Fundamentals of Operations Research (Part 1)

LP (Linear Programming) Introduction:

A linear programming problem is a problem of minimizing or maximizing a linear function in the presence of linear constraints of the inequality and/or the equality type.

Standard and Canonical Formats:

Standard Form -

A linear program in which all restrictions are equalities and all variables are non-negative. The simplex method is designed to be applied only after the problem is put in the standard form.

Canonical Form -

The canonical form is also useful especially in exploiting duality relationships. A minimization problem is in canonical form if all variables are nonnegative and all the constraints are of the $\geq$ type. A maximization problem is in canonical format if all the variables are non-negative and all the constraints are of the $\leq$ type.

Formulation of Linear Programming Model:

1. Identify the decision variables.
2. Identify the problem constraints and express the constraints as a series of linear equations / inequalities.
3. Identify the objective functions as a linear equation, and state whether the objective is maximization or minimization.

Given below are few examples:

- Consider a small manufacturer making n products A_1; A_2; …. ; An. Each of the n products require some part of m resources R_1; R_2; …. ; R_m. Each unit of product A_j, j = 1; …. ; n requires a_{ij} units of R_i; i = 1;….;m. The manufacturer has b_i units of resource R_i; i = 1; 2; …; m available. He makes a profit p_j ; j = 1; 2;…. ; n per unit of product A_j ; j = 1; 2; …..; n. The manufacturer needs to decide on number of units of each product A_j, j = 1; 2; ….; n. He needs to manufacture in order to maximize his profit.
    - Identify the decision variables

      x_j = Number of units of product A_j manufactured, j = 1; 2; … ; n
    - Identify the problem constraints

      $$\sum_{j=1}^{n} a_{ij} \leq b_i \qquad i = 1, 2, \ldots, m$$

      $x_j \geq 0$, j = 1, 2, …., n

      and express the constraints as a series of linear equations/inequalities.

- Identify the objective functions as a linear equation, and state whether the objective is maximization or minimization.
  Maximize Profit
  $$\text{maximize} \sum_{j=1}^{n} c_j x_j$$

❖ Consider a company making a single product.
The estimated demand for the product for the next four months are 1000, 800, 1200 and 900 respectively. Company has a regular time capacity of 800 per month and overtime capacity of 200 per month. Cost of regular time production is Rs. 20 per unit. Cost of overtime production is Rs. 25 per unit. Inventory holding cost is Rs. 3 per unit per month. Demand has to be met every month.
Formulate as a linear programming problem

- Identify the decision variables.
  x_j = Quantity produced using regular time production in month j
  y_j = Quantity produced using overtime production in month j.
  i_j = Quantity carried at end of month j to next month.
- Identify the problem constraints and express the constraints as a series of linear equations/inequalities.
  $x_1 + y_1 = 1000 + i_1$
  $i_1 + x_2 + y_2 = 800 + i_2$
  $i_2 + x_3 + y_3 = 1200 + i_3$
  $i_3 + x_4 + y_4 = 900$
  $x_j \leq 800 \quad j = 1, 2, 3, 4$
  $y_j \leq 200 \quad j = 1, 2, 3, 4$
  $x_j , y_j , i_j \geq 0$

- Identify the objective functions as a linear equation, and state whether the objective is maximization or minimization.
  $$\text{minimize } 20 \sum_{j=1}^{4} x_j + 25 \sum_{j=1}^{4} y_j + 3 \sum_{j=1}^{4} i_j$$

❖ Consider a restaurant that is open seven days a week. Based on past experience, the number of workers needed on a particular day is given as follows:

Day	M	T	W	Th	Fr	Sa	Su
Number	14	13	15	16	19	18	11

Every worker works five consecutive days, and then takes two days off, repeating this pattern indefinitely. How can we minimize the number of workers that staff the restaurant?

- Decision Variable: x_i: The number of workers who begin their five day shift on day i.
- Objective function: Minimize number of workers which is
  Minimize $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
- Constraints: number of workers on Monday is given by $x_1+x_4+x_5+x_7$

$x_1 + x_4 + x_5 + x_7 \geq 14$
$x_2 + x_5 + x_6 + x_1 \geq 13$
$x_3 + x_6 + x_7 + x_2 \geq 15$
$x_4 + x_7 + x_1 + x_3 \geq 16$
$x_5 + x_1 + x_2 + x_4 \geq 19$
$x_6 + x_2 + x_3 + x_5 \geq 18$
$x_7 + x_3 + x_4 + x_6 \geq 11$
$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$

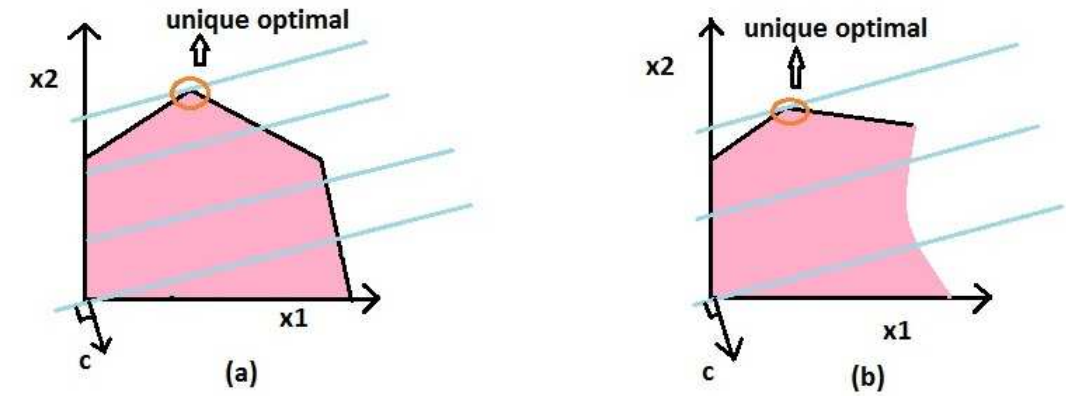
LP Geometry in two-dimensions:

