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L(A) = L(B)? A simplified decidability proof

Géraud Sénizergues

LaBRI and UFR Math-info, Université Bordeaux1, 351 Cours de la libération -33405- Talence Cedex, France

Abstract

We give a proof of decidability of the equivalence problem for deterministic pushdown automata, which simplifies that of Sénizergues (Theoret. Comput. Sci. 251 (2000) 1). © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Deterministic pushdown automata; Rational languages; Finite dimensional vector spaces; Matrix semi-groups; Complete formal systems

1. Introduction

1.1. Result

The so-called "equivalence problem for deterministic pushdown automata", is the following decision problem:

INSTANCE: two dpda A, B

QUESTION: $\mathrm{L(A) = L(B)}$?

i.e. do the given automata recognize the *same* language? This work consists of a simplified proof of the following

Theorem 93. *The equivalence problem for deterministic pushdown automata is decidable.*

This result was proved in [8; 11, Sections 1–9] and has been generalised in [11, Section 11; 9, 10] to other decision problems. Some simplifications of the method presented in

Fax: 05-56-84-66-69.

E-mail address: ges@labri.u-bordeaux.fr (G. Sénizergues).

URL: http://dept-info.labri.u-bordeaux.fr/∼ges/

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[8] have been found by Stirling in [12,13]. Nevertheless, Stirling was not using the same framework nor style. We hope this new exposition of the ideas from [11] and [13] will combine the qualities of both works.

Logics. Our solution consists in constructing a *complete* formal system, in the general sense taken by this word in mathematical logics i.e.: it consists of a set of well-formed assertions, a subset of basic assertions, the axioms, and a set of deduction rules allowing to derive new assertions from assertions which are already generated. The well-formed assertions we are considering are pairs (S, T) of rational boolean series over the non-terminal alphabet V of some strict-deterministic grammar $G = \langle X, V, P \rangle$. Such an assertion is true when the two series S, T generate the same language over the terminal alphabet X, via the rules of G.

1.2. Main tools

We still use the notions developed in [8,11] (1–4)

(1) the *deduction systems* (which were in turn inspired by [1]);
(2) the *deterministic boolean series* (which were in turn inspired by [3]);
(3) the notion of *linear independence* for such series (which is based on the ideas from [4,5]);
(4) the *analysis* of the proof-trees generated by a suitable strategy (which was somehow similar with the analysis of the parallel computations, interspersed with replacement-moves, done in [14,7,6]).

We bring into this framework some simplifications found by Stirling [12,13] (5–8):

(5) the technical notion of "N-stacking sequence" is replaced by the slightly simpler notion of "B-stacking sequence";
(6) the analysis of Section 8 uses a choice of "generating set" which is simpler than the choice given in [11,9];
(7) a main simplification linked with this more clever choice, is that one can restrict the proof to the case of a *proper*, *reduced* strict-deterministic grammar;
(8) some "second level" undeterminates corresponding bijectively to the deterministic rational series of [8,11] are introduced; this trick allows to treat polynomials instead of general rational series in the crucial part of the proof (Section 8).

1.3. Contents

We recall in Section 2 some basic definitions concerning automata, grammars, monoids and semi-rings.

We introduce in Section 3 a notion of *rational deterministic* series, vectors and matrices; we then develop their basic algebraic properties. The set of these series can be seen as a set of *automata-configurations* endowed with nice algebraic operations.

We introduce in Section 4 a kind of logical formal system and define a particular system $\mathscr{D}_0$ allowing to deduce equivalences between series.

In Section 5, we define a *triangulation process* for linear equations over series.

In Section 6 we give an overview of the *constants* used throughout the paper.

Section 7 is devoted to the definition of *strategies* allowing to construct a formal equivalence-proof from a given pair of equivalent series.

In Section 8 we analyze the trees produced by the strategies of Section 7.

In Section 9 we prove the formal system $\mathscr{D}_0$ is in fact *complete*; this implies that the equivalence problem for dpda is decidable.

In Section 10 we show that the infinite set of "second-level" undeterminates can be safely eliminated from the formal system (though they were useful to ease the first completeness proof).

2. Preliminaries

2.1. Pushdown automata

A *pushdown automaton* on the alphabet X is a 7-tuple $\mathscr{M} = \langle X, Z, Q, \delta, q_0, z_0, F\rangle$ where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol, F is a finite subset of QZ^*, the set of *final* configurations, and δ, the transition function, is a mapping $\delta : QZ \times (X \cup \{\varepsilon\}) \to \mathscr{P}_f(QZ^*)$.

Let $q, q' \in Q$, $\omega, \omega' \in Z^*$, $z \in Z$, $f \in X^*$ and $a \in X \cup \{\varepsilon\}$; we note $(qz\omega, af) \mapsto_{\mathscr{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\mapsto_{\mathscr{M}}^*$ is the reflexive and transitive closure of $\mapsto_{\mathscr{M}}$.

For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \xrightarrow{f}_{\mathscr{M}} q'\omega'$ iff $(q\omega, f) \mapsto_{\mathscr{M}}^* (q'\omega', \varepsilon)$.

$\mathscr{M}$ is said *deterministic* iff, for every $z \in Z$, $q \in Q$, $x \in X$:

$$\mathrm{Card}(\delta(qz, \varepsilon)) \in \{0, 1\}, \tag{1}$$

$$\mathrm{Card}(\delta(qz, \varepsilon)) = 1 \;\Rightarrow\; \mathrm{Card}(\delta(qz, x)) = 0, \tag{2}$$

$$\mathrm{Card}(\delta(qz, \varepsilon)) = 0 \;\Rightarrow\; \mathrm{Card}(\delta(qz, x)) \leqslant 1. \tag{3}$$

$\mathscr{M}$ is said *real-time* iff, for every $q \in Q$, $z \in Z$, $\mathrm{Card}(\delta(qz, \varepsilon)) = 0$.

A configuration $q\omega$ of $\mathscr{M}$ is said ε-bound iff there exists a configuration $q'\omega'$ such that $(q\omega, \varepsilon) \mapsto_{\mathscr{M}} (q'\omega', \varepsilon)$; $q\omega$ is said ε-free iff it is not ε-bound.

A pda $\mathscr{M}$ is said *normalized* iff, it fufills conditions (1), (2) (see above) and (4)–(6):

$$q_0 z_0 \quad \text{is } \varepsilon\text{-free}, \; F \subseteq Q \tag{4}$$

and for every $q \in Q$, $z \in Z$, $x \in X$:

$$q'\omega' \in \delta(qz, x) \;\Rightarrow\; |\omega'| \leqslant 2, \tag{5}$$

$$q'\omega' \in \delta(qz, \varepsilon) \;\Rightarrow\; |\omega'| = 0. \tag{6}$$

The *language recognized* by $\mathscr{M}$ is

$$\mathrm{L}(\mathscr{M}) = \{w \in X^* \mid \exists c \in F, \; q_0 z_0 \xrightarrow{w}_{\mathscr{M}} c\}.$$

It is a "folklore" result that, given a deterministic pda $\mathscr{M}$, one can effectively compute another dpda $\mathscr{M}'$ which is normalized and fulfills:

$$\mathrm{L}(\mathscr{M}) = \mathrm{L}(\mathscr{M}') - \{\varepsilon\}.$$

2.2. Deterministic context-free grammars

Let $\mathscr{M}$ be some deterministic pushdown automaton (we suppose here that $\mathscr{M}$ is normalized). The *variable* alphabet $V_{\mathscr{M}}$ associated to $\mathscr{M}$ is defined as

$$V_{\mathscr{M}} = \{[p,z,q] \mid p,q \in Q,\ z \in Z\}.$$

The *context-free* grammar $G_{\mathscr{M}}$ associated to $\mathscr{M}$ is then

$$G_{\mathscr{M}} = \langle X, V_{\mathscr{M}}, P_{\mathscr{M}} \rangle,$$

where $P_{\mathscr{M}}$ is the set of all the pairs of one of the following forms:

$$([p,z,q], x[p',z_1,p''][p'',z_2,q]), \tag{7}$$

where $p,q,p',p'' \in Q,\ x \in X,\ p'z_1z_2 \in \delta(pz,x)$

$$([p,z,q], x[p',z',q]), \tag{8}$$

where $p,q,p' \in Q,\ x \in X,\ p'z' \in \delta(pz,x)$

$$([p,z,q], a), \tag{9}$$

where $p,q, \in Q,\ a \in X \cup \{\varepsilon\},\ q \in \delta(pz,a)$. $G_{\mathscr{M}}$ is a *strict-deterministic* grammar (see Definition 313). A general theory of this class of grammars is exposed in [2] and used in [3].

2.3. Free monoids acting on semi-rings

Semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$. Let $(\mathsf{B},+,\cdot,0,1)$ where $\mathsf{B}=\{0,1\}$ denote the semi-ring of "booleans". Let W be some alphabet. By $(\mathsf{B}\langle\langle W\rangle\rangle,+,\cdot,\emptyset,\varepsilon)$ we denote the semi-ring of *boolean series* over W:
the set $\mathsf{B}\langle\langle W\rangle\rangle$ is defined as B^{W^*}; the sum and product are defined as usual; each word $w \in W^*$ can be identified with the element of B^{W^*} mapping the word w on 1 and every other word $w' \neq w$ on 0; every boolean series $S \in \mathsf{B}\langle\langle W\rangle\rangle$ can then be written in a unique way as

$$S = \sum_{w \in W^*} S_w \cdot w,$$

where, for every $w \in W^*$, $S_w \in \mathsf{B}$.

The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0\}.$$

In the particular case where the semi-ring of coefficients is B (which is the only case considered in this article) we sometimes identify the series S with its support. A series

$S \in \mathsf{B}\langle\langle W\rangle\rangle$ is called a boolean *polynomial* over W if and only if its support is *finite*. The set of all boolean polynomials over W is denoted by $\mathsf{B}\langle W\rangle$.

We recall that for every $S \in \mathsf{B}\langle\langle W\rangle\rangle$, S^* is the series defined by

$$S^* = \sum_{0 \leqslant n} S^n. \tag{10}$$

Given two alphabets W, W', a map $\psi : \mathsf{B}\langle\langle W\rangle\rangle \to \mathsf{B}\langle\langle W'\rangle\rangle$ is said σ-*additive* iff it fulfills: for every denumerable family $(S_i)_{i\in\mathbb{N}}$ of elements of $\mathsf{B}\langle\langle W\rangle\rangle$,

$$\psi\left(\sum_{i\in\mathbb{N}} S_i\right) = \sum_{i\in\mathbb{N}} \psi(S_i). \tag{11}$$

A map $\psi : \mathsf{B}\langle\langle W\rangle\rangle \to \mathsf{B}\langle\langle W'\rangle\rangle$ which is both a semi-ring homomorphism and a σ-additive map is usually called a *substitution*.

Actions of monoids. Given a semi-ring $(\mathsf{S}, +, \cdot, 0, 1)$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \to \mathsf{S}$ is called a *right-action* of the monoid M over the semi-ring S iff, for every $S, T \in \mathsf{S}$, $m, m' \in \mathsf{M}$:

$$\begin{aligned}&0 \circ m = 0, \quad S \circ 1_M = S, \quad (S+T)\circ m = (S\circ m) + (T \circ m) \quad \text{and}\\ &\quad S \circ (m \cdot m') = (S \circ m) \circ m'.\end{aligned} \tag{12}$$

In the particular case where $\mathsf{S} = \mathsf{B}\langle\langle W\rangle\rangle$, $\circ$ is said to be a σ-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i\in\mathbb{N}}$ of elements of S and $m \in \mathsf{M}$:

$$\left(\sum_{i\in\mathbb{N}} S_i\right) \circ m = \sum_{i\in\mathbb{N}} (S_i \circ m). \tag{13}$$

The action of W^* on $\mathsf{B}\langle\langle W\rangle\rangle$. We recall the following classical σ-right-action $\bullet$ of the monoid W^* over the semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$: for all $S, S' \in \mathsf{B}\langle\langle W\rangle\rangle$, $u \in W^*$

$$S \bullet u = S' \quad \Leftrightarrow \quad \forall w \in W^*,\ (S'_w = S_{u\cdot w})$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u, or the *residual* of S by u).

For every $S \in \mathsf{B}\langle\langle W\rangle\rangle$ we denote by $\mathsf{Q}(S)$ the set of residuals of S:

$$\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}.$$

We recall that S is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*.

The reduced grammar G_1. The classical reduced and ε-free grammar associated with $G_{\mathscr{M}}$ is $G_0 = \langle X, V_0, P_0\rangle$ where

$$\begin{aligned}&V_0 = \{v \in V_{\mathscr{M}} \mid \exists\, w \in X^+, v \xrightarrow{*}_{P_{\mathscr{M}}} w\},\\ &\varphi_0 : \mathsf{B}\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle V_0\rangle\rangle\end{aligned} \tag{14}$$

is the unique substitution such that, for every $v \in V$:

$$\varphi_0(v) = v \text{ (if } v \in V_0), \quad \varphi_0(v) = \varepsilon \text{ (if } v \xrightarrow{*}_{P_{\mathscr{M}}} \varepsilon), \quad \varphi_0(v) = \emptyset \text{ (otherwise)},$$

$$P_0 = \{(v, w') \in V_0 \times (X \cup V_0)^+ \mid v \in V_0, \exists w \in (X \cup V_{\mathscr{M}})^*, (v, w) \in P_{\mathscr{M}},$$
$$w' = \varphi_0(w)\}. \tag{15}$$

G_0 is the *reduced* and ε-*free* form of $G_{\mathscr{M}}$. It is well-known that, for all $v \in V_0$:

$$\exists w \in X^+, v \xrightarrow{*}_{P_0} w \text{ and}$$
$$\{w \in X^*, v \xrightarrow{*}_{P_{\mathscr{M}}} w\} = \{w \in X^*, v \xrightarrow{*}_{P_0} w\}.$$

For technical reasons (which will be made clear in Section 7), we introduce an alphabet of "marked variables" $\bar{V}_0$ together with a fixed bijection: $v \mapsto \bar{v}$ from V_0 to $\bar{V}_0$. Let $V_1 = V_0 \cup \bar{V}_0$. We denote by ρ_e (letter e stands here for "erasing the marks") the litteral morphism $V^* \to V_0^*$ defined by: for every $v \in V_0$,

$$\rho_e(v) = v, \quad \rho_e(\bar{v}) = v.$$

Similarly, $\bar{\rho}_e$ is the litteral morphism $V^* \to \bar{V}_0^*$ defined by: for every $v \in V_0$,

$$\bar{\rho}_e(v) = \bar{v}, \quad \bar{\rho}_e(\bar{v}) = \bar{v}.$$

We denote also by $\rho_e, \bar{\rho}_e$ the unique substitutions extending these monoid homomorphisms.

At last, the grammar G_1 is defined by, $G_1 = \langle X, V_1, P_1 \rangle$ where

$$P_1 = P_0 \cup \{(\bar{\rho}_e(v), \bar{\rho}_e(w) \mid (v, w) \in P_0\}.$$

In other words, the rules of G_1 consist of the rules of the usual proper and reduced grammar associated with $\mathscr{M}$ to-gether with their marked copies.

Let us consider the unique substitution $\varphi_1 : \mathsf{B}\langle\langle V_1 \rangle\rangle \to \mathsf{B}\langle\langle X \rangle\rangle$ fulfilling: for every $v \in V_1$,

$$\varphi_1(v) = \{u \in X^* \mid v \xrightarrow{*}_{P_1} u\},$$

(in other words, φ_1 maps every subset $L \subseteq V_1^*$ on the language generated by the grammar G_1 from the set of axioms L).

Clearly, the equivalence problem for two different dpda reduces to the equivalence problem for two different configurations of a single normalized dpda $\mathscr{M}$. This last problem reduces to the problem

$$\varphi_1(S) = \varphi_1(T)?$$

for some polynomials $S, T \in \mathsf{B}\langle V_1 \rangle$.

3. Series and matrices

3.1. Deterministic series, vectors and matrices

We introduce here a notion of *deterministic* series which, in the case of the alphabet $V_{\mathscr{M}}$ associated to a dpda $\mathscr{M}$, generalizes the classical notion of *configuration* of $\mathscr{M}$. The main advantage of this notion is that, unlike for configurations, we shall be able to define *nice algebraic operations* on these series (see, in particular, Section 3.3). Let us consider a pair $(W, \smile)$ where W is an alphabet (i.e. a set[1]) and $\smile$ is an equivalence relation over W. We call $(W, \smile)$ a *structured* alphabet. The most classical examples are:

- the case where $W = V_{\mathscr{M}}$, the variable alphabet associated to $\mathscr{M}$ and $[p, z, q] \smile [p', z', q']$ iff $p = p'$ and $z = z'$ (see [2])
- the case where $W = X$, the terminal alphabet of $\mathscr{M}$ and $x \smile y$ holds for every $x, y \in X$ (see [2]).

3.1.1. Definitions

Definition 31. Let $S \in \mathsf{B}\langle\langle W \rangle\rangle$. S is said *left-deterministic* iff either
(1) $S = \emptyset$ or
(2) $S = \varepsilon$ or
(3) $\exists\, i_0 \in [1, m],\ S_{i_0} \neq \emptyset$ and $\forall w, w' \in W^*$,

$$S_w = S_{w'} = 1 \ \Rightarrow\ [\exists A, A' \in W, w_1, w_1' \in W^*, A \smile A',$$
$$w = A \cdot w_1 \text{ and } w' = A' \cdot w_1'].$$

A left-deterministic series S is said to have the type $\emptyset$ (resp. ε, $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

Definition 32. Let $S \in \mathsf{B}\langle\langle W \rangle\rangle$. S is said *deterministic* iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.

This notion is the straighforward extension to the infinite case of the notion of (finite) *set of associates* defined in [3, Definition 32 p. 188].

We denote by $\mathsf{DB}\langle\langle W \rangle\rangle$ the subset of deterministic boolean series over W. Let us denote by $\mathsf{B}_{n,m}\langle\langle W \rangle\rangle$ the set of (n, m)-matrices with entries in the semi-ring $\mathsf{B}\langle\langle W \rangle\rangle$.

Definition 33. Let $m \in \mathbb{N}$, $S \in \mathsf{B}_{1,m}\langle\langle W \rangle\rangle$: $S = (S_1, \ldots, S_m)$. S is said *left-deterministic* iff either
(1) $\forall i \in [1, m],\ S_i = \emptyset$ or
(2) $\exists\, i_0 \in [1, m],\ S_{i_0} = \varepsilon$ and $\forall i \neq i_0,\ S_i = \emptyset$ or
(3) $\forall w, w' \in W^*,\ \forall i, j \in [1, m],\ (S_i)_w = (S_j)_{w'} = 1 \Rightarrow [\exists A, A' \in W,\ w_1, w_1' \in V^*,\ A \smile A',$ $w = A \cdot w_1$ and $w' = A' \cdot w_1']$.

[1] Notice that we do not suppose that W is finite.

A left-deterministic row-vector S is said to have the type $\emptyset$ (resp. (ε, i_0), $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

The right-action $\bullet$ on $\mathsf{B}\langle\langle W\rangle\rangle$ is extended componentwise to $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$: for every $S=(s_{i,j})$, $u\in W^*$, the matrix $T=S\bullet u$ is defined by

$$t_{i,j} = s_{i,j} \bullet u.$$

Definition 34. Let $S\in\mathsf{B}_{1,m}\langle\langle W\rangle\rangle$. S is said *deterministic* iff, for every $u\in W^*$, $S\bullet u$ is left-deterministic.

We denote by $\mathsf{DB}_{1,m}\langle\langle W\rangle\rangle$ the subset of deterministic row-vectors of dimension m over $\mathsf{B}\langle\langle W\rangle\rangle$.

Definition 35. Let $S\in\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$. S is said *deterministic* iff, for every $i\in[1,n]$, $S_{i,\cdot}$ is a deterministic row-vector.

Let us notice first an easy fact about deterministic series.

Fact 36. For every $S\in\mathsf{DB}\langle\langle W\rangle\rangle$, $u\in W^*$, $S\bullet u\in\mathsf{DB}\langle\langle W\rangle\rangle$.

Norm. Let us generalize the classical definition of *rationality* of series in $\mathsf{B}\langle\langle W\rangle\rangle$ to matrices. Given $M\in\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M:

$$\mathsf{Q}(M) = \{M\bullet u \mid u\in W^*\}.$$

Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M:

$$\mathsf{Q}_r(M) = \bigcup_{1\leqslant i\leqslant n} \mathsf{Q}(M_{i,*}).$$

M is said *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{RB}_{n,m}\langle\langle W\rangle\rangle$ (resp. $\mathsf{DRB}_{n,m}\langle\langle W\rangle\rangle$) the set of rational (resp. deterministic, rational) matrices over $\mathsf{B}\langle\langle W\rangle\rangle$. For every $M\in\mathsf{RB}_{n,m}\langle\langle W\rangle\rangle$, we define the norm of M as

$$\|M\| = \mathrm{Card}(\mathsf{Q}_r(M)).$$

3.1.2. Residuals

Lemma 37. *Let $S\in\mathsf{DB}\langle\langle W\rangle\rangle$, $T\in\mathsf{B}\langle\langle W\rangle\rangle$, $u\in W^*$. If $S\bullet u\neq\emptyset$ then $(S\cdot T)\bullet u = (S\bullet u)\cdot T$.*

Proof. Let $S\in\mathsf{DB}\langle\langle W\rangle\rangle$, $T\in\mathsf{B}\langle\langle W\rangle\rangle$, $u\in W^*$, such that $S\bullet u\neq\emptyset$. Let $u', u''\in W^*$ such that $u=u'\cdot u''$, $u''\neq\varepsilon$ and let $w\in\mathrm{supp}(S)$. If $w\bullet u'=\varepsilon$ then $S\bullet u'=\varepsilon$ (because $S\bullet u'$ is left-deterministic), hence $S\bullet u=\varepsilon\bullet u''=\emptyset$, which would contradict the hypothesis. It follows that

$$\forall u'\prec u,\ \forall w\in\mathrm{supp}(S),\quad w\bullet u'\neq\varepsilon.$$

Hence

$$\forall w_1 \in \mathrm{supp}(S),\ \forall w_2 \in \mathrm{supp}(T), \quad (w_1 \cdot w_2) \bullet u = (w_1 \bullet u) \cdot w_2.$$

This proves that $(S \cdot T) \bullet u = (S \bullet u) \cdot T$. $\square$

Lemma 38. *Let $S \in \mathsf{DB}\langle\langle W \rangle\rangle$, $T \in \mathsf{B}\langle\langle W \rangle\rangle$, $u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true*:

(1) $S \bullet u \neq \emptyset$;
in this case $U \bullet u = (S \bullet u) \cdot T$.

