第三节 定积分的计算

一、定积分的换元法

二、定积分的分部积分法

一、定积分的换元法积分法

定理. 设(1)函数 $f(x)$在$[a, b]$上连续;

(2) $x=\psi(t)$在区间$[\alpha, \beta]$上有一个连续导数;

(3)$a = \psi(\alpha)$, $b = \psi(\beta)$, $\psi(t) \in [a, b]$,

则 $\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(\psi(t))\psi'(t)\mathrm{d}t$.

——定积分的换元法

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(\psi(t))\psi'(t)\mathrm{d}t$$

说明：

(1) 当$\beta < \alpha$，换元公式仍成立.

(2)换元必换限，原函数中的变量不必代回.

例1 求 $\int_0^3 \frac{x}{\sqrt{1+x}}dx$

解 令 $\sqrt{1+x}=t$，则 $x=t^2-1$ ，$dx=2tdt$

当 $x=0$ 时，$t=1$ ，当 $x=3$ 时，$t=2$

于是 $\int_0^3 \frac{x}{\sqrt{1+x}}dx=\int_1^2 \frac{t^2-1}{t}\cdot 2tdt$

$$=2\int_1^2 (t^2-1)dt$$

$$=2[\frac{1}{3}t^3-t]_1^2=\frac{8}{3}$$

例2 计算 $\int_0^4 \frac{x+2}{\sqrt{2x+1}}dx$.

解 令 $t=\sqrt{2x+1}$, 则 $x=\frac{t^2-1}{2}$, $dx=t\,dt$,

且当 $x=0$ 时, $t=1$; $x=4$ 时, $t=3$.

$$原式=\int_1^3 \frac{\frac{t^2-1}{2}+2}{t}t\,dt=\frac{1}{2}\int_1^3 (t^2+3)\,dt$$

$$=\frac{1}{2}\left(\frac{1}{3}t^3+3t\right)\Big|_1^3=\frac{22}{3}.$$

例3 求 $\int_0^{\ln 2}\sqrt{e^x-1}dx$

解 令 $\sqrt{e^x-1}=t$， $x=\ln(1+t^2),dx=\frac{2t}{1+t^2}dt$

当 $x=0$ 时，$t=0$ ； 当 $x=\ln 2$ 时，$t=1$

于是 $\int_0^{\ln 2}\sqrt{e^x-1}dx=\int_0^1 t\cdot\frac{2t}{1+t^2}dt$

$$=2\int_0^1\frac{t^2}{1+t^2}dt=2\int_0^1(1-\frac{1}{1+t^2})dt$$

$$=2[t-\arctan t]_0^1=2-\frac{\pi}{2}$$

例4 计算 $\int_0^a \sqrt{a^2 - x^2}\, dx \ (a > 0)$.

解 令 $x = a\sin t$, 则 $dx = a\cos t\, dt$,

当 $x = 0$ 时, $t = 0$; $x = a$时, $t = \dfrac{\pi}{2}$.

$$\text{原式} = a^2 \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}.$$

例5 $\int_1^{e^2} \frac{1}{x(1+\ln x)}dx$

$$\int_1^{e^2} \frac{1}{x(1+\ln x)}dx = \int_1^{e^2} \frac{1}{(1+\ln x)}d(\ln x) = \ln|1+\ln x|\Big|_1^{e^2}$$

令 $\ln x = t$ ，$x = 1$时，$t = 0$; $x = e^2$时，$t = 2$;

$$\int_1^{e^2} \frac{1}{(1+\ln x)}d(\ln x) = \int_0^2 \frac{1}{(1+t)}dt$$

$$= \ln|1+t|\Big|_0^2 = \ln 3 - \ln 1 = \ln 3$$

例6 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx.$

解 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx = -\int_0^{\frac{\pi}{2}} \cos^5 x d\cos x$

$$= -\frac{1}{6}\cos^6 x\Big|_0^{\frac{\pi}{2}} = \frac{1}{6}.$$

注：(3) 不换元就不换限 .