Geometric procedure for solving a linear programming problem is only suitable for very small problems. It provides a great deal of insight into the linear programming problem.
Consider the following problem
Minimize c'x
Subject to $Ax \geq b$
$x \geq 0$

- Note that the feasible region consists of all vectors x satisfying $Ax \geq b$ and $x \geq 0$.
- Among all such points we wish to find a point with minimal cx value.
    - Note that points with the same objective z satisfy the equation cx = z, that is, $\sum_{j=1}^{n} c_j x_j = z$
    - Since z is to be minimized, then the plane (line in a two-dimensional space) $\sum_{j=1}^{n} c_j x_j = z$ must be moved parallel to itself in the direction that minimizes the objective most.
- This direction is -c, and hence the plane is moved in the direction -c as much as possible.(For a maximization problem you move it in the direction c).
- Note that as the optimal point $x^* = (x^*_1, x^*_2)'$ is reached, the line $c_1x_1 + c_2x_2 = z^*$, where $z^*= c_1x^*_1 + c_2x^*_2$, cannot be moved farther in the direction $c = (-c_1,-c_2)$ because this will lead to only points outside the feasible region.
- We therefore conclude that x* is indeed the optimal solution.
- If a linear program has a finite optimal solution, then it has an optimal corner (or extreme) solution.

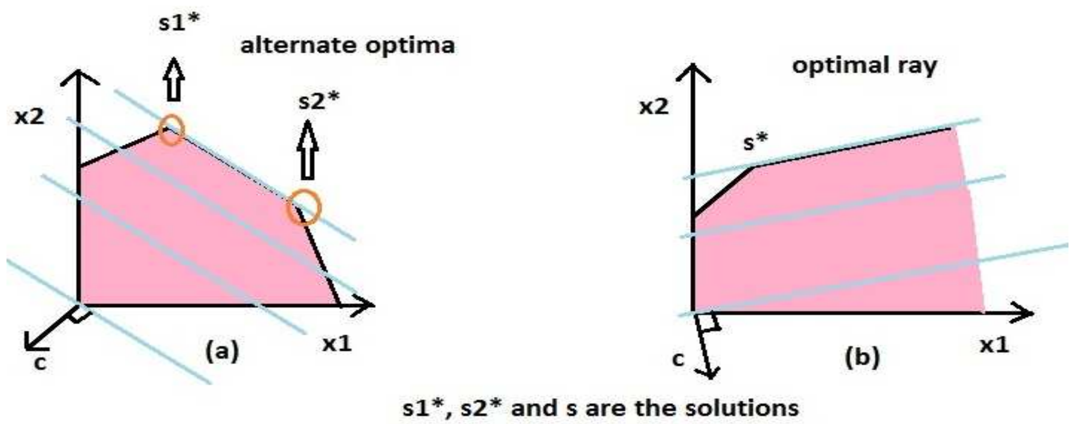
All the possible cases that may arise for a minimization problem are as follows:

Unique finite optimal solution –



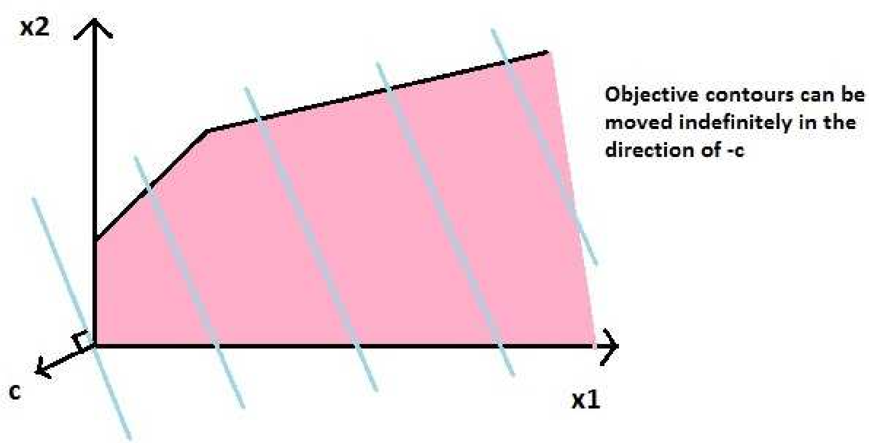
(a)= bounded region, (b)= unbounded region.

