Data Structures & Algorithms

《数据结构与算法》课程组
重庆大学计算机学院

16 SHORTEST PATH ALGORITHMS

Outline

16.1 Shortest Path Problems

16.2 Single Source Shortest Paths

16.3 All-Pairs Shortest Paths

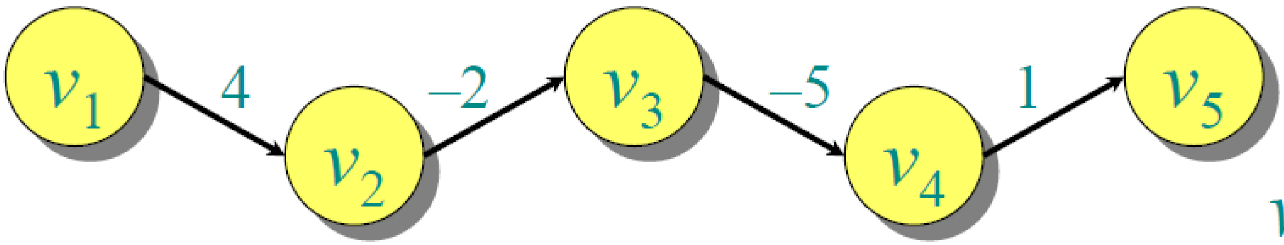
16.1 Shortest Path Problems

Paths in Graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \cdots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}) .$$

Example:



$w(p) = -2$

Shortest Paths

A ***shortest path*** from u to v is a path of minimum weight from u to v. The ***shortest-path weight*** from u to v is defined as

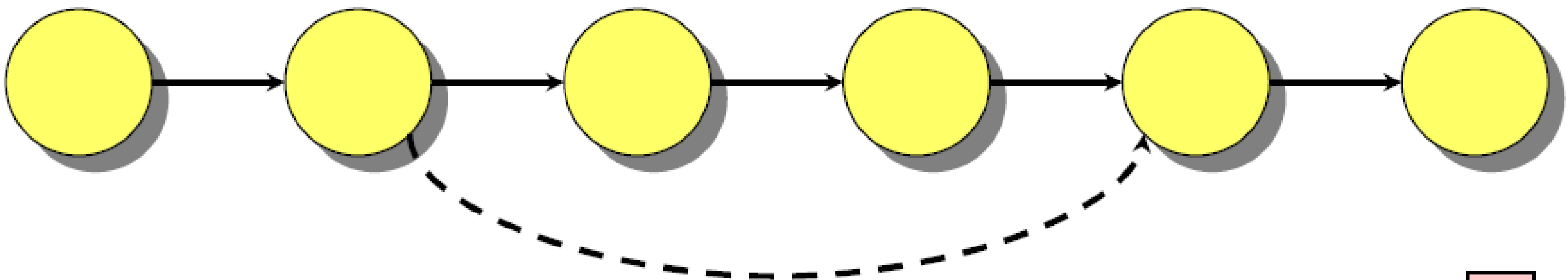
$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}.$$

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Optimal Sub-Structure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:



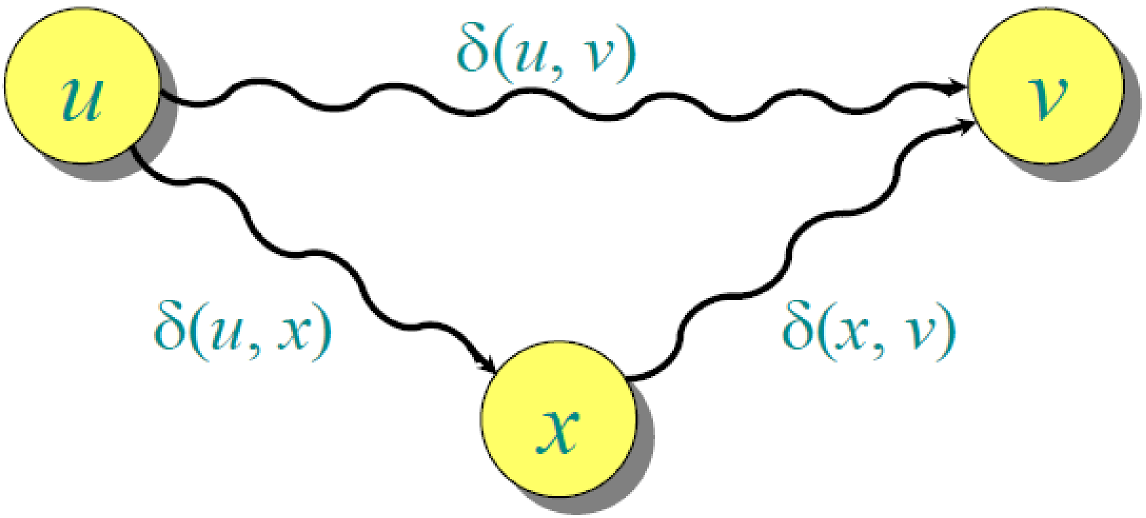
∎

Triangle Inequality

Theorem. For all $u, v, x \in V$, we have

$$\delta(u, v) \leq \delta(u, x) + \delta(x, v).$$

Proof.

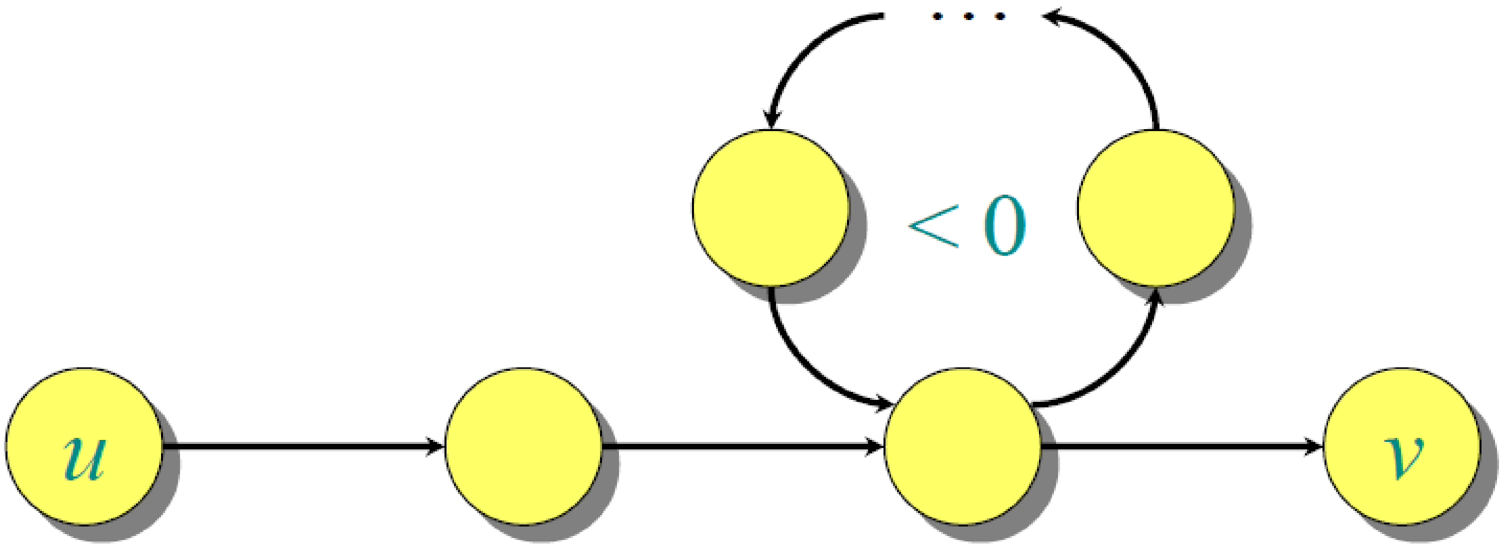


$\square$

Well-Defininedness of SP

If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:



16.2 Single-Source Shortest Paths

Single-Source Shortest Paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

Single-Source Shortest Paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.

Dijkstra’s Algorithm

Dijkstra, Edsger Wybe

• Legendary figure in computer science;
•1930.5.11~2002.8.6
• Supports teaching introductory computer courses without computers (pencil and paper programming)

• Supposedly wouldn’t (until recently) read his e-mail; so, his staff had to print out messages and put them in his box.

Single-Source Shortest Paths

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.

IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra's Algorithm

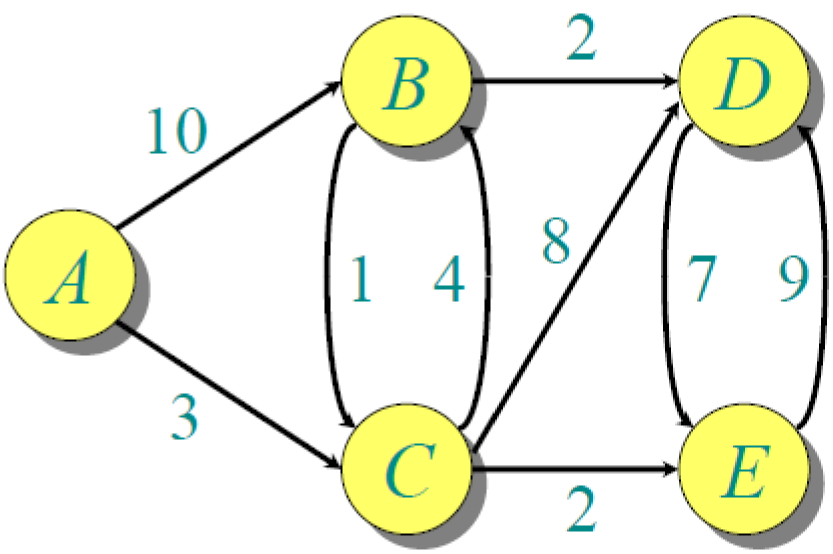
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ ▷ Q is a priority queue maintaining $V - S$
while $Q \neq \varnothing$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$
        for each $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
                then $d[v] \leftarrow d[u] + w(u, v)$

relaxation step

Implicit DECREASE-KEY

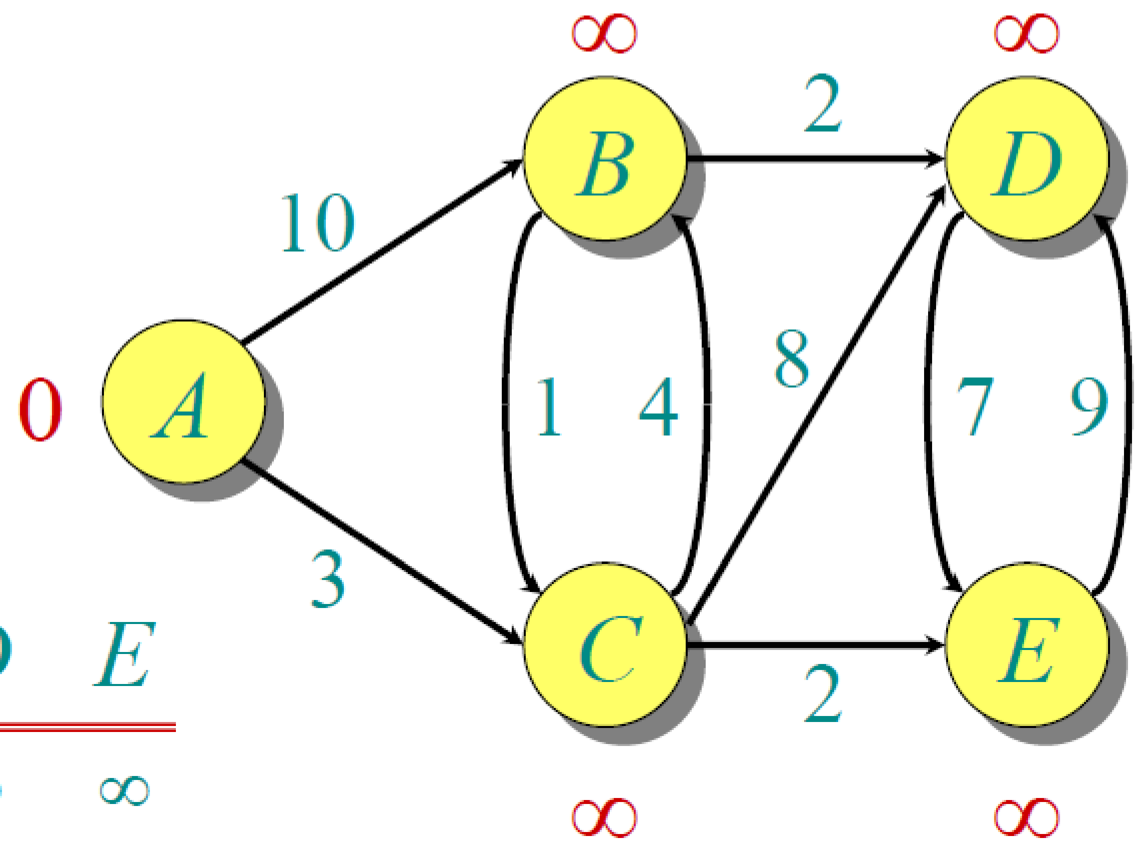
Example

Graph with nonnegative edge weights:



Example

Initialize:

