The Equivalence Problem for Deterministic Pushdown Automata is Decidable

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Abstract. The equivalence problem for deterministic pushdown automata is shown to be decidable. We exhibit a *complete formal system* for deducing equivalent pairs of deterministic rational series on the alphabet associated with a dpda $\mathcal{M}$.

Keywords: deterministic pushdown automata; rational series; finite dimensional vector spaces; matrix semigroups; complete formal systems.

1 Introduction

The so-called "equivalence problem for deterministic pushdown automata" (dpda for short), is the following decision problem:

$$\text{INSTANCE: two dpda } A, B. \quad \text{QUESTION: } \mathrm{L}(A) = \mathrm{L}(B)?$$

where $\mathrm{L}(A)$ (resp. $\mathrm{L}(B)$) is the language recognized by A (resp. B). (This problem is often denoted by $Eq(D, D)$, where D stands for the class of all dpda). The question of whether this problem is *decidable* or not is raised in [GG66] and has received much attention since this time. Beside the fact that this question was natural from the point of view of *formal language* theory, it appeared later as Turing-equivalent with other equivalence-problems for different types of recursive *program schemes* (see [Cou90] for a survey). Some other Turing-equivalent problems on *semi-Thue systems* were also found (see [Sén94] for a survey) and formulations in terms of bisimulation equivalence of infinite *graphs* (or *processes*) have been found too (see [Cau95] for a survey).

Among a large number of papers let us only quote [Val74, VP75, Bee76, Rom85, Oya87, Sti96] which proved decidability of $Eq(D', D')$ for subclasses D' of the full class D of dpda. (We refer the reader to the surveys ([Cou90, Cau95, Lis96]) for other results on problems related to $Eq(D, D)$). The work [Mei89, Mei92] is an attempt to solve the general problem. On account of its incompleteness (see for example the comment in [Lis96, p.219]) it does not provide a full solution; nevertheless it introduced a fundamental new idea: the notion of *linear independance* for languages.

We prove here that the equivalence problem for dpda is *decidable* (theorem 9.3).

We obtain this result by providing a *complete* formal system $\mathcal{D}_0$ for equivalence identities between *deterministic rational series* (we use here a type of formal system inspired by [Cou83] and a notion of deterministic series inspired by [HHY79]). The proof of this completeness property leans on three types of arguments:

- in section 3 we develop around the fundamental idea of [Mei89, Mei92] an *algebraic* theory of "d-spaces",
- in sections 5,7 these structure results are turned into a construction of *strategies* for the formal system $\mathcal{D}_0$,
- in section 8 we analyze the *infinite trees* generated by some strategies associated with $\mathcal{D}_0$.

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2 Preliminaries

2.1 Pushdown automata

A *pushdown automaton* on the alphabet X is a 6-tuple $\mathcal{M} = < X, Z, Q, \delta, q_0, z_0 >$ where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol and $\delta : QZ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}_f(QZ^*)$, is the transition mapping.

Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^*$ and $a \in X \cup \{\epsilon\}$; we note $(qz\omega, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$. For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', \epsilon)$. $\mathcal{M}$ is said *deterministic* iff, for every $z \in Z, q \in Q$:

$$\text{either } \mathrm{Card}(\delta(qz, \epsilon)) = 1 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz, x)) = 0, \tag{1}$$

$$\text{or } \mathrm{Card}(\delta(qz, \epsilon)) = 0 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz, x)) \leq 1. \tag{2}$$

$\mathcal{M}$ is said *real-time* iff, for every $qz \in QZ$, $\mathrm{Card}(\delta(qz, \epsilon)) = 0$. A dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ, x \in X$:

$$q'\omega' \in \delta(qz, x) \Rightarrow |\, \omega' \,| \leq 2, \text{ and } q'\omega' \in \delta(qz, \epsilon) \Rightarrow |\, \omega' \,| = 0 \tag{3}$$

Given some finite set $F \subseteq QZ^*$ of configurations, the *language recognized by $\mathcal{M}$ with final configurations* F is defined by $\mathrm{L}(\mathcal{M}, F) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \xrightarrow{w}_{\mathcal{M}} c\}$.

2.2 Deterministic context-free grammars

Let $\mathcal{M}$ be some deterministic pushdown automaton (for sake of simplicity [3], we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_{\mathcal{M}}$ associated to $\mathcal{M}$ is defined as: $V_{\mathcal{M}} = \{[p, z, q] \mid p, q \in Q, z \in Z\}$. The *context-free* grammar $G_{\mathcal{M}}$ associated to $\mathcal{M}$ is then $G_{\mathcal{M}} =< X, V, P >$ where $V = V_{\mathcal{M}}$ and P is the set of all the pairs of one of the following forms:

$$([p, z, q], x[p', z_1, p''][p'', z_2, q]) \text{ or } ([p, z, q], x'[p', z', q]) \text{ or } ([p, z, q], a) \tag{4}$$

where $p, q \in Q, z \in Z, x, x' \in X, a \in X \cup \{\epsilon\}, p'z_1z_2 \in \delta(pz, x), p'z' \in \delta(pz, x'), q \in \delta(pz, a)$. $G_{\mathcal{M}}$ is a *strict-deterministic* grammar. (A general theory of this class of grammars is exposed in [Har78] and used in [HHY79]). We call *mode* every element of $QZ \cup \{\epsilon\}$. For every $q \in Q, z \in Z$, qz is said *ϵ-bound* (respectively ϵ-free) iff condition (1) (resp. condition (2)) in the above definition of deterministic automata is realized. The mode ϵ is said ϵ-free. We define a mapping $\mu : V^* \rightarrow QZ \cup \{\epsilon\}$ by

$$\mu(\epsilon) = \epsilon \text{ and } \mu([p, z, q] \cdot \beta) = pz,$$

for every $p, q \in Q, z \in Z, \beta \in V^*$. For every $w \in V^*$ we call $\mu(w)$ the *mode* of the word w.
For technical reasons (which will be made clear in section 7), we suppose that Z contains a special symbol e such that, for every $q \in Q, \delta(qe, \epsilon) = \{q\}$ and $\mathrm{im}(\delta) \subseteq \mathcal{P}_f(Q(Z - \{e\})^*)$.

