8.1 In 1,000 tosses of a coin, 560 heads and 440 tails appear. Is it reasonable to assume that the coin is fair? Justify your answer.

8.3 Here, the LRT alluded to in Example 8.2.9 will be derived. Suppose that we observe m iid Bernoulli(θ) random variables, denoted by $Y_1, \ldots, Y_m$. Show that the LRT of $H_0: \theta \leq \theta_0$ versus $H_1: \theta > \theta_0$ will reject H_0 if $\sum_{i=1}^{m} Y_i > b$.

8.5 A random sample, $X_1, \ldots, X_n$, is drawn from a Pareto population with pdf

$$f(x|\theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I_{[\nu,\infty)}(x), \quad \theta > 0, \quad \nu > 0.$$

(a) Find the MLEs of θ and ν.
(b) Show that the LRT of

$$H_0: \theta = 1,\ \nu \text{ unknown}, \qquad \text{versus} \qquad H_1: \theta \neq 1,\ \nu \text{ unknown},$$

has critical region of the form $\{\mathbf{x}: T(\mathbf{x}) \leq c_1 \text{ or } T(\mathbf{x}) \geq c_2\}$, where $0 < c_1 < c_2$ and

$$T = \log \left[\frac{\prod_{i=1}^{n} X_i}{(\min_i X_i)^n} \right].$$

(c) Show that, under H_0, $2T$ has a chi squared distribution, and find the number of degrees of freedom. (*Hint:* Obtain the joint distribution of the $n-1$ nontrivial terms $X_i/(\min_i X_i)$ conditional on $\min_i X_i$. Put these $n-1$ terms together, and notice that the distribution of T given $\min_i X_i$ does not depend on $\min_i X_i$, so it is the unconditional distribution of T.)

8.7 We have already seen the usefulness of the LRT in dealing with problems with nuisance parameters. We now look at some other nuisance parameter problems.

(a) Find the LRT of

$$H_0: \theta \leq 0 \qquad \text{versus} \qquad H_1: \theta > 0$$

based on a sample $X_1, \ldots, X_n$ from a population with probability density function $f(x|\theta, \lambda) = \frac{1}{\lambda} e^{-(x-\theta)/\lambda} I_{[\theta,\infty)}(x)$, where both θ and λ are unknown.

(b) We have previously seen that the exponential pdf is a special case of a gamma pdf. Generalizing in another way, the exponential pdf can be considered as a special case of the Weibull(γ, β). The Weibull pdf, which reduces to the exponential if $\gamma = 1$, is very important in modeling reliability of systems. Suppose that $X_1, \ldots, X_n$ is a random sample from a Weibull population with both γ and β unknown. Find the LRT of $H_0: \gamma = 1$ versus $H_1: \gamma \neq 1$.

8.9 Stefanski (1996) establishes the arithmetic-geometric-harmonic mean inequality (see Example 4.7.8 and Miscellanea 4.9.2) using a proof based on likelihood ratio tests. Suppose that $Y_1, \ldots, Y_n$ are independent with pdfs $\lambda_i e^{-\lambda_i y_i}$, and we want to test H_0: $\lambda_1 = \cdots = \lambda_n$ vs. H_1: λ_i are not all equal.

(a) Show that the LRT statistic is given by $(\bar{Y})^{-n}/(\prod_i Y_i)^{-1}$ and hence deduce the arithmetic-geometric mean inequality.

(b) Make the transformation $X_i = 1/Y_i$, and show that the LRT statistic based on $X_1, \ldots, X_n$ is given by $[n/\sum_i(1/X_i)]^n/\prod_i X_i$ and hence deduce the geometric-harmonic mean inequality.

8.13 Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0: \theta = 0$ versus $H_1: \theta > 0$, we have two competing tests:

$$\phi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > .95,$$

$$\phi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

(a) Find the value of C so that ϕ_2 has the same size as ϕ_1.

(b) Calculate the power function of each test. Draw a well-labeled graph of each power function.

(c) Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

(d) Show how to get a test that has the same size but is more powerful than ϕ_2.

8.15 Show that for a random sample $X_1, \ldots, X_n$ from a n$(0, \sigma^2)$ population, the most powerful test of $H_0: \sigma = \sigma_0$ versus $H_1: \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi(\Sigma X_i^2) = \begin{cases} 1 & \text{if } \Sigma X_i^2 > c \\ 0 & \text{if } \Sigma X_i^2 \leq c. \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

8.17 Suppose that $X_1, \ldots, X_n$ are iid with a beta$(\mu, 1)$ pdf and $Y_1, \ldots, Y_m$ are iid with a beta$(\theta, 1)$ pdf. Also assume that the Xs are independent of the Ys.

(a) Find an LRT of $H_0: \theta = \mu$ versus $H_1: \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\Sigma \log X_i}{\Sigma \log X_i + \Sigma \log Y_i}.$$

(c) Find the distribution of T when H_0 is true, and then show how to get a test of size $\alpha = .10$.

8.19 The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0: \theta = 1$ versus $H_1: \theta = 2$ needs to be constructed. Find the UMP level $\alpha = .10$ test and compute the Type II Error probability.

8.23 Suppose X is one observation from a population with beta$(\theta, 1)$ pdf.

(a) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and sketch the power function of the test that rejects H_0 if $X > \frac{1}{2}$.
(b) Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.
(c) Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it. If not, prove so.

8.25 Show that each of the following families has an MLR.

(a) n(θ, σ^2) family with σ^2 known
(b) Poisson(θ) family
(c) binomial(n, θ) family with n known

8.27 Suppose $g(t|\theta) = h(t)c(\theta)e^{w(\theta)t}$ is a one-parameter exponential family for the random variable T. Show that this family has an MLR if $w(\theta)$ is an increasing function of θ. Give three examples of such a family.

8.29 Let X be one observation from a Cauchy(θ) distribution.

(a) Show that this family does not have an MLR.
(b) Show that the test

$$\phi(x) = \begin{cases} 1 & \text{if } 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is most powerful of its size for testing $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. Calculate the Type I and Type II Error probabilities.
(c) Prove or disprove: The test in part (b) is UMP for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$. What can be said about UMP tests in general for the Cauchy location family?