Alternative finite optimal solution –

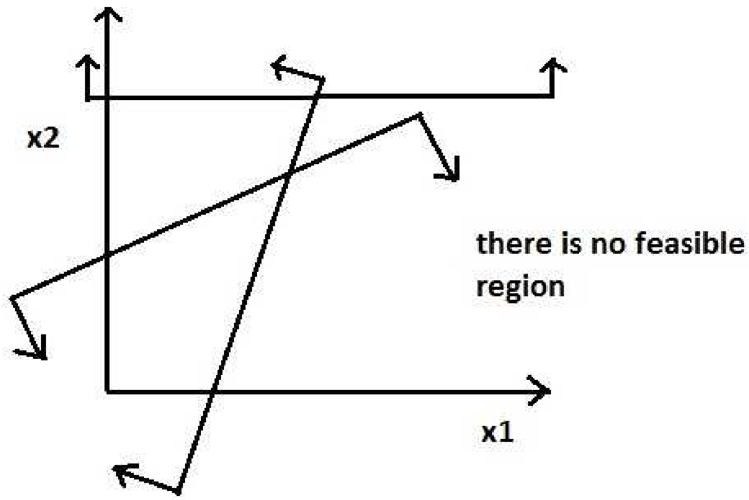


(a)= bounded region, (b)= unbounded region.

Unbounded optimal solution –



Empty feasible region –



Representation theorem:

Let $S = \{x : Ax \le b, x \ge 0\}$ be a nonempty polyhedral set. Any point in S can be represented as a convex combination of its extreme points plus a non-negative combination of its extreme directions. Let $x_1, x_2, \ldots x_k$ denote the extreme points and $d_1, d_2, \ldots\ldots, d_l$ denote the extreme directions of the set S. Then for any $x \in S$ there exists $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\mu_1, \mu_2, \ldots, \mu_l$ such that
$x = (\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_k x_k) + (\mu_1 d_1 + \mu_2 d_2 + \ldots + \mu_l d_l)$
where $\sum_{j=1}^{k} \lambda j = 1, \lambda_j \ge 0, j = 1,2,\ldots. k; \mu_j \ge 0, j = 1,2,\ldots,l.$

Basic Feasible Solutions:

In this section, the notion of basic feasible solutions will be introduced and its correspondence to extreme points of a polyhedra will be shown.
Consider the linear program with the constraints: $Ax = b$; $x \ge 0$; Where A is an m X n matrix and b is an m vector. Let rank(A) = m (generally, rank of the matrix is the number if rows of that matrix). Let A = [B,N]; where B is m X m invertible matrix, and N is m X (n - m) matrix. Then the solution

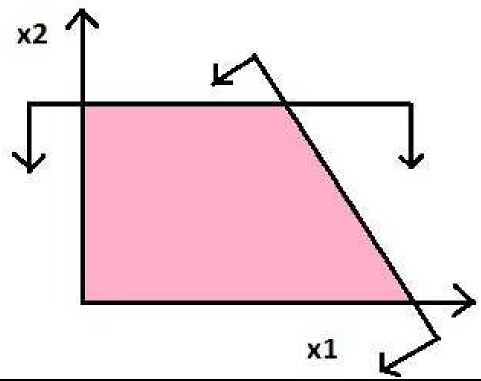
$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$$

to the equations $Ax = b$; $x \ge 0$; where $x_B = B^{-1}b$ and $x_N = 0$ is called a basic solution of the system.
If $x_B \ge 0$, then x is called a basic feasible solution (BFS) of the system.
A basic feasible solution x is called a degenerate basic feasible solution if at least one component of x_B is zero.
Note:

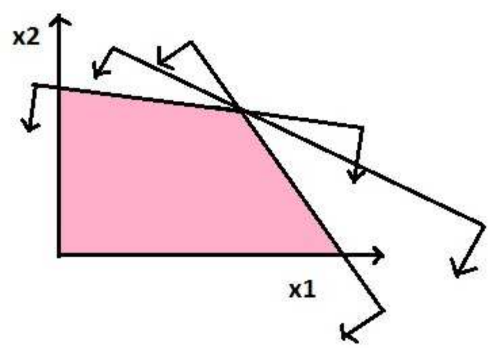
- B is called the basis matrix.
- Variables in x_B are called basic variables.
- Variables in x_N are called non basic variables.

- In general the possible number of basic feasible solutions is bounded by the number of ways of extracting m columns out of n columns and it is bounded by $\binom{n}{m}$ i.e., there are at most $\binom{n}{m}$ BFS. Hence there are finite number of BFS.

Example of Basic Feasible Solutions (BFS):



Example of Degenerate Basic Feasible Solutions:



Results regarding BFS:

- The collection of extreme points corresponds to the collection of basic feasible solutions, and both are non-empty if the feasible region is non-empty.
- Assume the feasible region S is non-empty. A finite optimal solution exists if and only if $cd_j \geq 0$ for $j = 1, 2, .., l$ where $d_1, .., d_l$ are the extreme directions of the feasible region S.
- If an optimal solution exists, then an optimal extreme point exists.
- For every extreme point there corresponds a basis and conversely for every basis there corresponds a unique basis. If an extreme point has more than one basis representing it, it is a degenerate extreme point.

The Simplex Method:

Simplex method is used to solve the linear programming problem. The simplex method is a procedure that moves from an extreme point (basic feasible solution) to another extreme point with a better (improved) objective function value.

Key to the Simplex Method:

Consider the following LP problem –
Min cx
subject to Ax = b
$x \geq 0$

where A is an m X n matrix of rank m. Suppose $x^* = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ is a basic feasible solution whose objective function value $z^* = c.\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} = (C_B, C_N)\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix} = C_B B^{-1} b$.

