《数据结构与算法》课程组
重庆大学计算机学院

Data Structures & Algorithms

15

ELEMENTARY GRAPH ALGORITHMS

outlines

15.1 Basic notions of graphs

15.2 Standard graph-traversal algorithms

15.3 Topological sorting

15.1 Basic notions of graphs

Definitions

- *Graph* $G = (V, E)$
  - V = set of vertices
  - E = set of edges $\subseteq (V \times V)$
- $|E| = O(|V|^2)$

Types of Graphs

- **Undirected:** edge $(u, v) = (v, u)$;
  for all v, $(v, v) \notin E$ (No self-loop)
- **Directed:** (u, v) is edge from u to v, denoted as $u \to v$. Self loops are allowed.
- **Weighted:** each edge has an associated weight, given by a weight function

$$w : E \to \mathbf{R}.$$

Definitions –continue-

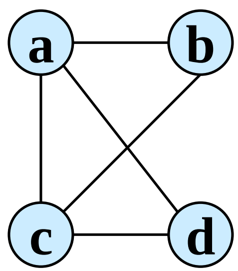
- If $(u, v) \in E$, then vertex v is adjacent to vertex u.

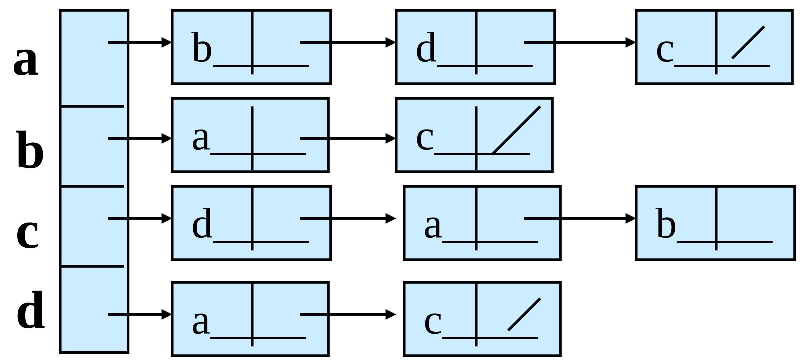
- Adjacency relationship is:
  - Symmetric if G is undirected.
  - Not necessarily so if G is directed.

- If G is connected:
  - There is a path between every pair of vertices.
  - $|E| \geq |V| - 1$.
  - Furthermore, if $|E| = |V| - 1$, then G is a tree.

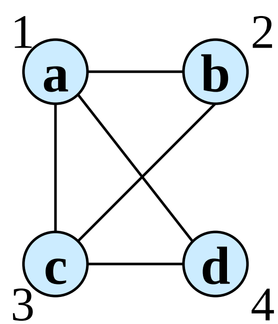
Representation of Graphs

- Two standard ways.
  - Adjacency Lists.





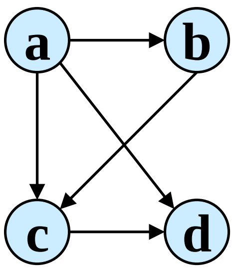
  - Adjacency Matrix.

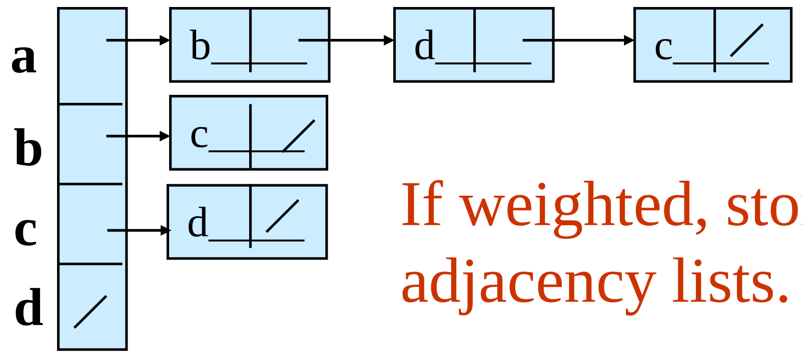


	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

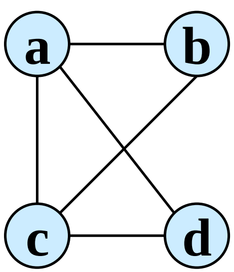
Adjacency Lists

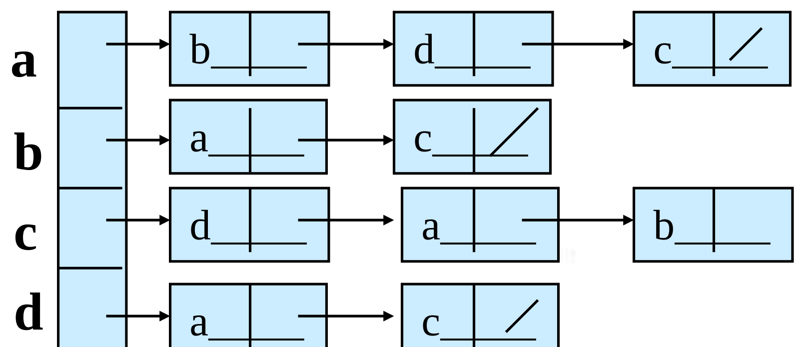
- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ = {all vertices adjacent to u}.





If weighted, store weights also in adjacency lists.





Storage Requirement

- **For directed graphs:**
  - **Sum of lengths of all adj. lists is**

    $$\sum_{v \in V} \text{out-degree}(v) = |E|$$

    Number of edges leaving v

  - **Total storage:** $\Theta(|V|+|E|)$

- **For undirected graphs:**
  - **Sum of lengths of all adj. lists is**

    $$\sum_{v \in V} \text{degree}(v) = 2|E|$$

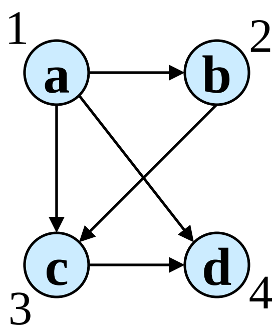
    Number of edges incident on v.

  - **Total storage:** $\Theta(|V|+|E|)$

Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$
- A is then given by:

$$A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$



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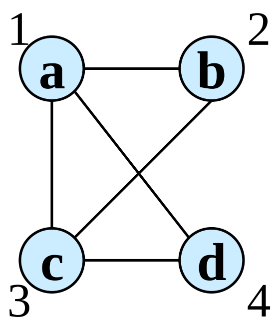
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
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4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs.

Space and Time

- **Space:** $\Theta(|V|^2)$.
  - Not memory efficient for large graphs.
- **Time:** **to list all vertices adjacent to u: $\Theta(|V|)$.**
- **Time:** **to determine if $(u, v) \in E$: $\Theta(1)$.**
- **Can store weights for weighted graph.**

15.2 Graph-Traversal Algorithms

Standard Algorithms

- **Searching a graph:**
  - Systematically follow the edges of a graph to visit the vertices of the graph
- **discovering the structure of a graph.**
- **Standard graph-searching algorithms.**
  - **Breadth-first Search (BFS).**
  - **Depth-first Search (DFS).**

Breadth-First Search

- **Input:**
  - Graph $G = (V, E)$, either directed or undirected,
  - source vertex $s \in V$.

- **Output:** **for all $v \in V$**

  - **$d[v]$ = length of shortest path from s to v**
    ($d[v] = \infty$ if v is not reachable from s).
  - **$\pi[v] = u$ if (u, v) is last edge on shortest path $s \rightsquigarrow v$.**
    - **u is v's predecessor.**
  - **breadth-first tree = a tree with root s that contains all reachable vertices.**

Definitions on BSF

- **Path between vertices u and v:**

  vertices $(v_1, v_2, \ldots, v_k)$ such that

  $u = v_1$ and $v = v_k$,

  $(v_i, v_{i+1}) \in E$, for all $1 \le i \le k-1$.

- **Length of the path: Number of edges in the path.**

- **Path is simple if no vertex is repeated.**

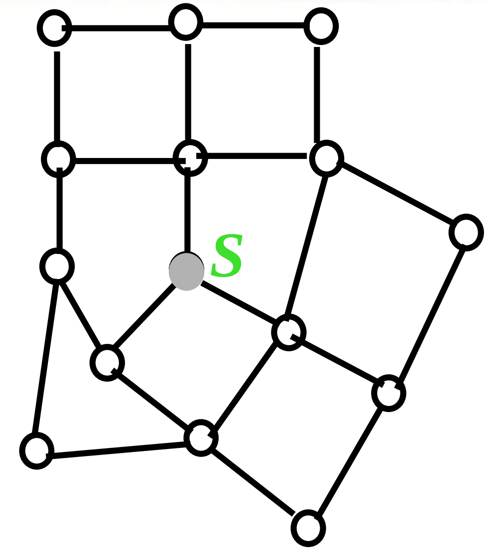
Principle of Breadth-First Search

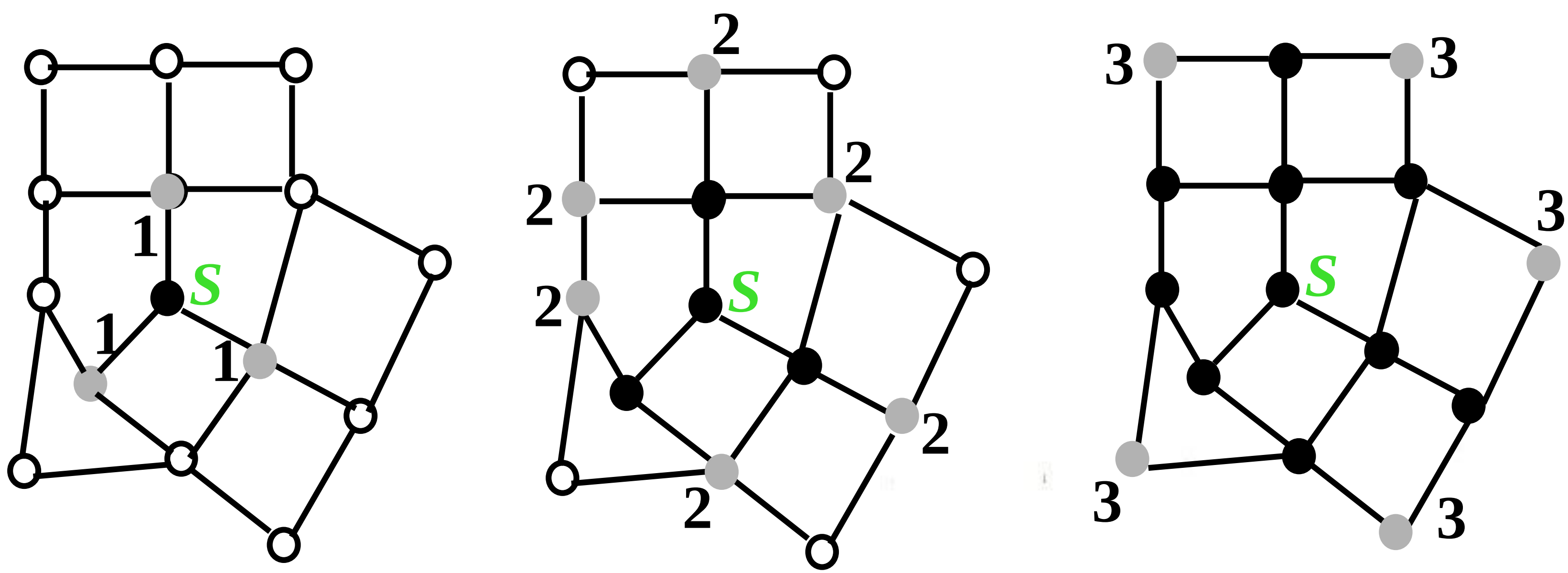
- Expands the frontier between discovered and undiscovered vertices **uniformly** across the **breadth** of the frontier.
  - A vertex is "**discovered**" the first time it is encountered during the search.
  - A vertex is "**finished**" if all vertices adjacent to it have been discovered.

BFS for Shortest Paths

<u>Colors the vertices to keep track of progress</u>.