(2) $S \bullet u = \emptyset$, $\exists u', u'', u = u' \cdot u''$, $S \bullet u' = \varepsilon$;
in this case $U \bullet u = T \bullet u''$.

(3) $S \bullet u = \emptyset$, $\forall u' \leqslant u$, $S \bullet u' \neq \varepsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof. Clearly, one of the hypotheses (1–3) must occur. Let us examine each one of these cases.

In case (1), by Lemma 37, $U \bullet u = (S \bullet u) \cdot T$.

In case (2), $U \bullet u = (U \bullet u') \bullet u''$ and by case (1), $U \bullet u' = (S \bullet u') \cdot T$. It follows that $U \bullet u = T \bullet u''$.

In case (3), if $S = \emptyset$, the conclusion of the lemma is clearly true. Let us suppose now that $S \neq \emptyset$ and let $u' \prec u$ be the maximum prefix of u such that $S \bullet u' \neq \emptyset$. Then, there exist some $A \in W$, $u'' \in W^*$such that $u = u' \cdot A \cdot u''$ and there exist some $B_1, \dots, B_q \in W, S_1, \dots, S_q \in \mathsf{B}\langle\langle W \rangle\rangle - \{\emptyset\}$ such that $S \bullet u' = \sum_{1 \leqslant i \leqslant q} B_q \cdot S_q$ and $B_1 \smile \cdots \smile B_i \smile \cdots \smile B_q$ (because $S \bullet u'$ is left-deterministic). By maximality of u', A does not belong to $\{B_1, \dots, B_q\}$, hence

$$U \bullet u = \left(\left(\sum_{1 \leqslant i \leqslant q} B_i \cdot S_i \cdot T \right) \bullet A \right) \bullet u'' = \emptyset \bullet u'' = \emptyset. \qquad \square$$

Lemma 39. *Let $S \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$, $T \in \mathsf{B}_{m,s}\langle\langle W \rangle\rangle$, $u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true*:

(1) $\exists j$, $S_j \bullet u \notin \{\emptyset, \varepsilon\}$
in this case $U \bullet u = (S \bullet u) \cdot T$.

(2) $\exists j_0$, $\exists u', u''$, $u = u' \cdot u''$, $S_{j_0} \bullet u' = \varepsilon$;
in this case $U \bullet u = T_{j_0} \bullet u''$.

(3) $\forall j$, $\forall u' \leqslant u$, $S_j \bullet u = \emptyset$, $S_j \bullet u' \neq \varepsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

This lemma is an easy extension of Lemma 38 to the matricial case. We leave to the reader the corresponding routine proof.

Lemma 310. *For every $S \in \mathsf{DB}_{n,m}\langle\langle W \rangle\rangle$, $T \in \mathsf{DB}_{m,s}\langle\langle W \rangle\rangle$,*

1. $S \cdot T \in \mathsf{DB}_{n,s}\langle\langle W \rangle\rangle$.
2. $\|S \cdot T\| \leqslant \|S\| + \|T\|$.

Proof. As the notion of deterministic matrix is defined row by row, it is sufficient to prove this lemma in the particular case where $n = 1$. Let $u \in W^*$. Let us consider every one of the 3 cases considered in Lemma 39. In case (1) or (3),

$$(S \cdot T) \bullet u = (S \bullet u) \cdot T$$

and in case (2),

$$(S \cdot T) \bullet u = T \bullet u''.$$

In both cases, $(S \cdot T) \bullet u$ is left-deterministic. Moreover, the total number of residuals of the first (resp. second) form is $\leqslant \|S\|$ (resp. $\leqslant \|T\|$). □

3.1.3. Operations on row-vectors

Let us introduce two new operations on row-vectors and prove some technical lemmas about them.

Given $A, B \in \mathsf{B}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$ we define the vector $C = A \nabla_{j_0} B$ as follows: if $A = (a_1, \ldots, a_j, \ldots, a_m)$, $B = (b_1, \ldots, b_j, \ldots, b_m)$ then $C = (c_1, \ldots, c_j, \ldots, c_m)$, where

$$c_j = a_j + a_{j_0} \cdot b_j \text{ if } j \neq j_0, \quad c_j = \emptyset \text{ if } j = j_0.$$

Lemma 311. *Let $A, B \in \mathsf{B}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$.*
(1) *if A, B are deterministic, then $A \nabla_{j_0} B$ is deterministic.*
(2) *if A, B are deterministic, then $\|A \nabla_{j_0} B\| \leqslant \|A\| + \|B\|$.*

Proof. Suppose that A, B are deterministic and let $C = A \nabla_{j_0} B$, $u \in W^*$.

Using Lemma 38 we obtain that one of the two following cases occurs:

$$C \bullet u = (A \bullet u) \nabla_{j_0} B, \tag{16}$$

$$C \bullet u = \langle (B \bullet u'') | \emptyset^m_{j_0} \rangle, \tag{17}$$

where $\emptyset^m_{j_0}$ is the row-vector $\varepsilon^m_{j_0}$ in which $\emptyset$ and ε have been exchanged and $\langle *, * \rangle$ is the "scalar product" defined by $\langle S, T \rangle = \sum_{j=1}^m S_j \cdot T_j$.

These formulas show that $C \bullet u$ is left-deterministic.

The number of residuals of the form (16) is bounded above by $\|A\|$ and the number of residuals of the form (17) is bounded above by $\|B\|$. Hence $\|C\| \leqslant \|A\| + \|B\|$. □

Given $A \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$ we define the vector $A' = \nabla^*_{j_0}(A)$ as follows: if $A = (a_1, \ldots, a_j, \ldots, a_m)$ then $A' = (a'_1, \ldots, a'_j, \ldots, a'_m)$ where

$$a'_j = a^*_{j_0} \cdot a_j \text{ if } j \neq j_0, \quad a'_j = \emptyset \text{ if } j = j_0.$$

Lemma 312. *Let $A \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$.*
Then $\nabla^*_{j_0}(A) \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$ and $\|\nabla^*_{j_0}(A)\| \leqslant \|A\|$.

Proof. Let us examine a residual $A' \bullet u$, for some $u \in W^*$. Let $u' = \max\{v \leqslant u \mid v \in a_{j_0}^*\}$. Let $u'' \in W^*$ such that $u = u' \cdot u''$. One can check that

$$A' \bullet u = (A \bullet u'')\nabla_{j_0} A'. \tag{18}$$

By formula (18), $A' \bullet u$ is left-deterministic.

Moreover, $\mathrm{Card}(\mathsf{Q}(A')) \leqslant \mathrm{Card}(\mathsf{Q}(A))$, i.e. $\|A'\| \leqslant \|A\|$. □

3.2. The infinite grammar G

3.2.1. Strict-deterministic grammars

Definition 313. Let $G = \langle X, V, P\rangle$ be a context-free grammar in Greibach normal form. G is said *strict-deterministic* iff there exists an equivalence relation $\smile$ over V fulfilling the following condition: for every $x \in X$, if $(E_k)_{1 \leqslant k \leqslant m}$ are distinct variables in V such that $\forall k \in [1, m]$, $E_1 \smile E_k$ and $H_k = \sum_{(E_k,h) \in P} h \bullet x$, then

$$(H_1, H_2, \ldots, H_m) \quad \text{is a deterministic vector.}$$

Any equivalence $\smile$ satisfying the above condition is said to be a *strict equivalence* for the grammar G.

This definition is a reformulation of [2, Definition 11.4.1 p. 347] adapted to the case of a Greibach normal-form.

Theorem 314. *Let $G = \langle X, V, P\rangle$ be a strict-deterministic grammar. Then its reduced form $G_0 = \langle X, V_0, P_0\rangle$, as defined in formulas* (14), (15), *is strict-deterministic too. Moreover*, *if $\smile$ is a strict equivalence for G*, *its restriction over V_0 is a strict equivalence for G_0.*

The proof would consist in slightly extending the proof of [2, Theorem 11.4.1 p. 350].

It is known that, given a dpda $\mathscr{M}$, its associated grammar $G_{\mathscr{M}}$ is strict-deterministic. By Theorem 314 G_0 is strict-deterministic too. Let us consider the minimal strict equivalence $\smile$ for G_0 and extend it to V_1 by, $\forall v, v' \in V_0$:

$$\bar{v} \smile \bar{v}' \quad\Leftrightarrow\quad v \smile v'; \qquad \bar{v} \not\smile v'.$$

Then $\smile$ is a strict equivalence for G_1 (the grammar G_1 is defined in Section 2.3). This ensures that G_1 is strict-deterministic.

Marks. A word $w \in V^*$ is said *marked* iff $w \in V^* \cdot \bar{V}_0 \cdot V^*$; it is said *fully marked* iff $w \in \bar{V}_0^*$.

A series $S \in \mathsf{B}\langle\langle V\rangle\rangle$ is said *marked* iff $\exists\, w \in \mathrm{supp}(S)$, w is marked; it is said *fully marked* iff $\forall w \in \mathrm{supp}(S)$, w is fully marked. It is said *unmarked* iff it is *not* marked. A matrix $S \in \mathsf{B}_{m,n}\langle\langle V\rangle\rangle$ is said marked (resp. fully marked, unmarked) iff, for every $i \in [1, m]$, the series $\sum_{j=1}^{n} S_{i,j}$ is marked (resp. fully marked, unmarked).

Definition 315. Let $d \in \mathbb{N}$. A vector $S \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$ is said d-marked iff there exists $q \in \mathbb{N}$, $\alpha \in \mathsf{DRB}_{1,q}\langle V\rangle$, $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle V\rangle\rangle$ such that

$$S = \sum_{k=1}^{q} \alpha_k \cdot \Phi_k \quad \text{and} \quad \|\alpha\| \leqslant d,$$

and Φ is unmarked.

Lemma 316. *For every* $S \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$ $\rho_e(S), \bar{\rho}_e(S) \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$

Proof (sketch). Let us notice that the homomorphism $\rho_e : V^* \to V^*$ preserves the equivalence $\smile$: for every $v, v' \in V$, if $v \smile v'$ then $\rho_e(v) \smile \rho_e(v')$. It follows that the corresponding substitution ρ_e preserves determinism. The same argument applies on $\bar{\rho}_e$. $\square$

Infinite alphabet V. Let us introduce an infinite alphabet

$$V_2 = \{\Psi_1, \Psi_2, \ldots, \Psi_n, \ldots\}$$

together with a bijection:

$$\varphi_2 : V_2 \cup \{\varepsilon\} \to \mathsf{DRB}\langle V_1\rangle - \{\emptyset\}$$

such that $\varphi_2(\varepsilon) = \varepsilon$.

We suppose that the map $n \mapsto \varphi_2(\Psi_n)$ is computable.

We then define the (infinite) structured alphabet $(V, \smile)$ as the following extension of $(V_1, \smile)$:

$$V = V_1 \cup V_2; \;\; \forall v, v' \in V, \; v \smile v' \Leftrightarrow (v, v' \in V_1 \text{ and } v \smile v') \text{ or } (v = v' \in V_2).$$

In other words, the Ψ_n are *second-level variables* which represent, bijectively, the deterministic rational boolean series over the set of *first-level* variables V_1. We call *linear polynomials* the row-vectors of the form:

$$(\alpha_1\beta_1, \ldots, \alpha_i\beta_i, \ldots, \alpha_m\beta_m) \in \mathsf{DB}_{1,m}\langle V\rangle \tag{19}$$

such that $\forall i \in [1, m]$, $\alpha_i \in \mathsf{B}_{1,m}\langle V_1\rangle$, $\beta_i \in V_2 \cup \{\varepsilon\}$.

We extend φ_2 as a substitution $\mathsf{B}\langle\langle V\rangle\rangle \to \mathsf{B}\langle V_1\rangle$ by setting

$$\forall v \in V_1, \quad \varphi_2(v) = v.$$

Let us define a set P_2 of productions by

if $\varphi_2(\Psi_n) = \sum_{k=1}^{q} E_k \cdot \Phi_k$, where $\forall k \in [1, q]$, $E_k \in V_1$, $E_1 \smile E_k$ then, for every $k \in [1, q]$

$$(\Psi_n, x \cdot (E_k \odot x) \cdot \varphi_2^{-1}(\Phi_k)) \tag{20}$$

is a rule of P_2. We extend now the set P_1 by defining $P = P_1 \cup P_2$.

One can check that the (infinite) grammar $G = \langle X, V, P\rangle$ is still strict-deterministic.

Let us consider the unique substitution $\varphi : \mathsf{B}\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle X\rangle\rangle$ fulfilling: for every $v \in V$,

$$\varphi(v) = \{u \in X^* \mid v \xrightarrow{*}_P u\}.$$

The action of X^* on $\mathsf{B}\langle\langle V\rangle\rangle$. Let $(V, \smile)$ be the infinite structured alphabet associated with $\mathscr{M}$ in the above paragraph. We define the right-action $\odot$ as the unique σ-right-action of the monoid X^* over the semi-ring $\mathsf{B}\langle\langle V\rangle\rangle$ such that for every $v \in V$, $\beta \in V^*, x \in X$

$$(v \cdot \beta) \odot x = \left(\sum_{(v,h) \in P} h \bullet x \right) \cdot \beta, \tag{21}$$

$$\varepsilon \odot x = \emptyset. \tag{22}$$

Lemma 317. *For every $S \in \mathsf{B}\langle\langle V\rangle\rangle$, $u \in X^*$,*
(1) $\varphi(S \odot u) = \varphi(S) \bullet u$ (*i.e. φ is a morphism of right-actions*).
(2) $\varphi = \varphi_1 \circ \varphi_2$

Proof. (1) Let $v \in V$, $\beta \in V^*$, $x \in X$. One can then check on formulas (21), (22) that

$$\varphi(\varepsilon \odot x) = \varphi(\varepsilon) \bullet x \quad \text{and} \quad \varphi((v \cdot \beta) \odot x) = \varphi(v \cdot \beta) \bullet x.$$

By σ-additivity of φ, point 1 of the lemma is true in the case where $|u| \leqslant 1$. By induction on $|u|$ the general case follows.

(2) It suffices to prove that, for every $\Psi \in V$,

$$\varphi(\Psi) = \varphi_1(\varphi_2(\Psi)). \tag{23}$$

(2.1) If $\Psi \in V_1$, (23) is true because $\varphi_2(\Psi) = \Psi$ and the only rules appearing in a derivation $\Psi \xrightarrow{*}_P u$ belong to P_1.

(2.2) Suppose $\Psi \in V_2$ and $\varphi_2(\Psi) = \sum_{k=1}^{q} E_k \cdot \Phi_k$. Let us show, by induction on $|u|$, that for every $u \in X^*$,

$$\varphi_2(\Psi \odot u) = \varphi_2(\Psi) \odot u. \tag{24}$$

If $u = \varepsilon$, this is clear.

If $u = x$ for some $x \in X$:

$$\Psi \odot x = \sum_{k=1}^{q} (E_k \odot x) \cdot \varphi_2^{-1}(\Phi_k)$$

while

$$\varphi_2(\Psi) \odot x = \left(\sum_{k=1}^{q} E_k \cdot \Phi_k \right) \odot x = \sum_{k=1}^{q} (E_k \odot x) \cdot \Phi_k.$$

Hence, equality (24) is true.

If $u = u'x$ for some $u' \in X^*$, $x \in X$:

$$\begin{aligned}\varphi_2(\Psi \odot u'x) &= \varphi_2((\Psi \odot u') \odot x)\\ &= (\varphi_2(\Psi \odot u')) \odot x \text{ (by the case above)}\\ &= (\varphi_2(\Psi) \odot u') \odot x \text{ (by induction hypothesis)}\\ &= \varphi_2(\Psi) \odot u'x.\end{aligned}$$

Equality (24) is established. Using point 1 of the lemma, we get that:

$u \in \varphi(\Psi) \Leftrightarrow \varepsilon \in \varphi(\Psi)\bullet u \Leftrightarrow \varepsilon \in \varphi(\Psi \odot u) \Leftrightarrow \Psi \odot u = \varepsilon$ (because G is proper) $\Leftrightarrow \varphi_2(\Psi \odot u) = \varepsilon \Leftrightarrow \varepsilon \in \varphi_2(\Psi) \bullet u \Leftrightarrow u \in \varphi(\varphi_2(\Psi)) \Leftrightarrow u \in \varphi_1(\varphi_2(\Psi))$ (because φ, φ_1 coincide on V_1). $\square$

Equivalence of vectors. We denote by $\equiv$ the kernel of φ i.e.: for every $S, T \in \mathsf{B}\langle\langle V\rangle\rangle$,

$$S \equiv T \quad \Leftrightarrow \quad \varphi(S) = \varphi(T).$$

One can notice that, by Lemma 317 point (2), for every $S \in \mathsf{B}\langle\langle V\rangle\rangle$,

$$S \equiv \varphi_2(S). \tag{25}$$

For every integer n, we denote by $\equiv_n$ the following approximation of $\equiv$:

$$S \equiv_n T \quad \Leftrightarrow \quad \varphi(S) \cap X^{\leqslant n} = \varphi(T) \cap X^{\leqslant n}.$$

We extend the substitutions $\varphi_1, \varphi_2, \varphi$, the action $\odot$ and the equivalence $\equiv$, componentwise, to $\mathsf{B}_{n,m}\langle\langle V\rangle\rangle$ (for every integers $n, m \geqslant 0$).

Right-action and product. Let us define here handful notations for some particular vectors or matrices. We use the *Kronecker symbol* $\delta_{i,j}$ meaning ε if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leqslant n, 1 \leqslant i \leqslant n$, we define the row-vector ε_i^n as:

$$\varepsilon_i^n = (\varepsilon_{i,j}^n)_{1 \leqslant j \leqslant n} \quad \text{where } \forall j,\ \varepsilon_{i,j}^n = \delta_{i,j}.$$

We call *unit row-vector* any vector of the form ε_i^n.

For every $1 \leqslant n$, we denote by $\emptyset^n \in \mathsf{DB}_{1,n}\langle\langle V\rangle\rangle$ the row-vector:

$$\emptyset^n = (\emptyset, \ldots, \emptyset).$$

Let us define

$$K_0 = \max\{\|(E_1, E_2, \ldots, E_n) \odot x\| \mid (E_i)_{1 \leqslant i \leqslant n} \text{ is a bijective numbering of some class in } V/\smile,\ x \in X\}. \tag{26}$$

Lemma 318. *For every* $S \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle,\ u \in X^*$,

(1) $S \odot u \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$

(2) $\|S \odot u\| \leqslant \|S\| + K_0 \cdot |u|$.

Proof. We treat first the case where u is just a letter.

Let $S \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$ and $x \in X$. If $S = \emptyset^\lambda$ or $S = \varepsilon_i^\lambda$ (for some $i \in [1, \lambda]$), then $S \odot x = \emptyset^\lambda$ and points (1), (2) are both true.

Otherwise

$$S = \sum_{k=1}^{q} E_k \cdot \Phi_k$$

for some $q \in \mathbb{N}$, $\Phi_k \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$, $(E_k)_{1 \leqslant k \leqslant q}$ bijective numbering of some class of $V/\smile$.

By Eq. (21) , which defines the right-action $\odot$,

$$S \odot x = \sum_{k=1}^{q} (E_k \odot x) \cdot \Phi_k$$

hence $S \odot x$ has the form $H \cdot \Phi$ where $H \in \mathsf{DB}_{1,q}\langle\langle V \rangle\rangle$ (see Definition 313), $\|H\| \leqslant K_0$ (see Eq. (26)) and $\Phi \in \mathsf{DB}_{q,\lambda}\langle\langle V \rangle\rangle$.

By Lemma 310, $H \cdot \Phi$ is deterministic and by Lemma 2 $\|H \cdot \Phi\| \leqslant \|\Phi\| + K_0$. As every $\Phi_k \in \mathsf{Q}_r(S)$ we obtain

$$\|H \cdot \Phi\| \leqslant \|\Phi\| + K_0 \leqslant \|S\| + K_0.$$

Both points (1), (2) are proved.

The general case where u is any word of X^* can be deduced by induction on $|u|$ from this particular case. □

Lemma 319. *Let* $\lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$, $u \in X^*$. *One of the three following cases must occur*:
(1) $S \odot u = \emptyset^{\lambda}$,
(2) $S \odot u = \varepsilon_j^{\lambda}$ *for some* $j \in [1, \lambda]$,
(3) $\exists u_1, u_2 \in X^*$, $v_1 \in V^*$, $q \in \mathbb{N}$, $E_1, \ldots, E_k, \ldots, E_q \in V$, $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle V \rangle\rangle$ *such that*

$$u = u_1 \cdot u_2,\ S \odot u_1 = S \bullet v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k,\ S \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k,\ \textit{and}$$

$$\forall k \in [1,q],\ E_k \smile E_1,\ E_k \odot u_2 \notin \{\varepsilon, \emptyset\}.$$

Proof. Let $u \in X^*$. Let us prove the lemma by induction on $|u|$. $u = \varepsilon$:
if $S \in \emptyset^{\lambda} \cup \{\varepsilon_j^{\lambda} \mid 1 \leqslant j \leqslant \lambda\}$ then clearly the conclusion of case (1) or (2) is realized.