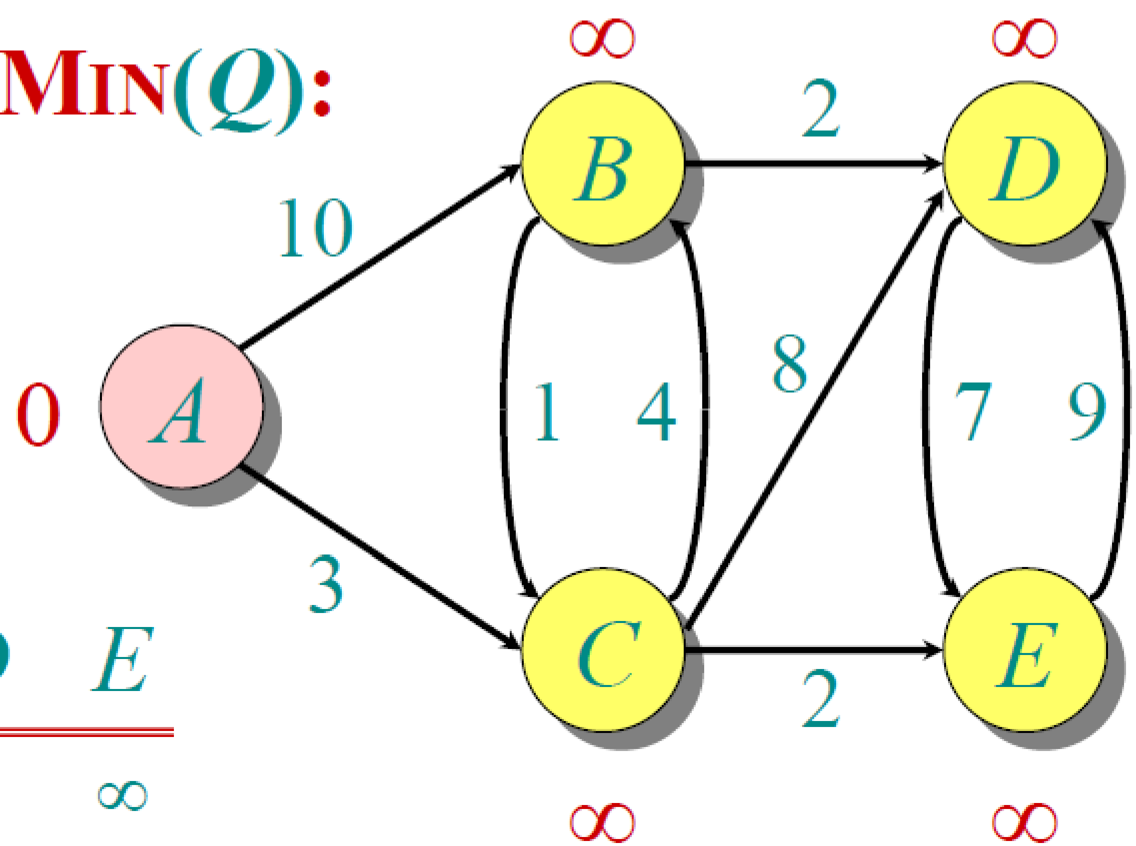


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: {}

Example

"A" $\leftarrow$ EXTRACT-MIN(Q):

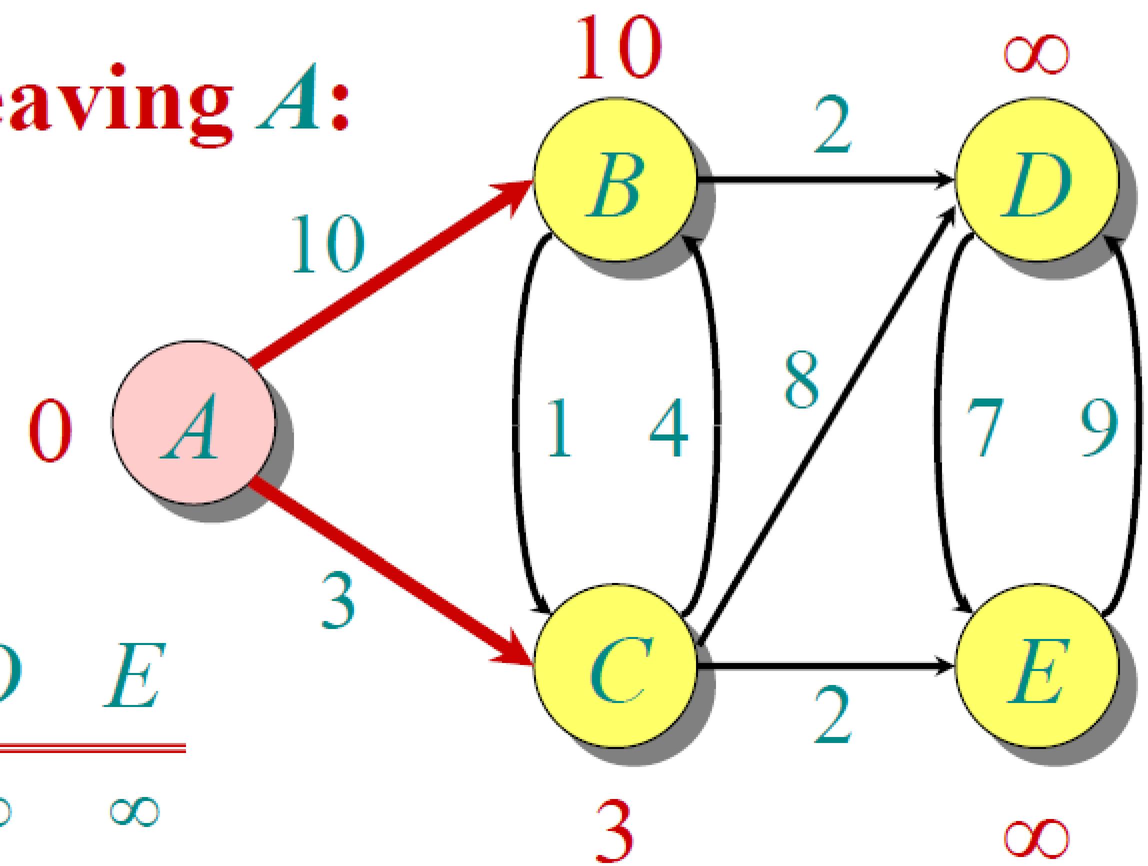


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: { A }

Example

Relax all edges leaving *A*:

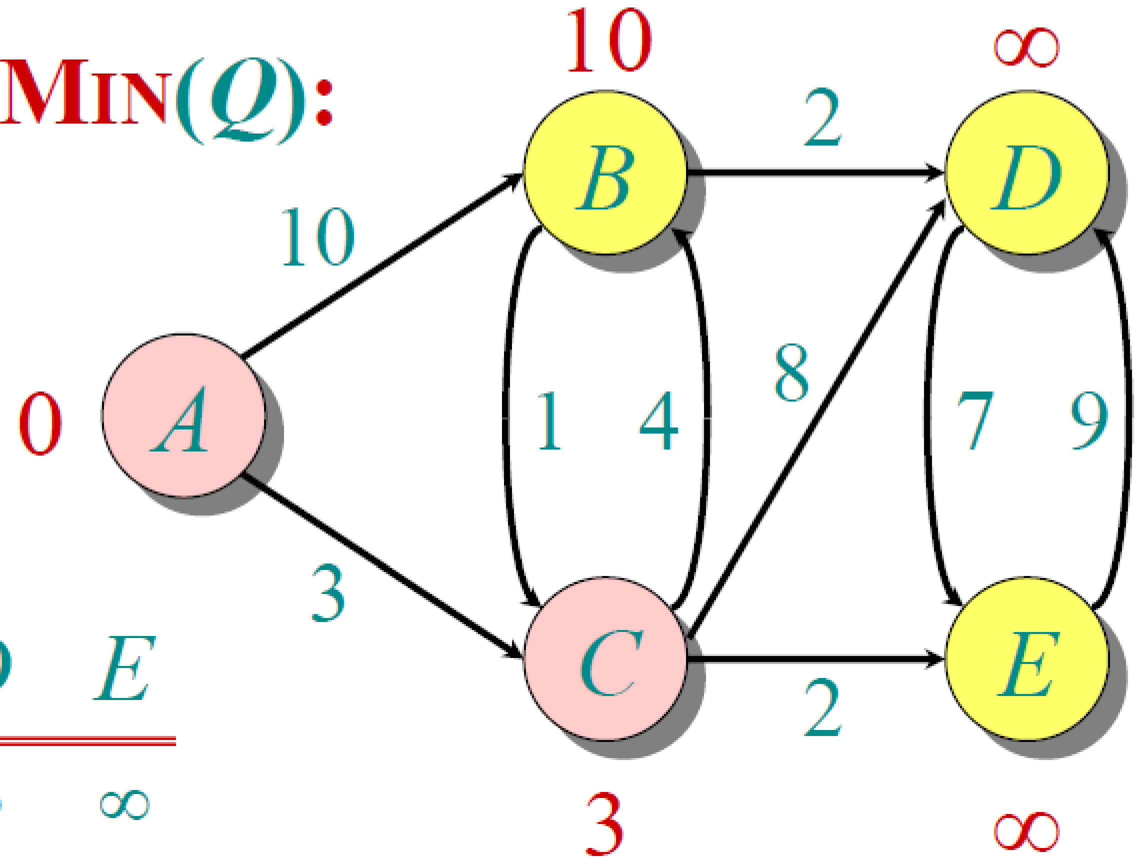


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–

S: { A }

Example

"C" $\leftarrow$ **EXTRACT-MIN(Q):**

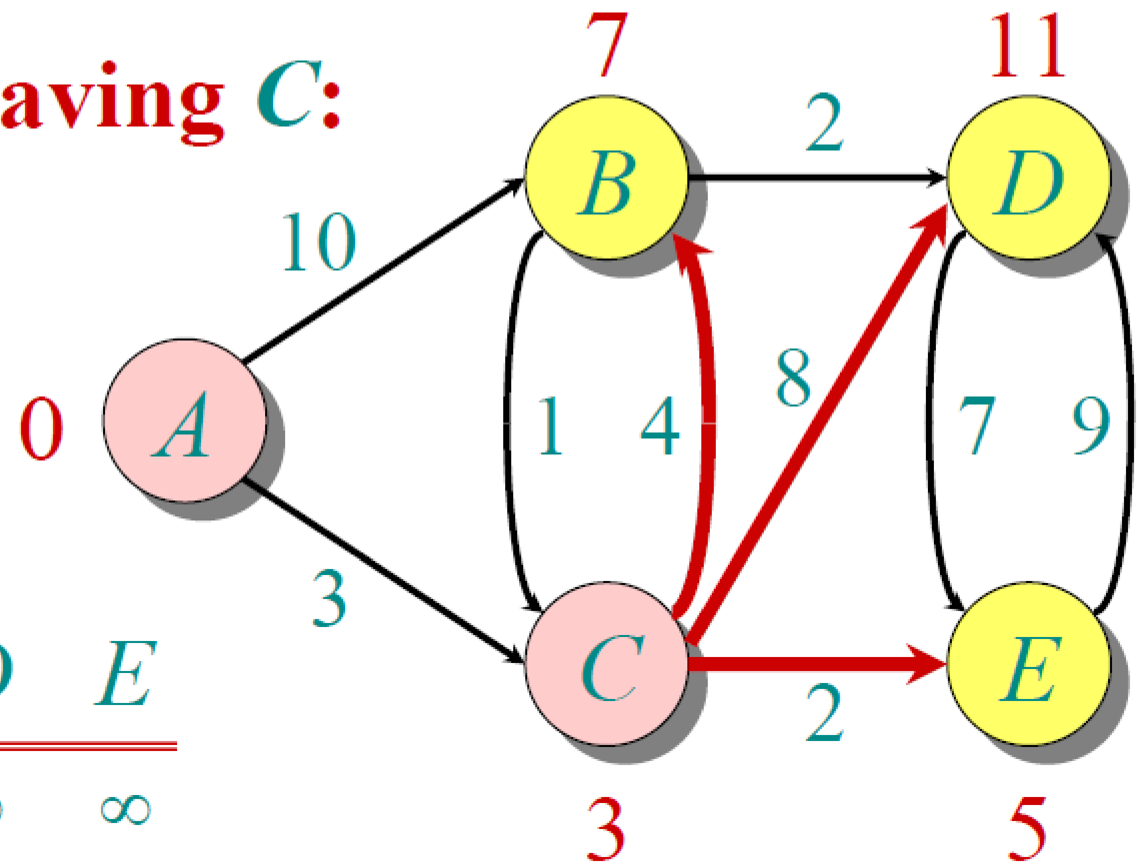


Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	–	–

S: { A, C }

Example

Relax all edges leaving *C*:

