第一类换元法（二）

1. 利用　$\mathrm{d}x=\frac{1}{a}\mathrm{d}(ax+b), a\neq 0$

2. 当被积函数中各因式之间具有求导关系

3. 利用三角函数的恒等式

平方关系： $\sin^2 x+\cos^2 x=1,$

$$1+\tan^2 x=\sec^2 x,$$

$$1+\cot^2 x=\csc^2 x;$$

积化和差：

$$\sin\alpha x\cos\beta x=\frac{1}{2}\left[\sin(\alpha+\beta)x+\sin(\alpha-\beta)x\right],$$

$$\cos\alpha x\sin\beta x=\frac{1}{2}\left[\sin(\alpha+\beta)x-\sin(\alpha-\beta)x\right]$$

$$\sin\alpha x\sin\beta x=-\frac{1}{2}\left[\cos(\alpha+\beta)x-\cos(\alpha-\beta)x\right],$$

$$\cos\alpha x\cos\beta x=\frac{1}{2}\left[\cos(\alpha+\beta)x+\cos(\alpha-\beta)x\right].$$

倍角公式： $2\sin\frac{x}{2}\cos\frac{x}{2}=\sin x$,

$$\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=1-2\sin^2\frac{x}{2}$$

$$=2\cos^2\frac{x}{2}-1=\cos x,$$

$$\sin^2 x=\frac{1-\cos 2x}{2},$$

$$\cos^2 x=\frac{1+\cos 2x}{2};$$

3.利用三角函数的恒等式

例1 求$\int \sin^2 xdx$.　　——偶次降幂

解 $\int \sin^2 xdx = \frac{1}{2}\int (1-\cos 2x)dx$

$$= \frac{x}{2} - \frac{1}{2}\int \cos 2xdx = \frac{x}{2} - \frac{1}{4}\int \cos 2xd(2x)$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

类似的，可求$\int \cos^2 xdx$.

例2 求 $\int \sin^2 x\cos^2 xdx$.

解: $\int \sin^2 x\cos^2 xdx = \frac{1}{4}\int \sin^2 2xdx$

$$= \frac{1}{8}\int (1-\cos 4x)dx$$

$$= \frac{1}{8}\int dx - \frac{1}{32}\int \cos 4xd(4x)$$

$$= \frac{x}{8} - \frac{1}{32}\sin 4x + C.$$

例3 求 $\int \sin^3 x\mathrm{d}x$. ——奇次凑微分

解 $\int \sin^3 x\mathrm{d}x = \int \sin^2 x \sin x\mathrm{d}x$

$$= -\int \sin^2 x\mathrm{d}\cos x$$

$$= -\int (1-\cos^2 x)\mathrm{d}\cos x$$

$$= \int (u^2 - 1)\mathrm{d}u$$

$$= \frac{1}{3}\cos^3 x - \cos x + C.$$

例4 求$\int \csc x dx$.

解 $$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = -\int \frac{1}{1-\cos^2 x} d(\cos x)$$

$$= \int \frac{1}{(\cos x - 1)(\cos x + 1)} d(\cos x)$$

$$= \frac{1}{2} \int \left(\frac{1}{\cos x - 1} - \frac{1}{\cos x + 1} \right) d(\cos x)$$

$$= \ln \left| \frac{\cos x - 1}{\cos x + 1} \right|^{\frac{1}{2}} + C$$

$$= \ln \left| \frac{\cos x - 1}{\sin x} \right| + C = \ln \left| \csc x - \cot x \right| + C$$

求 $\int \csc x dx$.

解（二）$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$$= \frac{1}{2}\int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} dx \quad = \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{\tan\frac{x}{2}} dx$$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$$

$= \ln\left|\tan\frac{x}{2}\right| + C$ （使用了三角函数恒等变形）

例5 求 $\int \sin^3 x\cos^2 x dx$.

解: $\int \sin^2 x\cos^2 x\cdot\sin x dx$

$$=-\int (1-\cos^2 x)\cos^2 x\cdot d(\cos x)$$

$$=\int (\cos^4 x-\cos^2 x)d(\cos x)$$

$$=\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C.$$

说明 当被积函数是同角三角函数相乘时，拆开奇次项去凑微分.