Otherwise, as S is left-deterministic, S has a decomposition as $S = \sum_{k=1}^{q} E_k \cdot \Phi_k$ such that the conclusion of case (3) is realized with $u_1 = u_2 = \varepsilon$, $v_1 = \varepsilon$, the given integer q and the letters $E_1 \smile \cdots \smile E_q \in V$. $u = u_0 \cdot a$, $a \in X$:

Let us consider the $u_1, u_2, v_1, q, (E_k)_{1 \leqslant k \leqslant q}, (\Phi_k)_{1 \leqslant k \leqslant q}$ given by the induction hypothesis on u_0.

$$(S \odot u) \odot a = \left(\sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k \right) \odot a \text{ and}$$

$$\forall k \in [1,q],\ \|E_k \odot u_2\| \geqslant 3.$$

Case 1: $\forall k \in [1,q], \|E_k \odot u_2 a\| \geqslant 3$.

Then $S \odot ua = \sum_{k=1}^{q} (E_k \odot u_2 a) \cdot \Phi_k$. Hence conclusion (3) of the lemma is fulfilled by $u_1' = u_1$, $u_2' = u_2 a$, $v_1' = v_1$, $q' = q$, $E_k' = E_k$, $\Phi_k' = \Phi_k$.

Case 2: $\exists r \in [1,q]$, $\|E_r \odot u_2 a\| = 2$.

In other words: there exists some $r \in [1,q]$ such that $E_r \odot u_2 a = \varepsilon$, hence

$$S \odot ua = \Phi_r.$$

Subcase 1: $\Phi_r \in \{\emptyset^\lambda\} \cup \{\varepsilon_j^\lambda \mid 1 \leqslant j \leqslant \lambda\}$.
Conclusion (1) or (2) of the lemma is then realized.
Subcase 2: $\Phi_r = \sum_{\ell=1}^{r'} F_\ell \cdot \Psi_\ell$ for some $r' \in \mathbb{N}$, $F_1 \smile \cdots F_{r'} \in V$, $\Psi \in \mathsf{DRB}_{r',\lambda}\langle\langle V \rangle\rangle$.
Then

$$S \odot ua = \sum_{\ell=1}^{r'} F_\ell \cdot \Psi_\ell; \quad S \bullet (v_1 E_r) = \Phi_r = \sum_{\ell=1}^{r'} F_\ell \cdot \Psi_\ell.$$

Conclusion (3) of the lemma is then realized by $u'_1 = ua$, $u'_2 = \varepsilon$, $v'_1 = v_1 E_r$, $q' = r'$, $E'_k = F_k$, $\Phi' = \Psi$.
Case 3: $\forall k \in [1,q]$, $\|E_k \odot u_2 a\| = 1$.
This means that $E \odot u_2 a = \emptyset^q$, hence that case (1) is realized. □

Corollary 320. *Let* $\lambda \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$. *Then* $S \equiv S'$ *if and only if*, $\forall u \in X^*$, $\forall j \in [1,\lambda]$,

$$S \odot u = \varepsilon_j^\lambda \;\; \Leftrightarrow \;\; S' \odot u = \varepsilon_j^\lambda.$$

Proof. (1) Let us show that for every $T \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$, $j \in [1,\lambda]$ and $u \in X^*$:

$$T \odot u = \varepsilon_j^\lambda \;\; \Leftrightarrow \;\; u \in \varphi(T_j). \tag{27}$$

Suppose $T \odot u = \varepsilon_j^\lambda$. Then $T_j \odot u = \varepsilon$. and by Lemma 317, point (1), $\varphi(T_j) \bullet u = \varepsilon$, hence $u \in \varphi(T_j)$.

Suppose now that $u \in \varphi(T_j)$. Then $T_j \odot u = \varepsilon$ (same arguments as above) and, by Lemma 318 point (1), $T \odot u$ is a deterministic vector, hence $T \odot u = \varepsilon_j^\lambda$.

(2) From equivalence (27), the lemma follows. □

We give now an adaptation of Lemma 39 to the action $\odot$ in place of $\bullet$.

Lemma 321. *Let* $S \in \mathsf{DB}_{1,m}\langle\langle V \rangle\rangle$, $T \in \mathsf{B}_{m,s}\langle\langle V \rangle\rangle$, $u \in X^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true*:
(1) $S \odot u \notin \{\emptyset^m\} \cup \{\varepsilon_j^m \mid 1 \leqslant j \leqslant m\}$
in this case $U \odot u = (S \odot u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u''$, $S \odot u' = \varepsilon_{j_0}^s$;
in this case $U \odot u = T_{j_0} \odot u''$.
(3) $\forall j, \forall u' \leqslant u$, $S \odot u = \emptyset^m$ *and* $S \odot u' \neq \varepsilon_j^m$;
in this case $U \odot u = \emptyset^s = (S \odot u) \cdot T$.

Proof. The arguments used in the proofs of Lemmas 37–39 can be adapted to $\odot$ in place of $\bullet$. The only non-trivial adaptation is that of lines 6–7 of the proof of Lemma 37: let us suppose that $u \in X^*$ is such that

$$\forall u' \prec u, \quad S \odot u' \neq \varepsilon \tag{28}$$

and let us prove that

$$(S \cdot T) \odot u = (S \odot u) \cdot T. \tag{29}$$

We prove by induction on $|u|$ that (28) implies (29).

$|u| = 0$: by definition of a right-action, $\forall S' \in \mathsf{DB}\langle\langle V \rangle\rangle$, $S' \odot \varepsilon = S'$. Hence conclusion (29) is true.

$u = u_0 \cdot a$, where $u_0 \in X^*$, $a \in X$:

Hypothesis (28) is fulfilled by u_0 too, hence, by induction hypothesis

$$(S \cdot T) \odot u_0 = (S \odot u_0) \cdot T. \tag{30}$$

If $S \odot u_0 = \emptyset$, then, by the above equality $(S \cdot T) \odot u_0 = \emptyset$ too, hence

$$(S \cdot T) \odot u_0 a = \emptyset = (S \odot u_0 a) \cdot T,$$

hence (29) is true.

Otherwise, by hypothesis (28) $S \odot u_0 \notin \{\emptyset, \varepsilon\}$, hence there exists $q \in \mathbb{N}$, $E_1 \smile \cdots \smile E_q \in V$, $\Phi \in \mathsf{DB}_{m,s}\langle\langle V \rangle\rangle$ such that

$$S \odot u_0 = \sum_{k=1}^{q} E_k \cdot \Phi_k. \tag{31}$$

By Definition (21) and the fact that $\odot$ is a σ-action

$$(E_k \cdot \Phi_k) \odot a = (E_k \odot a) \cdot \Phi_k$$

hence, by σ-additivity

$$\left(\sum_{k=1}^{q} E_k \cdot \Phi_k\right) \odot a = \sum_{k=1}^{q} (E_k \odot a) \cdot \Phi_k$$

and by product by T:

$$(S \odot u_0 a) \cdot T = \sum_{k=1}^{q} (E_k \odot a) \cdot \Phi_k \cdot T. \tag{32}$$

Let us examine now $(ST) \odot u_0 a$. By (30)

$$(S \cdot T) \odot u_0 = \sum_{k=1}^{q} E_k \cdot \Phi_k \cdot T. \tag{33}$$

By Definition (21) and the fact that $\odot$ is a σ-action

$$(E_k \cdot \Phi_k \cdot T) \odot a = (E_k \odot a) \cdot \Phi_k \cdot T$$

hence, by σ-additivity

$$\left(\sum_{k=1}^{q} E_k \cdot \Phi_k \cdot T\right) \odot a = \sum_{k=1}^{q} (E_k \odot a) \cdot \Phi_k \cdot T$$

Using (33) this last equality can be read

$$(ST) \odot u_0 a = \sum_{k=1}^{q} (E_k \odot a) \cdot \Phi_k \cdot T. \tag{34}$$

As equalities (34), (32) have the same right-hand side, we conclude that (29) is true. □

3.3. Linear independence

We adapt here the key-idea of [4,5] to deterministic series in $\mathsf{DRB}\langle\langle V \rangle\rangle$.

Let us call *linear combination* of the series $S_1, \ldots, S_j, \ldots, S_m$ any series of the form $\sum_{1 \leqslant j \leqslant m} \alpha_j \cdot S_j$ where $\boldsymbol{\alpha} \in \mathsf{DRB}_{1,m}\langle\langle V \rangle\rangle$. Following an analogy with classical linear algebra, we develop a notion corresponding to a kind of *linear independence* of the classes (mod $\equiv$) of the given series.

Lemma 322. *Let $S_1, \ldots, S_j, \ldots, S_m \in \mathsf{DRB}\langle\langle V \rangle\rangle$. The following are equivalent*
(1) $\exists \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{DRB}_{1,m}\langle\langle V \rangle\rangle, \boldsymbol{\alpha} \not\equiv \boldsymbol{\beta}$, *such that* $\sum_{1 \leqslant j \leqslant m} \alpha_j \cdot S_j \equiv \sum_{1 \leqslant j \leqslant m} \beta_j \cdot S_j$,
(2) $\exists j_0 \in [1, m]$, $\exists \boldsymbol{\gamma} \in \mathsf{DRB}_{1,m}\langle\langle V \rangle\rangle$, $\boldsymbol{\gamma} \not\equiv \varepsilon^m_{j_0}$, *such that* $S_{j_0} \equiv \sum_{1 \leqslant j \leqslant m} \gamma_j \cdot S_j$,
(3) $\exists j_0 \in [1, m]$, $\exists \boldsymbol{\gamma}' \in \mathsf{DRB}_{1,m}\langle\langle V \rangle\rangle$, $\gamma'_{j_0} \equiv \emptyset$, *such that* $S_{j_0} \equiv \sum_{1 \leqslant j \leqslant m} \gamma'_j \cdot S_j$.

The equivalence between (1), (2) and (3) was first proved in [4, Lemma 11 p. 589],[2] in the case where the S_j's are configurations $q_j\omega$, with the same ω.

Proof. Let use the notation $S = (S_j)_{1 \leqslant j \leqslant m} \in \mathsf{DRB}_{m,1}\langle\langle V \rangle\rangle$.
(1) ⇒ (2):
Let us consider

$$u = \min\{\varphi(\boldsymbol{\alpha}) \Delta \varphi(\boldsymbol{\beta})\}.$$

By Corollary 320, $\exists j_0 \in [1, m]$, such that

$$\boldsymbol{\alpha} \odot u = \varepsilon^m_{j_0} \quad \Leftrightarrow \quad \boldsymbol{\beta} \odot u \neq \varepsilon^m_{j_0}.$$

Let us suppose, for example, that $\boldsymbol{\alpha} \odot u = \varepsilon^m_{j_0}$ while $\boldsymbol{\beta} \odot u \neq \varepsilon^m_{j_0}$ and let $\boldsymbol{\gamma} = \boldsymbol{\beta} \odot u$. As $\equiv$ is preserved by the action $\odot$ (see Lemma 317)

$$(\boldsymbol{\alpha} \cdot S) \odot u \equiv (\boldsymbol{\beta} \cdot S) \odot u. \tag{35}$$

Using Lemma 321 we obtain

$$(\boldsymbol{\alpha} \cdot S) \odot u = S_{j_0}. \tag{36}$$

Let us examine now the right-hand side of equality (35). Let $u' \prec u$. By minimality of u, $\boldsymbol{\beta} \odot u'$ is a unit iff $\boldsymbol{\alpha} \odot u'$ is a unit. But if $\boldsymbol{\alpha} \odot u'$ is a unit, then $\boldsymbol{\alpha} \odot u = \emptyset^m$, which

[2] Numbering of the english version.

is false. Hence $\boldsymbol{\beta} \odot u'$ is not a unit. By Lemma 321

$$(\boldsymbol{\beta} \cdot S) \odot u = (\boldsymbol{\beta} \odot u) \cdot S. \tag{37}$$

Let us plug equalities (36) and (37) in equivalence (35) and let us define $\boldsymbol{\gamma} = \boldsymbol{\beta} \odot u$. We obtain

$$S_{j_0} \equiv \boldsymbol{\gamma} \cdot S, \quad \text{where } \boldsymbol{\gamma} \not\equiv \varepsilon_{j_0}^m.$$

$(2) \Rightarrow (3)$:

$$S_{j_0} \equiv \gamma_{j_0} \cdot S_{j_0} + \left(\sum_{j \neq j_0}, \gamma_j \cdot S_j \right), \quad \gamma_{j_0} \not\equiv \varepsilon.$$

By the well-known Arden's lemma (see Corollary 49 point (C1)), we can deduce that

$$S_{j_0} \equiv \sum_{j \neq j_0} \gamma_{j_0}^* \gamma_j \cdot S_j = \nabla_{j_0}^*(\gamma) \cdot S.$$

Taking $\gamma' = \nabla_{j_0}^*(\gamma)$ we obtain

$$S_{j_0} \equiv \gamma' \cdot S \quad \text{where } \gamma'_{j_0} = \emptyset.$$

$(3) \Rightarrow (1)$:

(3) asserts that

$$\sum_{1 \leqslant j \leqslant m} \delta_{j,j_0} S_j \equiv \sum_{1 \leqslant j \leqslant m} \gamma'_j \cdot S_j.$$

The fact that $\delta_{j_0,j_0} = \varepsilon \not\equiv \emptyset$ shows that $(\delta_{1,j_0}, \ldots, \delta_{m,j_0}) \not\equiv (\gamma'_1, \ldots, \gamma'_m)$. Hence (1) is true. □

3.4. Derivations

For every $u \in X^*$ we define the binary relation $\uparrow(u)$ over $\mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$ by: for every $S, S' \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$, $S \uparrow (u) S' \Leftrightarrow \exists q \in \mathbb{N}, \exists E_1, \ldots, E_k, \ldots, E_q \in V$, $\Phi \in \mathsf{DB}_{q,\lambda}\langle\langle V \rangle\rangle$ such that

$$S = \sum_{k=1}^{q} E_k \cdot \Phi_k, \quad S' = \sum_{k=1}^{q} (E_k \odot u) \cdot \Phi_k$$

and $\forall k \in [1, q]$, $E_1 \smile E_k$, $E_k \odot u \notin \{\emptyset, \varepsilon\}$.

It is clear that if $S \uparrow (u) S'$ then $S \odot u = S'$ and that the converse is not true in general. A sequence of deterministic row-vectors $S_0, S_1, \ldots, S_n$ is a *derivation* iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. The *length* of this derivation is n. If $u = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S, u). We denote this derivation by $S_0 \xrightarrow{u} S_n$.

A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow (u) S_n$. A derivation $S_0, S_1, \ldots, S_n$ is said to be a *sub-derivation* of a derivation $S'_0, S'_1, \ldots, S'_m$ iff there exists some $i \in [0, m]$ such that, $\forall j \in [1, n],\ S_j = S'_{i+j}$.

Lemma 323. *Let* $S \in \mathsf{DRB}_{1,\lambda}\langle V\rangle,\ w \in X^*$, *such that* $\forall u \leqslant w,\ \|S \odot u\| \geqslant \|S\|$.
Then the derivation $S \xrightarrow{w} S \odot w$ *is stacking.*

Proof. S is left-deterministic. If it has type $\emptyset$ or (ε, j), the lemma is trivially true. Otherwise

$$S = \sum_{k=1}^{q} E_k \cdot \Phi_k$$

for some class of letter $[E_1]_\smile = \{E_1, \ldots, E_q\}$ and some matrix $\Phi \in \mathsf{DRB}_{q,\lambda}\langle\langle V\rangle\rangle$. Suppose that for some prefix $u \leqslant w$ and $k \in [1, q]$,

$$E_k \odot u = \varepsilon. \tag{38}$$

Then, $S \odot u = \Phi_k$ so that $\|S \odot u\| \leqslant \|\Phi\| < \|S\|$ which contradicts the hypothesis.

Let us apply now Lemma 321 to the expression $(E \cdot \Phi) \odot w$: case (2) is impossible, hence

$$(E \cdot \Phi) \odot w = (E \odot w) \cdot \Phi,$$

which is equivalent to

$$S \uparrow (w) S \odot w. \qquad \square$$

Lemma 324. *Let* $S \in \mathsf{DRB}_{1,\lambda}\langle V\rangle,\ w \in X^*,\ k \in \mathbb{N}$, *such that* $\|S \odot w\| \geqslant \|S\| + k \cdot K_0 + 1$.
Then the derivation $S \xrightarrow{w} S \odot w$ *contains some stacking sub-derivation of length* k.

Proof (sketch). Let $S = S_0, \ldots, S_i, \ldots, S_n$ be the derivation associated to (S, w). Let $i_0 = \max\{i \in [0, n] \mid \|S_i\| = \min\{\|S_j\| \mid 0 \leqslant j \leqslant n\}\}$. Let $w = w_0 w'$ where $|w_0| = i_0$.

As $\|S_n\| - \|S_{i_0}\| \geqslant k \cdot K_0 + 1$, by Lemma 318 we must have $|w'| \geqslant k$. Let $w' = w_2 w_3$ with $|w_2| = k$. By definition of i_0, $\forall i \in [i_0 + 1, i_0 + k], \|S_i\| \geqslant \|S_{i_0}\| + 1$.

By Lemma 323, the sub-derivation $S_{i_0}, \ldots, S_{i_0+k}$ (associated to (S_{i_0}, w_2)) is stacking. $\square$

Lemma 325. *Let* $S, S' \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ w \in X^*,\ k, d, d' \in \mathbb{N}$, *such that* S *is* d*-marked and*:
(1) *the derivation* $S \xrightarrow{w} S'$ *contains no stacking sub-derivation of length* k.
(2) $|w| \geqslant d \cdot k$.
Then S' *is unmarked.*

Proof. By hypothesis

$$S = \sum_{k=1}^{q} \alpha_k \cdot \Phi_k$$

for some $\boldsymbol{\alpha} \in \mathsf{DRB}_{1,q}\langle V\rangle,\ \Phi \in \mathsf{DRB}_{q,1}\langle\langle V\rangle\rangle,\ \|\boldsymbol{\alpha}\| \leqslant d,\ \Phi$ unmarked.

Let $S \xrightarrow{w} S' = (S_0, \ldots, S_n)$. By induction on ℓ, using hypothesis (1) and Lemma 323 one can show that: for every $\ell \in [0, d]$, there exists some prefix w_ℓ of w, with length $|w_\ell| \leqslant k \cdot \ell$ such that either

$$S \odot w_\ell = \sum_{k=1}^{q} (\alpha_k \odot w_\ell) \cdot \Phi_k, \quad \text{with } \|\boldsymbol{\alpha}_\odot w_\ell\| < \|\alpha\| - \ell \tag{39}$$

or there exists an integer $k \in [1, q]$ such that

$$S \odot w_\ell = \Phi_k. \tag{40}$$

Let us apply this property to $\ell = d$: inequality (39) is not possible for this value of ℓ because, by hypothesis $\|\boldsymbol{\alpha}\| - \ell \leqslant 0$. Hence (40) is true and, as Φ is unmarked, Φ_k is unmarked, so that $S \odot w$ is unmarked. $\Box$

4. Deduction systems

4.1. General formal systems

We follow here the general philosophy of [3,1]. For any set $\mathscr{E}$, we denote by $\mathscr{P}(\mathscr{E})$ the set of its subsets and by $\mathscr{P}_f(\mathscr{E})$ the set of it *finite* subsets.

Let us call *formal system* any triple $\mathscr{D} = \langle \mathscr{A}, H, \mid\!\!\!-\!- \rangle$ where $\mathscr{A}$ is a denumerable set called the *set of assertions*, H, the *cost function* is a mapping $\mathscr{A} \to \mathbb{N} \cup \{\infty\}$ and $\mid\!\!\!-\!-$, the *deduction relation* is a subset of $\mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$.

$\mathscr{A}$ is given with a fixed bijection with $\mathbb{N}$ (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function,... over $\mathscr{A}, \mathscr{P}_f(\mathscr{A}), \ldots$ are defined, up to this fixed bijection; we assume that $\mathscr{D}$ satisfies the following axiom:

(A 1) $\forall (P, A) \in \mid\!\!\!-\!-$, $(\min\{H(p), p \in P\} < H(A))$ or $(H(A) = \infty)$.

(We let $\min(\emptyset) = \infty$). We call $\mathscr{D}$ a *deduction system* iff $\mathscr{D}$ is a formal system satisfying the additional axiom:

(A 2) $\mid\!\!\!-\!-$ is recursively enumerable.

In the sequel we use the notation $P \mid\!\!\!-\!- A$ for $(P, A) \in \mid\!\!\!-\!-$. We call *proof* in the system $\mathscr{D}$, *relative to the set of hypotheses* $\mathscr{H} \subseteq \mathscr{A}$, any subset $P \subseteq \mathscr{A}$ fulfilling:

$$\forall p \in P, \quad (\exists Q \subseteq P, Q \mid\!\!\!-\!- p) \quad \text{or} \quad (p \in \mathscr{H}).$$

We call P a *proof* iff

$$\forall p \in P, \quad (\exists Q \subseteq P, Q \mid\!\!\!-\!- p)$$

(i.e. iff P is a proof relative to $\emptyset$).

Let us define the total map $\chi : \mathscr{A} \to \{0, 1\}$ and the partial map $\bar{\chi} : \mathscr{A} \to \{0, 1\}$ by

$$\chi(A) = 1 \text{ if } H(A) = \infty, \quad \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\bar{\chi}(A) = 1 \text{ if } H(A) = \infty, \quad \bar{\chi} \text{ is undefined if } H(A) < \infty.$$

(χ is the "truth-value function", $\bar{\chi}$ is the "1-value function").

Lemma 41. *Let P be a proof relative to $\mathscr{H} \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.*

In other words: if an assertion is provable from true hypotheses, then it is true.

Proof. Let P be a proof. We prove by induction on n that

$$\mathscr{P}(n):\ \forall p \in P,\ H(p) \geqslant n.$$

It is clear that, $\forall p \in P,\ H(p) \geqslant 0$. Suppose that $\mathscr{P}(n)$ is true. Let $p \in P - \mathscr{H}$: $\exists Q \subseteq P$, $Q \models\!\!- p$. By induction hypothesis, $\forall q \in Q,\ H(q) \geqslant n$ and by (A1), $H(p) \geqslant n+1$. It follows that: $\forall p \in P - \mathscr{H},\ H(p) = \infty$. But by hypothesis, $\forall p \in \mathscr{H},\ H(p) = \infty$. $\square$

A formal system $\mathscr{D}$ will be said *complete* iff, conversely, $\forall A \in \mathscr{A},\ \chi(A) = 1 \Rightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathscr{D}$ is complete iff every true assertion is "finitely" provable).

