8.31 Let $X_1, \ldots, X_n$ be iid Poisson(λ).

(a) Find a UMP test of $H_0\colon \lambda \le \lambda_0$ versus $H_1\colon \lambda > \lambda_0$.

(b) Consider the specific case $H_0 : \lambda \le 1$ versus $H_1 : \lambda > 1$. Use the Central Limit Theorem to determine the sample size n so a UMP test satisfies $P(\text{reject } H_0|\lambda = 1) = .05$ and $P(\text{reject } H_0|\lambda = 2) = .9$.

8.33 Let $X_1, \ldots, X_n$ be a random sample from the uniform$(\theta, \theta+1)$ distribution. To test $H_0\colon \theta = 0$ versus $H_1\colon \theta > 0$, use the test

$$\text{reject } H_0 \text{ if } Y_n \ge 1 \text{ or } Y_1 \ge k,$$

where k is a constant, $Y_1 = \min\{X_1, \ldots, X_n\}, Y_n = \max\{X_1, \ldots, X_n\}$.

(a) Determine k so that the test will have size α.

(b) Find an expression for the power function of the test in part (a).

(c) Prove that the test is UMP size α.

(d) Find values of n and k so that the UMP .10 level test will have power at least .8 if $\theta > 1$.

8.37 Let $X_1, \ldots, X_n$ be a random sample from a n(θ, σ^2) population. Consider testing

$$H_0\colon \theta \le \theta_0 \qquad \text{versus} \qquad H_1\colon \theta > \theta_0.$$

(a) If σ^2 is known, show that the test that rejects H_0 when

$$\bar{X} > \theta_0 + z_\alpha \sqrt{\sigma^2/n}$$

is a test of size α. Show that the test can be derived as an LRT.

(b) Show that the test in part (a) is a UMP test.

(c) If σ^2 is unknown, show that the test that rejects H_0 when

$$\bar{X} > \theta_0 + t_{n-1,\alpha} \sqrt{S^2/n}$$

is a test of size α. Show that the test can be derived as an LRT.

8.39 Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$. We are interested in testing

$$H_0\colon \mu_X = \mu_Y \qquad \text{versus} \qquad H_1\colon \mu_X \neq \mu_Y.$$

(a) Show that the random variables $W_i = X_i - Y_i$ are iid $\mathrm{n}(\mu_W, \sigma_W^2)$.

(b) Show that the above hypothesis can be tested with the statistic

$$T_W = \frac{\bar{W}}{\sqrt{\frac{1}{n} S_W^2}},$$

where $\bar{W} = \frac{1}{n}\sum_{i=1}^n W_i$ and $S_W^2 = \frac{1}{(n-1)}\sum_{i=1}^n (W_i - \bar{W})^2$. Furthermore, show that, under H_0, $T_W \sim$ Student's t with $n-$ 1 degrees of freedom. (This test is known as the *paired-sample t test.*)

8.49 In each of the following situations, calculate the p-value of the observed data.

(a) For testing $H_0 : \theta \le \frac{1}{2}$ versus $H_1 : \theta > \frac{1}{2}$, 7 successes are observed out of 10 Bernoulli trials.

(b) For testing $H_0 : \lambda \le 1$ versus $H_1 : \lambda > 1, X = 3$ are observed, where $X \sim$ Poisson(λ).

(c) For testing $H_0 \colon \lambda \le 1$ versus $H_1 \colon \lambda > 1, X_1 = 3, X_2 = 5$, and $X_3 = 1$ are observed, where $X_i \sim$ Poisson(λ), independent.

8.51 Here is another common interpretation of p-values. Consider a problem of testing H_0 versus H_1. Let $W(\mathbf{X})$ be a test statistic. Suppose that for each α, $0 \le \alpha \le 1$, a critical value c_α can be chosen so that $\{\mathbf{x} : W(\mathbf{x}) \ge c_\alpha\}$ is the rejection region of a size α test of H_0. Using this family of tests, show that the usual p-value $p(\mathbf{x})$, defined by (8.3.9), is the smallest α level at which we could reject H_0, having observed the data $\mathbf{x}$.

8.53 In Example 8.2.7 we saw an example of a one-sided Bayesian hypothesis test. Now we will consider a similar situation, but with a two-sided test. We want to test

$$H_0: \theta = 0 \qquad \text{versus} \qquad H_1: \theta \neq 0,$$

and we observe $X_1, \ldots, X_n$, a random sample from a $\mathrm{n}(\theta, \sigma^2)$ population, σ^2 known. A type of prior distribution that is often used in this situation is a mixture of a point mass on $\theta = 0$ and a pdf spread out over H_1. A typical choice is to take $P(\theta = 0) = \frac{1}{2}$, and if $\theta \neq 0$, take the prior distribution to be $\frac{1}{2}\mathrm{n}(0, \tau^2)$, where τ^2 is known.

(a) Show that the prior defined above is proper, that is, $P(-\infty < \theta < \infty) = 1$.
(b) Calculate the posterior probability that H_0 is true, $P(\theta = 0 | x_1, \ldots, x_n)$.
(c) Find an expression for the p-value corresponding to a value of $\bar{x}$.
(d) For the special case $\sigma^2 = \tau^2 = 1$, compare $P(\theta = 0 | x_1, \ldots, x_n)$ and the p-value for a range of values of $\bar{x}$. In particular,

   (i) For $n = 9$, plot the p-value and posterior probability as a function of $\bar{x}$, and show that the Bayes probability is greater than the p-value for moderately large values of $\bar{x}$.
   (ii) Now, for $\alpha = .05$, set $\bar{x} = Z_{\alpha/2}/\sqrt{n}$, fixing the p-value at α for all n. Show that the posterior probability at $\bar{x} = Z_{\alpha/2}/\sqrt{n}$ goes to 1 as $n \to \infty$. This is *Lindley's Paradox*.

   Note that small values of $P(\theta = 0 | x_1, \ldots, x_n)$ are evidence *against* H_0, and thus this quantity is similar in spirit to a p-value. The fact that these two quantities can have very different values was noted by Lindley (1957) and is also examined by Berger and Sellke (1987). (See the Miscellanea section.)

8.55 Let X have a $\mathrm{n}(\theta, 1)$ distribution, and consider testing $H_0: \theta \geq \theta_0$ versus $H_1: \theta < \theta_0$. Use the loss function (8.3.13) and investigate the three tests that reject H_0 if $X < -z_\alpha + \theta_0$ for $\alpha = .1, .3,$ and $.5$.

(a) For $b = c = 1$, graph and compare their risk functions.
(b) For $b = 3, c = 1$, graph and compare their risk functions.
(c) Graph and compare the power functions of the three tests to the risk functions in parts (a) and (b).

8.57 Consider testing $H_0: \mu \leq 0$ versus $H_1: \mu > 0$ using 0–1 loss, where $X \sim \mathrm{n}(\mu, 1)$. Let δ_c be the test that rejects H_0 if $X > c$. For every test in this problem, there is a δ_c in the class of tests $\{\delta_c, -\infty \leq c \leq \infty\}$ that has a uniformly smaller (in μ) risk function. Let δ be the test that rejects H_0 if $1 < X < 2$. Find a test δ_c that is better than δ. (Either prove that the test is better or graph the risk functions for δ and δ_c and carefully explain why the proposed test should be better.)