2. This problem considers a *noninteracting* ($a = 0$ in Eq. 1) pair of indistinguishable oscillators.

(i) To respect the constraint of indistinguishability, the basis functions $\phi_A(x, y)$ must themselves be antisymmetric.

The basis states $|x_A, y_A\rangle$ we have used in the past, representing the function

$$\exp[-\alpha(x - x_A)^2 - \alpha(y - y_A)^2]$$

are not antisymmetric. Show that this is true.

For antisymmetry: $\hat{P}\phi_A(x,y) = -\phi_A(x,y)$.

$\Rightarrow \exp(-\alpha(y-x_A)^2 - \alpha(x-y_A)^2)$ ← always positive

$\neq -\exp(-\alpha(x-x_A)^2 - \alpha(y-y_A)^2)$ ← " negative.

(ii) Acceptable basis functions for the antisymmetric system cannot distinguish between x and y, and therefore cannot be centered on a specific point (x_A, y_A) in the xy-plane. Instead, a function centered at the point (u_A, v_A) could be paired with another function centered at the exchanged point (v_A, u_A):

$$\phi_A(x, y) = \exp[-\alpha(x - u_A)^2 - \alpha(y - v_A)^2] - \exp[-\alpha(x - v_A)^2 - \alpha(y - u_A)^2] \quad (4)$$

(We have switched the notation from (x_A, y_A) to (u_A, v_A) because these parameters can no longer be associated exclusively with x or y.) Show that the basis function in Eq. 4 is in fact antisymmetric, and make a contour plot of $\phi_A(x, y)$ in the xy-plane for the case $u_A = 2$ and $v_A = 1$.

$\hat{P}\phi_A(x,y) = \exp(-\alpha(y-u_A)^2 - \alpha(x-v_A)^2) - \exp(-\alpha(y-v_A)^2 - \alpha(x-u_A)^2)$

$= -\phi_A(x,y)$

$= -(\exp(-\alpha(x-u_A)^2 - \alpha(y-v_A)^2) - \exp(-\alpha(x-v_A)^2 - \alpha(y-u_A)^2))$

$= \exp(-\alpha(x-v_A)^2 - \alpha(y-u_A)^2) - \exp(-\alpha(x-v_A)^2 - \alpha(y-v_A)^2)$.

(iii) Let us denote the basis function of Eq. 4 as

$$|\phi_A\rangle = |u_A, v_A\rangle - |v_A, u_A\rangle$$

Show that the overlap between two such functions is

$$\begin{aligned} S_{AB} &= \langle\phi_A|\phi_B\rangle \\ &= 2s(u_A, u_B)s(v_A, v_B) - 2s(v_A, u_B)s(u_A, v_B), \end{aligned}$$

where

$$s(x_A, x_B) = \int dx\, e^{-\alpha(x-x_A)^2} e^{-\alpha(x-x_B)^2} = \sqrt{\frac{\pi}{2\alpha}}\ \exp\left[-\frac{\alpha}{2}(x_A - x_B)^2\right].$$

$$\langle\phi_A|\phi_B\rangle$$

$$= \big(\langle u_A, v_A| - \langle v_A, u_A|\big)\big(|u_B, v_B\rangle - |v_B, u_B\rangle\big).$$

$$= \langle u_A, v_A|u_B, v_B\rangle - \langle v_A, u_A|u_B, v_B\rangle - \langle u_A, v_A|v_B, u_B\rangle + \langle v_A, u_A|v_B, u_B\rangle.$$

Swap u_A, v_A for third & last term, $\because e^-$s are indistinguishable,

$$= \langle u_A, v_A|u_B, v_B\rangle + \langle u_A, v_A|u_B, v_B\rangle - \big(\langle v_A, u_A|u_B, v_B\rangle + \langle v_A, u_A|u_B, v_B\rangle\big).$$

$$\langle u_A, v_A|u_B, v_B\rangle = \iint dx\,dy\ \exp(-\alpha(x-u_A)^2 - \alpha(y-v_A)^2)\ \exp(-\alpha(x-u_B)^2 - \alpha(y-v_B)^2)$$

$$\int dx\, \exp(-\alpha(x-u_A)^2 - \alpha(x-v_A)^2)\int dy\, \exp(-\alpha(y-u_B)^2 - \alpha(y-v_B)^2)$$

$$= s(u_A, u_B)\, s(v_A, v_B)$$

$$\Rightarrow \langle\phi_A|\phi_B\rangle.$$

$$= 2s(u_A, u_B)\, s(v_A, v_B) - 2s(v_A, u_B)\, s(u_A, v_B)$$

(iv) Similarly, show that the single-particle matrix elements $\langle\phi_A|h(x)|\phi_B\rangle$ and $\langle\phi_A|h(y)|\phi_B\rangle$ evaluate to

$$\begin{aligned} h_{AB} &= \langle\phi_A|h(x)|\phi_B\rangle = \langle\phi_A|h(y)|\phi_B\rangle \\ &= f(u_A,u_B)s(v_A,v_B) + f(v_A,v_B)s(u_A,u_B) \\ &\quad -f(u_A,v_B)s(v_A,u_B) - f(v_A,u_B)s(u_A,v_B), \end{aligned}$$