2.3 Free monoids acting on semi-rings

Semi-ring $\mathsf{B} < W >$ Let $(\mathsf{B}, +, \cdot, 0, 1)$ where $\mathsf{B} = \{0, 1\}$ denote the semi-ring of "booleans". Let W be some alphabet. By $(\mathsf{B} < W >, +, \cdot, \emptyset, \epsilon)$ we denote the semi-ring of *boolean series* over W: every boolean series $S \in \mathsf{B} < W >$ can be written in a unique way as: $S = \Sigma_{w \in W^*} S_w \cdot w$, where, for every $w \in W^*$, $S_w \in \mathsf{B}$. The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0\}.$$

In the particular case where the semi-ring of coefficients is B (which is the only case considered in this article) we sometimes identify the series S with its support. We recall that for every $S \in \mathsf{B} < W >$, S^* is the series defined by: $S^* = \sum_{0 \leq n} S^n$. Given two alphabets W, W', a map $\psi : \mathsf{B} < W > \rightarrow \mathsf{B} < W' >$ is said *σ-additive* iff it fulfills: for every denumerable family $(S_i)_{i \in \mathbb{N}}$ of elements of $\mathsf{B} < W >$, $\psi(\sum_{i \in \mathbb{N}} S_i) = \sum_{i \in \mathbb{N}} \psi(S_i)$. A map $\psi : \mathsf{B} < W > \rightarrow \mathsf{B} < W' >$ which is both a semi-ring homomorphism and a σ-additive map is usually called a *substitution*.

[3] but without loss of generality for the equivalence problem

Actions of monoids Given a semi-ring $(\mathsf{S},+,\cdot,0,1)$ and a monoid $(\mathsf{M},\cdot,1_M)$, a map $\circ : \mathsf{S}\times\mathsf{M} \to \mathsf{S}$ is called a *right-action* of the monoid M over the semi-ring S iff, for every $S,T\in\mathsf{S}, m,m'\in\mathsf{M}$:

$$0\circ m = 0,\ \ S\circ 1_M = S,\ \ (S+T)\circ m = (S\circ m)+(T\circ m)\ \text{ and } S\circ(m\cdot m') = (S\circ m)\circ m' \quad (5)$$

In the particular case where $\mathsf{S} = \mathsf{B} < W >$, $\circ$ is said to be a σ-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i\in\mathbb{N}}$ of elements of S and $m\in\mathsf{M}$:

$$(\sum_{i\in\mathbb{N}} S_i)\circ m = \sum_{i\in\mathbb{N}}(S_i\circ m). \quad (6)$$

The action of W^* on $\mathsf{B} < W >$ We recall the following classical σ-right-action $\bullet$ of the monoid W^* over the semi-ring $\mathsf{B} < W >$: for all $S,S'\in\mathsf{B} < W >, u\in W^*$

$$S\bullet u = S' \Leftrightarrow \forall w\in W^*, (S'_w = 1 \text{ iff } S_{u\cdot w} = 1).$$

(i.e. $S\bullet u$ is the *left-quotient* of S by u , or the *residual* of S by u). For every $S\in\mathsf{B} < W >$ we denote by $\mathsf{Q}(S)$ the set of residuals of S: $\mathsf{Q}(S) = \{S\bullet u \mid u\in W^*\}$. We recall that S is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*. We define the *norm* of a series $S\in\mathsf{B} < W >$, denoted $\|S\|$ by: $\|S\| = \mathrm{Card}(\mathsf{Q}(S))\in\mathbb{N}\cup\{\infty\}$.

The action of X^* on $\mathsf{B} < V >$ Let us fix now a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar G. We define a σ-right-action $\otimes$ of the monoid $(X\cup\{e\})^*$ over the semi-ring $\mathsf{B} < V >$ by: for every $p,q\in Q, A\in Z, H\in V^*, \beta\in V^*, x\in X$

$$[p,A,q]\cdot\beta\otimes x = H\cdot\beta \text{ iff } ([p,A,q],x\cdot H)\in P,\ \ [p,A,q]\cdot\beta\otimes e = H\cdot\beta \text{ iff } ([p,A,q],H)\in P \quad (7)$$

$$\epsilon\otimes x = \emptyset,\ \ \epsilon\otimes e = \emptyset. \quad (8)$$

A series $S\in\mathsf{B} < V >$ is said *ϵ-free* iff $\forall w\in V^*, S_w = 1 \Rightarrow \mu(w)$ is $\epsilon-$free. We denote by $\mathsf{B}_\epsilon < V >$ the subset of ϵ-free series. We define the map $\rho_\epsilon : \mathsf{B} < V > \to \mathsf{B} < V >$ as the unique σ-additive map such that, for every $p\in Q, z\in Z, q\in Q, \beta\in V^*$,

$$\rho_\epsilon([p,z,q]\cdot\beta) = \rho_\epsilon(([p,z,q]\otimes e)\cdot\beta) \text{ if } pz \text{ is } \epsilon-\text{bound},\ \ \rho_\epsilon([p,z,q]\cdot\beta) = [p,z,q]\cdot\beta \text{ if } pz \text{ is } \epsilon-\text{free},$$

and $\rho_\epsilon(\epsilon) = \epsilon$. The above definition is sound because, by hypothesis (3), every $[p,z,q]\otimes e$ is either the unit series ϵ or the empty series $\emptyset$. One can notice that for every $w\in V^*$, $\rho_\epsilon(w)\in V^*\cup\{\emptyset\}$. We call ρ_ϵ the ϵ-reduction map. We then define $\odot$ as the unique right-action of the monoid X^* over the semi-ring $\mathsf{B} < V >$ such that: for every $S\in\mathsf{B} < V >, x\in X$, $S\odot x = \rho_\epsilon(\rho_\epsilon(S)\otimes x)$. One can notice that if $u\neq\epsilon$, then $S\odot u$ is ϵ-free. Let us consider the unique substitution $\varphi : \mathsf{B} < V > \to \mathsf{B} < X >$ fulfilling: for every $p,q\in Q, z\in Z$, $\varphi([p,z,q]) = \{u\in X^* \mid [p,z,q]\odot u = \epsilon\}$, (in other words, φ maps every subset $L\subseteq V^*$ on the language generated by the grammar G from the set of axioms L).

Lemma 2.1 *φ is a morphism of right-actions i.e. for every $S\in\mathsf{B} < V >, u\in X^*$, $\varphi(S\odot u) = \varphi(S)\bullet u$.*

We denote by $\equiv$ the kernel of φ i.e.: for every $S,T\in\mathsf{B} < V >$, $S\equiv T \Leftrightarrow \varphi(S) = \varphi(T)$.

