Sufficient and Necessary Conditions for Winning Strategies in the Banach-Mazur Game

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Abstract

The goal of this paper is to analyze the Banach-Mazur game and determine algorithms, given certain conditions, that would guarantee the existence of winning strategies for one of the two players by defining sufficient and necessary conditions for their existence. The authors first define a generalized version of the very first version of the game due. Then, utilizing the properties of sets that are countable unions of nowhere dense sets (viz., sets of first category), the authors shall make use of a greedy approach to formulate said winning strategies. For the sake of comparison, the authors will then also compare the use of these strategies to the minimax algorithm through an analysis of worst-case time complexity.

Keywords

Banach-Mazur Game, Topology, Perfect Information Game, Minimax

I Introduction

A topological game is a (potentially infinite) game of perfect information that is typically played between two players (cf. [Ber]). Moves in the game are done by either player through the picking of a certain topological object, whether it is a point, certain open sets, closed sets, and other types of objects with topological significance. As is with many two-player games like chess and tic-tac-toe, players alternate in taking turns. One of the earliest games to ever be defined was the Banach-Mazur Game, which was first introduced on the 4 August 1935 as the 43rd problem of the Scottish Book [Ulam, Rev].

The original Banach-Mazur game goes as follows: one begins with a predefined set of real numbers E. Then, the two players—call them α (for $A\lambda\iota\kappa\eta$, the "first player") and β (for $B\alpha\rho\beta\alpha\rho o\varsigma\ M\pi o\mu\pi$, the "second player")—take turns in choosing arbitrary intervals such that every chosen interval is contained in the previously chosen interval (except for the first move, which could be any interval). Player α wins if the intersection of all the chosen sets contains a point of E; otherwise, B wins.

In the paper, however, that is not the version of the game the authors would like to analyze. In the following sections, the authors would be studying a variation of that game introduced in a more general setting due to a paper by Oxtoby in 1957 [Oxt]. Instead of focusing on the set of real numbers, this paper would instead consider an arbitrary topological space. The precise modifications to the rules will be soon defined in a later section of this paper where the formal definition of the game itself would be given (section IV), but not before the introduction of the conventions and necessary concepts that this paper will make use of.

*Both authors contributed equally to this research.

II Notation

Let the following notational conventions be defined:

$A \subset B$ means "A is a subset of B" (not necessarily a proper subset).

$\mathring{A}$ denotes the interior of a set A.

$\overline{A}$ denotes the closure of a set A.

$\wp(X)$ denotes the power set of X, the set of subsets of X.

III Preliminary Definitions

A number of important concepts that shall facilitate further discussion. For a deeper understanding of these definitions or concepts related thereto, refer to [NB].

A **topological structure** or a **topology** on a set X is a set τ of subsets of X such that:

(1) any of sets from τ is also in τ;
(2) every finite intersection of sets from τ is also in τ.

A set endowed with a topological structure is then called a **topological space**; the sets that constitute the topological structure thereof are called **open sets**. Notice how condition (1) implies that the empty set $\varnothing$ must always be part of a topological structure; moreover, condition (2) suggests that the entire set X must also be always part of any topological structure defined thereon. Examples of topological structures are, given any set S, the power set $\wp(S)$ (the *discrete topology*) and the the set containing only S and the empty set (the *trivial topology*). Another more famous example is the real line, which consists of the set of real numbers whose topology is given by the set of all open intervals and their unions.

A set in a topological space X is called **closed** if its complement is open in X. For example, in the topology of the real line given previously, the set $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is a closed set in the topology of the real line since its complement in X is equal to $(-\infty, 0) \cup (1, \infty)$, the union of two open sets.

The **interior** of a set A in a topological space X is the union of all open sets contained in that set. The **closure** is the intersection of all closed sets that contain A. It is clear from the definitions that

$$\mathring{A} \subset A \subset \overline{A}$$

for any subset A in a topological space X. For example, in the real line, the interior of the set $[0, 1]$ is the open interval $(0, 1)$, and the interior of the singleton $\{1\}$ is the empty set since the only open set contained therein is just the empty set. The closure of the open set $(0,1)$ in the real line is equal to the closed interval $[0,1]$.

A set A is called **dense** if its closure is equal to the entire space X; in symbols: $\overline{A} = X$. Equivalently, a set A is dense if, given any nonempty open set U, $A \cap U \neq \emptyset$, A set is called **nowhere dense** if the interior of its closure is the empty set. An example of a dense set in the real line is the set $\mathbb{Q}$ of all rational numbers (a well-known result in real analysis; see [KF]). Examples of nowhere dense subsets in the real line is the set $\mathbb{Z}$ of integers and any singleton containing any real number.

A set A of a topological space X is said to be of **first category** if it is equal to a countable union of nowhere dense sets. Otherwise, it is said to be of **second category**. Both the set of integers $\mathbb{Z}$ and the rational numbers $\mathbb{Q}$ are actually examples of sets of first category in the real line.

