Keplerian Reduced 2-body Problem

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Abstract

This note summarizes the solution to the Keplerian 2-body problem, which is equivalent to a reduced mass μ orbiting around a fixed total mass M at radius r (the separation between the two bodies). The approach can be generalized to any 2-body problem with a central force that only depends on the separation.

Reduced 2-body Problem

Consider two point masses m_1 and m_2 moving in an inertial frame with the coordinate origin at the center of mass of the system. Their positions in this frame are denoted as $\mathbf{r}_1$ and $\mathbf{r}_2$. Let us define the reduced mass and total mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad M = m_1 + m_2, \tag{1}$$

and the separation vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{2}$$

The gravitational potential energy is given by

$$V = -\frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = -\frac{GM\mu}{|\mathbf{r}|}. \tag{3}$$

Since $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = 0$, we can show that the total kinetic energy of the system is

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 = \frac{1}{2}\mu\dot{\mathbf{r}}^2, \tag{4}$$

and that the total angular momentum of the system is

$$\mathbf{L} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 = \mu\mathbf{r} \times \dot{\mathbf{r}}. \tag{5}$$

The Lagrangian of the system is given by

$$\mathcal{L} = T - V = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{GM\mu}{|\mathbf{r}|}. \tag{6}$$

We know that the motion of the two is confined within a plane, so it is the best to adopt a spherical coordinate system with the polar axis along the angular momentum direction. In this way $\mathbf{r} = (r, \theta, \phi)$ and we have $\theta = \pi/2$ and $\dot{\theta} = 0$. Therefore, the magnitude of the velocity is given by $\dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2$, and the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{GM\mu}{r}. \tag{7}$$

The specific angular momentum is defined as

$$\ell \equiv \frac{L}{\mu} = r^2\dot{\phi}, \tag{8}$$

and the specific energy is defined as

$$\mathcal{E} \equiv \frac{E}{\mu} = \frac{T+V}{\mu} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GM}{r} = \frac{1}{2}(\dot{r}^2 + \ell^2/r^2) - \frac{GM}{r}. \tag{9}$$

It is sometimes convenient to use the effective potential which is defined as

$$V_{\rm eff}(r) = \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r}, \tag{10}$$

which gives $E = \mu\dot{r}^2/2 + V_{\rm eff}(r)$. For known energy E and angular momentum L, the pericenter or apocenter separations $r_{\rm p/a}$ are given by solving the quadratic equation $E = V_{\rm eff}(r_{\rm p/a})$.

The Euler-Lagrangian equations for any $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ and general coordinates $\mathbf{q} = \{q_j\}$ are

$$\frac{\rm d}{{\rm d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) = \left(\frac{\partial\mathcal{L}}{\partial q_j}\right). \tag{11}$$

Thus, we have the r-equation (describing the radial acceleration due to centrifugal and gravitational accelerations)

$$\ddot{r} = r\dot{\phi}^2 - \frac{GM}{r^2} = \frac{\ell^2}{r^3} - \frac{GM}{r^2}, \tag{12}$$

and the ϕ-equation (describing the angular momentum conservation)

$$\frac{\rm d}{{\rm d}t}\left(\mu r^2\dot{\phi}\right) = \frac{{\rm d}L}{{\rm d}t} = 0. \tag{13}$$

Finally, we derive the expression of the Keplerian orbit $r(\phi)$. This is possible because, knowing both $\dot{r}$ (from $\mathcal{E}$ and ℓ) and $\dot{\phi} = \ell/r^2$, we can compute ${\rm d}r/{\rm d}\phi = \dot{r}/\dot{\phi}$. The best way is to define $u \equiv 1/r$ and hence

$$\frac{{\rm d}u}{{\rm d}\phi} = \frac{\dot{u}}{\dot{\phi}} = \pm\sqrt{\frac{2\mathcal{E}}{\ell^2} - u^2 + \frac{2GM}{\ell^2}u}, \tag{14}$$