- **Undiscovered**
- **Discovered**
- **Finished**





BFS(G,s)

```
for each vertex u in V[G] – {s}
        do color[u] ← white
           d[u] ← ∝
           π[u] ← nil
color[s] ← gray
d[s] ← 0
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
        do u ← dequeue(Q)
                for each v in Adj[u]
                        do if color[v] = white
                                then color[v] ← gray
                                     d[v] ← d[u] + 1
                                     π[v] ← u
                                     enqueue(Q,v)
                color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

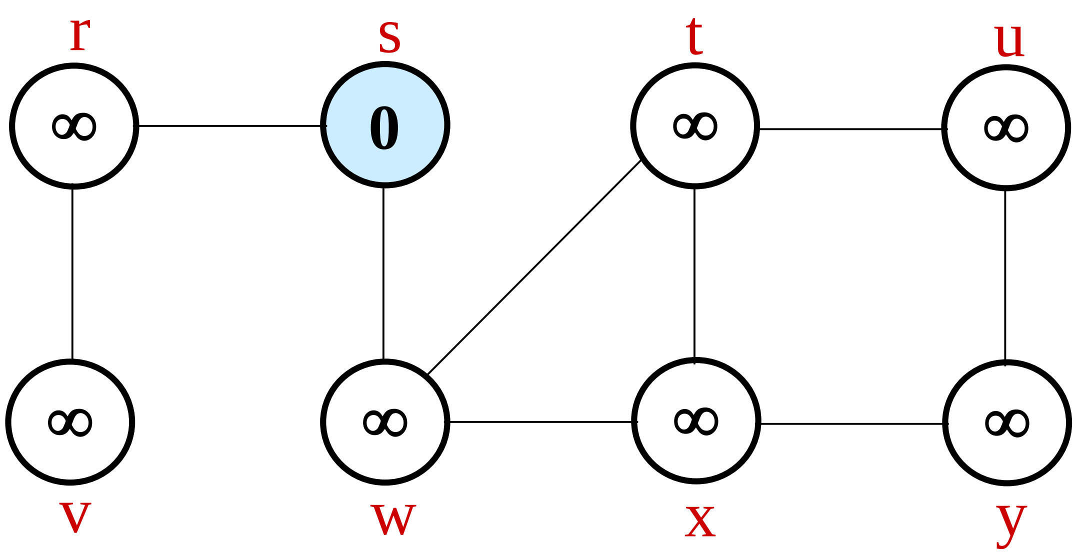
Q: a queue of discovered vertices

color[*v*]: color of v

d[*v*]: distance from s to v

π[*u*]: predecessor of v

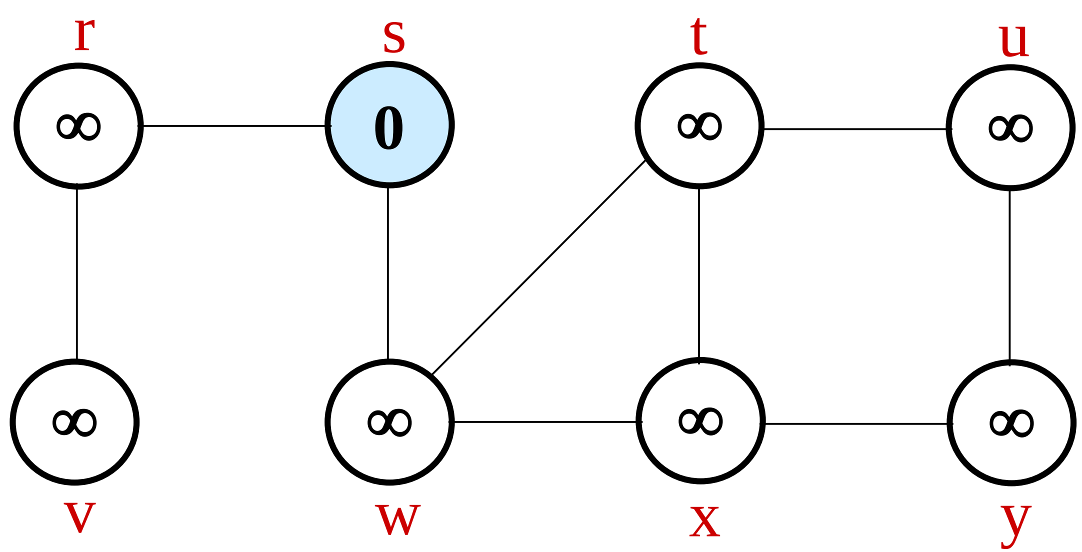
Example (BFS)



Q:	s
	0

frontier

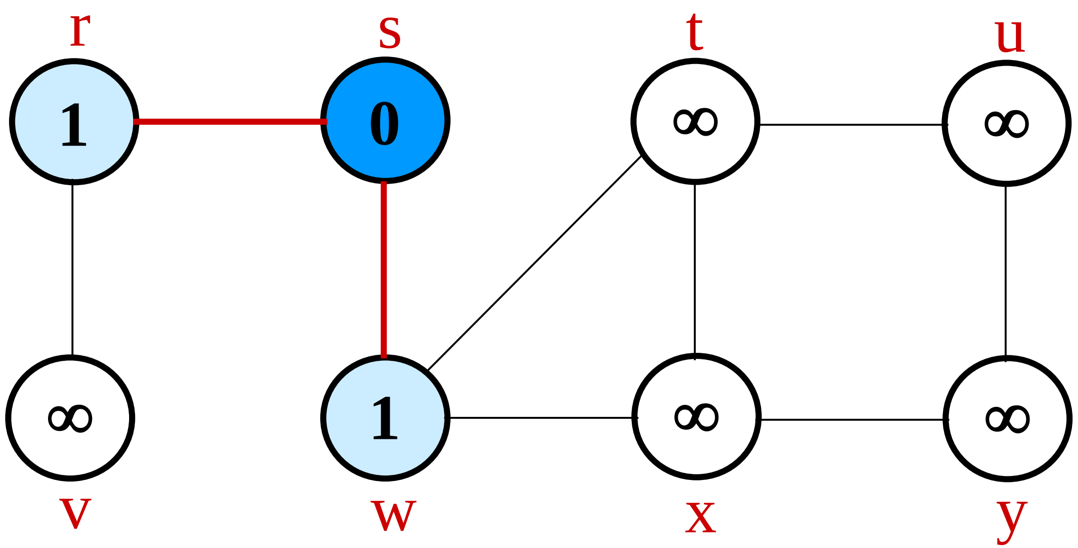
Example (BFS)



Q:	s
	0

frontier

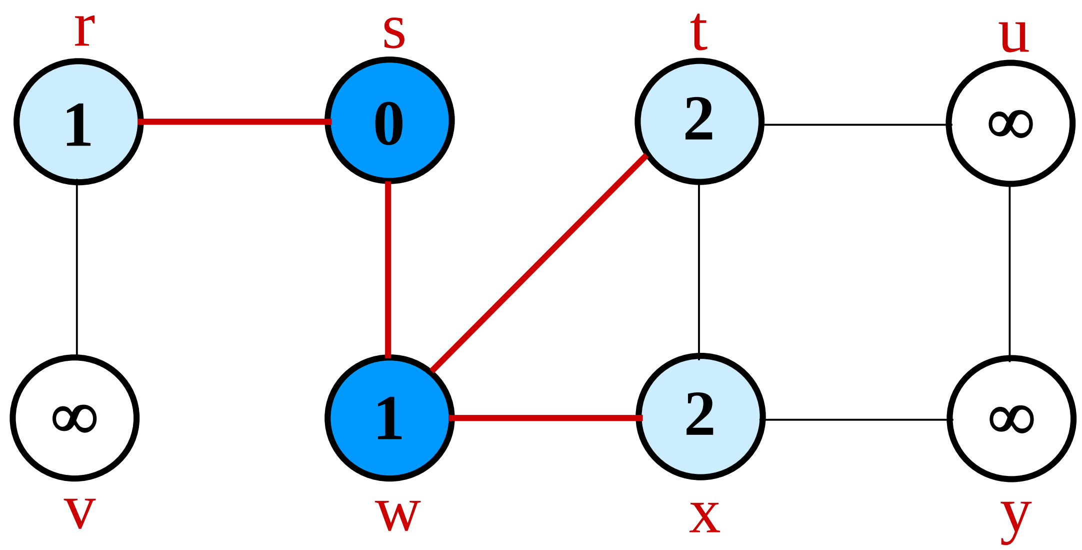
Example (BFS)



Q: w r

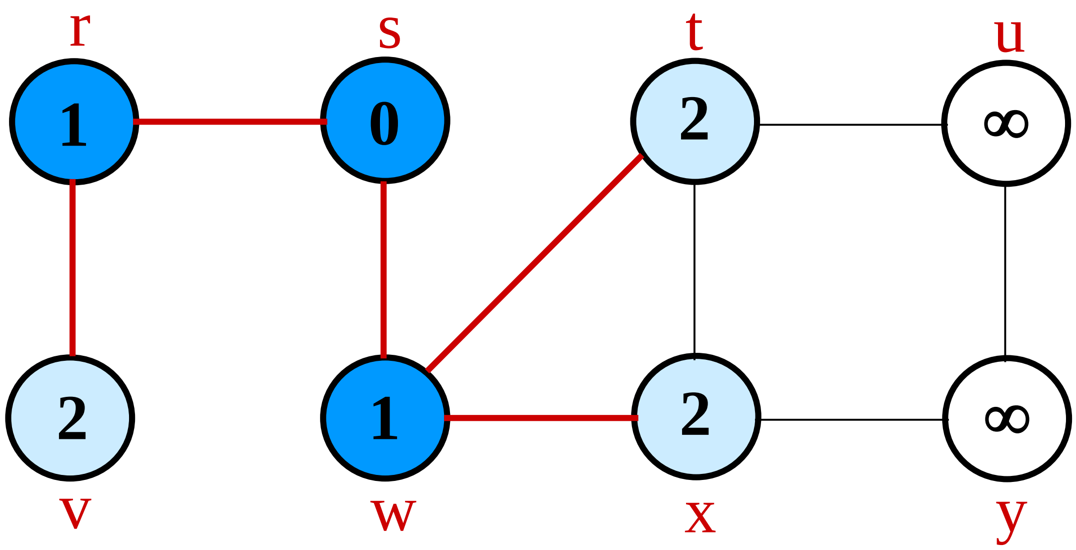
1 1

Example (BFS)



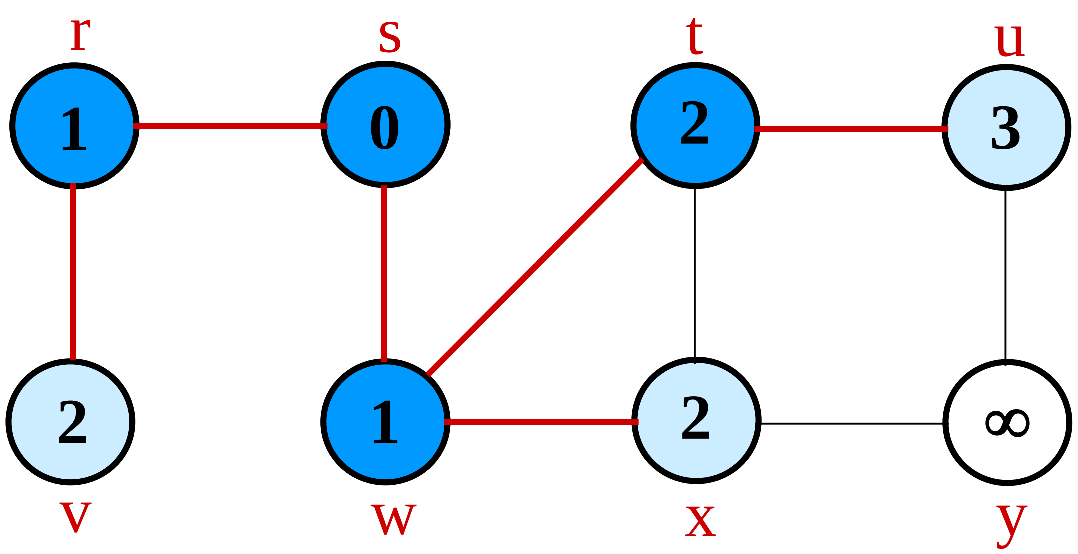
Q:	r	t	x
	1	2	2

Example (BFS)



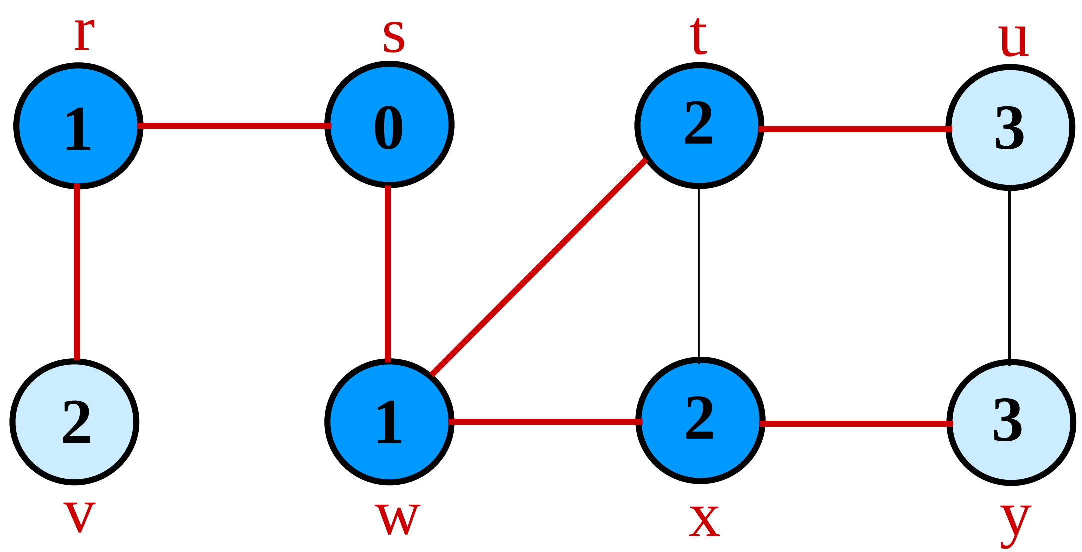
Q:	t	x	v
	2	2	2

Example (BFS)



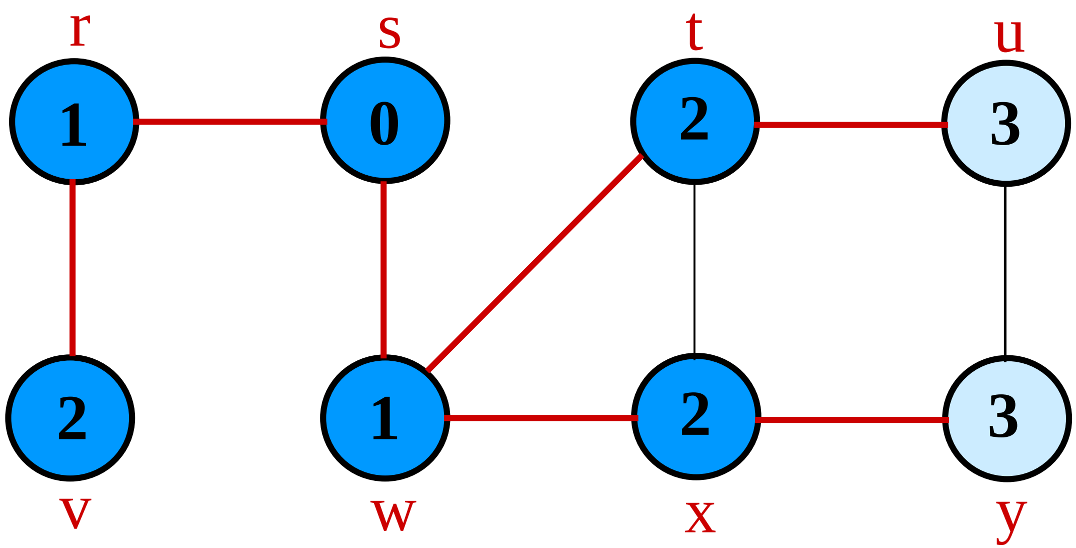
Q:	x	v	u
	2	2	3

Example (BFS)



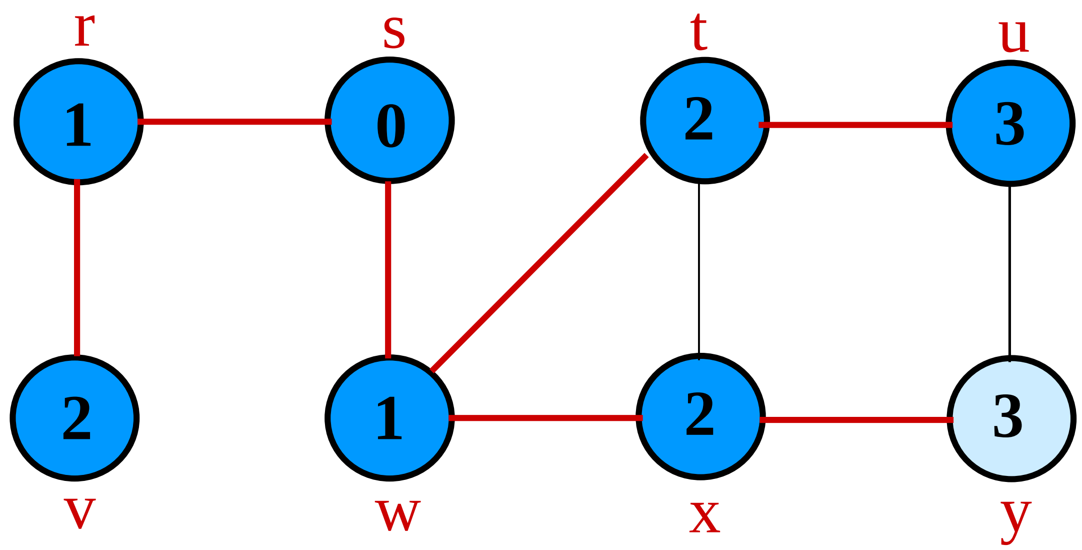
Q:	v	u	y
	2	3	3

Example (BFS)



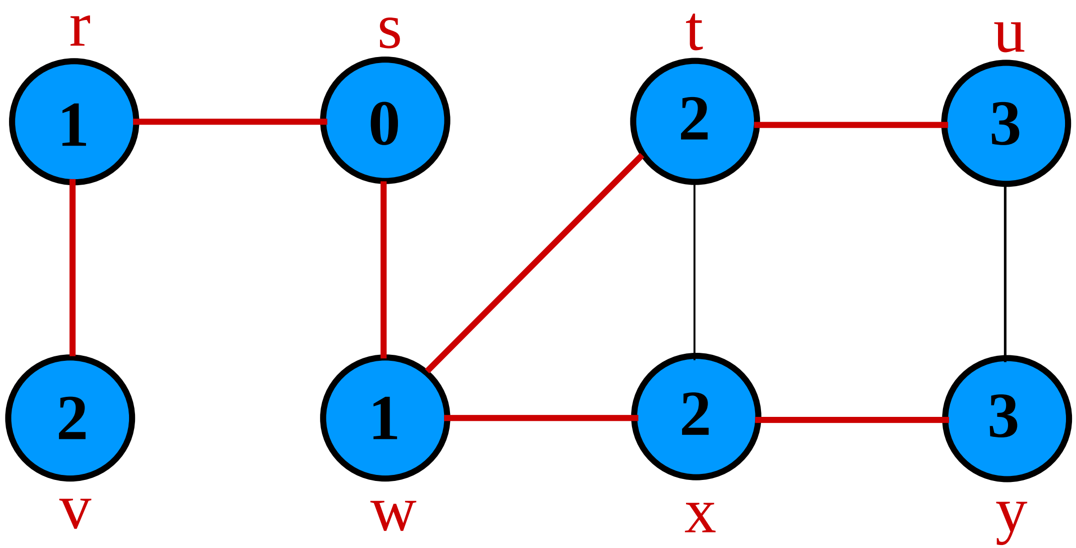
Q: u y
3 3

Example (BFS)



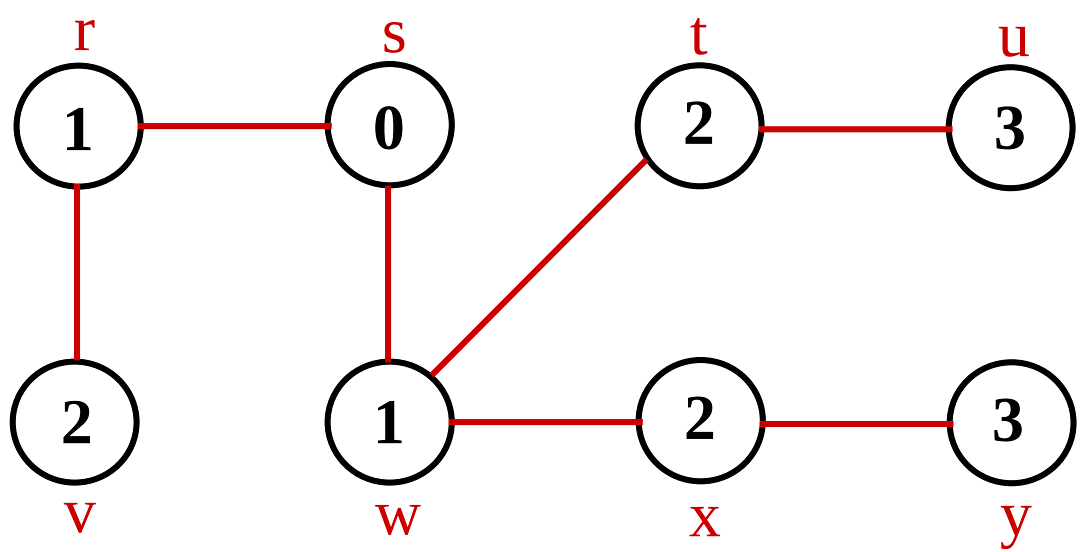
Q: y
3

Example (BFS)



Q: $\varnothing$

Example (BFS)



BF Tree

Breadth-First Tree

- **Predecessor sub-graph** of $G = (V, E)$ with source s is
  $G_\pi = (V_\pi, E_\pi)$ where
  - $V_\pi = \{v \in V : \pi[v] \neq \text{NIL}\} + \{s\}$
  - $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$

- G_π is a **breadth-first tree** if:
  - V_π consists of the vertices reachable from s
