From Plato's Diairesis To Shelah's Dividing Line

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Background

Fundamental questions

- Is mathematics a kind of knowledge about an objective world or a creation of the human mind?

Main thesis

- The objects of mathematics are mathematical concepts (structures).
- Mathematical concepts are mind-independent, they form an objective world.
- Mathematical theories provide true descriptions of this world.

Physical and abstract objects

After the 1960s, the philosophy of mathematics evolved into a more purely philosophical debate. Each side in the dispute developed its own theory regarding the ontological status of mathematical objects, basing their arguments on the distinction between physical and abstract objects. Simply put, a physical object is one that exists in space and time and exhibits causal relations. In contrast, objects that do not exist in space and time and lack any causal connections are considered abstract. Since mathematical objects fall into the latter category, the debate about their ontological status essentially becomes an argument about abstract objects.

Gödel's conceptional realism

Gödel is a key representative of Platonism during this period, and his stance is known as conceptual realism. The core idea of conceptual realism is that there exists a realm of concepts parallel to the world of physical objects. According to Gödel, these concepts, like the physical world, possess an objective reality—they are not subject to creation or alteration but can only be perceived and described. Unlike Plato's view, which regards the physical world as an imperfect image of the Forms (Idea), and Aristotle's understanding of a concept as merely the form imposed on substances, Gödel maintained that the two realms exist independently, though they are closely intertwined.

World of things and world of concepts

So one of the main features of his realism is the parallelism between the world of things and the world of concept: "Classes and concepts can also be regarded as real objects... Assuming that such objects are as legal as hypothetical physical bodies. There are just as many reasons to believe that they exist. They are necessary for obtaining a satisfactory mathematical system, just as physical objects are necessary for a satisfactory theory of our sense perception." [Gödel(1944)]

Fictionalism

Summarizing the views of philosophers with opposing positions is no easy task. Unlike Gödel, these philosophers do not present their ideas in a simple or clear manner. Moreover, even among anti-Platonist philosophers there appears to be a variety of positions, at least on the surface. Although referring to these positions as "fictionalism" may be somewhat imprecise, I am more inclined to use that term than alternatives such as naturalism, physicalism, or nominalism. Essentially, these positions all share the view that mathematical objects do not exist objectively but are instead products of human imagination. In this regard, fictionalism offers the least ambiguous stance.

Mathematical propositions are not true

As an example, let's briefly discuss the basic ideas of Hartry Field, the most important representative of fictionalism.

Towards that part of mathematics which does contain references to (or quantifications over) abstract entities—and this includes virtually all of conventional mathematics—I adopt a fictionalist attitude: that is, I see **no reason to regard this part of mathematics as true.** [Field(2016)] 1-2

Mahtematical objects are not indispensable

…… subatomic particles are theoretically indispensable; and I believe that that is as good an argument for their existence as we need. …… I will argue that **mathematical entities are not theoretically indispensable**: although they do play a role in the powerful theories of modern physics, we can give attractive reformulations of such theories in which mathematical entities play no role. [Field(2016)] 8

Mathematics yielding nothing new about non-mathematical entities

……logic (and that part of math that doesn't make reference to abstract entities) doesn't yield genuinely new conclusions, we can give a clear and precise sense to the idea that mathematics doesn't yield genuinely new conclusions: more precisely, we can show that the part of mathematics that does make reference to mathematical entities can be applied without yielding any genuinely new conclusions about non-mathematical entities.
[Field(2016)] 16-17

Objective and subjective

- Since no one can deny the existence of physical objects, the debate focuses on whether abstract objects, such as mathematical concepts, exist independently of human thought.
- For Platonists, trying to show that mathematical concepts are objective is a key part of their philosophical project.
- There plenty of arguments support or against the objectivity of concepts. Let us consider some of them.

Must concepts be subjective?

- Kant believed that our cognition is not simply a passive acceptance of the external world, but is composed of our internal a priori forms. In other words, the reason why objects show a specific structure is because our cognitive ability organizes and interprets experience.
- Our rational conception of the world (that is, the concept in thought) corresponds to the rationality itself (objective concept) that determines the world itself is what it is.([Maybee(2009)], P6.)
- reason exists in the world and mean by it that reason is the soul of the world, residing in it, immanent in it as its own most, innermost nature, its universal. ([Hegel(2010)], §24, Addition 1)

Subjective is constrained by objective

- Subjective means that we can form concepts arbitrarily by correct principles of formations of thought. Since the principles leading to the paradoxes seem to be quite correct in this sense, the paradoxes prove that subjectivism is mistaken.([Wang(1996)],307
- In other words, reality offers resistance and constraints to our subjective inclinations.

