State Extraction by Dimensionality Reduction

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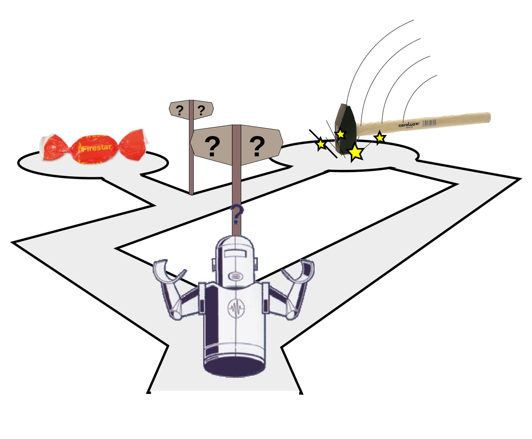
Abstract

Real world data (e.g. video images) are usually high dimensional and present information highly mixed and noise afflicted. Most applications therefore require a *dimensionality reduction* or *information filtering* beforehand. Such filters have to present the desired information in an applicable fashion and have to make sure few are lost. Several methods exist that *learn* these filters from the statistics of presented training data (e.g. PCA, CCA, PLS or SFA [1]).
We used the unsupervised method of *slow feature analysis* (SFA) [2] to extract the position of a robot from the images of its head-mounted camera. To catch higher order correlations in video images, we constructed a kernelized SFA algorithm analogous to *kernel PCA* [4], which outperformed its linear counterpart considerably. In order to deal with the huge number of training samples needed to catch the statistics of real images, we employed a sparse kernel matrix approximation method first introduced by Csató and Opper [5]. The resulting feature space (an approximation of the space of trigonometric polynomials [3]) is particularly well suited to approximate continuous functions with a linear model, e.g. *value functions* in *reinforcement learning*. We demonstrated this by learning the robots control in a simple navigation task.

Reinforcement Learning

Q-Learning approach:

- Execute *actions* a_t in *states* $x_t \in \mathcal{X}$
- Draw actions from *policy* $a_t \sim \pi(x_t)$
- Next state is drawn $x_{t+1} \sim P(x_t, a_t)$
- Receive *rewards* $r_t \sim R(x_t, a_t, x_{t+1})$
- Estimate the expected discounted future reward given a choice a in x:



$$Q^\pi(x,a) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t\, r_t \big| x_0 = x, a_0 = a\Big] \quad \text{(Q-value)}$$

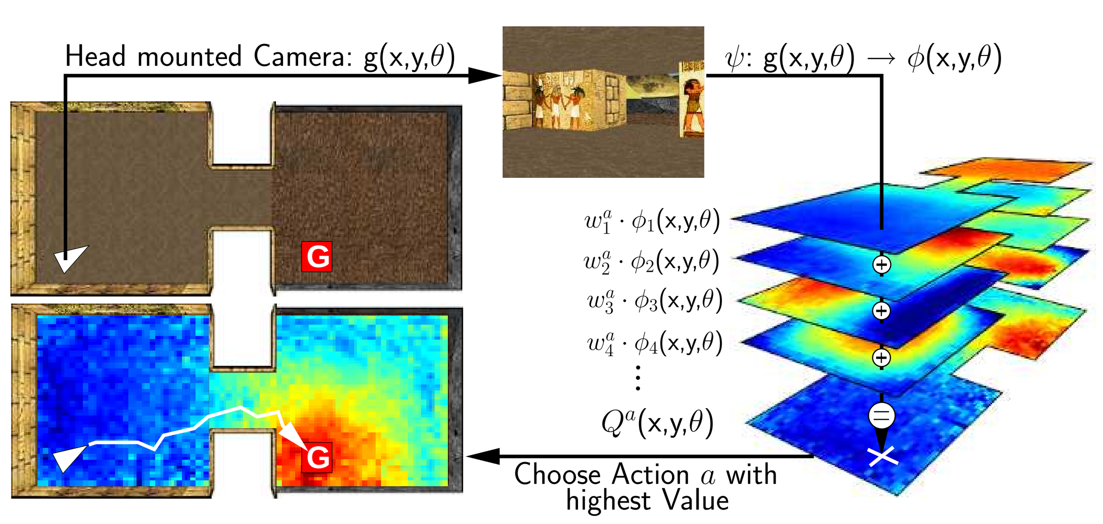
- Find the *optimal policy* $\pi^*(\cdot)$ that $\forall x \in \mathcal{X} : \max Q^{\pi^*}(x, \pi^*(x))$
- Define $\pi^*(x)$ by always choosing $a^* = \arg\max_a Q^{\pi^*}(x,a)$
- *Policy iteration* between Q-value estimation and policy improvement
- $\mathcal{X}$ is *continuous* $\Rightarrow Q^\pi(x,a)$ must be approximated, e.g. linear: $Q^\pi(x,a) \approx Q^a(x) = \boldsymbol{\varphi}(x)^\top \boldsymbol{w}^a$, with $\boldsymbol{\varphi} : \mathcal{X} \mapsto \Phi$

