# Polytropic Star

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**Abstract**

This note first discusses the density profile of a polytropic star. Then, it is shown that a radiative outer envelope of a star can be roughly described by a polytropic equation of state with $\gamma \simeq 4/3$. Finally, the negligible-mass outer layers of the star is studied.

## Lane-Emden Equation

Defining $\rho = \rho_c \theta^n$, $\gamma = 1 + 1/n$, $P = K\rho^\gamma = K\rho_c^{1+1/n}\theta^{n+1}$, $\alpha^2 = (n+1)K\rho_c^{1/n-1}/(4\pi G)$, and $\xi = r/\alpha$, the equation of hydrostatic equilibrium ($\mathrm{d}P/\mathrm{d}r = -4\pi G\rho m(r)/r^2$) and mass continuity ($\mathrm{d}m/\mathrm{d}r = 4\pi\rho r^2$) can be written in the following Lane-Emden form

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{1}$$

The boundary conditions are $\theta(\xi = 0) = 1$ and $\mathrm{d}\theta/\mathrm{d}\xi(\xi = 0) = 0$ (since $\mathrm{d}P/\mathrm{d}r = 0$ at $r = 0$).

The above equation can be solved by defining a vector $\vec{X} = (\theta, \phi)^\mathrm{T}$, where $\phi \equiv -\xi^2\mathrm{d}\theta/\mathrm{d}\xi$. The spatial gradient of $\vec{X}$ is given by

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}\xi} = (-\phi/\xi^2, \xi^2\theta^n)^\mathrm{T}, \tag{2}$$

and the boundary condition is $\vec{X}(\xi = 0) = (1, 0)$. We integrate the above vector equation from $\xi = 0$ to a maximum $\xi_\mathrm{max}$ where $\theta$ has decreased to 0. It can be seen that $\phi > 0$ is a monotonically increasing function of $\xi$ and $\theta(\xi)$ is a monotonically decreasing function.

It should be noted that since $\mathrm{d}\phi/\mathrm{d}\xi = \xi^2\theta^n \propto r^2\rho \propto \mathrm{d}m/\mathrm{d}r$, $\phi(\xi)$ is proportional to the mass coordinate:

$$\phi(\xi) = \frac{m(r)}{4\pi\alpha^3\rho_c}. \tag{3}$$

Once the functions $\theta(\xi)$ and $\phi(\xi)$ have been numerically obtained (including the value of $\xi_\mathrm{max}$), we can solve for the two constants $\rho_c$ and $K$ from the physical stellar mass $M$ and radius $R$ by

$$\phi_\mathrm{max} \equiv \phi(\xi_\mathrm{max}) = \frac{M}{4\pi\alpha^3\rho_c},\ \alpha\xi_\mathrm{max} = R. \tag{4}$$

The results are

$$\rho_c = \frac{M/\phi_\mathrm{max}}{4\pi(R/\xi_\mathrm{max})^3},\ K = \frac{G}{n+1}(4\pi)^{1/n}(M/\phi_\mathrm{max})^{1-1/n}(R/\xi_\mathrm{max})^{3/n-1}. \tag{5}$$

The central pressure is then given by

$$P_c = \frac{1}{n+1}\frac{G(M/\phi_\mathrm{max})^2}{4\pi(R/\xi_\mathrm{max})^4}. \tag{6}$$

For a constant $K$, we obtain a mass-radius relation

$$R \propto M^{(n-1)/(n-3)}, \quad \text{for } K = \text{const.} \tag{7}$$

Such a mass-radius relation applies to low-mass white dwarfs that are supported by non-relativistic degeneracy pressure as $K$ is related to fundamental constants. Another interesting situation is that, when a star with a convective envelope (where $K$ is uniform and $\gamma = 5/3$ or $n = 1.5$) undergoes mass loss on a timescale shorter than the thermal timescale of its envelope, we expect $K$ to remain constant despite the adjustment of the remaining star to reach hydrostatic equilibrium (as $K$ only evolves on a thermal timescale) and the mass-radius relation predicts $R \propto M^{-1/3}$. We find that a star with a convective envelope expands as it loses mass rapidly on timescales shorter than the thermal time.