Let x_B and x_N denote the set of basic and non-basic variables for the given basis. We know

$$
\begin{aligned}
x_B &= B^{-1}b - B^{-1}Nx_N \\
&= B^{-1}b - \sum_{j \in R} B^{-1}a_j x_j \\
&= \bar{b} - \sum_{j \in R} y_j x_j
\end{aligned}
$$

where R is the current set of indices of the non-basic variables, N is the matrix of columns of the non- basic variables and $y_j = B^{-1}a_j$. Let z denote the objective function value, we have

$$
\begin{aligned}
Z &= CX \\
&= C_B X_B + C_N X_N \\
&= C_B . \left(\bar{b} - \sum_{j \in R} y_j X_j\right) + C_N X_N \\
&= C_B \bar{b} - \sum_{j \in R} C_B y_j X_j + \sum_{j \in R} C_j X_j
\end{aligned}
$$

Define for each non basic variable, $z_j = c_B y_j$. Hence we get

$$Z = C_B \bar{b} - \sum_{j \in R} (Z_j - C_j).X_j = Z^* - \sum_{j \in R} (Z_j - C_j).X_j$$

The linear programming problem can be rewritten in the non-basic variable space as

$$
\begin{aligned}
\text{minimize } \; & Z = Z^* - \sum_{j \in R} (Z_j - C_j).X_j \\
\text{subject to } & \sum_{j \in R} y_j + X_B = \bar{b} \\
& X_j \geq 0 \; ; j \in R \\
& X_B \geq 0
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\text{minimize } \; & Z = Z^* - \sum_{j \in R} (Z_j - C_j).X_j \\
\text{subject to } & \sum_{j \in R} y_j x_j \leq \bar{b} \\
& X_j \geq 0 \; ; j \in R
\end{aligned}
$$

Key Result: If $(z_j - c_j) \leq 0$ for all $j \in R$, then the current basic feasible solution is optimal.

Algebra of simplex method:

The key result says that if $(z_j - c_j) \leq 0$ for all $j \in R$, then the current BFS $x_B = \bar{b}$; $x_j = 0$; $j \in R$ is optimal. Otherwise holding (p - 1) non-basic variables fixed at zero, the simplex method increases the remaining variable say x_k, where x_k is the variable corresponding to the most positive $z_j - c_j$ i.e., $z_k - c_k = \max \{z_j - c_j , j \in R \}$

Now fixing $x_j = 0$ for $j \in R - k$ we obtain equation (1) is equal to

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_r} \\ \vdots \\ x_{B_m} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_r \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ y_{2k} \\ \vdots \\ y_{rk} \\ \vdots \\ y_{mk} \end{bmatrix} x_k$$

- If $y_{ik} \leq 0$, then x_{Bi} increases as x_k increases and so x_{Bi} continues to be non-negative.
- If $y_{ik} > 0$ then x_{Bi} decreases as x_k increases. To maintain feasibility, x_k is increased until the first point at which a basic variable x_{Br} drops to zero. In other words, increase x_k until

  $$x_k = \frac{\bar{b}_r}{y_{rk}} \equiv Min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

  The above test to determine the basic variable that drops out of the basis is called **the Minimum ratio test.**

Note: After one iteration of the simplex method, the corresponding columns of the basis in A , a_{B1} , a_{B2} ,….. , a_{Br-1} ; a_k; $a_{Br+1}$are linearly independent since $y_{rk} \neq 0$.

Termination:

In the previous section we discussed a procedure that moves from one basis to an adjacent basis, by introducing a variable in the basis and removing another variable from the basis. The criteria for entering and leaving are summarized below:

1. Entering: x_k may enter if $z_k - c_k > 0$.
2. Leaving: x_{Br} may leave if

$$\frac{\bar{b}_r}{y_{rk}} = Min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

Unique and Alternative optimal solutions:

- If $z_j - c_j < 0$ for all $j \in R$, then the current optimal solution is unique.
- If $z_k - c_k = 0$ for at least one non basic variable x_k, then we get infinite number of alternative optimal solutions.

Simplex algorithm:

Minimize $c^T.x$
Subject to $Ax = b$
$X \geq 0$

Initial
Step: Start with basis B.

Main
Step 1: Solve $Bx_B = b$

Solution $x_B = B^{-1}b = \bar{b}$

Initial BFS is given below

$$X = \begin{pmatrix} xB = B^{-1}b \\ xN = 0 \end{pmatrix}$$

Step 2: Define -

R = set of indices of non-basic variables.

$y_j = B^{-1}a_j \ \forall\ j \in R$

$z_j = c_B{}^T B^{-1} a_j$

Compute –

$z_j - c_j \ \forall\ j \in R$

If $z_j - c_j \leq 0 \ \forall\ j \in R$, then current solution is optimal and the simplex method can be stopped.

Else, choose –

$z_k - c_k$: max. of $z_j - c_j \geq 0 \ \forall\ j \in R$, x_k enters the basis.

Note that: if all $y_{ik} \leq 0$, i=1, ….., m, then the solution is unbounded.

Step 3: choose index r of variable to leave by (minimum ratio test) –

$$\frac{\bar{b}_r}{y_{rk}} = Min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

So, x_{Br} leaves the basis and x_k enters the basis.

Repeat step 1.