Lemma 42. *If $\mathscr{D}$ is a complete deduction system, $\bar{\chi}$ is a recursive partial map.*

Proof. The property "$H(A) = \infty$" is semi-decidable just because the property "there exists a finite P such that P is a $\mathscr{D}$-proof and $A \in P$" is semi-decidable too. $\square$

In order to define deduction relations from more elementary ones, we set the following definitions.
Let $\models\!\!- \subseteq \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$. For every $P, Q \in \mathscr{P}_f(\mathscr{A})$ we set:
— $P \overset{[0]}{\models\!\!-} Q$ iff $P \supseteq Q$
— $P \overset{[1]}{\models\!\!-} Q$ iff $\forall q \in Q, \exists R \subseteq P,\ R \models\!\!- q$
— $P \overset{\langle 0\rangle}{\models\!\!-} Q$ iff $P \overset{[0]}{\models\!\!-} Q$
— $P \overset{\langle 1\rangle}{\models\!\!-} Q$ iff $\forall q \in Q,\ (\exists R \subseteq P,\ R \models\!\!- q)$ or $(q \in P)$
— $P \overset{\langle n+1\rangle}{\models\!\!-} Q$ iff $\exists R \in \mathscr{P}_f(\mathscr{A})$, $P \overset{\langle 1\rangle}{\models\!\!-} R$ and $R \overset{\langle n\rangle}{\models\!\!-} Q$ (for every $n \geqslant 0$).
— $\overset{\langle *\rangle}{\models\!\!-} = \bigcup_{n \geqslant 0} \overset{\langle n\rangle}{\models\!\!-}$.
Given $\models\!\!-_1, \models\!\!-_2 \subseteq \mathscr{P}_f(\mathscr{A}) \times \mathscr{P}_f(\mathscr{A})$, for every $P, Q \in \mathscr{P}_f(\mathscr{A})$ we set:

$$P(\models\!\!-_1 \circ \models\!\!-_2)Q \quad \text{iff} \quad \exists R \subseteq \mathscr{A}, (P \models\!\!-_1 R) \wedge (R \models\!\!-_2 Q).$$

The particular deduction systems $\mathscr{D}_i = \langle \mathscr{A}_i, H_i, \models\!\!-_{\mathscr{D}_i} \rangle$ $(i \in [0,1])$, that we shall introduce in Sections 4.3 and 10, will always be defined from simpler binary relations $|\!\models\!\!-_j$ by means of the above constructions.

The key-statement of this work is that a particular deduction system, $\mathscr{D}_0$ (defined in Section 4.3), is complete (Theorem 92). We prove this completeness result by exhibiting a "strategy" $\mathscr{S}$ which, for every true assertion constructs a finite $\mathscr{D}_0$-proof of this assertion.

4.2. Strategies

Let $\mathscr{D} = \langle \mathscr{A}, H, |\!-\!- \rangle$ be a deduction system. We call a *strategy* for $\mathscr{D}$ any partial map $\mathscr{S} : \mathscr{A}^+ \to \mathscr{A}^*$ such that:

(S1) if $\mathscr{S}(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ then $\exists\, Q \subseteq \{A_i \mid 1 \leqslant i \leqslant n-1\}$ such that

$$\{B_j \mid 1 \leqslant j \leqslant m\} \cup Q \,|\!-\!-\, A_n,$$

(S2) if $\mathscr{S}(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ then

$$\min\{H(A_i) \mid 1 \leqslant i \leqslant n\} = \infty \;\Rightarrow\; \min\{H(B_j) \mid 1 \leqslant j \leqslant m\} = \infty.$$

Given a strategy $\mathscr{S}$, we define $\mathscr{T}(\mathscr{S}, A)$, the proof-tree associated to the strategy $\mathscr{S}$ and the assertion A as the unique tree t such that:

$$\varepsilon \in \mathrm{dom}(t), \quad t(\varepsilon) = A$$

and, for every path $x_0 x_1, \cdots x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leqslant i \leqslant n-1$) if x_{n-1} has m sons $x_{n-1} \cdot 1, \cdots x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leqslant j \leqslant m$) then

$$\begin{aligned}
&(\forall i \in [1, n-1],\; A_i \neq A_n \text{ and } \mathscr{S}(A_1 \cdots A_n) = B_1 \cdots B_m) \text{ or}\\
&(\exists\, i \in [1, n-1],\; A_i = A_n \text{ and } m = 0) \text{ or}\\
&(A_1 \cdots A_n \notin \mathrm{dom}(\mathscr{S}) \text{ and } m = 0).
\end{aligned} \tag{41}$$

Notice that x_{n-1} is a leaf (i.e. $m = 0$) iff:

$$(\mathscr{S}(A_1 \cdots A_n) = \varepsilon) \text{ or } (\exists\, i \in [1, n-1], A_i = A_n) \text{ or } (A_1 \cdots A_n \notin \mathrm{dom}(\mathscr{S})). \tag{42}$$

Let us say that $\mathscr{S}$ *terminates* iff, $\forall A \in \chi^{-1}(1), \mathscr{T}(\mathscr{S}, A)$ is finite; $\mathscr{S}$ is said *closed* iff, $\forall W \in (\chi^{-1}(1))^+, W \in \mathrm{dom}(\mathscr{S})$ (i.e. $\mathscr{S}$ is defined on every non-empty sequence of true assertions). For every tree t let us define

$$\mathscr{L}(t) = \{t(x) \mid \forall y \in \mathrm{dom}(t), x \preccurlyeq y \Rightarrow x = y\},$$
$$\mathscr{I}(t) = \{t(x) \mid \exists\, y \in \mathrm{dom}(t), x \prec y\}.$$

(Here $\mathscr{L}$ stands for "leaves" and $\mathscr{I}$ stands for "internal labels").

Lemma 43. *If $\mathscr{S}$ is a strategy for the deduction-system $\mathscr{D}$ then, for every true assertion A:*

(1) *the set of labels of $\mathscr{T}(\mathscr{S}, A)$ is a $\mathscr{D}$-proof, relative to the set $\mathscr{L}(\mathscr{T}(\mathscr{S}, A)) - \mathscr{I}(\mathscr{T}(\mathscr{S}, A))$.*

(2) *every label of a leaf is true.*

Proof. Let us suppose that $H(A) = \infty$. Let $t = \mathscr{T}(\mathscr{S}, A)$, $P = \mathrm{im}(t)$ (the set of labels of t), $\mathscr{H} = \mathscr{L}(\mathscr{T}(\mathscr{S}, A)) - \mathscr{I}(\mathscr{T}(\mathscr{S}, A))$.

Using (S2), one can prove by induction on the depth of $x \in \mathrm{dom}(t)$ that, $H(t(x)) = \infty$. Point (2) is then proved. Let x be an internal node of t, with sons $x \cdot 1, x \cdot 2, \cdots, x \cdot m$ ($m \geqslant 0$), and with ancestors $y_1, y_2, \ldots, y_{n-1}, y_n = x$ ($n \geqslant 1$), such that

$$t(y_1) \cdots t(y_n) = A_1 \cdots A_n, \quad t(x \cdot 1) t(x \cdot 2) \cdots t(x \cdot m) = B_1 \cdot B_2 \cdots B_m.$$

By definition of $\mathscr{T}(\mathscr{S},A)$,

$$\mathscr{S}(A_1\cdots A_n)=B_1\cdots B_m$$

and by condition (S1):

$$\exists Q\subseteq\{A_i\mid 1\leqslant i\leqslant n-1\},\quad \text{such that } \{B_j\mid 1\leqslant j\leqslant m\}\cup Q\mid\!\!-\!\!-\, A_n.$$

It follows that for every $p\notin\mathscr{H}$, $\exists R\subseteq P$, $R\mid\!\!-\!\!-\, p$, hence

$$\forall p\in P,\quad (\exists R\subseteq P, R\mid\!\!-\!\!-\, p)\quad \text{or}\quad p\in\mathscr{H}.$$

Point (1) is proved. $\square$

Lemma 44. *If $\mathscr{S}$ is a closed strategy for $\mathscr{D}$, then, for every true assertion A, the set of labels of $\mathscr{T}(\mathscr{S},A)$ is a $\mathscr{D}$-proof.*

Proof. Let us suppose that $H(A)=\infty$. Let $t=\mathscr{T}(\mathscr{S},A)$ and let $P,\mathscr{H}$ be defined as above. By Lemma 43, P is a $\mathscr{D}$-proof relative to $\mathscr{H}$. By Lemma 43 point (2) and Lemma 41, every label of a node of t is true. By the definition of a closed strategy, if $p\in\mathscr{H}$ and x is a leaf of t such that $p=t(x)$ then, the only possible true assertion in clause (42) is "$\mathscr{S}(A_1\cdots A_n)=\varepsilon$", which implies that

$$\exists Q\subseteq P,\quad Q\mid\!\!-\!\!-\, t(x).$$

Lemma 43 point (1) and this fact show that P is a proof. $\square$

Lemma 45. *If $\mathscr{D}$ admits some terminating, closed strategy then $\mathscr{D}$ is complete.*

Proof. Clear from Lemma 44. $\square$

4.3. System $\mathscr{D}_0$

Let us define here a particular deduction system $\mathscr{D}_0$ "Taylored for the equivalence problem for dpda's".

Given a fixed dpda $\mathscr{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathscr{M}$ (see Section 3.1) and the set $\mathsf{DRB}\langle\langle V\rangle\rangle$ (the set of Deterministic Rational Boolean series over V^*). The set of assertions is defined by

$$\mathscr{A}=\mathbb{N}\times\mathsf{DRB}\langle\langle V\rangle\rangle\times\mathsf{DRB}\langle\langle V\rangle\rangle$$

i.e. an assertion is here a *weighted equation* over $\mathsf{DRB}\langle\langle V\rangle\rangle$.

The "cost-function" $H:\mathscr{A}\to\mathbb{N}\cup\{\infty\}$ is defined by

$$H(n,S,S')=n+2\cdot\mathrm{Div}(S,S'),$$

where $\mathrm{Div}(S,S')$, the *divergence* between S and S', is defined by

$$\mathrm{Div}(S,S')=\inf\{|u|\mid u\in\varphi(S)\triangle\varphi(S')\}.$$

Let us notice that here

$$\chi(n,S,S') = 1 \quad \Leftrightarrow \quad S \equiv S'.$$

We define a binary relation $|\!\!\vdash\!\!- \subset \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(R0)

$$\{(p,S,T)\} \,|\!\!\vdash\!\!-\, (p+1,S,T)$$

for $p \in \mathbb{N},\ S,T \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R1)

$$\{(p,S,T)\} \,|\!\!\vdash\!\!-\, (p,T,S)$$

for $p \in \mathbb{N},\ S,T \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R2)

$$\{(p,S,S'),(p,S',S'')\} \,|\!\!\vdash\!\!-\, (p,S,S'')$$

for $p \in \mathbb{N},\ S,S',S'' \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R3)

$$\emptyset \,|\!\!\vdash\!\!-\, (0,S,S)$$

for $S \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R′3)

$$\emptyset \,|\!\!\vdash\!\!-\, (0,S,T)$$

for $S \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ T \in \{\emptyset,\varepsilon\},\ S \equiv T$,

(R4)

$$\{(p+1,S \odot x,T \odot x) \mid x \in X\} \,|\!\!\vdash\!\!-\, (p,S,T)$$

for $p \in \mathbb{N},\ S,T \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ (S \not\equiv \varepsilon \wedge T \not\equiv \varepsilon)$,

(R5)

$$\{(p,S,S')\} \,|\!\!\vdash\!\!-\, (p+2,S \odot x,S' \odot x)$$

for $p \in \mathbb{N},\ S,T \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ x \in X$,

(R6)

$$\{(p,S \cdot T' + S',T')\} \,|\!\!\vdash\!\!-\, (p,S^* \cdot S',T')$$

for $p \in \mathbb{N},\ (S,S') \in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle,\ T' \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ S \not\equiv \varepsilon$,

(R7)

$$\{(p,S,S'),(p,T,T')\} \,|\!\!\vdash\!\!-\, (p,S+T,S'+T')$$

for $p \in \mathbb{N},\ (S,T),(S',T') \in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle$,

(R8)

$$\{(p, S, S')\} \mid\vdash\!\!- (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N}$, $S, S', T \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R9)

$$\{(p, T, T')\} \mid\vdash\!\!- (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}$, $S, T, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R10)

$$\emptyset \mid\vdash\!\!- (0, S, \rho_e(S))$$

for $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R11)

$$\emptyset \mid\vdash\!\!- (0, S, \varphi_2(S))$$

for $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$.

Remark 46. One can check that, by the results of Section 3, the above rules really belong to $\mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$.

Lemma 47. *Let* $P \in \mathscr{P}_f(\mathscr{A})$, $A \in \mathscr{A}$ *such that* $P \mid\vdash\!\!- A$. *Then* $\min\{H(p) \mid p \in P\} \leqslant H(A)$.

Proof. The only non-trivial checks of this property are for rules of type (R4) or (R6).

(R4) Let $S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $S \not\equiv \varepsilon$, $T \not\equiv \varepsilon$. If $\mathrm{Div}(S, T) = \infty$ the required inequality is true. If $\mathrm{Div}(S, T) = n \in \mathbb{N}$, let us consider some $u \in \varphi(S) \triangle \varphi(T)$, $|u| = n$. We can suppose, for example, that $S \odot u = \varepsilon$, $T \odot u \neq \varepsilon$. As $S \not\equiv \varepsilon$, $\exists x_0 \in X$, $\exists v \in X^*, u = x_0 \cdot v$. Hence $(S \odot x_0) \odot v = \varepsilon, (T \odot x_0) \odot v \neq \varepsilon, \mathrm{Div}(S \odot x_0, T \odot x_0) \leqslant |v|$. Hence we have

$$\begin{aligned}
\min\{H(p+1, S \odot x, T \odot x) \mid x \in X\} &\leqslant H(p+1, S \odot x_0, T \odot x_0) \\
&\leqslant (p+1) + 2|v| \\
&\leqslant (p+1) + 2 \cdot \mathrm{Div}(S, T) - 2 \\
&< H(p, S, T).
\end{aligned}$$

(R6) We are reduced to prove that, for every $n \geqslant 0$, $(S, S') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle$, $T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $S \not\equiv \varepsilon$,

$$S \cdot T' + S' \equiv_n T' \quad \Rightarrow \quad S^* \cdot S' \equiv_n T'. \tag{43}$$

Let us suppose that $S \cdot T' + S' \equiv_n T'$. By definition of the star operation:

$$S^{n+1} \cdot S^* \cdot S' + \sum_{k=0}^{n} S^k \cdot S' = S^* \cdot S'. \tag{44}$$

And by the properties established in the treatment of (R7), (R9):

$$S^{n+1} \cdot T' + \sum_{k=0}^{n} S^k \cdot S' \equiv_n T'. \tag{45}$$

Let $u \in X^{\leqslant n}$. As $S \not\equiv \varepsilon$, $\forall u' \preccurlyeq u$, $S^{n+1} \odot u' \neq \varepsilon$. By Lemma 321, for every $U \in \mathsf{B}\langle\langle V \rangle\rangle$,

$$(S^{n+1} \cdot U) \odot u = (S^{n+1} \odot u) \cdot U \neq \varepsilon.$$

Using now equations (44), (45) we obtain that

$$S^* \cdot S' \odot u = \varepsilon \;\Leftrightarrow\; \sum_{k=0}^{n} S^k \cdot S' \odot u = \varepsilon \;\Leftrightarrow\; T' \odot u = \varepsilon.$$

hence $S^* \cdot S' \equiv_n T'$. This ends the proof of implication (43). □

Let us define $\models\!\!-$ by: for every $P \in \mathscr{P}_f(\mathscr{A})$, $A \in \mathscr{A}$,

$$P \models\!\!- A \;\Leftrightarrow\; P \overset{\langle * \rangle}{||\!\!\!-\!\!-} \circ \overset{[1]}{||\!\!\!-\!\!-}_{0,3,4,10,11} \circ \overset{\langle * \rangle}{||\!\!\!-\!\!-} \{A\},$$

where $||\!\!\!-\!\!-_{0,3,4,10,11}$ is the relation defined by (R0),(R3),(R′3),(R4),(R10),(R11) only. We let

$$\mathscr{D}_0 = \langle \mathscr{A}, H, \models\!\!- \rangle.$$

Lemma 48. *$\mathscr{D}_0$ is a deduction system.*

Proof. From the well-known decidability properties of finite automata and the hypothesis that φ_2 is a *computable* bijection, we conclude that $\models\!\!-$ is recursively enumerable. Using Lemma 47, one can show by induction on n that

$$P \overset{\langle n \rangle}{||\!\!\!-\!\!-} Q \;\Rightarrow\; \forall q \in Q, \quad \min\{H(A) \mid A \in P\} \leqslant H(q).$$

The proof of Lemma 47 also reveals that

$$P \,||\!\!\!-\!\!-_{\{0,3,4,10,11\}} q \;\Rightarrow\; (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

It follows that, for every $m, n \geqslant 0$:

$$P \overset{\langle n \rangle}{||\!\!\!-\!\!-} Q \overset{[1]}{||\!\!\!-\!\!-}_{0,3,4,10,11} R \overset{\langle m \rangle}{||\!\!\!-\!\!-} q \;\Rightarrow\; (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

Both axioms $(A1), (A2)$ are fulfilled. □

Let us remark the following algebraic corollaries of Lemma 47.

Corollary 49. (C1) $\forall (S,S') \in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle,\ T' \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ S \not\equiv \varepsilon,$

$$S \cdot T' + S' \equiv T' \quad \Rightarrow \quad S^* \cdot S' \equiv T'$$

(C2) $\forall S,S' \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ T \in \mathsf{DRB}\langle\langle V\rangle\rangle,$

$$[S \cdot T \equiv S' \cdot T \ \textit{and}\ T \not\equiv \emptyset] \quad \Rightarrow \quad S \equiv S'.$$

Proof. Statement (C1) is a direct corollary of the fact that the value of H at the left-hand side of rule (R6) is smaller or equal to the value of H at the right-hand side of rule (R6). Let us prove (C2): let us consider $S,S' \in \mathsf{DRB}\langle\langle V\rangle\rangle,\ T \in \mathsf{DRB}\langle\langle V\rangle\rangle$, such that

$$S \cdot T \equiv S' \cdot T \quad \text{and} \quad S \not\equiv S'. \tag{46}$$

Let

$$u = \min\{v \in X^* \mid (S \odot v = \varepsilon) \ \Leftrightarrow\ (S' \odot v \neq \varepsilon)\}.$$

From the hypothesis that $S \cdot T \equiv S' \cdot T$, we get that, for every $v \in X^*$,

$$(S \cdot T) \odot v \equiv (S' \cdot T) \odot v$$

and by the choice of u we obtain that

$$T \equiv (S' \odot u) \cdot T \quad \text{or} \quad (S \odot u) \cdot T \equiv T,$$

which, by (C1), implies

$$T \equiv (S' \odot u)^* \cdot \emptyset \quad \text{or} \quad (S \odot u)^* \cdot \emptyset \equiv T,$$

i.e.

$$T \equiv \emptyset. \tag{47}$$

We have proved that (46) implies (47), hence (C2). □

5. Triangulations

Let $S_1, S_2, \ldots, S_d$ be a family of deterministic rational boolean series over the structured alphabet V (i.e. $S_i \in \mathsf{DRB}\langle\langle V\rangle\rangle$). We recall V is the infinite alphabet associated with some dpda $\mathscr{M}$ as defined in Section 2.2.

Let us consider a sequence $\mathscr{S}$ of n "weighted" linear equations

$$(\mathscr{E}_i):\ \ p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j,\ \sum_{j=1}^{d} \beta_{i,j} S_j, \tag{48}$$

where $p_i \in \mathbb{N} - \{0\}$, and $A = (\alpha_{i,j}),\ B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n,d), with indices $m \leqslant i \leqslant m+n-1, 1 \leqslant j \leqslant d$. The coefficients $\alpha_{i,j}, \beta_{i,j}$, are supposed to belong to $\mathsf{DRB}\langle\langle V_1\rangle\rangle$ (i.e. they do not contain any occurrence of the "second-level" variables).

For any weighted equation, $\mathscr{E}=(p,S,S')$, we recall the "cost" of this equation is: $H(\mathscr{E})=p+2\cdot\mathrm{Div}(S,S')$.

We associate to every system (48) another system of weighted equations, $\mathrm{INV}(\mathscr{S})$, which "translates the equations of $\mathscr{S}$ into equations over the coefficients $(\alpha_{i,j},\beta_{i,j})$ only".[3] The general idea of the construction of INV consists in iterating the transformation used in the proof of $(1)\Rightarrow(2)\Rightarrow(3)$ in Lemma 322, i.e. the classical idea of *triangulating* a system of linear equations.

We assume here that

$$\forall j\in[1,d],\quad S_j\not\equiv\emptyset. \tag{49}$$

Let us define $\mathrm{INV}(\mathscr{S})$, $\mathrm{W}(\mathscr{S})\in\mathbb{N}\cup\{\bot\}$, $\mathrm{D}(\mathscr{S})\in\mathbb{N}$, by induction on n. $\mathrm{W}(\mathscr{S})$ is the *weight* of $\mathscr{S}$ while $\mathrm{D}(\mathscr{S})$ is the *weak codimension* of $\mathscr{S}$.

