Chaotic Behavior and Feedback Control of Magnetorheological Suspension System With Fractional-Order Derivative

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The fractional differential equations of the single-degree-of-freedom (DOF) quarter vehicle with a magnetorheological (MR) suspension system under the excitation of sine are established, and the numerical solution is acquired based on the predictor–corrector method. The analysis of phase trajectory, time domain response, and Poincaré section shows that the nonlinear dynamic characteristics between fractional and integer-order suspension systems are quite different, which proves the superiority of using fractional order to describe the physical properties. By discussing the influence of each parameter on the vibration, the range of parameters to avoid the chaotic vibration is obtained. The variable feedback control is used to control the chaotic vibration effectively.
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Keywords: MR suspension, fractional-order derivative, predictor–corrector method, chaos

1 Introduction

The vehicle suspension system is an important part of the car chassis with a lot of nonlinear factors in its suspension spring, damping, and tire. Due to the complexity of the mutual excitation between the various components and the road, a chaotic vibration state is easily achieved. With the development and progress of automobile technology, people have more and more demands on the comfort and stability of automobile performance [1,2]. The further study of the nonlinear dynamic behavior of the suspension system is needed as the complex chaotic vibration that has significant influence over the performance. Scholars from various fields of research began to pay close attention to the chaotic suspension system: Naik and Singru [3] studied the chaotic vibration of a quarter-car vehicle model with time-delay feedback, Fakhraei et al. [4] discussed the nonlinear dynamic behavior of a heavy articulated vehicle with magnetorheological (MR) dampers, and Abtahi [5] analyzed the chaotic dynamics along with the control of chaos in a half-car model with the semi-active suspension system. In addition, a magnetic flux feedback controller [6] was designed which achieved both excellent ride comfort and road handling characteristics.

Magnetorheological suspension system is a complex system with multiparameter input. The MR damper, which makes up its main structure is developed based on the principle of MR fluid. This fluid can be changed from a Newtonian fluid to a viscoelastic body with high shear yield stress in a millisecond under an applied magnetic field. Chen and Huang [7] showed that the damping effect of the MR suspension has nonlinear characteristics, Turnip et al. [8] studied the control of an MR-damper semi-active suspension with a sixth-order polynomial model, and Liem et al. [9] put forward an MR damper model using self-tuning Lyapunov-based fuzzy approach. The integer differential equation is widely used to describe the MR suspension, which can neither accurately depict the dynamic behavior nor reveal the actual physical characteristics.

Fractional calculus is a mathematical theory to study the characteristics of any order differential and integral operator with its application. It is a generalization of integer calculus to noninteger one in the traditional sense. In the general case, describing the nature by an integer order is only an idealized process. However, sometimes nature can be described by the fractional-order differential more accurately, and the real physical properties of the system are able to be truly reflected [10–13]. The application of fractional order in chaotic systems and the mathematical research of nonlinear dynamical systems are of great significance. For example, the MR fluid is neither an ideal solid nor an ideal fluid, but rather a material between the two. The theory of viscoelasticity is widely used in fractional calculus [10]. A large number of experiments show that the integral equation is more accurate than the differential equation, and the definition of the fractional derivative is a definite integral which is more suitable for describing the characteristics of viscoelastic materials. Therefore, the application of fractional order in the chaotic system and the mathematical research of nonlinear dynamical system are of great significance [11]. MR fluids show low viscosity characteristics under no magnetic field conditions and possess viscoelastic properties of high viscosity and low flow under a strong magnetic field. Therefore, the controllability of the MR fluid makes it possible to realize the continuous variable of the damping force, and to control the magnetic field based on the data collected by the displacement sensor, so as to achieve the active control of the vibration. Additionally, fractional calculus has characteristics of genetic and infinite memory function [14,15]. As the fact that the stress–strain response of MR fluid depends on the memory of time, strain rate and is also related to the load and deformation history, while the fractional-order viscoelastic calculus model can accurately describe the dynamic characteristics of a large number of complex viscoelastic materials in a wide frequency range with fewer parameters [12,16]. It has been proved that the experimental data have a good fit as the kinetic behavior of viscoelastic damping materials is described by fractional calculus [13]. Therefore, the fractional calculus is attracting more and more attention, and also showing a broader engineering application prospects than the integral one [12]. With the development of chaos theory, the study of integer chaotic systems has been extended to fractional chaotic systems: Atan et al. [17] discussed the fractional-order proportional–integral–derivative and its application, and Dimeas et al. [18] verified the fractional-order filters using a reconfigurable fractional-order impedance emulator. In 2002, Kai et al. [19] proposed an estimate-correction numerical

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method for calculating fractional differential equations, i.e., the generalized Adams–Bashforth–Moulton method, which provides a great convenience for the direct calculation of fractional differential equations.

In this paper, the chaotic characteristics of single-degree-of-freedom (DOF) quarter-car MR suspension system under the single-frequency sinusoidal road excitation are analyzed and discussed by introducing the fractional differential equation, which is different from the most of the existing literature. The numerical integration is carried out based on the predictor–corrector method [19,20]; then, the phase trajectories, time domain responses, and Poincaré sections under different fractional orders are used to identify the chaos. By comparing with the dynamic characteristics of the integer-order suspension system, it shows that there are significant differences between the integer and fractional suspension system characteristics, which proves that the use of the fractional suspension system is necessary. In order to control the chaotic vibration of the fractional suspension system, the influence of each parameter of the MR suspension on the vibration is discussed by using the control variable method, and the parameters ranges of the chaos vibration are determined. Furthermore, the variable feedback control method [21] is used, and the chaotic vibration is effectively controlled by introducing the feedback variables.

2 The Definitions of the Fractional Derivatives

In the course of the development of fractional calculus theory, scholars have proposed a number of different definitions of fractional derivatives, and among them, the most widely used is the *Grünwald–Letnikov* derivative and the *Riemann–Liouville* derivative.