  - for all $v \in V_\pi$, there is a unique simple path from s to v in G_π
  - the path is also a shortest path from s to v in G.

- The edges in E_π are called **tree edges**.
  $|E_\pi| = |V_\pi| - 1$.

Analysis of BFS

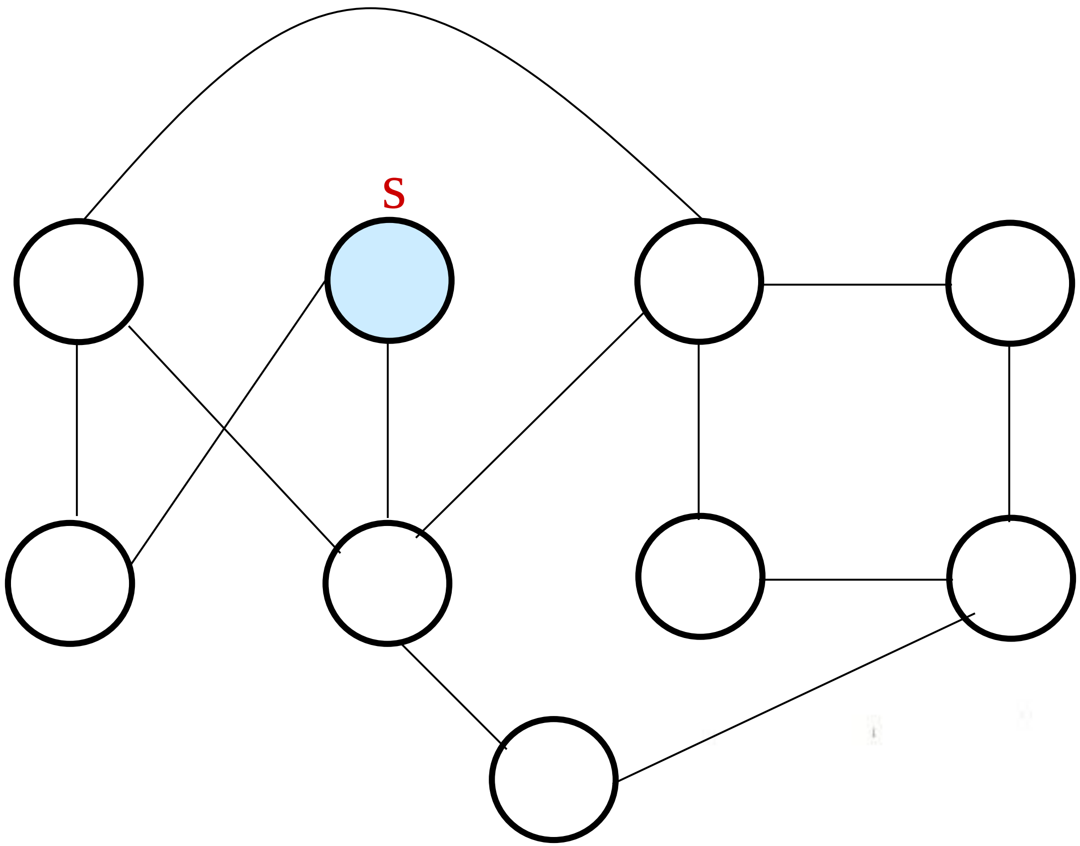
- **Initialization takes $O(|V|)$.**

- **Traversal Loop**
  - **Each vertex is enqueued and dequeued at most once, so the total time for queuing is $O(|V|)$.**
  - **The adjacency list of each vertex is scanned at most once.**
  - **The sum of lengths of all adjacency lists is $\Theta(|E|)$.**

- **Total running time of BFS is $O(|V|+|E|)$**
- **Correctness of BFS (see Dijkstra later)**

Short Test in Class

Compute the shortest distances of each vertex from the source vertex s, and give the BSF tree of the graph below.



Depth-First Search (DFS)

- **Explore edges out of the most recently discovered vertex *v*.**
- **When all edges of *v* have been explored, backtrack to its *predecessor* to explore other edges**
- **“Search as deep as possible first.”**
- **Continue until all vertices reachable from the original source are discovered.**

Depth-First Search

- **Input:** $G = (V, E)$, directed or undirected.
  No source vertex given!

- **Output:**
  - 2 time stamps on each vertex.
    - $d[v]$ = *discovery time* (v turns from white to gray)
    - $f\,[v]$ = *finishing time* (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.

Program

<u>DFS(G)</u>

1. **for each vertex $u \in V[G]$**
2. **do $color[u] \leftarrow$ white**
3. **$\pi[u] \leftarrow$ NIL**
4. **$time \leftarrow 0$**
5. **for each vertex $u \in V[G]$**
6. **do if $color[u]$ = white**
7. **then DFS-Visit(u)**

<u>DFS-Visit(u)</u>

1. $color[u] \leftarrow$ GRAY

   ∇ u has been discovered

1. $time \leftarrow time + 1$
2. $d[u] \leftarrow time$
3. **for** each $v \in Adj[u]$
4. **do if** $color[v]$ = WHITE