3 Series and languages

3.1 Deterministic series and matrices

We introduce here a notion of *deterministic* series which, in the case of the alphabet V associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. The main advantage of this notion is that, unlike for configurations, we shall be able to define *nice algebraic operations* on these series (this is done in section 3.2). Let us consider a pair $(W,\sim)$ where W is an alphabet and $\sim$ is an equivalence relation over W. We call $(W,\sim)$ a *structured* alphabet. The two examples we have in mind are:

the case where $W = V$, the variable alphabet associated to $\mathcal{M}$ and $[p,A,q]\sim[p',A',q']$ iff $p = p'$ and $A = A'$ (see [Har78])

the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x\sim y$ holds for every $x,y\in X$ (see [Har78]).

Definition 3.1 *Let $S \in \mathsf{B} < W >$. S is said* left-deterministic *iff either* (1) $S = \emptyset$ *or* (2) $S = \epsilon$ *or* (3) $\forall w, w' \in W^*, S_w = S_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in W^*, A \sim A', w = A \cdot w_1$ *and* $w' = A' \cdot w_1'$.

Definition 3.2 *Let $S \in \mathsf{B} < W >$. S is said* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

This notion is the straighforward extension to the infinite case of the notion of (finite) *set of associates* defined in [HHY79].
We denote by $\mathsf{DB} < W >$ the subset of deterministic boolean series over W. Let us denote by $\mathsf{B}_{n,m} < W >$ the set of (n, m)-matrices with entries in the semi-ring $\mathsf{B} < W >$.

Definition 3.3 *Let $m \in \mathbb{N}, S \in \mathsf{B}_{1,m} < W >$: $S = (S_1, \cdots, S_m)$. S is said* left-deterministic *iff either* (1) $\forall i \in [1, m], S_i = \emptyset$ *or* (2) $\exists i_0 \in [1, m], S_{i_0} = \epsilon$ *and* $\forall i \neq i_0, S_i = \emptyset$ *or* (3) $\forall w, w' \in W^*, \forall i, j \in [1, m], (S_i)_w = (S_j)_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in V^*, A \sim A', w = A \cdot w_1$ *and* $w' = A' \cdot w_1'$.

The right-action $\bullet$ on $\mathsf{B} < W >$ is extended componentwise to $\mathsf{B}_{n,m} < W >$: for every $S = (s_{i,j})$, $u \in W^*$, the matrix $T = S \bullet u$ is defined by $t_{i,j} = s_{i,j} \bullet u$.

Definition 3.4 *Let $S \in \mathsf{B}_{1,m} < W >$. S is said* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

We denote by $\mathsf{DB}_{1,m} < W >$ the subset of deterministic row-vectors of dimension m over $\mathsf{B} < W >$.

Definition 3.5 *Let $S \in \mathsf{B}_{n,m} < W >$. S is said* deterministic *iff, for every $i \in [1, n]$, $S_{i,*}$ is a deterministic row-vector.*

The following property is crucial for establishing a correct theory of *deterministic spaces* (see §3.2 below).

Lemma 3.6 *For every $S \in \mathsf{DB}_{n,m} < W >, T \in \mathsf{DB}_{m,s} < W >, S \cdot T \in \mathsf{DB}_{n,s} < W >$.*

W=V Let $(W, \sim)$ be the structured alphabet $(V, \sim)$ associated with $\mathcal{M}$ and let us consider a bijective numbering of the elements of Q: $(q_1, q_2, \ldots, q_{n_Q})$. Some particular "vectorial" notions turn out to be useful:

- we define a *Q-series* to be a family $(S_q)_{q \in Q}$ such that the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ is deterministic
- we define a *Q-form* to be a family $\Phi = (\Phi_q)_{q \in Q}$ of deterministic series.

Given a Q-series S and a Q-form Φ, their Q-product $S * \Phi$ is the deterministic series defined by $S * \Phi = \sum_{q \in Q} S_q \cdot \Phi_q$. If the Q-series $(S_q)_{q \in Q}$ is identified with the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ and the Q-form $(\Phi_q)_{q \in Q}$ with the column-vector $(\Phi_{q_j})_{j \in [1, n_Q]}$, then the Q-product appears to be just the ordinary product of matrices.

Let us define here handful notations for some particular row-vectors or Q-series. Let us use the *Kronecker symbol* $\delta_{i,j}$ meaning ϵ if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leq n, 1 \leq i \leq n$, we define the row-vector ϵ_i^n as: $\epsilon_i^n = (\epsilon_{i,j}^n)_{1 \leq j \leq n}$ where $\forall j, \epsilon_{i,j}^n = \delta_{i,j}$. We call *unit row-vector* any vector of the form ϵ_i^n.
For every $\omega \in Z^*, p, q \in Q$, $[p\omega q]$ is the deterministic series defined inductively by:

$$[p\epsilon q] = \emptyset \text{ if } p \neq q, [p\epsilon q] = \epsilon \text{ if } p = q,$$
$$[p\omega q] = \sum_{r \in Q} [pAr] \cdot [r\omega' q] \text{ if } \omega = A \cdot \omega' \text{ for some } A \in Z, \omega' \in Z^*.$$

By $[p\omega]$ we denote the Q-series: $[p\omega] = ([p\omega q])_{q \in Q}$. (In particular $[q_i] = \epsilon_i^{n_Q}$). By $[\omega]$ we denote the Q-matrix: $[\omega] = ([p\omega q])_{p \in Q, q \in Q}$. The next lemma relates the right-action $\odot$ with the right-action $\bullet$.

Lemma 3.7 *Let $S \in \mathsf{DB} < V >, u \in X^*$. One of the three following cases must occur:* (1) $S \odot u = \emptyset$, *or* (2) $S \odot u = \epsilon$, *or* (3) $\exists u_1, u_2 \in X^*, v_1 \in V^*, q \in Q, A \in Z, \Phi$ $Q-$*form such that* $u = u_1 \cdot u_2, S \odot u_1 = S \bullet v_1 = [qA] * \Phi$ *and* $S \odot u = ([qA] \odot u_2) * \Phi$.

