Pigeonhole principle

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An image of pigeons in holes. Here there are n = 10 pigeons in m = 9 holes. Since 10 is greater than 9, the pigeonhole principle says that at least one hole has more than one pigeon.

In mathematics, the **pigeonhole principle** states that if n items are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item. This theorem is exemplified in real-life by truisms like "there must be at least two left gloves or two right gloves in a group of three gloves". It is an example of a counting argument, and despite seeming intuitive it can be used to demonstrate possibly unexpected results; for example, that two people in London have the same number of hairs on their heads (see below).

The first formalization of the idea is believed to have been made by Peter Gustav Lejeune Dirichlet in 1834 under the name *Schubfachprinzip* ("drawer principle" or "shelf principle"). For this reason it is also commonly called **Dirichlet's box principle**, **Dirichlet's drawer principle** or simply "Dirichlet principle" — a name that could also refer to the minimum principle for harmonic functions. The original "drawer" name is still in use in French ("principe des tiroirs"), Polish ("zasada szufladkowa"), Hungarian ("skatulyaelv"), Italian ("principio dei cassetti"), German ("Schubfachprinzip"), Danish ("Skuffeprincippet"), and Chinese ("抽屜原理").

Though the most straightforward application is to finite sets (such as pigeons and boxes), it is also used with infinite sets that cannot be put into one-to-one correspondence. To do so requires the formal statement of the pigeonhole principle, which is "*there does not exist an injective function on finite sets whose codomain is smaller than its domain*". Advanced mathematical proofs like Siegel's lemma build upon this more general concept.

Contents

Examples

Softball team

Imagine five people who want to play softball ($n = 5$ items), with a limitation of only four softball teams ($m = 4$ holes) to choose from. The pigeonhole principle tells us that they cannot all play for different teams. At least 2 must play on the same team.

Sock-picking

Assume you have a mixture of black socks and blue socks, what is the minimum number of socks needed before a pair of the same color can be guaranteed. Using the pigeonhole principle, to have at least one pair of the same color ($m = 2$ holes, one per color) using one pigeonhole per color, you need only three socks ($n = 3$ items).

Hand-shaking

If there are n people who can shake hands with one another (where $n > 1$), the pigeonhole principle shows that there is always a pair of people who will shake hands with the same number of people. As the 'holes', or m, correspond to number of hands shaken, and each person can shake hands with anybody from 0 to $n - 1$ other people, this creates $n - 1$ possible holes. This is because either the '0' or the '$n - 1$' hole must be empty (if one person shakes hands with everybody, it's not possible to have another person who shakes hands with nobody; likewise, if one person shakes hands with no one there cannot be a person who shakes hands with everybody). This leaves n people to be placed in at most $n - 1$ non-empty holes, guaranteeing duplication.

Hair-counting

We can demonstrate there must be at least two people in London with the same number of hairs on their heads as follows. Since a typical human head has an average of around 150,000 hairs; it is reasonable to assume (as an upper bound) that no one has more than 1,000,000 hairs on their head ($m = 1$ million holes). There are more than 1,000,000 people in London (n is bigger than 1 million items). Assigning a pigeonhole to each number of hairs on a person's head, and assign people to pigeonholes according to the number of hairs on their head, there must be at least two people assigned to the same pigeonhole by the 1,000,001 assignment (because they have the same number of hairs on their heads) (or, $n > m$). For the average case (m = 150,000) with the constraint: fewest overlaps, there will be at most one person assigned to every pigeonhole and the 150,001st person assigned to the same pigeonhole as someone else. In the absence of this constraint, there may be empty pigeonholes because the "collision" happens before we get to the 150,001st person. The principle just proves the existence of an overlap; it says nothing of the number of overlaps (which falls under the subject of Probability Distribution).

The birthday problem

The birthday problem asks, for a set of n randomly chosen people, what is the probability that some pair of them will have the same birthday. By the pigeonhole principle, if there are 367 people in the room, we know that there is at least one pair who share the same birthday, as there are only 366 possible birthdays to choose from (including February 29, the standard leap day).

Uses and applications

The pigeonhole principle arises in computer science. For example, collisions are inevitable in a hash table because the number of possible keys exceeds the number of indices in the array. A hashing algorithm, no matter how clever, cannot avoid these collisions.

The principle can be used to prove that any lossless compression algorithm, provided it makes some inputs smaller (as the name compression suggests), will also make some other inputs larger. Otherwise, the set of all input sequences up to a given length L could be mapped to the (much) smaller set of all sequences of length less than L, and do so without collisions (because the compression is lossless), which possibility the pigeonhole principle excludes.

A notable problem in mathematical analysis is, for a fixed irrational number a, to show that the set $\{[na]: n$ is an integer$\}$ of fractional parts is dense in $[0, 1]$. One finds that it is not easy to explicitly find integers n, m such that $|na - ma| < e$, where $e > 0$ is a small positive number and a is some arbitrary irrational number. But if one takes M such that $1/M < e$, by the pigeonhole principle there must be $n_1, n_2 \in \{1, 2, ..., M + 1\}$ such that $n_1 a$ and $n_2 a$ are in the same integer subdivision of size $1/M$ (there are only M such subdivisions between consecutive integers). In particular, we can find n_1, n_2 such that $n_1 a$ is in $(p + k/M, p + (k + 1)/M)$, and $n_2 a$ is in $(q + k/M, q + (k + 1)/M)$, for some p, q integers and k in $\{0, 1, ..., M - 1\}$. We can then easily verify that $(n_2 - n_1)a$ is in $(q - p - 1/M, q - p + 1/M)$. This implies that $[na] < 1/M < e$, where $n = n_2 - n_1$ or $n = n_1 - n_2$. This shows that 0 is a limit point of $\{[na]\}$. We can then use this fact to prove the case for p in $(0, 1]$: find n such that $[na] < 1/M < e$; then if $p \in (0, 1/M]$, we are done. Otherwise p in $(j/M, (j + 1)/M]$, and by setting $k = \sup\{r \in N : r[na] < j/M\}$, one obtains $|[(k + 1)na] - p| < 1/M < e$.