The moment of inertia of a spherical shell of thickness d$r$ is $\mathrm{d}I = (8\pi/3)\rho(r)r^4\mathrm{d}r$, which can be obtained by differentiating $I = (2/5)MR^2$ for a uniform sphere. Then, the moment of inertia of the entire star is given by

$$I = \frac{8\pi}{3}\rho_c\alpha^5\tilde{I} = \frac{2}{3}\frac{MR^2}{\phi_{\max}\xi_{\max}^2}\tilde{I}, \tag{8}$$

where

$$\tilde{I} \equiv \int_0^{\xi_{\max}} \theta^n\xi^4\mathrm{d}\xi. \tag{9}$$

It is common to define a convenient constant

$$k = (2/3)\tilde{I}/(\phi_{\max}\xi_{\max}^2), \tag{10}$$

such that $I = kMR^2$.

The central potential of the star is given by the sum of the contribution from all spherical shells[1]

$$\Phi_c = -G\int_0^R \frac{\mathrm{d}m}{r} = -4\pi\alpha^2 G\rho_c \int_0^{\xi_{\max}} \frac{\mathrm{d}\phi}{\xi} = -\frac{GM}{R}\frac{\xi_{\max}}{\phi_{\max}}\tilde{\Phi}_c, \tag{11}$$

where we have used $\mathrm{d}\phi/\mathrm{d}\xi = \xi^2\theta^n$, $M/R = 4\pi\alpha^2\rho_c\phi_{\max}/\xi_{\max}$, and

$$\tilde{\Phi}_c = \int_0^{\xi_{\max}} \theta^n\xi\mathrm{d}\xi. \tag{12}$$

Polynomial fits to the numerical results are

$$\begin{aligned}
\xi_{\max} &\approx 0.05583n^4 - 0.2165n^3 + 0.5656n^2 + 0.1914n + 2.544,\\
\phi_{\max} &\approx 0.02393n^4 - 0.2363n^3 + 0.989n^2 - 2.402n + 4.767,\\
k = I/(MR^2) &\approx 2.112\times10^{-4}n^4 - 1.969\times10^{-3}n^3 + 0.02006n^2 - 0.1561n + 0.3992,\\
\Phi_c/(-GM/R) &\approx 0.03297n^4 - 0.1246n^3 + 0.3245n^2 + 0.2092n + 1.557,
\end{aligned} \tag{13}$$

[1]Remember that the potential inside a spherical shell of mass d$m$ and radius $r$ is constant and equal to the surface value of $-G\mathrm{d}m/r$.

which are accurate to fractional errors $< 0.2\%$ for $n \in (0.5, 3)$ (the above $k$ fit has maximum fractional error of $2 \times 10^{-4}$).

The gravitational potential energy of the star is given by

$$U = -G \int_0^M \frac{m(r)\mathrm{d}m}{r} = -\frac{GM^2}{R} \frac{\xi_{\max}}{\phi_{\max}^2} \tilde{U}, \tag{14}$$

where

$$\tilde{U} = \int_0^{\xi_{\max}} \phi(\xi) \theta^n \xi \mathrm{d}\xi. \tag{15}$$

The specific thermal energy is given by $e = P/[\rho(\gamma - 1)]$, so the total thermal energy is

$$T = \int_0^M e\, \mathrm{d}m = \frac{n}{n+1} \frac{GM^2}{R} \frac{\xi_{\max}}{\phi_{\max}^2} \tilde{T}, \tag{16}$$

where

$$\tilde{T} = \int_0^{\xi_{\max}} \theta^{n+1} \xi^2 \mathrm{d}\xi. \tag{17}$$

Miraculously, the condition of hydrostatic equilibrium dictates the potential and thermal energies of a star to be (cf. Chandrasekhar)

$$U = -\frac{3}{5-n} \frac{GM^2}{R},\ \ T = \frac{n}{5-n} \frac{GM^2}{R} \ \ \Rightarrow\ \ \frac{3T}{n} + U = 0 \text{ (virial equilibrium)}. \tag{18}$$

The total binding energy of a star is (physical systems correspond to $n < 3$)

$$U + T = -\frac{3-n}{5-n} \frac{GM^2}{R}. \tag{19}$$

Finally, let us consider how the star's radius responds as it loses mass rapidly such that the entropy profile of the remaining layers stays unchanged. Fixing the polytropic index $n$ and enforcing $\mathrm{d}K = 0$, we obtain

$$\frac{\mathrm{d}\ln R}{\mathrm{d}\ln M} = \frac{n-3}{n-1} = \frac{2-\gamma}{4-3\gamma}. \tag{20}$$

## Radiative Envelope where $m(r) \approx M$ and $L(r) \approx L$

The polytropic equation of state works well for a convective envelope where the entropy is constant — and we have $P \propto \rho^{5/3}$ across the entire convective envelope.

