Jump Conditions for Relativistic Adiabatic Shocks

Wenbin Lu

July 3, 2025

This note considers a fast wind running into a static medium and derive the jump conditions for the forward and reverse shocks. An arbitrary wind velocity is considered. The only assumption here is that the unshocked gas is initially dynamically cold everywhere.

1 Forward Shock and Reverse Shock Structure

There are 4 regions: unshocked medium (1), shocked medium (2), shocked wind (3), and unshocked wind. Our lab frame is chosen such that the unshocked medium (region 1) is at rest. Regions (1) and (2) are separated by the forward shock (FS), which causes a pressure jump and a density jump between them. we assume the unshocked medium to be pressureless or dynamically cold, so the Mach number of the FS approaches infinity. Similarly, regions (3) and (4) are separated by the reverse shock (RS). In the lab frame, the unshocked wind has speed v_4, and we assume that the unshocked wind (region 4) to be pressureless, so the Mach number of the RS also approaches infinity.

Between region (2) and (3), there is a contact discontinuity that produces no pressure jump. If the system is in a steady state[1], these two regions move at the same speed $v_2 = v_3$ and they have the same pressure $P_2 = P_3$. Hereafter, we denote the Lorentz factor of the shocked regions (2 and 3) as $\Gamma = \Gamma_2 = \Gamma_3$, and their velocity as $v = v_2 = v_3$. However, there is a density jump between (2) and (3), i.e., $\rho_2 \neq \rho_3$.

Hereafter, pressure P, thermal energy density e, and density ρ are all measured in the rest frame of a given fluid; whereas the velocity v_i and Lorentz factor Γ_i ($i = 1, 2, 3, 4$) are measured in the lab frame (= the comoving frame of region 1).

At the two shocks, there are jump conditions that connect the gas properties behind and ahead of each shock. These two sets of jump conditions allow us to solve all properties of

[1] In realistic physical systems, the density of the unshocked wind may vary with time (e.g., on a dynamical timescale of the wind), and the density of the unshocked medium may vary with position. Here, we ignore such complexities. We assume that the unshocked medium has uniform density $\rho_1 = \text{const}$ and that the wind has uniform density $\rho_4 = \text{const}$ and steady velocity $v_4 = \text{const}$.

the shocked gas (regions 2 and 3) given the properties of the unshocked gas: ρ_1, ρ_4 and v_4. To close the system of equations, we need an equation of state.

For simplicity, we assume that the gas is made of pure hydrogen and the pressure in the shocked regions is dominated by hot protons — ignoring the contribution from electrons[2]. We further assume that the hot protons follow a Maxwell-Jüttner distribution in Lorentz factor,

$$f(\gamma) = \frac{\gamma^2\beta}{\theta K_2(1/\theta)} \mathrm{e}^{-\gamma/\theta}, \quad \theta \equiv k_{\mathrm{B}}T/(m_{\mathrm{p}}c^2) \tag{1}$$

where β is the thermal velocity in units of the speed of light c, and $\gamma = 1/\sqrt{1-\beta^2}$ is the corresponding Lorentz factor, K_2 is the modified Bessel function of the second kind, θ is a dimensionless temperature, and the distribution function has been normalized such that $\int_1^\infty f(\gamma)\mathrm{d}\gamma = 1$. The Lorentz factor distribution can be converted into a distribution in the dimensionless 4-velocity[3] $u = \gamma\beta$,

$$f(u) = \frac{u^2}{\theta K_2(1/\theta)} \mathrm{e}^{-\gamma/\theta}, \tag{2}$$

which is normalized such that $\int_0^\infty f(u)\mathrm{d}u = 1$. The thermal energy density of the gas is then given by

$$e = \rho c^2 \int_0^\infty (\gamma - 1) f(u)\mathrm{d}u, \tag{3}$$

where $\rho = n m_{\mathrm{p}}$ is the rest-mass density measured in the comoving frame, and n is the proton number density. The pressure is given by

$$P = \frac{1}{3}\rho c^2 \int_0^\infty u\beta f(u)\mathrm{d}u. \tag{4}$$

It is possible to show that the pressure is always given by $P = nk_{\mathrm{B}}T$ (valid for all θ), so we have $P = \theta\rho c^2$. Let us define the average thermal Lorentz factor as $\bar{\gamma}$, which is a function of temperature θ as given by

