第四节 多元复合函数的求导法则

一、链式法则

二、全微分形式不变性

三、小结 思考题

复合函数:

$$y = f(x), x = g(t) \Rightarrow y = f(g(t)).$$

复合函数求导公式:

y —— x —— t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

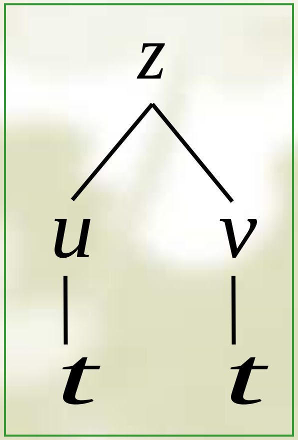
链式法则

一、链式法则

1.【中间变量均为一元函数】

【定理 1】若 $u=\varphi(t)$ 及 $v=\psi(t)$ 都在点 t 可导，$z=f(u,v)$在对应点(u,v)具有连续偏导数，则复合函数$z=f[\varphi(t),\psi(t)]$在对应点t可导，且其导数可用下列公式计算：

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}.$$



证明 设 t 取增量 Δt，则相应中间变量有增量 $\Delta u,\Delta v,$

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\qquad(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2})$$

上式同除以 Δt，得

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t},\ \Rightarrow\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

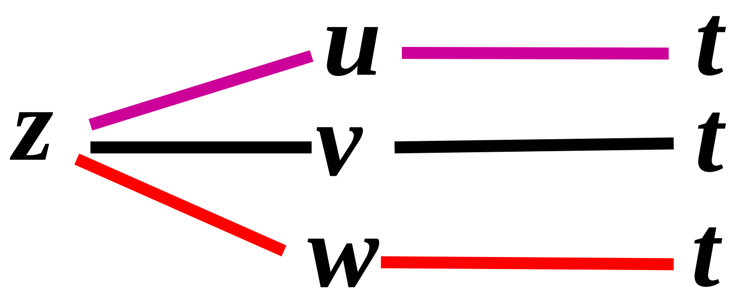
(全导数公式)

令 $\Delta t\to 0,$

$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\to 0$$

上定理的结论可推广到中间变量多于两个的情况.

如



$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$

以上公式中的导数 $\frac{dz}{dt}$ 称为**全导数.**

【例1】 设 $z = e^{x-2y}, x = \sin t, y = t^3$, 求 $\frac{dz}{dt}$.

【解】

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= e^{x-2y} \cdot \cos t + e^{x-2y} \cdot (-2) \cdot 3t^2$$

$$= e^{\sin t - t^3}(\cos t - 6t^2)$$

【例2】 设 $u = f(R\cos t, R\sin t, vt)$,求 $\dfrac{du}{dt}$.

【提示】由于含抽象函数, 一般要先设中间变量.

【解】 令 $x = R\cos t,\quad y = R\sin t,\quad z = vt,$

则 $u = f(x, y, z) \quad x = R\cos t, y = R\sin t, z = vt,$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= f_1'\cdot(-R\sin t) + f_2'\cdot R\cos t + f_3'\cdot v$$

(记 $f_1' = \dfrac{\partial f(x, y, z)}{\partial x}, f_2' = \dfrac{\partial f(x, y, z)}{\partial y}, \ f_3' = \dfrac{\partial f(x, y, z)}{\partial z}$)

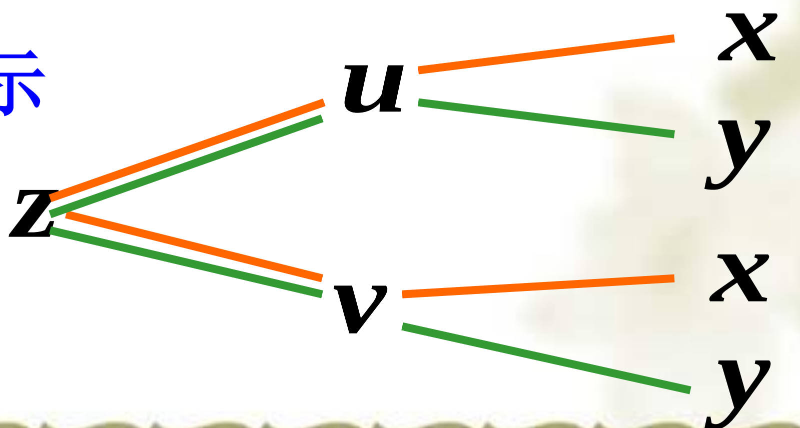
2.【中间变量均为多元函数】

上定理还可推广到中间变量不是一元函数而是多元函数的情况：$z = f[\varphi(x,y),\psi(x,y)]$.