where the $\pm$ sign indicates the direction of motion (the two objects are either approaching or moving away from each other). We carry out one more derivative wrt. ϕ and obtain

$$\frac{{\rm d}^2u}{{\rm d}\phi^2} = -u + \frac{GM}{\ell^2}. \tag{15}$$

This clearly shows that $u - GM/\ell^2$ is a sinusoidal function of ϕ. Let us first find the turning points where $\dot{r} = 0$ or $\dot{u} = 0$ using the energy equation,

$$u^2 - \frac{2GM}{\ell^2}u - \frac{2\mathcal{E}}{\ell^2} = 0. \tag{16}$$

Later we will show that $\ell^2/(GM) = r_{\rm p}(1+e)$ as given by the pericenter separation $r_{\rm p}$ and eccentricity e. For finite angular momentum $\ell \neq 0$, the quadratic equation has two (or one) real solutions when $\mathcal{E} < 0$ (or > 0), and the solutions are given by

$$u_\pm = \frac{GM}{\ell^2}\left[1 \pm \sqrt{1 + \frac{2\mathcal{E}}{\ell^2}\left(\frac{\ell^2}{GM}\right)^2}\,\right]. \tag{17}$$

It is easy to show that

$$u_+ + u_- = \frac{r_{\rm p} + r_{\rm a}}{r_{\rm p} r_{\rm a}} = \frac{2GM}{\ell^2}, \quad u_+ u_- = \frac{1}{r_{\rm p} r_{\rm a}} = \frac{-2\mathcal{E}}{\ell^2}. \tag{18}$$

Therefore, we obtain the semi-major axis a from

$$2a = r_{\rm p} + r_{\rm a} = -\frac{GM}{\mathcal{E}}, \tag{19}$$

which is only physically meaningful if $\mathcal{E} < 0$ or the orbit is bound. The eccentricity of the orbit e is defined such that $r_{\rm p} = a(1-e)$ and $r_{\rm a} = a(1+e)$, and hence $r_{\rm p} r_{\rm a} = a^2(1-e^2) = \ell^2/(-2\mathcal{E}) = a\ell^2/(GM)$ (making use of eqs. 18 and 19), so we obtain

$$a(1-e^2) = r_{\rm p}(1+e) = \frac{\ell^2}{GM}. \tag{20}$$

The semi-minor axis is given by $b = a\sqrt{1-e^2}$, so we have $b^2/a = \ell^2/(GM)$. Furthermore, the *eccentricity* is given by the energy and angular momentum as follows

$$1 - e^2 = -\frac{2\mathcal{E}\ell^2}{G^2M^2}, \text{ and } e = \sqrt{1 - \frac{\ell^2}{GMa}}. \tag{21}$$

When $\mathcal{E} < 0$, we have a bound orbit with $e < 1$; whereas the orbit is unbound when $\mathcal{E} > 0$ or $e > 1$ (eccentricity in this case is only mathematically meaningful).

The general solution of eq. (15) is

$$u - \frac{GM}{\ell^2} = u - \frac{1}{r_{\rm p}(1+e)} = B\cos(\phi - \phi_0), \tag{22}$$

and the constants B and ϕ_0 can be calculated by making use of the information at the pericenter, i.e. $r(\phi = \phi_0) = r_{\rm p}$ or $u(\phi = \phi_0) = u_+ = 1/r_{\rm p}$ (note that u_+ is always real for finite ℓ). Then, we obtain

$$B = \frac{1}{r_{\rm p}} - \frac{1}{r_{\rm p}(1+e)}. \tag{23}$$

Let us define $\Phi = \phi - \phi_0$ which is called the *true anomaly*, and then the full solution of the orbit is given by

$$r = \frac{r_{\rm p}(1+e)}{1 + e\cos\Phi}. \tag{24}$$

Note that eq. (24) can be used for bound and unbound orbits, as long as the eccentricity is defined by eq. (21). The line joining the reduced mass μ and the central object M sweeps an equal amount of area per unit time, because $\mathrm{d}A/\mathrm{d}t = r^2\dot{\phi}/2 = \ell/2$. For an elliptical orbit, the total area of an ellipse is πab, so the *orbital period* is given by

$$P_{\rm orb} = \frac{2\pi ab}{\ell} = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{25}$$