Quotations from Frege

- A proposition may be thought, and again it may be true, let us never confuse these two things. We must remind ourselves, it seems, that a proposition no more ceases to exist when I cease to think of it than the sun ceases to eixst when I shut my eye.

Concepts and knowledge of concepts

- What is known as the history of concepts is really a history either of our knowledge of concepts or of the meanings of words. Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind. [Frege(1953)]

The conception of Concept

- Concepts are the Forms in which things exist.
- A Concept is a (Class of) Structure(s): a non-empty domain with properties, relations, operations on it.
- The distinction between concept and object is relatively.
- Never to lose sight of the distinction between objective concept and subjective concept.

To be is to be under a concept

Quotations from Hegel

- There is no object does not fall under any concept, there is no concept wihtout any object fall under.
- Existence [Dasein] is being with a determinacy that is immediate or that simply is, i.e. quality. Existence qua reflected into itself in this its determinacy is an existent [Daseiendes], something [Etwas]. ([Hegel(2010)], §90)

- ……there is no ontological gap between the sort of thing one can mean, or generally the sort of thing one can think, and the sort of thing that can be the case. When one thinks truly, what one thinks is what is the case. So since the world is everything that is the case (as he himself once wrote), there is no gap between thought, as such, and the world. Of course thought can be distanced from the world by being false, but there is no distance from the world implicit in the very idea of thought.
- All the point comes to is that one can think, for instance, that spring has begun, and that very same thing, that spring has begun, can be the case. [McDowell(1994)] 27

Methodology

- Investigate the practical activities of mathematicians to find evidence of objectivity.

What could be considered 'Evidence'?

- Concepts discovered independently in different areas for different motivations turn out to be identical or equivalent (e.g., the concept of 'computability').
- Unexpected orders or hierarchies are found (e.g., the well-ordering of large cardinals).
- Applications of concepts in unexpected areas (e.g., the use of model theory in algebra, algebraic geometry, and other fields).

Understanding a concept

The understanding of a concept includes two directions:

- One is to identify the characteristics that fully describe the concept, distinguishing it from other entities.
- The second is to explore the internal structure of the concept, often leading to a structural theory in mathematics.

Thesis

Plato's Division or Shelah's dividing line represents the most significant, if not the only, method to understand a concept.

Dialectic: division and collection

- For Plato, a philosopher is an expert in dialectics, the latter is an art of division and collection.
- ...... he'll be capable of adequately discriminating a single **form** spread out all through a lot of other things, each of which stands separate from the others. In addition he can discriminate **forms** that are different from each other but are included within a single **form** that's outside them, or a single **form** that's connected as a unit throughout many wholes, or many **forms** that are completely separate from others. *That's what it is to know how to discriminate by kinds how things can associate and how they can't.* (Sophist, 253d-e, [Plato(1997)])

Idea, Class and kind

- Class, real class: eidos, genos.
    - which is used synonymously in this role in Statesman.
- Form or type: eidos; kind: genos. (Sophist)
- Idea: eidos, [character].
    - It can be used to refer to what distinguishes a given class of things from others—its character—but can also substitute for eidos and genos as class/real class. Conversely, eidos itself can be used synonymously with idea in the sense of character.

(C. J. Rowe, a footnote in the translation of Statesman, [Plato(1997)], 297.)

Dialecticians and philosophers

- I am myself a lover of these divisions and collections, so that I may be able to think and to speak; and if I believe that someone else is capable of discerning a single thing that is also by nature capable of encompassing many, …… I have always called them 'dialecticians'.(Phaedrus, 266b-d)
- And you'll assign this **dialectical activity only to someone who has a pure and just love of wisdom.** (Sophist,253e)

To divid is to divid by ideas

- Aren't we going to say that it takes expertise in dialectic to **divide things by kinds** and not to think that the same form is a different one or that a different form is the same? (Sophist, 253d)
- And this means that the division should be made in accordance with **actual divisions, existing in things themselves**. (C. J. Rowe, [Plato(1997)], 297, footnote)

Class and part

But what happened when a wrong division was made?

- That whenever there is a class of something, it is necessarily also a part of whatever thing it is called a class of, but it is not at all necessary that a part is a class. (Statesman, 263b)
- An arbitraty combination of things is a part.
- A class could be divide into to two classes if and only if both sides of the division are themself classes. Ohterwise, the division can only gets parts.