【定理 2】若$u=\varphi(x,y)$及$v=\psi(x,y)$都在点(x,y)具有对x和y的偏导数，且$z=f(u,v)$在对应点(u,v)具有连续偏导数，则复合函数$z=f[\varphi(x,y),\psi(x,y)]$在对应点$(x,y)$的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

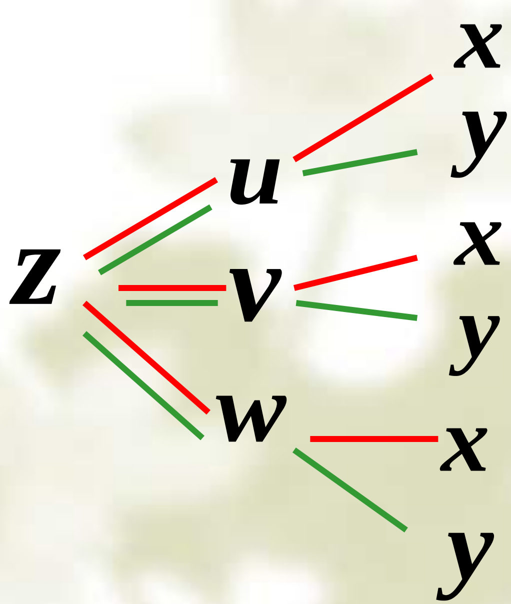
链式图如右所示



类似地再推广，设$u=\varphi(x,y)$、$v=\psi(x,y)$、$w=w(x,y)$都在点(x,y)具有对 x和 y的偏导数，函数$z=f(u,v,w)$在对应点(u,v,w)处具有连续偏导数，则复合函数$z=f[\varphi(x,y),\psi(x,y),w(x,y)]$在对应点$(x,y)$的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}.$$



注意：$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$

【教材例 3】 设 $z = e^u \sin v$，而 $u = xy$，$v = x + y$，

求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

【解】 $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial x} = e^u \sin v \cdot y + e^u \cos v \cdot 1$

$$= e^u (y \sin v + \cos v) = e^{xy}[y \sin(x+y) + \cos(x+y)],$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^u (x \sin v + \cos v) = e^{xy}[x \sin(x+y) + \cos(x+y)].$$

【练习】 设$z=\arctan\dfrac{v}{u}, u=x-y, v=x+y$,求$\dfrac{\partial z}{\partial y}$.

【解】
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

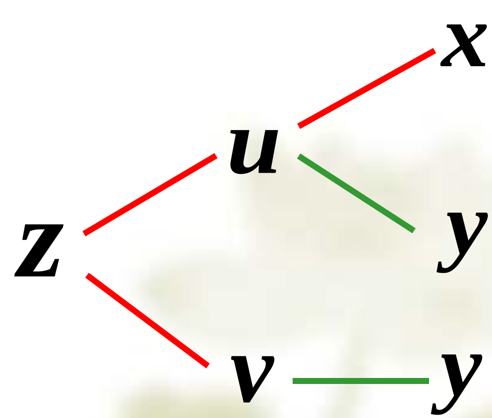
$$=\frac{1}{1+\dfrac{v^2}{u^2}}\cdot\left(-\frac{v}{u^2}\right)\cdot(-1)+\frac{1}{1+\dfrac{v^2}{u^2}}\cdot\frac{1}{u}$$

$$=\frac{u+v}{u^2+v^2}=\frac{x}{x^2+y^2}$$

3.【中间变量既有一元又有多元函数的情形】

【定理3】$z=f(u,v), u=\varphi(x,y), v=\psi(y),$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$$



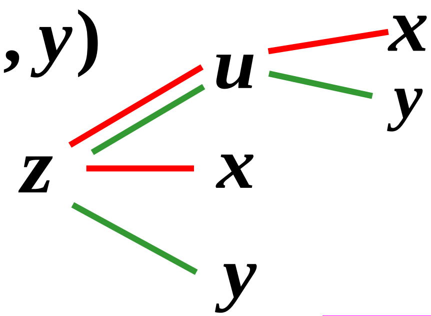
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{dv}{dy}$$

$$=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{dv}{dy}$$

口诀：分段用乘，分叉用加，单路全导，叉路偏导.