The Simplex Method in Tableau Format:

The following equations were obtained in the section "**key to the simplex method**" earlier.

$$x_B = B^{-1}b - B^{-1}Nx_N \quad \text{.... (1)}$$

$$Z = C_BX_B + C_NX_N \quad \text{.... (2)}$$

Multiplying equation 1 with C_B and adding it to equation 2, we get –

$$Z + 0X_B + (C_BB^{-1}N - C_N) + X_N = C_BB^{-1}b$$

In the tableau the objective row is referred to as row 0 and remaining rows are rows 1 through m. The right hand side denotes the values of the basic variables including the objective function. The basic variables are identified in the far left column. The tableau is given below:

	Z	X_B	X_N	RHS	
Z	1	0	$C_B B^{-1} N - C_N$	$C_B B^{-1} b$	Row 0
X_B	0	I	$B^{-1} N$	$B^{-1} b$	Rows 1 to m

Now, we operate the table and find the optimal solution similar to the earlier simplex algorithm. We first identify the basic variables and complete the initial tableau. Later, we perform pivoting which is described below.

In case of minimization problem, we identify the highest value of $z_j - c_j$ column (say $z_k - c_k$: max. of $z_j - c_j$), thus x_k enters the basis. Now, in that column, choose index r (from 1 to m) of variable to leave by (minimum ratio test) –

$$\frac{b_r}{y_{rk}} = Min_{1 \le i \le m} \left\{ \frac{b_i}{y_{ik}} : y_{ik} > 0 \right\}$$

Thereby, x_{Br} leaves the basis. Divide row r by y_{rk}. For i = 1,…,m $i \neq r$, update the i^{th} row by adding to it $-y_{ik}$ times the new r^{th} row. Update the row zero by adding to it $c_k - z_k$ times the new r^{th} row. In this way, we keep updating the table until we get optimal solution, that is all the values of $z_j - c_j \le 0$.

Problems:

1. Feed is manufactured for cattle, sheep, and chickens. This is done by mixing the following main ingredients: corn, limestone, soybeans, and fish meal. These ingredients contain the following nutrients: vitamins, protein, calcium, and crude fat. The contents of the nutrients in each kilogram of the ingredients are given in the table below –

Ingredient	**Vitamin**	**Protein**	**Calcium**	**Crude fat**
Corn	8	10	6	8
Limestone	6	5	10	6
Soyabean	10	12	6	6
Fish meal	4	8	6	9

It is required to produce 10, 6, 8 (metric) tons of cattle feed, sheep feed, and chicken feed. Amount of the ingredients available are namely, 6 tons of corn, 10 tons of limestone, 4 tons of soybeans, and 5 tons of fish meal. The price per kilogram of these ingredients is respectively 0.20, 0.12, 0.24 and 0.12. The minimal and maximal units of the various nutrients that are permitted is summarized for a kilogram of the cattle feed, the sheep feed, and the chicken feed.

	Vitamins		Proteins		Calcium		Crude fat	
Product	Min	Max	Min	Max	Min	Max	Min	Max
Cattle	6	∞	6	∞	7	∞	4	8
Sheep	6	∞	6	∞	6	∞	4	6
Chicken	4	6	6	∞	6	∞	4	6

Formulate into a linear programming model so that the total cost is minimized.

Soln:

Decision variables-

Let X_{ij} be the amount of ingredient i in 1kg of feed j

where, i = 1(corn), 2(limestone), 3(soyabeans), 4(fish meal)

j = 1(cattle), 2(sheep), 3(chicken)

Constraints-

$10000X_{11} + 6000X_{12} + 8000X_{13} \leq 6000$

$10000X_{21} + 6000X_{22} + 8000X_{23} \leq 10000$

$10000X_{31} + 6000X_{31} + 8000X_{33} \leq 4000$

$10000X_{41} + 6000X_{42} + 8000X_{43} \leq 5000$

Vitamins

$6 \leq 8X_{11} + 6X_{21} + 10X_{31} + 4X_{41}$

$6 \leq 8X_{12} + 6X_{22} + 10X_{32} + 4X_{42}$

$4 \leq 8X_{13} + 6X_{23} + 10X_{33} + 4X_{43} \leq 6$

Proteins

$6 \leq 10X_{11} + 5X_{21} + 12X_{31} + 8X_{41}$

$6 \leq 10X_{12} + 5X_{22} + 12X_{32} + 8X_{42}$

$6 \leq 10X_{13} + 5X_{23} + 12X_{33} + 8X_{43}$

Calcium

$7 \leq 6X_{11} + 10X_{21} + 6X_{31} + 6X_{41}$

$6 \leq 6X_{12} + 10X_{22} + 6X_{32} + 6X_{42}$

$6 \leq 6X_{13} + 10X_{23} + 6X_{33} + 6X_{43}$

Crude fat

$4 \leq 8X_{11} + 6X_{21} + 6X_{31} + 9X_{41} \leq 8$

$4 \leq 8X_{12} + 6X_{22} + 6X_{32} + 9X_{42} \leq 6$

$4 \leq 8X_{13} + 6X_{23} + 6X_{33} + 9X_{43} \leq 6$

Objective function-

min ($a\times0.2 + b\times0.12 + c\times0.24 + d\times0.12$)

where, $a = 10000X_{11} + 6000X_{12} + 8000X_{13}$

$b = 10000X_{21} + 6000X_{22} + 8000X_{23}$

$c = 10000X_{31} + 6000X_{31} + 8000X_{33}$

$d = 10000X_{41} + 6000X_{42} + 8000X_{43}$

2. The company operates three plants, namely A, B, and C. The capacities (in thousands of square feet) and unit manufacturing costs (in rupees per thousand square feet) of these plants are shown below:

Plant	Capacity	Manufacturing costs per unit
A	1000	380
B	1200	400
C	1400	360

The product is distributed within 3 marketing regions. Their requirements (in thousands of square feet) are given in the table below.