Division as a method to search for concept

- Division is (normally) an iterated dichotomy.
- At each step, one needs to determine which side the concept you are looking for falls on.
- Then continue to divide the side under which the concept falls.
- Ideally, one side of the final division is exactly the concept you are looking for.

What the Visitor and Young Socrates appear to be doing when they 'divide' in each case——as here, with knowledge——is to divide a more generic grouping or 'class' into more specific sub-groups or '(sub-)classes' . (C. J. Rowe, Ibid.)

A figure of division

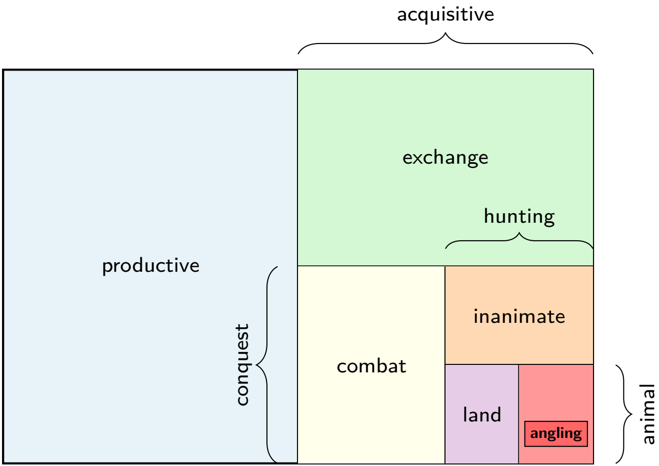


Figure: Division of Arts

Dividing things by ideas, but How?

- According to Plato's requirements, the ultimate basis of division is the idea. However, when it is used as a method for discovering objective concepts, we often begin without knowing precisely which concept we seek.
- In the Statesman, Plato outlines a single principle of division—cutting through the middle—which is the only principle he proposes for the process.

Cutting through the middle (CTM)

- Let's not take off one small part on its own, leaving many large ones behind, and without reference to real classes; let the part bring a real class along with it. (Statesman, 262b)
- but in fact, my friend, it's not safe to make thin cuts; it's safer to go along **cutting through the middle of things**, and that way one will be more likely to encounter real classes. This makes all the difference in relation to philosophical investigations. (Statesman, 262b-c)

Examples of wrong division

- to divide the human race into two and made the cut in the way that most people here carve things up, taking the Greek race away as one, separate from all the rest, and to all the other races together, which are unlimited in number, which don't mix with one another, and don't share the same language—calling this collection by the single appellation 'barbarian'.
- Another example would be if someone thought that he was dividing number into two real classes by cutting off the number ten-thousand from all the rest, separating it off as a single class, and in positing a single name for all the rest supposed here too that through getting the name this class too came into existence, a second single one apart from the other. (Statesman, 262d-e)

Examples of right divisions

- But I imagine the division would be done better, more by real classes and more into two, if one cut number by means of even and odd, and the human race in its turn by means of male and female, and only split off Lydians or Phrygians or anyone else and ranged them against all the rest when one was at a loss as to how to split in such a way that each of the halves split off was simultaneously a real class and a part. (Statesman, 262e-263a)

- Gilbert Ryle criticized Plato's method of division, arguing that the number of sub-kinds a kind can be divided into is an 'empirical' question, not something that can be determined a priori. ([Ryle(1939)], 322)
- However, J.L. Ackrill countered this view, suggesting that the inability to conduct divisions a priori does not mean they are purely empirical. Instead, they are a matter of philosophical reflection. ([Ackrill(1997)], 105)
- Ackrill further argued that Ryle's dismissal of Plato's method amounts to rejecting conceptual analysis as a legitimate part of philosophy.
- This raises a deeper question: How should we define the distinction between a priori and a posteriori? Is this distinction itself a valid dividing line?

This reminds me of Gödel

If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics. The fact is that in mathematics we still have the same attitude today that in former times one had toward all science, namely, we try to derive everything by cogent proofs from the definitions (that is, in ontological terminology, from the essences of things). Perhaps this method, if it claims monopoly, is as wrong in mathematics as it was in physics. ([Gödel(1951)], 313.)

Is CTM necessarily dichotomous?