特殊的， $z=f(u,x,y)$ 其中 $u=\varphi(x,y)$

即 $z=f[\varphi(x,y),x,y]$，变量关系为：



$$\boxed{\frac{\partial z}{\partial x}}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\boxed{\frac{\partial f}{\partial x}},\qquad \boxed{\frac{\partial z}{\partial y}}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\boxed{\frac{\partial f}{\partial y}}.$$

区别类似

两者的区别

把复合函数 $z=f[\varphi(x,y),x,y]$ 中的 y 看作不变而对 x 的偏导数

把 $z=f(u,x,y)$ 中的 u 及 y 看作不变而对 x 的偏导数

【教材例 4】 设$z = uv + \sin t$，而$u = e^t$，$v = \cos t$，

求全导数 $\dfrac{dz}{dt}$.

【解】

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial t}$$

$$= ve^t - u\sin t + \cos t$$

$$= e^t \cos t - e^t \sin t + \cos t$$

$$= e^t (\cos t - \sin t) + \cos t.$$

【练习题】1. 设$z=f(x,u)$,$u=\Phi(x,y)$,求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

【解】 $\dfrac{\partial z}{\partial x}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x}$ $\qquad \dfrac{\partial z}{\partial y}=\dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial y}$

2. 设$z=f(xy,\dfrac{x}{y})+g(\dfrac{y}{x})$，$f$、$g$ 均可微，求 $\dfrac{\partial z}{\partial x}$.

令 $u=xy, v=\dfrac{x}{y}$ $\qquad \dfrac{\partial z}{\partial x}=\dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial x}+g'(\dfrac{y}{x})\cdot(-\dfrac{y}{x^2})$

$$=f_1'\cdot y+f_2'\cdot\frac{1}{y}+g'(\frac{y}{x})\cdot(-\frac{y}{x^2})$$

【教材例 5】设$w=f(x+y+z,xyz)$，f 具有二阶连续偏导数，求$\dfrac{\partial w}{\partial x}$和$\dfrac{\partial^2 w}{\partial x\partial z}$.

【分析】求抽象函数的偏导数，一般要先设中间变量.

【解】 令 $u=x+y+z,\quad v=xyz;$

记 $f_1'=\dfrac{\partial f(u,v)}{\partial u},\qquad f_{12}''=\dfrac{\partial^2 f(u,v)}{\partial u\partial v},$

同理有 $f_2',\quad f_{11}'',\quad f_{22}''.$

$$\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'+yz\,f_2';$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}(f_1' + yz f_2') = \frac{\partial f_1'}{\partial z} + y f_2' + yz\frac{\partial f_2'}{\partial z};$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{11}'' + xy f_{12}'';$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{21}'' + xy f_{22}'';$$

于是 $$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + xy f_{12}'' + y f_2' + yz(f_{21}'' + xy f_{22}'')$$

【练习题】设$z=f(u,x,y),\ u=xe^y$，f 二阶偏导连续，

求$\dfrac{\partial z}{\partial y},\dfrac{\partial^2 z}{\partial y\partial x}$.

【解】 $$\frac{\partial z}{\partial y}=f_1'\cdot\frac{\partial u}{\partial y}+f_3'=xe^y f_1'+f_3'$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=e^y f_1'+xe^y[f_{11}''\cdot e^y+f_{12}''\cdot 1]$$

$$+[f_{31}''\cdot e^y+f_{32}''\cdot 1]$$

$$=e^y f_1'+xe^{2y}f_{11}''+xe^y f_{12}''+e^y f_{31}''+f_{32}''$$

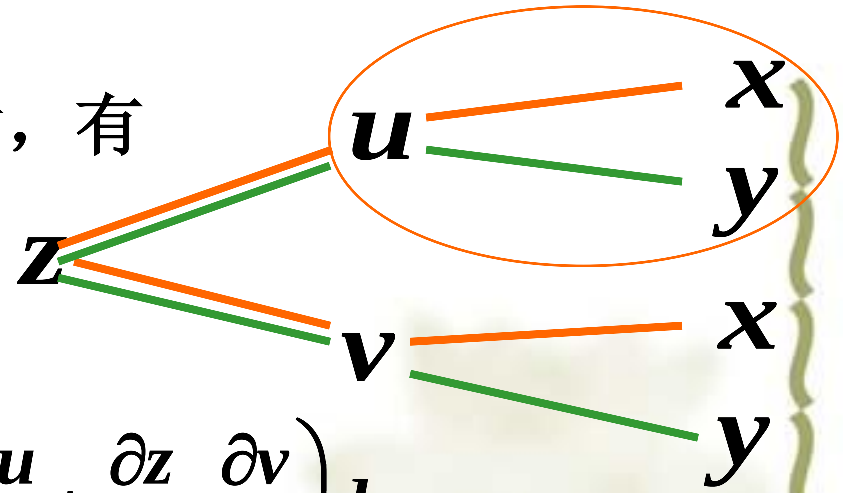
二、全微分形式不变性

设函数$z = f(u,v)$具有连续偏导数，则有全微分

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv;$$

(u、v 自变量情形)

当$u = \varphi(x,y)$、$v = \psi(x,y)$时，有



$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

(u、v 中间变量情形)

1.【全微分形式不变性的实质】

无论 z 是自变量 u、v 的函数或中间变量 u、v 的函数，它的全微分形式是一样的.