Corollary 3.8 *Let $S \in \mathsf{DB} < V >, u \in X^*$. Then $S \odot u \in \mathsf{DB} < V >$*

The particular letters $[p, e, q]$ for $p, q \in Q$ play a special role in sections 7 and 8: we use them as *marks* in the series (somehow like the ceilings of [Val74]). We define below a map ρ_e which removes the marks in the series. Let us define $\rho_e : \mathsf{DB} < V > \longrightarrow \mathsf{B} < V >$ as the unique substitution such that:

$$\rho_e([p, e, q]) = \epsilon \quad \text{if } p = q, \quad \rho_e([p, e, q]) = \emptyset \quad \text{if } p \neq q.$$

Lemma 3.9 *For every $S \in \mathsf{DB} < V >$, $\rho_e(S) \in \mathsf{DB} < V >$ and $\|\rho_e(S)\| \leq \|S\|$.*

Rational series, norm Let us generalize the definition of *rationality* of series in $\mathsf{B} < W >$ to matrices. Given $M \in \mathsf{B}_{n,m} < W >$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M: $\mathsf{Q}(M) = \{M \bullet u \mid u \in W^*\}$. Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M: $\mathsf{Q}_r(M) = \bigcup_{1 \leq i \leq n} \mathsf{Q}(M_{i,*})$. M is said *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{DRB}_{n,m} < W >$ the set of deterministic, rational matrices over $\mathsf{B} < W >$. For every $M \in \mathsf{DRB}_{n,m} < W >$, we define the norm of M as: $\|M\| = \mathrm{Card}(\mathsf{Q}_r(M))$.

Lemma 3.10 *Let* $A \in \mathsf{DB}_{n,m} < W >, B \in \mathsf{DB}_{m,s} < W >$. *Then* $\|A \cdot B\| \leq \|A\| + \|B\|$.

3.2 Deterministic spaces

We adapt here the key-idea of [Mei89, Mei92] to series.

Definitions Let $(W, \sim)$ be some structured alphabet and let us consider the set $E = \mathsf{DRB} < W >$. A series $U = \sum_{i=1}^{n} \gamma_i \cdot U_i$ where $\gamma \in \mathsf{DRB}_{1,n} < W >$, $U_i \in \mathsf{DRB} < W >$ is called a *linear combination* of the U_i's. We call *deterministic space* of rational series (d-space for short) any subset V of E which is closed under finite linear combinations. Given any set $\mathcal{G} = \{U_i | i \in I\}$, one can check that the set V of all (finite) linear combinations of elements of $\mathcal{G}$ is a d-space (by lemma 3.6) and that it is the smallest d-space containing $\mathcal{G}$. Therefore we call V the d-space *generated* by $\mathcal{G}$ and we call $\mathcal{G}$ a *generating set* of V (we note $\mathsf{V} = \mathsf{V}(\{U_i | i \in I\})$). (Similar definitions can be given for *families* of series).
We let now $W = V$. Following an analogy with classical linear algebra, we develop now a notion corresponding to a kind of *linear independence* of the images by φ of the given series Let us extend the equivalence relation $\equiv$ to d-spaces by: for every d-spaces $\mathsf{V}_1, \mathsf{V}_2$, $\mathsf{V}_1 \equiv \mathsf{V}_2 \Leftrightarrow \forall i, j \in \{1,2\}, \forall S \in \mathsf{V}_i, \exists S' \in \mathsf{V}_j, S \equiv S'$.

Lemma 3.11 *Let* $S_1, \ldots, S_j, \ldots, S_m \in \mathsf{DRB} < V >$. *The following are equivalent*

1. $\exists \alpha, \beta \in \mathsf{DRB}_{1,m} < V >, \alpha \not\equiv \beta$, *such that* $\sum_{1 \leq j \leq m} \alpha_j \cdot S_j \equiv \sum_{1 \leq j \leq m} \beta_j \cdot S_j$,
2. $\exists j_0 \in [1,m], \exists \gamma \in \mathsf{DRB}_{1,m} < V >, \gamma \not\equiv \epsilon_{j_0}^m$, *such that* $S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma_j \cdot S_j$,
3. $\exists j_0 \in [1,m], \exists \gamma' \in \mathsf{DRB}_{1,m} < V >, \gamma'_{j_0} \equiv \emptyset$, *such that* $S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma'_j \cdot S_j$,
4. $\exists j_0 \in [1,m]$, *such that* $\mathsf{V}((S_j)_{1 \leq j \leq m}) \equiv \mathsf{V}((S_j)_{1 \leq j \leq m, j \neq j_0})$.

The equivalence between (1),(2) and (3) was first proved in [Mei89, Mei92], in the case where the S_j's are configurations $q_j\omega$, with the same ω.

4 Deduction systems

4.1 General deduction systems

We follow here the general philosophy of [HHY79, Cou83]. Let us call *deduction system* any triple $\mathcal{D} = < \mathcal{A}, H, \vdash\!\!- >$ where $\mathcal{A}$ is a denumerable set called the *set of assertions*, H, the *cost function* is a mapping $\mathcal{A} \rightarrow \mathbb{N} \cup \{\infty\}$ and $\vdash\!\!-$, the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$; $\mathcal{A}$ is given with a fixed bijection with $\mathbb{N}$ (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function, ... over $\mathcal{A}, \mathcal{P}_f(\mathcal{A})$,... are defined, up to this fixed bijection ; we assume that $\mathcal{D}$ satisfies the following axioms:
(A 1) $\vdash\!\!-$ is recursively enumerable
(A 2) $\forall (P, A) \in \vdash\!\!-$, (min $\{H(p), p \in P\} < H(A)$) or ($H(A) = \infty$). (We let $\min(\emptyset) = \infty$).

In the sequel we use the notation $P \vdash\!\!- A$ for $(P, A) \in \vdash\!\!-$. We call *proof* in the system $\mathcal{D}$, any subset $P \subseteq \mathcal{A}$ fulfilling : $\forall p \in P, (\exists Q \subseteq P, Q \vdash\!\!- p)$. Let us define the total map $\chi : \mathcal{A} \rightarrow \{0,1\}$ and the partial map $\overline{\chi} : \mathcal{A} \rightarrow \{0,1\}$ by :
$\chi(A) = 1$ if $H(A) = \infty, \chi(A) = 0$ if $H(A) < \infty$, $\overline{\chi}(A) = 1$ if $H(A) = \infty, \overline{\chi}$ is undefined if $H(A) < \infty$.
(χ is the "truth-value function", $\overline{\chi}$ is the "1-value function").

Lemma 4.1 *Let* P *be a proof and* $A \in P$. *Then* $\chi(A) = 1$.

In other words : every provable assertion is true. The deduction system $\mathcal{D}$ will be said *complete* iff, conversely, $\forall A \in \mathcal{A}, \chi(A) = 1 \Longrightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is "finitely" provable).