- No. CTM is not necessarily dichotomous.
  - And once we have grasped it, we must look for two, as the case would have it, or if not, for three or some other number. …… For we must not grant the form of the unlimited to the plurality before we know the exact number of every plurality that lies between the unlimited and the one. (Philebus,16d-e)
- But it is as near as possible to two.
  - Then let's divide them limb by limb, like a sacrificial animal, since we can't do it into two. For we must always cut into the nearest number so far as we can.(Statesman, 287c)

Why should we try two or very few divisions?

- This helps us focus on the concept to be studied. ([Ackrill(1997)],103)
  - In fact, this contradicts division by kinds.
- Maybe the very reason is that there are just few ideas over there.
  - If we accept that a concept represents the very structure of the world, then it is reasonable to believe that not every arbitrary combination of things corresponds to a genuine concept—in other words, not every division produces a real class.
  - Many results in modern model theory seem to confirm this view.

Two sides of a division

Among the division I have in mind are those in which one side is defined by reference to what is genuine or real or original or pure or perfect, while the other side is defined by reference to the bogus, the apparent, the imitative, the impure, the defective (e.g. Sophist 236, 265-6; Politicus 293; Philebus 55-7). ([Ackrill(1997)],106)

Summary

- Division is made in accordance with idea.
- It's safer to go along cutting through the middle of things.
- Finding the correct division is a practical process, even empirical, and there is no first principle of division.
- Division is not necessarily always dichotomous, but we should always cut into the number as near as possible to two.
- Two sides of a division are quite different.

What is a dividing line?

- A property P of theories;
- We have information on both side of P: those theories do have P and those do not;
- The positive side is analyzable: there is a structure theory for the class of models;
- The negative side is unanalyzable: their models are complicated.

A dividing line is a property of theories which is *robust* and *successful*. ([Shelah(2021)])

Criteria of becoming a dividing line

- Robust: There are both internal and external definitions that are equivalent.
- Successful:
  - There exists a significant structure theory on the positive side.
  - It aids in proving the existence of complex models on the negative side.
- Additional desirable properties:
  - Fruitful: It has applications in other areas of mathematics.
  - Versatile: It is useful in contexts that do not fall within the standard framework.

A figure of dividing line by Shelah

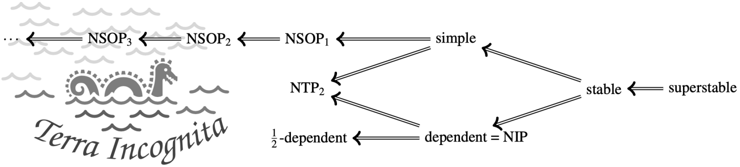


FIGURE 1. A map of dividing lines and candidates.

([Shelah(2021)])

How to find a dividing line? Test problem

- A test problem serves as a guiding question to identify meaningful dividing lines. ([Shelah(2021)])
- The goal is to solve the test problem while uncovering dividing lines that lead to a deeper understanding of the mathematical landscape. ([Shelah(2009)], 5)
- This approach mirrors Plato's method of division, emphasizing empirical and iterative refinement to uncover objective concepts.

Let's see an example of test problem and the associated dividing line.

Test Problem: Morley's Conjecture

- $I(\lambda, T)$=: the number of models of T with cadinality λ upto imsomorphism.
- $\hat{I}(f, T)$ is a relation, where $f(\lambda) =: I(\lambda, T)$.
- Morley Theorem: If $I(\lambda, T) = 1$ for some uncountable λ, then $I(\lambda, T) = 1$ for all uncountable λ.

Morley's Conjecture

for all uncountable cardinlas λ, κ, $\lambda \leq \kappa$ implies $I(\lambda, T) \leq I(\kappa, T)$.

Shelah's Strategy: Dividing line

- Shelah proposed solving this problem by finding a series of dividing lines.
- To do this, he defined the uncountable spectrum function $I(\lambda, T)$.
- Then he got an amazing result in terms of it.

Main Gap

Main Gap Theorem

Exactly one of the following holds:

- $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$ for all $\alpha > 0$;
- $I(\aleph_\alpha, T) < \beth_{\aleph_1}(\max\{|\alpha|, \omega\})$ for all $\alpha > 0$. And T has a structure theory with countable depth.

The so-called "Main Gap" refers to the behaviour below the maximum (the thesis being that if f is below the maximum then it is quite small), which is crucial for the general case of this test problem. ([Shelah(2021)])

Stable Theory

- A theory T is **stable** if **no** formula $\varphi(x; y)$ has the following order property:
  - there is a model $\mathcal{M} \models T$ and sequences $\{a_i\}, \{b_j\}$ such that $\mathcal{M} \models \varphi(a_i; b_j)$ if and only if $i < j$.
- A theory T is λ-**stable** if and only if for any M with cardinality λ, $|S_n(M)| \leq \lambda$.
- A theory T is **superstable** if it is λ-stable for all sufficiently large cardinals λ.

