Isotropic independent random walks

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Abstract

This note shows that the probability distribution of the position of a random walker after $N(\gg 1)$ steps is an d-dimensional Gaussian, provided that the steps are uncorrelated and isotropic in d dimensions.

General Consideration

At each step, a walker moves by $\boldsymbol{r}$ which is randomly drawn from a probability distribution $p(\boldsymbol{r})$. This probability distribution is assumed to be isotropic and hence $p(\boldsymbol{r})$ only depends on the magnitude of $r = |\boldsymbol{r}|$, and it has been normalized such that $\int p(\boldsymbol{r})\,\mathrm{d}^d\boldsymbol{r} = 1$, where $\mathrm{d}^d\boldsymbol{r}$ is the d-dimensional volume element. The probability distribution $p(\boldsymbol{r})$ can be an arbitrary scalar function, and the results in this note apply as long as the mean-squared stepsize is well defined

$$\langle r^2 \rangle = \int r^2\, p(\boldsymbol{r})\,\mathrm{d}^d\boldsymbol{r}. \tag{1}$$

Suppose the probability distribution of the walker's position $\boldsymbol{R}$ after N steps is $P_N(R)$. Then, the following recurrence relation must hold

$$P_{N+1}(\boldsymbol{R}) = \int p(\boldsymbol{r}) P_N(\boldsymbol{R}-\boldsymbol{r})\,\mathrm{d}^d\boldsymbol{r}. \tag{2}$$

In the limit $N \gg 1$, we expect that $P_N(\boldsymbol{R})$ varies on lengthscales much longer than the typical step size $\boldsymbol{r}$, so it is reasonable to Taylor expand the $P_N(\boldsymbol{R}-\boldsymbol{r})$ term as follows

$$P_N(\boldsymbol{R}-\boldsymbol{r}) \approx P_N(\boldsymbol{R}) - \sum_i r_i \partial_{R_i} P_N(\boldsymbol{R}) + \frac{1}{2}\sum_{ij} r_i r_j \partial_{R_i}\partial_{R_j} P_N(\boldsymbol{R}) + \ldots, \tag{3}$$

where i and j goes over all dimensions. The linear term does not contribute to the integral on the RHS of eq. (2) because $p(\boldsymbol{r})$ is forward-backward symmetric along any axis $i = 1, 2, \ldots, d$. As for the quadratic term involving $\sum_{ij} r_i r_j$, we only need to consider the terms with $i = j$ because each step is isotropic (meaning that there is no correlation between i and j directions if $i \neq j$). Thus, we obtain $\int p(\boldsymbol{r}) \left(\sum_{ij} r_i r_j\right) \mathrm{d}^d\boldsymbol{r} = \int p(\boldsymbol{r}) \left(\sum_i r_i^2\right) \mathrm{d}^d\boldsymbol{r} = \langle r^2\rangle$. We also know that $P_N(\boldsymbol{R})$ is isotropic along all axes $i = 1, 2, \ldots, d$, so we write $\partial^2_{R_i} P_N(\boldsymbol{R}) = d^{-1}\sum_i \partial^2_{R_i} P_N(\boldsymbol{R}) = d^{-1}\nabla^2 P_N(\boldsymbol{R})$. Therefore, the probability distribution at step $N+1$ is given by

$$P_{N+1}(\boldsymbol{R}) = P_N(\boldsymbol{R}) + \frac{\langle r^2\rangle}{2d}\nabla^2 P_N(\boldsymbol{R}). \tag{4}$$

In the limit $N \gg 1$, we expect that $P_N(\boldsymbol{R})$ varies on timescales much longer than the typical time step Δt, so we can approximate $P_N(\boldsymbol{R})$ as a continuous, time-dependent probability distribution $\rho(\boldsymbol{R}, t)$, which satisfies the following diffusion equation

$$\partial_t \rho(\boldsymbol{R}, t) = D\nabla^2 \rho\,(\boldsymbol{R}, t)\,, \quad D = \frac{\langle r^2\rangle}{2d\Delta t}. \tag{5}$$

Here D is called a time-independent diffusion coefficient.

For a given initial condition $\rho(\boldsymbol{R}, t=0) = \delta(\boldsymbol{R})$, the above diffusion equation can be solved by separation of variables. It can be shown that $\rho(\boldsymbol{R}, t)$ has the following d-dimensional Gaussian form

$$\rho(\boldsymbol{R}, t) \propto \exp\left[-\frac{R^2}{2\sigma^2(t)}\right], \quad \sigma^2 = 2Dt = \frac{\langle r^2\rangle}{d\Delta t}t = \frac{\langle r^2\rangle}{d}N, \tag{6}$$

and it is normalized such that $\int \rho \, \mathrm{d}^d\boldsymbol{R} = 1$. Here $\Delta t = t/N$ is the time step. We see that the characteristic radius of the probability density d-dimensional "cloud" grows as the square root of time t or the number of steps N, i.e., $\sigma \propto t^{1/2} \propto N^{1/2}$. Sometimes, we also want to know the PDF for the distance from the origin R and it is given by

$$\rho(R, t) = A_d R^{d-1} \rho(\boldsymbol{R}, t), \tag{7}$$

where $A_d = 2\pi^{d/2}/\Gamma(d/2)$ (where $\Gamma(x)$ is the Gamma function) is the surface area of the unit sphere in d dimensions ($A_1 = 2, A_2 = 2\pi, A_3 = 4\pi, A_4 = 2\pi^2, \ldots$).

Example — 2D

For $d = 2$, we obtain the *Rayleigh distribution*

$$P_N(\boldsymbol{R}) = \frac{1}{\pi N\langle r^2\rangle}\exp\left(-\frac{R^2}{N\langle r^2\rangle}\right), \quad P_N(R) = \frac{2R}{N\langle r^2\rangle}\exp\left(-\frac{R^2}{N\langle r^2\rangle}\right). \tag{8}$$

Let us further consider various moments of the PDF of the distance R,

$$\langle R^n\rangle = \int_0^\infty P_N(R) R^n \, \mathrm{d}R = \Gamma(n/2+1)\left(N\langle r^2\rangle\right)^{n/2}, \tag{9}$$

The average distance from the origin is $\langle R\rangle = 2^{-1}\sqrt{\pi N\langle r^2\rangle}$. The variance of the distance to the origin is

$$\sqrt{\langle (R - \langle R\rangle)^2\rangle} = \sqrt{\langle R^2\rangle - \langle R\rangle^2} = \sqrt{1 - \frac{\pi}{4}}\left(N\langle r^2\rangle\right)^{1/2}, \tag{10}$$

which is of the same order as the average distance to the origin (meaning that the uncertainty in R is quite large). Another interesting result is

$$\langle (R^2 - \langle R^2\rangle)^2\rangle = \langle R^4\rangle - \langle R^2\rangle^2 = \langle R^2\rangle^2 = \left(N\langle r^2\rangle\right)^2, \tag{11}$$

which means that the variance in the intensity (which is proportional to the square of the amplitude R) is equal to the average intensity — this is a nice property of the Rayleigh distribution.