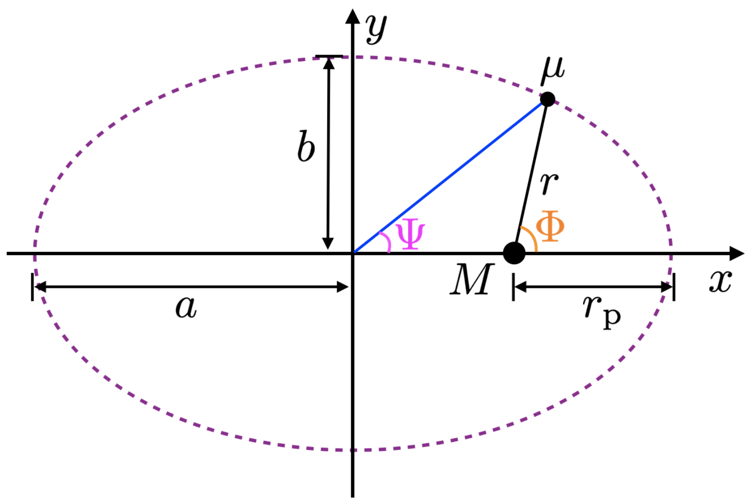

Fig. 1.— A Keplerian orbit with semi-major axis a, semi-minor axis $b = a(1-e^2)$, eccentricity e, pericenter radius $r_{\rm p}$, eccentric anomaly $\Psi(t)$, and true anomaly $\Phi(t)$. In the reduced 2-body problem, the position of the central object M is fixed and the motion of a test particle μ is described by an ellipse.

It is not possible to obtain a simple analytical expression for $\Phi(t)$ and $r(t)$. The best way of getting a numerical solution at a given time is to make use of the *eccentric anomaly* Ψ, which relates to both r and Φ analytically

$$r = a(1 - e\cos\Psi), \quad \cos\Phi = \frac{\cos\Psi - e}{1 - e\cos\Psi}. \tag{26}$$

The relation between eccentric anomaly and time is (which must be solved numerically)

$$\Psi - e\sin\Psi = \frac{2\pi}{P_{\rm orb}}t = \Omega_{\rm orb}t, \tag{27}$$

where the quantity on the right-hand side is called the *mean anomaly* and $\Omega_{\rm orb}$ is called the *mean motion*. Sometimes, we need to compute a numerical integral over the orbit $\int \mathrm{d}t\, A[t, r(t), \Phi(t), \Psi(t)]$, and this can be converted to an integral over Ψ or Φ, because

$$\dot{\Psi} = \frac{\Omega_{\rm orb}}{1 - e\cos\Psi}, \quad \dot{\Phi} = \frac{\ell}{r_{\rm p}^2(1+e)^2}(1 + e\cos\Phi)^2. \tag{28}$$

A linear Φ grid has better resolution near the pericenter.