First, the general calculus operator which includes both fractional and integer orders is defined [22]

$$ {}_aD_t^q x(t) = \begin{cases} \dfrac{d^q}{dt^q} x(t), & \mathrm{Re}(q) > 0 \\ x^{(n)}(t), & q = n \\ \displaystyle\int_a^t x(t)(d\tau)^{-q}, & \mathrm{Re}(q) < 0 \end{cases} \tag{1} $$

where a is the lower limits of the operation, t is the upper limits, q denotes the calculus operator order which can be any complex number, and when $q = n \in N$ the operator represents the general sense of integer derivative.

When $\mathrm{Re}(q) > 0$, ${}_\alpha D_t^q$ represents fractional derivative, and ${}_\alpha D_t^{-q}$ represents fractional integral. Then, the definition of the fractional derivative can be given according to the operator [14].

The Grünwald–Letnikov derivative is defined as

$$ {}_\alpha D_t^q x(t) = \sum_{k=0}^{m} \frac{x^{(k)}(a)(t-a)^{-q-k}}{\Gamma(n-q+k+1)} + \frac{1}{\Gamma(-q+m+1)} \int_a^t (t-x)^{m-q} x^{(m-1)}(\tau) d\tau \tag{2} $$

The Riemann–Liouville derivative is defined as

$$ {}_\alpha D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{q-n+1}} d\tau \quad (n-1 \le q < n) \tag{3} $$

where, $\Gamma(\cdot)$ represents the Gamma function.

It is proved that the Grünwald–Letnikov derivative is completely equivalent to the Riemann–Liouville derivative for a concrete practical function. For the Riemann–Liouville form definition is the most commonly used definition in the research, therefore, it is used by the fractional differential definition in this paper.

3 The Quarter-Car Suspension System

The factors that affect the vibration state of the vehicle suspension system are mainly found in the suspension spring, suspension damping, and tire elastic damping characteristics. These properties are used according to the corresponding physical components to establish the vehicle suspension model in order to facilitate the analysis of its kinematic characteristics. The vehicle suspension model can be simplified by taking the rotational DOF of pitch and roll, as well as the vertical vibrational DOF of the body and the four wheels into account [23]. To facilitate the calculation, the following assumptions are made:

(1) Left and right wheels are subjected to the road excitation simultaneously, regardless the roll rotational DOF of the body.
(2) It is considered that the vibrational DOFs of the front and rear suspensions are independent of each other. As the negligible difference between the front and rear suspension displacements, the pitching rotational DOF of the body can be ignored.
(3) Due to the slight deformation and damping of the tire, it is considered as rigid, and then, the unsprung mass is ignored.
(4) In the case of driving, since the passenger has zero relative motion with respect to the vehicle and does not provide the driving force for the vibration of the suspension system, the influence of the person on the vehicle is ignored and the mass of the person is included in the suspension.
(5) The damping is considered to only occur in the damper.

In the suspension system, vibration is generated due to the impact of the elastic element. To improve the stability of the vehicle, the damper is installed in parallel with the elastic element in the suspension system to attenuate the vibration. Based on the above assumptions, the suspension model is simplified to a single-DOF quarter car suspension model containing only a spring and damper (as shown in Fig. 1) by extracting the physical characteristics of the suspension system, where m is the spring mass of the quarter car, k is the spring stiffness coefficient, c is the damping coefficient, $f_{(t)}$ is the road excitation function, and x is the displacement of the suspension in the vertical direction relative to the equilibrium position. In this paper, the nonlinear dynamic characteristics of the suspension system based on MR suspension are analyzed. The system of motion differential equations is established according to the D'Alembert principle [24]

$$ m\ddot{x} + k_1(x - f_{(t)}) + k_2(x - f_{(t)})^3 + c_1(\dot{x} - \dot{f}_{(t)}) + c_2(\dot{x} - \dot{f}_{(t)})^3 = 0 \tag{4} $$

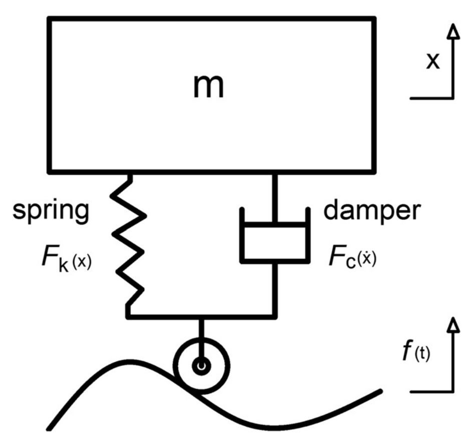

Fig. 1 Single DOF quarter-car suspension model

4 Nonlinear Dynamics Behavior Analysis of Suspension System

Vehicle dynamics is a study of the relationship between the state of movement of the car and time under the excitation. The suspension system has strong nonlinear characteristics; therefore, using MATLAB, the numerical integration of fractional differential suspension system is realized, and the nonlinear dynamic behavior of the suspension system is analyzed.

4.1 Dynamic Characteristics of Integer Suspension System. Define a new variable for the relative displacement as $y = x - f_{(t)}$, where $f_{(t)} = A_0 \sin(\omega t)$, ω is the frequency of the road excitation, and t is the time of the excitation. By substituting into Eq. (4), then, the integer-order differential suspension system under single-frequency sinusoidal excitation can be expressed as

$$\ddot{y} + Ay + By^3 + C\dot{y} + D\dot{y}^3 = A_0\omega^2 \sin(\omega t) \tag{5}$$

where $A = (k_1/m)$, $B = (k_2/m)$, $C = (c_1/m)$, $D = (c_2/m)$. That is the integer differential expression of the single-DOF quarter-car suspension system. In the system given above, the values of the parameters are [24]: $m = 240\,\mathrm{kg}$, $k_1 = 16{,}000\,\mathrm{N\cdot m^{-1}}$, $k_2 = -30{,}000\,\mathrm{N\cdot m^{-3}}$, $c_1 = 250\,\mathrm{N\cdot s\cdot m^{-1}}$, $c_2 = -25\,\mathrm{N\cdot s^3\cdot m^{-3}}$, correspondingly, $A = 66.6667\,\mathrm{s^{-2}}$, $B = -125\,\mathrm{s^{-2}\cdot m^{-2}}$, $C = 1.0417\,\mathrm{s^{-1}}$, $D = -0.1042\,\mathrm{s\cdot m^{-2}}$. The parameters of the single-frequency sinusoidal excitation are as follows: $A_0 = 0.05\,\mathrm{m}$, $\omega = 8\,\mathrm{rad\cdot s^{-1}}$.