$$\bar{\gamma}(\theta) = \int_0^\infty \gamma f(u)\mathrm{d}u. \tag{5}$$

[2] There are two caveats here: (1) it is widely believed that collisionless shocks can accelerate non-thermal protons and electrons (which are observed as cosmic rays), which take roughly $\sim$10% of the energy density in the shocked regions; (2) collisionless plasma processes can rapidly heat up thermal electrons to a temperature $T_{\mathrm{e}} \sim 0.3T_{\mathrm{p}}$ [PCS15], so pressure contribution by thermal electrons is not negligible. These complexities, as well as the assumption of pure hydrogen composition (ignoring helium), cause our model to have errors of the order a few 10% of percent. In many astrophysical applications, we do not hope to pin down the plasma properties to better than a factor of 2, so these assumptions may be OK. If better accuracies are needed (e.g., in laboratory applications), the model will be more complicated.

[3] This is the magnitude of the spatial components of the 4-velocity in special relativity.

Then, the thermal energy density can be written as $e = (\bar{\gamma} - 1)\rho c^2$. It is possible to show that $\bar{\gamma} - 1 \approx 1.5\theta$ in the non-relativistic limit ($\theta \ll 1$) and $\bar{\gamma} - 1 \approx 3\theta$ in the ultra-relativistic limit ($\theta \gg 1$).

We then define an adiabatic index[4] that is given by the ratio between the pressure and energy density

$$\kappa(\theta) = \frac{P}{e} = \frac{\theta}{\bar{\gamma} - 1} \in (1/3, 2/3). \tag{6}$$

We use the following approximation for the adiabatic index [Uhm11]

$$\kappa \approx \frac{1}{3}\left(1 + \frac{1}{\bar{\gamma}}\right), \tag{7}$$

as it is reasonably accurate (with a maximum fractional error of 4.8%) and drastically simplifies the expression for the jump conditions. Fig. 1 comparison between the accurate and approximate adiabatic indices.

Some readers might still worry about the inaccuracy of our approximation for κ. However, one should keep in mind that we have ignored the contribution to pressure and thermal energy density by thermal electrons, and non-thermal cosmic rays. There are large uncertainties in these components (as their energy shares depend on the physics of collisionless plasma near the shock front), and we expect them to share $\gtrsim 10\%$ of the total thermal energy. Our model also ignore the contribution from ions other than hydrogen (e.g. helium), which may share $\sim 10\%$ of the total thermal energy. Thus, our model is rather crude anyway and is certainly not to be trusted to better than 10% accuracy. In the following, we proceed with our approximate κ in eq. (7), which reasonably captures the transition from non-relativistic limit ($\kappa \approx 2/3$) to the ultra-relativistic limit ($\kappa \approx 1/3$).

Using the approximate adiabatic index, the jump conditions for the forward shock between region 1 and 2 become

$$P_2 = (4/3)(\Gamma^2 - 1)\rho_1 c^2, \tag{8}$$

$$\rho_2 = 4\Gamma\rho_1, \tag{9}$$

$$e_2 = 4(\Gamma^2 - 1)\rho_1 c^2, \tag{10}$$

where Γ is the bulk Lorentz factor of regions 2 and 3 (they have the same Lorentz factor). It is important to note that $e_2 = (\Gamma - 1)\rho_2 c^2$, so the proton mean Lorentz factor in region 2 is $\bar{\gamma}_2 = \Gamma$.

More generally, for any shock, the mean thermal Lorentz factor in the shocked region is equal to the relative Lorentz factor between the unshocked and shocked region. The forward

[4] The relationship between the polytropic adiabatic index $\hat{\gamma} \equiv (\partial \ln P/\partial \ln \rho)_{\rm ad}$ and our new definition $\kappa \equiv P/e$ is: $\kappa = \hat{\gamma} - 1$. This can be shown from the first law of thermodynamics: $\mathrm{d}(eV) = \kappa^{-1}\mathrm{d}(PV) = \kappa^{-1}(V\mathrm{d}P + P\mathrm{d}V) = -P\mathrm{d}V$ (for $\mathrm{d}Q = 0$), and then $V\mathrm{d}P = -(\kappa+1)P\mathrm{d}V$ or $\mathrm{d}\ln P = -(\kappa+1)\,\mathrm{d}\ln V$.

