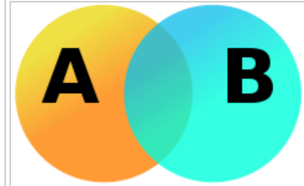
Inclusion–exclusion principle

From Wikipedia, the free encyclopedia
(Redirected from Inclusion-exclusion principle)

In combinatorics (combinatorial mathematics), the **inclusion–exclusion principle** is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where A and B are two finite sets and $|S|$ indicates the cardinality of a set S (which may be considered as the number of elements of the set, if the set is finite). The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection. The principle is more clearly seen in the case of three sets, which for the sets A, B and C is given by

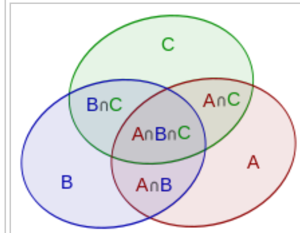


The union of sets A and B

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

This formula can be verified by counting how many times each region in the Venn diagram figure is included in the right-hand side of the formula. In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.

Generalizing the results of these examples gives the principle of inclusion–exclusion: to find the cardinality of the union of n sets, i. include the cardinalities of the sets, ii. exclude the cardinalities of the pairwise intersections, iii. include the cardinalities of the triple-wise intersections, iv. exclude the cardinalities of the quadruple-wise intersections, v. include the cardinalities of the quintuple-wise intersections, vi. and continue, until the cardinality of the n-tuple-wise intersection is included (if n is odd) or excluded (n even).



Inclusion–exclusion illustrated by a Venn diagram for three sets

The name comes from the idea that the principle is based on over-generous *inclusion*, followed by compensating *exclusion*. This concept is attributed to Abraham de Moivre (1718);[1] but it first appears in a paper of Daniel da Silva (1854),[2] and later in a paper by J. J. Sylvester (1883).[3] Sometimes the principle is referred to as the formula of Da Silva, or Sylvester due to these publications. The principle is an example of the sieve method extensively used in number theory and is sometimes referred to as the *sieve formula*,[4] though Legendre already used a similar device in a sieve context in 1808.

As finite probabilities are computed as counts relative to the cardinality of the probability space, the formulas for the principle of inclusion–exclusion remain valid when the cardinalities of the sets are replaced by finite probabilites. More generally, both versions of the principle can be put under the common umbrella of measure theory.

In a very abstract setting, the principle of inclusion–exclusion amounts to no more than the calculation of the inverse of a certain matrix.[5] From this point of view, there is nothing mathematically interesting about the principle. However, the wide applicability of the principle makes it an extremely valuable technique in combinatorics and related areas of mathematics. As Gian-Carlo Rota put it:[6]

> "One of the most useful principles of enumeration in discrete probability and combinatorial theory is the celebrated principle of inclusion–exclusion. When skillfully applied, this principle has yielded the solution to many a combinatorial problem."

Contents

- 1 Statement
- 2 Examples
- 3 A special case
- 4 A generalization
- 5 In probability
  - 5.1 Special case
- 6 Other forms
- 7 Applications
  - 7.1 Counting derangements
  - 7.2 Counting intersections
  - 7.3 Graph coloring
  - 7.4 Bipartite graph perfect matchings
  - 7.5 Number of onto functions
  - 7.6 Permutations with forbidden positions
  - 7.7 Stirling numbers of the second kind
  - 7.8 Rook polynomials
  - 7.9 Euler's phi function
- 8 Diluted inclusion–exclusion principle
- 9 Proof
  - 9.1 Alternative proof
- 10 See also
- 11 Notes
- 12 References

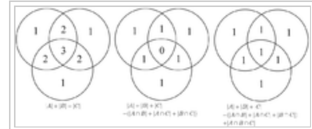
Statement

In its general form, the principle of inclusion–exclusion states that for finite sets $A_1, \ldots, A_n$, one has the identity

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{i=1}^n |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .$$

This can be compactly written as

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{1\le i_1<\cdots<i_k\le n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right).$$



Each term of the inclusion–exclusion formula gradually corrects the count until finally each portion of the Venn diagram is counted exactly once.

In words, to count the number of elements in a finite union of finite sets, first sum the cardinalities of the individual sets, then subtract the number of elements which appear in more than one set, then add back the number of elements which appear in more than two sets, then subtract the number of elements which appear in more than three sets, and so on. This process naturally ends since there can be no elements which appear in more than the number of sets in the union.