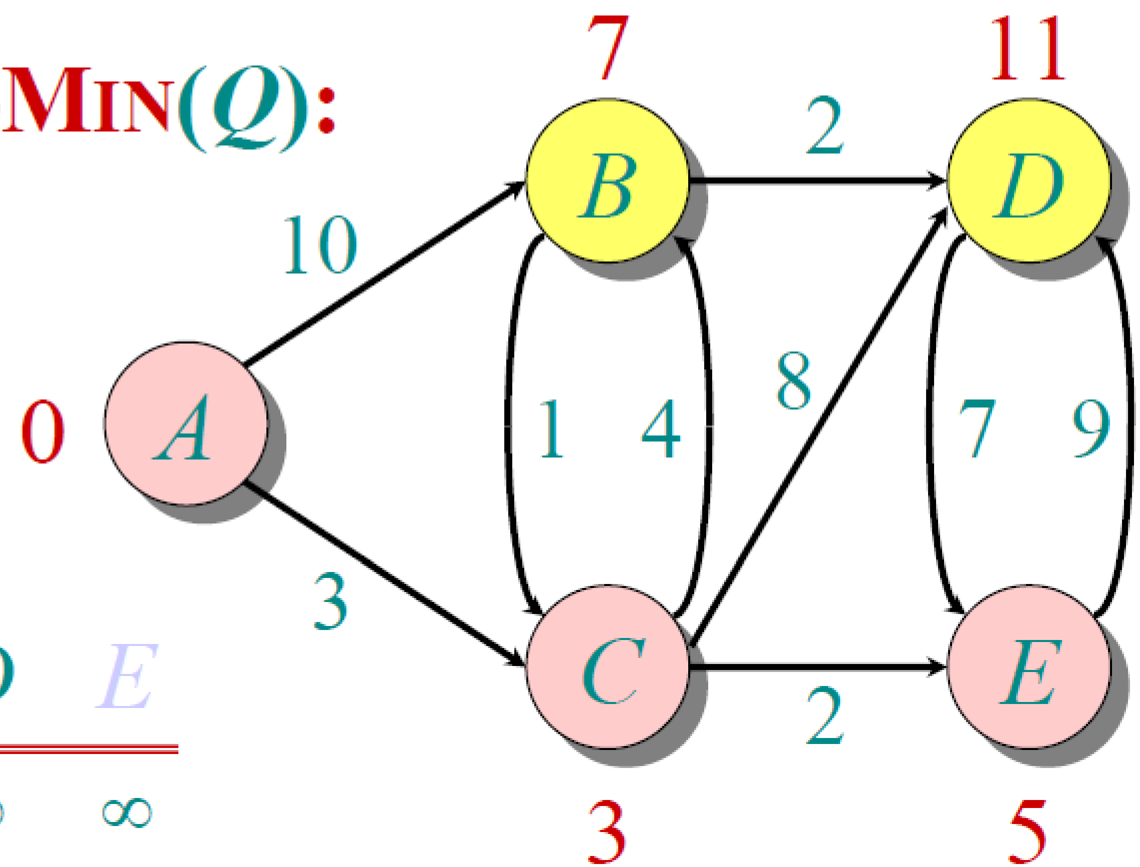


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { *A*, *C* }

Example

"E" $\leftarrow$ **EXTRACT-MIN(Q):**

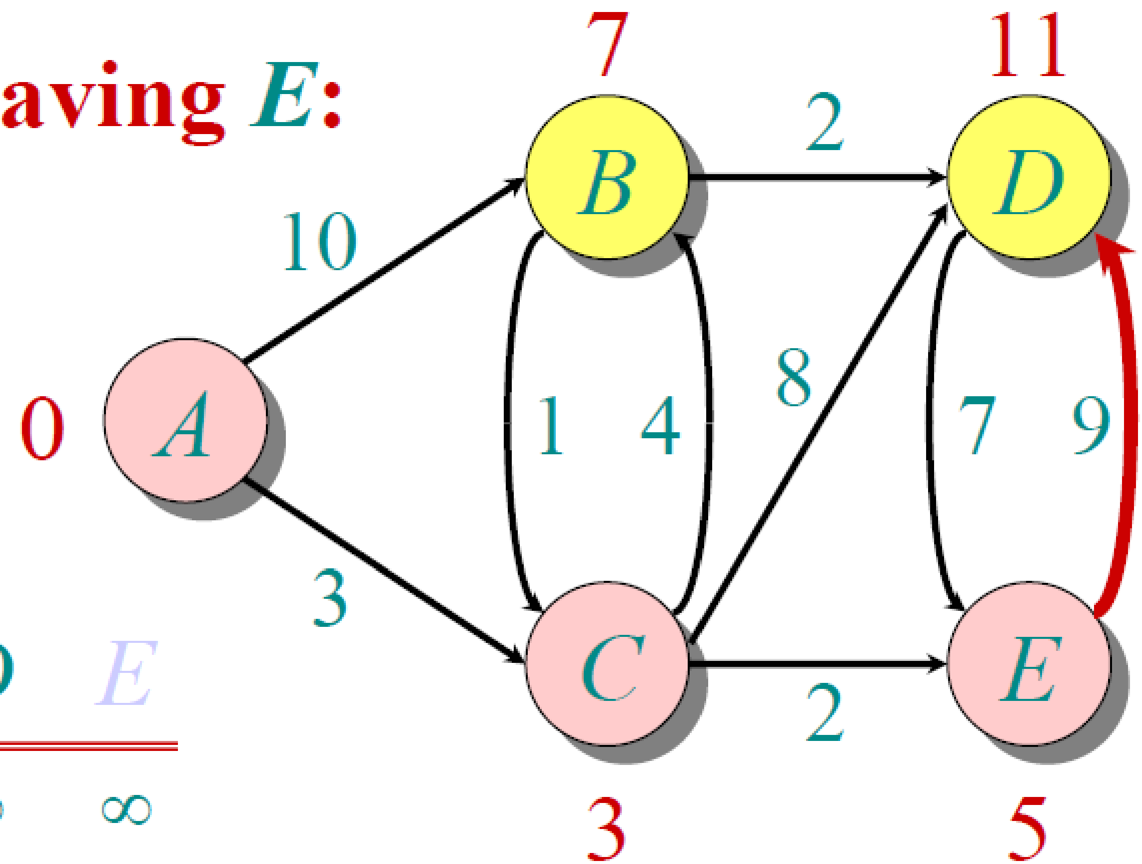


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { A, C, E }

Example

Relax all edges leaving *E*:

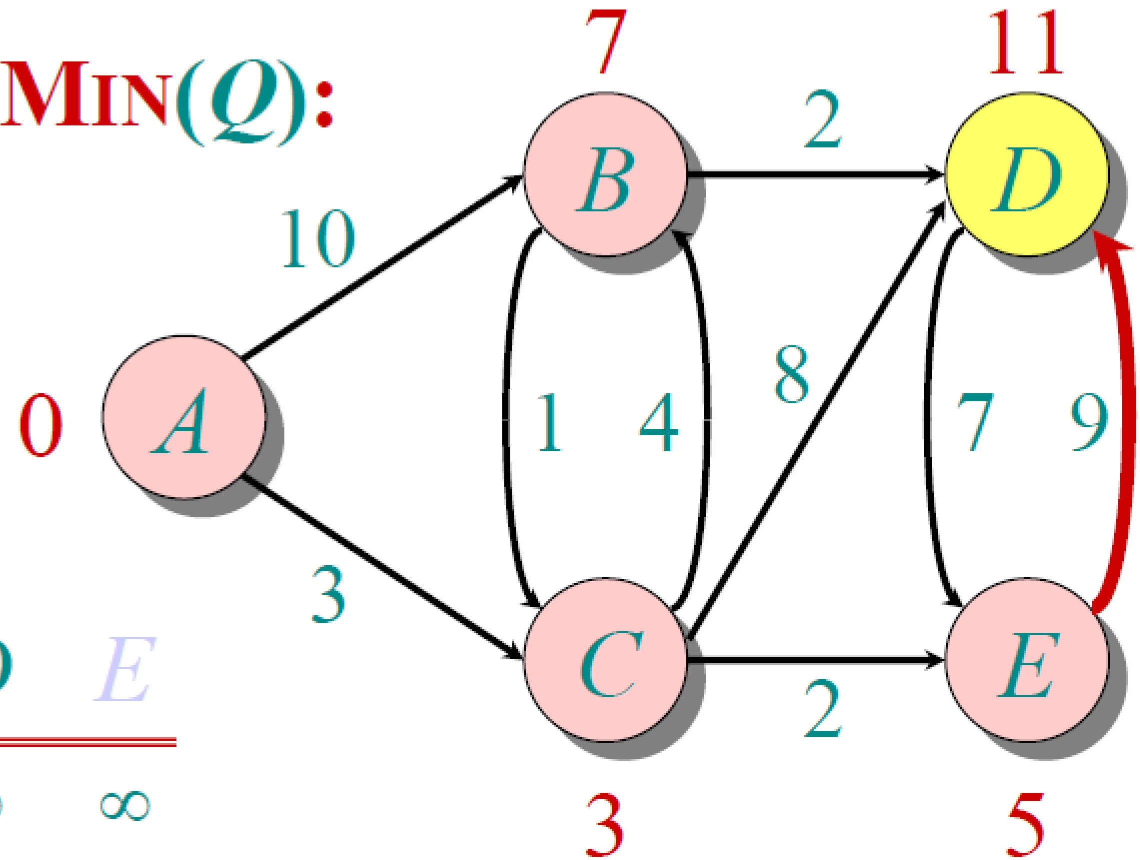


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E }

Example

"*B*" ← **Extract-Min(*Q*):**

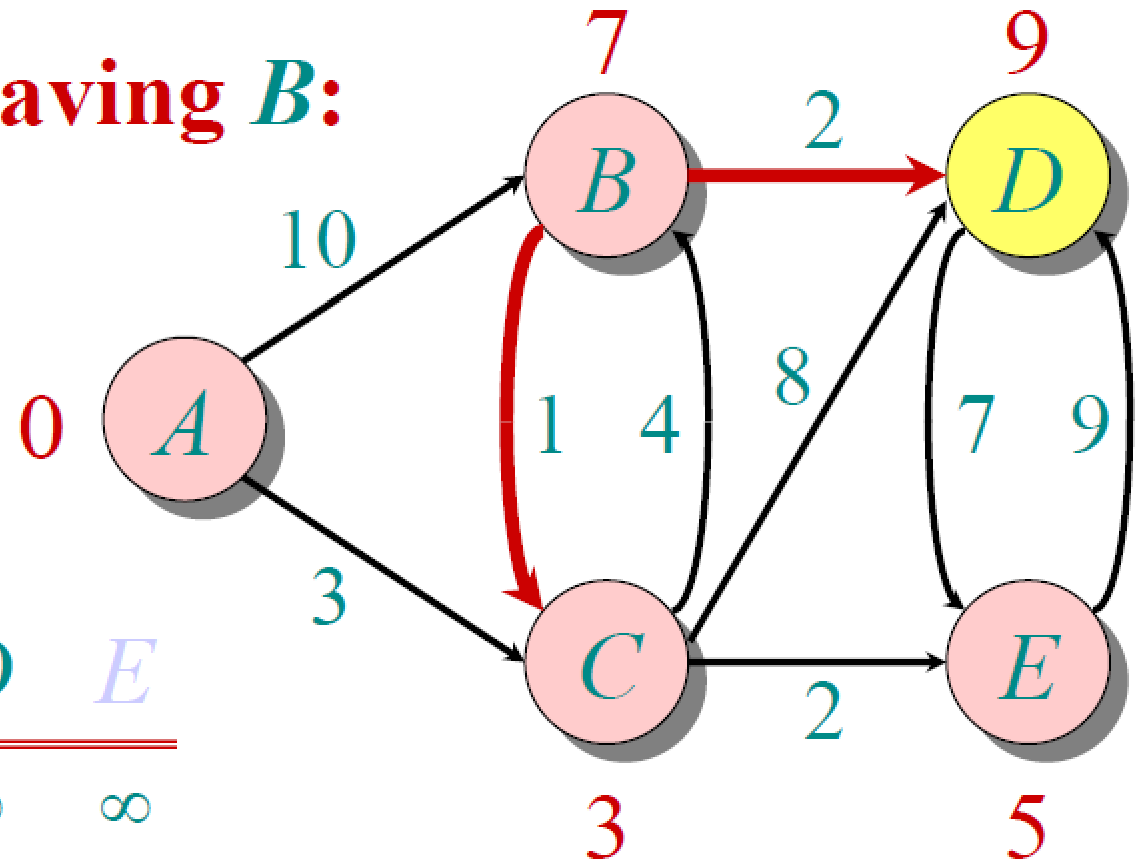


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

Example

Relax all edges leaving *B*:

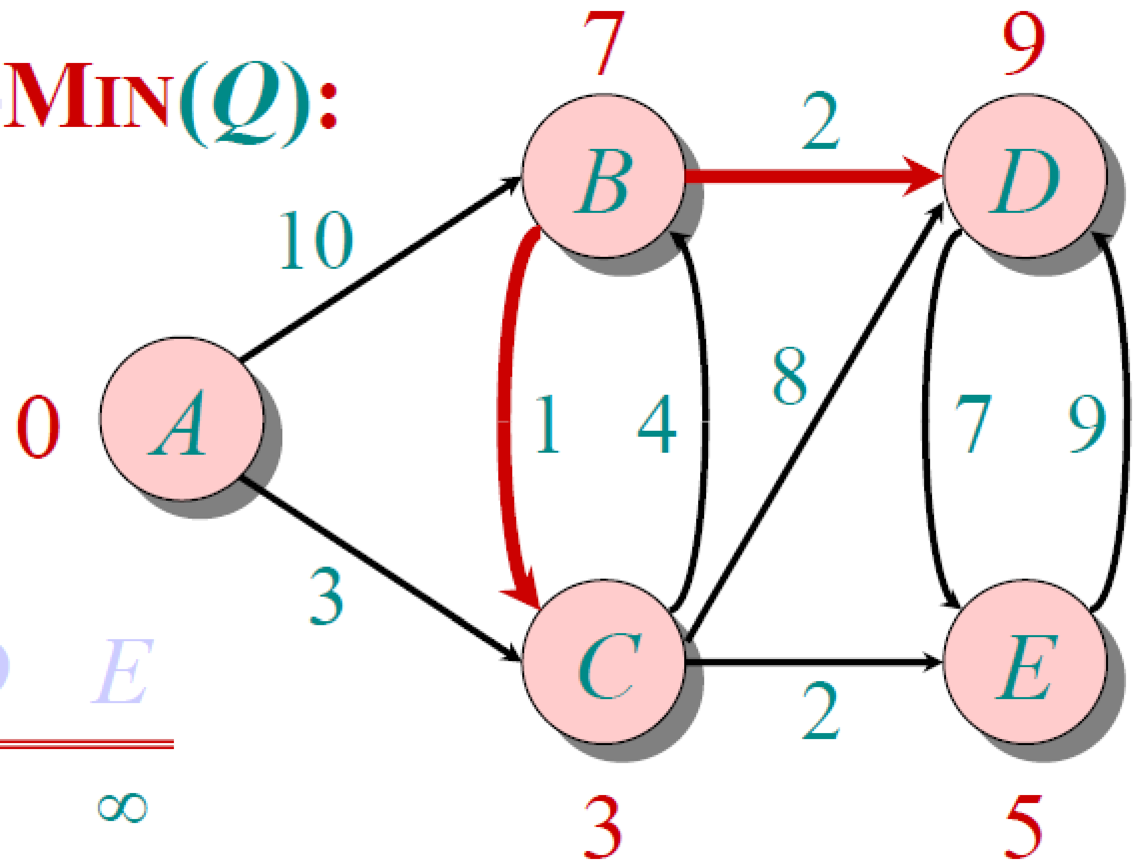


Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B }

Example

"D" $\leftarrow$ EXTRACT-MIN(Q):



Q:	A	B	C	D	E
	$\boxed{0}$	∞	∞	∞	∞
		10	$\boxed{3}$	∞	∞
		7		11	$\boxed{5}$
		$\boxed{7}$		11	
				$\boxed{9}$	

S: { A, C, E, B, D }

Correctness-I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Correctness-I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Proof. Suppose not. Let v be the first vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,