$\Rightarrow$ **Feature space Φ must facilitate function approximation!**

Robot Navigation

- Hidden state $(x, y, \theta) \in \mathcal{X}$ (position x, y and orientation θ)
- Observation $z = g(x, y, \theta) \in \mathcal{Z}$ (images from head mounted camera)

$\Rightarrow$ **Requires a mapping $\boldsymbol{\phi} : \mathcal{Z} \mapsto \Phi$, with $\boldsymbol{\phi}(g(x,y,\theta)) \approx \boldsymbol{\varphi}(x,y,\theta)$**



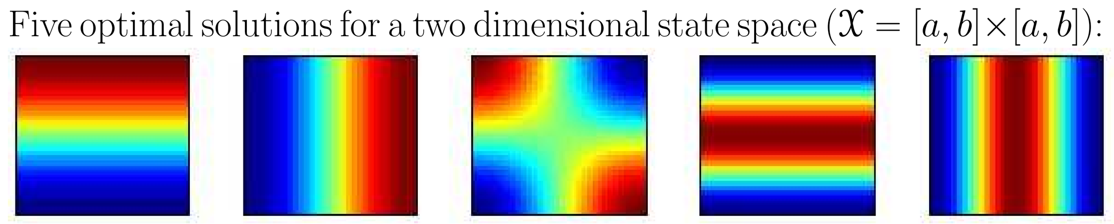
Slow Feature Analysis

- SFA [2] analyzes second order statistics of *time series* $z^{(1)}, \ldots, z^{(n)} \in \mathcal{Z}$
- Learns a set of mappings $\boldsymbol{\phi}_i : \mathcal{Z} \mapsto \mathbb{R}$ that change slowly over time
- Infinite time series and unrestricted model class $\Rightarrow$ Mappings become *trigonometric polynomials* in the domain of a *hidden state* $x^{(t)} \in \mathcal{X}$
- $x^{(t)}$ is the *minimal explanation* of $z^{(t)}$, in the sense that there exists an unknown invertible function $g : \mathcal{X} \mapsto \mathcal{Z}$ such that $\forall t : z^{(t)} = g(x^{(t)})$

Optimization Problem

$$\begin{aligned}
\min \quad s(\boldsymbol{\phi}_i) &:= \mathbb{E}[\dot{\boldsymbol{\phi}}_i^2] && \text{(Slowness)}\\
\text{s.t.} \quad \mathbb{E}[\boldsymbol{\phi}_i] &= 0 && \text{(Zero Mean)}\\
\mathbb{E}[\boldsymbol{\phi}_i^2] &= 1 && \text{(Unit Variance)}\\
\forall j \neq i : \mathbb{E}[\boldsymbol{\phi}_i \boldsymbol{\phi}_j] &= 0 && \text{(Decorrelation),}\\
\forall j > i : s(\boldsymbol{\phi}_i) &\leq s(\boldsymbol{\phi}_j) && \text{(Order)}
\end{aligned}$$

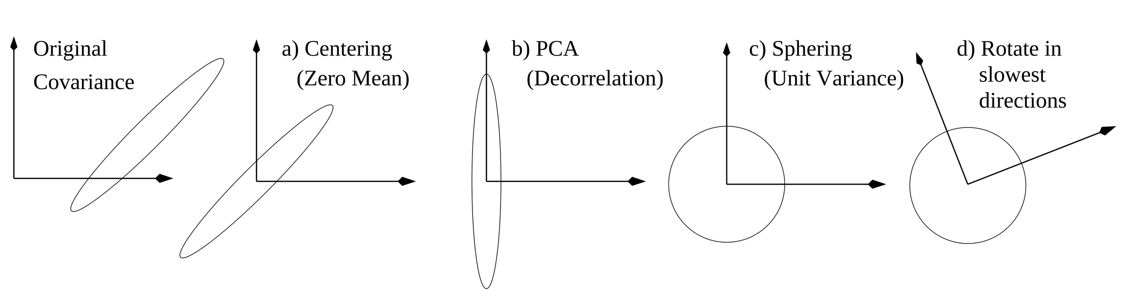
Five optimal solutions for a two dimensional state space ($\mathcal{X} = [a,b] \times [a,b]$):



Features of equal slowness can also exhibit an arbitrary rotation in their subspace, i.e. $s(\boldsymbol{\phi}_i) = s(\boldsymbol{\phi}_j) = s(a\boldsymbol{\phi}_i + b\boldsymbol{\phi}_j)$ with $a^2 + b^2 = 1$.