Case 1: $\alpha_{m,*}\equiv\beta_{m,*}$

$$\mathrm{INV}(\mathscr{S})=((\mathrm{W}(\mathscr{S}),\alpha_{m,j},\beta_{m,j}))_{1\leqslant j\leqslant d},\quad \mathrm{W}(\mathscr{S})=p_m-1,\quad \mathrm{D}(\mathscr{S})=0.$$

Case 2: $\alpha_{m,*}\not\equiv\beta_{m,*},\ n\geqslant 2,\ p_{m+1}-p_m\geqslant 2\cdot\mathrm{Div}(\alpha_{m,*},\beta_{m,*})+1$

Let us consider

$$u=\min\{v\in X^*\mid \exists j\in[1,d],(\alpha_{m,*}\odot v=\varepsilon_j^d)\ \Leftrightarrow\ (\beta_{m,*}\odot v\neq\varepsilon_j^d)\}. \tag{50}$$

(Corollary 320 ensures the existence of such a word u).

Let $j_0\in[1,n]$ such that $(\alpha_{m,*}\odot u=\varepsilon_{j_0}^d)\Leftrightarrow(\beta_{m,*}\odot u\neq\varepsilon_{j_0}^d)$.

Subcase 1: $\alpha_{m,j_0}\odot u=\varepsilon,\beta_{m,j_0}\odot u\neq\varepsilon$.

Let us consider the equation

$$(\mathscr{E}'_m):\ \ p_m+2\cdot|u|,S_{j_0},\quad \sum_{\substack{j=1\\ j\neq j_0}}^{d}(\beta_{m,j_0}\odot u)^*(\beta_{m,j}\odot u)S_j$$

and define a new system of weighted equations $\mathscr{S}'=(\mathscr{E}'_i)_{m+1\leqslant i\leqslant m+n-1}$ by

$$(\mathscr{E}'_i):\ \ p_i,\sum_{j\neq j_0}[\alpha_{i,j}+\alpha_{i,j_0}(\beta_{m,j_0}\odot u)^*(\beta_{m,j}\odot u)]\cdot S_j,$$
$$\sum_{j\neq j_0}[\beta_{i,j}+\beta_{i,j_0}(\beta_{m,j_0}\odot u)^*(\beta_{m,j}\odot u)]\cdot S_j.$$

(The above equation is seen as an equation between two linear combinations of the S_i's, $1\leqslant i\leqslant d$, where the j_0th coefficient is $\emptyset$ on both sides). We then define

$$\mathrm{INV}(\mathscr{S})=\mathrm{INV}(\mathscr{S}'),\quad \mathrm{W}(\mathscr{S})=\mathrm{W}(\mathscr{S}'),\quad \mathrm{D}(\mathscr{S})=\mathrm{D}(\mathscr{S}')+1.$$

[3] This function INV is an "elaborated version" of the *inverse* systems defined in [5, Eq. (2.8), p. 586, english version] or [5, Eq. (2.8), p. 677, english version] in the case of a single equation.

Subcase 2: $\alpha_{m,j_0} \odot u \neq \varepsilon$, $\beta_{m,j_0} \odot u = \varepsilon$.
(analogous to subcase 1).
Case 3: $\alpha_{m,*} \not\equiv \beta_{m,*}$, $n = 1$.
We then define

$$\mathrm{INV}(\mathscr{S}) = \perp, \quad \mathrm{W}(\mathscr{S}) = \perp, \quad \mathrm{D}(\mathscr{S}) = 0,$$

where $\perp$ is a special symbol which can be understood as meaning "undefined".
Case 4: $\alpha_{m,*} \not\equiv \beta_{m,*}$, $n \geqslant 2$, $p_{m+1} - p_m \leqslant 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$.
We then define

$$\mathrm{INV}(\mathscr{S}) = \perp, \quad \mathrm{W}(\mathscr{S}) = \perp, \quad \mathrm{D}(\mathscr{S}) = 0.$$

Lemma 51. *Let $\mathscr{S}$ be a system of weighted linear equations with deterministic rational coefficients. If* $\mathrm{INV}(\mathscr{S}) \neq \perp$ *then*, $\mathrm{INV}(\mathscr{S})$ *is a system of weighted linear equations with deterministic rational coefficients.*

Proof. Follows from Lemmas 311, 312 and the formula defining $\mathscr{S}'$ from $\mathscr{S}$. $\square$

From now on, and up to the end of this section, we simply write "linear equation" to mean weighted linear equation with deterministic rational coefficients.

Lemma 52. *Let $\mathscr{S}$ be a system of linear equations. If* $\mathrm{INV}(\mathscr{S}) \neq \perp$ *then* $\mathrm{INV}(\mathscr{S}) = (\bar{\mathscr{E}}_j)_{1 \leqslant j \leqslant d}$ *fulfills*:
(1) $\{\bar{\mathscr{E}}_j \mid 1 \leqslant j \leqslant d\} \cup \{\mathscr{E}_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \mid\!\!-\!\!- \mathscr{E}_{m+\mathrm{D}(\mathscr{S})}$
(2) $\min\{H(\mathscr{E}_i) \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})\} = \infty \Rightarrow \min\{H(\bar{\mathscr{E}}_j) \mid 1 \leqslant j \leqslant d\} = \infty$.

In what follows we sometimes write $\mathrm{INV}(\mathscr{S})$ to mean the set $\{\bar{\mathscr{E}}_j \mid 1 \leqslant j \leqslant d\}$ (i.e. we do not distinguish between the family of equations $\mathrm{INV}(\mathscr{S})$ and the corresponding set of equations). We also denote by $H(\mathrm{INV}(\mathscr{S}))$ the element $\min\{H(\bar{\mathscr{E}}_j) \mid 1 \leqslant j \leqslant d\} \in \mathbb{N} \cup \{\infty\}$.

Proof. See in Fig. 1 the "graph of the deductions" we use for proving point (1). Let us prove by induction on $\mathrm{D}(\mathscr{S})$ the following strengthened version of point (1):

$$\mathrm{INV}(\mathscr{S}) \cup \{\mathscr{E}_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \ \mid\!\mid\!\!\overset{\langle * \rangle}{-\!\!-} \ \tau_{-1}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})}), \tag{51}$$

where, for every integer $k \in \mathbb{Z}$, $\tau_k : \{(p, S, S') \in \mathscr{A} \mid p \geqslant -k\} \to \mathscr{A}$ is the *translation* map on the weights: $\tau_k(p, S, S') = (p + k, S, S')$.
if $\mathrm{D}(\mathscr{S}) = 0$: as $\mathrm{INV}(\mathscr{S}) \neq \perp, \mathscr{S}$ must fulfill the hypothesis of case 1.

$$\mathscr{E}_m = \left(p_m, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j \right) = \mathscr{E}_{m+\mathrm{D}(\mathscr{S})}$$

$$\mathrm{INV}(\mathscr{S}) = ((p_m - 1, \alpha_{m,j}, \beta_{m,j}))_{1 \leqslant j \leqslant d}.$$

Using rules (R7), (R8) we obtain

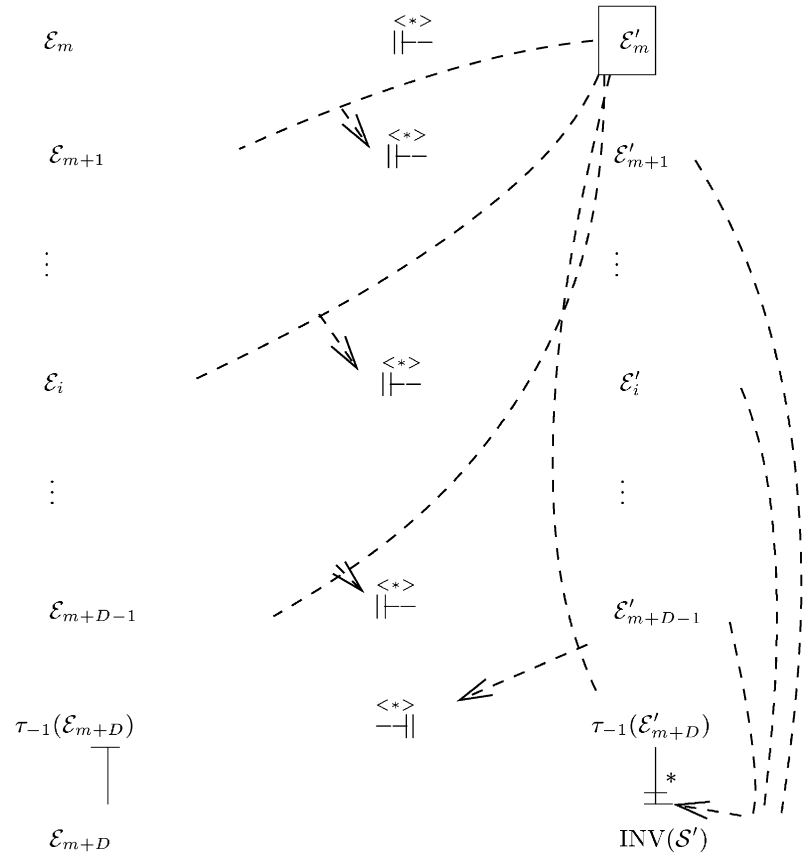

Fig. 1. Proof of Lemma 5.2.

$$\mathrm{INV}(\mathscr{S}) \overset{\langle * \rangle}{||\!\!-\!\!\!-} \left(p_m - 1, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j \right) = \tau_{-1}(\mathscr{E}_m).$$

if $\mathrm{D}(\mathscr{S}) = n+1,\ n \geqslant 0$: $\mathscr{S}$ must fulfill case 2.

• Suppose case 2, subcase 1 occurs.

Using $|u|$ times (R5) and then (R6) (this is possible because $\beta_{m,j_0} \odot u \not\equiv \varepsilon$), we obtain a deduction

$$\mathscr{E}_m \overset{\langle |u|+1 \rangle}{||\!\!-\!\!\!-} \mathscr{E}'_m. \tag{52}$$

Using (R7)–(R9) we get that, for every $i \in [m+1, m+\mathrm{D}(\mathscr{S})]$,

$$\{\mathscr{E}_i, \mathscr{E}'_m\} \overset{\langle * \rangle}{||\!\!-\!\!\!-} (\max\{p_i, p_m + 2|u|\},$$

$$\sum_{j\neq j_0}(\alpha_{i,j}+\alpha_{i,j_0}(\beta_{m,j_0}\odot u)^*(\beta_{m,j}\odot u))\cdot S_j,$$

$$\sum_{j\neq j_0}(\beta_{i,j}+\beta_{i,j_0}(\beta_{m,j_0}\odot u)^*(\beta_{m,j}\odot u))\cdot S_j).$$

The hypothesis of case 2 implies that $\max\{p_{m+1}, p_m+2|u|\}=p_{m+1}$ and the fact that $\mathrm{INV}(\mathscr{S}')$ is defined implies that $\forall i\in[m+1, m+\mathrm{D}(\mathscr{S})]$, $p_i\geqslant p_{m+1}$, hence, $\max\{p_i, p_m+2|u|\}=p_i$ and the right-hand side of the above deduction is exactly $\mathscr{E}'_i$. Hence

$$\forall i\in[m+1, m+\mathrm{D}(\mathscr{S})],\quad \{\mathscr{E}_i,\mathscr{E}'_m\}\ \overset{\langle *\rangle}{||\!\!-\!\!-}\ \mathscr{E}'_i. \tag{53}$$

Using deductions (52) and (53), we obtain that

$$\{\mathscr{E}_i\,|\,m\leqslant i\leqslant m+\mathrm{D}(\mathscr{S})-1\}\ \overset{\langle *\rangle}{||\!\!-\!\!-}\ \{\mathscr{E}'_i\,|\,m\leqslant i\leqslant m+\mathrm{D}(\mathscr{S})-1\}. \tag{54}$$

By induction hypothesis

$$\mathrm{INV}(\mathscr{S}')\cup\{\mathscr{E}'_i\,|\,m+1\leqslant i\leqslant m+1+\mathrm{D}(\mathscr{S}')-1\}\ \overset{\langle *\rangle}{||\!\!-\!\!-}\ \tau_{-1}(\mathscr{E}'_{m+1+\mathrm{D}(\mathscr{S}')})$$

which is equivalent to

$$\mathrm{INV}(\mathscr{S})\cup\{\mathscr{E}'_i\,|\,m+1\leqslant i\leqslant m+\mathrm{D}(\mathscr{S})-1\}\ \overset{\langle *\rangle}{||\!\!-\!\!-}\ \tau_{-1}(\mathscr{E}'_{m+\mathrm{D}(\mathscr{S})}). \tag{55}$$

As $p_m+2\cdot|u|\leqslant p_{m+1}-1\leqslant p_{m+\mathrm{D}(\mathscr{S})}-1$, we have also the following inverse deduction (which is similar to deduction (53))

$$\{\mathscr{E}'_m,\tau_{-1}(\mathscr{E}'_{m+\mathrm{D}(\mathscr{S})})\}\ \overset{\langle *\rangle}{||\!\!-\!\!-}\ \tau_{-1}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})}). \tag{56}$$

Combining together deductions (54)–(56), we have proved (51). Using rule (R0), this last deduction leads to point (1) of the lemma.

• Suppose that case 2, subcase 2 occurs:
this case can be treated in the same way as subcase 1 just by exchanging the roles of $\boldsymbol{\alpha},\boldsymbol{\beta}$.

Let us prove statement (2) of the lemma.

We prove by induction on $\mathrm{D}(\mathscr{S})$ the statement:

$$\min\{H(\mathscr{E}_i)\,|\,m\leqslant i\leqslant m+\mathrm{D}(\mathscr{S})\}=\infty\ \Rightarrow\ H(\mathrm{INV}(\mathscr{S}))=\infty. \tag{57}$$

if $\mathrm{D}(\mathscr{S})=0$: as $\mathrm{INV}(\mathscr{S})\neq\bot$, case 1 must occur. $\alpha_{m,*}\equiv\beta_{m,*}$ implies that $H(\mathrm{INV}(\mathscr{S}))=\infty$, hence the statement is true.

if $\mathrm{D}(\mathscr{S})=p+1$, $p\geqslant 0$: as $\mathrm{D}(\mathscr{S})\geqslant 1$ and $\mathrm{INV}(\mathscr{S})\neq\bot$, case 2 must occur.

Using deductions (52) and (53) established above we obtain that

$$\{\mathscr{E}_i\,|\,m\leqslant i\leqslant m+\mathrm{D}(\mathscr{S})\}\ \overset{\langle *\rangle}{||\!\!-\!\!-}\ \{\mathscr{E}'_i\,|\,m+1\leqslant i\leqslant m+1+\mathrm{D}(\mathscr{S}')\},$$

which proves that

$$\min\{H(\mathscr{E}_i)\,|\,m \leqslant i \leqslant m+\mathrm{D}(\mathscr{S})\} \\ \leqslant \min\{H(\mathscr{E}'_i)\,|\,m+1 \leqslant i \leqslant m+1+\mathrm{D}(\mathscr{S}')\}. \tag{58}$$

As $\mathrm{D}(\mathscr{S}')=\mathrm{D}(\mathscr{S})-1$, we can use the induction hypothesis:

$$\min\{H(\mathscr{E}'_i)\,|\,m+1 \leqslant i \leqslant m+1+D(\mathscr{S}')\} = \infty \;\Rightarrow\; H(\mathrm{INV}(\mathscr{S}')) = \infty. \tag{59}$$

As $\mathrm{INV}(\mathscr{S})=\mathrm{INV}(\mathscr{S}')$, (58), (59) imply statement (57). □

Lemma 53. *Let $\mathscr{S}$ be a system of linear equations satisfying the hypothesis of case* 2. *Then*, $\forall i \in [m+1, m+n-1]$, $\|\alpha'_{i,*}\| \leqslant \|\alpha_{i,*}\| + \|\beta_{m,*}\| + K_0|u|$, $\|\beta'_{i,*}\| \leqslant \|\beta_{i,*}\| + \|\beta_{m,*}\| + K_0|u|$.

Proof. The formula defining $\mathscr{S}'$ from $\mathscr{S}$ show that

$$\alpha'_{i,*} = \alpha_{i,*}\nabla_{j_0}(\nabla^*_{j_0}(\beta_{m,*}\odot u)); \quad \beta'_{i,*} = \beta_{i,*}\nabla_{j_0}(\nabla^*_{j_0}(\beta_{m,*}\odot u)).$$

From these equalities and Lemmas 311, 312, 318, the inequalities on the norm follow. □

Let us consider the function F defined by

$$F(d,n) = \max\{\mathsf{Div}(A,B)\,|\,A,B \in \mathsf{DRB}_{1,d}\langle\langle V_1\rangle\rangle, \|A\| \leqslant n, \|B\| \leqslant n, A \not\equiv B\}. \tag{60}$$

For every integer parameters $K_0, K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leqslant i \leqslant m+n-1}$ by

$$\delta_m = 0, \quad \ell_m = 0, \quad L_m = K_2, \quad s_m = K_3 \cdot K_2 + K_4, \quad S_m = 0, \quad \Sigma_m = 0, \tag{61}$$

$$\begin{aligned} \delta_{i+1} &= 2 \cdot F(d, s_i + \Sigma_i) + 1, \\ \ell_{i+1} &= 2 \cdot \delta_{i+1} + 3, \\ L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2, \\ s_{i+1} &= K_3 \cdot L_{i+1} + K_4, \\ S_{i+1} &= s_i + \Sigma_i + K_0 F(d, s_i + \Sigma_i), \\ \Sigma_{i+1} &= \Sigma_i + S_{i+1} \end{aligned} \tag{62}$$

for $m \leqslant i \leqslant m+n-2$.

These sequences are intended to have the following meanings when K_0, K_1, K_2, K_3, K_4 are chosen to be the constants defined in Section 6 and the equations $(\mathscr{E}_i)$ are labelling nodes of a B-stacking sequence (see Section 8.2):
$\delta_{i+1} \leqslant$ increase of weight between $\mathscr{E}_i, \mathscr{E}_{i+1}$,
$\ell_{i+1} \geqslant$ increase of depth between $\mathscr{E}_i, \mathscr{E}_{i+1}$,
$L_{i+1} \geqslant$ increase of depth between $\mathscr{E}_m, \mathscr{E}_{i+1}$,
$s_{i+1} \geqslant$ size of the coefficients of $\mathscr{E}_{i+1}$,

$$
\begin{array}{lllll}
\mathcal{E}_m = \mathcal{E}_m^{(0)} & \boxed{(\mathcal{E}_m^{(0)})'} & & & \\
\mathcal{E}_{m+1} = \mathcal{E}_{m+1}^{(0)} & \mathcal{E}_{m+1}^{(1)} & \ddots & & \\
\vdots & \vdots & & \boxed{(\mathcal{E}_{i-1}^{(i-1-m)})'} & \\
\mathcal{E}_i = \mathcal{E}_i^{(0)} & \mathcal{E}_i^{(1)} & \cdots & \mathcal{E}_i^{(i-m)} & \ddots \\
 & & & & \boxed{(\mathcal{E}_{m-1+D}^{(D-1)})'} \\
\vdots & \vdots & & \vdots & \vdots \\
\mathcal{E}_{m+d-1} = \mathcal{E}_{m+d-1}^{(0)} & \mathcal{E}_{m+d-1}^{(1)} & \cdots & \mathcal{E}_{m+d-1}^{(i-m)} & \mathcal{E}_{m+d-1}^{(D)}
\end{array}
$$

Fig. 2. Proof of Lemma 5.4.

$S_{i+1} \geqslant$ size of the coefficients of $\mathscr{E}_{i+1}^{(i+1-m)}$ (these systems are introduced below in the proof of Lemma 54),
$\Sigma_{i+1} \geqslant$ increase of the coefficients between $\mathscr{E}_k^{(i-m)}, \mathscr{E}_k^{(i+1-m)}$ (for $k \geqslant i+1$).

For every linear equation $\mathscr{E} = (p, \sum_{j=1}^{d} \alpha_j S_j,\ \sum_{j=1}^{d} \beta_j S_j)$, we define

$$|||\mathscr{E}||| = max\{\|(\alpha_1, \ldots, \alpha_d)\|, \|(\beta_1, \ldots, \beta_d)\|\}.$$

Lemma 54. *Let* $\mathscr{S} = (\mathscr{E}_i)_{m \leqslant i \leqslant m+d-1}$ *be a system of* d *linear equations such that* $H(\mathscr{E}_i) = \infty$ *(for every* i*) and*
(1) $\forall i \in [m, m+d-1],\ |||\mathscr{E}_i||| \leqslant s_i$,
(2) $\forall i \in [m, m+d-2],\ \mathrm{W}(\mathscr{E}_{i+1}) - \mathrm{W}(\mathscr{E}_i) \geqslant \delta_{i+1}$.
Then $\mathrm{INV}(\mathscr{S}) \neq \perp$, $\mathrm{D}(\mathscr{S}) \leqslant d-1$, $\forall \mathscr{E} \in \mathrm{INV}(\mathscr{S})$, $|||\mathscr{E}||| \leqslant \Sigma_{m+\mathrm{D}(\mathscr{S})} + s_{m+\mathrm{D}(\mathscr{S})}$.

Proof. (Fig. 2 might help the reader to follow the definitions below). Let us define a sequence of systems $\mathscr{S}^{(i-m)} = (\mathscr{E}_k^{(i-m)})_{m \leqslant i \leqslant k \leqslant m+d-1}$, where $i \in [m, m + \mathrm{D}(\mathscr{S})]$, by induction:

— $\mathscr{E}_k^{(0)} = \mathscr{E}_k$ for $m \leqslant k \leqslant m+d-1$;
— if case 1 or case 3 or case 4 is realized, $\mathrm{D}(\mathscr{S}) = 0$, hence $\mathscr{S}^{(i-m)}$ is well-defined for $m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})$;
— if case 2 is realized then we set: $\forall i \geqslant m+1$, $\mathscr{E}_k^{(i-m)} = (\mathscr{E}_k')^{(i-m-1)}$, for $m+1 \leqslant k \leqslant m+d-1$.