Fact

A theory T is stable if and only T is λ-stable for some infinite λ.

Stable IS the first dividing line

- For any unstable theory T, for any uncountable λ, $I(\lambda, T) = 2^{\lambda}$, that is, above the main gap.
- For any stable theory T, for any uncountable λ, $I(\aleph_\alpha, T) < \beth_{\aleph_1}(\max\{|\alpha|, \omega\})$, that is, below the main gap.

The theory developed out of the test problem of the determination of the number of models of given cardinality has been the most successful and productive to date, in most respects.([Shelah(2021)])

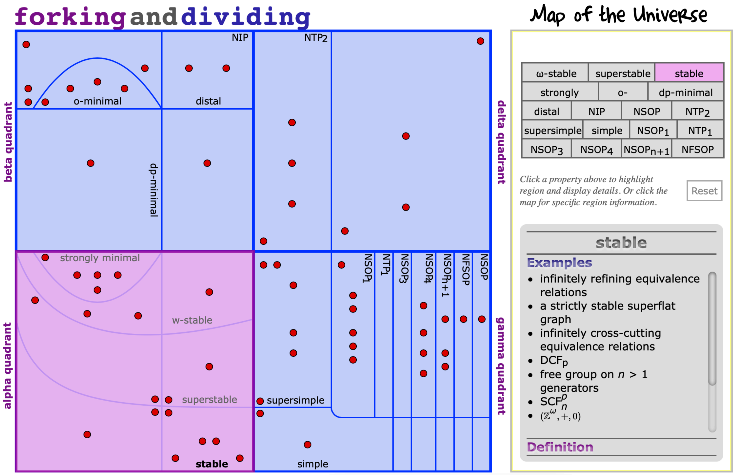
The stability Hierarchy

Theorem

For any countable completely theory T, exactly one of the followings holds:

- There is no infinite λ such that T is λ-stable. (unstable)
- T is λ-stable whenever $\lambda^\omega = \lambda$. (strictly stable)
- T is λ-stable whenever $\lambda \geq 2^{\aleph_0}$. (strictly superstable)
- T is λ-stable for all infinite λ. (ω-stable)

Moreover, stable is not the only one. There are many other dviding lines have been found.


forkinganddividing
NIP
NTP2
o-minimal
distal
dp-minimal
beta quadrant
delta quadrant
alpha quadrant
gamma quadrant
strongly minimal
w-stable
superstable
stable
supersimple
simple
NSOP1
NTP1
NSOP3
NSOP4
NSOPn+1
NFSOP
NSOP
Map of the Universe
ω-stable
superstable
stable
strongly
o-
dp-minimal
distal
NIP
NSOP
NTP2
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NSOP3
NSOP4
NSOPn+1
NFSOP
Click a property above to highlight region and display details. Or click the map for specific region information.
Reset
stable
Examples
infinitely refining equivalence relations
a strictly stable superflat graph
infinitely cross-cutting equivalence relations
DCFp
free group on n > 1 generators
SCF
(ℤ^ω, +, 0)
Definition

Cherlin's observation

- Cherlin's observation: this strategy is a version for mathematics of Plato's strategy of definition, 'cutting through the middle'.
- Baldwin's comparison
    - cutting through the middle is interpreted as both sides are virtuous, i.e., have significant mathematical consequences for any theory falling on.
    - Plato aims at finding the definition of a single concept, while Shelah's procedure is a strategy for constructing a taxonomy.

([Baldwin(2021)],[Baldwin(2018)].)

This is a very illuminating observation. In addressing the question of how to understand objective concepts, both mathematicians and philosophers seem to have ultimately converged on remarkably similar methods. This should not be regarded as a mere coincidence, and the reasons behind it are certainly worthy of further philosophical inquiry.

Definition and classification

- On the surface, Plato's method is indeed looking for the definition of a specific concept, and Shelah pays more attention to the problem of classification.
- Moreover, they divide in the opposite direction: Plato is constantly narrowing the scope, while Shelah is expanding the side with the property P.
- However, Plato's method can often shed light on a whole area: 'Often, though, Platonic division is concerned not to elucide a particular term, but to illuminate a whole area, to lay bare the structure of some 'genus'.' ([Ackrill(1997)], 194)

Why is there a dividing line?