Linear SFA

- Given $\boldsymbol{z}^{(t)} \in \mathbb{R}^d$, assume $\boldsymbol{\phi}_i(\boldsymbol{z}) := \boldsymbol{w}_i^\top \boldsymbol{z} - c_i$ to be an affine function
- Fulfill constrains (a-c); requires eigenvalue decomposition of $\mathbb{E}[\boldsymbol{z}\boldsymbol{z}^\top]$
- Minimize objective (d) by rotation to lowest eigenvectors of $\mathbb{E}[\dot{\boldsymbol{z}}\dot{\boldsymbol{z}}^\top]$
- Overall time complexity $\mathcal{O}(d^2 n)$



Kernel SFA

- Implicit projection $\boldsymbol{\psi}(\cdot)$ in high dimensional feature space
- *Positive semidefinite kernel function* $k(\boldsymbol{z}, \boldsymbol{z}') = \boldsymbol{\psi}(\boldsymbol{z})^\top \boldsymbol{\psi}(\boldsymbol{z}')$
- Steps analogous to linear SFA (for projected data $\boldsymbol{\Psi}_{it} = \boldsymbol{\psi}_i(\boldsymbol{z}^{(t)})$):

a) Centered kernel matrix $\mathbf{K}_c = \boldsymbol{\Psi}_c^\top \boldsymbol{\Psi}_c = (\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^\top)\boldsymbol{\Psi}^\top\boldsymbol{\Psi}(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^\top)$

b) Analogous to kernel PCA [4]: Let $\mathrm{cov}(\psi(x)) = \frac{1}{n}\boldsymbol{\Psi}_c\boldsymbol{\Psi}_c^\top := \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$ and $\mathbf{K}_c = \boldsymbol{\Psi}_c^\top\boldsymbol{\Psi}_c := \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top$, then $\mathbf{U} = \frac{1}{\sqrt{n}}\boldsymbol{\Psi}_c\mathbf{V}\boldsymbol{\Lambda}^{-1/2}$ [1]

c) Sphered expanded data $\boldsymbol{\Psi}_s = \frac{1}{\sqrt{n}}\boldsymbol{\Lambda}^{-1}\mathbf{V}^\top\mathbf{K}_c$

d) Rotation to eigenvectors $\mathbf{R}$ of $\dot{\boldsymbol{\Psi}}_s\dot{\boldsymbol{\Psi}}_s^\top = \frac{1}{n}\boldsymbol{\Lambda}^{-1}\mathbf{V}^\top\dot{\mathbf{K}}_c\dot{\mathbf{K}}_c\mathbf{V}\boldsymbol{\Lambda}^{-1}$

$$\Rightarrow \quad \boldsymbol{\phi}(\boldsymbol{z}) = \mathbf{W}^\top \boldsymbol{k}(\boldsymbol{z}) - \boldsymbol{c}$$

$$\text{where } \mathbf{W} = \tfrac{1}{\sqrt{n}}(\mathbf{I} - \tfrac{1}{n}\mathbf{1}\mathbf{1}^\top)\mathbf{V}\boldsymbol{\Lambda}^{-1}\mathbf{R} \quad \text{and} \quad \boldsymbol{c} = \tfrac{1}{n}\mathbf{W}^\top\mathbf{K}\mathbf{1}$$