Let us prove by induction on $i \in [m, m + \mathrm{D}(\mathscr{S})]$ that, $\forall k \in [i, m + d - 1]$:

$$|||\mathscr{E}_k^{(i-m)}||| \leqslant s_k + \Sigma_i. \tag{63}$$

$i = m$: in this case

$$|||\mathscr{E}_k^{(i-m)}||| = |||\mathscr{E}_k||| \leqslant s_k = s_k + \Sigma_m.$$

$i + 1 \leqslant m + \mathrm{D}(\mathscr{S})$: in this case, by Lemma 53

$$|||\mathscr{E}_k^{(i+1-m)}||| \leqslant |||\mathscr{E}_k^{(i-m)}||| + |||\mathscr{E}_i^{(i-m)}||| + K_0|u_i|,$$

where

$$u_i = \min\{v \in X^* \mid \exists j \in [1, d], (\alpha_{i,*}^{(i-m)} \odot v = \varepsilon_j^d) \iff (\beta_{i,*}^{(i-m)} \odot v \neq \varepsilon_j^d)\}. \tag{64}$$

By definition of F and the induction hypothesis

$$|u_i| \leqslant F(d, |||\mathscr{E}_i^{(i-m)}|||) \leqslant F(d, s_i + \Sigma_i).$$

Hence

$$\begin{aligned}|||\mathscr{E}_k^{(i+1-m)}||| \leqslant (s_k + \Sigma_i) + (s_i + \Sigma_i) + K_0F(d, s_i + \Sigma_i) &= (s_k + \Sigma_i) + S_{i+1}\\ &= s_k + \Sigma_{i+1}.\end{aligned}$$

Let us notice that $\mathrm{D}(\mathscr{S})$ is always an integer and that this proof is valid for $m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})$, $i \leqslant k \leqslant m + d - 1$.

Let us prove now that $\mathrm{INV}(\mathscr{S}) \neq \perp$. Let us consider the system $(\mathscr{E}_k^{(\mathrm{D}(\mathscr{S}))})_{m+\mathrm{D}(\mathscr{S}) \leqslant k \leqslant m+d-1}$.

If $\mathrm{D}(\mathscr{S}) = d - 1$, as the system $(\mathscr{E}_{\mathrm{D}(\mathscr{S})}^{(\mathrm{D}(\mathscr{S}))})$ consists of a single equation, it must fulfill either case 1 or case 3 of the definition of INV.

Using the successive deductions (52), (53) established in the proof of Lemma 52 we get that

$$\{\mathscr{E}_i \mid m \leqslant i \leqslant m + d - 1\} \mathrel{||\!\!\overset{\langle * \rangle}{\vdash\!\!-}} \{\mathscr{E}_{m+d-1}^{(d-1)}\}.$$

Using now the hypothesis that $H(\mathscr{E}_i) = \infty$ (for $m \leqslant i \leqslant m + d - 1$), we obtain

$$H(\mathscr{E}_{m+d-1}^{(d-1)}) = \infty. \tag{65}$$

For any system of equations $\mathscr{S}$, let us define the *column-support* of the system as

$$\mathrm{csupp}(\mathscr{S}) = \left\{ j \in [1, d] \mid \sum_{i=m}^{m+n-1} \alpha_{i,j} + \beta_{i,j} \neq \emptyset \right\}.$$

Let us consider $\delta = \mathrm{Card}(\mathrm{csupp}(\mathscr{S}^{(d-1)})$. One can prove by induction on i that

$$\mathrm{Card}(\mathrm{csupp}(\mathscr{S}^{(i-m)})) \leqslant d - i + m$$

hence

$$\delta = \mathrm{Card}(\mathrm{csupp}(\mathscr{S}^{(d-1)})) \leqslant d-(d-1) = 1.$$

— If $\delta = 1, \mathrm{csupp}(\mathscr{S}^{(d-1)}) = \{j_0\}$, for some $j_0 \in [1,d]$.
By Corollary 49 point (C2), and hypothesis (49), the implication

$$[\alpha^{(d-1)}_{m+d-1,j_0} S_{j_0} \equiv \beta^{(d-1)}_{m+d-1,j_0} S_{j_0}] \;\Rightarrow\; \alpha^{(d-1)}_{m+d-1,j_0} \equiv \beta^{(d-1)}_{m+d-1,j_0}$$

holds. Hence, by (65), $\alpha^{(d-1)}_{m+d-1,j_0} \equiv \beta^{(d-1)}_{m+d-1,j_0}$, i.e. $\mathscr{S}^{(d-1)}$ fulfills case 1, so that

$$\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\mathscr{S}^{(d-1)}) \neq \perp.$$

— If $\delta = 0$, $\mathrm{csupp}(\mathscr{S}) = \emptyset$.
Then $\alpha^{(d-1)}_{m+d-1,*} = \beta^{(d-1)}_{m+d-1,*} = \emptyset^d$. Here also $\mathscr{S}^{(d-1)}$ fulfills case 1.

If $\mathrm{D}(\mathscr{S}) < d-1$, by hypothesis

$$\mathrm{W}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})+1}) - \mathrm{W}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})}) \geqslant \delta_{m+\mathrm{D}(\mathscr{S})+1} = 2F(d, s_{m+\mathrm{D}(\mathscr{S})} + \Sigma_{m+\mathrm{D}(\mathscr{S})}) + 1.$$

If $\alpha^{\mathrm{D}(\mathscr{S})}_{m+\mathrm{D}(\mathscr{S}),*} \equiv \beta^{\mathrm{D}(\mathscr{S})}_{m+\mathrm{D}(\mathscr{S}),*}$, then $\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}$ fulfills case 1 of the definition of INV, hence $\mathrm{INV}(\mathscr{S}) \neq \perp$.

Otherwise, let us consider:

$$u = \min\{v \in X^* \mid \exists j \in [1,d], (\alpha^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S}),*} \odot v = \varepsilon^d_j) \Leftrightarrow (\beta^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S}),*} \odot v \neq \varepsilon^d_j)\}. \tag{66}$$

By definition of F and inequality (63)

$$|u| \leqslant F(d, |||\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}|||) \leqslant F(d, s_{m+\mathrm{D}(\mathscr{S})} + \Sigma_{m+\mathrm{D}(\mathscr{S})}).$$

Hence $p_{m+\mathrm{D}(\mathscr{S})+1} - p_{m+\mathrm{D}(\mathscr{S})} \geqslant 2|u|+1$, i.e. the hypothesis of case 2 is realized. This proves that $\mathrm{D}(\mathscr{S}^{(\mathrm{D}(\mathscr{S}))}) \geqslant 1$ while in fact, $\mathrm{D}(\mathscr{S}^{(\mathrm{D}(\mathscr{S}))}) = 0$. This contradiction shows that this last case ($\mathrm{D}(\mathscr{S}) < d-1$ and $\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}$ not fulfilling case 1 of definition of INV) is impossible. We have proved point (2) of the lemma. □

6. Constants

Let us *fix* a dpda $\mathscr{M}$. This short section is devoted to the definition of some integer *constants*: these integers depend on the dpda $\mathscr{M}$ only. The *motivation* of each of these definitions will appear later on, in different places for the different constants. The equations below provide an overview of the dependencies between these constants and allow to check that the definitions are sound (i.e. there is no hidden loop in these dependencies).

$$k_0 = \max\{\nu(v) \mid v \in V\}, \quad k_1 = \max\{2k_0+1, 3\}, \tag{67}$$

$$K_0 = \max\{\|(E_1, E_2, \ldots, E_n) \odot x\| \mid (E_i)_{1 \leqslant i \leqslant n} \text{ is a bijective numbering of some class in } V/\smile,\ x \in X\}. \tag{68}$$

K_0 serves as an upper-bound on the possible increase of norm under the right-action of a single letter $x \in X$, see Lemma 318.

$$D_1 = k_0 \cdot K_0 + |V_1| + 2, \quad k_2 = D_1 \cdot k_1 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + K_0. \tag{69}$$

k_1 is used in the definition of strategy T_B (Section 7), D_1 appears as an upper-bound on the marked part of series and k_2 is used in Lemma 85.

$$k_3 = k_2 + k_1 \cdot K_0, \quad k_4 = (k_3 + 1) \cdot k_0 + k_1. \tag{70}$$

k_3 appears in Lemma 86, k_4 is used in definition (98) of the polynomial space V_0.

$$\begin{aligned} K_1 &= k_1 \cdot K_0 + 1, \\ K_2 &= k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + 2 \cdot k_1 + 4. \end{aligned} \tag{71}$$

These constants K_1, K_2 appear in Lemma 88.

$$K_3 = k_0 |V_1|, \quad K_4 = D_1. \tag{72}$$

These constants K_3, K_4 appear in Lemma 89.

$$d_0 = \mathrm{Card}(X^{\leqslant k_4}), \tag{73}$$

d_0 appears as an upper-bound on the dimension of the polynomial space V_0 defined by Eq. (98) and used in Lemma 88. We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leqslant i \leqslant m+n-1}$ defined by the relations (62) of Section 5 where the parameters $K_0, K_1, \ldots, K_4$ are chosen to be the above constants and $m = 1, n = d = d_0$. Equivalently, they are defined by

$$\delta_1 = 0, \quad \ell_1 = 0, \quad L_1 = K_2, \quad s_1 = K_3 \cdot K_2 + K_4, \quad S_1 = 0, \quad \Sigma_1 = 0, \tag{74}$$

$$\begin{aligned} \delta_{i+1} &= 2 \cdot F(d_0, s_i + \Sigma_i) + 1, \\ \ell_{i+1} &= 2 \cdot \delta_{i+1} + 3, \\ L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2, \\ s_{i+1} &= K_3 \cdot L_{i+1} + K_4, \\ S_{i+1} &= s_i + \Sigma_i + K_0 \cdot F(d_0, s_i + \Sigma_i), \\ \Sigma_{i+1} &= \Sigma_i + S_{i+1} \end{aligned} \tag{75}$$

for $1 \leqslant i \leqslant d_0 - 1$. The function F is defined in Section 5. The constants

$$s_i, \delta_i \quad (1 \leqslant i \leqslant d_0) \tag{76}$$

appear in the hypothesis of Lemma 54 when we take $d = d_0$. The constant $\Sigma_{d_0} + s_{d_0}$ serves as an upper-bound on the norm of the equations produced by the strategy T_C (proof of Lemma 91).

$$N_0 = k_3 + 4. \tag{77}$$

N_0 appears as a lower bound for the norm in the definition of a B-stacking sequence (Section 8.2, condition (88)).

7. Strategies for $\mathscr{D}_0$

Let us define strategies for the particular system $\mathscr{D}_0$.

We define first auxiliary strategies $T_{cut}, T_\emptyset, T_\varepsilon, T_A, T_B, T_C$ and then derive some closed strategies from them. Let us fix here some total ordering on X and also some total ordering $\leqslant$ of type ω on $\mathscr{A}$ (inherited from the usual well-ordering of $\mathbb{N}$ by the fixed encoding). From these orderings one can construct in the usual way an ordering of type ω on the sets $X^*, \mathscr{A}^*$ and $\mathbb{N}^* \times (\mathsf{DRB}\langle\langle V\rangle\rangle)^*$.

T_{cut}: $T_{cut}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $\exists i \in [1, n-1], \exists S, T,$

$$A_i = (p_i, S, T), \quad A_n = (p_n, S, T), \quad p_i < p_n \quad \text{and} \quad m = 0.^4$$

$T_\emptyset$: $T_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T,\ A_n = (p, S, T),\ p \geqslant 0,\ S = T = \emptyset$ and $m = 0$

T_ε: $T_\varepsilon(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $A_n = (p, S, T),\ p \geqslant 0,\ S = T = \varepsilon$ and $m = 0$

T_A: $T_A(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, S, T), \quad m = |X|, \quad B_1 = (p+1, S \odot x_1, T \odot x_1), \ldots,$$
$$B_m = (p+1, S \odot x_m, T \odot x_m),$$

where $S \not\equiv \varepsilon,\ T \not\equiv \varepsilon$

T_B^+: $T_B^+(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $n \geqslant k_1 + 1, A_{n-k_1} = (\pi, \bar{U}, U')$, (where $\bar{U}$ is unmarked)

$$U' = \sum_{k=1}^{q} E_k \cdot \Phi_k \quad \text{for some } q \in \mathbb{N},\ E_k \in V_1,$$

$(E_k)_{1 \leqslant k \leqslant}$ bijective numbering of a class in $V_1/\smile$, $\Phi_k \in \mathsf{DRB}\langle\langle V\rangle\rangle$ $A_i = (\pi + k_1 + i - n, U_i, U_i')$ for $n - k_1 \leqslant i \leqslant n$, $(U_i)_{n-k_1 \leqslant i \leqslant n}$ is a derivation, $(U_i')_{n-k_1 \leqslant i \leqslant n}$ is a "stacking derivation" (see definitions in Section 3.4),

$$U_n' = \sum_{k=1}^{q} (E_k \odot u) \cdot \Phi_k, \quad \text{for some } u \in X^*,$$

$m = 1,\ B_1 = (\pi + k_1 - 1, V, V'),\ V = U_n,$

$$V' = \sum_{k=1}^{q} \bar{\rho}_e(E_k \odot u) \cdot (\bar{U} \odot u_k),$$

where $\forall k \in [1, q],\ u_k = \min(\varphi(E_k))$.

T_B^-: T_B^- is defined in the same way as T_B^+ by exchanging the left series (S^-) and right series (S^+) in every assertion (p, S^-, S^+).

T_C: $T_C(A_1 \cdots A_n) = B_1 \cdots B_m$ iff there exists $d \in [1, d_0],\ D \in [0, d-1],\ S_1, S_2, \ldots, S_d \in \mathsf{DRB}\langle\langle V\rangle\rangle - \{\emptyset\},\ 1 \leqslant \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n$, such that,

(C1) every equation $\mathscr{E}_i = A_{\kappa_i} = (p_{\kappa_i} S^-_{p_{\kappa_i}}, S^+_{p_{\kappa_i}})$ is a weighted equation over $S_1, S_2, \ldots, S_d$, with coefficients in $\mathsf{DB}_{1,d}\langle V_1\rangle$ and $p_{\kappa_i} \geqslant 1$,

---

[4] I.e. $B_1 \cdots B_m = \varepsilon$.

(C2) $\mathscr{S} = (\mathscr{E}_i)_{1 \leqslant i \leqslant D+1}$ is such that,
$\mathrm{INV}(\mathscr{S}) \neq \perp$, $\mathrm{D}(\mathscr{S}) = D$,
$|||\mathscr{E}_i||| \leqslant s_i$ (for $1 \leqslant i \leqslant D+1$) and $p_{\kappa_{i+1}} - p_{\kappa_i} \geqslant \delta_{i+1}$ (for $1 \leqslant i \leqslant D$),

(C3) $(\kappa_1, \kappa_2, \ldots, \kappa_{D+1}, S_1, \ldots, S_d) \in \mathbb{N}^* \times (\mathsf{DRB}\langle\langle V \rangle\rangle)^*$ is the minimal vector satisfying conditions (C1),(C2) for the given sequence $(A_1 \cdots A_n)$ and

(C4) $B_1 \cdots B_m = \varphi_2^{-1}(\mathrm{INV}(\mathscr{S}))$ (where φ_2^{-1} is the obvious extension of φ_2^{-1} to pairs of series and then to sequences of weighted equations; in other words the result of T_C is $\mathrm{INV}(\mathscr{S})$ where the series are replaced by the corresponding second-level variables).

Lemma 71. *$T_{cut}, T_{\emptyset}, T_{\varepsilon}, T_A$ are $\mathscr{D}_0$-strategies.*

Proof. T_{cut}: (S1) is true by rule R0. (S2) is trivially true.
$T_{\emptyset}, T_{\varepsilon}$: (S1) is true by rule R3. (S2) is trivially true.
T_A: by rule (R4), $\{B_j \mid 1 \leqslant j \leqslant m\} \mid\!\!\vdash\!\!-_4 A_n$, which proves (S1). Suppose $H(A_n) = \infty$, i.e. $S \equiv T$. Then, $\forall j \in [1, m]$, $S \odot x_j \equiv T \odot x_j$, so that $\min\{H(B_j) \mid 1 \leqslant j \leqslant m\} = \infty$. (S2) is proved. $\square$

Lemma 72. *T_B^+, T_B^- are $\mathscr{D}_0$-strategies.*

Proof. Let us show that T_B^+ is a $\mathscr{D}_0$-strategy.

Let us use the notation of the definition of T_B^+. Let $\mathscr{H} = \{(\pi, \bar{U}, U'), (\pi + k_1 - 1, V, V')\}$. Let us show that

$$\mathscr{H} \;\; \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{D}_0} (\pi + k_1 - 1, U_n, U'_n). \tag{78}$$

Using rule (R5) we obtain: $\forall k \in [1, q]$,

$$\begin{aligned}\{(\pi, \bar{U}, U')\} = \left\{ \left(\pi, \bar{U}, \sum_{j=1}^{q} E_j \cdot \Phi_j \right) \right\} &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{R5} (\pi + 2 \cdot |u_k|, \bar{U} \odot u_k, U' \odot u_k) \\ &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{R0} (\pi + 2 \cdot k_0, \bar{U} \odot u_k, U' \odot u_k) \\ &= \quad (\pi + 2 \cdot k_0, \bar{U} \odot u_k, \Phi_k).\end{aligned} \tag{79}$$

Using rule (R1), (R10), for every $k \in [1, q]$:

$$\emptyset \;\; \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{D}_0} (0, E_k \odot u, \bar{\rho}_e(E_k \odot u)). \tag{80}$$

Using (80), (79) and rules (R0), (R2), (R7), (R8), we obtain

$$\begin{aligned}\{(\pi, \bar{U}, U')\} \;\; &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{D}_0} \left(\pi + 2k_0, \sum_{k=1}^{q} (E_k \odot u) \cdot \Phi_k, \sum_{k=1}^{q} \bar{\rho}_e(E_k \odot u) \cdot (\bar{U} \odot u_k) \right) \\ = \quad &\{(\pi, \bar{U}, U')\} \;\; \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{D}_0} (\pi + 2k_0, U'_n, V').\end{aligned} \tag{81}$$

Let us recall that $U_n = V$. Hence, by (R0), (R1), (R2)

$$\{(\pi + k_1 - 1, V, V'), (\pi + 2k_0, U_n', V')\} \;||\!\overset{\langle * \rangle}{\vdash\!\!-}_{\mathscr{D}_0} (\pi + k_1 - 1, U_n, U_n'). \tag{82}$$

By (81), (82), (78) is proved. Using now (78) and rule (R0), we obtain

$$\mathscr{H} \;||\!\overset{\langle * \rangle}{\vdash\!\!-}_{\mathscr{D}_0} (\pi + k_1 - 1, U_n, U_n') \vdash\!\!-_{R0} (\pi + k_1, U_n, U_n'), \tag{83}$$

i.e. T_B^+ fulfills (S1).

Let us suppose now that $\forall i \in [n - k_1, n]$, $U_i \equiv U_i'$. Then, by (81), $U_n' \equiv V'$ and by hypothesis $V = U_n \equiv U_n'$. Hence $V \equiv V'$. This shows that T_B^+ fulfills (S2).

An analogous proof can obviously be written for T_B^-. □

Lemma 73. *Let (p, S, S') be a weighted equation, i.e. $p \in \mathbb{N}$, $S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle$. Then $\{(p, S, S')\} \;||\!\overset{\langle * \rangle}{\vdash\!\!-}_{\mathscr{D}_0} \{(p, \varphi_2(S), \varphi_2(S'))\}$ and $\{(p, \varphi_2(S), \varphi_2(S'))\} \;||\!\overset{\langle * \rangle}{\vdash\!\!-}_{\mathscr{D}_0} \{(p, S, S')\}$.*

Proof. Follows easily from (R1), (R2), (R11). □

Lemma 74. *T_C is a $\mathscr{D}_0$-strategy.*

Proof. By Lemma 52 point (1), combined with Lemma 73, (S1) is proved. By Lemma 52 point (2), combined with Lemma 73, (S2) is proved. □

Let us define the strategy $\mathscr{S}_{AB}$ by: for every $W = A_1 A_2 \cdots A_n$,

(0) if $W \in \mathrm{dom}(T_{cut})$, then $\mathscr{S}_{AB}(W) = T_{cut}(W)$ (1) elsif $W \in \mathrm{dom}(T_\emptyset)$, then $\mathscr{S}_{AB}(W) = T_\emptyset(W)$,

(2) elsif $W \in \mathrm{dom}(T_\varepsilon)$, then $\mathscr{S}_{AB}(W) = T_\varepsilon(W)$, (4) elsif $W \in \mathrm{dom}(T_B^+)$, then $\mathscr{S}_{AB}(W) = T_B^+(W)$,

(5) elsif $W \in \mathrm{dom}(T_B^-)$, then $\mathscr{S}_{AB}(W) = T_B^-(W)$, (6) elsif $W \in \mathrm{dom}(T_A)$, then $\mathscr{S}_{AB}(W) = T_A(W)$,

(7) else $\mathscr{S}_{AB}(W)$ is undefined.

The strategy $\mathscr{S}_{ABC}$ is obtained by inserting "(3) elsif $W \in \mathrm{dom}(T_C)$, then $\mathscr{S}_{ABC}(W) = T_C(W)$" in the above list of cases.

Lemma 75. *$\mathscr{S}_{ABC}, \mathscr{S}_{AB}$ are closed.*

Proof. Given any true assertion $A_n = (\pi, S, T)$ and any word $W = A_1 \cdots A_n$, at least one of T_ε, T_A is defined on W. □

8. Tree analysis

This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathscr{S}_{AB}$ defined in Section 7. The main results are Lemmas 810 and 811 whose combination asserts that if some branch of τ is infinite, then there exists some finite prefix on which

T_C is defined. This key technical result will ensure termination of strategy $\mathscr{S}_{ABC}$ (see Section 9).