Lemma 4.2 *: If $\mathcal{D}$ is complete, $\overline{\chi}$ is a recursive partial map.*

In order to define deduction relations from more elementary ones, we set the following definitions. Let $\vdash\!\!\!-\ \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$. For every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set:
$P \overset{[0]}{\vdash\!\!\!-} Q$ iff $P \supseteq Q$; $P \overset{[1]}{\vdash\!\!\!-} Q$ iff $\forall q \in Q, \exists R \subseteq P, R \vdash\!\!\!- q$; $P \overset{<0>}{\vdash\!\!\!-} Q$ iff $P \overset{[0]}{\vdash\!\!\!-} Q$; $P \overset{<1>}{\vdash\!\!\!-} Q$ iff $\forall q \in Q, (\exists R \subseteq P, R \vdash\!\!\!- q)$ or $(q \in P)$; $P \overset{<n+1>}{\vdash\!\!\!-} Q$ iff $\exists R \in \mathcal{P}_f(\mathcal{A}), P \overset{<1>}{\vdash\!\!\!-} R$ and $R \overset{<n>}{\vdash\!\!\!-} Q$ (for every $n \geq 1$).;
$\overset{<*>}{\vdash\!\!\!-} = \bigcup_{n \geq 0} \overset{<n>}{\vdash\!\!\!-}$.
Given $\vdash\!\!\!-_1, \vdash\!\!\!-_2 \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(\mathcal{A})$, for every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set : $P(\vdash\!\!\!-_1 \circ \vdash\!\!\!-_2)Q$ iff $\exists R \subseteq \mathcal{A}, (P \vdash\!\!\!-_1 R) \wedge (R \vdash\!\!\!-_2 Q)$.

4.2 System $\mathcal{D}_0$

Let us define here a particular deduction system $\mathcal{D}_0$ "Taylored for the equivalence problem for dpda's".

Given a fixed dpda $\mathcal{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathcal{M}$ (see section 3.1) and the set $\mathsf{DRB} < V >$ (the set of Deterministic Rational Boolean series over V^*). The set of assertions is defined by : $\mathcal{A} = \mathbb{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V >$ i.e. an assertion is here a *weighted equation* over $\mathsf{DRB} < V >$.
The "cost-function" $H : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is defined by : $H(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S')$, where $\mathrm{Div}(S, S')$, the *divergence* between S and S', is defined by : $\mathrm{Div}(S, S') = \min\{| u \| \mid u \in \Delta(\varphi(S), \varphi(S'))\}$. (We recall $\min(\emptyset) = \infty$).
Let us notice that here : $\chi(n, S, S') = 1 \Longleftrightarrow S \equiv S'$.
We define a binary relation $|\!\!\vdash\!\!\!-\ \subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

$(R0)$	$\{(p, S, T)\}$	$\lvert\!\vdash\!\!\!-\ (p+1, S, T)$	
$(R1)$	$\{(p, S, T)\}$	$\lvert\!\vdash\!\!\!-\ (p, T, S)$	
$(R2)$	$\{(p, S, S'), (p, S', S'')\}$	$\lvert\!\vdash\!\!\!-\ (p, S, S'')$	
$(R3)$	$\emptyset$	$\lvert\!\vdash\!\!\!-\ (0, S, S)$	
$(R'3)$	$\emptyset$	$\lvert\!\vdash\!\!\!-\ (0, [qzr], \epsilon)$	(for $q, r \in Q, z \in Z, [qzr] \equiv \epsilon$)
$(R4)$	$\{(p+1, S \odot x, T \odot x) \mid x \in X\}$	$\lvert\!\vdash\!\!\!-\ (p, S, T)$	(for $S \not\equiv \epsilon \wedge T \not\equiv \epsilon$)
$(R5)$	$\{(p, S, S')\}$	$\lvert\!\vdash\!\!\!-\ (p+2, S \odot x, S' \odot x)$	(for $x \in X$)
$(R6)$	$\{(p, S \cdot T' + T, T')\}$	$\lvert\!\vdash\!\!\!-\ (p, S^* \cdot T, T')$	(for $S \not\equiv \varepsilon$)
$(R7)$	$\{(p, S, S')\}$	$\lvert\!\vdash\!\!\!-\ (p, S + T, S' + T)$	
$(R8)$	$\{(p, S, S')\}$	$\lvert\!\vdash\!\!\!-\ (p, S \cdot T, S' \cdot T)$	
$(R9)$	$\{(p, S, S')\}$	$\lvert\!\vdash\!\!\!-\ (p, U \cdot S, U \cdot S')$	

where $p \in \mathbb{N}, S, S', T, T' \in \mathsf{DRB} < V >, U \in \mathsf{RB} < V >$. (By set of "all" these pairs we mean, all the pairs which fulfill both properties "to belong to $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$" and "to have one of these 11 possible forms" ; but of course, for example, not all the triples $(p, S + T, S' + T)$ belong to $\mathcal{A}$ because $\mathsf{DRB} < V >$ is not closed under sum).

Lemma 4.3 *: Let $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$ such that $P \,|\!\!\vdash\!\!\!-\, A$. Then $min\{H(p) \mid p \in P\} \leq H(A)$.*

Let us define $\vdash\!\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$, $P \vdash\!\!\!- A \Longleftrightarrow P \overset{<*>}{|\!\!\vdash\!\!\!-} \circ \overset{[1]}{|\!\!\vdash\!\!\!-}_{0,3,4} \circ \overset{<*>}{|\!\!\vdash\!\!\!-} \{A\}$, where $|\!\!\vdash\!\!\!-_{0,3,4}$ is the relation defined by R_0, R_3, R'_3, R_4 only. We let $\mathcal{D}_0 =< \mathcal{A}, H, \vdash\!\!\!- >$.

Lemma 4.4 *: $\mathcal{D}_0$ is a deduction system.*

The key-statement of this work is that $\mathcal{D}_0$ is complete (theorem 9.2). We prove this completeness result by exhibiting a "strategy" $\mathcal{S}$ which, for every true assertion (n, S, S'), constructs a finite $\mathcal{D}_0$-proof of this assertion. Notice that, by lemma 4.2, we do not need to prove that $\mathcal{S}$ is computable in any sense to establish that $\overline{\chi}$ is partial-recursive.