Region 1	Region 2	Region 3
900	1000	1500

The transportation costs (in rupees) per thousand square feet from plants to marketing regions are given in the table below.

Plant	Region 1	Region 2	Region 3
A	200	300	150
B	250	400	280
C	300	175	500

Formulate a linear programming problem to determine the quantities the company should transport from each plant to each marketing region in order to minimize the total manufacturing and transportation cost for the year.

Soln:

Decision variable-

X_{ij} = amount (in thousands square feet) to be transported from plant i to region j.

where, i = 1 (A), 2 (B), 3 (C)

j = 1, 2, 3.

Constraints-

$X_{11} + X_{12} + X_{13} \leq 1000$

$X_{21} + X_{22} + X_{23} \leq 1200$

$X_{31} + X_{32} + X_{33} \leq 1400$

$X_{11} + X_{21} + X_{31} \geq 900$

$X_{12} + X_{22} + X_{32} \geq 1000$

$X_{13} + X_{23} + X_{33} \geq 1500$

Objective function-

min $(200X_{11} + 300X_{12} + 150X_{13} + 250X_{21} + 400X_{22} + 280X_{23} + 300X_{31} + 175X_{32} + 500X_{33} + 380(X_{11} + X_{12} + X_{13}) + 400(X_{21} + X_{22} + X_{23}) + 360(X_{31} + X_{32} + X_{33}))$

3. Two types of devices are to be produced from device 3. Device 1 requires 9 hours of labor and 10 feet of rubber material. The unit profit from Device 1 is Rs. 325. To produce one device 2, 6 hours of labor and 21 feet of rubber material is required. The unit profit from device 2 is Rs. 400. There are a total of 200 devices 3, 1600 hours of labor, and 3000 feet of rubber material available. If the profit is to be maximized, then formulate the above as a linear programming problem.

Soln:
Decision variables-
X_1 = number of devices 1,
X_2 = number of devices 2.
Constraints-
$9X_1 + 6X_2 \leq 1600$
$10X_1 + 21X_2 \leq 3000$
$X_1 + X_2 \leq 200$
$X_1, X_2 \geq 0$
Objective function-
Max $(325X_1 + 400X_2)$

4. A steel manufacturer produces three sizes of beams: small, medium and large. These beams can be produced on any one of the three machine types: A, B, and C. The length in feet of the beams that can be produced on the machines per hour are summarized.

Beam	A	B	C
Small	500	400	300
Medium	400	300	200
Large	300	200	100

The hourly operating costs of the machines are Rs. 50, 30 and 25 respectively and each machine can be used up to 20 hrs per week. Assume that 1000, 700, 650 feet of beams of different sizes are required weekly. Formulate linear program for this machine scheduling problem.
Soln:
Decision variables-
X_{ij} = number of hours required for machine i to produce j type beams.
i, j = 1, 2, 3.
Constraints-
$X_{i1} + X_{i2} + X_{i3} \leq 20$, i = 1, 2, 3.
$500X_{11} + 400X_{21} + 300X_{31} \geq 1000$
$400X_{12} + 300X_{22} + 200X_{32} \geq 700$
$300X_{13} + 200X_{23} + 100X_{33} \geq 100$
Objective function-
Min $(50(X_{11} + X_{12} + X_{13}) + 30(X_{21} + X_{22} + X_{23}) + 25(X_{31} + X_{32} + X_{33}))$

5. The following quantities of gasoline, kerosene and jet fuel are produced per barrel of each type of oil (light and heavy crude oil).

	Gasoline	Kerosene	Jet fuel
Light crude oil	0.35	0.45	0.31
Heavy crude oil	0.21	0.33	0.45

The cost per barrel of light and heavy crude oil is 10 and 8 respectively. It is required to deliver 500000 barrels of gasoline, 300000 barrels of kerosene and 450000 barrels of jet fuel. Formulate a linear program to find the number of barrels of crude oil and minimizes the total cost.

Soln:
<u>Decision variables-</u>
X_1 – number of barrels of light crude oil.
X_2 – number of barrels of heavy crude oil.
<u>Constraints-</u>
$0.35X_1 + 0.21X_2 \geq 500000$
$0.45X_1 + 0.33X_2 \geq 300000$
$0.31X_1 + 0.45X_2 \geq 450000$
<u>Decision variables-</u>
Min $(10X_1 + 8X_2)$

6. Consider an iron roll from which sheets of same length but different widths have to be cut. Let the iron roll be 15cm wide and following sizes should be made from it.
   - 7 cm = 286 numbers
   - 6 cm = 112 numbers
   - 5 cm = 345 numbers
   - 4 cm = 209 numbers

   Only one dimension cutting is allowed. Formulate a LP such that wastage of the material is minimized.