$d[v] < \delta(s, v)$	supposition
$\leq \delta(s, u) + \delta(u, v)$	triangle inequality
$\leq \delta(s,u) + w(u, v)$	sh. path $\leq$ specific path
$\leq d[u] + w(u, v)$	v is first violation

Correctness-II

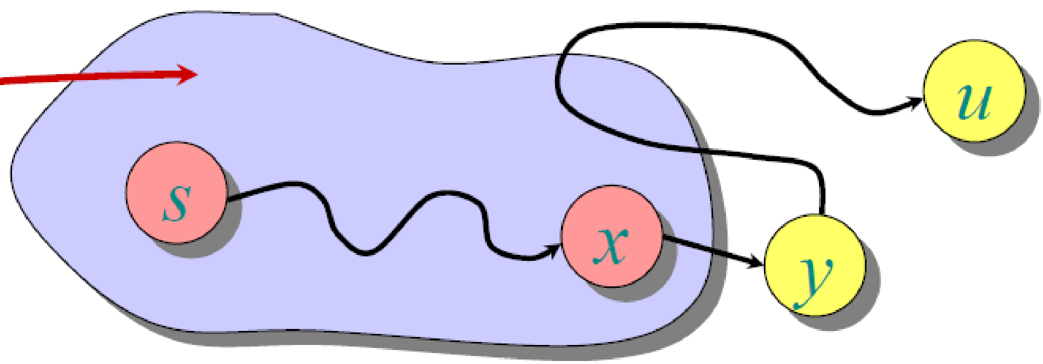
Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Correctness-II

Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

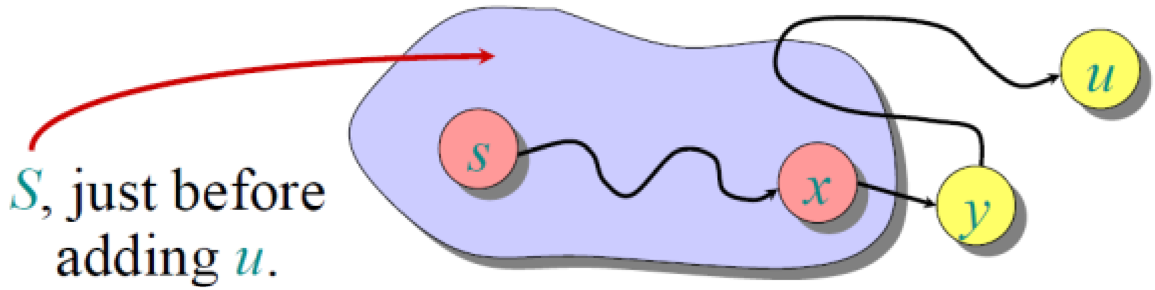
Proof. It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S. Suppose u is the first vertex added to S for which $d[u] \neq \delta(s, u)$. Let y be the first vertex in $V - S$ along a shortest path from s to u, and let x be its predecessor:

S, just before adding u.



Correctness-II

Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.



Since u is the first vertex violating the claimed invariant,

$$d[x] = \delta(s, x).$$

Since subpaths of shortest paths are shortest paths

$$d[y] = \delta(s, x) + w(x, y) = \delta(s, y) \le \delta(s, u) \le d[u].$$

Non-negative weight

But, $d[u] \le d[y]$ by our choice of u

$$\Rightarrow \quad d[y] = \delta(s, u) = d[u]$$

Record the Shortest paths

- The algorithm described above does not record the shortest paths. It can not output the shortest paths.

- The algorithm can be modified to record the paths by building an array **pre[]**. If **pre[i]=k**, this represents that the shortest path from v_0 to v_i is $(v_0,..,v_k,v_i)$. It is easy to prove that if $(v_0,..,v_k,v_i)$ is the shortest path from v_0 to v_i, the path $(v_0,..,v_k)$ is the shortest path from v_0 to v_k. We can output the shortest path from v_0 to v_i by outputting the shortest path from v_0 to v_k recursively and vertex to v_i

- **The pre[i]** is initiated by v_0. It is updated while the minimum distance is modified.

Dijkstra's Algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$ $prev[v] \leftarrow s$
$S \leftarrow \emptyset$
$Q \leftarrow V$ $\triangleright$ Q is a priority queue maintaining $V - S$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$
          $prev[v] \leftarrow u$

relaxation step

Analysis of Dijkstra

$|V|$ times

while $Q \neq \emptyset$

  do $u \leftarrow$ EXTRACT-MIN(Q)

  $S \leftarrow S \cup \{u\}$

$degree(u)$ times

  for each $v \in Adj[u]$

    do if $d[v] > d[u] + w(u, v)$

      then $d[v] \leftarrow d[u] + w(u, v)$

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

$$\text{Time} = \Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$$

Analysis of Dijkstra

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

Dijkstra for Unweighted Graphs

Suppose $w(u, v) = 1$ for all $(u, v) \in E$. Can the code for Dijkstra be improved?

Dijkstra for Unweighted Graphs

- Use a simple FIFO queue instead of a priority queue.
- ***Breadth-first search***

$$
\begin{array}{l}
\textbf{while } Q \neq \varnothing \\
\quad \textbf{do } u \leftarrow \textsc{Dequeue}(Q) \\
\qquad \textbf{for each } v \in Adj[u] \\
\qquad\quad \textbf{do if } d[v] = \infty \\
\qquad\qquad \textbf{then } d[v] \leftarrow d[u] + 1 \\
\qquad\qquad\qquad \textsc{Enqueue}(Q, v)
\end{array}
$$

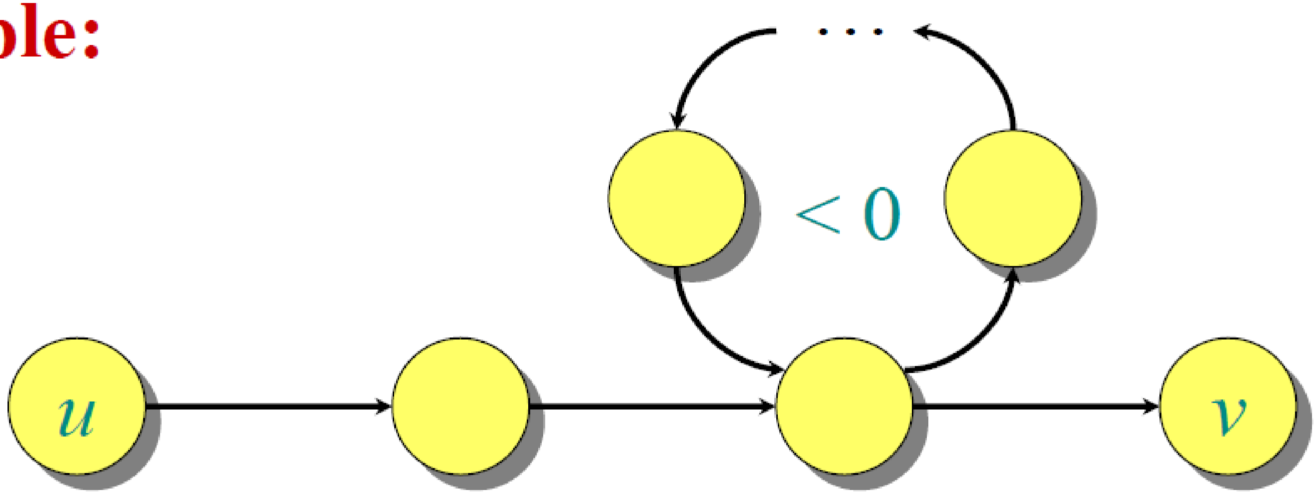
Analysis: Time $= O(V + E)$.

Bellman-Ford Algorithm

Negative-Weight Cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Bellman-Ford algorithm: Finds all shortest-path lengths from a ***source*** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.

Bellman-Ford Algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$

initialization

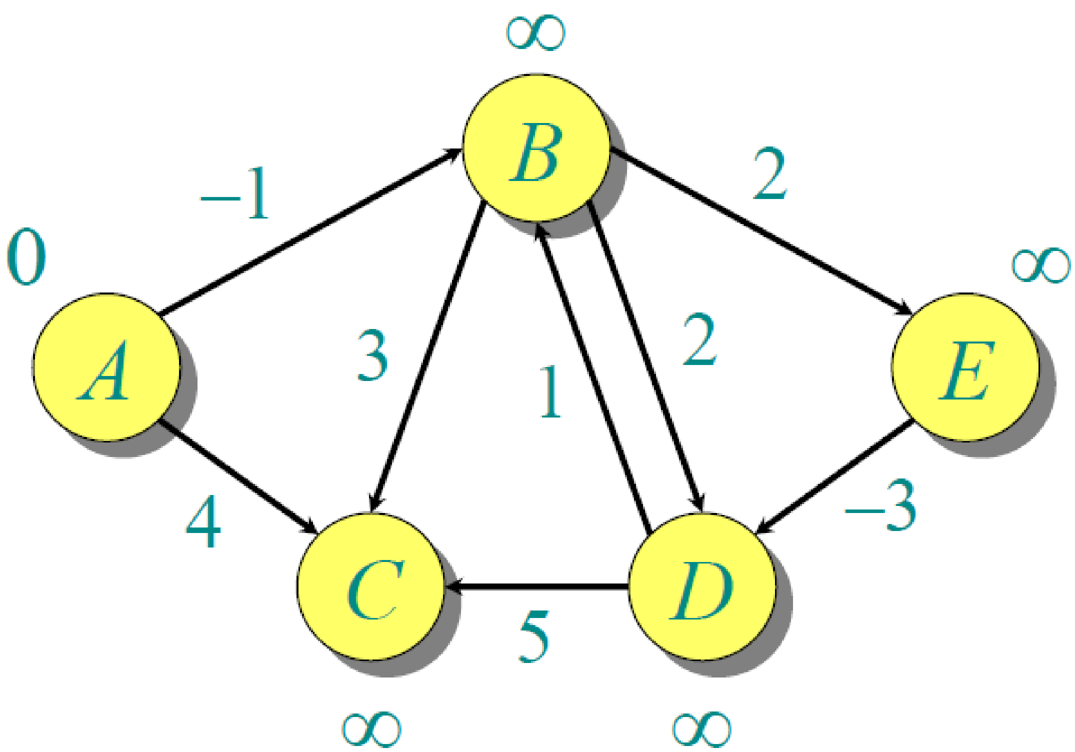
for $i \leftarrow 1$ **to** $|V| - 1$
    do for each edge $(u, v) \in E$
        do if $d[v] > d[u] + w(u, v)$
            then $d[v] \leftarrow d[u] + w(u, v)$

relaxation step

for each edge $(u, v) \in E$
    do if $d[v] > d[u] + w(u, v)$
        then report that a negative-weight cycle exists

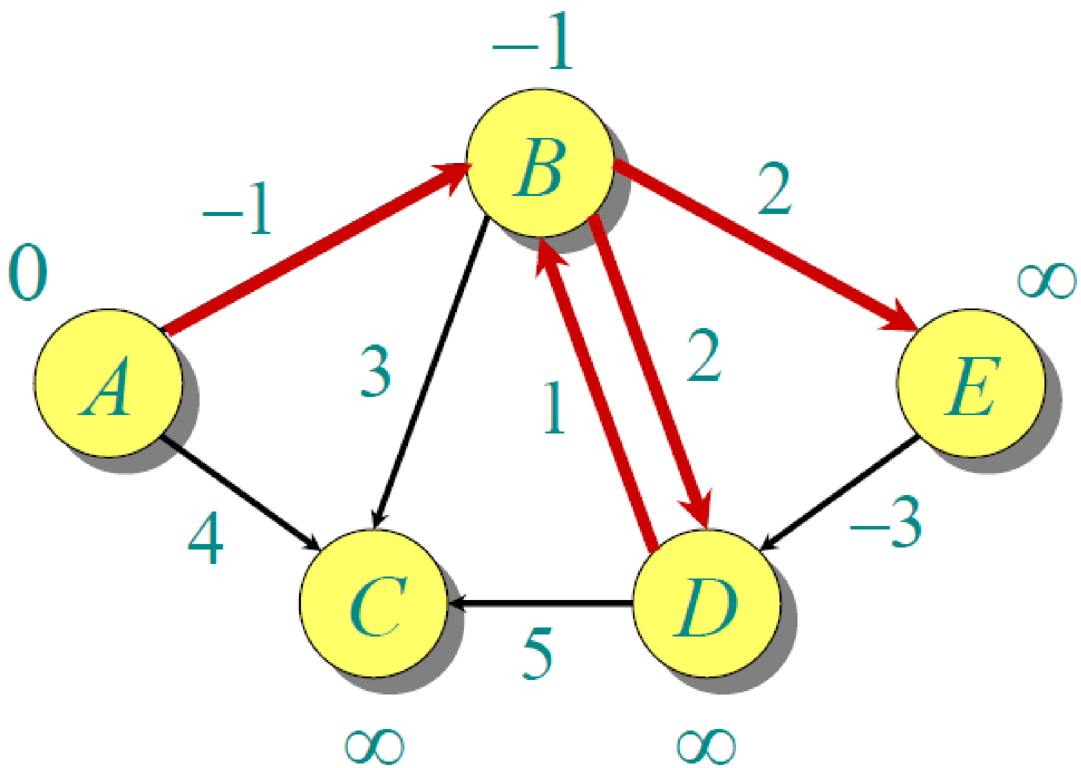
At the end, $d[v] = \delta(s, v)$. Time $= O(VE)$.