4.3 Strategies

Let $\mathcal{D} =< \mathcal{A}, H, \longmapsto >$ be a deduction system. We call a *strategy* for $\mathcal{D}$ any partial map $\mathcal{S} : \mathcal{A}^+ \longrightarrow \mathcal{A}^*$ such that :
(S1) if $\mathcal{S}(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ then $\exists Q \subseteq \{A_i \mid 1 \leq i \leq n-1\}$ such that

$$\{B_j \mid 1 \leq j \leq m\} \cup Q \longmapsto A_n,$$

(S2) if $\mathcal{S}(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ then

$$\min\{H(A_i) \mid 1 \leq i \leq n\} = \infty \Longrightarrow \min\{H(B_j) \mid 1 \leq j \leq m\} = \infty.$$

Given a strategy $\mathcal{S}$, we define $\mathcal{T}(\mathcal{S}, A)$, the proof-tree associated to the strategy $\mathcal{S}$ and the assertion A as the unique tree t such that :
$\varepsilon \in dom(t)$, $t(\varepsilon) = A$, and, for every path $x_0 x_1, \cdots x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if x_{n-1} has m sons $x_{n-1} \cdot 1, \cdots .x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$\mathcal{S}(A_1 \cdots A_n) = B_1 \cdots B_m \text{ or } (m = 0 \text{ and } A_1 \cdots A_n \notin \mathrm{dom}(\mathcal{S})).$$

Let us say that $\mathcal{S}$ *terminates* iff, $\forall A \in \chi^{-1}(1), \mathcal{T}(\mathcal{S}, A)$ is finite; $\mathcal{S}$ is said *closed* iff, $\forall W \in \mathcal{A}^+, W \in (\chi^{-1}(1))^+ \Longrightarrow W \in dom(\mathcal{S})$ (i.e. $\mathcal{S}$ is defined on every non-empty sequence of true assertions).

Lemma 4.5 : *If $\mathcal{S}$ is a closed strategy for $\mathcal{D}$, then, for every true assertion A, the set of labels of $\mathcal{T}(\mathcal{S}, A)$ is a $\mathcal{D}$-proof.*

Lemma 4.6 : *If $\mathcal{D}$ admits some terminating, closed strategy then $\mathcal{D}$ is complete.*

5 Triangulations

Let $S_1, S_2, \cdots, S_d$ be a family of deterministic series over the structured alphabet V (we recall V is the alphabet associated with some dpda $\mathcal{M}$ as defined in section 2.2).
Let us consider a sequence $\mathcal{S}$ of n "weighted" linear equations :

$$(\mathcal{E}_i) : p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j \ , \ \sum_{j=1}^{d} \beta_{i,j} S_j$$

where $p_i \in \mathbb{N}$, and $A = (\alpha_{i,j}), B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n, d), with indices $m \leq i \leq m+n-1, 1 \leq j \leq d$. For any weighted equation, $\mathcal{E} = (p, S, S')$, we recall the "cost" of this equation is : $H(\mathcal{E}) = p + 2 \cdot \mathrm{Div}(\varphi(S), \varphi(S'))$.
We associate to such a system another system of equations, $\mathrm{INV}(\mathcal{S})$, which "translates the equations of $\mathcal{S}$ into equations over $(\alpha_{i,j}, \beta_{i,j})$ only". This function INV is in some sense an "elaborated version" of the *inverse* systems defined in [Mei89, Mei92]. The general idea of the construction of INV consists in iterating the transformation used in the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in lemma 3.11, i.e. the classical idea of *triangulating* a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathcal{D}_0$. Let us assume here that

$$\forall j \in [1, d], S_j \not\equiv \emptyset. \tag{9}$$

For every $S \in \mathsf{B} < X >$ (resp. $S' \in \mathsf{B}_{1,d} < X >$), we define $\nu(S) = \min\{|u|, u \in \mathrm{supp}(S)\}$ (resp. $\nu(S') = \min\{|u|, u \in \cup_{1 \leq j \leq d} \mathrm{supp}(S'_j)\}$). Let us define $\mathrm{INV}(\mathcal{S})$, $\mathrm{W}(\mathcal{S}) \in \mathbb{N} \cup \{\perp\}$, $\mathrm{D}(\mathcal{S}) \in \mathbb{N}$ by induction on n. $\mathrm{W}(\mathcal{S})$ is the *weight* of $\mathcal{S}$. $\mathrm{D}(\mathcal{S})$ is the *weak codimension* of $\mathcal{S}$.
Case 1 : $\varphi(\alpha_{m,*}) = \varphi(\beta_{m,*})$ or $n = 1$

$$\mathrm{INV}(\mathcal{S}) = ((W(\mathcal{S}), \alpha_{m,j}, \beta_{m,j}))_{1 \leq j \leq d}, \ \mathrm{W}(\mathcal{S}) = p_m - 1, \ \mathrm{D}(\mathcal{S}) = 0.$$

Case 2 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n \geq 2, p_{m+1} - p_m \geq 2 \cdot \nu(\Delta(\varphi(\alpha_{m,*}), \varphi(\beta_{m,*}))) + 1$
Let $u = \min\Delta(\varphi(\alpha_{m,*}), \varphi(\beta_{m,*}))$. Suppose $u \in \Delta(\varphi(\alpha_{m,j_0}), \varphi(\beta_{m,j_0}))$.
Subcase 1 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u = \emptyset$.

Let us consider the equation $(p_m, S_{j_0}, \sum_{j=1}^{d}(\beta_{m,j} \odot u)S_j)$ and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0}(\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j} \odot u))S_j \ , \ \sum_{j \neq j_0}(\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j} \odot u))S_j$$

where the above equation is seen as as an equation between two linear combinations of the S_i's where the j_0-th coefficient is $\emptyset$ on both sides. We then define :

$$\mathrm{INV}(\mathcal{S}) = \mathrm{INV}(\mathcal{S}'), \mathrm{W}(\mathcal{S}) = \mathrm{W}(\mathcal{S}'), \mathrm{D}(\mathcal{S}) = \mathrm{D}(\mathcal{S}') + 1. \tag{10}$$

Subcase 2 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u \neq \emptyset$.
Let us consider the w-equation $(p_m, S_{j_0} \ , \ \sum_{j=1}^{d}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)S_j)$ and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0}[(\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)]S_j \ , \ \sum_{j \neq j_0}[(\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)]S_j.$$

We then set the same definitions (10) as above.
Subcase 3 : $\alpha_{m,j_0} \odot u = \emptyset, \beta_{m,j_0} \odot u = \varepsilon$. (Analogous to subcase 1).
Subcase 4 : $\alpha_{m,j_0} \odot u \neq \emptyset, \beta_{m,j_0} \odot u = \varepsilon$. (Analogous to subcase 2).
Case 3 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n \geq 2, p_{m+1} - p_m \leq 2 \cdot \nu(\Delta(\varphi(\alpha_{m,*}), (\varphi(\beta_{m,*})))$.
We then define: $\mathrm{INV}(\mathcal{S}) = \perp, \mathrm{W}(\mathcal{S}) = \perp, \mathrm{D}(\mathcal{S}) = 0$, where $\perp$ is a special symbol which can be understood as meaning "undefined".