In applications it is common to see the principle expressed in its complementary form. That is, letting S be a finite universal set containing all of the A_i and letting $\bar{A}_i$ denote the complement of A_i in S, by De Morgan's laws we have

$$\left|\bigcap_{i=1}^n \bar{A}_i\right| = \left|S - \bigcup_{i=1}^n A_i\right| = |S| - \sum_{i=1}^n |A_i| + \sum_{1\le i<j\le n} |A_i \cap A_j| - \ldots + (-1)^n |A_1 \cap \cdots \cap A_n| .$$

As another variant of the statement, let $P_1, P_2, \ldots, P_n$ be a list of properties that elements of a set S may or may not have, then the principle of inclusion–exclusion provides a way to calculate the number of elements of S which have none of the properties. Just let A_i be the subset of elements of S which have the property P_i and use the principle in its complementary form. This variant is due to J.J. Sylvester.[1]

Examples

As a simple example of the use of the principle of inclusion–exclusion, consider the question:[7]

> How many integers in {1,...,100} are not divisible by 2, 3 or 5?

Let $S = \{1,\ldots,100\}$ and P_1 the property that an integer is divisible by 2, P_2 the property that an integer is divisible by 3 and P_3 the property that an integer is divisible by 5. Letting A_i be the subset of S whose elements have property P_i we have by elementary counting: $|A_1| = 50$, $|A_2| = 33$, and $|A_3| = 20$. There are 16 of these integers divisible by 6, 10 divisible by 10 and 6 divisible by 15. Finally, there are just 3 integers divisible by 30, so the number of integers not divisible by any of 2, 3 or 5 is given by:

$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

A more complex example is the following.

Suppose there is a deck of n cards, each card is numbered from 1 to n. Suppose a card numbered m is in the correct position if it is the mth card in the deck. How many ways, W, can the cards be shuffled with at least 1 card being in the correct position?

Begin by defining set A_m, which is all of the orderings of cards with the mth card correct. Then the number of orders, W, with *at least* one card being in the correct position, m, is

$$W = \left|\bigcup_{m=1}^n A_m\right|.$$

Apply the principle of inclusion–exclusion,

$$W = \sum_{m_1=1}^n |A_{m_1}| - \sum_{1\le m_1<m_2\le n} |A_{m_1} \cap A_{m_2}| + \sum_{1\le m_1<m_2<m_3\le n} |A_{m_1} \cap A_{m_2} \cap A_{m_3}| - \cdots + (-1)^{p-1} \sum_{1\le m_1<\cdots<m_p\le n} \left|A_{m_1} \cap \cdots \cap A_{m_p}\right| \cdots .$$

Each value $A_{m_1} \cap \cdots \cap A_{m_p}$ represents the set of shuffles having p values $m_1, \ldots, m_p$ in the correct position. Note that the number of shuffles with p values correct only depends on p, not on the particular values of m. For example, the number of shuffles having the 1st, 3rd, and 17th cards in the correct position is the same as the number of shuffles having the 2nd, 5th, and 13th cards in the correct positions. It only matters that of the n cards, 3 were chosen to be in the correct position. Thus there are $\binom{n}{p}$ terms in each summation (see combination).

$$W = \binom{n}{1} |A_1| - \binom{n}{2} |A_1 \cap A_2| + \binom{n}{3} |A_1 \cap A_2 \cap A_3| - \cdots + (-1)^{p-1} \binom{n}{p} |A_1 \cap \cdots \cap A_p| \cdots .$$

$|A_1 \cap \cdots \cap A_p|$ is the number of orderings having p elements in the correct position, which is equal to the number of ways of ordering the remaining $n - p$ elements, or $(n - p)!$. Thus we finally get:

$$W = \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{p-1}\binom{n}{p}(n-p)! \cdots W = \sum_{p=1}^n (-1)^{p-1} \binom{n}{p} (n-p)!.$$

Noting that $\binom{n}{p} = \frac{n!}{p!(n-p)!}$, this reduces to

$$W = \sum_{p=1}^n (-1)^{p-1} \frac{n!}{p!}.$$

A permutation where *no* card is in the correct position is called a derangement. Taking $n!$ to be the total number of permutations, the probability Q that a random shuffle produces a derangement is given by

$$Q = 1 - \frac{W}{n!} = \sum_{p=0}^{n} \frac{(-1)^p}{p!},$$

a truncation to n+1 terms of the Taylor expansion of e^{-1}. Thus the probability of guessing an order for a shuffled deck of cards and being incorrect about every card is approximately $1/e$ or 37%.