However, for a radiative envelope, one needs to consider the radiative diffusion

$$\frac{L}{4\pi r^2} = -\frac{c}{3\rho\kappa} \frac{\mathrm{d}(aT^4)}{\mathrm{d}r}. \tag{21}$$

This should be combined with the equation for hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{GM}{r^2}. \tag{22}$$

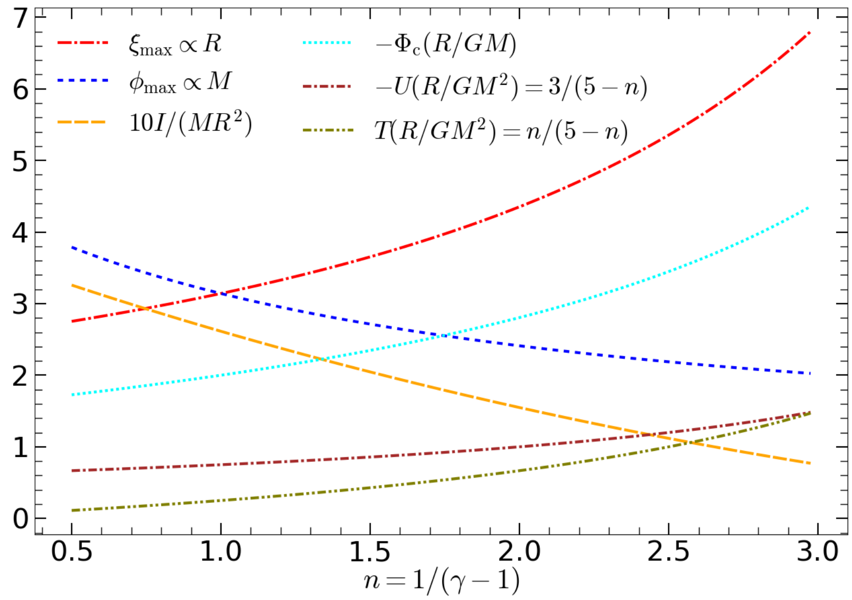


Fig. 1.— The integral quantities $\xi_{\rm max}$ (related to stellar radius), $\phi_{\rm max}$ (related to stellar mass), $k \equiv I/(MR^2)$ (related to the moment of inertia), $\Phi_{\rm c}$ (central potential), $U$ (gravitational potential energy), and $T$ (thermal energy), for a polytropic star. Here, we show $10k$ (instead of $k$) for clarity.

Let us assume that the pressure is dominated by the gas (instead of radiation), which means $P \simeq \rho kT/(\mu m_{\rm p})$. This allows us to elliminate $\rho$ in the above two equations. Taking the ratio between these two equations, we obtain

$$\frac{T^3}{\kappa}\frac{{\rm d}T}{{\rm d}P} = \frac{3L}{16\pi acGM}. \tag{23}$$

Since we are considering the outer envelope, the right-hand side of the above equation is nearly a constant.

Let us then consider a particular form of Rosseland-mean opacity $\kappa = \kappa_0 P^q T^{-q-s}$, and for Kramer's law, one has $q = 1$ and $s = 3.5$. Then, the relation between $T$ and $P$ is given by

$${\rm d}T^{4+q+s} = C\,{\rm d}P^{q+1},\ C = \frac{4+q+s}{q+1}\frac{3\kappa_0 L}{16\pi acGM}. \tag{24}$$

One can integrate the above equation from the outer boundary (near the photosphere) where $T = T_{\rm ph}$ and $P = P_{\rm ph}$. Then, in the regions far below the photosphere where $T \gg T_{\rm ph}$ and $P \gg P_{\rm ph}$, one obtains the following

$$T^{4+q+s} = C\,P^{q+1}, \tag{25}$$

which means (for the gas pressure dominated case)

$$P \propto \rho^{1+(q+1)/(s+3)}. \tag{26}$$

For Kramer's law ($q = 1$ and $s = 3.5$), we have $P \propto \rho^{1+4/13\approx 1.308}$ or $n = 13/4 \approx 3.25$, which is very close to the case of a $\gamma = 4/3$ (or $n = 3$) polytrope. The proportional constant $K = P/\rho^\gamma$

depends on the stellar luminosity and mass. This applies to stars on the upper main-sequence with masses greater than about $1M_\odot$ (for lower-mass stars, there is a substantial convective envelope due to higher opacity in the outer layers).