2.【全微分形式不变性的简单应用】

(1)求函数的全微分会更简便些.

(2)利提供了隐函数求导的方法.

【例 5】已知$e^{-xy}-2z+e^{z}=0$，求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

【注意】此为隐函数的偏导数的计算(§5)

【解】 $\because d(e^{-xy}-2z+e^{z})=0, \Rightarrow de^{-xy}-2dz+de^{z}=0,$

$$\therefore e^{-xy}d(-xy)-2dz+e^{z}dz=0,$$

$$-e^{-xy}(xdy+ydx)-2dz+e^{z}dz=0$$

$$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$$

$$dz=\frac{ye^{-xy}}{(e^{z}-2)}dx+\frac{xe^{-xy}}{(e^{z}-2)}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

不管 x, y, z 各变量之间的关系，最后化简求即可.

三、小结

1、链式法则（分三种情况）

（特别要注意课中所讲的特殊情况）

2、复合函数偏导数存在的充分条件

（外层函数偏导连续、内层函数偏导存在）

3、全微分形式不变性

（理解其实质）

【思考题】

设$z=f(u,v,x)$，而$u=\varphi(x)$，$v=\psi(x)$，

则$\frac{dz}{dx}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}+\frac{\partial f}{\partial x}$，试问$\frac{dz}{dx}$与$\frac{\partial f}{\partial x}$是否相同？为什么？

【思考题解答】 不相同.

等式左端的z是作为一个自变量x的函数，

而等式右端最后一项f是作为u,v,x的三元函数，

写出来为

$$\left.\frac{dz}{dx}\right|_x=\left.\frac{\partial f}{\partial u}\right|_{(u,v,w)}\cdot\left.\frac{du}{dx}\right|_x+\left.\frac{\partial f}{\partial v}\right|_{(u,v,x)}\cdot\left.\frac{dv}{dx}\right|_x+\left.\frac{\partial f}{\partial x}\right|_{(u,v,x)}$$

【补充练习题】

1.设$z = xyf(\frac{y}{x})$, $f(u)$可导，求$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

2.设$u = f(x, xy, xyz)$,求$\frac{\partial u}{\partial x}$.

3.设$z = f(2x - y) + g(x, xy)$,其中f二阶可导，g二阶偏导数连续，求$\frac{\partial^2 z}{\partial x \partial y}$.

1.设$z = xyf\left(\frac{y}{x}\right), f(u)$可导，求$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$.

解 $\because \frac{\partial z}{\partial x} = yf\left(\frac{y}{x}\right) + xy \cdot f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)$

$$\frac{\partial z}{\partial y} = xf\left(\frac{y}{x}\right) + xy \cdot f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2xyf\left(\frac{y}{x}\right) = 2z$$

2.设$u = f(x, xy, xyz)$,求$\frac{\partial u}{\partial x}$.

解 $\frac{\partial u}{\partial x} = f_1' \cdot 1 + f_2' \cdot y + f_3' \cdot yz$

3.设$z = f(2x-y)+g(x,xy)$,其中f二阶可导，

g二阶偏导数连续，求$\dfrac{\partial^2 z}{\partial x\partial y}$.

解 $\because \dfrac{\partial z}{\partial x} = f'(2x-y)\cdot 2 + g_1'\cdot 1 + g_2'\cdot y$

$$\therefore \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial y}[2f'(2x-y)+g_1'(x,xy)+yg_2'(x,xy)]$$

$$= -2f'' + xg_{12}'' + g_{22}''\cdot xy + g_2'$$

【练习】设$z = f(\sin x, \cos y, e^{x+y})$,求$\dfrac{\partial z}{\partial x}, \dfrac{\partial^2 z}{\partial x \partial y}$.

【解】 $$\frac{\partial z}{\partial x} = f_1' \cdot \cos x + f_3' \cdot e^{x+y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \cos x[f_{12}'' \cdot (-\sin y) + f_{13}'' \cdot e^{x+y}]$$

$$+[f_{32}'' \cdot (-\sin y) + f_{33}'' \cdot e^{x+y}]\, e^{x+y} + f_3' \cdot e^{x+y}$$