A special case

The situation that appears in the derangement example above occurs often enough to merit special attention.[8] Namely, when the size of the intersection sets appearing in the formulas for the principle of inclusion–exclusion depend only on the number of sets in the intersections and not on which sets appear. More formally, if the intersection

$$A_J := \bigcap_{j \in J} A_j$$

has the same cardinality, say $\alpha_k = |A_J|$, for every k-element subset J of $\{1, ..., n\}$, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \alpha_k.$$

Or, in the complementary form, where the universal set S has cardinality α_0,

$$\left|S \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \alpha_k.$$

A generalization

Given a family (repeats allowed) of subsets $A_1, A_2, ..., A_n$ of a universal set S, the principle of inclusion–exclusion calculates the number of elements of S in none of these subsets. A generalization of this concept would calculate the number of elements of S which appear in exactly some fixed m of these sets.

Let $N = [n] = \{1,2,...,n\}$. If we define $A_\emptyset = S$, then the principle of inclusion–exclusion can be written as, using the notation of the previous section; the number of elements of S contained in none of the A_i is:

$$\sum_{J \subseteq [n]} (-1)^{|J|} |A_J|.$$

If I is a fixed subset of the index set N, then the number of elements which belong to A_i for all i in I and for no other values is:[9]

$$\sum_{I \subseteq J} (-1)^{|J|-|I|} |A_J|.$$

Define the sets

$B_k = A_{I \cup \{k\}}$ for k in $N \setminus I$.

We seek the number of elements in none of the B_k which, by the principle of inclusion–exclusion (with $B_\emptyset = A_I$), is

$$\sum_{K \subseteq N \setminus I} (-1)^{|K|} |B_K|.$$

The correspondence $K \leftrightarrow J = I \cup K$ between subsets of $N \setminus I$ and subsets of N containing I is a bijection and if J and K correspond under this map then $B_K = A_J$, showing that the result is valid.

In probability

In probability, for events $A_1, ..., A_n$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the inclusion–exclusion principle becomes for $n = 2$

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2),$$

for $n = 3$

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) \quad - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) \quad + \mathbb{P}(A_1 \cap A_2 \cap A_3)$$

and in general

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) \quad + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right),$$

which can be written in closed form as

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{\substack{I\subset\{1,\ldots,n\} \\ |I|=k}} \mathbb{P}(A_I) \right),$$

where the last sum runs over all subsets I of the indices 1, ..., n which contain exactly k elements, and

$$A_I := \bigcap_{i\in I} A_i$$

denotes the intersection of all those A_i with index in I.

According to the Bonferroni inequalities, the sum of the first terms in the formula is alternately an upper bound and a lower bound for the LHS. This can be used in cases where the full formula is too cumbersome.

For a general measure space (S,Σ,μ) and measurable subsets $A_1, \ldots, A_n$ of finite measure, the above identities also hold when the probability measure $\mathbb{P}$ is replaced by the measure μ.

Special case

If, in the probabilistic version of the inclusion–exclusion principle, the probability of the intersection A_I only depends on the cardinality of I, meaning that for every k in $\{1, \ldots, n\}$ there is an a_k such that

$$a_k = \mathbb{P}(A_I) \quad \text{for every} \quad I \subset \{1,\ldots,n\} \quad \text{with} \quad |I| = k,$$

then the above formula simplifies to

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} a_k$$

due to the combinatorial interpretation of the binomial coefficient $\binom{n}{k}$.

An analogous simplification is possible in the case of a general measure space (S,Σ,μ) and measurable subsets $A_1, \ldots, A_n$ of finite measure.