We fix throughout this section a tree $\tau = \mathscr{T}(\mathscr{S}_{AB}, (\pi_0, U_0^-, U_0^+))$ (i.e. the proof tree associated to the assertion (π_0, U_0^-, U_0^+) by the strategy $\mathscr{S}_{AB}$). We suppose that

$$U_0^-, U_0^+ \text{ are both linear polynomials.} \tag{84}$$

(i.e. elements of $\mathsf{DB}\langle V_1\rangle \cdot (V_2 \cup \{\varepsilon\})$)

$$U_0^-, U_0^+ \text{ are both unmarked} \tag{85}$$

$$U_0^- \equiv U_0^+. \tag{86}$$

We recall that, formally, τ is a map $\mathrm{dom}(\tau) \to \mathbb{N} \times \mathsf{DRB}\langle\langle V\rangle\rangle \times \mathsf{DRB}\langle\langle V\rangle\rangle$ such that $\mathrm{dom}(\tau) \subseteq \{1, \ldots, |X|\}^*$ is closed under prefix and under "left-brother" (i.e. $w \cdot (i+1) \in \mathrm{dom}(\tau) \Rightarrow w \cdot i \in \mathrm{dom}(\tau)$).

8.1. Depth and weight

In this paragraph we check that the *weight* and the *depth* of a given node are closely related. Let us say that the strategy T "occurs at" node x iff,

$$\tau(x) = T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x| - 1])),$$

i.e. the label of x belongs to the image of the path from ε (included) to x (excluded) by the strategy T.

Lemma 81. *Let* $\alpha \in \{-, +\}, A_1, \ldots, A_n \in \mathscr{A}$ *such that* $T_B^\alpha(A_1 \cdots A_n)$ *is defined. Then,* $\forall i \in [n - k_1 + 1, n],\ A_i \neq T_B(A_1 \cdots A_{i-1})$.

In other words: if T_B occurs at node x of τ, it cannot occur at any of its k_1 above immediate ancestors.

Proof. Suppose that $\exists i \in [n - k_1 + 1, n],\ A_i = T_B(A_1 \cdots A_{i-1})$. Hence $\pi_i = \pi_{i-1} - 1 < \pi_{n-k_1} + i$, contradicting one of the hypothesis under which $T_B(A_1 \cdots A_n)$ is defined. □

Lemma 82. *Let* τ *be a proof-tree associated to the strategy* $\mathscr{S}_{AB}$. *Let* $x, x' \in \mathrm{dom}(\tau)$, $x \leqslant x'$. *Then* $|W(x') - W(x)| \leqslant |x'| - |x| \leqslant 2 \cdot (W(x') - W(x)) + 3$.

(We recall the *depth* of a node x is just its length $|x|$). We denote by $W(x)$ the weight of x which we define as the first component of $\tau(x)$, i.e. the weight of the equation labelling x).

Proof. Let x, x' be such that $|x'| = |x| + 1$. Then $W(x') - W(x) \in \{-1, +1\}$, hence the inequality $|W(x') - W(x)| \leqslant |x'| - |x|$ is fulfilled by such nodes. The general case follows by induction on $(|x'| - |x|)$.

Lemma 81 ensures that, in every branch $(x_i)_{i\in I}$ and for every interval $[n+1, n+4] \subseteq I$, at most one integer j is such that T_B occurs at j. The second inequality follows from this remark. $\square$

Lemma 83. *There exists some finite subset $\bar{V}_2 \subseteq V_2$, such that, for every label (π, U^-, U^+) of the tree τ, $U^-, U^+ \in \mathsf{DB}\langle V_1\rangle \cdot (\bar{V}_2 \cup \{\varepsilon\})$.*

Proof. Let Ψ^-, Ψ^+ be the variables in V_2 appearing in the two polynomials U_0^-, U_0^+. Let us define $\bar{V}_2 = \{\varphi_2^{-1}(\varphi_2(\Psi^-) \bullet v) \mid v \in V_1^*\} \cup \{\varphi_2^{-1}(\varphi_2(\Psi^+) \bullet v) \mid v \in V_1^*\}$. This set $\bar{V}_2$ is finite because $\varphi_2(\Psi^-), \varphi_2(\Psi^+)$ are rational. The property is true at the root of τ by hypothesis (84), and this property is preserved by the maps T_A, T_B^-, T_B^+. $\square$

8.2. B-stacking sequences

We establish here that every infinite branch must contain an infinite suffix (a "B-stacking sequence") where at least d_0 labels (U, U') are belonging to the same polynomial space V_0 of dimension $\leqslant d_0$ with coordinates not greater than s_{d_0} (over some fixed generating family of cardinality $\leqslant d_0$).

Let $\sigma = (x_i)_{i\in I}$ be a path in τ, where $I = [i_0, \infty[$ and let $(x_i)_{i\geqslant 0}$ be the unique branch of τ containing σ. Let us note $\tau(x_i) = (\pi_i, U_i^-, U_i^+)$.

We call σ a *B-stacking sequence* iff: there exists some $\alpha_0 \in \{-, +\}$ such that

$$T_B^{\alpha_0} \text{ occurs at } x_{i_0+k_1+1} \tag{87}$$

and, for every $i \in I$, $\alpha \in \{-, +\}$, if T_B^{α} occurs at x_{i+k_1+1} then

$$\|U_i^{-\alpha}\| \geqslant \|U_{i_0}^{-\alpha_0}\| \geqslant N_0. \tag{88}$$

From now on and until Lemma 811, we fix a B-stacking sequence $\sigma = (x_i)_{i\in I}$ and we denote by S_0 the series $U_{i_0}^{-\alpha_0}$.

Lemma 84. *There exists some word $u_0 \in X^*$ and some sign $\alpha_0' \in \{-, +\}$ such that $S_0 = U_0^{\alpha_0'} \odot u_0$.*

Proof. One can prove by induction on $i \in \mathbb{N}$ that, for every $\alpha \in \{-, +\}$, U_i^{α} has one of the two following forms:
(1) $U_i^{\alpha} = U_0^{\alpha'} \odot u$ for some $\alpha' \in \{-, +\}, |u| \leqslant i$,
(2) $U_i^{\alpha} = \sum_{k=1}^{q} \beta_k \cdot (U_0^{\alpha'} \odot u u_k)$
for some deterministic polynomial vector $\boldsymbol{\beta} \in \mathsf{DB}_{1,q}\langle V_1\rangle$, $\alpha' \in \{-, +\}$, $|u \cdot u_k| \leqslant i$, $|u_k| \leqslant k_0$. $\square$

Lemma 85. *Suppose that $i_0 \leqslant j < i$, no T_B occurs in $[j+1, i]$, $U_j^{-\alpha}$ is D_1-marked and U_j^{α} is unmarked. Then, for every $j' \in [j, i]$, $\|U_{j'}^{\alpha}\| \geqslant \|U_i^{\alpha}\| - k_2$.*

Proof. Let i, j fulfill the hypothesis of the lemma.

(1) Let us treat first the case where $j' = j$.

If $(i-j) \leqslant (D_1+1)k_1$ then, by Lemma 318

$$\|U_i^\alpha\| \leqslant \|U_j^\alpha\| + (D_1+1)\cdot k_1 \cdot K_0 \leqslant k_2$$

hence the lemma is true.

Let suppose now that $(i-j) \geqslant (D_1+1)k_1+1$. We can then define the integers $j < i_1 < i_2 < i$ by

$$i_1 = j + D_1 \cdot k_1, \quad i_2 = i - k_1 - 1.$$

By Lemma 318 we know that

$$\|U_{i_1}^\alpha\| \leqslant \|U_j^\alpha\| + D_1 \cdot k_1 \cdot K_0 \quad \text{and} \quad \|U_i^\alpha\| \leqslant \|U_{i_2}^\alpha\| + (k_1+1)\cdot K_0. \tag{89}$$

If there was some stacking subderivation of length k_1 in $U_j^{-\alpha} \to U_{i_1}^{-\alpha}$, as all the U_k^α (for $k \in [j,i]$) are unmarked, T_B would occur at some integer in $[j+k_1+1, i_1+1]$, which is untrue. Hence there is no such stacking subderivation, and by Lemma 325 $U_{i_1}^{-\alpha}$ is unmarked.

If there was some stacking subderivation of length k_1 in $U_{i_1}^\alpha \to U_{i_2}^\alpha$, as all the $U_k^{-\alpha}$ (for $k \in [i_1,i]$) are unmarked, T_B would occur at some integer in $[i_1+k_1+1, i]$, which is untrue. Hence there is no such stacking subderivation, and by Lemma 324

$$\|U_{i_2}^\alpha\| \leqslant \|U_{i_1}^\alpha\| + k_1 \cdot K_0. \tag{90}$$

Adding inequalities (89), (90) we obtain

$$\|U_i^\alpha\| \leqslant \|U_j^\alpha\| + (D_1\cdot k_1 + 2\cdot k_1 + 1)\cdot K_0 = \|U_j^\alpha\| + k_2,$$

which was to be proved.

(2) Let us suppose now that $j \leqslant j' \leqslant i$.

If $(i-j) \leqslant (D_1+1)k_1$, the same inequality is true for $i-j'$ and the conclusion is true for j'.

Otherwise, if $j' \leqslant i_1$, (89), (90) are still true for j' instead of j, hence the conclusion too.

Otherwise, by the arguments of part 1, $U_{j'}^{-\alpha}, U_{j'}^\alpha$ are both unmarked. Hence the hypothesis of part 1 are met by (j', i) instead of (j, i), hence the conclusion is met too. (We illustrate our argument in Fig. 3). $\square$

Lemma 86. *Let $i \in I$, $\alpha \in \{-,+\}$ such that T_B^α occurs at $i+k_1+1$. Then, there exists $u \in X^*$, $|u| \leqslant (i-i_0)$, $U_i^{-\alpha} = S_0 \odot u$ and, for every prefix $w \leqslant u$,*

$$\|S_0 \odot w\| \geqslant \|S_0\| - k_3.$$

Proof. We prove the lemma by induction on $i \in [i_0, \infty[$.
Basis. $i = i_0$.
Choosing $u = \varepsilon$, the lemma is true.
Induction step. $i_0 \leqslant i' < i$, $T_B^{\alpha'}$ occurs at $i'+k_1+1$, T_B^α occurs at $i+k_1+1$ and T_B does not occur in $[i'+k_1+2, i+k_1]$.

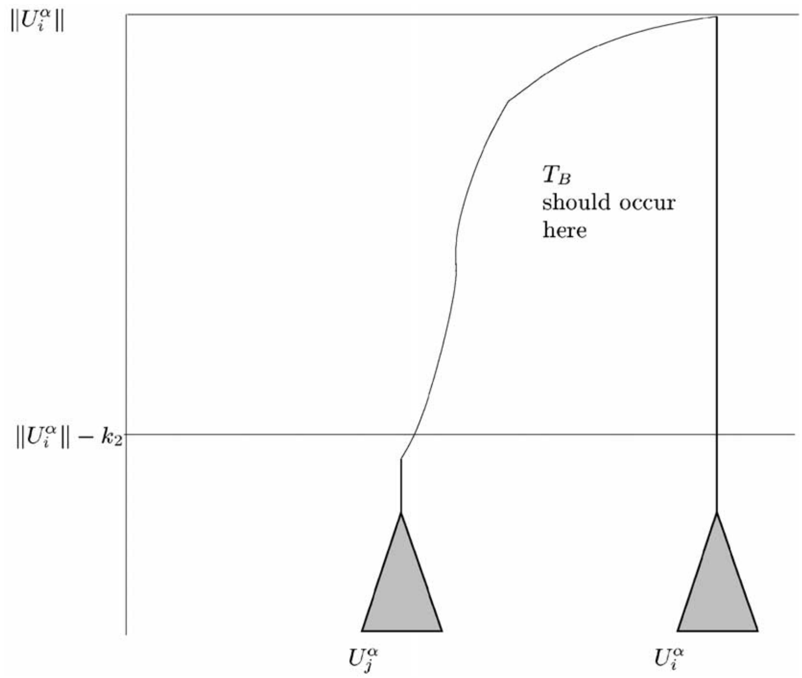

Fig. 3. $\|U_j^\alpha\|$ too small is impossible.

By induction hypothesis, there exists some $u' \in X^*$, $|u'| \leqslant (i' - i_0)$ fulfilling

$$U_{i'}^{-\alpha'} = S_0 \odot u', \tag{91}$$

$$\forall w' \leqslant u', \|S_0 \odot w'\| \geqslant \|S_0\| - k_3. \tag{92}$$

Let us define $j = i' + k_1 + 1$.

Let $\bar{u} \in X^*$ be the word such that

$$U_j^{-\alpha} \xrightarrow{\bar{u}} U_i^{-\alpha} \tag{93}$$

is the derivation described by the $-\alpha$ component of the path from x_j to x_i.

Case 1. $\alpha' = \alpha$.

$$U_j^{-\alpha} = U_{i'}^{-\alpha'} \odot u_1$$

for some $u_1 \in X^*$, $|u_1| = k_1$ and U_j^α is D_1-marked. Let us choose $u = u' \cdot u_1 \cdot \bar{u}$. Hence

$$U_i^{-\alpha} = S_0 \odot u. \tag{94}$$

Let us consider some prefix w of u.

Subcase 1. $w \leqslant u'$.

By (92) we know that $\|S_0 \odot w\| \geqslant \|S_0\| - k_3$.

Subcase 2. $w = u' \cdot u_1 \cdot u''$, for some $u'' \leqslant \bar{u}$.

By Lemma 85 we know that $\|S_0 \odot w\| \geqslant \|U_i^{\alpha}\| - k_2$, and by definition of a B-stacking sequence we also know that $\|U_i^{\alpha}\| \geqslant \|S_0\|$. Hence

$$\|S_0 \odot w\| \geqslant \|S_0\| - k_2.$$

Subcase 3. $w = u' \cdot u'_1$, where u'_1 is a prefix of u_1.

Then, by Lemma 318 and the above inequality we get

$$\|S_0 \odot w\| \geqslant \|S_0 \odot u'u_1\| - k_1 \cdot K_0 \geqslant \|S_0\| - k_3.$$

Case 2: $-\alpha' = \alpha$.

$$U_j^{-\alpha} = \sum_{k=1}^{q} \beta_k \cdot (U_{i'}^{\alpha} \odot u_k),$$

where β is a deterministic polynomial which is fully marked and every $|u_k| \leqslant k_0$.

By Lemma 321 either $U_i^{-\alpha} = \sum_{k=1}^{q} (\beta_k \odot \bar{u}) \cdot (U_{i'}^{\alpha} \odot u_k)$ or there exists a decomposition

$$\bar{u} = \bar{u}_1 \cdot \bar{u}_2 \tag{95}$$

and an integer $k \in [1, q]$ such that

$$U_i^{-\alpha} = U_{i'}^{\alpha} \odot u_k \bar{u}_2. \tag{96}$$

But, as $U_i^{-\alpha}$ is unmarked (by definition of T_B^{α}), the first formula is impossible unless $\beta \odot \bar{u}$ is unitary or nul. Hence (95), (96) is the only possibility.

Let us choose $u = u' \cdot u_k \cdot \bar{u}_2$. It is clear from (96) that $U_i^{-\alpha} = S_0 \odot u$.

Let us consider some prefix w of u.

Subcase 1: $w \leqslant u'$.

Same arguments as in case1, subcase1.

Subcase 2: $w = u' \cdot u_k \cdot u''$, for some $u'' \leqslant \bar{u}_2$.

By Lemma 85 applied on the interval $[j + |\bar{u}_1| + 1, i]$, we can conclude that

$$\|S_0 \odot w\| \geqslant \|S_0\| - k_3.$$

Subcase 3: $w = u' \cdot u'_k$, where u'_k is a prefix of u_k.

Same arguments as in case1, subcase3. $\square$

Given any subset $\mathscr{G} \subseteq \mathsf{DRB}\langle\langle V \rangle\rangle$, we define the *polynomial space* generated by $\mathscr{G}$ as the set

$$\mathsf{V}(\mathscr{G}) = \left\{ \sum_{j=1}^{m} \alpha_j \cdot S_j \mid m \geqslant 0,\ \forall j \in [1, m]\ S_j \in \mathscr{G},\ \alpha \in \mathsf{DB}_{1,m}\langle V_1 \rangle \right\}.$$

It follows from Lemma 310 that $\mathsf{V}(\mathscr{G})$ is closed under linear combinations with coefficients in $\bigcup_{\lambda \geqslant 0} \mathsf{DB}_{1,\lambda}\langle V_1 \rangle$.

The following generating set of series and polynomial space of series will be used in the sequel.

$$\mathscr{G}_0 = \{S_0 \odot u \mid u \in X^*,\ |u| \leqslant k_4\}, \tag{97}$$

$$\mathsf{V}_0 = \mathsf{V}(\mathscr{G}_0). \tag{98}$$

Lemma 87. *Let $i \geqslant i_0$ such that T_B occurs at i. Then, $U_i^-, U_i^+ \in \mathsf{V}_0$.*

Proof. Let us suppose that T_B^α occurs at i. By Lemma 86, $U_{i-k_1-1}^{-\alpha} = S_0 \odot u$ and, for every prefix $w \preccurlyeq u$,

$$\|S_0 \odot w\| \geqslant \|S_0\| - k_3 \geqslant 4. \tag{99}$$

By Lemma 319, $\exists u_1, u_2 \in X^*,\ v_1 \in V^*,\ E_1, \ldots, E_k \in V,\ E_1 \smile E_2 \ldots \smile E_k,\ \Phi \in \mathsf{DRB}_{q,1}\langle\langle V\rangle\rangle$, such that $u = u_1 \cdot u_2$,

$$S_0 \odot u_1 = S_0 \bullet v_1 = \sum_{k=1}^{q} E_k \cdot \Phi_k \tag{100}$$

$$S_0 \odot u = \sum_{k=1}^{q} (E_k \odot u_2) \cdot \Phi_k. \tag{101}$$

As S_0 is a polynomial (by Lemma 83)

$$|v_1| \leqslant \|S_0\| - \|S_0 \bullet v_1\| \leqslant k_3.$$

Formula (101) can be rewritten

$$U_{i-k_1-1}^{-\alpha} = \sum_{k=1}^{q} (E_k \odot u_2) \cdot (S_0 \bullet v_1 E_k) = \sum_{k=1}^{q} (E_k \odot u_2) \cdot (S_0 \odot \bar{u}_k),$$

where $\bar{u}_k \in X^*$ is the minimal word such that $v_1 E_k \odot \bar{u}_k = \varepsilon$.

The fact that inequality (99) holds for $S_0 \odot u_1$ implies that no E_k belongs to V_2, hence that all $v_1 E_k$ belong to V_1^*. It follows that

$$|\bar{u}_k| \leqslant (k_3 + 1) \cdot k_0.$$

Using Lemmas 321 and 310 we can deduce from the above form of $U_{i-k_1-1}^{-\alpha}$ that

$$U_i^\alpha \in \mathsf{V}(\{S_0 \odot w \mid w \in X^*, |w| \leqslant (k_3+1) \cdot k_0 + k_0\}),$$
$$U_i^{-\alpha} \in \mathsf{V}(\{S_0 \odot w \mid w \in X^*, |w| \leqslant (k_3+1) \cdot k_0 + k_1\}),$$

hence that both $U_i^{-\alpha}, U_i^\alpha$ belong to V_0. $\square$

We recall that

$$K_1 = k_1 \cdot K_0 + 1, \quad K_2 = k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 + D_1 \cdot k_1 + 2 \cdot k_1 + 4.$$

Lemma 88. *For every $L \geqslant 0$ there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, U_i^-, $U_i^+ \in \mathsf{V}_0$.*

Proof. Let us establish that

$$\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2 - k_1 - 1], \exists \alpha \in \{-, +\},$$
$$T_B^\alpha \text{ occurs at } i + k_1 + 1. \tag{102}$$

Let $L \geqslant 0$ and let $i' \geqslant i_0$ be the greatest integer in $[i_0, i_0 + L]$ such that T_B occurs at $i' + k_1 + 1$. Let $j = i' + k_1 + 1$. We then have

$$U_j^{\alpha'} = \sum_{k=1}^{q} \beta_k \cdot (U_{i'}^{-\alpha'} \odot u_k),$$

where $\|\beta\| \leqslant D_1$ and $U_j^{-\alpha'}$ is unmarked.

Case 1: there exists $i \in [j, j + k_1 \cdot D_1]$, such that T_B occurs at $i + k_1 + 1$.

In this case the small constants $K_1 = 0$, $K_2 = k_1 \cdot D_1 + k_1 + 1$ would be sufficient to satisfy (102). A fortiori the given constants satisfy (102).

Case 2: there exists no $i \in [j, j + k_1 \cdot D_1]$, such that T_B occurs at $i + k_1 + 1$.

Then, there is no stacking subderivation of length k_1 in $U_j^{\alpha'} \to U_{j+k_1 \cdot D_1}^{\alpha'}$. By Lemma 325 it follows that both $U_{j+D_1 \cdot k_1}^{\alpha}$ are unmarked.

(1) Let $j_1 = j + D_1 \cdot k_1$ and let us show that there exists some $i \geqslant j_1$ such that T_B occurs at $i + k_1 + 1$.

If such an i does not exist then, for every $\alpha \in \{-, +\}$, the infinite derivation

$$U_{j_1}^{\alpha} \to U_{j_1+1}^{\alpha} \to \cdots$$

does not contain any stacking sequence of length k_1. By Lemma 324 we would have

$$\forall k \geqslant j_1, \quad \|U_k^\alpha\| \leqslant \|U_{j_1}^\alpha\| + k_1 \cdot K_0.$$

As the set $\{\|U_k^\alpha\|,\ k \geqslant j_1,\ \alpha \in \{-, +\}\}$ is finite, by Lemma 83, the set $\{U_k^\alpha,\ k \geqslant j_1,\ \alpha \in \{-, +\}\}$ is finite too, hence there would be a repetition

$$(U_k^-, U_k^+) = (U_{k'}^-, U_{k'}^+) \quad \text{with } j_1 \leqslant k < k' \text{ and } \pi_k < \pi_{k'},$$

so that T_{cut} would have been defined on some finite prefix of the branch, contradicting the hypothesis that the branch is infinite.

(2) Let $i > i'$ be the smallest integer (in $[j_1, \infty[$) fulfilling point 1 above and suppose that T_B^α occurs at $i + k_1 + 1$.