Lemma 5.1 *: Let $\mathcal{S}$ be a system of linear equations. If $INV(\mathcal{S}) \neq \perp$ then $INV(\mathcal{S}) = (\bar{\mathcal{E}}_j)_{1 \leq j \leq d}$ fulfills:*

1. *$\forall j \in [1, d], \bar{\mathcal{E}}_j$ is a linear equation with deterministic coefficients,*
2. *$\{\bar{\mathcal{E}}_j \mid 1 \leq j \leq d\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + D(\mathcal{S}) - 1\} \models \mathcal{E}_{m+D(\mathcal{S})}$,*
   If, in addition, $n \geq d$ then :
3. *$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + D(\mathcal{S})\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty$.*

Let us consider the function F defined by :

$$F(n) = max\{\nu(\varphi(A)\Delta\varphi(B)) \mid A, B \in \mathsf{DRB}_{1,d} < V >, \| A \| \leq n, \| B \| \leq n, \varphi(A) \neq \varphi(B)\}.$$

For every integer parameters $K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ by :

$$\delta_m = 0, \ell_m = 0, L_m = K_2, s_m = K_3 \cdot K_2 + K_4, S_m = 0, \Sigma_m = 0, \tag{11}$$

and for every $m \leq i \leq m + n - 2$,

$$\begin{aligned} \delta_{i+1} = 2 \cdot F(s_i + \Sigma_i) + 1, \qquad & \ell_{i+1} = 5 \cdot \delta_{i+1} + 14, \qquad L_{i+1} = K_1 \cdot (L_i + \ell_{i+1}) + K_2, \\ s_{i+1} = K_3 \cdot L_{i+1} + K_4, \quad & S_{i+1} = s_i + \Sigma_i + \mid Q \mid F(s_i + \Sigma_i), \quad \Sigma_{i+1} = \Sigma_i + S_{i+1}. \end{aligned} \tag{12}$$

For every weighted, deterministic rational linear equation $\mathcal{E} = (p, \sum_{j=1}^{d} \alpha_j S_j \ . \ \sum_{j=1}^{d} \beta_j S_j)$, we define

$$||| \mathcal{E} ||| = max\{\| \alpha \|, \| \beta \|\}.$$

Lemma 5.2 *Let $\mathcal{S} = (\mathcal{E}_i)_{m \leq i \leq m+d-1}$ be a system of d weighted linear equations such that :*

(1) $\forall i \in [m, m + d - 1], ||| \mathcal{E}_i ||| \leq s_i$
(2) $\forall i \in [m, m + d - 2], \mathrm{W}(\mathcal{E}_{i+1}) - \mathrm{W}(\mathcal{E}_i) \geq \delta_{i+1}$.

Then $\mathrm{INV}(\mathcal{S}) \neq \perp, \mathrm{D}(\mathcal{S}) \leq d - 1, \forall \mathcal{E} \in \mathrm{INV}(\mathcal{S}), ||| \mathcal{E} ||| \leq \Sigma_{m+\mathrm{D}(\mathcal{S})} + s_{m+\mathrm{D}(\mathcal{S})}$.

6 Constants

The following constants will be used in the sequel.

$k_0 = \max\{\nu([pAq]) \mid p,q \in Q, A \in Z, [pAq] \not\equiv \emptyset\}$, $k_1 = \max\{2k_0+1, 3\}$, $k_2 = 4k_1 + 2(k_1)^2 + k_0$,
$D_1 = 4k_0 + 2$, $K_1 = k_1 + 1$, $K_2 = 2(k_1)^3 + 3(k_1)^2 + k_1 + 1$,
$K_3 = k_0|Q|$, $K_4 = k_0|Q|^2 + (k_2+6)|Q|$,
$d_0 = 2 \cdot |Q| \cdot \mathrm{Card}(X^{\leq k_1})$.

We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ defined by the relations (11,12) of section 5 where the parameters $K_1, \ldots, K_4$ are chosen to be the above constants, the functions F is associated with $d = d_0$ and $m = 1, n = d_0$.

$$D_2 = \Sigma_{d_0} + s_{d_0}.$$

7 Strategies for $\mathcal{D}_\circ$

Let us define strategies for the particular system $\mathcal{D}_0$.
We define first auxiliary strategies $T_{cut}, T_\emptyset, T_\varepsilon, T_A, T_B, T_C$ and then derive some closed strategies from them. Let us fix here some total ordering on $X: x_1 < x_2 < \cdots < x_\alpha$ and also some total ordering $\leq$ of type ω on $\mathcal{A}$ (inherited from the usual well-ordering of $\mathbb{N}$ by the fixed encoding). From these orderings one can construct in the usual way an ordering of type ω on the sets $X^*, \mathcal{A}^*$ and $\mathbb{N}^* \times (\mathsf{DRB} < V >)^*$.

Let us adapt the usual notion of ***stacking derivation*** to derivations of series. For every $u \in X^*$ we define the binary relation $\uparrow(u)$ over $\mathsf{DB} < V >$ by: for every $S, S' \in \mathsf{DB} < V >, S \uparrow (u) S' \Leftrightarrow \exists A \in Z, \omega \in Z^+, p, q \in Q, \Psi \in \mathsf{DB}_{Q,1} < V >$ such that

$$S = [pA] * \Psi, [pA] \odot u = [q\omega], S' = [q\omega] * \Psi.$$

A sequence of deterministic series $S_0, S_1, \ldots, S_n$ is a ***derivation*** iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. If $u = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S, u). A derivation $S_0, S_1, \ldots, S_n$ is said to be ***stacking*** iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow (u) S_n$.

