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# Pigeonhole Principle and Extensions

The ***Pigeonhole Principle*** is one of almost obvious mathematical concepts which are both simple and ***powerful***:

(1) *If n > m pigeons are put into m pigeonholes, there's a hole with more than one pigeon.*

For the proof, assume that the statment is wrong: i.e., assume there are m holes each with at most 1 pigeon. If that's really the case, then summing up the birds across the holes, we would have at most 1 + ... + 1 = m pigeons. However, the given number of pigeons n > m. A contradiction that proves the statement.

The proof, as the principle itself, is very simple and embodies an idea that can be used to prove a generalized statement:

(2) *If nk + 1 pigeons, where k is a positive integer, have been put into n holes, then at least one of the holes is crowded with at least k+1 pigeons.*

Indeed, let's again assume that the statement is wrong. Then each of the holes houses not more than k birds, which means that the total number of birds can't exceed nk. A contradiction.

In the same vein we can establish the following extension:

(3) *If $p_1 + p_2 + \ldots + p_n - n + 1$ pigeons are placed into n holes, then, for some k, hole k has more than $p_k$ pigeons.*

Once more, assume that the statement is wrong, i.e., assume that, for k = 1, ..., n, hole k contains at most $p_k - 1$ birds. Summing up over all n holes, we find that the total number of the pigeons can't exceed

$$\Sigma(p_k - 1) = \Sigma p_k - n.$$

A contradiction.

Finally, (2) admits a reformulation:

(2') *If m pigeons are found in n holes, then at least one of the holes contains at least $p = [(m-1)/n] + 1$ pigeons,*

where [x] is the ***floor function***. Indeed, assuming that every hole contains at most p pigeons, we arrive at the contradiction:

$$\begin{aligned} m &\le np \\ &\le n\cdot(m - 1)/n \\ &= m - 1. \\ &< m. \end{aligned}$$

A straightforward reformulation of ***(2')*** has been given by ***E. W. Dijkstra***

*For a non-empty, finite bag of numbers, the maximum value is at least the average value.*

(To which we can add the obvious: the average and the maximum values coincide iff the bag only contains equal numbers. Also, the above is obviously equivalent to the assertion that, for a non-empty, finite bag of numbers, the minimum value is at most the average value.)

### Example

[***Sharygin***, p. 12]. Assume in a class of students each of the number of committees contains more than half of all the students. Prove that there is a student who is a member in more than half of the committees.

Let's n be the number of students and m the number of committees in the class. The total committee membership T exceeds m·n/2: T > nm/2. On average, a student is a member in T/n

committees and we find that T/n > m/2. Since the maximum value is at least the average value, there is indeed a student who is a member in more than m/2 committees.

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