5. **then** $\pi[v] \leftarrow u$
6. DFS-Visit(v)
7. $color[u] \leftarrow$ BLACK

   ∇ Blacken u; it is finished.

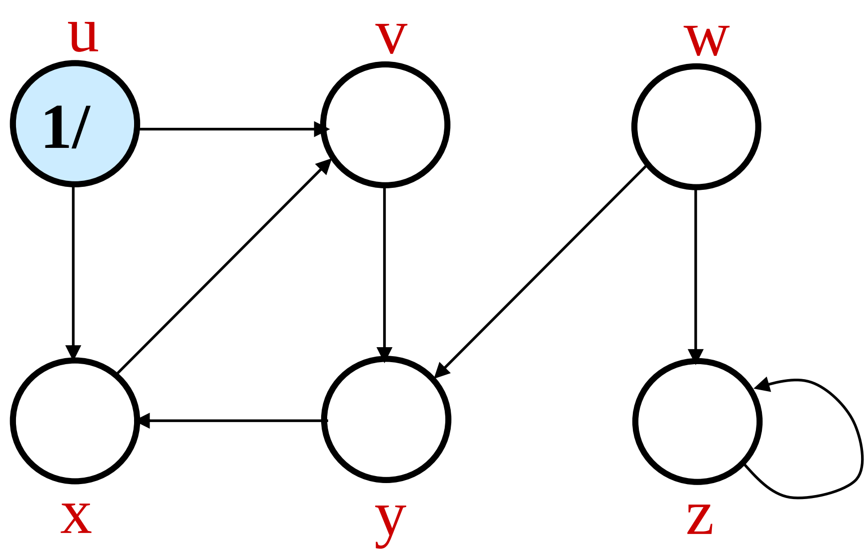
1. $f[u] \leftarrow time \leftarrow time + 1$

Uses a global timestamp ***time***.

DFS: Kinds of edges

- **DFS introduces an important distinction among edges in the original graph:**
  - ***Tree edge*: encounter new (white) vertex**
  - ***Back edge*: from descendent to ancestor**
  - ***Forward edge*: from ancestor to descendent**
  - ***Cross edge*: between a tree or subtrees**

Example (DFS)



Tree edge: encounter new (white) vertex

Back edge: from descendent to ancestor

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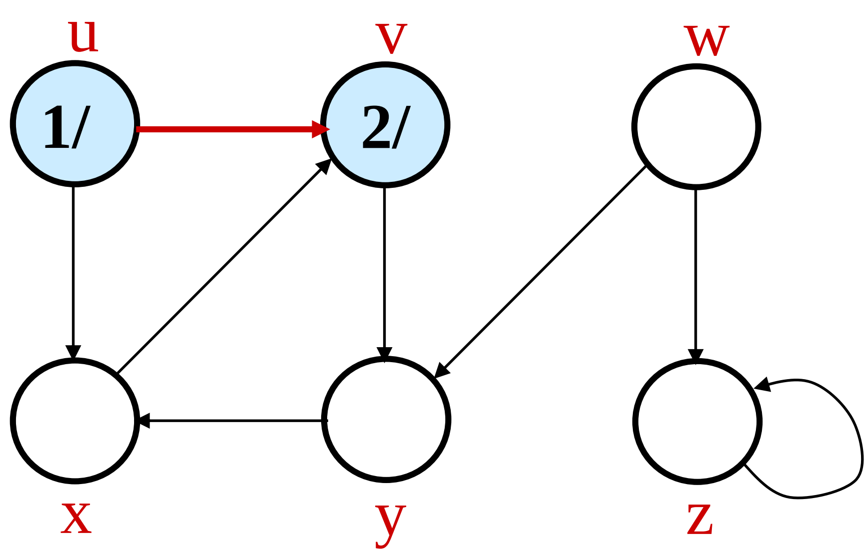
Example (DFS)



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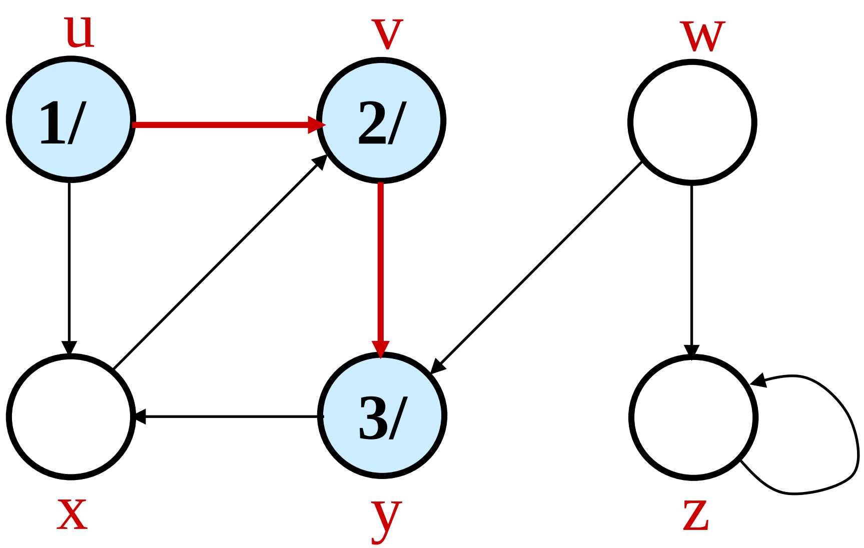

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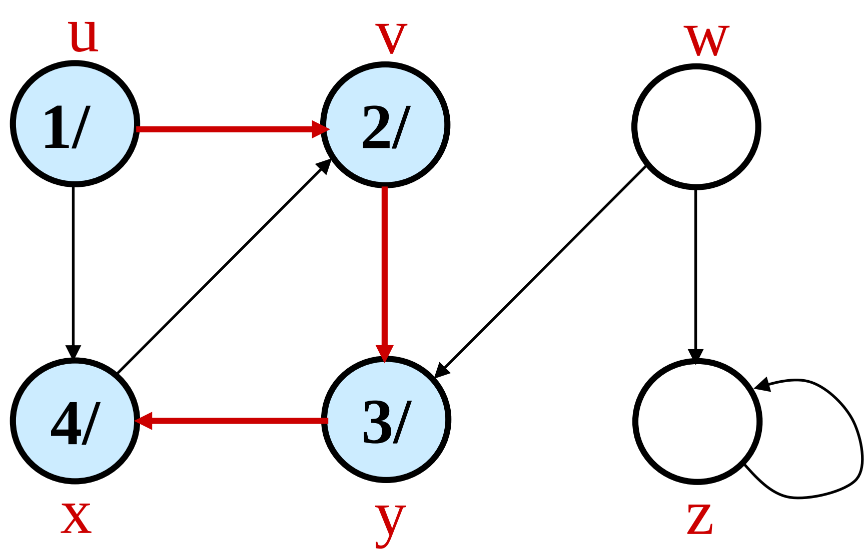

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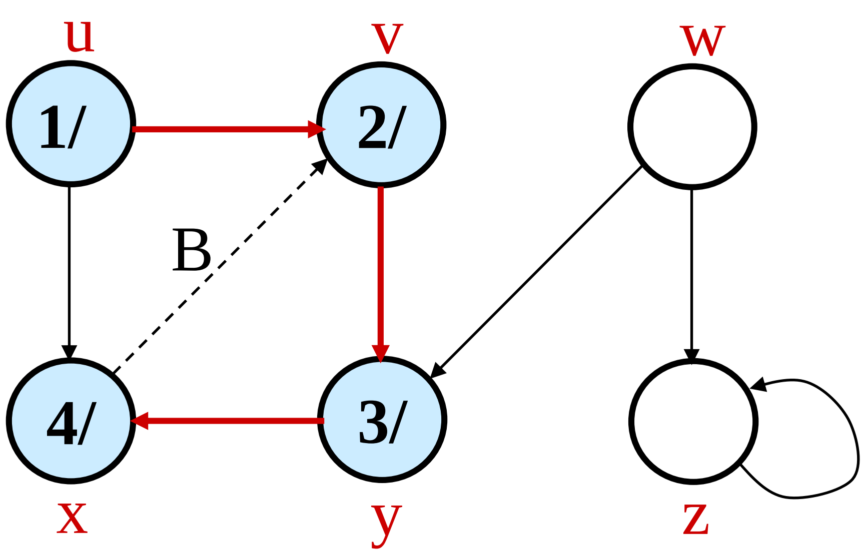

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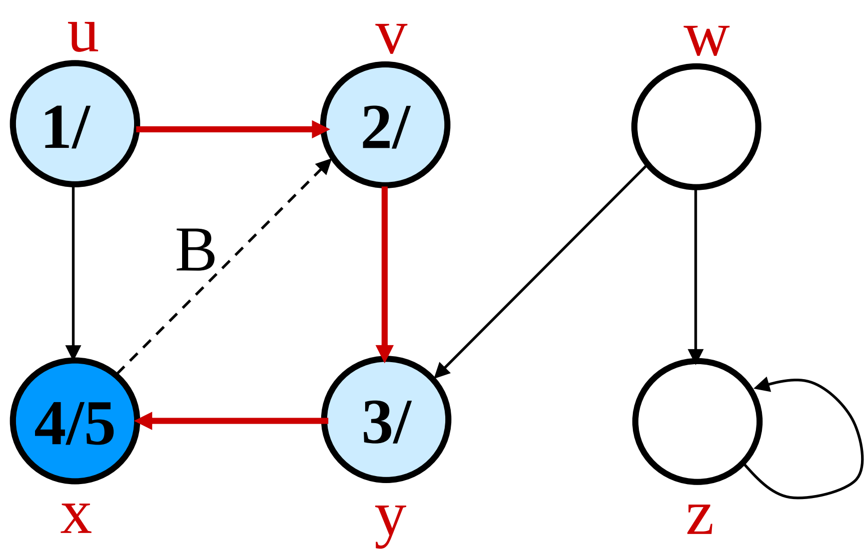

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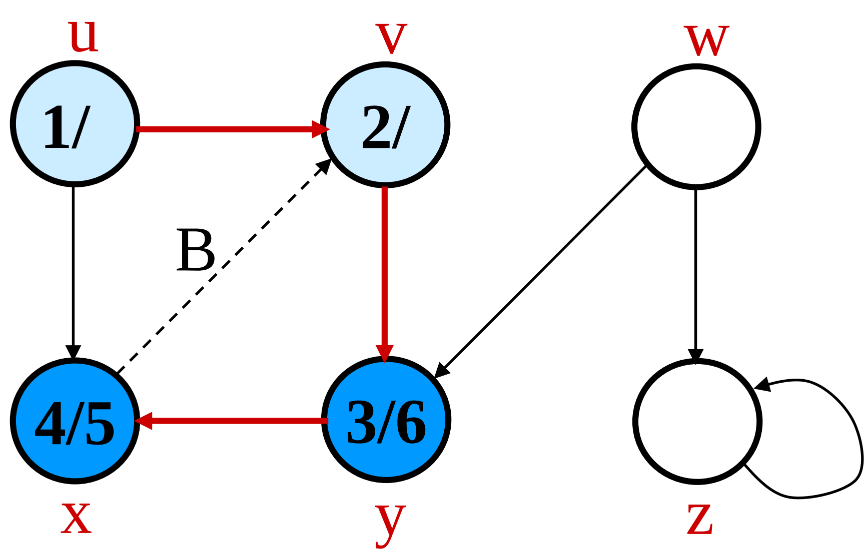

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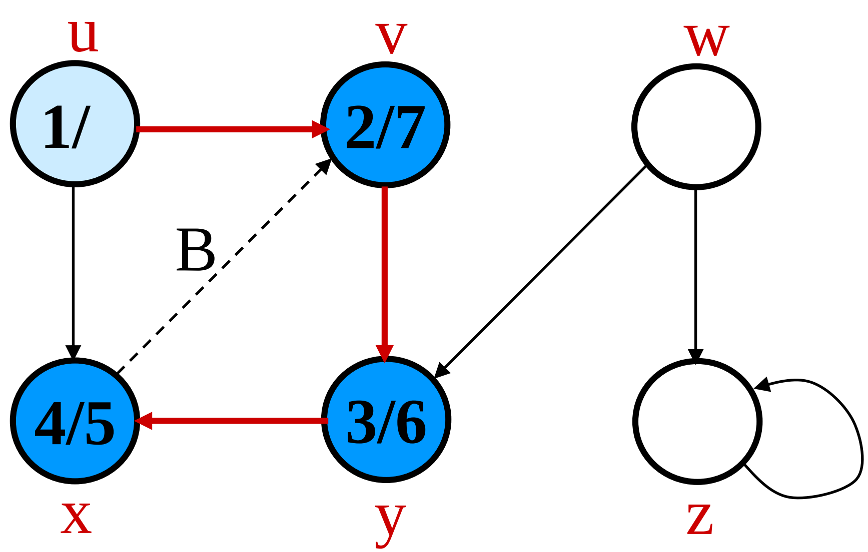

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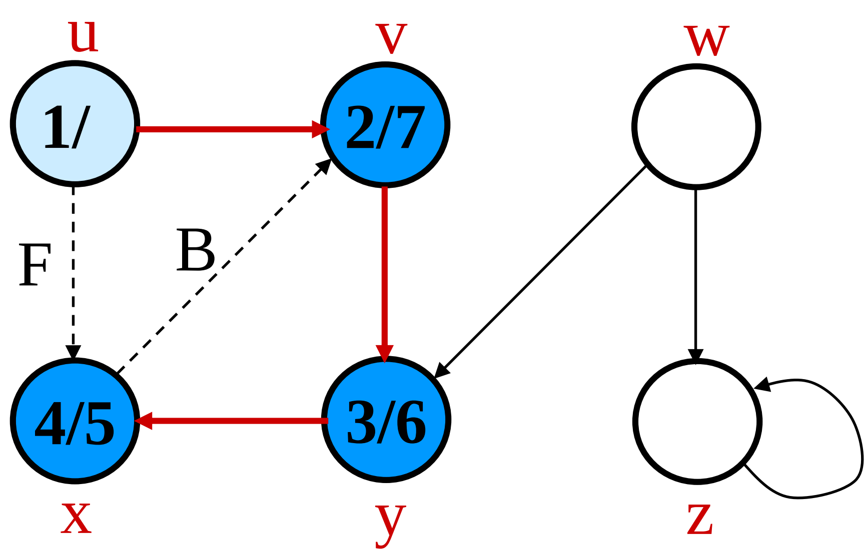

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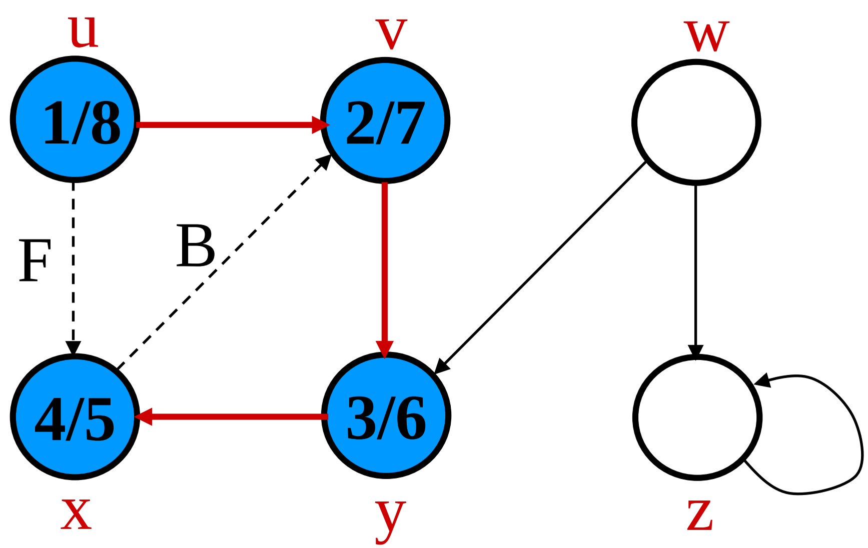

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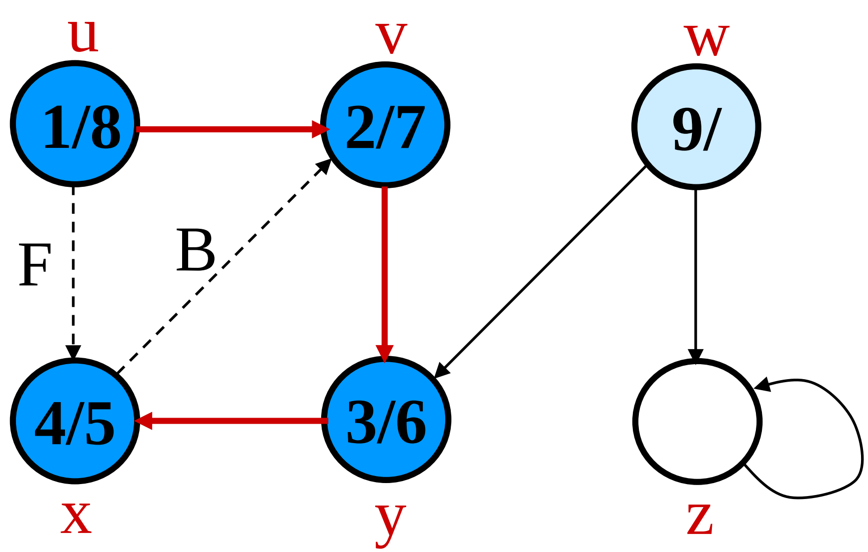

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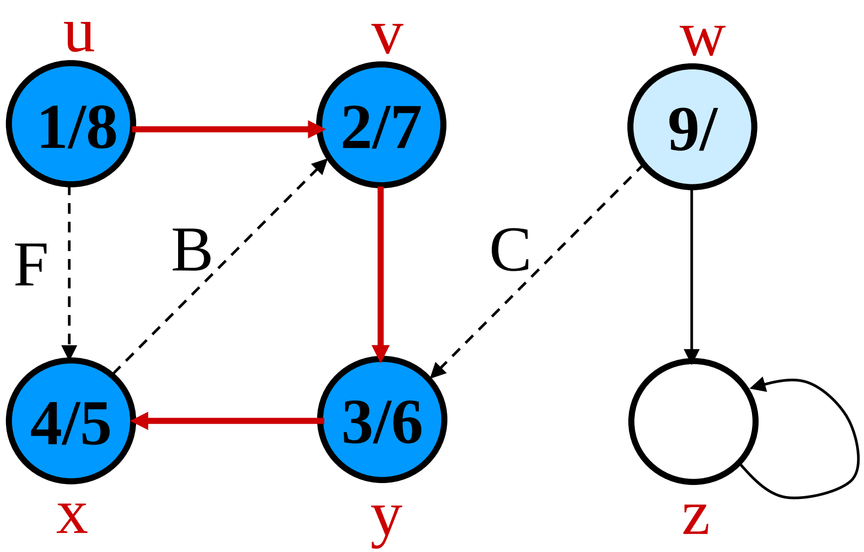

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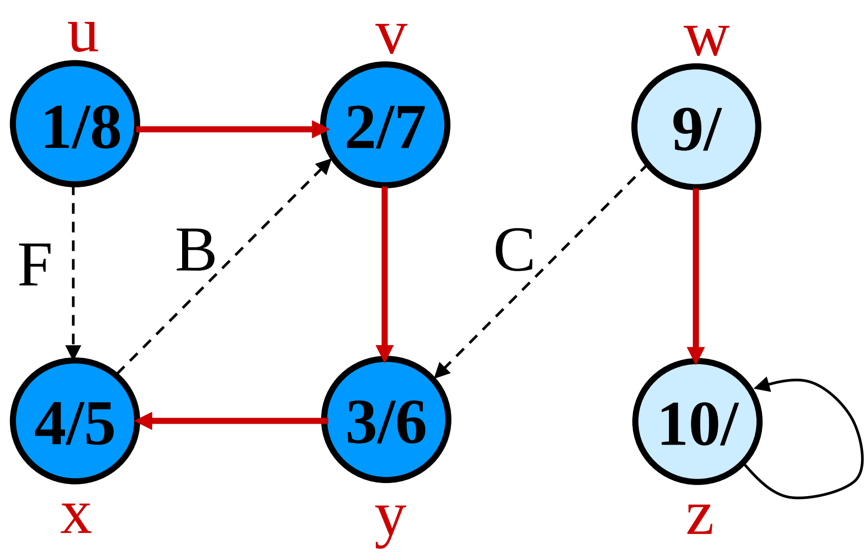

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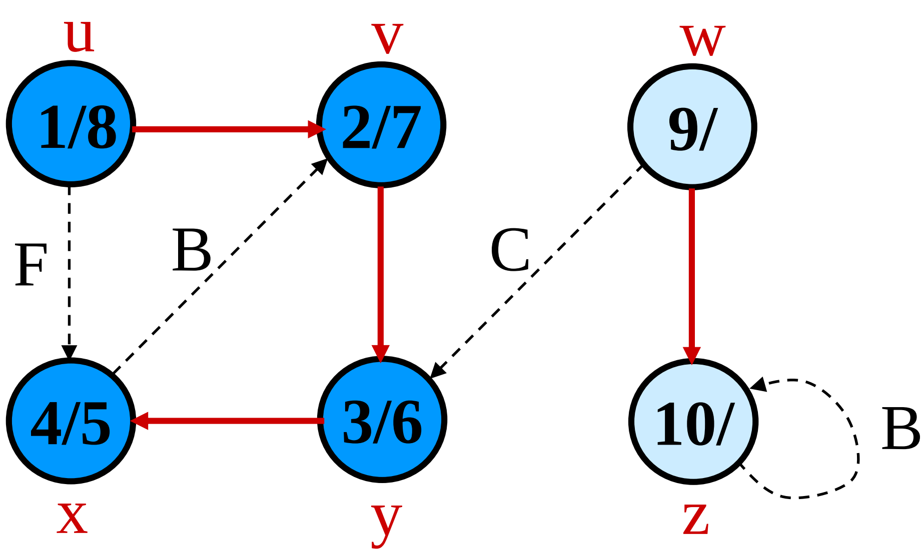