Next, define a new variable as to reduce order, and then, the second-order differential equation is transformed into first-order differential equations

$$\begin{cases} \dot{y} = z \\ \dot{z} = A_0\omega^2 \sin(\omega t) - Ay - By^3 - Cz - Dz^3 \end{cases} \tag{6}$$

For the integer-order system with a time step of $h = 0.005$, the numerical integration of the differential equations can be carried out by MATLAB, then, the value of x is obtained by $x = y + f_{(t)}$. The chaotic identification of the system is given by time domain response, phase trajectory, Poincaré section, and Lyapunov exponents [25], as shown in Fig. 2.

It can be seen from the figure that the chaotic vibration will appear under the given road excitation condition. The motion characteristics of the system are as follows: the time domain response is irregularly changed, the phase trajectory is composed of infinite rings, the Poincaré section is represented by an irregular graph which is composed of numerous points in a certain area, meanwhile, the maximum Lyapunov exponent of the system is greater than zero. From these characteristics, it can be determined that under the given single-frequency sinusoidal road excitation, the integer-order dynamical properties of the single DOF quarter car suspension system is manifested as chaotic motion.

4.2 Dynamic Characteristics of Fractional Suspension System. Considering a fractional-order modified suspension system, the conventional derivatives in Eq. (6) are replaced by the fractional derivatives, and then, the system can be written as follows:

$$\begin{cases} \dfrac{d^{q_1} y}{dt^{q_1}} = z \\ \dfrac{d^{q_2} z}{dt^{q_2}} = A_0\omega^2 \sin(\omega t) - Ay - By^3 - Cz - Dz^3 \end{cases} \tag{7}$$

Then, $q_1 = q_2 = q \in (0.5, 1)$ and a time step of $h = 0.01$ is selected. The fractional differential system is discretized by the predictor–corrector method (i.e., the generalized Adams–Bashforth–Moulton method) [19,20], the discretized system is shown below:

$$\begin{cases} y_{n+1} = y_0 + \dfrac{h^{q_1}}{\Gamma(q_1+2)}\left(z^*_{n+1} + \displaystyle\sum_{j=0}^{n} \alpha_{1,j,n+1} z_j\right) \\ z_{n+1} = z_0 + \dfrac{h^{q_2}}{\Gamma(q_2+2)}\Big\{A_0\omega^2 \sin[\omega(n+1)h] - Ay^{*3}_{n+1} - By^{*3}_{n+1} - Cz^*_{n+1} - Dz^{*3}_{n+1} \\ \qquad + \displaystyle\sum_{j=0}^{n} \alpha_{2,j,n+1}\left[A_0\omega^2 \sin(\omega jh) - Ay_j - By_j^3 - Cz_j - Dz_j^3\right]\Big\} \end{cases} \tag{8}$$

where

$$\begin{cases} y^*_{n+1} = y_0 + \dfrac{1}{\Gamma(q_1)} \displaystyle\sum_{j=0}^{n} \beta_{1,j,n+1} z_j \\ z^*_{n+1} = z_0 + \dfrac{1}{\Gamma(q_2)} \displaystyle\sum_{j=0}^{n} \beta_{2,j,n+1} z_j \left[A_0\omega^2 \sin(\omega jh) - Ay_j - By_j^3 - Cz_j - Dz_j^3\right] \end{cases} \tag{9}$$

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, & j = 0 \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1} & 1 \le j \le n \\ 1\,, & j = n+1 \end{cases} \tag{10}$$

$$\beta_{i,j,n+1} = \frac{h^{q_i}}{q_i}\left[(n-j+1)^{q_i} - (n-j)^{q_i}\right], \quad 0 \le j \le n, \quad i = 1,2 \tag{11}$$