where

$$\begin{aligned} f(x_A,x_B) &= \int dx\, e^{-\alpha(x-x_A)^2} h(x) e^{-\alpha(x-x_B)^2} \\ &= \frac{s(x_A,x_B)}{2}\left[\alpha + \frac{1}{4\alpha} + \frac{1}{4}(x_A+x_B)^2 - \alpha^2(x_A-x_B)^2\right] \end{aligned}$$

$$\begin{aligned} \langle\phi_A|h(x)|\phi_B\rangle &= \big(\langle u_A,v_A| - \langle v_A,u_A|\big)h(x)\big(|u_B,v_B\rangle - |v_B,u_B\rangle\big) \\ &= \langle u_A,v_A|h|u_B,v_B\rangle - \langle v_A,u_A|h|u_B,v_B\rangle - \langle u_A,v_A|h|v_B,u_B\rangle + \langle v_A,u_A|h|v_B,u_B\rangle \\ &= \big(\langle u_A,v_A|h|u_B,v_B\rangle + \langle v_A,u_A|h|v_B,u_B\rangle\big) - \big(\langle u_A,v_A|h|v_B,u_B\rangle + \langle v_A,u_A|h|u_B,v_B\rangle\big) \end{aligned}$$

$$\begin{aligned} \langle u_A,v_A|h(x)|u_B,v_B\rangle &= \iint dx\,dy\, \exp(-\alpha(x-u_A)^2 - \alpha(y-v_A)^2)\, h(x)\, \exp(-\alpha(x-u_B)^2 - \alpha(y-v_B)^2) \\ &= \iint dx\,dy\, \exp(-\alpha(x-u_A)^2)\, h(x)\, \exp(-\alpha(x-u_B)^2) \cdot \\ &\qquad \exp(-\alpha(y-v_A)^2 - \alpha(y-v_B)^2) \\ &= \int dx\, \exp(-\alpha(x-u_A)^2)\, h(x)\, \exp(-\alpha(x-u_B)^2) \int dy\, \exp(-\alpha(y-v_A)^2 - \alpha(y-v_B)^2) \\ &= f(u_A,u_B)\, s(v_A,v_B). \end{aligned}$$

$$\begin{aligned} \therefore\ \langle\phi_A|h(x)|\phi_B\rangle &= f(u_A,u_B)S(v_A,v_B) + f(v_A,v_B)S(u_A,u_B) \\ &\quad - f(u_A,v_B)S(v_A,u_B) - f(v_A,u_B)S(u_A,v_B) \end{aligned}$$

3. In this problem we will add interactions ($a > 0$ in Eq. 1) to the antisymmetric oscillators.

(i) For the interacting system we need matrix elements of the operator x^4y^4. Show that

$$\begin{aligned} G_{AB} &= \langle\phi_A|x^4y^4|\phi_B\rangle \\ &= 2g(u_A,u_B)g(v_A,v_B) - 2g(v_A,u_B)g(u_A,v_B), \end{aligned}$$

where

$$\begin{aligned} g(x_A,x_B) &= \int_{-\infty}^{\infty} dx\, e^{-\alpha(x-x_A)^2}\, x^4\, e^{-\alpha(x-x_B)^2} \\ &= s(x_A,x_B)\left[\frac{3}{16\alpha^2} + \frac{3}{8\alpha}(x_A+x_B)^2 + \frac{1}{16}(x_A+x_B)^4\right]. \end{aligned}$$

$$\langle\phi_A|x^4y^4|\phi_B\rangle$$

$$= \left(\langle u_A,v_A| - \langle v_A,u_A|\right) x^4y^4 \left(|u_B,v_B\rangle - |v_B,u_B\rangle\right)$$

$$= \langle u_A,v_A|x^4y^4|u_B,v_B\rangle - \langle v_A,u_A|x^4y^4|u_B,v_B\rangle - \langle u_A,v_A|x^4y^4|u_B,v_B\rangle + \langle v_A,u_A|x^4y^4|v_B,u_B\rangle$$

swap the symbols for the last 2 terms since e's are indistinguishable

$$= \left(\langle u_A,v_A|x^4y^4|u_B,v_B\rangle + \langle u_A,v_A|x^4y^4|u_B,v_B\rangle\right) - \left(\langle v_A,u_A|x^4y^4|u_B,v_B\rangle + \langle v_A,u_A|x^4y^4|u_B,v_B\rangle\right)$$

$$= 2\langle u_A,v_A|x^4y^4|u_B,v_B\rangle - 2\langle v_A,u_A|x^4y^4|u_B,v_B\rangle$$

$$\langle u_A,v_A|x^4y^4|u_B,v_B\rangle = \iint dx\,dy\, \exp(-\alpha(x-u_A)^2-\alpha(y-v_A)^2)\cdot x^4y^4 \exp(-\alpha(x-u_B)^2-\alpha(y-v_B)^2)$$

$$= \int dx\, \exp(-\alpha(x-u_A)^2)\, x^4 \exp(-\alpha(x-u_B)^2) \int dy\, \exp(-\alpha(y-v_A)^2)\, y^4 \exp(-\alpha(y-v_B)^2)$$

$$= g(u_A,u_B)\, g(v_A,v_B)$$

$$\Rightarrow \langle\phi_A|x^4y^4|\phi_B\rangle = 2g(u_A,u_B)g(v_A,v_B) - 2g(v_A,u_B)g(u_A,v_B).$$