- Raises time complexity to $\mathcal{O}(n^3)$

Kernel Matrix Sparsification

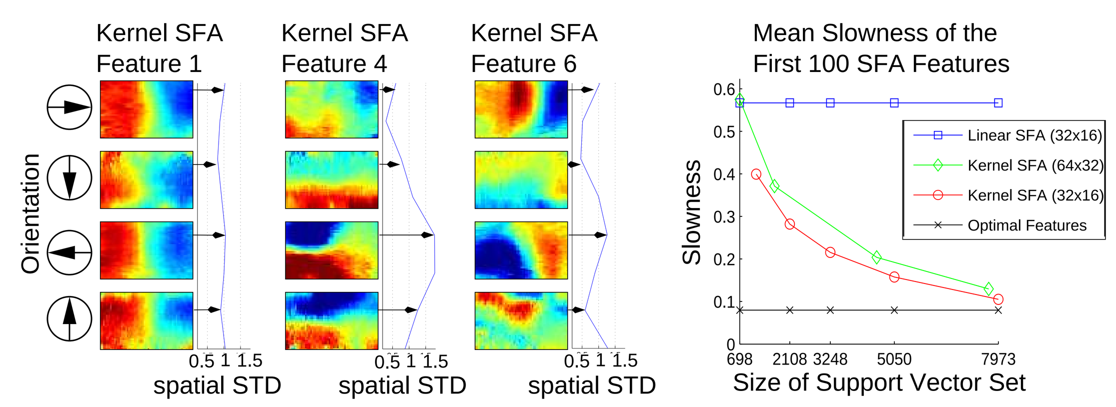
- $\boldsymbol{\phi}(\boldsymbol{z})$ depends on *all* training samples $\boldsymbol{k}(\boldsymbol{z}) = [k(\boldsymbol{z}^{(1)}, \boldsymbol{z}), \ldots, k(\boldsymbol{z}^{(n)}, \boldsymbol{z})]^\top$
- Express $\boldsymbol{\phi}(\cdot)$ with subset $\{\boldsymbol{s}^{(i)}\}_{i=1}^m \subset \{\boldsymbol{z}^{(i)}\}_{i=1}^n$
- Select only samples that contribute to $\mathcal{A} := \mathbf{aff}\{\boldsymbol{\psi}(\boldsymbol{s}^{(i)})\}_{i=1}^m$ largely [5]
- After one pass, all $\boldsymbol{\psi}(\boldsymbol{z}^{(i)})$ lie within a *sparsity threshold* ϵ of $\mathcal{A}$, i.e.

$$\min_\alpha \|\boldsymbol{\psi}(\boldsymbol{z}) - \sum_{i=1}^m \alpha_i \boldsymbol{\psi}(\boldsymbol{s}^{(i)})\|_2^2 = k(\boldsymbol{z},\boldsymbol{z}) - \boldsymbol{k}(\boldsymbol{z})^\top\mathbf{K}^{-1}\boldsymbol{k}(\boldsymbol{z}) \leq \epsilon^2$$

- The algorithm keeps track of $\mathbf{K}^{-1}$ and has time complexity $\mathcal{O}(m^2 n)$

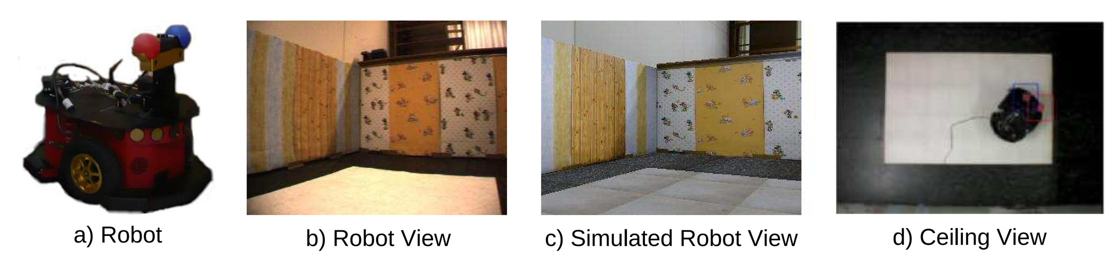
Sparse Kernel SFA

- Using m *support vectors*, time complexity reduces to $\mathcal{O}(m^2 n)$
- Vary m to find a tradeoff between complexity and slowness
- Experiments with ego perspective images as inputs (see next section):



Left: Spatial sparse kernel SFA feature output with 8,000 out of 35,000 support vectors

