7.1 One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$. Find the MLE of θ.

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	$\frac{1}{3}$	$\frac{1}{4}$	0
1	$\frac{1}{3}$	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
4	$\frac{1}{6}$	0	$\frac{1}{4}$

7.3 Given a random sample $X_1, \ldots, X_n$ from a population with pdf $f(x|\theta)$, show that maximizing the likelihood function, $L(\theta|\mathbf{x})$, as a function of θ is equivalent to maximizing $\log L(\theta|\mathbf{x})$.

7.5 Consider estimating the binomial parameter k as in Example 7.2.9.

(a) Prove the assertion that the integer $\hat{k}$ that satisfies the inequalities and is the MLE is the largest integer less than or equal to $1/\hat{z}$.

(b) Let $p = \frac{1}{2}$, $n = 4$, and $X_1 = 0$, $X_2 = 20$, $X_3 = 1$, and $X_4 = 19$. What is $\hat{k}$?

7.7 Let $X_1, \ldots, X_n$ be iid with one of two pdfs. If $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

while if $\theta = 1$, then

$$f(x|\theta) = \begin{cases} 1/(2\sqrt{x}) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the MLE of θ.

7.9 Let $X_1, \dots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0.$$

Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. Which one should be preferred and why?

7.11 Let $X_1, \dots, X_n$ be iid with pdf

$$f(x|\theta) = \theta x^{\theta - 1}, \quad 0 \le x \le 1, \quad 0 < \theta < \infty.$$

(a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.
(b) Find the method of moments estimator of θ.

7.13 Let $X_1, \dots, X_n$ be a sample from a population with double exponential pdf

$$f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Find the MLE of θ. (*Hint:* Consider the case of even n separate from that of odd n, and find the MLE in terms of the order statistics. A complete treatment of this problem is given in Norton 1984.)

7.15 Let $X_1, X_2, \dots, X_n$ be a sample from the *inverse Gaussian* pdf,

$$f(x|\mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\lambda(x-\mu)^2/(2\mu^2 x)\right\}, \quad x > 0.$$

(a) Show that the MLEs of μ and λ are

$$\hat{\mu}_n = \bar{X} \quad \text{and} \quad \hat{\lambda}_n = \frac{n}{\sum_i \frac{1}{X_i} - \frac{1}{\bar{X}}}.$$

(b) Tweedie (1957) showed that $\hat{\mu}_n$ and $\hat{\lambda}_n$ are independent, $\hat{\mu}_n$ having an inverse Gaussian distribution with parameters μ and $n\lambda$, and $n\lambda/\hat{\lambda}_n$ having a χ^2_{n-1} distribution. Schwarz and Samanta (1991) give a proof of these facts using an induction argument.

(i) Show that $\hat{\mu}_2$ has an inverse Gaussian distribution with parameters μ and 2λ, $2\lambda/\hat{\lambda}_2$ has a χ^2_1 distribution, and they are independent.

(ii) Assume the result is true for $n = k$ and that we get a new, independent observation x. Establish the induction step used by Schwarz and Samanta (1991), and transform the pdf $f(x, \hat{\mu}_k, \hat{\lambda}_k)$ to $f(x, \hat{\mu}_{k+1}, \hat{\lambda}_{k+1})$. Show that this density factors in the appropriate way and that the result of Tweedie follows.

7.17 The Borel Paradox (Miscellanea 4.9.3) can also arise in inference problems. Suppose that X_1 and X_2 are iid exponential(θ) random variables.

(a) If we observe only X_2, show that the MLE of θ is $\hat{\theta} = X_2$.

(b) Suppose that we instead observe only $Z = (X_2 - 1)/X_1$. Find the joint distribution of (X_1, Z), and integrate out X_1 to get the likelihood function.

(c) Suppose that $X_2 = 1$. Compare the MLEs for θ from parts (a) and (b).

(d) Bayesian analysis is not immune to the Borel Paradox. If $\pi(\theta)$ is a prior density for θ, show that the posterior distributions, at $X_2 = 1$, are different in parts (a) and (b).

(*Communicated by L. Mark Berliner, Ohio State University.*)

7.19 Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathrm{n}(0, \sigma^2)$, σ^2 unknown.

(a) Find a two-dimensional sufficient statistic for (β, σ^2).

(b) Find the MLE of β, and show that it is an unbiased estimator of β.

(c) Find the distribution of the MLE of β.

7.21 Again, let $Y_1, \ldots, Y_n$ be as defined in Exercise 7.19.

(a) Show that $\left[\sum(Y_i/x_i)\right]/n$ is also an unbiased estimator of β.

(b) Calculate the exact variance of $\left[\sum(Y_i/x_i)\right]/n$ and compare it to the variances of the estimators in the previous two exercises.

7.23 If S^2 is the sample variance based on a sample of size n from a normal population, we know that $(n-1)S^2/\sigma^2$ has a χ^2_{n-1} distribution. The conjugate prior for σ^2 is the *inverted gamma* pdf, $\mathrm{IG}(\alpha, \beta)$, given by

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\frac{1}{(\sigma^2)^{\alpha+1}}e^{-1/(\beta\sigma^2)}, \quad 0 < \sigma^2 < \infty,$$

where α and β are positive constants. Show that the posterior distribution of σ^2 is $\mathrm{IG}(\alpha + \frac{n-1}{2}, [\frac{(n-1)S^2}{2} + \frac{1}{\beta}]^{-1})$. Find the mean of this distribution, the Bayes estimator of σ^2.

7.25 We examine a generalization of the hierarchical (Bayes) model considered in Example 7.2.16 and Exercise 7.22. Suppose that we observe $X_1, \ldots, X_n$, where

$$\begin{aligned} X_i|\theta_i &\sim \mathrm{n}(\theta_i, \sigma^2), & i = 1, \ldots, n, &\quad \text{independent,} \\ \theta_i &\sim \mathrm{n}(\mu, \tau^2), & i = 1, \ldots, n, &\quad \text{independent.} \end{aligned}$$

(a) Show that the marginal distribution of X_i is $\mathrm{n}(\mu, \sigma^2 + \tau^2)$ and that, marginally, $X_1, \ldots, X_n$ are iid. (*Empirical Bayes analysis* would use the marginal distribution of the X_is to estimate the prior parameters μ and τ^2. See Miscellanea 7.5.6.)

(b) Show, in general, that if

$$\begin{aligned} X_i|\theta_i &\sim f(x|\theta_i), & i = 1, \ldots, n, &\quad \text{independent,} \\ \theta_i &\sim \pi(\theta|\tau), & i = 1, \ldots, n, &\quad \text{independent,} \end{aligned}$$

then marginally, $X_1, \ldots, X_n$ are iid.