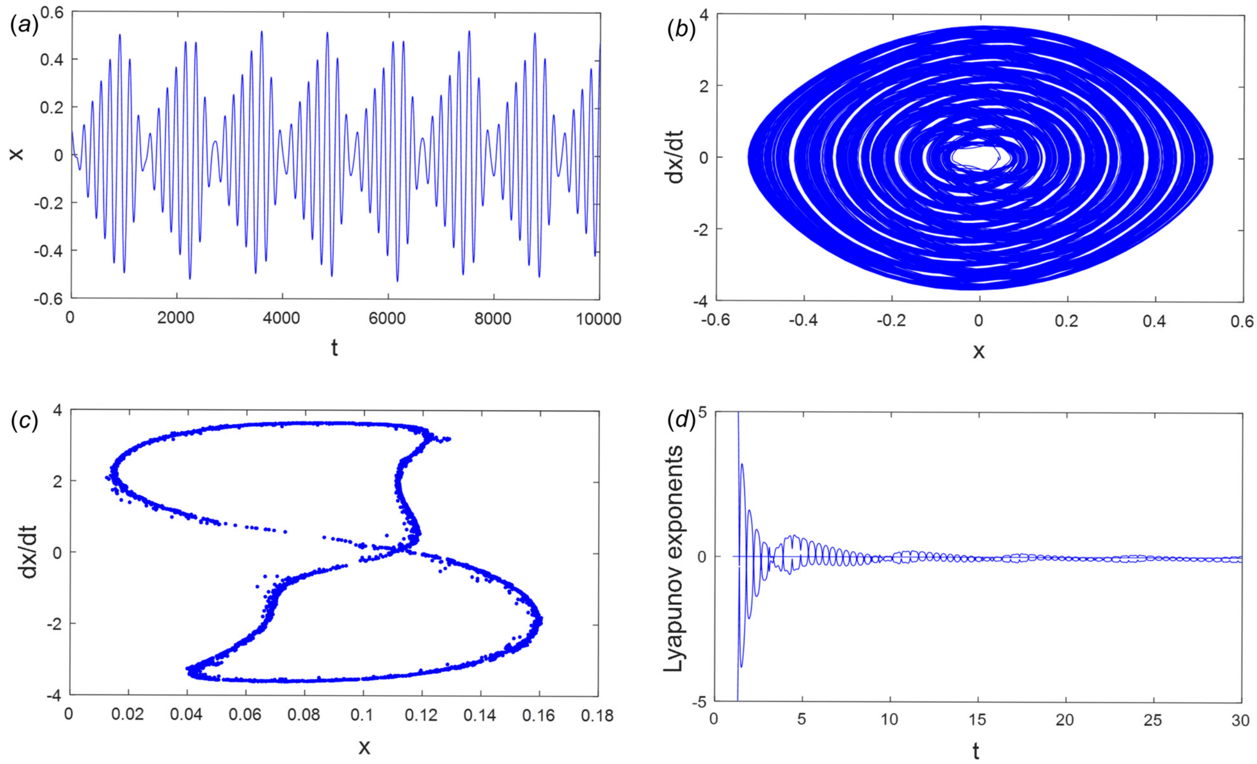

Fig. 2 Time domain response, phase trajectory, Poincaré section, and Lyapunov exponents of the integer-order differential suspension system: (*a*) time domain response, (*b*) phase trajectory, (*c*) Poincaré section, and (*d*) Lyapunov exponents

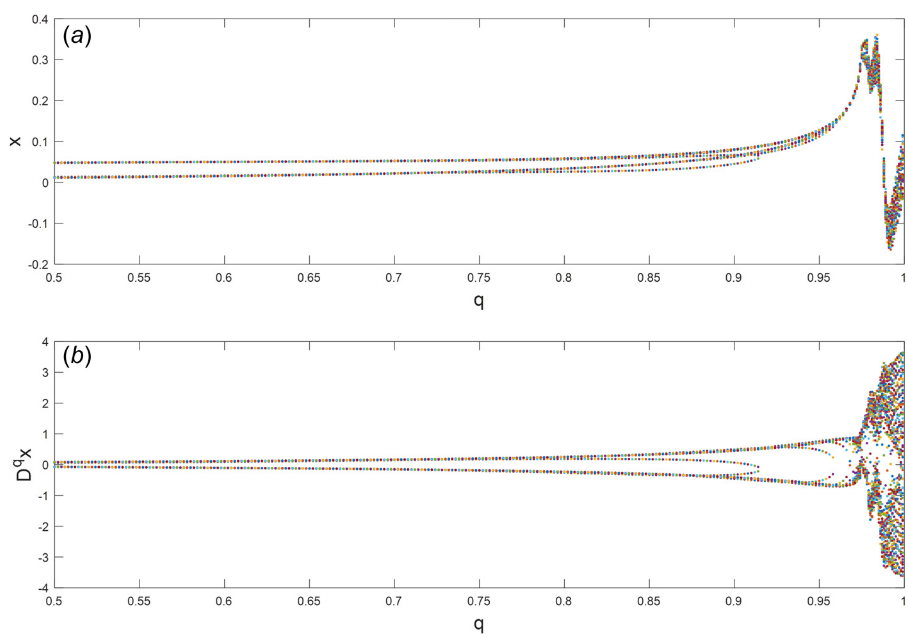

Fig. 3 Bifurcation diagram of the suspension system: (*a*) *x*–*q* chaotic bifurcation diagram of suspension system and (*b*) $D^q x$–*q* chaotic bifurcation diagram of suspension system

By solving Eq. (9) with MATLAB, the dynamic response of the suspension differential system can be obtained when the fractional order is q and under the condition of single frequency sinusoidal excitation.

The $x-q$ chaotic bifurcation diagram is obtained by continuously changing the fractional order q as the abscissa parameter and taking the x value corresponding to the point on the Poincaré section as the ordinate parameter when the system is stable, as shown in Fig. 3. It can be seen that when the fractional-order q is in the range of 0.500–0.974, the points on Poincaré section are countable, and the main mode of motion is period-doubling motion. When q is in the range of 0.976–1.000, the points on