练习： $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx. = \int_0^{\frac{\pi}{2}} \sin^2 x d(\sin x) = \frac{1}{3}\sin^3 x\Big|_0^{\frac{\pi}{2}}$

$$= \frac{1}{3}.$$

例7 计算 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}dx$

解 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}dx=\int_0^{\pi}\sin^{\frac{3}{2}}x|\cos x|dx$

$$=\int_0^{\frac{\pi}{2}}\sin^{\frac{3}{2}}x\cos xdx-\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac{3}{2}}x\cos xdx$$

$$=\int_0^{\frac{\pi}{2}}\sin^{\frac{3}{2}}xd\sin x-\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac{3}{2}}xd\sin x$$

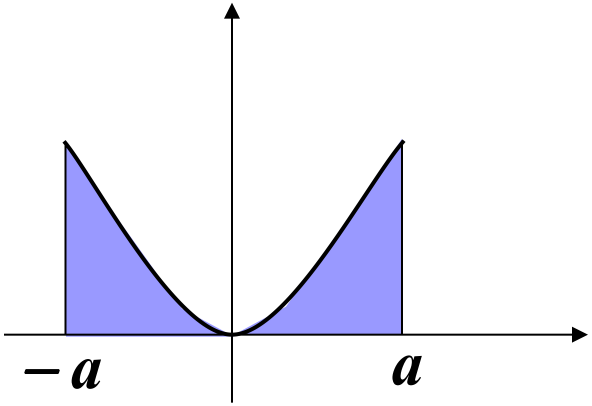
$$=[\frac{2}{5}\sin^{\frac{5}{2}}x]_0^{\frac{\pi}{2}}-[\frac{2}{5}\sin^{\frac{5}{2}}x]_{\frac{\pi}{2}}^{\pi}$$

$$=\frac{2}{5}-(-\frac{2}{5})=\frac{4}{5}$$

例8. 设$f(x)\in R([-a,a])$，证明

(1) 若$f(x)$为偶函数，则

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$$



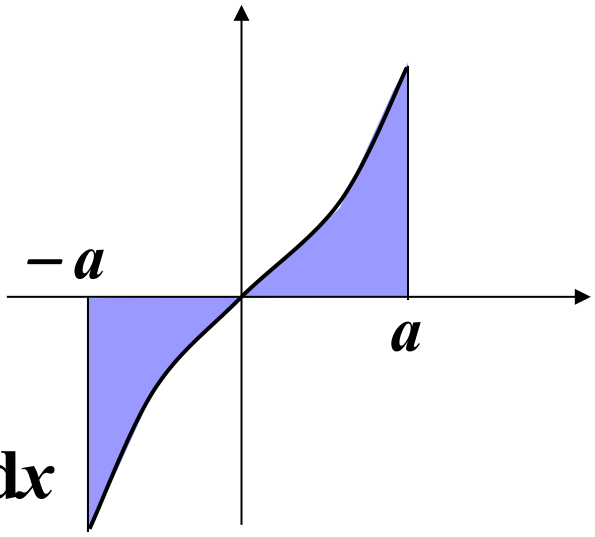
证: (1) $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{0} f(x)\mathrm{d}x + \int_{0}^{a} f(x)\mathrm{d}x$

$$\overset{\text{在第一个积分中}}{\underset{\text{令}x=-t}{=\!=\!=\!=\!=}} \int_{a}^{0} f(-t)\mathrm{d}(-t) + \int_{0}^{a} f(x)\mathrm{d}x$$

$$= \int_{0}^{a} f(t)\mathrm{d}t + \int_{0}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x;$$

(2) 若$f(x)$为奇函数, 则

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0.$$



分析: $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{0} f(x)\mathrm{d}x + \int_{0}^{a} f(x)\mathrm{d}x$

$$\xlongequal[\text{令}x=-t]{\text{在第一个积分中}} \int_{a}^{0} f(-t)\mathrm{d}(-t) + \int_{0}^{a} f(x)\mathrm{d}x$$

$$= -\int_{0}^{a} f(t)\mathrm{d}t + \int_{0}^{a} f(x)\mathrm{d}x = 0.$$

例9. 设 $f(x) \in C([0,1])$, 则

(1) $\boxed{\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x.}$

证: 令 $x = \dfrac{\pi}{2} - t$, 则

$$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} f(\cos t)\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$$