T_{cut}: $T_{cut}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $\exists i \in [1, n-1], \exists S, T,$

$$A_i = (p_i, S, T), A_n = (p_n, S, T), p_i < p_n \text{ and } m = 0$$

$T_\emptyset$: $T_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T,\ A_n = (p, S, T), p \geq 0, S \equiv T \equiv \emptyset$ and $m = 0$
T_ε: $T_\varepsilon(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $A_n = (p, S, T), p \geq 0, S \equiv T \equiv \varepsilon$ and $m = 0$
T_A: $T_A(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, S, T), m = |X|, B_1 = (p+1, S \odot x_1, T \odot x_1), \cdots, B_m = (p+1, S \odot x_m, T \odot x_m),$$

where $S \not\equiv \varepsilon, T \not\equiv \varepsilon$
T_B^+: $T_B^+(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $n \geq k_1, A_{n-k_1} = (\pi, \overline{U}, U')$, (where $\overline{U}$ is unmarked)

$$U' = \sum_{q \in Q} [\overline{p}Aq] \cdot V_q \quad (\text{ for some } (\overline{p} \in Q)$$

$A_i = (\pi + k_1 + i - n, U_i, U_i')$ for $n - k_1 \leq i \leq n$, $(U'i)_{n-k_1 \leq i \leq n}$ is a "stacking derivation" (see the above definition),

$$U_n' = \sum_{q \in Q} [p\tau q] \cdot V_q, \quad \text{for some } p \in Q, \tau \in Z^+,$$

$m = 1, B_1 = (\pi + k_1 - 1, V, V'), V = U_n,\ V' = \sum_{q \in Q'} [p\tau q] \cdot [qeq] \cdot (\overline{U} \odot u_q)$,
where $Q' = \{q \in Q \mid [\overline{p}Aq] \not\equiv \emptyset\}, \forall q \in Q', u_q = \min(\varphi([\overline{p}Aq]))$.
T_B^-: T_B^- is defined in the same way as T_B^- by exchanging the left series (S^-) and right (S^+) series in every assertion (p, S^-, S^+).

T_C: $T_C(A_1 \cdots A_n) = B_1 \cdots B_m$ iff there exists $d \in [1, d_0], S_1, S_2, \cdots, S_d \in \mathsf{DRB} < V >, 1 \leq \kappa_1 \leq \kappa_2 < \cdots < \kappa_d = n$, such that,

(C1) every equation $\mathcal{E}_i = A_{\kappa_i}$ is a weighted equation over $S_1, S_2, \cdots, S_d$,

(C2) $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq d}$ fulfills the hypothesis of lemma 5.2,

(C3) $(\kappa_1, \kappa_2, \cdots, \kappa_d, S_1, \cdots, S_d) \in \mathbb{N}^* \times (\mathsf{DRB} < V >)^*$ is the minimal vector satisfying conditions (C1,C2) for the given sequence $(A_1 \cdots A_n)$ and

(C4) $B_1 \cdots B_m = \rho_e(INV(\mathcal{S}))$ (where ρ_e is the obvious extension of ρ_e to pairs of series and then to sequences of weighted equations; in other words the result of T_C is $INV(\mathcal{S})$ where the marks have been removed).

Let us notice that, by lemma 5.2 and lemma 3.9, for every $j \in [1, m]$, $\||\, B_j \,\|| \leq \Sigma_{m+\mathrm{D}(\mathcal{S})} + s_{m+\mathrm{D}(\mathcal{S})} \leq \Sigma_{m+d_0} + s_{m+d_0} = D_2$. This inequality is *independant* of the sizes of the series appearing as lefthand sides (or rhs) of the initial equations $A_1 \cdots A_n$.

Lemma 7.1 : *$T_{cut}, T_\emptyset, T_\varepsilon, T_A, T_B, T_C$ are $\mathcal{D}_0$ strategies.*

Let us define the strategy $\mathcal{S}_{AB}$ by : for every $W = A_1 A_2 \cdots A_n$,

(0) if $W \in \mathrm{dom}(T_{cut})$, then $\mathcal{S}_{AB}(W) = T_{cut}(W)$ (1) elsif $W \in \mathrm{dom}(T_\emptyset)$, then $\mathcal{S}_{AB}(W) = T_\emptyset(W)$
(2) elsif $W \in \mathrm{dom}(T_\varepsilon)$, then $\mathcal{S}_{AB}(W) = T_\varepsilon(W)$ (4) elsif $W \in \mathrm{dom}(T_B^+)$, then $\mathcal{S}_{AB}(W) = T_B^+(W)$
(5) elsif $W \in \mathrm{dom}(T_B^-)$, then $\mathcal{S}_{AB}(W) = T_B^-(W)$ (6) elsif $W \in \mathrm{dom}(T_A)$, then $\mathcal{S}_{AB}(W) = T_A(W)$
(7) else $\mathcal{S}_{AB}(W)$ is undefined.

The strategy $\mathcal{S}_{ABC}$ is obtained by inserting "(3) elsif $W \in \mathrm{dom}(T_C)$, then $\mathcal{S}_{ABC}(W) = T_C(W)$" in the above list of cases.

Lemma 7.2 *$\mathcal{S}_{ABC}, \mathcal{S}_{AB}$ are closed.*

8 Tree analysis

This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathcal{S}_{AB}$ defined in section 7. The main results are [Sén97, lemma 8.14 , 8.15] whose combination asserts that if some path (from a node x to a node y) of τ is such that its origin has a "small norm" and its length is "large enough", then the transformation T_C is defined at some ancestor of y. [4]

9 Completeness of $\mathcal{D}_0$

Lemma 9.1 : *$\mathcal{S}_{ABC}$ is terminating.*

The proof leans on the two delicate lemmas [Sén97, lemma 8.14 , 8.15] mentioned above.

Theorem 9.2 *The system $\mathcal{D}_0$ is complete.*

Proof: By lemma 7.1 $\mathcal{S}_{ABC}$ is a strategy for $\mathcal{D}_0$, by lemma7.2 $\mathcal{S}_{ABC}$ is closed , by lemma 9.1 it is terminating and by lemma 4.6, $\mathcal{D}_0$ is complete. □

Theorem 9.3 *The equivalence problem for deterministic pushdown automata is decidable.*

Proof: Let $\mathcal{M}$ be some dpda. The equivalence relation $\equiv$ on $\mathsf{DRB} < V >$ (where V is the structured alphabet associated to the given $\mathcal{M}$) has a recursively enumerable complement (this is well-known). By theorem 9.2 and lemma 4.2 $\equiv$ is recursively enumerable too. Hence $\equiv$ is recursive. In addition, the system $\mathcal{D}_0$ associated with $\mathcal{M}$ is computable from $\mathcal{M}$, hence the theorem follows. □

[4] Technically speaking, this is the most difficult part of the full proof; we cannot sketch it here due to the lack of space.

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