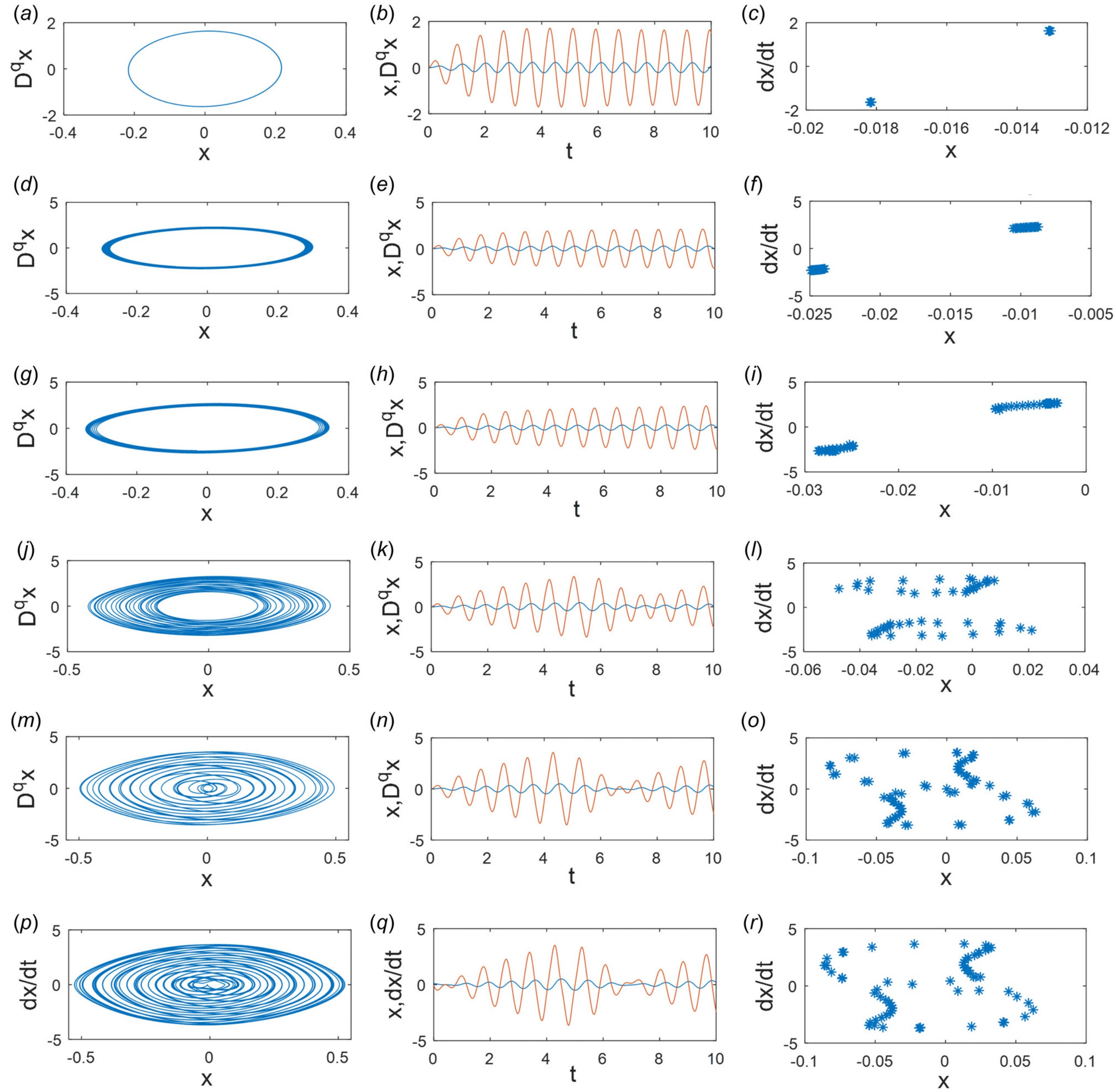

Fig. 4 The phase trajectory, time domain response, and Poincaré section of MR suspension under different fractional orders (=0.600,0.974, 0.976, 0.990, 0.997,1.000): (*a*) phase trajectory $q = 0.60$, (*b*) time domain response $q = 0.60$, (*c*) Poincaré section $q = 0.60$, (*d*) phase trajectory $q = 0.974$, (*e*) time domain response $q = 0.974$, (*f*) Poincaré section $q = 0.974$, (*g*) phase trajectory $q = 0.976$, (*h*) time domain response $q = 0.976$, (*i*) Poincaré section $q = 0.976$, (*j*) phase trajectory $q = 0.990$, (*k*) time domain response $q = 0.990$, (*l*), (*m*) phase trajectory $q = 0.997$, (*n*) time domain response $q = 0.997$, (*o*) Poincaré section $q = 0.997$, (*p*) phase trajectory $q = 1.000$, (*q*) time domain response $q = 1.000$, and (*r*) Poincaré section $q = 1.000$

Poincaré section are uncountable which indicates that chaotic motion occurs. It is observed that there is a similar motion behavior when the system changes within a certain range. Therefore, only certain fractional orders are taken as the representative values, and the nonlinear dynamic behaviors of the suspension system under these orders are analyzed.

According to the region which divided by the bifurcation diagram, q is taken as 0.600, 0.974, 0.976, 0.990, 0.997, and 1.000. Then chaotic identification of fractional suspension system is carried out by phase trajectory, time domain response, and Poincaré section, as shown in Fig. 4.

Figures 4(*a*)–4(*f*) correspond to the phase trajectory, the time domain response, and the Poincaré section when the fractional order is taken as 0.600 and 0.974. The image shows that the phase trajectories are single- and multiple-phase loops, the time domain responses show periodic changes, and the Poincaré sections are composed of countable points. It can be seen that the nonlinear dynamic behavior of the fractional differential suspension system will be periodic motion when q is in the range 0.500–0.974.

Figures 4(*g*)–4(*h*) correspond to the fractional order $q = 0.976$. The phase trajectory begins to appear in a large number of phase loops which is different from the periodic motion, the time domain response begins to break the previous periodicity, and there are countless points in the Poincaré section that make up the irregular pattern. According to the above, the system begins to exhibit chaotic vibration.

Figures 4(*j*)–4(*o*) correspond to the phase trajectory, the time domain response, and the Poincaré section when the fractional

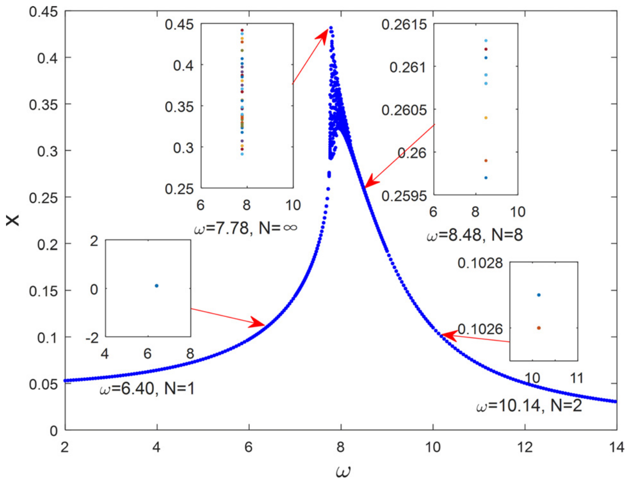

Fig. 5 $x-\omega$ **bifurcation diagram of the suspension system**

order is taken as 0.990 and 0.997. The phase trajectory is randomly distributed in a certain area and is never closed, the time domain response changes irregularly, and there are uncountable points in the Poincaré section that do not constitute a closed set of points. It can be seen that when q is in the range 0.976–0.999, the dynamic behavior of the suspension system will be chaotic vibration.