Example



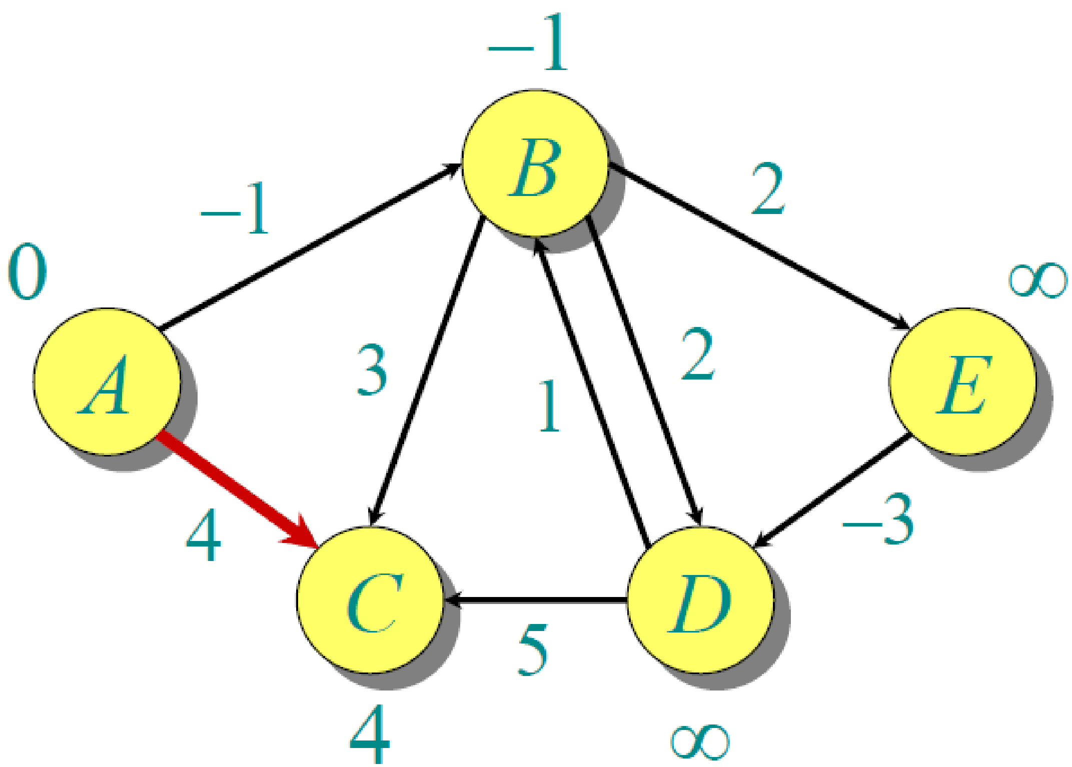
A	B	C	D	E
0	∞	∞	∞	∞

Example



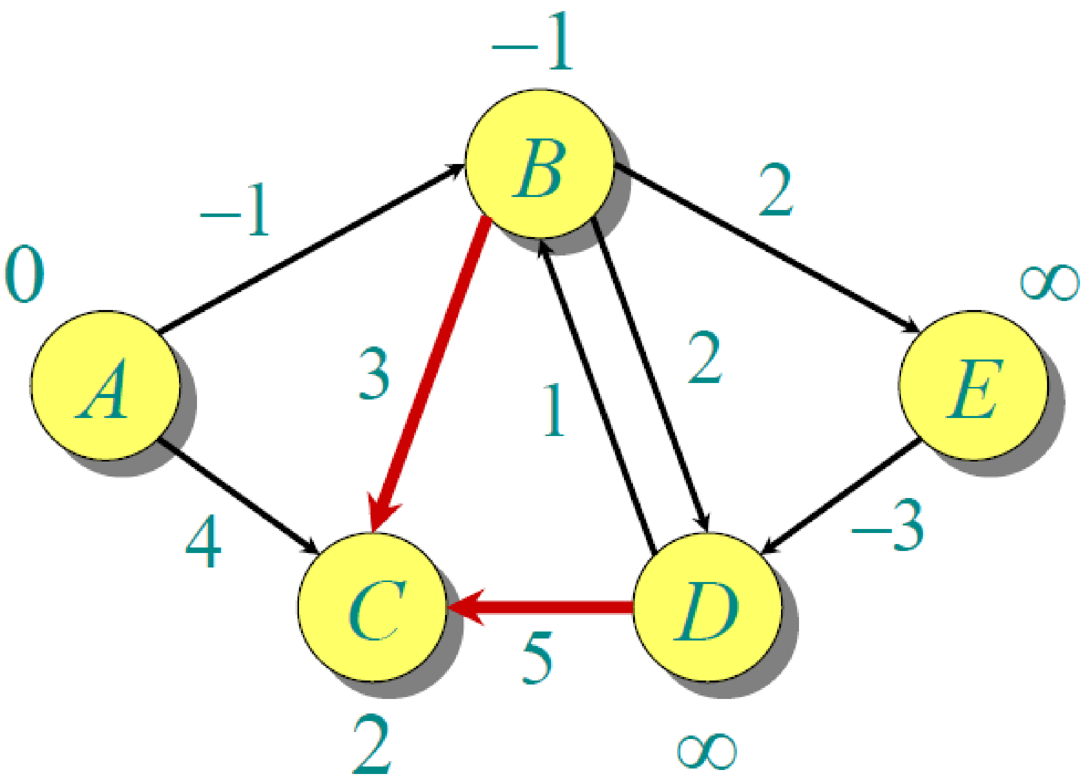
A	*B*	*C*	*D*	*E*
0	∞	∞	∞	∞
0	−1	∞	∞	∞

Example



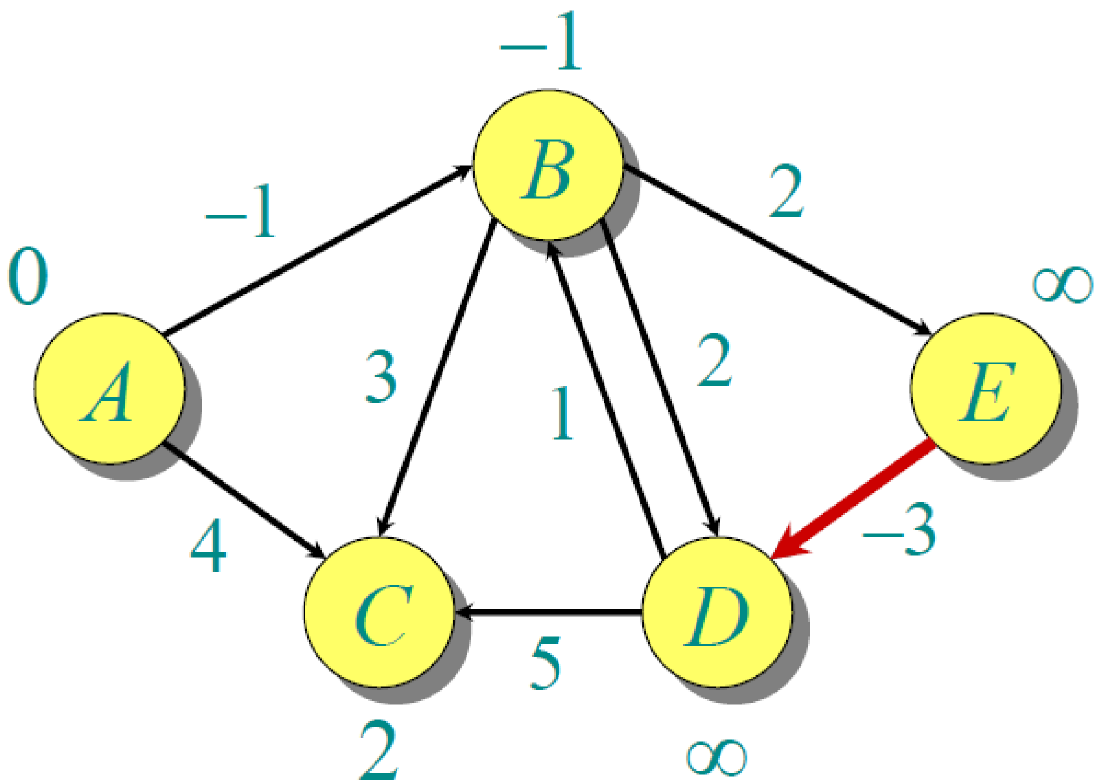
A	B	C	D	E
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞

Example



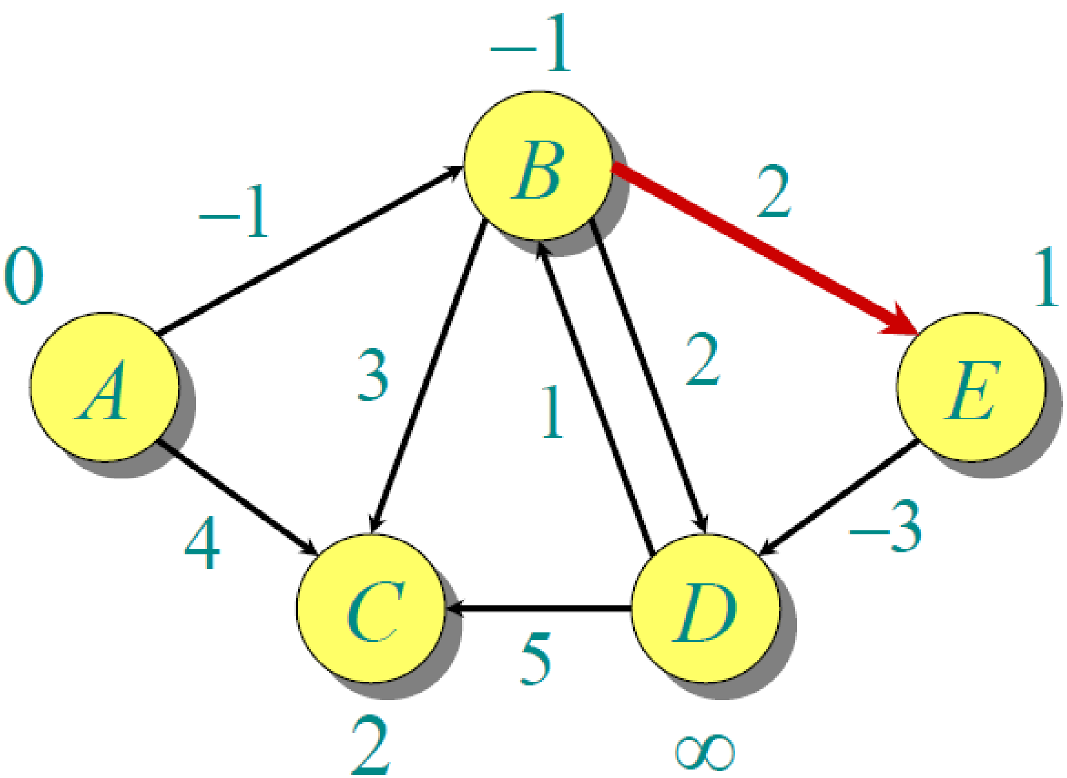
A	B	C	D	E
0	∞	∞	∞	∞
0	−1	∞	∞	∞
0	−1	4	∞	∞
0	−1	2	∞	∞

Example



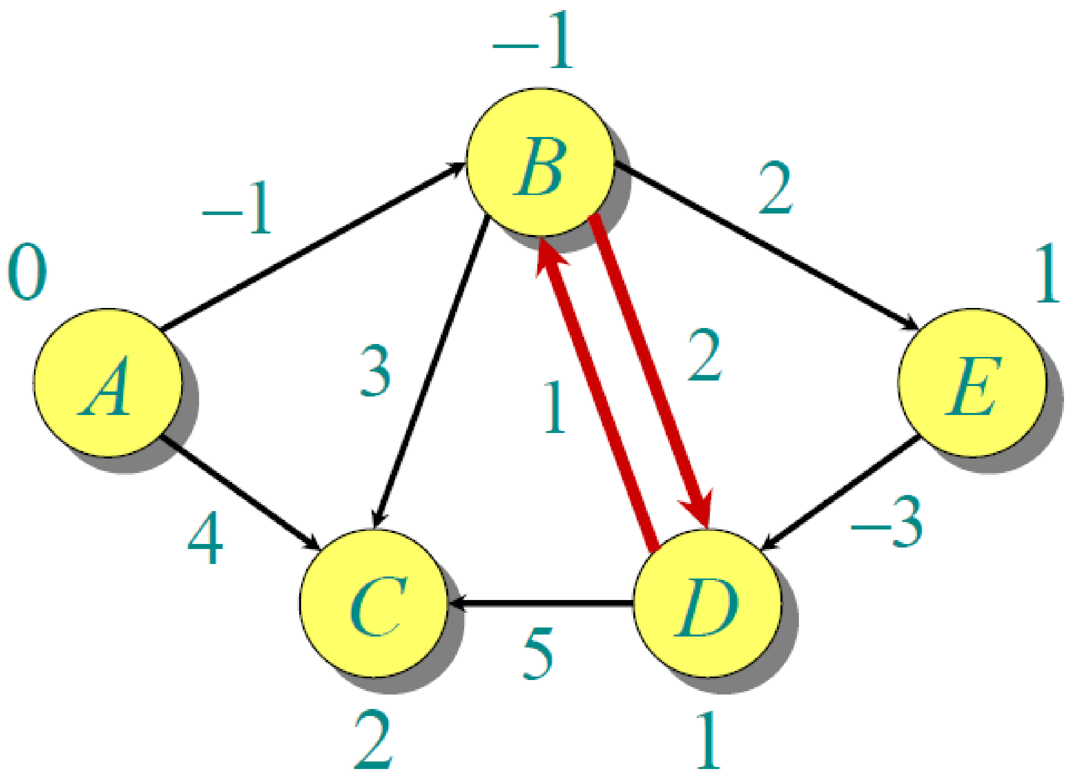
A	B	C	D	E
0	∞	∞	∞	∞
0	-1	∞	∞	∞
0	-1	4	∞	∞
0	-1	2	∞	∞