Mildly Eccentric Orbits and Epicyclic Motion

In the limit $e \ll 1$, the description of the orbit can be simplified if we only retain linear terms $O(e)$. The eccentric anomaly is given by

$$\Psi(t) = \Omega_{\rm orb}t + e\sin\Psi \approx \Omega_{\rm orb}t + e\sin(\Omega_{\rm orb}t) + \mathrm{O}(e^2). \tag{29}$$

The true anomaly can be calculated as follows (using $\Delta\cos\theta = -\sin\theta\Delta\theta$ as Φ and Ψ are close to each other)

$$\Phi - \Psi \approx \frac{\cos\Psi - \cos\Phi}{\sin\Psi} = \frac{\cos\Psi - (\cos\Psi - e)/(1 - e\cos\Psi)}{\sin\Psi} = \frac{e\sin\Psi}{1 - e\cos\Psi} \approx e\sin\Psi + \mathrm{O}(e^2), \tag{30}$$

which then gives

$$\Phi(t) \approx \Psi + e\sin\Psi \approx \Omega_{\rm orb}t + 2e\sin(\Omega_{\rm orb}t) + \mathrm{O}(e^2). \tag{31}$$

The separation r is given by

$$r(t)/a = 1 - e\cos\Psi \approx 1 - e\cos(\Omega_{\rm orb}t) + O(e^2). \tag{32}$$

We find that the separation is close to $r \sim a$ and the deviation $\Delta r = r - a$ is given by

$$\frac{\Delta r(t)}{a} = \frac{r - a}{a} = -e\cos(\Omega_{\rm orb}t) + O(e^2). \tag{33}$$

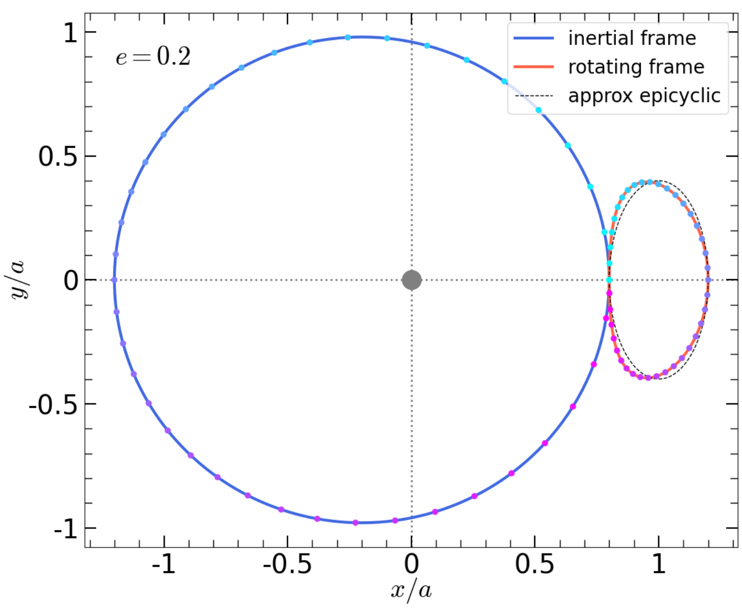

Fig. 2.— Motion of a test particle as viewed in an inertial frame (blue) and rotating frame (red). The black dashed line shows the approximate epicyclic motion in the limit $e \ll 1$ (eq. 35). The semi-major axis is a.

Since the orbit is nearly circular, it is convenient to go to the a rotating frame that co-rotates with the mean motion (i.e., at an angular frequency of $\Omega_{\rm orb}$). In this rotating frame, the true anomaly Φ' is given by

$$\Phi'(t) = \Phi(t) - \Omega_{\rm orb}t \approx 2e\sin(\Omega_{\rm orb}t) + \mathrm{O}(e^2). \tag{34}$$

We see that the test particle stays close to the position $(x = a, y = 0)$, which is called the *epicenter*. In the rotating frame, the deviation from the epicenter $\Delta\vec{r}(t) = (\Delta x, \Delta y)$ is given by

$$\begin{aligned} \Delta x(t)/a &= (r/a)\cos\Phi' - 1 \approx -e\cos\Omega_{\rm orb}t + \mathrm{O}(e^2),\\ \Delta y(t)/a &= (r/a)\sin\Phi' \approx 2e\sin\Omega_{\rm orb}t + O(e^2). \end{aligned} \tag{35}$$

We find that, in the rotating frame, the test particle's motion around the epicenter roughly traces an ellipse (although not exactly) going in the opposite direction as the sense of rotation in the inertial frame, with major axis (along $\hat{y}$) twice longer than the minor axis (along $\hat{x}$). This is called the *epicyclic motion*, which has the *epicyclic frequency* that is equal to the mean motion $\Omega_{\rm orb}$. Note that if the interaction potential is not the point-mass Newtonian potential as considered here (e.g., if the mass distribution is extended), the epicyclic frequency may not be equal to the mean motion.