Figures 4(*p*)–4(*r*) correspond to the integer-order system, whose Poincaré section is taken from the same section as the fractional-order system. It indicates that the results of the fractional-order system converge into those of the integer-order system when the fractional order approaches 1.

From the earlier analysis, it shows that as the fractional-order q approaches 1, the system will show chaotic characteristics which are similar to the integer-order system; however, when the fractional order q is deviated from 1, completely different dynamics characteristics will appear. It can be shown that the use of integer differential equation to describe the dynamic characteristics of the system in some cases will deviate far away from the nature of the system and produce a greater error [12]. Therefore, it is accurate and reasonable to describe the vibration characteristics of the suspension system by using the fractional differential equation.

5 Chaos Control

The performance of the vehicle suspension system is directly related to the driving experience. The vibration reduction performance of the suspension system directly determines both stability and comfort. In addition, the long-term chaotic vibration will impact the dynamic load which will accelerate the fatigue of the parts resulting in safety risks. It is necessary to control this harmful chaotic vibration and to avoid potential safety risks.

In order to control the chaotic vibration of the fractional differential suspension system, $q = 0.980$ is taken, whose behavior is chaotic.

5.1 Chaos Control Based on System Parameter Adjustment. Under the given road conditions, the excitation frequency of the road surface is positively correlated with the vehicle speed. Therefore, in order to study the corresponding vehicle speed range when the chaotic vibration occurs, the frequency can be set as the abscissa parameter to show the chaotic characteristics on the bifurcation diagram, which is shown as Fig. 5.

According to the frequency bifurcation diagram, where N represents the number of points on Poincaré section, it can be seen that the stable periodic motion occurs in the vehicle at the low-speed state. With the increase in speed, however, the behavior of periodic motion is exacerbated. When the vehicle is driven at medium

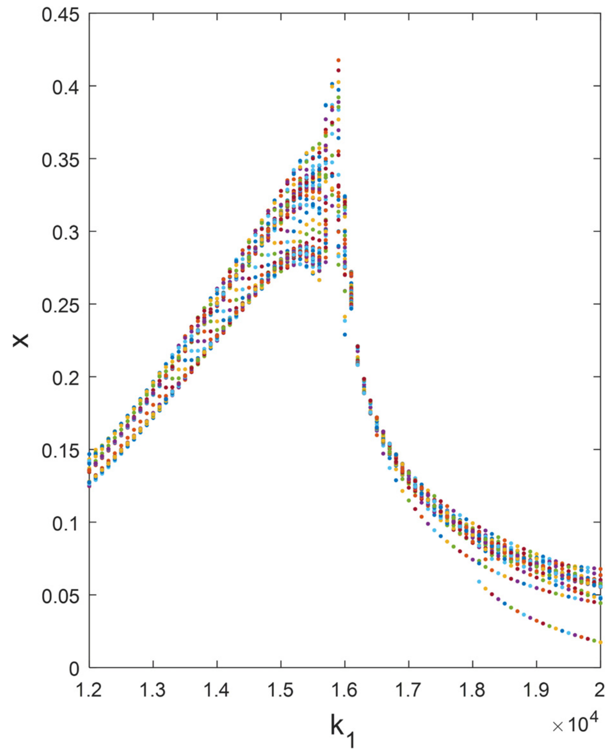

Fig. 6 $x-k_1$ **bifurcation diagram of the suspension system**

speed corresponding to the excitation frequncy be $\omega = 7.78\,\mathrm{rad/s}$, the motion of the suspension system will rapidly enter the chaotic vibration state and show complex dynamic characteristics. With the further acceleration, the chaotic vibration is alleviated and gradually steps into the period-doubling motion. Therefore, in the low-speed driving state, the suspension will not exhibit chaotic vibration, which has a key role in controlling the chaotic vibration and taking further measures. To achieve the aim of chaos control, first, the suspension parameters are adjusted. This makes the critical frequency of vibration high enough so as to ensure that the road excitation frequency generated in the safe driving speed range, is less than the critical frequency. Second, chaos control can be realized by using the active suspension to introduce variable feedback for road excitation. In the following, the road excitation frequency is taken as $\omega = 7.78\,\mathrm{rad/s}$ which is when the initial severe chaotic vibration is caused to analyze.

The elastic and damping elements in the car suspension system play a vital role in improving vehicle ride comfort, operational stability, and reducing damage to parts caused by dynamic loads. Therefore, it is necessary to analyze and adjust the parameters of the elastic and damping elements in the suspension system so as to provide the theoretical reference of the range of parameters for the design of the suspension.

5.1.1 Spring Parameters. Parameters in a reasonable range with reference value should be taken for analysis

Let $k_1 \in [12000, 20000]$, $k_2 \in [-50000, -20000]$.

(1) k_1 Bifurcation Diagram:
As shown in Fig. 6, period-doubling motion shows when k_1 falls within the range of [12,000,15,700) and (16,190,20,000]. At the point $k_1 = 15,700$, there are uncountable and irregular points on the Poincaré section, chaotic vibration exhibits. In the range of [15,700,16,190], the motion behavior of the system is dominated by chaotic vibration, and the number of points on Poincaré section when $k_1 = 16,191$ becomes countable again, which indicates that the period-doubling motion shows again.