特别地, $\boxed{\int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\mathrm{d}x.}$

(2) $\boxed{\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx,}$ 设 $x = \pi - t$,则

$$\int_0^{\pi} xf(\sin x)dx = -\int_{\pi}^{0}(\pi - t)f\big(\sin(\pi - t)\big)dt$$

$$= \int_0^{\pi}(\pi - t)f(\sin t)dt,$$

$$= \pi\int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt$$

故 $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$

计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx.$

解 $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x} d(\cos x)$$

$$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

例10 设函数$f(x)=\begin{cases} xe^{-x^2} & x\ge 0 \\ \dfrac{1}{1+\cos x} & -1<x<0 \end{cases}$，计算 $\int_1^4 f(x-2)dx$

解 设 $x-2=t$ ，则$dx=dt$;

当$x=1$ 时，$t=-1$ ；当$x=4$ 时$t=2$

$$\int_1^4 f(x-2)dx=\int_{-1}^2 f(t)dt=\int_{-1}^0 \frac{1}{1+\cos t}dt+\int_0^2 te^{-t^2}dt$$

$$=[\tan\frac{t}{2}]_{-1}^0-[\frac{1}{2}e^{-t^2}]_0^2=\tan\frac{1}{2}-\frac{1}{2}e^{-4}+\frac{1}{2}$$

练习1 $\int_{\frac{3}{4}}^{1} \frac{dx}{\sqrt{1-x}-1}$.

解 $\sqrt{1-x}=t, x=1-t^2, dx=-2tdt.$

$$x=\frac{3}{4}, t=\frac{1}{2}; x=1, t=0.$$

$$\int_{\frac{3}{4}}^{1} \frac{dx}{\sqrt{1-x}-1}=\int_{\frac{1}{2}}^{0} \frac{-2tdt}{t-1}=-2\int_{\frac{1}{2}}^{0} \frac{t-1+1}{t-1}dt$$

$$=-2\int_{\frac{1}{2}}^{0} (1+\frac{1}{t-1})dt$$

$$=(-2t-2\ln|t-1|)\Big|_{\frac{1}{2}}^{0} =1-2\ln 2.$$

练习3 (1)$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\arcsin x)^2}{\sqrt{1-x^2}} dx.$ (2)$\int_{-5}^{5} \frac{x^2 \sin x^3}{x^4+2x^2+1} dx.$

偶函数　　　奇函数

（1）原式 $= 2\int_0^{\frac{1}{2}} \frac{(\arcsin x)^2}{\sqrt{1-x^2}} dx$

$$= 2\int_0^{\frac{1}{2}} (\arcsin x)^2 d(\arcsin x)$$

$$= \frac{2}{3}(\arcsin x)^3 \Big|_0^{\frac{1}{2}} = \frac{2}{3}\left(\frac{\pi}{6}\right)^3 = \frac{\pi^3}{324}.$$

练习4 $\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}dx.$

解 原式$=\int_{-1}^{1}\underbrace{\boxed{\frac{2x^2}{1+\sqrt{1-x^2}}}}_{\text{偶函数}}dx+\int_{-1}^{1}\underbrace{\boxed{\frac{x\cos x}{1+\sqrt{1-x^2}}}}_{\text{奇函数}}dx$

$$=4\int_0^1\frac{x^2}{1+\sqrt{1-x^2}}dx=4\int_0^1\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}dx$$

$$=4\int_0^1(1-\sqrt{1-x^2})dx=4-\underbrace{\boxed{4\int_0^1\sqrt{1-x^2}dx}}_{\text{单位圆的面积}}$$

$$=4-\pi.$$

内容小结：

1. 定积分的换元法

$$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

2. 几个特殊积分、定积分的几个等式