   Soln:
   Let us now first find out the ways of cutting pattern and find the waste obtained from that pattern.
   [2 0 0 0] means from 15cm wide sheet, two 7cm wide sheet is cut. Waste = 1
   [0 2 0 0] means from 15cm wide sheet, two 6cm wide sheet is cut. Waste = 3
   [0 0 3 0] means from 15cm wide sheet, three 5cm wide sheet is cut. Waste = 0
   [0 0 0 3] means from 15cm wide sheet, three 4cm wide sheet is cut. Waste = 3
   [1 1 0 0] means from 15cm wide sheet, one 7cm and one 6cm wide sheet is cut. Waste= 2
   [1 0 1 0] means from 15cm wide sheet, one 7cm and one 5cm wide sheet is cut. Waste= 3
   [1 0 0 2] means from 15cm wide sheet, one 7cm and two 4cm wide sheet is cut. Waste= 0
   [0 1 1 1] means from 15cm wide sheet, one 6cm, 5cm, 4cm wide sheet is cut. Waste = 0
   [0 0 2 1] means from 15cm wide sheet, two 5cm and one 4cm wide sheet is cut. Waste= 1
   [0 1 0 2] means from 15cm wide sheet, one 6cm and two 4cm wide sheet is cut. Waste= 1
   [0 0 1 2] means from 15cm wide sheet, one 5cm and two 4cm wide sheet is cut. Waste= 2
   There are 11 patterns possible where the wastage is less than the minimum of the given sizes to be made.
   <u>Decision variables:</u>
   X_j = number of seats cut using pattern j.
   <u>Constraints:</u>
   For getting 7cm sheets,
   $2X_1 + X_5 + X_6 + X_7 = 286$

Similarly,
$2X_2 + X_5 + X_8 + X_{10} = 112$
$3X_3 + X_6 + X_8 + 2X_9 + X_{11} = 345$
$3X_4 + 2X_7 + X_8 + X_9 + 2X_{10} + 2X_{11} = 209$
$X_j \geq 0$, j = 1 to 11.
<u>Objective function:</u>
Minimize $(X_1 + 3X_2 + 3X_4 + 2X_5 + 3X_6 + X_9 + X_{10} + 2X_{11})$

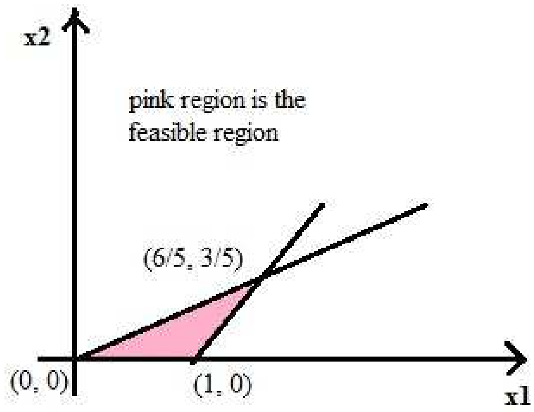
7. Game theory problem: two manufacturers A and B are competitors for the same product. Each wants to maximize their market share and adopt 2 strategies. The gain or pay-off for A when A adopts i and B adopts strategy j is given by a_{ij}. A 2×2 matrix would like $\begin{pmatrix} 4 & -2 \\ -1 & 6 \end{pmatrix}$. During a given time period T, both A and B have to mix their strategies. If A plays strategy 1, then B would play strategy 2 to gain, which A does not want. Each therefore wants to mix their strategies so that they gain maximum (or the other loses maximum). So, for time period T, what is the proportion that A plays strategy 1 and 2. Assume A and B are equally intelligent.

   Soln:
   <u>Decision variables:</u>
   P_1 and P_2 be the proportions of times A plays strategy 1 and 2 respectively.
   <u>Constraints:</u>
   $P_1 + P_2 = 1$
   If B plays strategy 1 all the time, then A's gain would be $4P_1 – P_2$ ….(1)
   If B plays strategy 2 all the time, then A's gain would be $-2P_1 + 6P_2$….(2)
   As A and B are equally intelligent, A would try to maximize his profit and B would try to minimize A's profit, thus A would try to maximize the minimum profit allowed by B (max (min profit)). Let min profit be u, therefore
   Max u
   But B would allow u to be minimum of (1) and (2).
   $u \leq 4P_1 – P_2$
   $u \leq -2P_1 + 6P_2$
   $P_1, P_2 \geq 0$ u = unrestricted.
   <u>Objective function:</u>
   Max u

8. Consider the following problem
   Max $x_1 – x_2$
   Subject to $–x_1 + 2x_2 \leq 0$
   $-3x_1 + x_2 \geq -3$
   $x_1, x_2 \geq 0$
   a) Sketch the feasible region
   b) Solve the problem geometrically

Soln:

a)



b) Since one of the extreme points gives the optimality, it is required to check with each of the corner points.
With (0, 0), we get the value of objective function to be 0.
With (1, 0), we get the value of objective function to be 1.
With (6/5, 3/5), we get the value of objective function to be 3/5.
As the maximum value of objective function is 1, therefore $(x_1, x_2) = (1, 0)$.

9. Find the basic feasible solutions of the following LP.
Max $x_1 + x_2$
Subject to $-x_1 \leq 1$
$20x_1 + x_2 \leq 100$
$x_1, x_2 \geq 0$
Soln:
Let $x_2, x_3 \geq 0$ be the slack variables.
Therefore,
$-x_1 + x_3 = 1$
$20x_1 + x_2 + x_4 = 100$
We can form 6 types of basis B.

- Basis from a_1 and a_2
  $B = \begin{pmatrix} -1 & 0 \\ 20 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} -1 & 0 \\ 20 & 1 \end{pmatrix}$. On calculating $B^{-1}b$,
  we get $(x_1, x_2, x_3, x_4) = (-1, 120, 0, 0)$, bfs does not exist as $x_1 < 0$.
- Basis from a_1 and a_3
  $B = \begin{pmatrix} -1 & 1 \\ 20 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 0 & 1/20 \\ 1 & 1/20 \end{pmatrix}$. On calculating $B^{-1}b$,
  we get $(x_1, x_2, x_3, x_4) = (5, 0, 6, 0)$, bfs exists.
- Basis from a_1 and a_4
  $B = \begin{pmatrix} -1 & 0 \\ 20 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} -1 & 0 \\ 20 & 1 \end{pmatrix}$. On calculating $B^{-1}b$,

we get $(x_1, x_2, x_3, x_4) = (-1, 0, 0, 120)$, bfs does not exist as $x_1 < 0$.