In the extreme limit where the pressure is dominated by radiation $P \simeq aT^4/3$ (although this is usually not fully realized), then the radiative transfer equation can be written as

$$\rho \frac{L}{4\pi r^2} = -\frac{c}{\kappa}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{27}$$

Taking the ratio between the above equation and the hydrostatic equilibrium equation, one obtains

$$L = \frac{4\pi GMc}{\kappa} = L_{\rm Edd}. \tag{28}$$

In order to maintain locally Eddington luminosity everywhere, the opacity must be nearly constant over the envelope. This can be achieved if $\kappa$ is dominated by electron scattering.

## Structure of the surface layers where $r \approx R$

The structure of the outermost layers of the star is very simple when we ignore the self-gravity. From hydrostatic equilibrium and the equation of state, we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2}, P = K\rho^\gamma \;\Rightarrow\; \mathrm{d}\rho^{\gamma-1} = \frac{(\gamma-1)GM}{\gamma K}\mathrm{d}r^{-1}. \tag{29}$$

This can be easily integrated using the boundary condition of $\rho(r = R) = 0$,

$$\rho^{\gamma-1}(r) = \frac{(\gamma-1)GM}{\gamma K}\left(\frac{1}{r} - \frac{1}{R}\right), \text{ or } \rho(r) = \left[\frac{GM}{(n+1)K}\left(\frac{1}{r} - \frac{1}{R}\right)\right]^n. \tag{30}$$

The pressure profile is given by

$$P(r) = K\left[\frac{GM}{(n+1)K}\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{n+1}. \tag{31}$$

Using $K$ in eq. (5), we obtain

$$P(r) = \frac{GM^2}{4\pi R^4}\frac{(R/r-1)^{n+1}}{n+1}\phi_{\rm max}^{n-1}\xi_{\rm max}^{3-n}. \tag{32}$$

This means that the pressure scaleheight near the surface is of the order $H \sim R - r$ (for $r \approx R$). Another consequence is that the isothermal sound speed $c_{\rm s} \equiv \sqrt{P/\rho}$ has the following simple form

$$c_{\rm s}^2 = K\rho^{\gamma-1} = \frac{GM\,(R-r)}{(n+1)rR}. \tag{33}$$

The exterior mass is given by

$$\begin{aligned} M_{\rm ex}(r) &= \int_r^R 4\pi r^2 \rho(r)\mathrm{d}r \\ &= 4\pi\left(\frac{(\gamma-1)GM}{\gamma K}\right)^{\frac{1}{\gamma-1}} R^{\frac{3\gamma-4}{\gamma-1}}\int_{r/R}^1 x^{\frac{2\gamma-3}{\gamma-1}}(1-x)^{\frac{1}{\gamma-1}}\mathrm{d}x, \end{aligned} \tag{34}$$

or

$$M_{\rm ex}(r) = 4\pi \left( \frac{GM}{(n+1)K} \right)^n R^{3-n} \left[ B(3-n, n+1, 1) - B(3-n, n+1, r/R) \right], \tag{35}$$

where $B(a,b,x) = \int_0^x t^{a-1}(1-t)^{b-1}{\rm d}t$ is the incomplete Beta-function. Using the entropy constant $K$ in eq. (5), we obtain

$$\frac{M_{\rm ex}(r)}{M} = \phi_{\rm max}^{n-1}\xi_{\rm max}^{3-n} \left[ B(3-n, n+1, 1) - B(3-n, n+1, r/R) \right]. \tag{36}$$

In the limit $r/R \approx 1$, this can be further simplified into

$$\frac{M_{\rm ex}(r)}{M} \approx \phi_{\rm max}^{n-1}\xi_{\rm max}^{3-n} \frac{(1-r/R)^{n+1}}{n+1}, \text{ for } 1 - r/R \ll 1. \tag{37}$$

In the same limit, the pressure profile is given by

$$\frac{P(r)}{\bar{P}} \approx \phi_{\rm max}^{n-1}\xi_{\rm max}^{3-n} \frac{(1-r/R)^{n+1}}{n+1}, \text{ for } 1 - r/R \ll 1, \tag{38}$$

where $\bar{P} \equiv GM^2/(4\pi R^4)$ is a rough estimate of the mean pressure inside the star. Thus, we have arrived at the following interesting result

$$\frac{P(r)}{\bar{P}} = \frac{M_{\rm ex}(P)}{M}, \tag{39}$$

where $M_{\rm ex}(P)$ is the mass exterior to a critical radius specified by a given pressure $P$.