Example



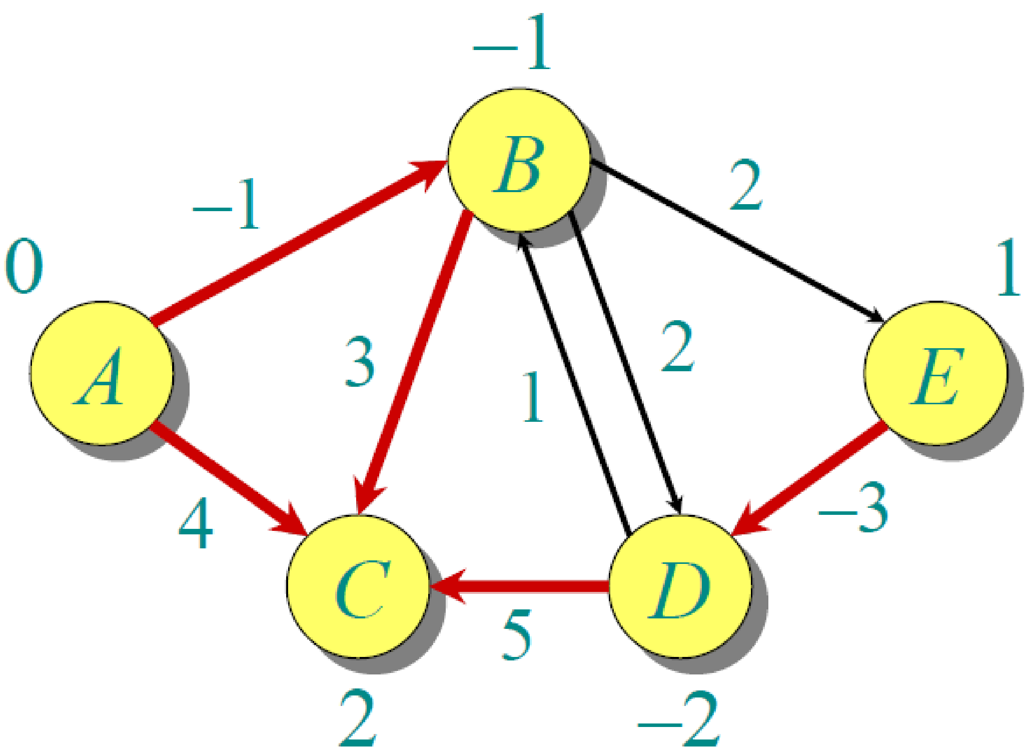
A	B	C	D	E
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1

Example



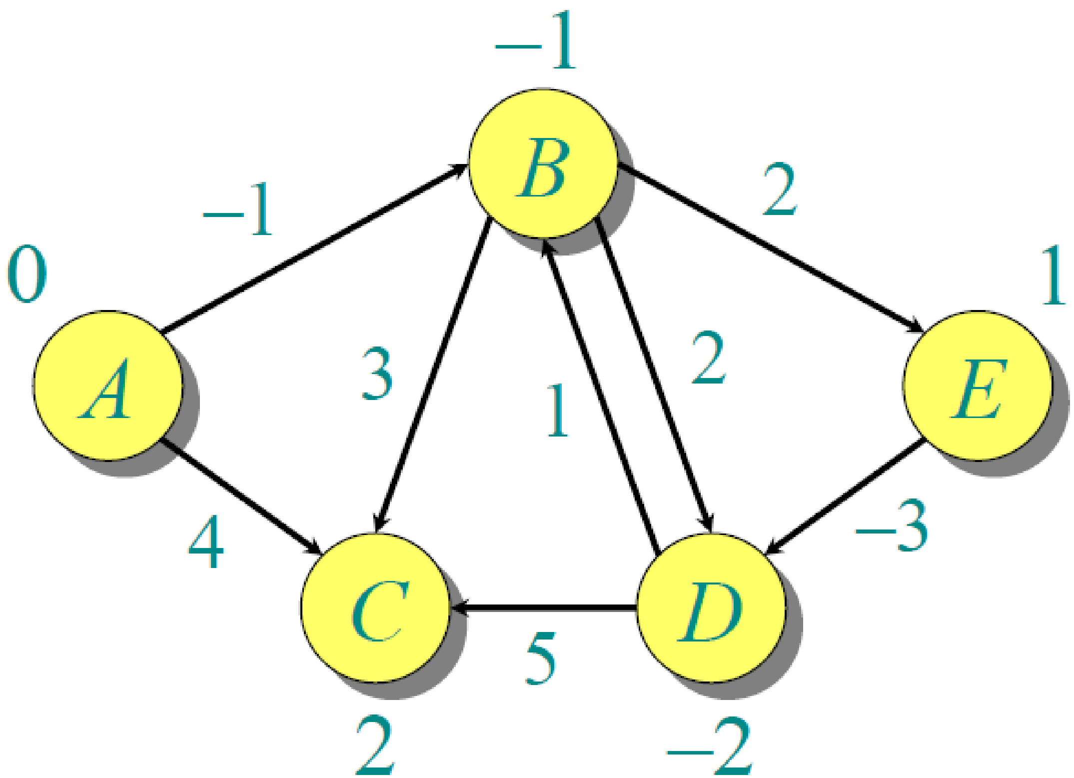
A	B	C	D	E
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1

Example



A	B	C	D	E
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1
0	–1	2	–2	1

Example



Note: Values decrease monotonically.

A	B	C	D	E
0	∞	∞	∞	∞
0	–1	∞	∞	∞
0	–1	4	∞	∞
0	–1	2	∞	∞
0	–1	2	∞	1
0	–1	2	1	1
0	–1	2	–2	1

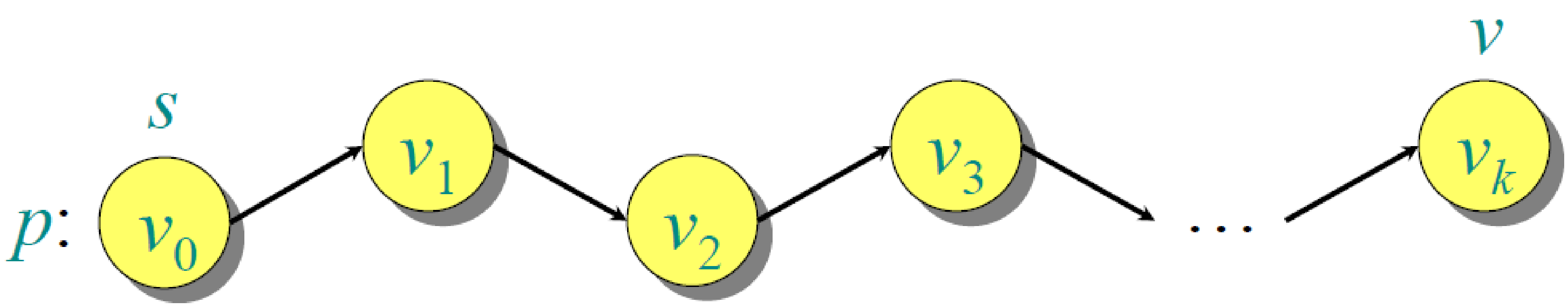
Correctness

Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

Correctness

Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. Let $v \in V$ be any vertex, and consider a shortest path p from s to v with the minimum number of edges.

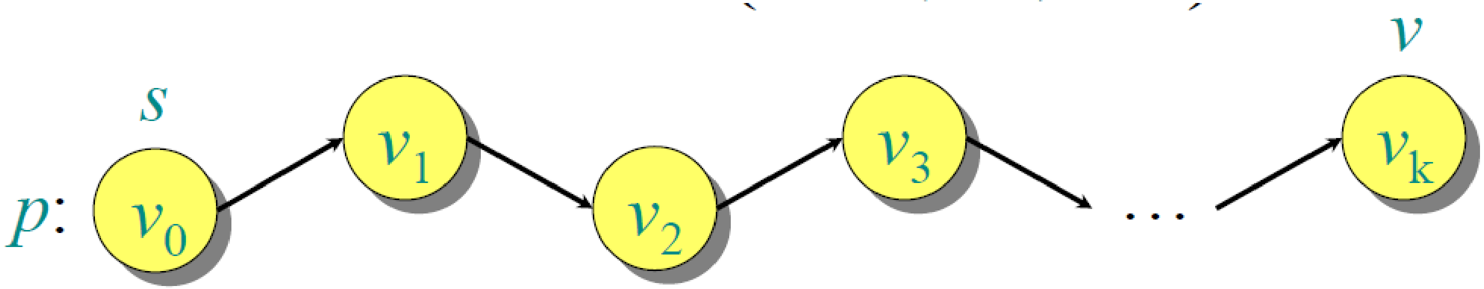


Since p is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) .$$

Correctness

Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.



Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[s]$ is unchanged by relaxations

- After 1 pass through E, we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through E, we have $d[v_2] = \delta(s, v_2)$.

⋮

- After k passes through E, we have $d[v_k] = \delta(s, v_k)$.

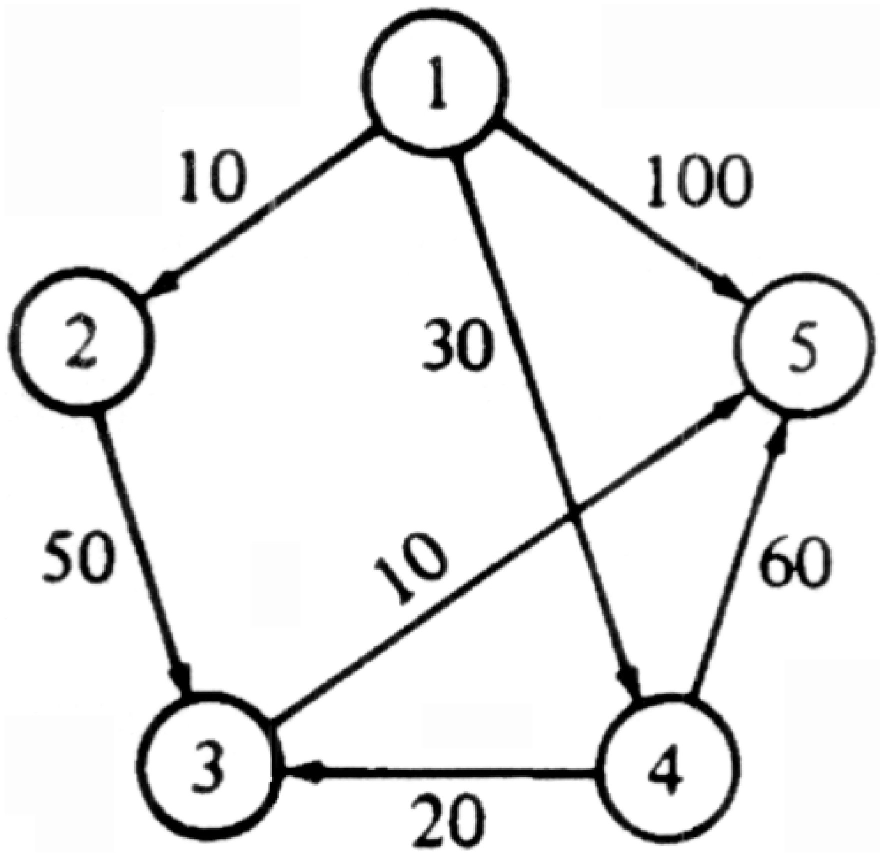
Since G contains no negative-weight cycles, p is simple. Longest simple path has $\leq |V| - 1$ edges. □

Detection of Negative-Weighted Cycles

Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s. $\square$

Short Test in Class

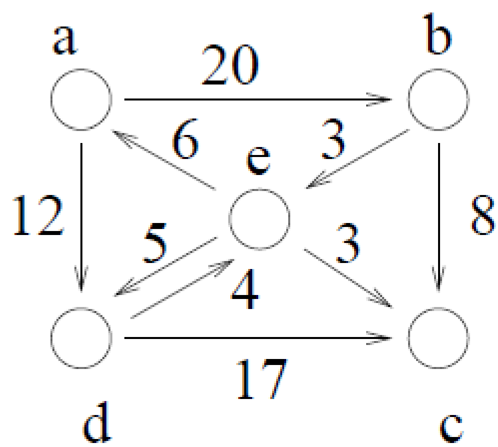
Work out the shortest distances of each vertex from vertex 1.



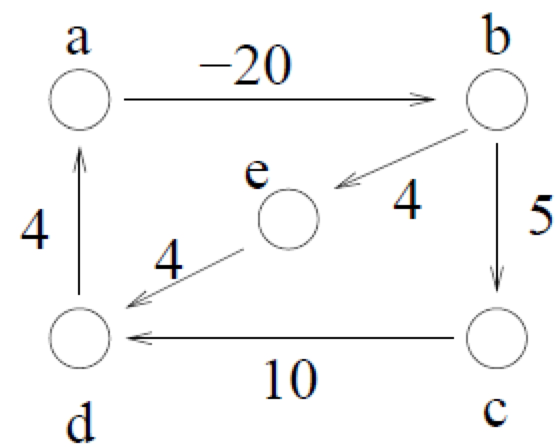
16.3 All-Pairs Shortest Paths

All-Pairs Shortest Paths

Given a weighted digraph $G = (V, E)$ with weight function $w : E \to \mathbf{R}$, (R is the set of real numbers), determine the length of the shortest path (i.e., distance) between all pairs of vertices in G.



without negative cost cycle



with negative cost cycle

Solution 1: Dijkstra's Algorithm

If there are no negative cost edges apply Dijkstra's algorithm to each vertex (as the source) of the digraph.

- Recall that D's algorithm runs in $\Theta((n+e)\log n)$. This gives a

$$\Theta(n(n+e)\log n) = \Theta(n^2 \log n + ne \log n)$$

time algorithm, where $n = |V|$ and $e = |E|$.

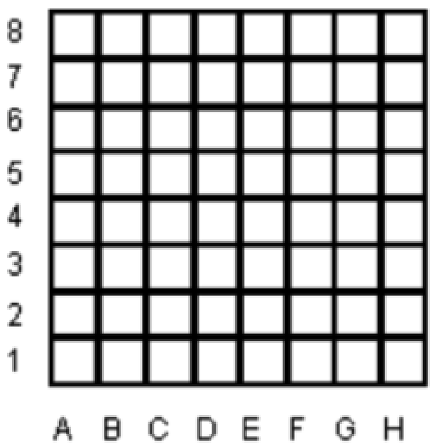
Application: Dijkstra’s Algorithm

< 返回

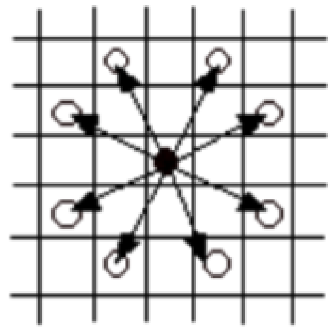
7-43 3.3.3 Camelot (190 分)

很久以前,亚瑟王和他的骑士习惯每年元旦去庆祝他们的友谊.在回忆中,我们把这些是看作是一个有一人玩的棋盘游戏. 有一个国王和若干个骑士被放置在一个由许多方格组成的棋盘上,没有两个骑士在同一个方格内.