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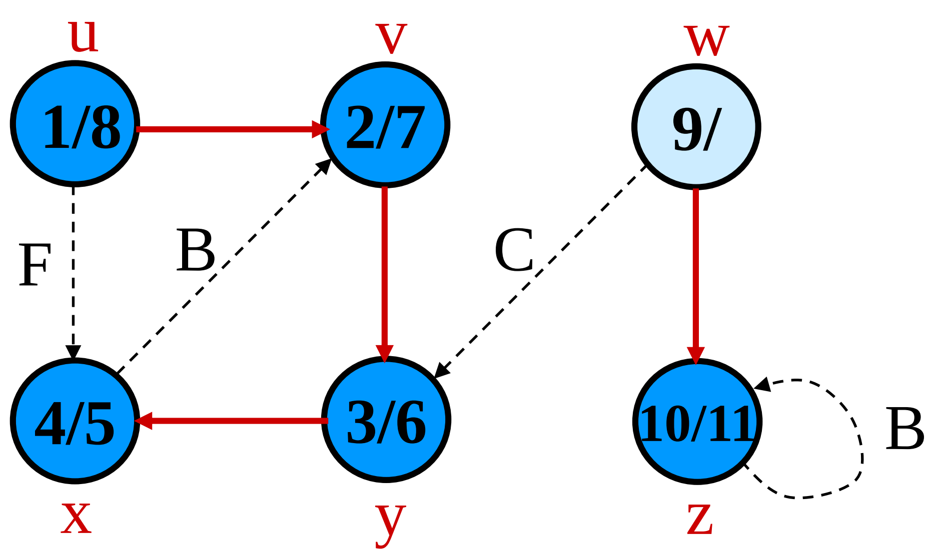

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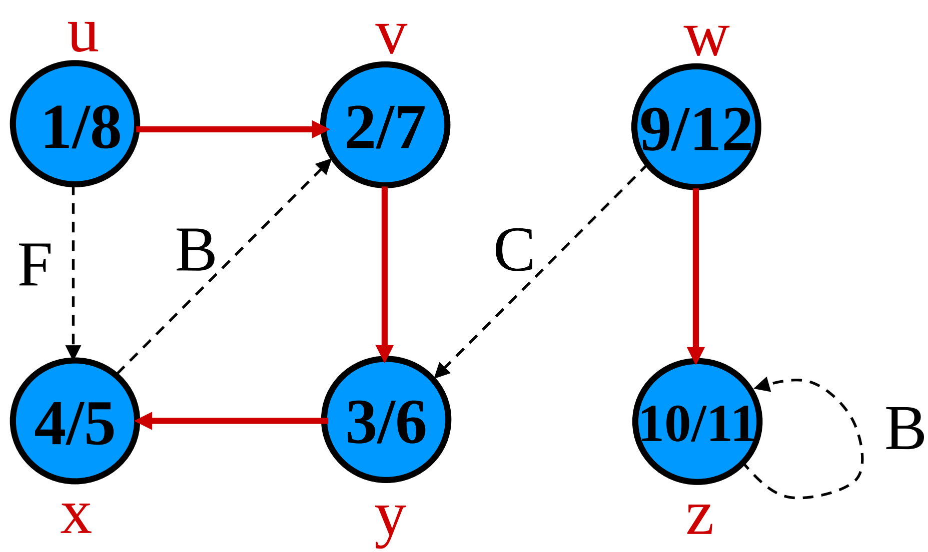


Tree edge: encounter new (white) vertex
Back edge: from descendent to ancestor
Forward edge: from ancestor to descendent
Cross edge: between a tree or subtrees

Classification of Edges

- **Tree edge:** in the **depth-first forest**, by exploring (*u*, *v*).
- **Back edge:** (*u*, *v*), where *u* is a descendant of *v* (in the depth-first tree). (include self-loop)
- **Forward edge:** (*u*, *v*), where *v* is a descendant of *u*, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Identification of Edges

- **Edge type for edge (*u*, *v*) can be identified when it is first explored by DFS.**

- **Identification is based on the color of *v*.**
  - **White – tree edge.**
  - **Gray – back edge.**
  - **Black – forward or cross edge.**

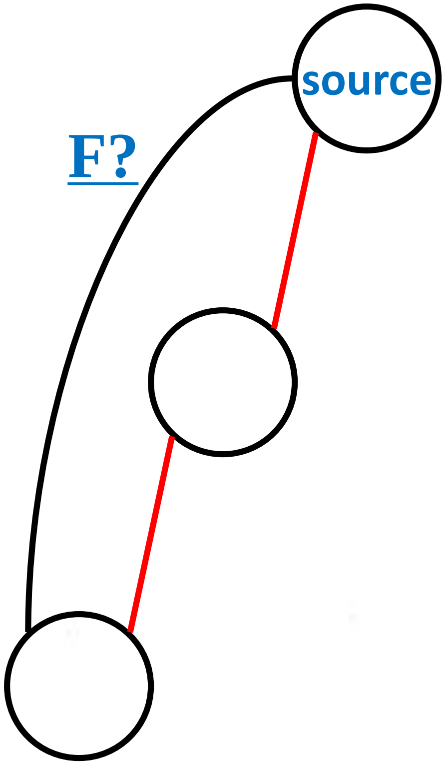
Identification of Edges

Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Proof by contradiction:

Assume there's a forward edge

But F? edge must actually be a back edge (*why?*)



Identification of Edges

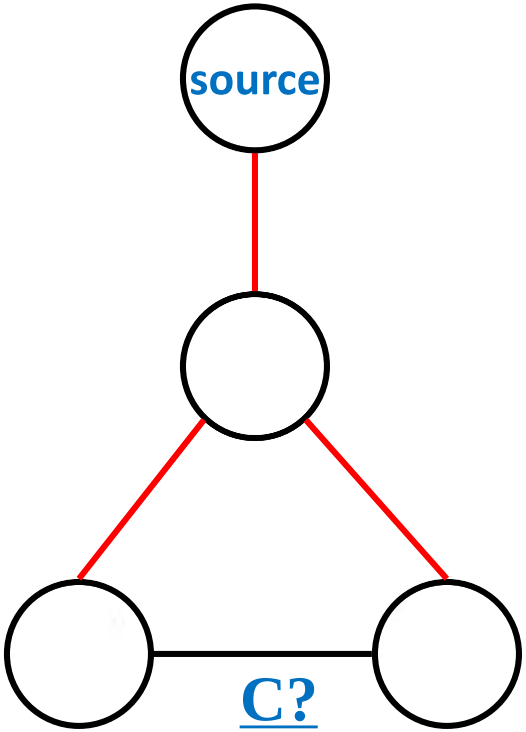
Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Proof by contradiction:

Assume there's a cross edge

But C? edge cannot be cross!

So in fact the picture is wrong…both lower tree edges cannot in fact be tree edges



Depth-First Trees

- **Predecessor subgraph is slightly different from that of BFS.**

- **The predecessor subgraph of DFS**

$$G_\pi = (V, E_\pi)$$

$$E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL}\}.$$

  - **G_π forms a *depth-first forest* composed of several *depth-first trees*. E_π consists of *tree edges*.**

Definition: Forest: An acyclic graph G that may be disconnected.

Analysis of DFS

DFS(*G*)