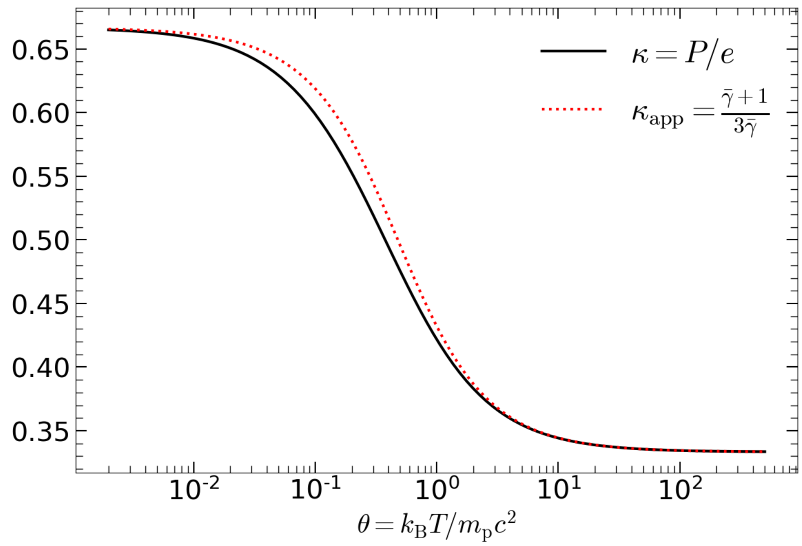

Figure 1: Adiabatic index $\kappa = P/e$ for a mono-atomic gas with Maxwell-Jüttner distribution. The black solid line is exact, and the red dotted line is our adopted approximation.

shock is a special case as the pre-shock gas is at rest and hence the relative Lorentz factor is $\Gamma_{\rm r,fs} = \Gamma$. For the reverse shock, the relative Lorentz factor between the two adjacent regions across the shock is

$$\Gamma_{\rm r,rs} = (1 - \beta_4\beta)\Gamma_4\Gamma, \tag{11}$$

and the relative speed between the two layers is

$$\beta_{\rm r,rs} = (\beta_4 - \beta)/(1 - \beta_4\beta). \tag{12}$$

Taking product of the two expressions above, we obtain

$$\Gamma_{\rm r,rs}\,\beta_{\rm r,rs} = \Gamma_4\Gamma(\beta_4 - \beta). \tag{13}$$

Thus, the jump conditions between region 3 and 4 are given by

$$P_3 = (4/3)(\Gamma_{\rm r,rs}^2 - 1)\rho_4 c^2, \tag{14}$$

$$\rho_3 = 4\Gamma_{\rm r,rs}\,\rho_4, \tag{15}$$

$$e_2 = 4(\Gamma_{\rm r,rs}^2 - 1)\rho_4 c^2. \tag{16}$$

Finally, we can use the pressure balance at the contact discontinuity $P_2 = P_3$ to solve for the velocity of the shocked regions (both 2 and 3)

$$\beta = \frac{\beta_4}{1 + (\rho_1/\rho_4)^{1/2}/\Gamma_4}. \tag{17}$$

The 4-velocity of the shocked regions is given by

$$\Gamma\beta = \frac{\Gamma_4\beta_4}{\left[1 + \rho_1/\rho_4 + 2\Gamma_4\left(\rho_1/\rho_4\right)^{1/2}\right]^{1/2}}. \tag{18}$$

Another simpler way of calculating the Lorentz factor of the shocked regions is to consider the balance of momentum fluxes. In the comoving frame of the shocked regions, the momentum flux from the wind (region 4) is given by $\rho_4 c^2(\Gamma_{\rm r,rs}\,\beta_{\rm r,rs})^2$, which can be obtained since each proton carries a momentum of $\Gamma_{\rm r,rs}\,\beta_{\rm r,rs} m_{\rm p} c$, and the number density of protons is $\Gamma_{\rm r,rs} n_4 = \Gamma_{\rm r,rs}\rho_4/m_{\rm p}$, and the protons are all moving at velocity $\beta_{\rm r,s}$. Similarly, the momentum flux of the medium (region 1) moving in the opposite direction is given by $\rho_1 c^2(\Gamma\beta)^2$. The balance between these two momentum fluxes means that

$$\rho_4 c^2(\Gamma_{\rm r,rs}\,\beta_{\rm r,rs})^2 = \rho_1 c^2(\Gamma\beta)^2 \quad\Rightarrow\quad \beta = \frac{\beta_4 - \beta}{\Gamma_4\left(\rho_1/\rho_4\right)^{1/2}}, \tag{19}$$

and this leads to eq. (17).