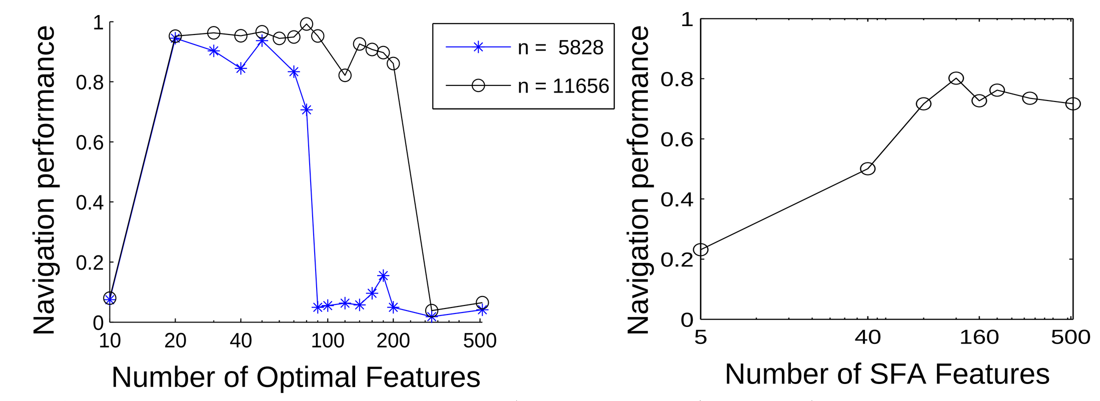
Robot Experiments



- Robots environment is a small rectangular room (wallpapered walls)
- Ego perspective images (head mounted camera) as states
- Choose between 3 actions: *move forward* and *turn left/right* 45°
- *Goal area* is not marked, but receives positive reward
- Positions close to walls are punished
- *Least Squares Policy Iteration* (LSPI) algorithm to learn policy π
- Estimate *navigation performance* of policy π: $\mathbb{E}\left[\frac{Q^\pi(x,\pi(x))}{Q^{\pi^*}(x,\pi^*(x))}\right]$

Simulated Experiment

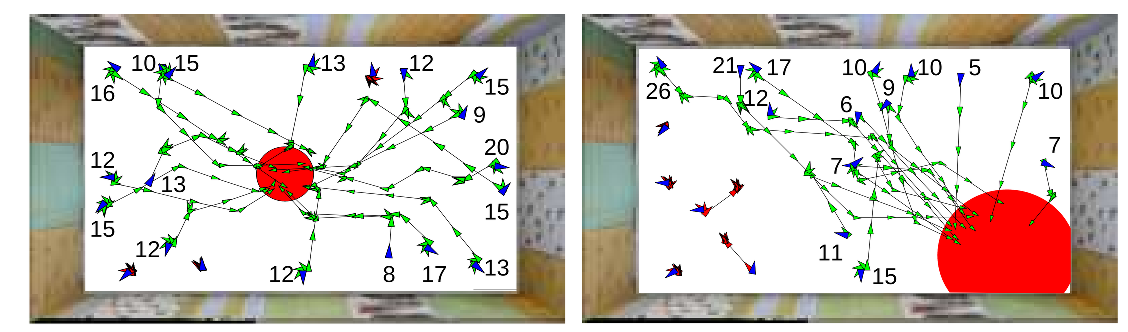
- *Control experiment* with optimal features (left figure)
- 3D-model of real world experiment (photographed textures)
- Learn sparse kernel SFA features with 35,000 rendered images
- Learn policy for centered goal area (ca. 12,000 seen transitions)
- Policy needs more features to reach working regime (right figure)
- Navigation performance is lower (ca. 2-3 unnecessary steps)



Left: Control experiment on optimal features (1000 trajectories/estimation). n ≙ # training samples.
Right: Simulated experiment with sparse kernel SFA features (8k/35k) (200 trajectories/estimation).

Real-World Experiment

- Same trajectories as in the simulated experiment (real images)
- Too time expensive to estimate navigation performance
- Two experiments with different reward (red goal areas):



Numbers indicate needed actions. 128 sparse kernel SFA features (8k/35k) are used for approximation.

- 75% success rate, but much more unnecessary steps
- Oscillatory behavior in areas of similar Q-value ($\Rightarrow$ strong noise)

$\Rightarrow$ **Learned feature space works but not yet noise robust**

Outlook

- **Approximation in noise afflicted feature spaces**
  - External influences and a weak model class appear as *noise*
  - ▹ Can we give *bounds* and *convergence statements*?
  - SFA features do not take past observations into account
  - ▹ Can we construct a *recurrent filter* to reduce noise?
- **Computational efficiency and alternative feature spaces**
  - Kernel expansion is computational very expensive
  - Trigonometric polynomials need support on the complete domain $\mathcal{X}$
  - ▹ Can we learn a *sparse feature space* (e.g. Gaussian bells)?
  - Hidden state might be high dimensional (e.g. moving humans)
  - ▹ Can we filter out state dimensions that are *uncorrelated* to the target?

References

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