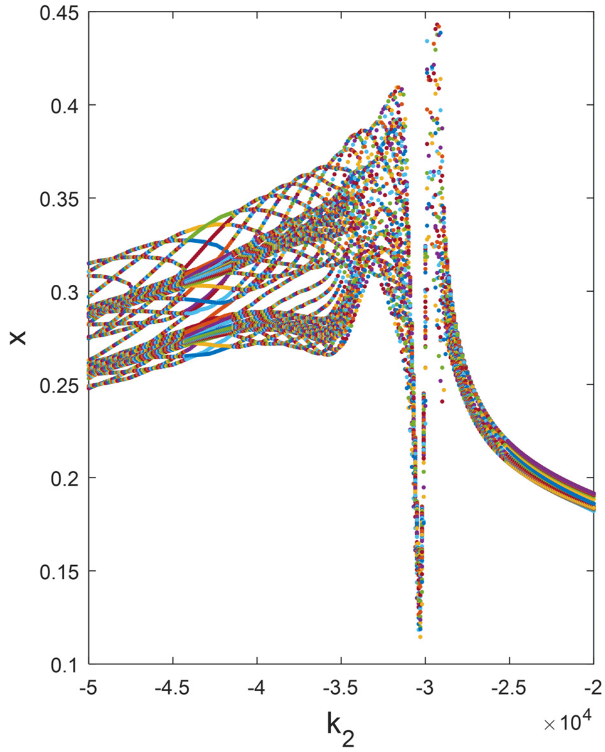

Fig. 7 $x-k_2$ **bifurcation diagram of the suspension system**

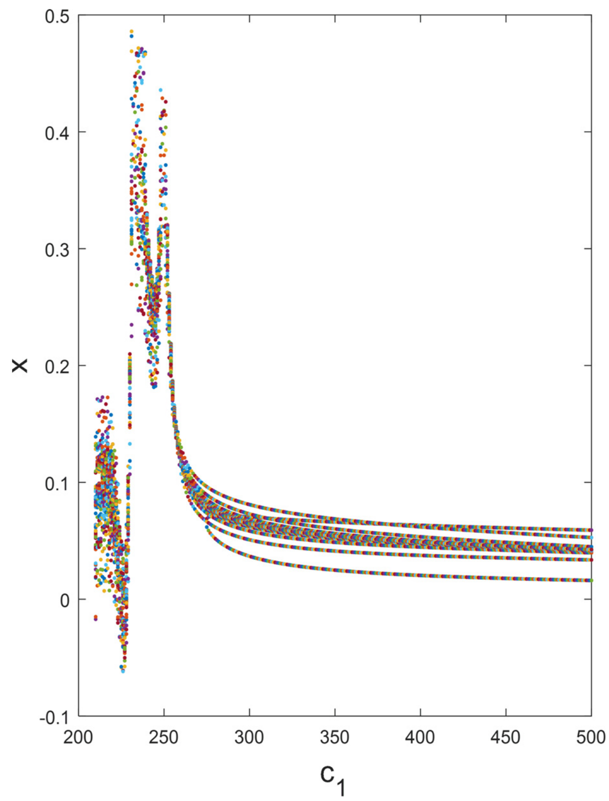

Fig. 8 $x-c_1$ **bifurcation diagram of the suspension system**

(2) k_2 *Bifurcation Diagram*:
As shown in Fig. 7, the main dynamic behavior is chaotic vibration when k_2 is in the range of [−50,000, −28,800]. As for the range of [−28,800, −20,000], the chaotic motion is transformed into period-doubling motion, and the points in the bifurcation diagram become countable.

5.1.2 Damper Damping Parameters. In order to make the damping value reasonable, values are taken from the significant intervals $c_1 \in [210, 500]$, $c_2 \in [-29, -5]$ for analysis.

(1) c_1 *Bifurcation Diagram*:
As shown in Fig. 8, it is observed that the chaotic vibration occurs in the interval of [210,253). At the point $c_1 = 253$, the chaotic motion begins to change to period-doubling motion and the uncountable points set in the bifurcation diagram changes into countable points set. This indicates that the dynamic behavior at the range of (253,500] is period-doubling motion.

(2) c_2 *Bifurcation Diagram*:
As shown in Fig. 9, the dynamic behavior of the system in the range of [−29, −23) is manifested as chaotic vibration which changes to a period-doubling motion at $c_2 = -23$.

According to the above analysis, the system can be separated from the chaotic vibration state and stay in the period-doubling motion state when the parameters satisfy $k_1 \in [12,000, 15,700) \cup (16,190, 20,000]$, $k_2 \in [-28,800, -20,000]$, $c_1 \in (253, 500]$ and $c_2 \in [-23, -5)$. The given value intervals can provide a theoretical reference for parameter selection of suspension system design. In order to verify the effect of chaos control, the parameters $k_1 = 13,000$, $k_2 = -25,000$, $c_1 = 260$, $c_2 = -20$ are selected to test of dynamic characteristics of the suspension system.

As shown in Fig. 10, under the condition of the selected parameter, the phase trajectory is a regular periodic phase ring, its time domain response is stable and regular, and there are countable, discrete points on the Poincaré section. Therefore, it indicates that the dynamic behavior of the suspension system is period-doubling