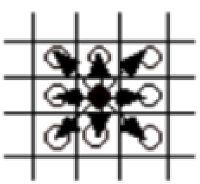
- 这个例子是标准的 8*8 棋盘



一个骑士可以从黑点移动到白点（如下图），但前提是他不掉出棋盘之外.



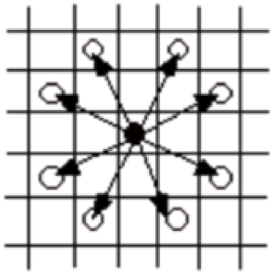
国王可以移动到任何一个相邻的方格,从黑点移动到白点（如下图）,但前提是他不掉出棋盘之外.



玩家的任务就是把所有的棋子移动到同一个方格里——用最小的步数. 为了完成这个任务,他必须按照上面所说的规则去移动棋子. 玩家必须选择一个骑士跟国王一起行动,其他的单独骑士则自己一直走到集中点. 骑士和国王一起走的时候,只算一个人走的步数.

Application: Dijkstra's Algorithm

一个骑士可以从黑点移动到白点（如下图），但前提是他不掉出棋盘之外.



$Dist[x][y][s]$表示某个骑士走到棋盘位置(x, y)的最小步数，$s \in \{0,1\}$, 0表示自己单独到达，1表示带着king一起到达。

$$Dist[x][y][0] = \min \begin{cases} \min(\{\, Dist[x+a][y+b][0] \mid a, b \in \{1, -1, 2, -2\} \,\}) + 1 \\ Dist[x][y][0] \end{cases}$$

$$Dist[x][y][1] = min \begin{cases} \min(\{\, Dist[x+a][y+b][1] \mid a, b \in \{1, -1, 2, -2\} \,\}) + 1 \\ Dist[x][y][0] + kingDist[x][y] \end{cases}$$

用DP计算，但bottom-up顺序不明确，直接迭代困难！

用Dijkstra Algorithm追踪bottom-up顺序

Solution 2：Dynamical Programming

To make DP work:

(1) How do we decompose the all-pairs shortest paths problem into subproblems?

(2) How do we express the optimal solution of a subproblem in terms of optimal solutions to some subsubproblems?

(3) How do we use the recursive relation from (2) to compute the optimal solution in a bottom-up fashion?

(4) How do we construct all the shortest paths?

Matrix multiplication

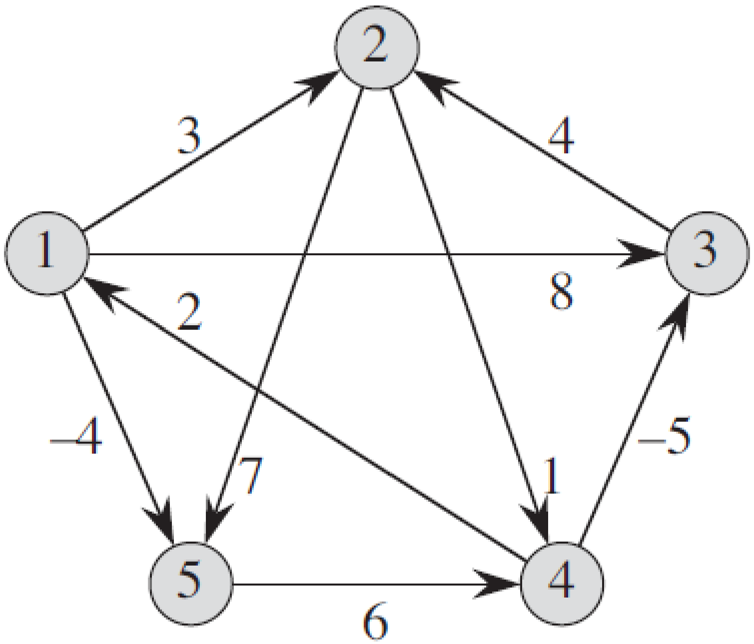
To simplify the notation, we assume that $V = \{1, 2, \ldots, n\}$.

Assume that the graph is represented by an $n \times n$ matrix with the weights of the edges:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ w(i, j) & \text{if } i \neq j \text{ and } (i, j) \in E, \\ \infty & \text{if } i \neq j \text{ and } (i, j) \notin E. \end{cases}$$

Output Format: an $n \times n$ matrix $D = [d_{ij}]$ where d_{ij} is the length of the shortest path from vertex i to j.

Example



Without negative circle

Input
$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Output
$$\begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

How to decompose the problem

- Subproblems with smaller sizes should be easier to solve.
- An optimal solution to a subproblem should be expressed in terms of the optimal solutions to subproblems with smaller sizes.

These are guidelines ONLY.

Step 1: Decompose in a Natural Way

- Define $d_{ij}^{(m)}$ to be the length of the shortest path from i to j that contains at most m edges.
  Let $D^{(m)}$ be the $n \times n$ matrix $[d_{ij}^{(m)}]$.

- $d_{ij}^{(n-1)}$ is the true distance from i to j (see next page for a proof this conclusion).

- Subproblems: compute $D^{(m)}$ for $m = 1, \cdots, n-1$.

Question: Which $D^{(m)}$ is easiest to compute?

$d_{ij}^{(n-1)}$ = True Distance from i to j

Proof: We prove that any shortest path P from i to j contains at most $n-1$ edges.

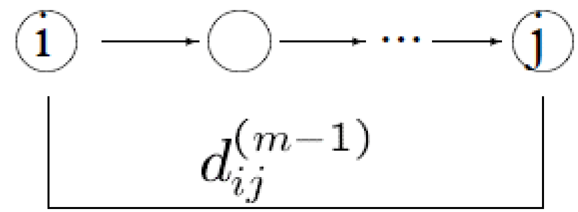
First note that since all cycles have positive weight, a shortest path can have no cycles (if there were a cycle, we could remove it and lower the length of the path).

A path without cycles can have length at most $n-1$ (since a longer path must contain some vertex twice, that is, contain a cycle).

Step 2： Recursive Formula

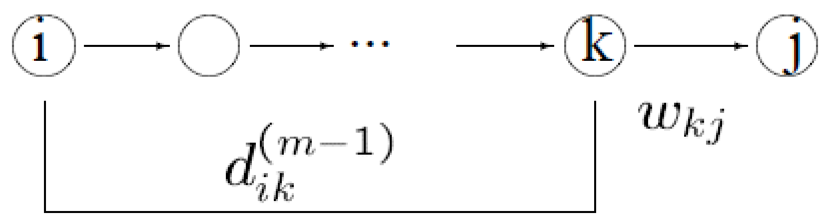
Consider a shortest path from i to j of length $d_{ij}^{(m)}$.

Case 1: It has at most $m-1$ edges.



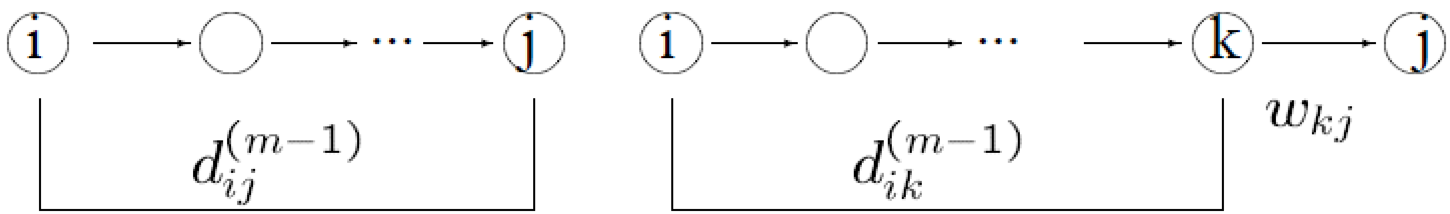
Then $d_{ij}^{(m)} = d_{ij}^{(m-1)} = d_{ij}^{(m-1)} + w_{jj}$.

Case 2: It has m edges. Let k be the vertex before j on a shortest path.



Then $d_{ij}^{(m)} = d_{ik}^{(m-1)} + w_{kj}$.

Step 2: Recursive Formula



Case 1: at most $m-1$ edges shortest path

Case 2: exactly m edges shortest path

Combining the two cases,

$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

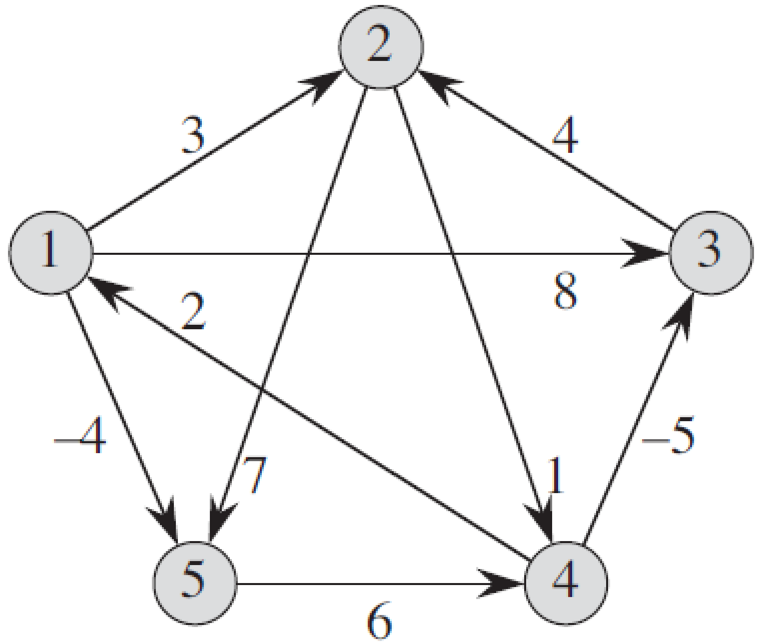
Step 3: Bottom-Up Computation

- Bottom: $D^{(1)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(m)}$ from $D^{(m-1)}$, for $m = 2, \ldots, n-1$, using

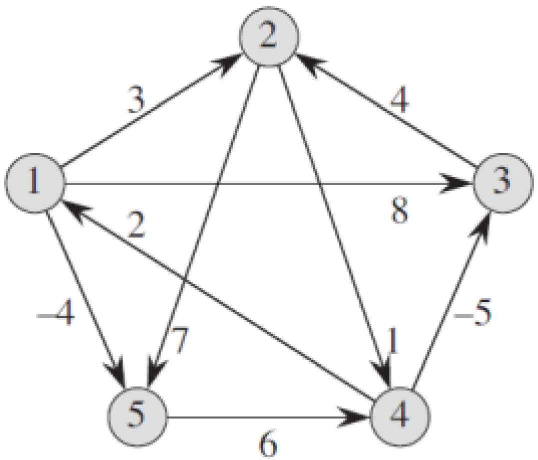
$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

Example



$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$ weight matrix

Example



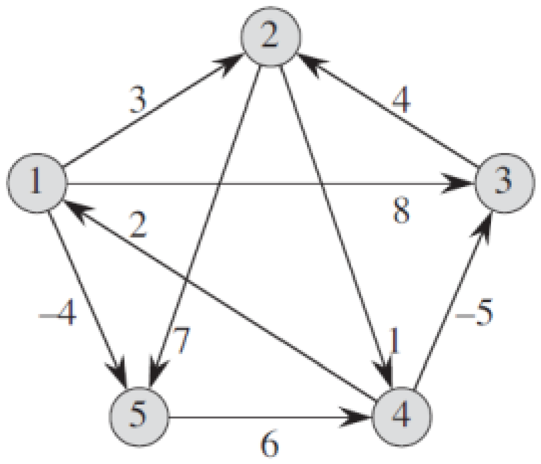
$$D^{(1)} \qquad\qquad D^{(1)}$$

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(2)} = \min_{1 \le k \le 5} \{ d_{ik}^{(1)} + d_{kj}^{(1)} \}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

Example

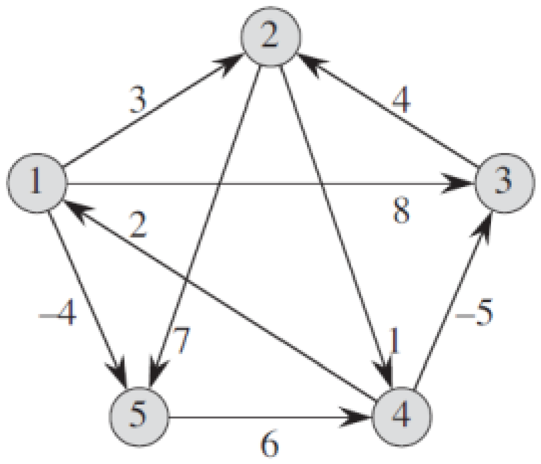


$$D^{(2)} \quad \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix} \times \quad D^{(1)} \quad \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(3)} = \min_{1 \le k \le 5} \{ d_{ik}^{(2)} + d_{kj}^{(1)} \}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Example



$D^{(3)}$ $D^{(1)}$

$$\begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \text{ x } \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(4)} = \min_{1 \le k \le 5} \{ d_{ik}^{(3)} + d_{kj}^{(1)} \}$$

$$D^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

The shortest distances between any pair of vertices

Algorithm

for $m = 1$ to $n - 1$

for $i = 1$ to n

for $j = 1$ to n

$\{$

$min = \infty;$

for $k = 1$ to n

$\{$

$new = d_{ik}^{(m-1)} + w_{kj};$

if $(new < min)\ min = new;$

$\}$

$d_{ij}^{(m)} = min;$

$\}$

Comments

- Algorithm uses $\Theta(n^3)$ space; how can this be reduced down to $\Theta(n^2)$?

- How can we extract the actual shortest paths from the solution?

- Running time $O(n^4)$, much worse than the solution using Dijkstra's algorithm. Can we improve this?

Improvement: Repeated Squaring

$$D^{(n-1)} = D^i, \text{ for all } i \geq n.$$

In particular, this implies that $D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n-1)}$.