The motion of the test particle with eccentricity $e = 0.2$ as viewed in an inertial frame and in the rotating frame are shown in Fig. 2.

Application — internal shocks in wind

Consider that one of the objects ejects a wind that is isotropic in its comoving frame and the wind speed is much higher than the orbital speeds. However, this wind will not be isotropic in the center-of-mass frame, and periodic modulation of the wind velocity field will lead to internal shocks. Here, we estimate the energy dissipation rate in the internal shocks in the limit of high eccentricity.

When the orbit is highly eccentric, only the component of the velocity perturbation along the major axis will be dissipated. Let us denote the velocity component along the major axis as v_x (relative velocity between the two objects), which is given by

$$v_x = \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\Phi) = r_{\rm p}(1+e)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cos\Phi}{1+e\cos\Phi} = -r_{\rm p}(1+e)\frac{\sin\Phi}{(1+e\cos\Phi)^2}\dot{\Phi}, \tag{36}$$

where $\dot{\Phi}$ is given by eq. (28). Suppose the constant mass loss rate is $\dot{M}$, then the average energy dissipation rate by internal shocks (which smooth out the velocity perturbations) will be

$$\dot{E}_{\rm sh} = \frac{0.5\dot{M}\int_0^{P_{\rm orb}} v_x^2\,\mathrm{d}t}{P_{\rm orb}} = \frac{\dot{M}}{P_{\rm orb}}\int_0^\pi v_x^2\dot{\Phi}^{-1}\mathrm{d}\Phi = \frac{\dot{M}\ell}{P_{\rm orb}}\int_0^\pi \frac{\sin^2\Phi}{(1+e\cos\Phi)^2}\mathrm{d}\Phi. \tag{37}$$

The above equation is missing a factor of $[m_1/(m_1+m_2)]^2$ if only one of the objects is losing mass, because the x-component velocity in the center-of-mass frame is $m_1 v_x/(m_1+m_2)$. The energy dissipation per unit mass is given by

$$\Delta u = \frac{\dot{E}_{\rm sh}}{\dot{M}} = \frac{m_1^2}{(m_1+m_2)^2}\frac{\ell}{P_{\rm orb}}\int_0^\pi \frac{\sin^2\Phi}{(1+e\cos\Phi)^2}\mathrm{d}\Phi. \tag{38}$$

In the limit $e \approx 1$, the integral roughly equals to $2/(1-e)^{1/2}$, so and for $m_1 \simeq m_2$, we obtain $\Delta u \simeq \ell/(2P_{\rm orb}) = \sqrt{1-e^2}GM/(2\pi a)$. The factor of $(1-e)^{1/2}$ is essentially due to the fact that the system spends a large fraction of time near the apocenter. For smaller eccentricities, one has to consider the full orbital velocity (as opposed to v_x only), and it is easy to show that

$$\Delta u = \frac{0.5 \int_0^{P_{\rm orb}} v^2 {\rm d}t}{P_{\rm orb}} = \frac{m_1^2}{(m_1 + m_2)^2} \frac{2\ell}{P_{\rm orb}} \int_0^{\pi} \left[\frac{1}{1 + e \cos \Phi} - \frac{1 - e^2}{2} \frac{1}{(1 + e \cos \Phi)^2} \right] {\rm d}\Phi, \quad (39)$$

which gives a result that is close to eq. (38) in the high eccentricity limit.