance σ^2, plot the power function of the following LRTs. Take $\alpha = .05$.

(a) $H_0: \mu \leq 0$ versus $H_1: \mu > 0$

(b) $H_0: \mu = 0$ versus $H_1: \mu \neq 0$

8.14 For a random sample $X_1, \ldots, X_n$ of Bernoulli(p) variables, it is desired to test

$$H_0: p = .49 \qquad \text{versus} \qquad H_1: p = .51.$$

Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about .01. Use a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.

8.16 One very striking abuse of α levels is to choose them *after* seeing the data and to choose them in such a way as to force rejection (or acceptance) of a null hypothesis. To see what the *true* Type I and Type II Error probabilities of such a procedure are, calculate size and power of the following two trivial tests:

(a) Always reject H_0, no matter what data are obtained (equivalent to the practice of choosing the α level to force rejection of H_0).

(b) Always accept H_0, no matter what data are obtained (equivalent to the practice of choosing the α level to force acceptance of H_0).

8.18 Let $X_1, \ldots, X_n$ be a random sample from a $n(\theta, \sigma^2)$ population, σ^2 known. An LRT of $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$ is a test that rejects H_0 if $|\bar{X} - \theta_0|/(\sigma/\sqrt{n}) > c$.

(a) Find an expression, in terms of standard normal probabilities, for the power function of this test.

(b) The experimenter desires a Type I Error probability of .05 and a maximum Type II Error probability of .25 at $\theta = \theta_0 + \sigma$. Find values of n and c that will achieve this.

8.20 Let X be a random variable whose pmf under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_1)$	.06	.05	.04	.03	.02	.01	.79

Use the Neyman–Pearson Lemma to find the most powerful test for H_0 versus H_1 with size $\alpha = .04$. Compute the probability of Type II Error for this test.

8.22 Let $X_1, \ldots, X_{10}$ be iid Bernoulli(p).

(a) Find the most powerful test of size $\alpha = .0547$ of the hypotheses $H_0: p = \frac{1}{2}$ versus $H_1: p = \frac{1}{4}$. Find the power of this test.

(b) For testing $H_0: p \leq \frac{1}{2}$ versus $H_1: p > \frac{1}{2}$, find the size and sketch the power function of the test that rejects H_0 if $\sum_{i=1}^{10} X_i \geq 6$.

(c) For what α levels does there exist a UMP test of the hypotheses in part (a)?