Shelah's dividing lines program is a beautiful mathematical theory. From a philosophical point of view, a natural question is:

Why are there dividing lines?

For Plato, the answer to this question is obvious: Concepts exist objectively, so these dividing lines exists in things themselves. The duty of philosophers is to find these dividing lines, that is, to understand Eidos.

Objectivity of the dividing line

- Yes, I am indeed suggesting that the best position to understand the philosophical significance of the dividing line program is some kind of Platonic objectivism. However, not everyone agrees with this perspective.
- Baldwin notes that Plato assumes some sort of 'natural kind,' but he argues that such an assumption is not necessary in mathematics.
- He cites Langlands to support the position that this assumption has psychological value but is not essential for mathematical practice.

it [the understanding of mathematics] often comes in the form of intimations, a word that suggests that mathematics, and not only its basic concepts, exists independently of us. This is a notion that is hard to credit, but hard for a professional mathematician to do without.' ([Langlands(2010)]; [Baldwin(2018)], 297)

Extreme classification

- Shelah's classification will allow us to obtain different forms of characterization of some very specific concepts. 'It is expected that this will eventually help in understanding even specific classes and even specific structures.' ([Shelah(2013)])
- Let's narrow stable theories down to strongly minimal theories, to strong minimal theories. Then we have the following conjecture.

The trichotomy conjecture

Every strongly minimal structure is either trivial, or is mutually interpretable with a vector space over a division ring, or is mutually interpretable with an algebraically closed field.

As near as possible to Two

- However, the trichotomy conjecture was disproved. But if you further narrow down to, e.g., so called Zaraski Geometry, the conjecture is true. In this case, we eventually get 3 kinds, which is as near as possible to two.
- In the other direction, if we expand stable theories to NIP theories, we have the following theorem:

Theorem (Johnson, 2015)

The only 1-dimension NIP theories of fields are ACF, RCF, and (a long but explicit list).

A belief in 'natural kinds'

- If we interpret Plato's requirement 'as few as possible' to mean that a dividing line is not an arbitrary combination, then these theorems and conjectures shows that this requirement is reasonable.
- In other words, we divide according to ideas, but there are few ideas. Again, "few" here means that not every part corresponds to a real classes.
- 'Allowing some philsophical licence here, this was also **a belief in a strong logical predetermination of basic mathematical structures**.' ([Zilber(2010)])

What makes a property a dividing line?

- If the answer is that a property is a well-established dividing line because it is useful, then we need to ask what makes it useful? Or more formally: what intrinsic properties of P make P useful but, e.g., Q not useful.
- If the answer is that this is our convention or choice, then we will continue to ask if we have other choices? Or can we choose that Q is the dividing line?

We might completely avoid discussing the metaphysical question, "Do mathematical objects exist?" However, we cannot ignore the question, "Is mathematics objective?" regardless of whether you consider it metaphysical.

The last example

- Cherlin's conjecture: Every simple group of finite Morley rank is, in fact, an algebraic group over an algebraically closed field.
- Zilber's conjecture: Every field of finite Morely rank is a pure (finite or algebraically closed) field.
- Nirvana Principle: No awkward situation can happen in a ranked universe.

Mathematical concepts are not the product of chance

- If you believe Cherlin's conjecture, you will believe it is possible to define the notion of algebraic group abstractly, without reference to geometry...The objective is to show that the algebraic groups, which are such important objects in our contemporary mathematics, are neither the product of chance nor arbitrary. . .
- As for Zilber's conjecture, it is even more ambitious, since it asserts that the only algebraic geometries are those which we already know: It seeks to explain to the mathematician, in mathematical language, what mathematics consists of!

([Poizat(2001)], 8-9.)

Summary

- For Plato, as well as for Shelah, the division is an endeavor to search for concepts or a structures or orders in Philosophy or mathematics.
- No dividing line is arbitraty or by chance.
- The way to find a dividingline is a practical one, or even empirical one, depends on how one understand this word.
- Plato kept dividing the positive side and finally reached a precise description of a concept. Shelah kept dividing the negative side, constantly searching for new order in the darkness.

Summary

- For both of them, the ultimate dividing line signifies a comprehensive understanding of a specific domain. While achieving such a goal remains uncertain, every incremental progress is valuable and meaningful.
- 'While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically, finding meaningful dividing lines among general families of structures. This means that there are meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones.' ([Shelah(2013)])

Thank You!

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