Other forms

The principle is sometimes stated in the form[10] that says that if

$$g(A) = \sum_{S\subseteq A} f(S)$$

then

$$f(A) = \sum_{S\subseteq A} (-1)^{|A|-|S|} g(S) \qquad (**)$$

We show now that the combinatorial and the probabilistic version of the inclusion–exclusion principle are instances of (**). Take $\underline{m} = \{1, 2, \ldots, m\}$, $f(\underline{m}) = 0$, and

$$f(S) = \left| \bigcap_{i\in \underline{m}\setminus S} A_i \middle\backslash \bigcup_{i\in S} A_i \right| \qquad \text{and} \qquad f(S) = \mathbb{P}\left(\bigcap_{i\in \underline{m}\setminus S} A_i \middle\backslash \bigcup_{i\in S} A_i \right)$$

respectively for all sets S with $S \subsetneq \underline{m}$. Then we obtain

$$g(A) = \left| \bigcap_{i\in \underline{m}\setminus A} A_i \right|, \; g(\underline{m}) = \left| \bigcup_{i\in \underline{m}} A_i \right| \qquad \text{and} \qquad g(A) = \mathbb{P}\left(\bigcap_{i\in \underline{m}\setminus A} A_i \right), \; g(\underline{m}) = \mathbb{P}\left(\bigcup_{i\in \underline{m}} A_i \right)$$

respectively for all sets A with $A \subsetneq \underline{m}$. This is because elements a of $\bigcap_{i\in \underline{m}\setminus A} A_i$ can be contained in other A_i's (A_i's with $i \in A$) as well, and the $\bigcap\backslash\bigcup$-formula runs exactly through all possible extensions of the sets $\{A_i \mid i \in \underline{m}\setminus A\}$ with other A_i's, counting a only for the set that matches the membership behavior of a, if S runs through all subsets of A (as in the definition of $g(A)$).

Since $f(\underline{m}) = 0$, we obtain from (**) with $A = \underline{m}$ that

$$\sum_{\underline{m}\supseteq T \supsetneq \varnothing} (-1)^{|T|-1} g(\underline{m}\setminus T) = \sum_{\varnothing \subseteq S \subsetneq \underline{m}} (-1)^{m-|S|-1} g(S) = g(\underline{m})$$

and by interchanging sides, the combinatorial and the probabilistic version of the inclusion–exclusion principle follow.

If one sees a number n as a set of its prime factors, then (**) is a generalization of Möbius inversion formula for square-free natural numbers. Therefore, (**) is seen as the Möbius inversion formula for the incidence algebra of the partially ordered set of all subsets of A.

For a generalization of the full version of Möbius inversion formula, (**) must be generalized to multisets. For multisets instead of sets, (**) becomes

$$f(A) = \sum_{S \subseteq A} \mu(A - S) g(S) \qquad (***)$$

where $A - S$ is the multiset for which $(A - S) \uplus S = A$, and

- $\mu(S) = 1$ if S is a set (i.e. a multiset without double elements) of even cardinality.
- $\mu(S) = -1$ if S is a set (i.e. a multiset without double elements) of odd cardinality.
- $\mu(S) = 0$ if S is a proper multiset (i.e. S has double elements).

Notice that $\mu(A - S)$ is just the $(-1)^{|A|-|S|}$ of (**) in case $A - S$ is a set.

Proof of (***): Substitute

$$g(S) = \sum_{T \subseteq S} f(T)$$

on the right hand side of (***). Notice that $f(A)$ appears once on both sides of (***). So we must show that for all T with $T \subsetneq A$, the terms $f(T)$ cancel out on the right hand side of (***). For that purpose, take a fixed T such that $T \subsetneq A$ and take an arbitrary fixed $a \in A$ such that $a \notin T$.

Notice that $A - S$ must be a set for each positive or negative appearance of $f(T)$ on the right hand side of (***) that is obtained by way of the multiset S such that $T \subseteq S \subseteq A$. Now each appearance of $f(T)$ on the right hand side of (***) that is obtained by way of S such that $A - S$ is a set that contains a cancels out with the one that is obtained by way of the corresponding S such that $A - S$ is a set that does not contain a. This gives the desired result.

Applications

The inclusion–exclusion principle is widely used and only a few of its applications can be mentioned here.

Counting derangements

Main article: Derangement

A well-known application of the inclusion–exclusion principle is to the combinatorial problem of counting all derangements of a finite set. A *derangement* of a set *A* is a bijection from *A* into itself that has no fixed points. Via the inclusion–exclusion principle one can show that if the cardinality of *A* is *n*, then the number of derangements is $[n! / e]$ where $[x]$ denotes the nearest integer to *x*; a detailed proof is available here and also see the examples section above.