Generalizations of the pigeonhole principle

A generalized version of this principle states that, if n discrete objects are to be allocated to m containers, then at least one container must hold at least $\lceil n/m \rceil$ objects, where $\lceil x \rceil$ is the ceiling function, denoting the smallest integer larger than or equal to x. Similarly, at least one container must hold no more than $\lfloor n/m \rfloor$ objects, where $\lfloor x \rfloor$ is the floor function, denoting the largest integer smaller than or equal to x.

A probabilistic generalization of the pigeonhole principle states that if n pigeons are randomly put into m pigeonholes with uniform probability $1/m$, then at least one pigeonhole will hold more than one pigeon with probability

$$1 - \frac{(m)_n}{m^n},$$

where $(m)_n$ is the falling factorial $m(m - 1)(m - 2)...(m - n + 1)$. For $n = 0$ and for $n = 1$ (and $m > 0$), that probability is zero; in other words, if there is just one pigeon, there cannot be a conflict. For $n > m$ (more pigeons than pigeonholes) it is one, in which case it coincides with the ordinary pigeonhole principle. But even if the number of pigeons does not exceed the number of pigeonholes ($n \leq m$), due to the random nature of the assignment of pigeons to pigeonholes there is often a substantial chance that clashes will occur. For example, if 2 pigeons are randomly assigned to 4 pigeonholes, there is a 25% chance that at least one pigeonhole will hold more than one pigeon; for 5 pigeons and 10 holes, that probability is 69.76%; and for 10 pigeons and 20 holes it is about 93.45%. If the number of holes stays fixed, there is always a greater probability of a pair when you add more pigeons. This problem is treated at much greater length at birthday paradox.

A further probabilistic generalisation is that when a real-valued random variable X has a finite mean $E(X)$, then the probability is nonzero that X is greater than or equal to $E(X)$, and similarly the probability is nonzero that X is less than or equal to $E(X)$. To see that this implies the standard pigeonhole principle, take any fixed arrangement of n pigeons into m holes and let X be the number of pigeons in a hole chosen uniformly at random. The mean of X is n/m, so if there are more pigeons than holes the mean is greater than one. Therefore, X is sometimes at least 2.

Infinite sets

The pigeonhole principle can be extended to infinite sets by phrasing it in terms of cardinal numbers: if the cardinality of set A is greater than the cardinality of set B, then there is no injection from A to B. However in this form the principle is tautological, since the meaning of the statement that the cardinality of set A is greater than the cardinality of set B is exactly that there is no injective map from A to B. What makes the situation of finite sets interesting is that adding at least one element to a set is sufficient to ensure that the cardinality increases.

Another way to phrase the pigeonhole principle is similar to the principle that finite sets are Dedekind finite: Let A and B be finite sets. If there is a surjection from A to B that is not injective, then no surjection from A to B is injective. In fact no function of any kind from A to B is injective.

The above principle is not true for infinite sets: Consider the function on the natural numbers that sends 1 and 2 to 1, 3 and 4 to 2, 5 and 6 to 3... and so on.

There is a similar principle for infinite sets: If uncountably many pigeons are stuffed into countably many pigeonholes, there will exist at least one pigeonhole having uncountably many pigeons stuffed into it.

This principle is not a generalisation of the pigeonhole principle for finite sets however: It is in general false for finite sets. In technical terms it says that if A and B are finite sets such that any surjective function from A to B is not injective, then there exists an element of b of B such that there exists a bijection between the preimage of b and A. This is a quite different statement, and is absurd for large finite cardinalities.

See also

- Combinatorial principles
- Combinatorial proof
- Dedekind-infinite set
- Hilbert's paradox of the Grand Hotel
- Multinomial theorem
- Pumping lemma for regular languages
- Ramsey's theorem

References

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External links

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- "The strange case of The Pigeon-hole Principle (http://www.cs.utexas.edu/users/EWD/transcriptions/EWD09xx/EWD980.html)"; Edsger Dijkstra investigates interpretations and reformulations of the principle.
- "The Pigeon Hole Principle (http://zimmer.csufresno.edu/~larryc/proofs/proofs.pigeonhole.html)"; Elementary examples of the principle in use by Larry Cusick.
- "Pigeonhole Principle from Interactive Mathematics Miscellany and Puzzles (http://www.cut-the-knot.org/do_you_know/pigeon.shtml)"; basic Pigeonhole Principle analysis and examples by Alexander Bogomolny.
- "16 fun applications of the pigeonhole principle (http://mindyourdecisions.com/blog/2008/11/25/16-fun-applications-of-the-pigeonhole-principle)"; Interesting facts derived by the principle.

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