Note that both eqs. (17) and (18) applies to arbitrary β_4 across the non-relativistic to relativistic transition. We find that the shocked regions only becomes non-relativistic when $\rho_1 \gg \Gamma_4^2\rho_4$, meaning that a weak wind running into a very dense medium. With the velocity of the shocked regions, we can then plug it back to eq. (11) to obtain the relative Lorentz factor across the reverse shock, and then it is possible to calculate the properties of the gas in region 3.

Finally, we would like to know the velocities of the shock fronts in the lab frame. A general rule from the jump conditions is that, in the comoving frame of the shocked region, the shock front is always moving away from the shocked region at a velocity $\beta'_{\rm s} = \beta_{\rm r}/3$, where $\beta_{\rm r}$ is the relative speed between the two fluids on each side of the shock (i.e., $\beta_{\rm r}$ is the speed at which the unshocked gas in the comoving frame of the shocked region). This shows that the speed of shock front, as viewed from the shocked region, is always non-relativistic $\beta'_{\rm s} < 1/3$ (as $\beta_{\rm r} < 1$).

Let us apply the above rule to the forward shock. In this case, $\beta_{\rm r,fs} = \beta$, so we obtain $\beta'_{\rm fs} = \beta/3$ in the comoving frame of region 2. In the lab frame, the speed and Lorentz factor of the forward shock are then given by

$$\beta_{\rm fs} = \frac{\beta'_{\rm fs} + \beta}{1 + \beta'_{\rm fs}\beta} = \frac{4\beta}{3 + \beta^2}, \quad \Gamma_{\rm fs} = \gamma'_{\rm s}\Gamma(1 + \beta'_{\rm s}\beta) = \frac{4\Gamma^2 - 1}{\sqrt{8\Gamma^2 - 1}}. \tag{20}$$

Then, for the reverse shock, we use $\beta_{\rm r,rs} = (\beta_4 - \beta)/(1 - \beta_4\beta)$ and the shock speed in the comoving frame of region 3, $\beta'_{\rm rs} = \beta_{\rm r,rs}/3$, and then the lab frame speed and Lorentz factor of the reverse shock front are given by

$$\beta_{\rm rs} = \frac{\beta - \beta'_{\rm rs}}{1 - \beta'_{\rm rs}\beta} = \frac{3\beta - \beta_{\rm r,rs}}{3 - \beta\beta_{\rm r,rs}}, \quad \Gamma_{\rm rs} = (1 - \beta'_{\rm rs}\beta)\Gamma'_{\rm rs}\Gamma. \tag{21}$$