By an argument analogous to that used in Lemma 84 we see that $U_{j_1}^{-\alpha} = S_0 \odot u$ for some $|u| \leqslant (j_1 - i_0)$, and by Lemma 318 we get

$$\|U_{j_1}^{-\alpha}\| \leqslant (j_1 - i_0) \cdot K_0 + \|S_0\|. \tag{103}$$

We also know that

$$\|S_0\| \leqslant \|U_i^{-\alpha}\| \leqslant \|U_{i-1}^{-\alpha}\| + K_0. \tag{104}$$

As the derivation $U_{j_1}^{-\alpha} \to U_{i-1}^{-\alpha}$ contains no stacking sub-derivation of length k_1, by Lemma 323 we obtain

$$\|U_{i-1}^{-\alpha}\| \leqslant \|U_{j_1}^{-\alpha}\| - (i - j_1 - 2)/k_1. \tag{105}$$

Combining the three inequalities (103), (104), (105) we get successively

$$\|S_0\| \leqslant \|S_0\| + (j_1 - i_0 + 1) \cdot K_0 - (i - j_1 - 2)/k_1,$$

$$(i - j_1 - 2) \leqslant (j_1 - i_0 + 1) \cdot k_1 K_0.$$

$$\begin{aligned}(i - i') &= (i - j_1 - 2) + (j_1 - i' + 2) \leqslant (j_1 - i_0 + 1) \cdot k_1 \cdot K_0 \\ &\quad + D_1 \cdot k_1 + k_1 + 3 \\ &= (i' - i_0) \cdot k_1 \cdot K_0 + k_1^2 \cdot D_1 \cdot K_0 + k_1^2 \cdot K_0 + 2 \cdot k_1 \cdot K_0 \\ &\quad + D_1 \cdot k_1 + k_1 + 3 \\ &= (K_1 - 1)(i' - i_0) + K_2 - k_1 - 1. \end{aligned} \tag{106}$$

(3) By the choice of i', i, we know that $i' \leqslant i_0 + L \leqslant i$. Using (106) we obtain

$$i \leqslant i' + (K_1 - 1)(i' - i_0) + K_2 - k_1 - 1,$$

$$i \leqslant i_0 + K_1 \cdot L + K_2 - k_1 - 1.$$

Assertion (102) is now established for case 2 as well as for case 1. From (102) and Lemma 87 the lemma follows. (We illustrate our argument in Fig. 4). $\square$

Let us give now a stronger version of Lemma 88 where we analyze the *size of the coefficients* of the linear combinations whose existence is proved in Lemma 88.

We recall that

$$K_3 = K_0|V_1|, \quad K_4 = D_1.$$

Let us fix a total ordering on $\mathcal{G}_0$:

$$\mathcal{G}_0 = \{\theta_1, \theta_2, \ldots, \theta_d\}, \quad \text{where } d = \mathrm{Card}(\mathcal{G}_0).$$

Let us remark that $d \leqslant \mathrm{Card}(X^{\leqslant k_4}) = d_0$.

Lemma 89. *Let $L \geqslant 0$. There exists $i \in [i_0+L, i_0+K_1 \cdot L+K_2]$ and, for every $\alpha \in \{-,+\}$, there exists a deterministic polynomial family $(\beta_{i,j}^{\alpha})_{1 \leqslant j \leqslant d}$ fulfilling*

(1) $U_i^{\alpha} = \sum_{j=1}^{d} \beta_{i,j}^{\alpha} \cdot \theta_j$,

(2) $\|\beta_{i,*}^{\alpha}\| \leqslant K_3 \cdot (i - i_0) + K_4$.

Proof. By Lemma 88 there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ and $\alpha \in \{-,+\}$ such that T_B^{α} occurs at i. Let us use the notation of the proof of Lemma 87 and compute

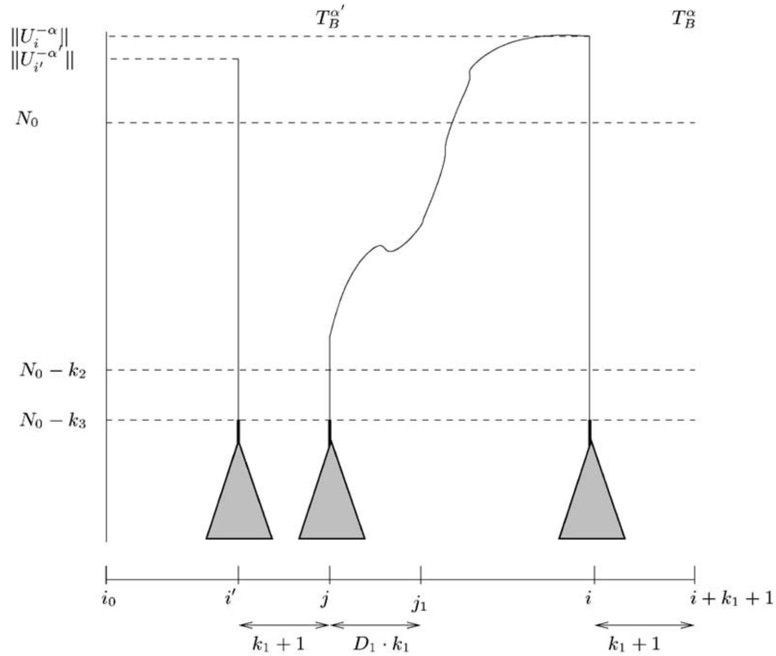

Fig. 4. Two successive T_B.

upper-bounds on the coefficients of $U_i^{-\alpha}, U_i^{\alpha}$ expressed as linear combinations of the vectors of $\mathscr{G}_0$.

Coefficients of $U_i^{-\alpha}$: $U_i^{-\alpha} = U_{i-k_1-1}^{-\alpha} \odot u'$, for some $u' \in X^*$, $|u'| = k_1$. By Lemma 321, $U_i^{-\alpha}$ can be expressed in one of the two following forms:

$$U_i^{-\alpha} = S_0 \odot (\bar{u}_k u'') \quad \text{where } u'' \text{ is a suffix of } u', \tag{107}$$

$$U_i^{-\alpha} = \sum_{k=1}^{q} (E_k \odot u_2 u') \cdot (S_0 \odot \bar{u}_k). \tag{108}$$

In case (107) we can choose as vector of coordinates: $\beta_{i,\star}^{-\alpha} = \varepsilon_{j_0}^{d}$. We then have $\|\beta_{i,\star}\| = 2 \leqslant K_4$.

In case (108), we can choose: $\beta_{i,\star}^{-\alpha} = E \odot u_2 u'$ (completed with $\emptyset$ in all the columns j not corresponding to some vector $S_0 \odot \bar{u}_k$ of $\mathscr{G}_0$). We then have

$$\|\beta_{i,\star}\| = \|E \odot u_2 u'\| \leqslant K_0 \cdot (i - i_0) \leqslant K_3 \cdot (i - i_0).$$

Coefficients of U_i^{α}: By definition of T_B^{α}

$$U_i^{\alpha} = \sum_{\ell=1}^{r} \tau_\ell \cdot (U_{i-k_1-1}^{-\alpha} \odot \bar{w}_\ell), \tag{109}$$

where $\|\tau\| \leqslant D_1$, $|\bar{w}_\ell| \leqslant k_0$.

Replacing u' by $\bar{w}_\ell$ in the above analysis, we get

$$\forall \ell \in [1,r], \quad U^{-\alpha}_{i-k_1-1} \odot \bar{w}_\ell = \sum_{j=1}^{d} \gamma_{\ell,j} \cdot \theta_j \tag{110}$$

with $\|\gamma_{\ell,\star}\| \leqslant K_0 \cdot (i - i_0)$.

Equalities (109), (110) show that

$$U^\alpha_i = \tau \cdot \gamma \cdot \theta,$$

where τ, γ, θ are deterministic rational matrices of dimensions, respectively $(1,r),(r,d)$, $(d,1)$. Let us choose $\beta_{i,\star} = (\tau \cdot \gamma)$.

$$\begin{aligned} \|\beta_{i,\star}\| &\leqslant \|\tau\| + \|\gamma\| \leqslant D_1 + r \cdot K_0 \cdot (i - i_0) \\ &\leqslant D_1 + |V_1| \cdot K_0 \cdot (i - i_0) = K_3 \cdot (i - i_0) + K_4. \qquad \square \end{aligned}$$

Lemma 810. *There exists $i_0 \leqslant \kappa_1 < \kappa_2 < \cdots < \kappa_d$ and deterministic rational vectors $(\beta^\alpha_{i,j})_{1\leqslant j\leqslant d}$ (for every $i \in [1,d]$) such that*

(0) $W(\kappa_1) \geqslant 1$,
(1) $\forall i, \forall \alpha,\ U^\alpha_{\kappa_i} = \sum_{j=1}^d \beta^\alpha_{i,j}\theta_j \in \mathsf{V}_0$,
(2) $\forall i, \forall \alpha,\ \|\beta^\alpha_{i,*}\| \leqslant s_i$,
(3) $\forall i,\ W(\kappa_{i+1}) - W(\kappa_i) \geqslant \delta_{i+1}$,

where the sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)$ are those defined by relations (74), (75) in Section 6.

Proof. Let us consider the additional property
(4) $\kappa_i - i_0 \leqslant L_i$.

We prove by induction on i the conjunction $(1) \wedge (2) \wedge (3) \wedge (4)$.

i = 1:

By Lemma 89, there exists $\kappa_1 \in [i_0, i_0 + K_2]$ such that $\forall \alpha \in \{-,+\}$, $\exists$ a deterministic vector $(\beta^\alpha_{1,j})_{1\leqslant j\leqslant d}$, such that

$$U^\alpha_{\kappa_1} = \sum_{j=1}^{d} \beta^\alpha_{1,j}\theta_j$$

and in addition $\|\beta^\alpha_{1,*}\| \leqslant K_3K_2 + K_4 = s_1$.

i → i + 1:

Suppose that $\kappa_1 < \kappa_2 < \cdots < \kappa_i$ are fulfilling $(1) \wedge (2) \wedge (3) \wedge (4)$. By Lemma 89, there exists $\kappa_{i+1} \in [i_0 + L_i + \ell_{i+1}, i_0 + K_1(L_i + \ell_{i+1}) + K_2]$ such that $\forall \alpha \in \{-,+\}, \exists$ a deterministic polynomial vector $(\beta^\alpha_{i+1,j})_{1\leqslant j\leqslant d}$, such that

$$U^\alpha_{\kappa_{i+1}} = \sum_{j=1}^{d} \beta^\alpha_{i+1,j}\theta_j \tag{111}$$

and in addition

$$\|\beta^{\alpha}_{i+1,*}\| \leqslant K_3(K_1(L_i + \ell_{i+1}) + K_2) + K_4 = K_3 L_{i+1} + K_4$$
$$= s_{i+1} \tag{112}$$

By Lemma 82

$$2(W(\kappa_{i+1}) - W(\kappa_i)) + 3 \geqslant \kappa_{i+1} - \kappa_i \geqslant \ell_{i+1} = 2\delta_{i+1} + 3$$

hence

$$W(\kappa_{i+1}) - W(\kappa_i) \geqslant \delta_{i+1}. \tag{113}$$

At last

$$\kappa_{i+1} - i_0 \leqslant K_1(L_i + l_{i+1}) + K_2 = L_{i+1}. \tag{114}$$

The above properties (111)–(114) prove the required conjunction.

It remains to prove point (0): the integer κ_1 introduced by Lemma 89 is such that T_B occurs at κ_1, hence

$$W(\kappa_1) = W(\kappa_1 - k_1 - 1) + k_1 - 1$$
$$\geqslant W(\kappa_1 - k_1 - 1) + 2 \geqslant 1. \qquad \square$$

Lemma 811. *Let $b = (x_i)_{i \in \mathbb{N}}$ be an infinite branch of τ. Then there exists some $i_0 \in \mathbb{N}$ such that $(x_i)_{i \geqslant i_0}$ is a B-stacking sequence.*

Proof. Let us distinguish, a priori, several cases, and see that only the case where b admits a B-stacking sequence is possible.

Case 1: T_B occurs finitely often on τ.

Let j be the largest integer such that T_B occurs at j. By the arguments used in the proof of Lemma 88, assertion (102), case 2, we know that $U^{-}_{j+k_1 \cdot D_1}, U^{+}_{j+k_1 \cdot D_1}$ are both unmarked, and that

$$\forall k \geqslant j + k_1 \cdot D_1,\ \forall \alpha \in \{-,+\}, \quad \|U^{\alpha}_k\| \leqslant \|U^{\alpha}_{j+k_1 \cdot D_1}\| + k_1 \cdot K_0.$$

By Lemma 83 the set of values of (U^{-}_k, U^{+}_k) would be finite. Hence the branch b would have a finite prefix on which T_{cut} is defined: this is impossible on an infinite branch.

Case 2: For some sign α, there are infinitely many integers i such that $[T^{\alpha}_B$ occurs at $i + k_1 + 1$ and $\|U^{-\alpha}_i\| < N_0]$.

In this case there would exist an infinite sequence of integers $i_1 < i_2 < \cdots < i_\ell <$ such that

$$\forall \ell \geqslant 1, \quad U^{-\alpha}_{i_1} = U^{-\alpha}_{i_\ell}.$$

For a given $U_i^{-\alpha}$, only a finite number of values are possible for the pair $(U^-_{i+k_1+1}, U^+_{i+k_1+1})$. Hence there exist integers $\ell < \ell'$ such that

$$\ell < \ell', \quad \pi_\ell < \pi_{\ell'} \quad \text{and} \quad (U^-_{\ell+k_1+1}, U^+_{\ell+k_1+1}) = (U^-_{\ell'+k_1+1}, U^+_{\ell'+k_1+1}).$$

Here again T_{cut} would have a non-empty value on some prefix of τ, which is impossible.

Case 3: T_B occurs infinitely often on τ and, for every sign α, there are only finitely many integers i such that $[T_B^\alpha$ occurs at $i + k_1 + 1$ and $\|U_i^{-\alpha}\| < N_0]$.

Let us consider the set I_0 of the integers i such that, there exists a sign α_i such that

$$[T_B^{\alpha_i} \text{ occurs at } i + k_1 + 1 \text{ and } \|U_i^{-\alpha_i}\| \geqslant N_0].$$

By the hypothesis of case 3, $I_0 \neq \emptyset$. Let i_0 such that

$$\|U_{i_0}^{-\alpha_{i_0}}\| = \min\{\|U_i^{-\alpha_i}\| \mid i \in I_0\}.$$

Then $(x_i)_{i \geqslant i_0}$ is a B-stacking sequence. □

9. Completeness of $\mathscr{D}_0$

We show that, up to some slight details, $\mathscr{S}_{ABC}$ is terminating.

An assertion $A = (\Pi, S^-, S^+)$ is said *simple* iff it fulfills the 3 conditions below

$$S^-, S^+ \text{ are both linear polynomials,} \tag{115}$$

$$S^-, S^+ \text{ are both unmarked,} \tag{116}$$

$$S^- \equiv S^+. \tag{117}$$

Lemma 91. *Let A_0 be some simple assertion. Then the tree $\mathscr{T}(\mathscr{S}_{ABC}, A_0)$ is finite.*

Proof. Suppose $A_0 = (\Pi_0, S_0^-, S_0^+)$ is simple and $t = \mathscr{T}(\mathscr{S}_{ABC}, A_0)$ is infinite.

By Koenig's lemma, t contains an infinite branch whose (infinite) labelling word is $A_0 A_1 \cdots A_n \cdots$.

Condition (C2) in the definition of T_C together with Lemma 54, applied to $m = 1$ and $d \leqslant d_0$, shows that the equations $B_j = (\pi_j, T_j, U_j)$ produced by T_C must fulfill:

$$\|\varphi_2(T_j)\| \leqslant \Sigma_{d_0} + s_{d_0}, \quad \|\varphi_2(U_j)\| \leqslant \Sigma_{d_0} + s_{d_0}$$

hence that the number of possible pairs (T_j, U_j) produced by T_C is *finite*. Hence T_C occurs only a finite number of times on this branch (otherwise T_{cut} would occur on this branch, which is impossible on an infinite branch). Let n_0 be the last point where T_C occurs or $n_0 = 0$ if T_C does not occur on this branch.

$(A_{n_0+i})_{i \geqslant 0}$ is a branch of the tree $t' = \mathscr{T}(\mathscr{S}_{AB}, A_{n_0})$. If $n_0 > 0$, A_{n_0} is a result of T_C, hence t' fulfills hypotheses (84), (85) of Section 8.2. As A_0 is true and the strategies T_A, T_B, T_C preserve truth, A_{n_0} is also true, hence t' fulfills hypothesis (86) of Section 8.2. If $n_0 = 0$, as A_0 is simple, t' fulfills the hypotheses (84), (85), (86) assumed in Section 8.2. We can apply now the results of Section 8.2.

By Lemma 811, the branch $(A_{n_0+i})_{i\geqslant 0}$ must contain a B-stacking sequence σ. Let us remark that, as $T_\emptyset$ does not occur (otherwise the branch would be finite) every equation (π, U^-, U^+) labelling this branch is such that $U^- \not\equiv \emptyset$, $U^+ \not\equiv \emptyset$. By Lemma 810 such a B-stacking sequence contains a subsequence $(A_{\kappa_1}, A_{\kappa_2}, \ldots, A_{\kappa_i}, \ldots, A_{\kappa_d})$ with $d \leqslant d_0$, fulfilling hypotheses $(1,2)$ of Lemma 54. Hence some prefix $(A_{\kappa_1}, A_{\kappa_2}, \ldots, A_{\kappa_{D+1}}$ fulfills conditions (C1,C2) of the definition of T_C (it suffices to choose $D = \mathrm{D}(A_{\kappa_1}, A_{\kappa_2}, \ldots, A_{\kappa_i}, \ldots, A_{\kappa_d}))$. Hence the sequence of assertions $(A_i)_{0\leqslant i\leqslant \kappa_{D+1}}$ belongs to $\mathrm{dom}(T_C)$. The priority ordering given in the definition of $\mathscr{S}_{ABC}$ then implies that either $T_{cut}, T_\emptyset, T_\varepsilon$ or T_C occurs at some $n_0 + i$. But the three first cases cannot occur on an infinite branch and the fourth one contradicts the maximality of n_0. $\square$

Theorem 92. *The system $\mathscr{D}_0$ is complete.*

Proof. By Lemma 75 $\mathscr{S}_{ABC}$ is a strategy for $\mathscr{D}_0$ which is closed and by Lemma 91 $\mathscr{S}_{ABC}$ is terminating on every simple assertion. By a slight variant of Lemma 45, every simple assertion has a $\mathscr{D}_0$-proof.

Let $A = (\Pi, S^-, S^+)$ be some true assertion. Let $A' = (\Pi, \varphi_2^{-1}(\varphi_2(S^-)), \varphi_2^{-1}(\varphi_2(S^+)))$. As A' is simple, there exists a finite $\mathscr{D}_0$-proof P of A'. By rule (R11), $P \cup \{A\}$ is a finite $\mathscr{D}_0$-proof of A. $\square$

Theorem 93. *The equivalence problem for deterministic pushdown automata is decidable.*

Proof. Let $\mathscr{M}$ be some dpda. The equivalence relation $\equiv$ on $\mathsf{DRB}\langle\langle V\rangle\rangle$ (where V is the structured alphabet associated to the given $\mathscr{M}$) has a recursively enumerable complement (this is well-known). By Theorem 92 and Lemma 42 $\equiv$ is recursively enumerable too. Hence $\equiv$ is recursive. In addition, the system $\mathscr{D}_0$ associated with $\mathscr{M}$ is computable from $\mathscr{M}$, hence the theorem follows. $\square$

10. Elimination

Let $\mathscr{D}_1 = \langle \mathscr{A}_1, H_1, \mid\!\!-\!\!-_{\mathscr{D}_1}\rangle$ where $\mathscr{A}_1 = \mathbb{N} \times \mathsf{DRB}\langle\langle V_1\rangle\rangle \times \mathsf{DRB}\langle\langle V_1\rangle\rangle$, H_1 is the restriction of H to $\mathscr{A}_1$, and the *elementary deduction relation* $||\!\!-\!\!-_{\mathscr{D}_1}$ is the set of all instances of the metarules (R0), (R1), (R2), (R3), (R4), (R6), (R7), (R8), (R9), where the series are taken in $\mathsf{DRB}\langle\langle V_1\rangle\rangle$ (i.e. we eliminate all the second-order variables and also the rule dealing with the marks).

The deduction relation $\mid\!\!-\!\!-_{\mathscr{D}_1}$ is now defined by

$$\mid\!\!-\!\!-_{\mathscr{D}_1} = \overset{\langle *\rangle}{||\!\!-\!\!-}_{\mathscr{D}_1} \circ \overset{[1]}{||\!\!-\!\!-}_{R0,R3,R4} \circ \overset{\langle *\rangle}{||\!\!-\!\!-}_{\mathscr{D}_1}.$$

Lemma 101. *$\mathscr{D}_1$ is a deduction system.*

Proof (*sketch*). As $\models\!\!-_{\mathscr{D}_1} \subseteq \models\!\!-_{\mathscr{D}_0}$, property (A1) is fulfilled by $\models\!\!-_{\mathscr{D}_1}$. By the well-known decidability properties for finite-automata, rules (R0)–(R9), are recursively enumerable. Hence property (A2) is fulfilled by $\mathscr{D}_1$. □

Theorem 102. *$\mathscr{D}_1$ is a complete deduction system.*

Proof. Let $A = (\Pi, S^-, S^+)$ be some true assertion with $S^-, S^+ \in \mathsf{DRB}\langle\langle V_1\rangle\rangle$. A admits some finite $\mathscr{D}_0$-proof P. Let $P' = \rho_e(\varphi_2(P))$. One can check that every elementary rule of $\mathscr{D}_0$ is mapped, by $\rho_e \circ \varphi_2$, into an elementary rule of $\mathscr{D}_1$. □

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