- Basis from a_2 and a_3

  $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On calculating $B^{-1}b$,
  we get $(x_1, x_2, x_3, x_4) = (0, 100, 1, 0)$, bfs exists.
- Basis from a_2 and a_4

  $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, basis does not exist.
- Basis from a_3 and a_4

  $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On calculating $B^{-1}b$,
  we get $(x_1, x_2, x_3, x_4) = (0, 0, 1, 100)$, bfs exists.

10. Solve the following problem using simplex method.
Min $-x_1-3x_2$
Subject to $2x_1 + 3x_2 \leq 6$
$-x_1 + x_2 \leq 1$
$x_1, x_2 \geq 0$
Soln:
Let x_3 and x_4 be the slack variables such that,
$2x_1 + 3x_2 + x_3 \leq 6$
$-x_1 + x_2 + x_4 \leq 1$
$x_1, x_2, x_3, x_4 \geq 0$

<u>Iteration 1-</u>

Let $B = [a_3, a_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = [a_1, a_2] = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$. Solving the equation $Bx_B = b$ gives $x_3 = 6$ and $x_4 = 1$.
Now,
$z_1 - c_1 = 1$
$z_2 - c_2 = 3$ (this value is more, thus x_2 enters the basis.)
since $By_2 = a_2$, we get $y_{12} = 3$, and $y_{22} = 1$.
x_4 leaves the basis (by minimum ratio test).

<u>Iteration 2-</u>
Variable x_2 enters the basis and x_4 leaves the basis.
Now,

$B = [a_3, a_2] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $N = [a_1, a_4] = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$. By again solving the equation $Bx_B = b$, we get $x_3 = 3$, $x_2 = 1$. After calculating the values of $z_1 - c_1$ and $z_4 - c_4$, it is found that $z_4 - c_4 < 0$ and $z_1 - c_1 = 4$. Thus, x_1 enters the basis.
Since, $By_1 = a_1$, we get $y_{11} = 5$, $y_{21} = -1$. As $y_{21} < 0$, x_3 leaves the basis.

<u>Iteration 3-</u>
Variable x_1 enters the basis and x_3 leaves the basis.

$B = [a_1, a_2] = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $N = [a_3, a_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By solving the equation $Bx_B = b$, we get $x_1 = 3/5$ and $x_2 = 8/5$. Now, on calculating the values of $z_3 - c_3$ and $z_4 - c_4$, it is found that both the values are less than zero. Thus, the current solution is optimal.

Thus, the optimal solution is

$(x_1, x_2, x_3, x_4) = (3/5, 8/5, 0, 0)$ and the value of objective function is – (27/5).

11. Solve the following problem using simplex tableau format:

Minimize $x_1 + x_2 - 4x_3$

Subject to $x_1 + x_2 + 2x_3 \leq 9$

$x_1 + x_2 - x_3 \leq 2$

$-x_1 + x_2 + x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

Soln:

Let x4, x5, x6 be the slack variables. Then the constraints become the following-

$x_1 + x_2 + 2x_3 + x_4 = 9$

$x_1 + x_2 - x_3 + x_5 = 2$

$-x_1 + x_2 + x_3 + x_6 = 4$

Initial basis = $[a_4, a_5, a_6] = I$

Iteration 1:

	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-1	-1	4	0	0	0	0
x_4	0	1	1	2	1	0	0	9
x_5	0	1	1	-1	0	1	0	2
x_6	0	-1	1	1	0	0	1	4

Iteration 2:

	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	3	-5	0	0	0	-4	-16
x_4	0	3	-1	0	1	0	-2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	-1	1	1	0	0	1	4

Iteration 3:

	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	-4	0	-1	0	-2	-17
x_1	0	1	-1/3	0	1/3	0	-2/3	1/3
x_5	0	0	2	0	0	1	1	6
x_3	0	0	2/3	1	1/3	0	1/3	13/3

This is the optimal tableau since $z_j - c_j \leq 0$ for all non-basic variables. The optimal solution is given by
$x_1 = 1/3$, $x_2 = 0$, $x_3 = 13/3$ and $z = -17$.

12. The starting and current tableaux of a given problem are shown. Find the values of a to h.

Starting tableau:

z	x_1	x_2	x_3	x_4	x_5	RHS
1	2	1	-3	0	0	0
0	a	b	c	1	0	6
0	-1	2	d	0	1	1

Current tableau:

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	-1/3	2/3	-2/3	i	-4
0	e	2/3	2/3	1/3	0	h
0	f	g	-1/3	1/3	1	3

Soln:
$y_j = B^{-1}a_j$

$$B^{-1} = \begin{pmatrix} 1/3 & 0 \\ 1/3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} a \\ -1 \end{pmatrix}$$

From the current tableau, we come to know that x_1 and x_5 form basis, hence

$\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, therefore, a = 3 and also i = 0 (x_5 is in the basis).

Similarly, by applying the above logic, we get
b = 2, c = 2, d = -1, g = 2/3 and h = 2.