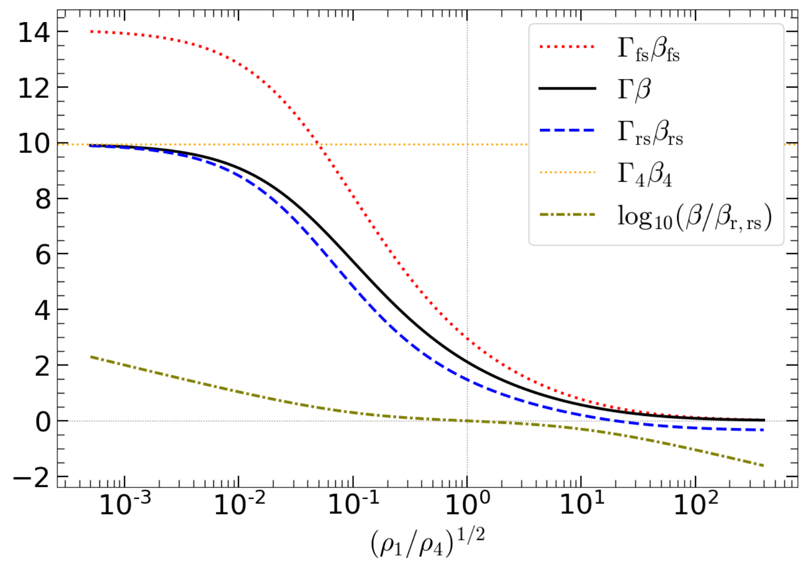

Figure 2: Four-velocities of the forward shock front, shocked regions, and reverse shock front. The Lorentz factor of the wind is fixed to be $\Gamma_4 = 10$. When $(\rho_1/\rho_4)^{1/2} \gtrsim 2\Gamma_4 = 20$, we see that $\beta_{\rm rs}$ becomes negative and the reverse shock front is propagating backwards in the lab frame, and this is because $\beta < 1/3 \approx \beta_{\rm r,rs}/3$ (since $\beta_{\rm r,rs} \approx 1$) and hence $\beta_{\rm rs} < 0$.

When the shocked regions have a high speed $\beta > 1/3$, we see that $\beta_{\mathrm{rs}} > 0$ and hence the reverse shock front always moving in the same direction as the wind (β_4). However, when the shocked regions have a very low speed (meaning that the wind is running into a very dense medium), it is possible that $\beta_{\mathrm{rs}} < 0$ and hence the reverse shock front may propagate backwards. These speeds are shown in Fig. 2.

Finally, we would like to know which of the two shocks dissipates more energy per unit time. To make this comparison, we notice that the thermal energy density in the two shocked regions are comparable (as $P_2 = P_3$), so the ratio between the thermal energies in these two regions is roughly given by the volume ratio. For a one-dimensional system, we would then look at the expansion rate of regions 2 and 3.

In the comoving frame of the shocked regions, the volume of region 2 is growing at a rate that is proportional to $\beta'_{\mathrm{fs}} = \beta/3$ whereas the growth rate of the volume of region 3 is proportional to $\beta'_{\mathrm{rs}} = \beta_{\mathrm{r,rs}}/3$. Thus, we are interested in the following ratio

$$\frac{E_2}{E_3} \simeq \frac{\beta}{\beta_{\mathrm{r,rs}}} = \frac{1 + a^{-1}/\Gamma_4}{1 + a/\Gamma_4}, \quad a \equiv (\rho_1/\rho_4)^{1/2}\,, \tag{22}$$

where we have used eqs. (12) and (17). When $\rho_1 = \rho_4$ (or $a = 1$), we obtain $\beta = \beta_{\mathrm{r,rs}}$ and hence the energies in regions 2 and 3 are equal to each other (this can also be seen from eq. 19). When $\rho_1 > \rho_4$ (a lighter wind running into a dense medium), we have $\beta < \beta_{\mathrm{r,rs}}$ or $E_2 < E_3$ and hence the reverse shock dissipates more energy than the forward shock. For the case of $\Gamma_4 = 10$, the ratio $\beta/\beta_{\mathrm{r,rs}}$ is shown by the dark green line in Fig. 2.

If we stare at eq. (22), it is possible to find that $E_2/E_3 \sim 1$ as long as $\Gamma_4^{-1} \ll (\rho_1/\rho_4)^{1/2} \ll \Gamma_4$. In the ultra-relativistic limit ($\Gamma_4 \gg 1$), we find that the two shocked regions often have comparable amount of energies, unless the density contrast between regions 1 and 4 is extremely large. However, in the non-relativistic limit ($\Gamma_4 \approx 1$), we find $E_2/E_3 \simeq 1/a = (\rho_1/\rho_4)^{-1/2}$, which means that the two shocked regions can have very different energies even for moderate density contrast between the wind and surrounding medium (the shock propagating to the less dense gas dissipates nearly all the energy).

References

[Uhm11] Z. Lucas Uhm. "A Semi-analytic Formulation for Relativistic Blast Waves with a Long-lived Reverse Shock". In: *ApJ* 733.2, 86 (June 2011), p. 86. DOI: 10.1088/0004-637X/733/2/86. arXiv: 1003.1115 [astro-ph.HE].

[PCS15] Jaehong Park, Damiano Caprioli, and Anatoly Spitkovsky. "Simultaneous Acceleration of Protons and Electrons at Nonrelativistic Quasiparallel Collisionless Shocks". In: *Phys. Rev. Lett* 114.8, 085003 (Feb. 2015), p. 085003. DOI: 10.1103/PhysRevLett.114.085003. arXiv: 1412.0672 [astro-ph.HE].