The first occurrence of the problem of counting the number of derangements is in an early book on games of chance: *Essai d'analyse sur les jeux de hazard* by P. R. de Montmort (1678 – 1719) and was known as either "Montmort's problem" or by the name he gave it, "*problème des rencontres*."[11] The problem is also known as the *hatcheck problem*.

The number of derangements is also known as the subfactorial of *n*, written $!n$. It follows that if all bijections are assigned the same probability then the probability that a random bijection is a derangement quickly approaches $1/e$ as *n* grows.

Counting intersections

The principle of inclusion–exclusion, combined with De Morgan's law, can be used to count the cardinality of the intersection of sets as well. Let $\overline{A}_k$ represent the complement of A_k with respect to some universal set *A* such that $A_k \subseteq A$ for each *k*. Then we have

$$\bigcap_{i=1}^{n} A_i = \overline{\bigcup_{i=1}^{n} \overline{A}_i}$$

thereby turning the problem of finding an intersection into the problem of finding a union.

Graph coloring

The inclusion exclusion principle forms the basis of algorithms for a number of NP-hard graph partitioning problems, such as graph coloring.[12]

A well known application of the principle is the construction of the chromatic polynomial of a graph.[13]

Bipartite graph perfect matchings

The number of perfect matchings of a bipartite graph can be calculated using the principle.[14]

Number of onto functions

Given finite sets *A* and *B*, how many surjective functions (onto functions) are there from *A* to *B*? Without any loss of generality we may take $A = \{1,2,...,k\}$ and $B = \{1,2,...,n\}$, since only the cardinalities of the sets matter. By using *S* as the set of all functions from *A* to *B*, and defining, for each *i* in *B*, the property P_i as "the function misses the element *i* in *B*" (*i* is not in the image of the function), the principle of inclusion–exclusion gives the number of onto functions between *A* and *B* as:[15]

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^k.$$

Permutations with forbidden positions

A permutation of the set $S = \{1,2,...,n\}$ where each element of S is restricted to not being in certain positions (here the permutation is considered as an ordering of the elements of S) is called a *permutation with forbidden positions*. For example, with $S = \{1,2,3,4\}$, the permutations with the restriction that the element 1 can not be in positions 1 or 3, and the element 2 can not be in position 4 are: 2134, 2143, 3124, 4123, 2341, 2431, 3241, 3421, 4231 and 4321. By letting A_i be the set of positions that the element i is not allowed to be in, and the property P_i to be the property that a permutation puts element i into a position in A_i, the principle of inclusion–exclusion can be used to count the number of permutations which satisfy all the restrictions.[16]

In the given example, there are 12 = 2(3!) permutations with property P_1, 6 = 3! permutations with property P_2 and no permutations have properties P_3 or P_4 as there are no restrictions for these two elements. The number of permutations satisfying the restrictions is thus:

$$4! - (12 + 6 + 0 + 0) + (4) = 24 - 18 + 4 = 10.$$

The final 4 in this computation is the number of permutations having both properties P_1 and P_2. There are no other non-zero contributions to the formula.

Stirling numbers of the second kind

Main article: Stirling numbers of the second kind

The Stirling numbers of the second kind, $S(n,k)$ count the number of partitions of a set of n elements into k non-empty subsets (indistinguishable *boxes*). An explicit formula for them can be obtained by applying the principle of inclusion–exclusion to a very closely related problem, namely, counting the number of partitions of an n-set into k non-empty but distinguishable boxes (ordered non-empty subsets). Using the universal set consisting of all partitions of the n-set into k (possibly empty) distinguishable boxes, $A_1, A_2, ..., A_k$, and the properties P_i meaning that the partition has box A_i empty, the principle of inclusion–exclusion gives an answer for the related result. Dividing by $k!$ to remove the artificial ordering gives the Stirling number of the second kind:[17]

$$S(n,k) = \frac{1}{k!}\sum_{t=0}^{k}(-1)^t\binom{k}{t}(k-t)^n.$$

Rook polynomials

Main article: Rook polynomial

A rook polynomial is the generating function of the number of ways to place non-attacking rooks on a *board B* that looks like a subset of the squares of a checkerboard; that is, no two rooks may be in the same row or column. The board B is any subset of the squares of a rectangular board with n rows and m columns; we think of it as the squares in which one is allowed to put a rook. The coefficient, $r_k(B)$ of x^k in the rook polynomial $R_B(x)$ is the number of ways k rooks, none of which attacks another, can be arranged in the squares of B. For any board B, there is a complementary board B' consisting of the squares of the rectangular board that are not in B. This complementary board also has a rook polynomial $R_{B'}(x)$ with coefficients $r_k(B')$.

