Analytic fit to binary GW merger time

Wenbin Lu

Abstract

This note provides an accurate (fractional error < 1% for all eccentricities) analytic fit to the post-Newtonian gravitational-wave merger time for two point masses.

Background

The post-Newtonian merger time for two point masses m_1 and m_2 in an initial orbit with semimajor axis a_0 and eccentricity e_0 is given by Peters (1964):

$$\frac{T_{\rm gw}}{T_0} = \frac{24}{19} \frac{(1-e_0^2)^{1/2}}{e_0^{48/19}(1+\frac{121}{304}e_0^2)^{3480/2299}} \int_0^{e_0^2} \frac{x^{5/19}(1+\frac{121}{304}x)^{1181/2299}}{(1-x)^{3/2}} \mathrm{d}x, \tag{1}$$

where the R.H.S. is a factor of order unity and the denominator on the L.H.S. is the lowest-order (sometimes unsatisfactory) approximation

$$T_0 = T_{\rm c}(1-e_0^2)^{7/2}, \tag{2}$$

where $T_{\rm c}$ is the merger time for a circular orbit

$$T_{\rm c} = \frac{5c^5 a_0^4}{256 G^3 m_1 m_2 (m_1+m_2)}. \tag{3}$$

The ratio $T_{\rm gw}/T_0$ has the following asymptotic behaviors

$$\frac{T_{\rm gw}}{T_0} \approx \begin{cases} 1 & \text{if } e_0 \ll 1, \\ 768/425 & \text{if } e_0 \approx 1. \end{cases} \tag{4}$$

The behavior of $T_{\rm gw}/T_0$ as a function of e_0 is such that the deviation from 1 is only significant when $e_0 \gtrsim 0.8$ (see Fig. 1).

Thus, to find an analytic fit for this ratio, the key is to identify the asymptotic behavior of $(T_{\rm gw}/T_0 - 768/425)$ as $(1-e_0) \to 0$. Then, it is also possible to identify the asymptotic behavior of $(T_{\rm gw}/T_0 - 1)$ as $e_0 \to 0$. Combining these two, one should be able to find a good analytic expression for the ratio. The math is left for future fun.

Numerically, we find the following to be a good fit (to within 1% for all e_0's)

$$\frac{T_{\rm gw,app}}{T_0} \approx \frac{768}{425} - p\left(1-e_0^2\right)^{0.5} + \left(p - \frac{343}{425}\right)\left(1-e_0^2\right)^{0.8}, \text{ with } p = 2.66. \tag{5}$$

The comparison between the analytic expression and the results from numerical integration is shown in Fig. 1.

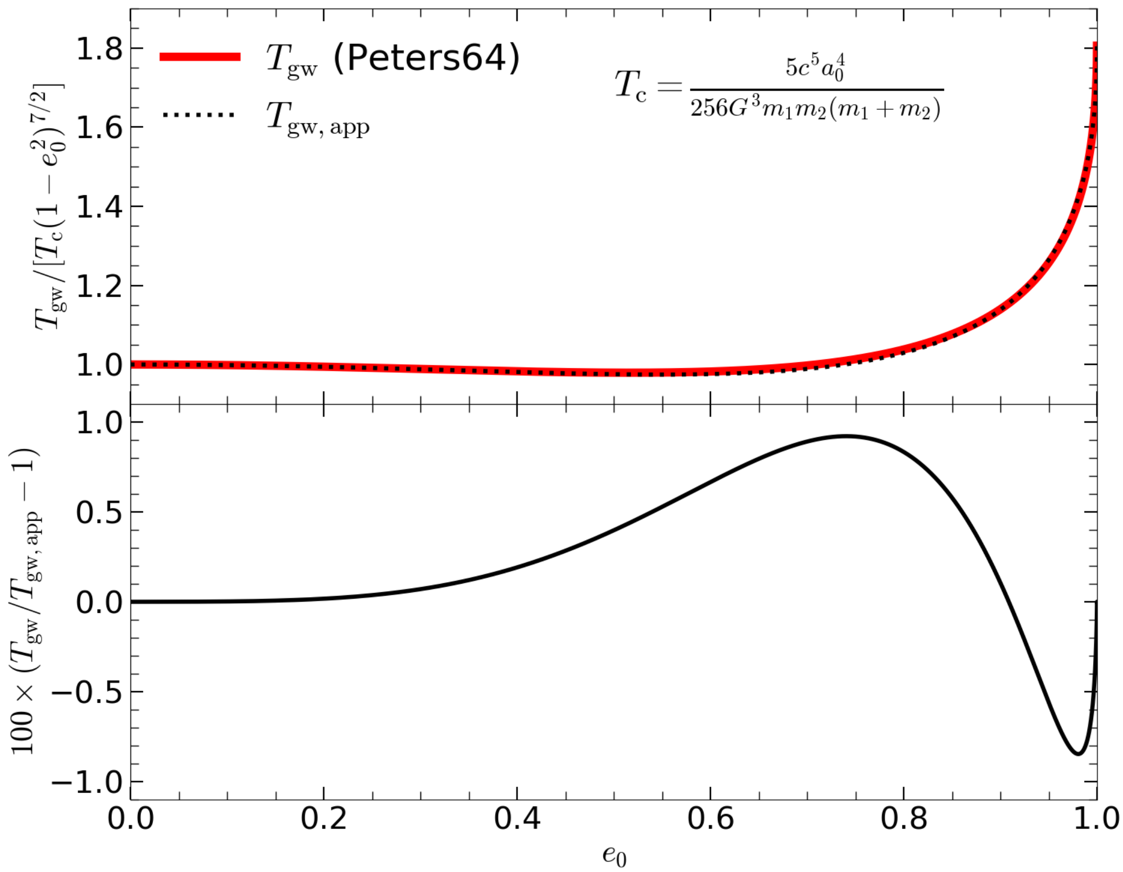

Fig. 1.— The upper panel shows the ratio between the GW inspiral time and the rough estimate of $T_0 = T_c(1-e_0^2)^{7/2}$. The red solid line is obtained by numerically integrating eq. (1) and the black dotted line is the analytic approximation of eq. (5). The bottom panel shows the fractional difference between the two results (multiplied by 100).

During orbital decay, the following quantity is conserved

$$\frac{a(1-e^2)}{e^{12/19}}\left(1+\frac{121}{304}e^2\right)^{-870/2299} = \text{const} \approx 1.762\, r_{\rm p,0}, \tag{6}$$

where $r_{\rm p,0}$ is the initial pericenter radius and the approximation applies if the initial eccentricity is very close to unit, $e_0 \approx 1$.

References

Peters P. C., 1964, Physical Review, 136, 1224