10.2 Calculus with Parametric Curves

1. $x = 2t^3 + 3t,\ y = 4t - 5t^2 \ \Rightarrow\ \dfrac{dx}{dt} = 6t^2 + 3,\ \dfrac{dy}{dt} = 4 - 10t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4 - 10t}{6t^2 + 3}$.

2. $x = t - \ln t,\ y = t^2 - t^{-2} \ \Rightarrow\ \dfrac{dx}{dt} = 1 - t^{-1},\ \dfrac{dy}{dt} = 2t + 2t^{-3}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 2t^{-3}}{1 - t^{-1}} \cdot \dfrac{t^3}{t^3} = \dfrac{2t^4 + 2}{t^3 - t^2}$.

3. $x = te^t,\ y = t + \sin t \ \Rightarrow\ \dfrac{dx}{dt} = te^t + e^t = e^t(t + 1),\ \dfrac{dy}{dt} = 1 + \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + \cos t}{e^t(t + 1)}$.

4. $x = t + \sin(t^2 + 2),\ y = \tan(t^2 + 2) \ \Rightarrow\ \dfrac{dx}{dt} = 1 + 2t\cos(t^2 + 2),\ \dfrac{dy}{dt} = 2t\sec^2(t^2 + 2)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t\sec^2(t^2 + 2)}{1 + 2t\cos(t^2 + 2)}$.

5. $x = t^2 + 2t,\ y = 2^t - 2t;\ (15, 2).\ \dfrac{dy}{dt} = 2^t \ln 2 - 2,\ \dfrac{dx}{dt} = 2t + 2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2^t \ln 2 - 2}{2t + 2}$.

At $(15, 2)$, $x = t^2 + 2t = 15 \ \Rightarrow\ t^2 + 2t - 15 = 0 \ \Rightarrow\ (t + 5)(t - 3) = 0 \ \Rightarrow\ t = -5$ or $t = 3$. Only $t = 3$ gives $y = 2$. With $t = 3$, $\dfrac{dy}{dx} = \dfrac{2^3 \ln 2 - 2}{2(3) + 2} = \dfrac{4\ln 2 - 1}{4} = \ln 2 - \dfrac{1}{4} \approx 0.44$.

6. $x = t + \cos \pi t,\ y = -t + \sin \pi t;\ (3, -2).\ \dfrac{dy}{dt} = -1 + \pi \cos \pi t,\ \dfrac{dx}{dt} = 1 - \pi \sin \pi t$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1 + \pi\cos \pi t}{1 - \pi \sin \pi t}$. When $x = 3$, we have $t + \cos \pi t = 3 \ \Rightarrow\ \cos^2 \pi t = (3 - t)^2$ **(1)**. When $y = -2$, we have $-t + \sin \pi t = -2 \ \Rightarrow\ \sin^2 \pi t = (t - 2)^2$ **(2)**. Adding **(1)** and **(2)** gives $\sin^2 \pi t + \cos^2 \pi t = (t - 2)^2 + (3 - t)^2 \ \Rightarrow$ $1 = t^2 - 4t + 4 + 9 - 6t + t^2 \ \Rightarrow\ 0 = 2t^2 - 10t + 12 \ \Rightarrow\ 0 = 2(t - 2)(t - 3) \ \Rightarrow\ t = 2$ or $t = 3$. Only $t = 2$ gives $y = -2$. With $t = 2$, $\dfrac{dy}{dx} = \dfrac{-1 + \pi\cos 2\pi}{1 - \pi \sin 2\pi} = \dfrac{-1 + \pi}{1 - 0} = \pi - 1 \approx 2.14$.

7. $x = t^3 + 1,\ y = t^4 + t;\ \ t = -1.\ \dfrac{dy}{dt} = 4t^3 + 1,\ \dfrac{dx}{dt} = 3t^2$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t^3 + 1}{3t^2}$. When $t = -1$, $(x, y) = (0, 0)$ and $dy/dx = -3/3 = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $y - 0 = -1(x - 0)$, or $y = -x$.

8. $x = \sqrt{t},\ y = t^2 - 2t;\ \ t = 4.\ \dfrac{dy}{dt} = 2t - 2,\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = (2t - 2)2\sqrt{t} = 4(t - 1)\sqrt{t}$. When $t = 4$, $(x, y) = (2, 8)$ and $dy/dx = 4(3)(2) = 24$, so an equation of the tangent to the curve at the point corresponding to $t = 4$ is $y - 8 = 24(x - 2)$, or $y = 24x - 40$.

9. $x = \sin 2t + \cos t,\ y = \cos 2t - \sin t;\ t = \pi.\ \dfrac{dy}{dt} = -2\sin 2t - \cos t,\ \dfrac{dx}{dt} = 2\cos 2t - \sin t$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin 2t - \cos t}{2\cos 2t - \sin t}$. When $t = \pi$, $(x, y) = (-1, 1)$, and $\dfrac{dy}{dx} = \dfrac{1}{2}$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $y - 1 = \frac{1}{2}[x - (-1)]$, or $y = \frac{1}{2}x + \frac{3}{2}$.