It is sometimes convenient to be able to calculate the highest coefficient of a rook polynomial in terms of the coefficients of the rook polynomial of the complementary board. Without loss of generality we can assume that $n \le m$, so this coefficient is $r_n(B)$. The number of ways to place n non-attacking rooks on the complete $n \times m$ "checkerboard" (without regard as to whether the rooks are placed in the squares of the board B) is given by the falling factorial:

$$(m)_n = m(m-1)(m-2)\cdots(m-n+1).$$

Letting P_i be the property that an assignment of n non-attacking rooks on the complete board has a rook in column i which is not in a square of the board B, then by the principle of inclusion–exclusion we have:[18]

$$r_n(B) = \sum_{t=0}^{n}(-1)^t(m-t)_{n-t}\, r_t(B').$$

Euler's phi function

Main article: Euler's totient function

Euler's totient or phi function, $\phi(n)$ is an arithmetic function that counts the number of positive integers less than or equal to n that are relatively prime to n. That is, if n is a positive integer, then $\phi(n)$ is the number of integers k in the range $1 \le k \le n$ which have no common factor with n other than 1. The principle of inclusion–exclusion is used to obtain a formula for $\phi(n)$. Let S be the set $\{1,2,...,n\}$ and define the property P_i to be that a number in S is divisible by the prime number p_i, for $1 \le i \le r$, where the prime factorization of

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$

Then,[19]

Diluted inclusion–exclusion principle

See also: Bonferroni inequalities

In many cases where the principle could give an exact formula (in particular, counting prime numbers using the sieve of Eratosthenes), the formula arising doesn't offer useful content because the number of terms in it is excessive. If each term individually can be estimated accurately, the accumulation of errors may imply that the inclusion–exclusion formula isn't directly applicable. In number theory, this difficulty was addressed by Viggo Brun. After a slow start, his ideas were taken up by others, and a large variety of sieve methods developed. These for example may try to find upper bounds for the "sieved" sets, rather than an exact formula.

Let $A_1, ..., A_n$ be arbitrary sets and $p_1, ..., p_n$ real numbers in the closed unit interval [0,1]. Then, for every even number k in $\{0, ..., n\}$, the indicator functions satisfy the inequality:[20]

Proof

Let A denote the union of the sets A_1, ..., A_n. To prove the inclusion–exclusion principle in general, we first have to verify the identity

for indicator functions, where

There are at least two ways to do this:

First possibility: It suffices to do this for every x in the union of A_1, ..., A_n. Suppose x belongs to exactly m sets with $1 \le m \le n$, for simplicity of notation say A_1, ..., A_m. Then the identity at x reduces to

$$1=\sum_{k=1}^m (-1)^{k-1} \sum_{\scriptstyle I\subset\{1,\ldots,m\}\atop\scriptstyle|I|=k} 1.$$

The number of subsets of cardinality k of an m-element set is the combinatorical interpretation of the binomial coefficient . Since $1=\binom{m}{0}$, we have

$$\binom{m}{0}=\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}.$$

Putting all terms to the left-hand side of the equation, we obtain the expansion for $(1 – 1)^m$ given by the binomial theorem, hence we see that (*) is true for x.

Second possibility: The following function is identically zero

$$(1_A-1_{A_1})(1_A-1_{A_2})\cdots(1_A-1_{A_n})\,=\,0,$$

because: if x is not in A, then all factors are $0 – 0 = 0$; and otherwise, if x does belong to some A_m, then the corresponding mth factor is $1 – 1 = 0$. By expanding the product on the left-hand side, equation (*) follows.

Use of (*): To prove the inclusion–exclusion principle for the cardinality of sets, sum the equation (*) over all x in the union of A_1, ..., A_n. To derive the version used in probability, take the expectation in (*). In general, integrate the equation (*) with respect to μ. Always use linearity.

Alternative proof

Pick an element contained in the union of all sets and let be the individual sets containing it. (Note that $t > 0$.) Since the element is counted precisely once by the left-hand side of the equation, we need to show that it is counted precisely once by the right-hand side. By the binomial theorem,

.