We can calculate $D^{\left(2^{\lceil \log_2 n \rceil}\right)}$ using "repeated squaring" to find

$$D^{(2)}, D^{(4)}, D^{(8)}, \ldots, D^{\left(2^{\lceil \log_2 n \rceil}\right)}$$

Improvement: Repeated Squaring

- Bottom: $D^{(1)} = [w_{ij}]$, the weight matrix.

- For $s \geq 1$ compute $D^{(2s)}$ using

$$d_{ij}^{(2s)} = \min_{1 \leq k \leq n} \left\{ d_{ik}^{(s)} + d_{kj}^{(s)} \right\}.$$

Given this relation we can calculate $D^{(2^i)}$ from $D^{(2^{i-1})}$ in $O(n^3)$ time. We can therefore calculate all of

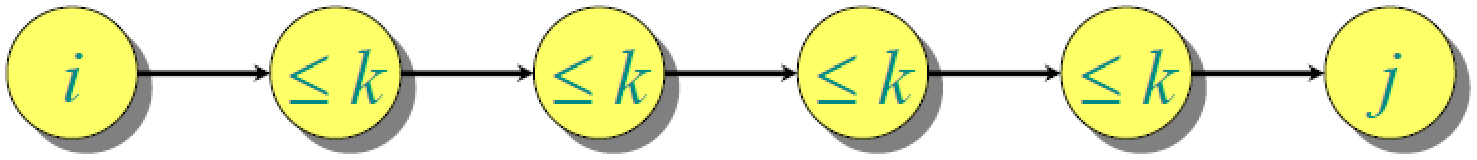
$$D^{(2)}, D^{(4)}, D^{(8)}, \ldots, D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n)}$$

in $O(n^3 \log n)$ time, improving our running time.

Floyd-Warshell Algorithm

Definition: The vertices $v_2, v_3, ..., v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, ..., v_{l-1}, v_l \rangle$.

- Let $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that *all* intermediate vertices on the path (if any) are in set $\{1, 2, \ldots, k\}$.



$d_{ij}^{(0)}$ is set to be w_{ij}, i.e., no intermediate vertex.

Let $D^{(k)}$ be the $n \times n$ matrix $[d_{ij}^{(k)}]$.

Floyd-Warshall Algorithm

Definition: The vertices $v_2, v_3, \ldots, v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, \ldots, v_{l-1}, v_l \rangle$.

- Claim: $d_{ij}^{(n)}$ is the distance from i to j. So our aim is to compute $D^{(n)}$.

- Subproblems: compute $D^{(k)}$ for $k = 0, 1, \cdots, n$.

Similar to a 0-1 knapsack problem!

The Structure of Shortest Paths

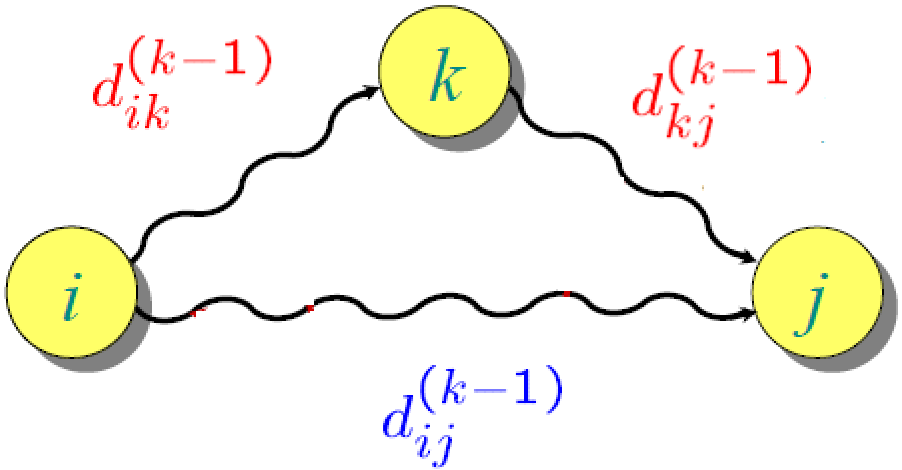
Observation 1: A shortest path does not contain the same vertex twice.

Non-negative circle!

Step 2: The Structure of Shortest Paths

Observation 2: For a shortest path from i to j such that any intermediate vertices on the path are chosen from the set $\{1, 2, \ldots, k\}$, there are two possibilities:

k is a vertex on the path.



k is not a vertex on the path,

$$d_{ij}^{(k)} = \min\left\{ d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right\}.$$

Step 3： Bottom-Up Computation

- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

for $k = 1, \ldots, n$.

Step 3： Bottom-Up Computation

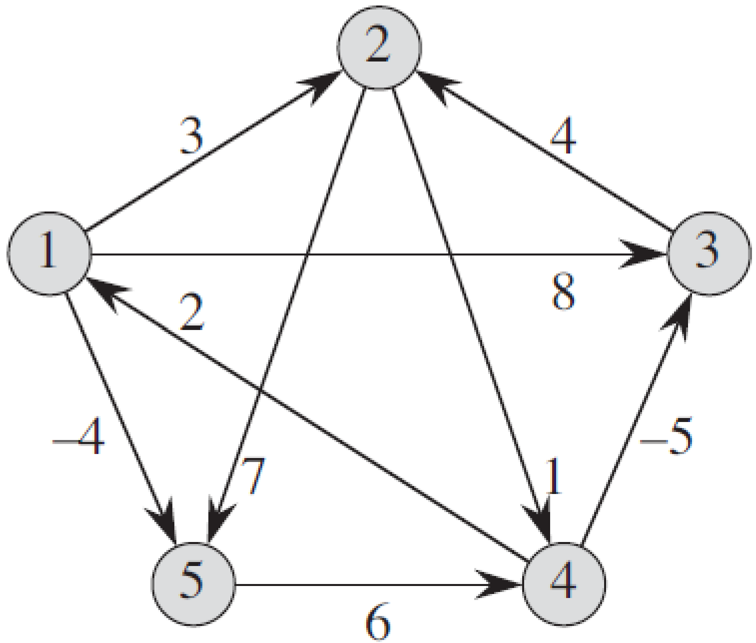
- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

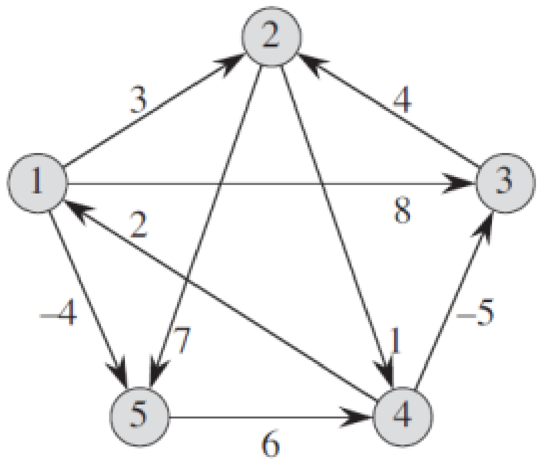
for $k = 1, \ldots, n$.

Example



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$ weight matrix

Example



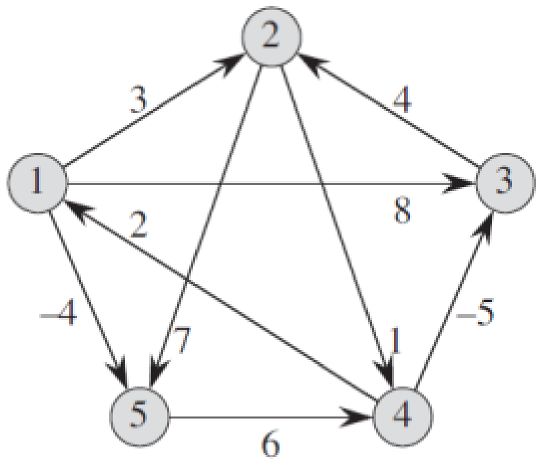
$$D^{(0)}$$

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(1)} = min\{d_{ij}^{(0)},\ d_{i1}^{(0)} + d_{1j}^{(0)}\}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Example



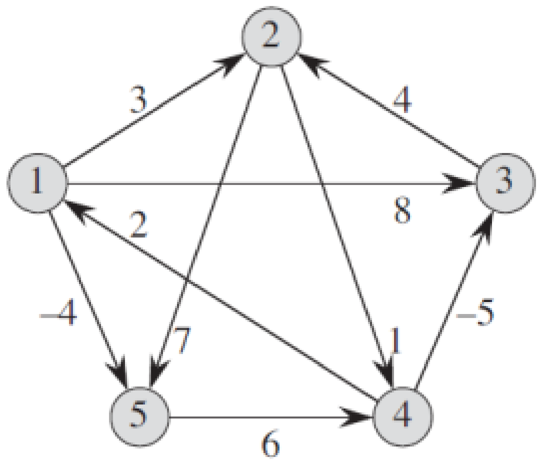
$$D^{(1)}$$

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(2)} = min\{d_{ij}^{(1)},\ d_{i2}^{(1)} + d_{2j}^{(1)}\}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Example



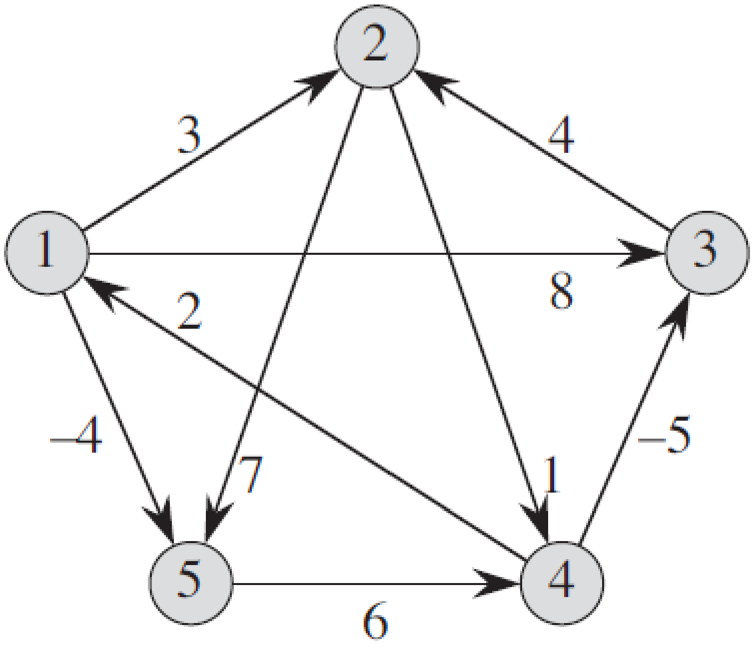
$$D^{(2)}$$

$$\begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(3)} = min\{d_{ij}^{(2)},\ d_{i3}^{(2)} + d_{3j}^{(2)}\}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Example



$$d_{ij}^{(5)} = min\{d_{ij}^{(4)},\ d_{i5}^{(4)} + d_{5j}^{(4)}\}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

The shortest distances between any pair of vertices

Algorithm

```
Floyd-Warshall(w, n)
{ for i = 1 to n do                                initialize
     for j = 1 to n do
     { d[i, j] = w[i, j];
       pred[i, j] = nil;
     }

  for k = 1 to n do                                dynamic programming
     for i = 1 to n do
        for j = 1 to n do
           if (d[i, k] + d[k, j] < d[i, j])
                 {d[i, j] = d[i, k] + d[k, j];
                  pred[i, j] = k;}
  return d[1..n, 1..n];
}
```

Comments

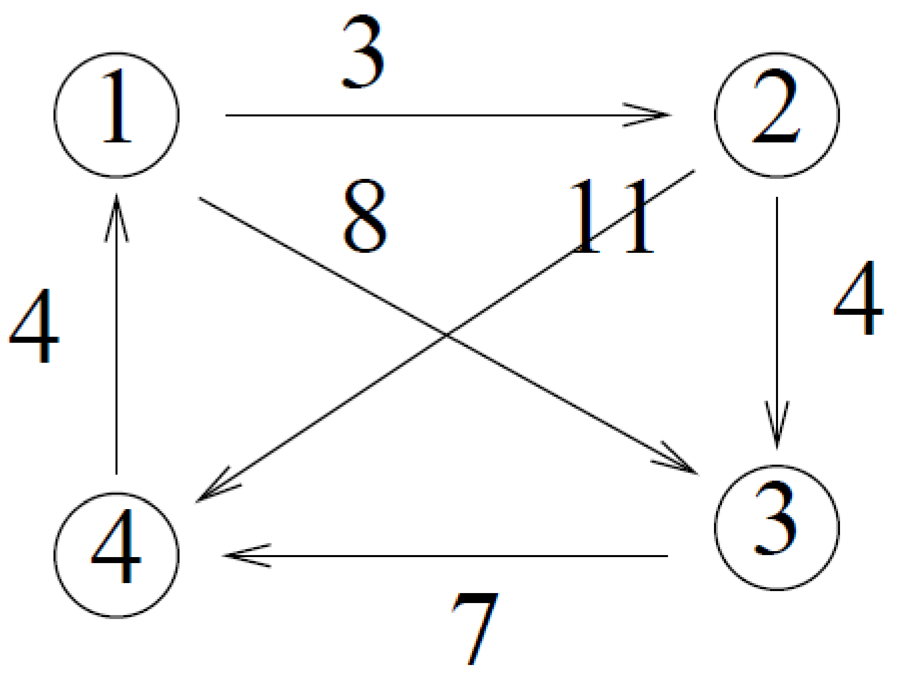
- The algorithm's running time is clearly $\Theta(n^3)$.
- The predecessor pointer `pred[i,j]` can be used to extract the final path (see later).
- Problem: the algorithm uses $\Theta(n^3)$ space.
  It is possible to reduce this down to $\Theta(n^2)$ space by keeping only one matrix instead of n.

Extracting The Shortest Paths

To find the shortest path from i to j, we consult $pred[i, j]$. If it is nil, then the shortest path is just the edge (i, j). Otherwise, we recursively compute the shortest path from i to $pred[i, j]$ and the shortest path from $pred[i, j]$ to j.

Short Test in Class

Give $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ with matrix multiplication algorithm, or $D^{(0)}$, $D^{(1)}$, $D^{(2)}$ by Floyd-Warshell algorithm.



数据结构与算法课程组
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End of Section.