10. $x = e^t \sin \pi t,\ y = e^{2t};\quad t = 0.\ \dfrac{dy}{dt} = 2e^{2t},\ \dfrac{dx}{dt} = e^t(\pi \cos \pi t) + (\sin \pi t)e^t = e^t(\pi \cos \pi t + \sin \pi t)$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{e^t(\pi \cos \pi t + \sin \pi t)} = \dfrac{2e^t}{\pi \cos \pi t + \sin \pi t}$. When $t = 0$, $(x, y) = (0, 1)$ and $\dfrac{dy}{dx} = \dfrac{2}{\pi}$, so an equation of the tangent to the curve at the point corresponding to $t = 0$ is $y - 1 = \frac{2}{\pi}(x - 0)$, or $y = \frac{2}{\pi}x + 1$.

11. (a) $x = \sin t,\ y = \cos^2 t;\ \left(\frac{1}{2}, \frac{3}{4}\right).\ \dfrac{dy}{dt} = 2\cos t(-\sin t),\ \dfrac{dx}{dt} = \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin t \cos t}{\cos t} = -2\sin t$.

At $\left(\frac{1}{2}, \frac{3}{4}\right)$, $x = \sin t = \frac{1}{2} \ \Rightarrow\ t = \frac{\pi}{6}$, so $dy/dx = -2\sin\frac{\pi}{6} = -2\left(\frac{1}{2}\right) = -1$, and an equation of the tangent is $y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$, or $y = -x + \frac{5}{4}$.

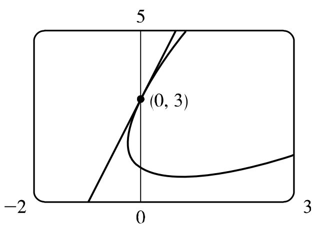
(b) $x = \sin t \ \Rightarrow\ x^2 = \sin^2 t = 1 - \cos^2 t = 1 - y$, so $y = 1 - x^2$, and $y' = -2x$. At $\left(\frac{1}{2}, \frac{3}{4}\right)$, $y' = -2 \cdot \frac{1}{2} = -1$, so an equation of the tangent is $y - \frac{3}{4} = -1\left(x - \frac{1}{2}\right)$, or $y = -x + \frac{5}{4}$.

12. (a) $x = \sqrt{t + 4},\ y = 1/(t + 4);\ \left(2, \frac{1}{4}\right).\ \dfrac{dy}{dt} = -\dfrac{1}{(t + 4)^2},\ \dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t + 4}}$, and

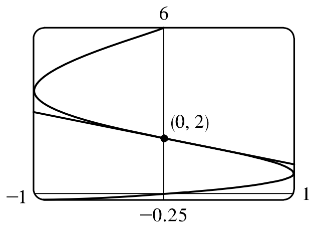
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-1/(t + 4)^2}{1/\left(2\sqrt{t + 4}\right)} = -2(t + 4)^{-3/2}$. At $\left(2, \frac{1}{4}\right)$, $x = \sqrt{t + 4} = 2 \ \Rightarrow\ t + 4 = 4 \ \Rightarrow\ t = 0$ and $dy/dx = -2(4)^{-3/2} = -\frac{1}{4}$, so an equation of the tangent is $y - \frac{1}{4} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{3}{4}$.

(b) $x = \sqrt{t + 4} \ \Rightarrow\ x^2 = t + 4 \ \Rightarrow\ t = x^2 - 4$, so $y = \dfrac{1}{t + 4} = \dfrac{1}{x^2 - 4 + 4} = \dfrac{1}{x^2}$, and $y' = -\dfrac{2}{x^3}$. At $\left(2, \frac{1}{4}\right)$, $y' = -2/2^3 = -1/4$, so an equation of the tangent is $y - \frac{1}{4} = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{3}{4}$.

13. $x = t^2 - t,\ y = t^2 + t + 1;\ (0, 3).\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 1}{2t - 1}$. To find the value of t corresponding to the point $(0, 3)$, solve $x = 0 \ \Rightarrow$ $t^2 - t = 0 \ \Rightarrow\ t(t - 1) = 0 \ \Rightarrow\ t = 0$ or $t = 1$. Only $t = 1$ gives $y = 3$. With $t = 1$, $dy/dx = 3$, and an equation of the tangent is $y - 3 = 3(x - 0)$, or $y = 3x + 3$.



14. $x = \sin \pi t,\ y = t^2 + t;\ (0, 2).\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 1}{\pi \cos \pi t}$. To find the value of t corresponding to the point $(0, 2)$, solve $y = 2 \ \Rightarrow$ $t^2 + t - 2 = 0 \ \Rightarrow\ (t + 2)(t - 1) = 0 \ \Rightarrow\ t = -2$ or $t = 1$. Either value gives $dy/dx = -3/\pi$, so an equation of the tangent is $y - 2 = -\frac{3}{\pi}(x - 0)$, or $y = -\frac{3}{\pi}x + 2$.



15. $x = t^2 + 1,\ y = t^2 + t \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + 1}{2t} = 1 + \dfrac{1}{2t} \ \Rightarrow\ \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-1/(2t^2)}{2t} = -\dfrac{1}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$.