Using the fact that and rearranging terms, we have

$$1=\binom{t}{1}-\binom{t}{2}+\cdots+(-1)^{t+1}\binom{t}{t}=|\{A_i\mid 1\le i\le t\}|-|\{A_i\cap A_j\mid 1\le i<j\le t\}|+\cdots+(-1)^{t+1}|\{A_1\cap A_2\cap\cdots\cap A_t\}|,$$

and so the chosen element is indeed counted only once by the right-hand side of the proposed equation.

See also

- Combinatorial principles
- Boole's inequality
- Necklace problem
- Schuette–Nesbitt formula
- Maximum-minimums identity

Notes

1. ^ ***a b*** Roberts & Tesman 2009, pg. 405
2. ^ Mazur 2010, pg. 94
3. ^ van Lint & Wilson 1992, pg. 77
4. ^ van Lint & Wilson 1992, pg. 77
5. ^ Stanley 1986, pg. 64
6. ^ Rota, Gian-Carlo (1964), "On the foundations of combinatoial theory I. Theory of Möbius functions", *Zeitschrift fur Wahrscheinlichkeitstheorie* **2**: 340–368
7. ^ Mazur 2010, pp. 83–4, 88
8. ^ Brualdi 2010, pp. 167–8
9. ^ Cameron 1994, pg. 78
10. ^ Graham, Grotschel & Lovasz 1995, pg. 1049
11. ^ van Lint & Wilson 1992, pp. 77-8
12. ^ Björklund, Husfeldt & Koivisto 2009
13. ^ Gross 2008, pp. 211–13
14. ^ Gross 2008, pp. 208–10
15. ^ Mazur 2008, pp.84-5, 90
16. ^ Brualdi 2010, pp. 177–81
17. ^ Brualdi 2010, pp. 282–7

18. ^ Roberts & Tesman 2009, pp.419–20
19. ^ van Lint & Wilson 1992, pg. 73
20. ^ (Fernández, Fröhlich & Alan D. 1992, Proposition 12.6)

References

- Allenby, R.B.J.T.; Slomson, Alan (2010), *How to Count: An Introduction to Combinatorics* (http://www.crcpress.com/product/isbn/9781420082609), Discrete Mathematics and Its Applications (2 ed.), CRC Press, pp. 51–60, ISBN 9781420082609
- Björklund, A.; Husfeldt, T.; Koivisto, M. (2009), "Set partitioning via inclusion–exclusion", *SIAM Journal on Computing* **39** (2): 546–563, doi:10.1137/070683933 (http://dx.doi.org/10.1137%2F070683933)
- Brualdi, Richard A. (2010), *Introductory Combinatorics* (5th ed.), Prentice–Hall, ISBN 9780136020400
- Cameron, Peter J. (1994), *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, ISBN 0-521-45761-0
- Fernández, Roberto; Fröhlich, Jürg; Alan D., Sokal (1992), *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*, Texts an Monographs in Physics, Berlin: Springer-Verlag, pp. xviii+444, ISBN 3-540-54358-9, MR 1219313 (//www.ams.org/mathscinet-getitem?mr=1219313), Zbl 0761.60061 (http://www.zentralblatt-math.org/zmath/en/search/?format=complete&q=an:0761.60061)
- Graham, R.L.; Grotschel, M.; Lovasz, L. (1995), *Hand Book of Combinatorics (volume-2)* (http://www.amazon.com/Handbook-Combinatorics-Vol-Ronald-Graham/dp/0262071711), MIT Press – North Holland, ISBN 9780262071710
- Gross, Jonathan L. (2008), *Combinatorial Methods with Computer Applications*, Chapman&Hall/CRC, ISBN 9781584887430
- Hazewinkel, Michiel, ed. (2001), "Inclusion-and-exclusion principle" (http://www.encyclopediaofmath.org/index.php?title=p/i050430), *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Mazur, David R. (2010), *Combinatorics A Guided Tour*, The Mathematical Association of America, ISBN 9780883857625
- Roberts, Fred S.; Tesman, Barry (2009), *Applied Combinatorics* (2nd ed.), CRC Press, ISBN 9781420099829
- Stanley, Richard P. (1986), *Enumerative Combinatorics Volume I*, Wadsworth & Brooks/Cole, ISBN 0534065465
- van Lint, J.H.; Wilson, R.M. (1992), *A Course in Combinatorics*, Cambridge University Press, ISBN 0521422604