总结：1. 形如 $\int \sin^m x \cos^n x dx.$

(1) m，n中有一个为奇数

$$\int \sin^m x \cos^n x dx \xrightarrow{m\text{为正奇数}} \int f(\cos x) d\cos x,$$

$$\int \sin^m x \cos^n x dx \xrightarrow{n\text{为正奇数}} \int f(\sin x) d\sin x.$$

(2) m, n均为正偶数 ——降幂

由 $\cos^2 x = \dfrac{1+\cos 2x}{2}, \sin^2 x = \dfrac{1-\cos 2x}{2}$

例6 求 $\int \sin^2 x \cdot \cos^5 x dx$.

解 $\int \sin^2 x \cdot \cos^5 x dx = \int \sin^2 x \cdot \cos^4 x d(\sin x)$

$= \int \sin^2 x \cdot (1-\sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

2. 形如 $\int \sin mx \cos nx dx, \int \sin mx \sin nx dx,$

$\int \cos mx \cos nx dx.$ ——积化和差.

例7 求 $\int \cos 3x \cos 2x dx.$

解 $\int \cos 3x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 5x) dx$

$$= \frac{1}{2}\sin x + \frac{1}{10}\int \cos 5x d(5x)$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

3. 形如 $I=\int \tan^m x\sec^n x dx$. ($m,n$为正整数)

(1) m为奇数时

$$I=\int \tan^{m-1} x\sec^{n-1} x d\sec x=\int f(\sec x)d\sec x.$$

(2) n为偶数时

$$I=\int \tan^{m} x\sec^{n-2} x d\tan x=\int f(\tan x)d\tan x.$$

例8 求 $\int \tan^5 x \sec^3 x dx$

解 $\int \tan^5 x \sec^3 x dx = \int \tan^4 x \sec^2 x d\sec x$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^2 x d\sec x$$

$$= \int \left(\sec^4 x - 2\sec^2 x + 1\right) \sec^2 x d\sec x$$

$$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x) d\sec x$$

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.$$

例9　求 $\int \sec^6 x dx$

解 $\int \sec^6 x dx = \int \sec^4 x d\tan x$

$$= \int \left(1+\tan^2 x\right)^2 d\tan x$$

$$= \int \left(1+2\tan^2 x+\tan^4 x\right) d\tan x$$

$$= \tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x+C.$$

4. 其他类型的积分

例10 求 $\int \frac{1}{1+e^x}dx.$

解 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

$$\int \frac{du}{1+u^2} = \arctan u + C$$

例11 求 $\int \frac{1}{a^2+x^2}dx.$

解 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a} + C.$$

例12 求 $\int \frac{1}{x^2-8x+25}dx.$

解 $\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

$$\int \frac{\mathrm{d}\,u}{\sqrt{1-u^2}} = \arcsin u + C$$

例13 $\int \frac{1}{\sqrt{a^2-x^2}}dx \quad (a>0)$

解 $$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}dx$$

$$= \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}d\left(\frac{x}{a}\right) = \arcsin\frac{x}{a} + C$$

例14 $\int \frac{1}{a^2-x^2}dx$

解 $\int \frac{1}{a^2-x^2}dx = \int \frac{1}{(a+x)(a-x)}dx$

$$= \frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx$$

$$= \frac{1}{2a}(\ln|a+x|-\ln|a-x|)+C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$$

例15 求$\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-1}}dx.$

原式$=\int \frac{\sqrt{2x+3}-\sqrt{2x-1}}{\left(\sqrt{2x+3}+\sqrt{2x-1}\right)\left(\sqrt{2x+3}-\sqrt{2x-1}\right)}dx$

$$=\frac{1}{4}\int \sqrt{2x+3}dx-\frac{1}{4}\int \sqrt{2x-1}dx$$

$$=\frac{1}{8}\int \sqrt{2x+3}d(2x+3)-\frac{1}{8}\int \sqrt{2x-1}d(2x-1)$$

$$=\frac{1}{12}\left(\sqrt{2x+3}\right)^3-\frac{1}{12}\left(\sqrt{2x-1}\right)^3+C.$$