```
for each vertex u ∈ V[G]
    do color[u] ← white
       π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
    do if color[u] = white
          then DFS-Visit(u)
```

DFS-Visit(*u*)

```
color[u] ← GRAY
            ∇ u has been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]
    do if color[v] = WHITE
          then π[v] ← u
               DFS-Visit(v)
color[u] ← BLACK
            ∇ Blacken u;  it is finished.
f[u] ← time ← time + 1
```

Uses a global timestamp ***time***.

Analysis of DFS

- **Loops on lines 1-2 & 5-7 take $\Theta(|V|)$ time, excluding time to execute DFS-Visit.**
- **DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 3-6 of DFS-Visit is executed |Adj[v]| times. The total cost of executing DFS-Visit is**

$$\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(|E|)$$

- **Total running time of DFS is $\Theta(|V|+|E|)$.**

Parenthesis Theorem

<u>Theorem 22.7</u>

For all u, v, exactly one of the following holds:

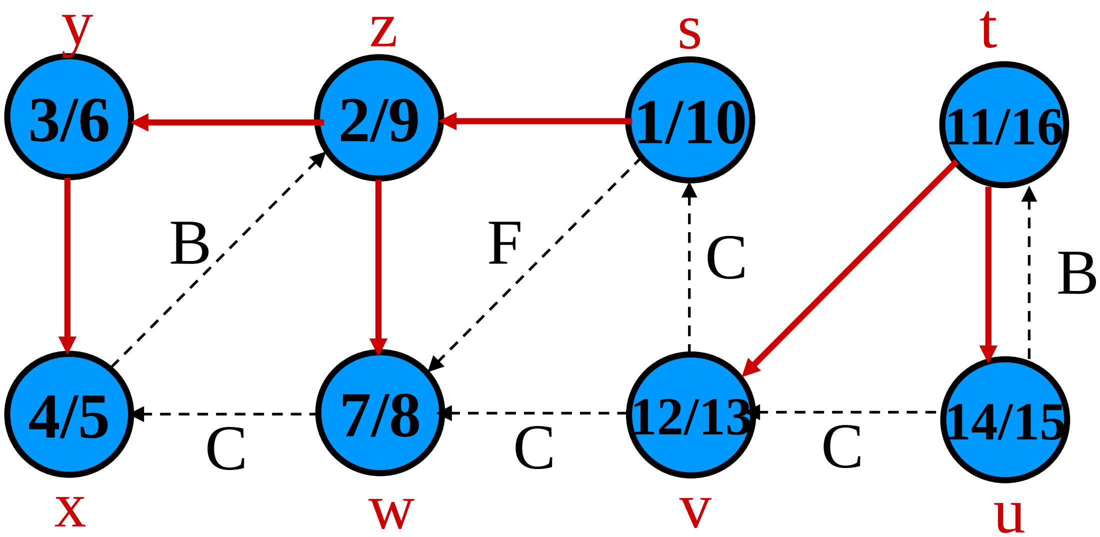
1. $d[u] < f[u] < d[v] < f[v]$ **or** $d[v] < f[v] < d[u] < f[u]$ **and neither** u **nor** v **is a descendant of the other.**
2. $d[u] < d[v] < f[v] < f[u]$ **and** v **is a descendant of** u**.**
3. $d[v] < d[u] < f[u] < f[v]$ **and** u **is a descendant of** v**.**

- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

Corollary

v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$.

Example (Parenthesis Theorem)



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

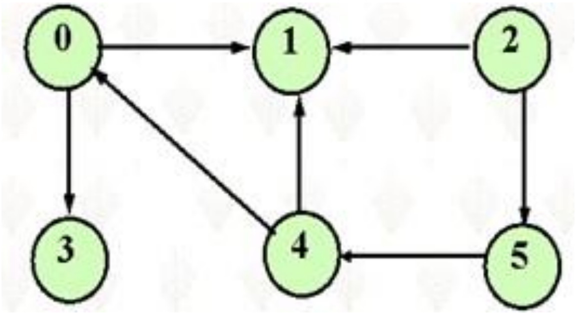
15.3 Topological Sorting

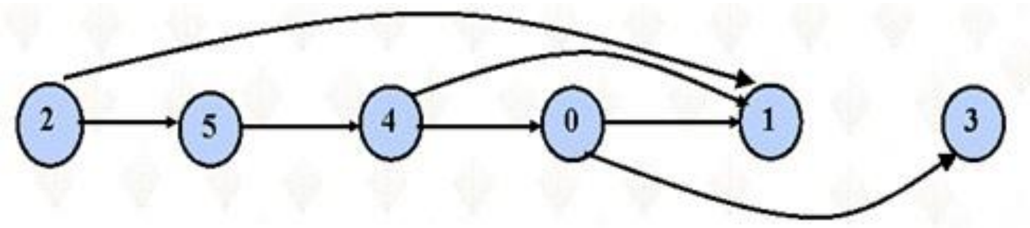
Directed Acyclic Graph

- **DAG – Directed graph with no cycles.**
- **Good for modeling processes and structures that have a partial order:**
  - $a > b$ **and** $b > c \Rightarrow a > c$. **(transitive closure)**
  - **But may have** a **and** b **such that neither** $a > b$ **nor** $b > a$.
- **Can always make a total order (either** $a > b$ **or** $b > a$ **for all** $a \neq b$**) from a partial order.**

Topological Ordering

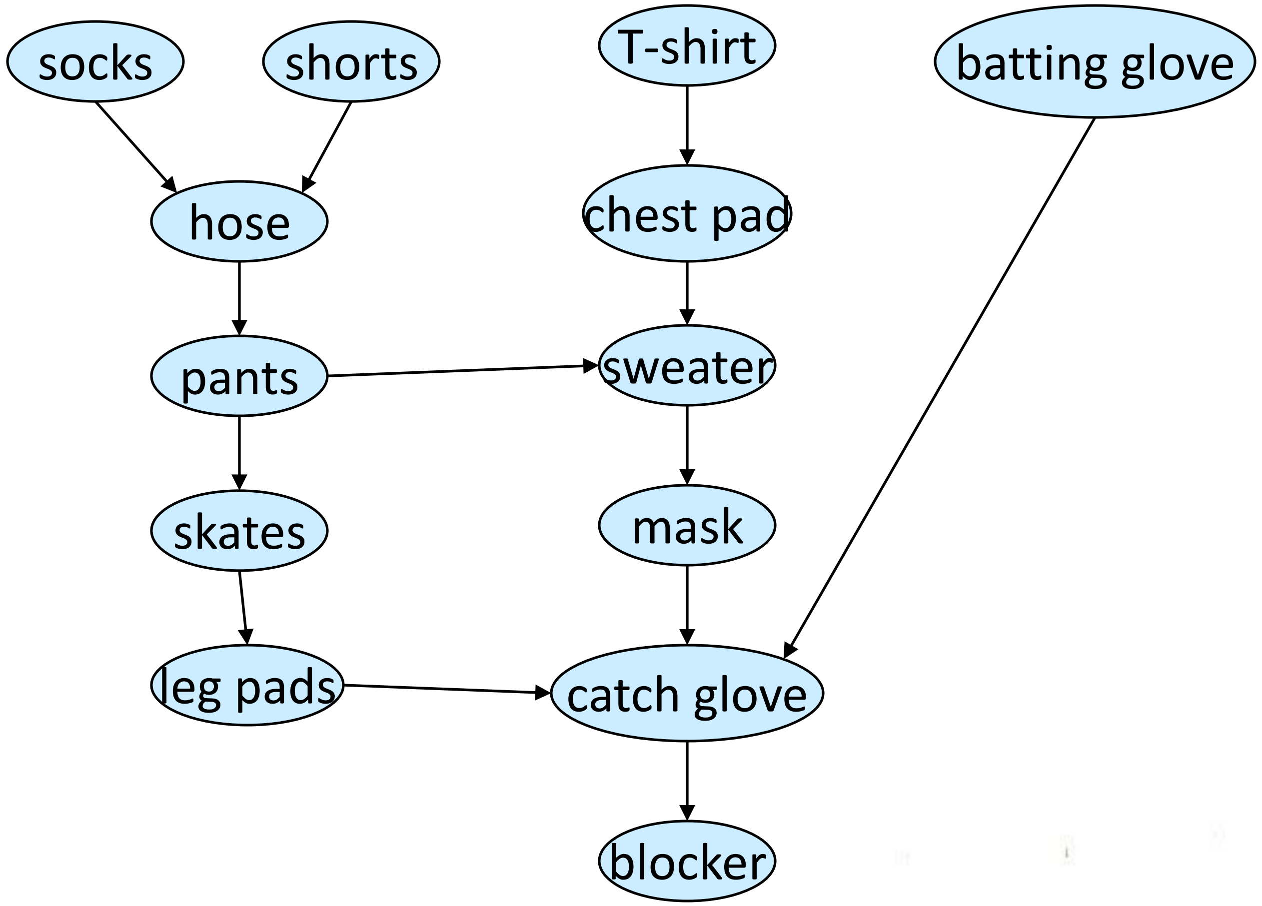
- Suppose that G is a directed graph which contains no directed cycles.
- Then a **topological ordering** of the vertices in G is a sequential listing of the vertices such that for any pair of vertices, v and w in G, if <v,w> is an edge in G then v precedes w in the sequential listing.



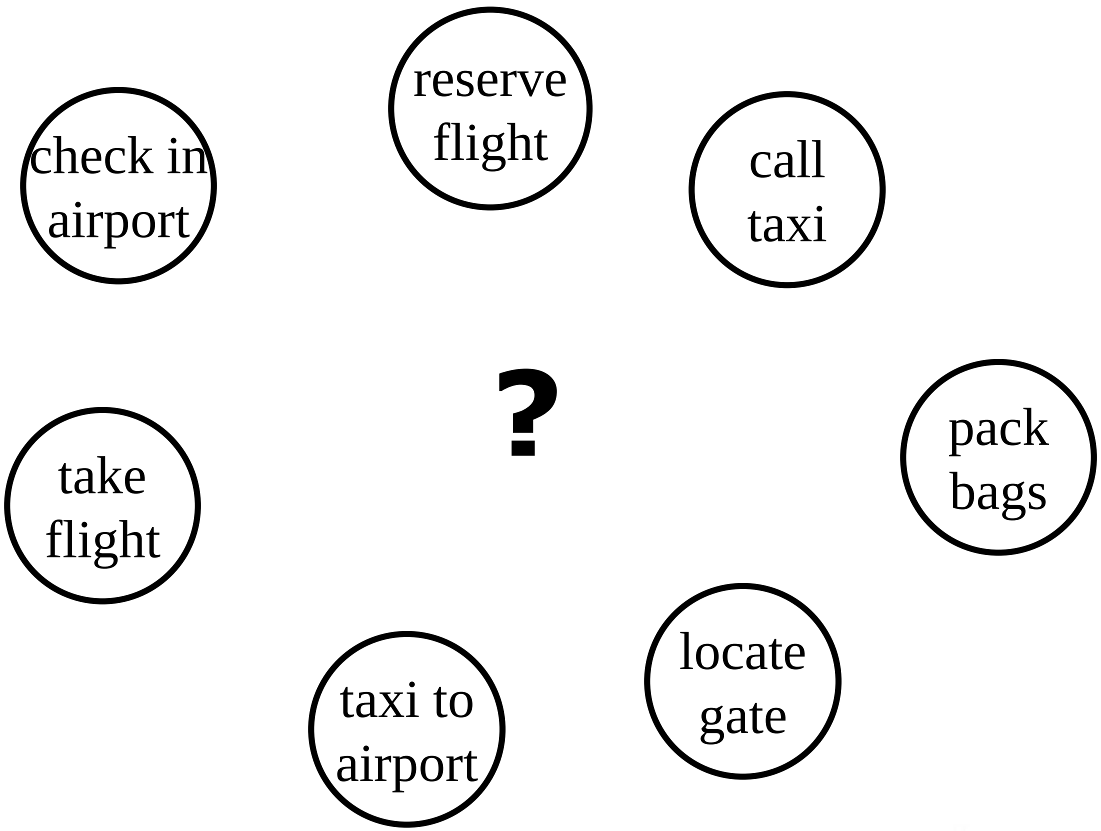


Example

DAG of dependencies for putting on goalie equipment.

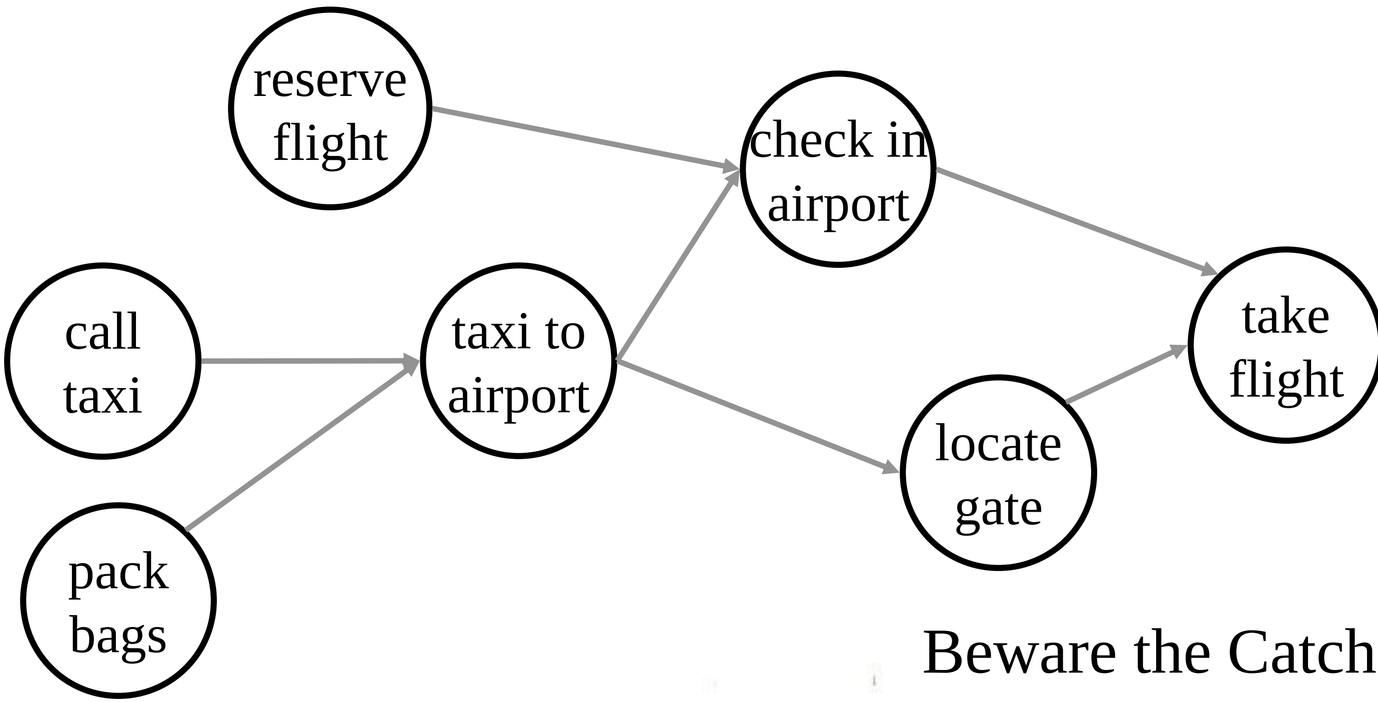


Partial Order: Planning a Trip with GF!



Partial Order: Planning a Trip with GF!

- **Given a graph, `G = (V, E)`, output all the vertices in `V` such that no vertex is output before any other vertex with an edge to it.**



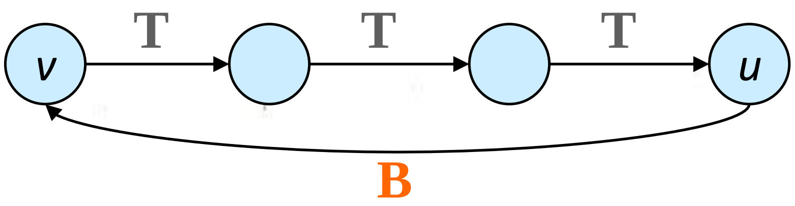
Beware the Catch-22!

Characterizing a DAG

Lemma 22.11 A directed graph *G* is acyclic iff a DFS of G yields no back edges.

Proof:

- **⇒: Show that back edge ⇒ cycle.**
  - **Suppose there is a back edge <*u, v*>. Then *v* is ancestor of *u* in depth-first forest.**