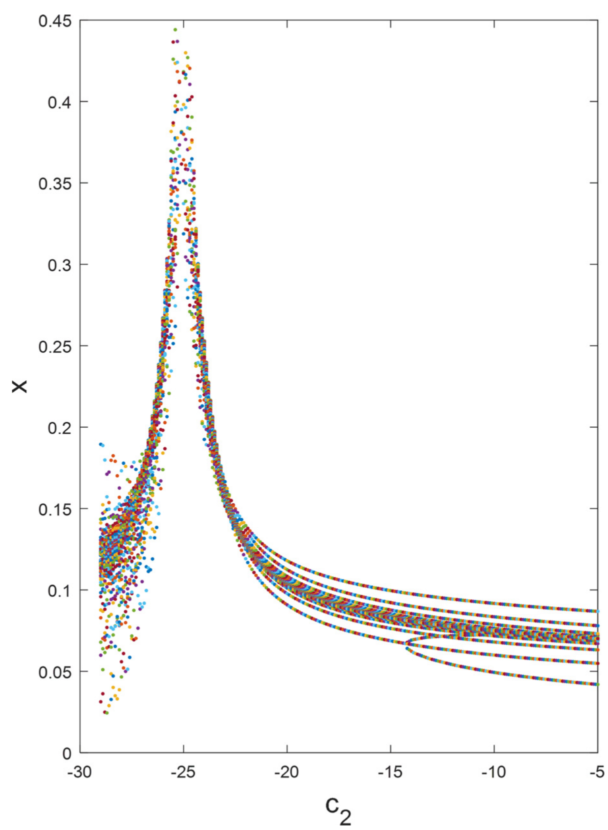

Fig. 9 $x-c_2$ **bifurcation diagram of the suspension system**

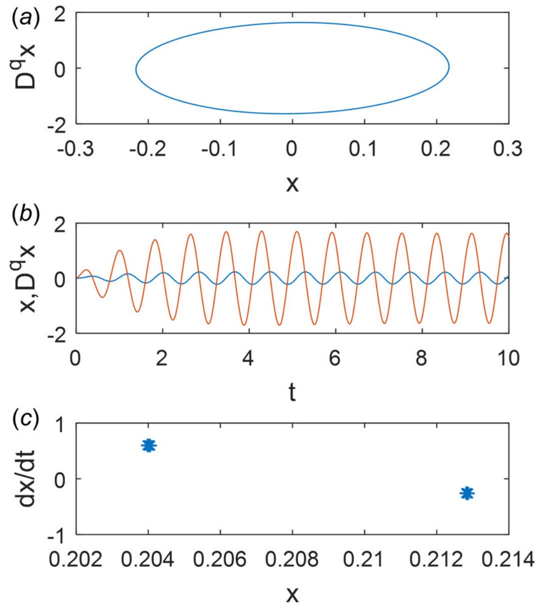

Fig. 10 Chaos control based on system parameter adjustment: (*a*) phase trajectory (controlled), (*b*) time domain response (controlled), and (*c*) Poincare section (controlled)

motion, that is, the chaotic vibration has been effectively controlled.

5.2 Chaos Control Based on Variable Feedback. In order to make the internal oil of MR damper magnetized, the coil can be controlled by the computer to induce magnetism, so that the damping coefficient can be changed to adjust the effect of the shock absorber. By collecting data from the displacement sensor, the magnetic field can be changed with the controlling current which directly impacts the damping coefficient and controls the chaotic vibration.

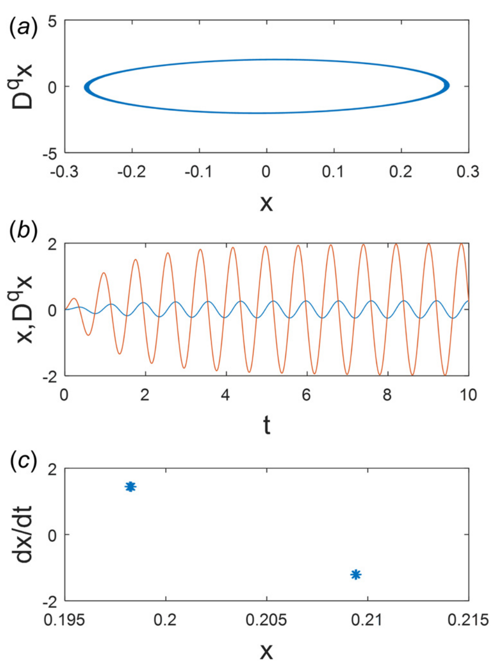

Fig. 11 Chaos control based on variable feedback: (*a*) phase trajectory controlled, (*b*) time domain response controlled, and (*c*) Poincare section controlled

In order to study the effect of introducing the variable feedback [21] on the nonlinear motion behavior of the vehicle suspension system, the feedback variables are added to the fractional-order differential suspension system in Eq. (8) which is rewritten as

$$\begin{cases} \dfrac{d^{q_1}y}{dt^{q_1}} = z - K_1 y \\ \dfrac{d^{q_2}z}{dt^{q_2}} = A_0\omega^2\sin(\omega t) - Ay - By^3 - Cz - Dz^3 - K_2 z \end{cases} \tag{12}$$

with parameters of the single-frequency sinusoidal road excitation $f_{(t)} = A_0\sin(\omega t)$ of $A_0 = 0.05\,\mathrm{m}$ and $\omega = 7.78\,\mathrm{rad}\cdot\mathrm{s}^{-1}$.

The appropriate feedback coefficient is selected, and the fractional differential equations are solved by MATLAB based on the predictor–corrector method. The state of the system after the feedback control is shown in Fig. 11, which indicates that the chaotic-phase trajectory of the original system is controlled and shows the periodic trajectory. The time domain response is controlled to a stable periodic state. Additionally, the Poincare section consists of discrete points which indicate that the system has been controlled and shows single periodic motion. This provides a theoretical basis for the chaos control of suspension system based on the displacement feedback controller.

6 Conclusion

This paper investigated the chaotic characteristics of the MR suspension system with the fractional-order derivative. The analysis of Lyapunov exponential spectrum, phase trajectory, time domain response, and Poincaré section indicates that the fractional-order and integer-order system are quite different. When the fractional order approaches integer order 1, their dynamic behaviors are similar, however, when the fractional order is reduced to 0.974 and below, a doubling period motion is generated that is completely different from the chaotic behavior of the integer order. It is proved that the fractional order can describe the dynamic characteristics of the system more comprehensively than the integer order. Two methods are used to control the chaotic vibration of the fractional MR suspension system. A variable control method is used to study the influence of each parameter of the system on the vibration and the ranges of parameters without chaotic vibration under normal driving speed are obtained. The variable feedback control method is also used to introduce a feedback variable into the active suspension, which has an excellent control effect on chaotic vibration.

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