  - **Therefore, there is a path *v* $\rightsquigarrow$ *u*, so *v* $\rightsquigarrow$ *u* $\rightsquigarrow$ *v* is a cycle.**

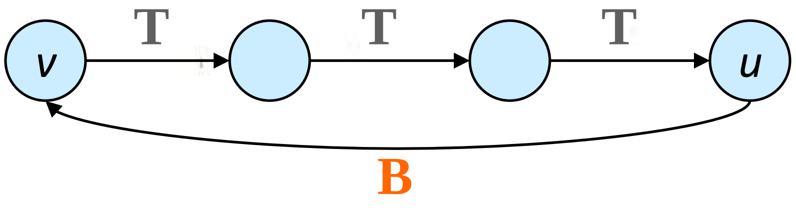


Characterizing a DAG

Proof (Contd.):

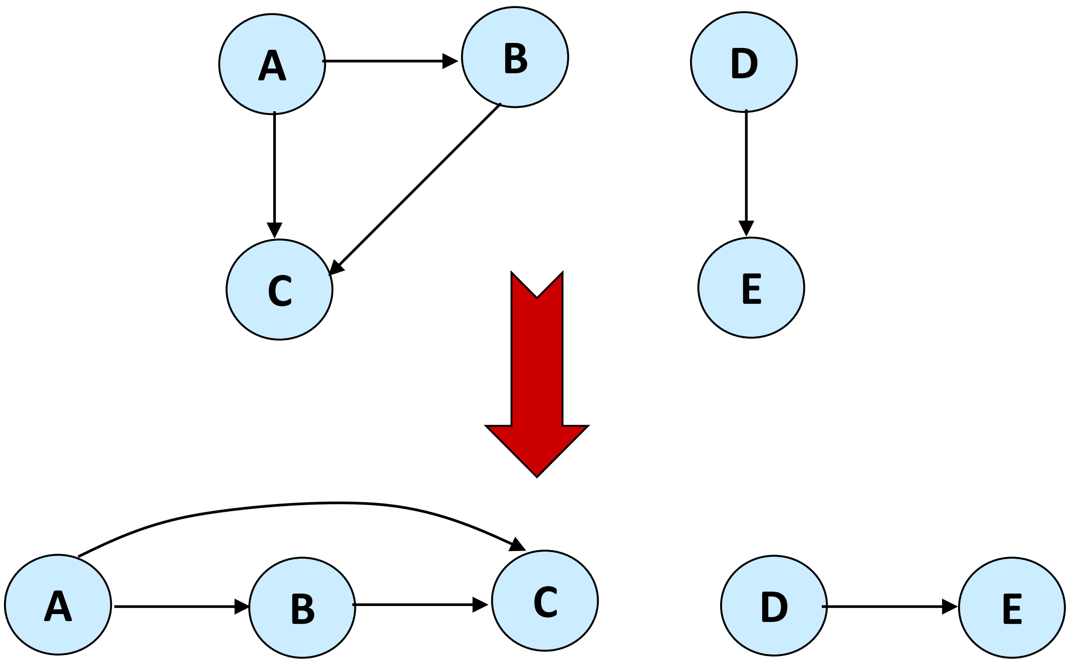
- **$\Leftarrow$: Show that a cycle implies a back edge.**

  - ***c*** **: cycle in *G*, *v* : first vertex discovered in *c*, <*u*, *v*> : *v*'s preceding edge in *c*.**
  - **At time *d*[*v*], vertices of *c* form a white path $v \rightsquigarrow u$. Why?**
  - **By white-path theorem, *u* is a descendent of *v* in depth-first forest.**
  - **Therefore, <*u*, *v*> is a back edge.**



Topological Sort

Want to "sort" a directed acyclic graph (DAG).



Think of original DAG as a **partial order**.

Want a **total order** that extends this partial order.

Topo-Sort Take One

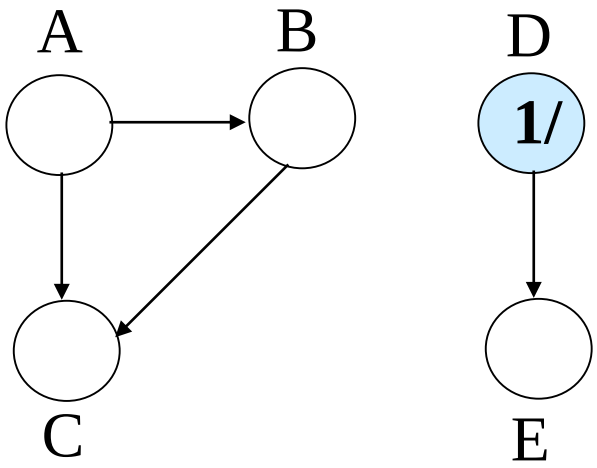
- **Performed on a DAG.**
- **Linear ordering of the vertices of G such that if $<u, v> \in E$, then u appears somewhere before v.**

Topological-Sort (G)

1. Call DFS(G) to compute $f[v]$ for all $v \in V$
2. As each vertex is finished, insert it onto the front of a linked list
3. **Return** the linked list of vertices

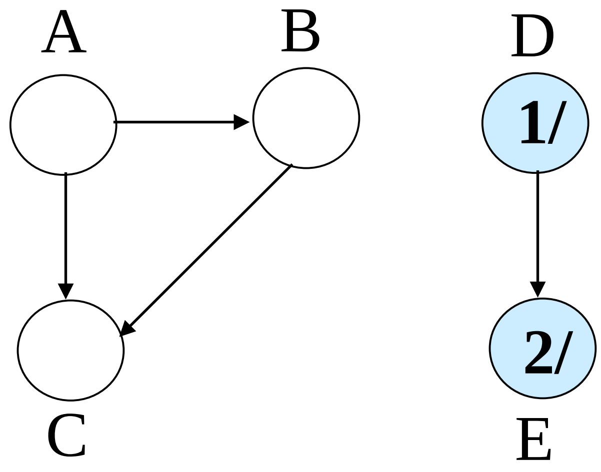
Time: $\Theta(V + E)$.

Example



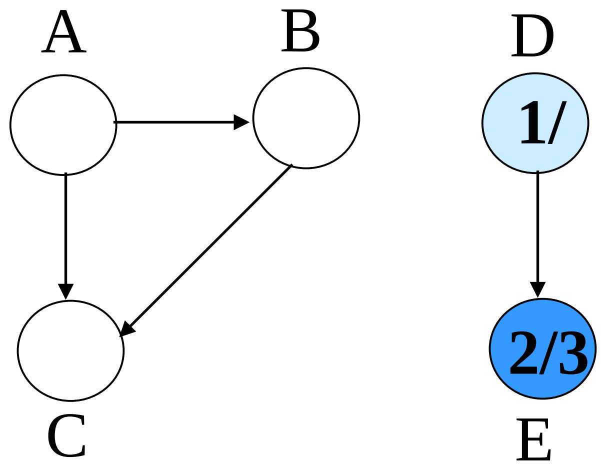
Linked List:

Example

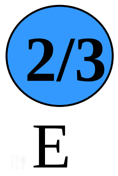


Linked List:

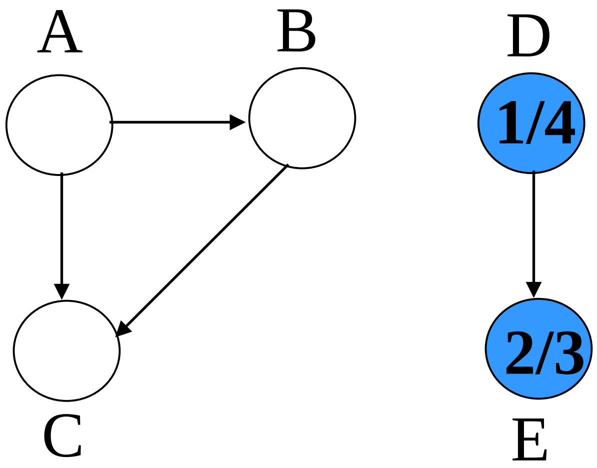
Example



Linked List:



Example



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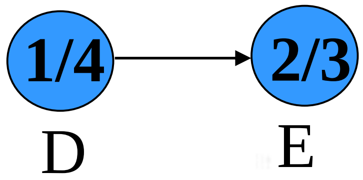
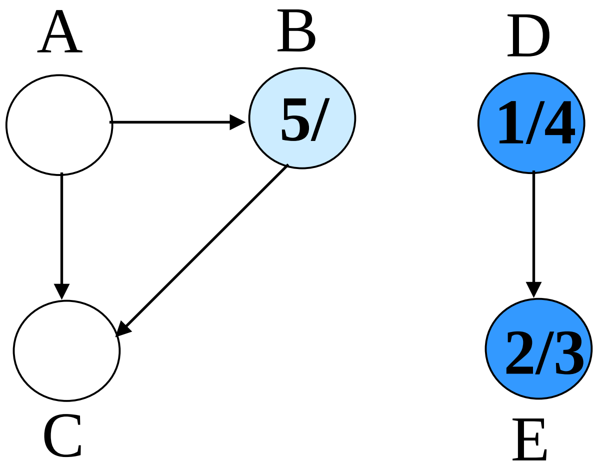


Example



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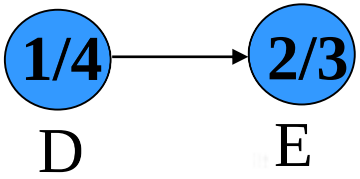
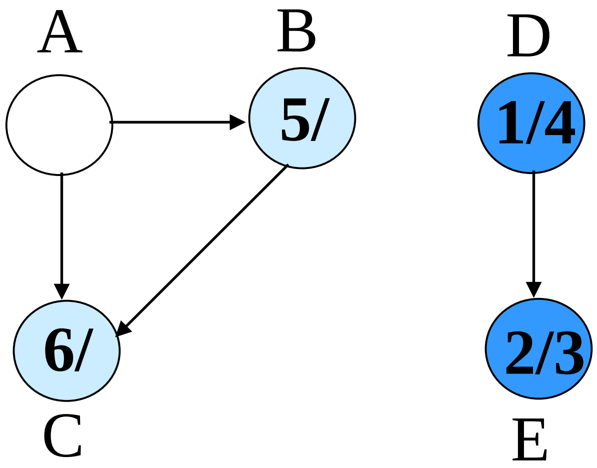


Example



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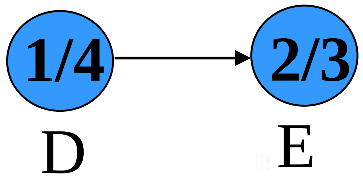
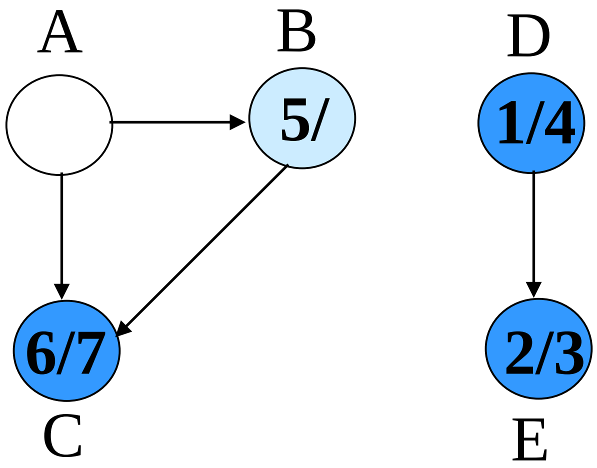


Example



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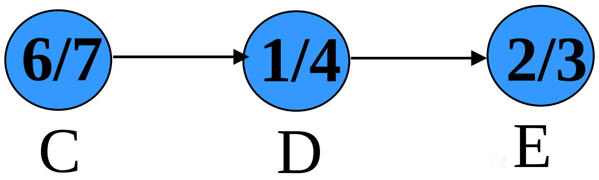
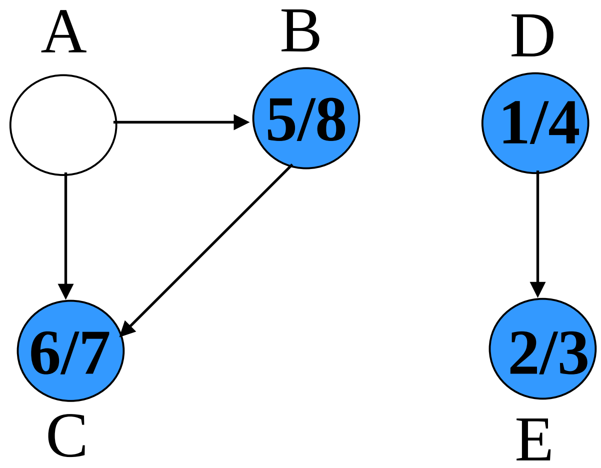


Example



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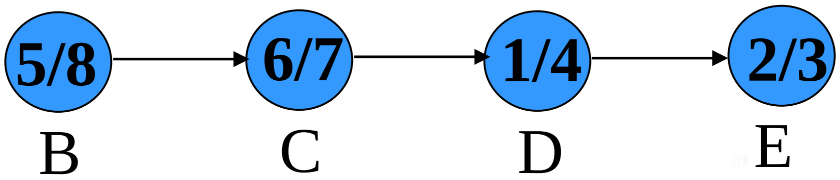
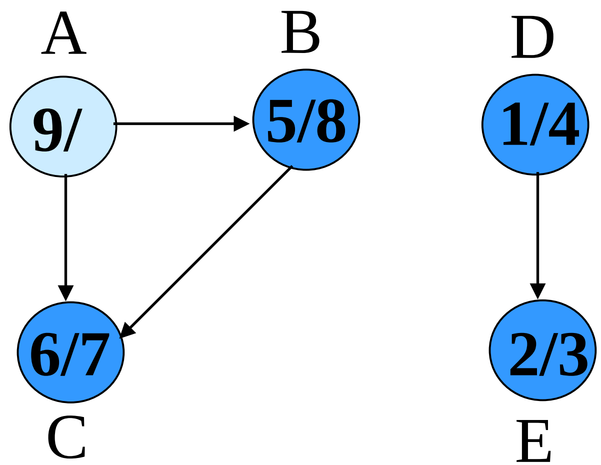


Example



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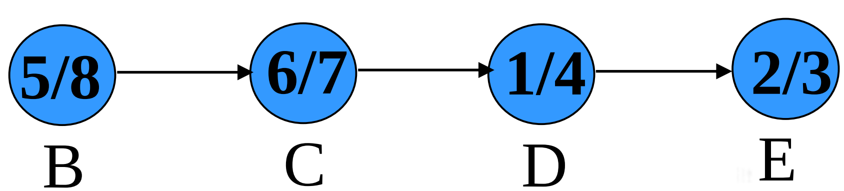
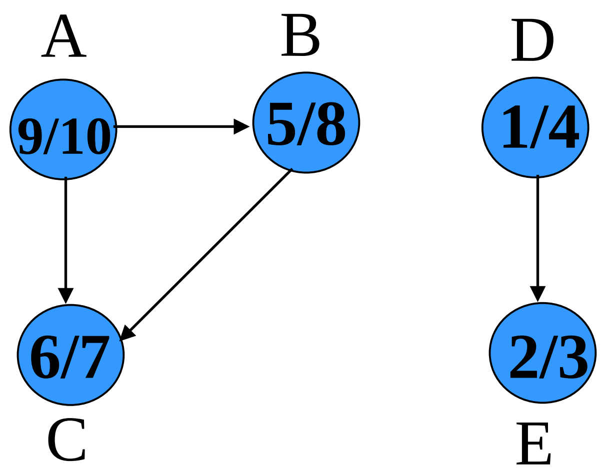


Example



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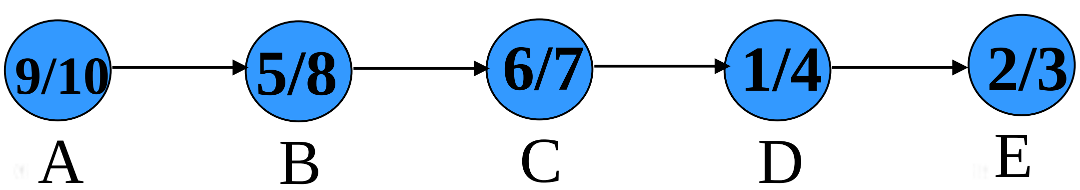

Correctness Proof

- Show if $\langle u, v \rangle \in E$, then $f[v] < f[u]$.
- When we explore $\langle u, v \rangle$, what are the colors of u and v?
  - u is gray.
  - Is v gray, too?
    - *No*, because then v would be ancestor of u.
    - $\Rightarrow \langle u, v \rangle$ is a back edge.
    - $\Rightarrow$ contradiction of Lemma 22.11 (dag has no back edges).
  - Is v white?
    - Then becomes descendant of u.
    - By parenthesis theorem, $d[u] < d[v] < \underline{f[v] < f[u]}$.
  - Is v black?
    - Then v is already finished.
    - Since we're exploring $\langle u, v \rangle$, we have not yet finished u.
    - Therefore, $f[v] < f[u]$.

```
void topsort(Graph* G) {   // Topological sort: recursive
  int i;
  for (i=0; i<G->n(); i++) // Initialize Mark array
    G->setMark(i, UNVISITED);
  for (i=0; i<G->n(); i++) // Process all vertices
    if (G->getMark(i) == UNVISITED)
      tophelp(G, i);       // Call recursive helper function
}

void tophelp(Graph* G, int v) { // Process vertex v
  G->setMark(v, VISITED);
  for (int w=G->first(v); w<G->n(); w = G->next(v,w))
    if (G->getMark(w) == UNVISITED)
      tophelp(G, w);
  printout(v);                  // PostVisit for Vertex v
}
```

Topo-Sort Take Two

- **Label each vertex's *in-degree* (# of inbound edges)**
- **While there are vertices remaining**
  - **Pick a vertex with in-degree of zero and output it**
  - **Reduce the in-degree of all vertices adjacent to it**
  - **Remove it from the list of vertices**

Runtime?

```
// Topological sort: Queue
void topsort(Graph* G, Queue<int>* Q) {
  int Count[G->n()];
  int v, w;
  for (v=0; v<G->n(); v++) Count[v] = 0; // Initialize
  for (v=0; v<G->n(); v++)   // Process every edge
    for (w=G->first(v); w<G->n(); w = G->next(v,w))
       Count[w]++;           // Add to v2's prereq count
  for (v=0; v<G->n(); v++)   // Initialize queue
    if (Count[v] == 0)       // Vertex has no prerequisites
      Q->enqueue(v);
  while (Q->length() != 0) { // Process the vertices
    v = Q->dequeue();
    printout(v);             // PreVisit for "v"
    for (w=G->first(v); w<G->n(); w = G->next(v,w)) {
      Count[w]--;            // One less prerequisite
      if (Count[w] == 0)     // This vertex is now free
        Q->enqueue(w);
    }
  }
}
```

数据结构与算法课程组
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End of Section.