IV The Game

Let X be a a set endowed with a topological structure τ, and let $\mathscr{W} \subset \wp(X)$ a collection of subsets of X such that

- $\forall W \in \mathscr{W}\ (\overset{\circ}{W} \neq \emptyset)$,
- $\forall U \in \tau\ [U \neq \emptyset \Rightarrow \exists W \in \mathscr{W}\ (W \subset U)]$.

That is to say, the interiors of sets from $\mathscr{W}$ all must have nonempty interiors; and for any nonempty open set in X, there is a member of W that is contained in that set.

Now, to set up the game, one chooses disjoint A, B such that $A \sqcup B = X$.

Definition. The *Banach-Mazur Game $BM\langle A, B, X, \mathscr{W}\rangle$* is a game between two players α and β, who each take turns choosing sets from $\mathscr{W}$ such that $W_n \supset W_{n+1}$ for all $n = 1, 2, \ldots$. Player α chooses sets of odd indices, and player β chooses sets of even indices. Player α wins if

$$A \cap \left(\bigcap_{n \geq 1} W_n \right) \neq \emptyset.$$

Otherwise, player β wins.

Definition. A *play* is any sequence $\{W_n\}$ (n runs over all the positive integers) of sets from $\mathscr{W}$ such that

$$W_1 \supset W_2 \supset W_3 \supset \ \ldots$$

Definition. A *strategy for player α* is a pair $(W, \{f_n\})$ where W is some set of $\mathscr{W}$ and $\{f_n\}$ is a sequence of functions $f_n : \mathscr{W}^{2n} \to \mathscr{W}$ that is, for all n, defined for every sequence of $2n$ sets $W_1, \ldots, W_{2n}$ belonging to $\mathscr{W}$ and $f_n(W_1, \ldots, W_{2n}) \subset W_{2n} \in \mathscr{W}$. It is denoted s_α.

Similarly, A *strategy for player β* is a sequence $\{g_n\}$ of functions $g_n : \mathscr{W}^{2n-1} \to \mathscr{W}$ that is, for all n, defined for every sequence of $2n-1$ sets $W_1, \ldots, W_{2n-1}$ from $\mathscr{W}$ and $g_n(W_1, \ldots, W_{2n-1}) \subset W_{2n-1} \in \mathscr{W}$. It is denoted s_β.

Definition. A play $\{W_i\}$ is *consistent with strategy $s_\alpha = (W, \{f_n\})$* if $W_1 = W$ and $W_{2n+1} = f_n(W_1, \ldots, W_{2n})$ $(n = 1, 2, 3, \ldots)$.

Similarly, a play is then said to be *consistent with strategy $s_\beta = \{g_n\}$* if $W_{2n} = g_n(W_1, \ldots, W_{2n-1})$ $(n = 1, 2, 3, \ldots)$.

In the definition of a strategy, one can think of the set W to be the "opening move"; and one can think of the functions f_n and g_n to be ways for players α and β respectively to make a move depending on the plays made so far.

Any pair (s_α, s_β) uniquely determines a play consistent with both strategies.

Definition. A strategy s_α is *winning for α* if

$$A \cap \left(\bigcap W_n \right) \neq \emptyset$$

for every play consistent with strategy s_α. Then, a strategy s_β is *winning for β* if

$$\bigcap W_n \subset B = X \setminus A$$

for every play consistent with strategy s_β.

Definition. The game $BM\langle A, B, X, \mathscr{W}\rangle$ is *determined in favor of α* (resp. of β) if there exists a winning strategy for *alpha* (resp. β).

V Winning Strategy for the Second Player

Here, let us provide a sufficient condition that guarantees the existence of a winning strategy for player β.

Theorem. The game $BM\langle A, B, X, \mathscr{W}\rangle$ is determined in favor of β if and only if A is of first category of X.

Proof. Suppose that A is of first category. Then A is the countable union of nowhere dense sets in X, i.e., $A = \bigcup_{n \in \mathbb{N}} A_n$ where each A_i is nowhere dense in X. Suppose player α begins the game by choosing a set W_1. Then, player β can pick a set W_2 such that W_2 is contained in the set $\overset{\circ}{W_1} \setminus \overline{A_1}$, which must be open. Then if player β continues the game by picking a set W_3, player β can respond by choosing a set W_4 such that W_4 is contained in the set $\overset{\circ}{W_3} \setminus \overline{A_2}$. In general, let the mapping be defined as follows

$$\gamma_n : (W_1, \ldots, W_{2n-1}) \mapsto W_{2n} \subset \overset{\circ}{W}_{2n-1} \setminus \overline{A_n}$$

for any sequence of $2n-1$ sets $W_1, \ldots, W_{2n-1}$ from $\mathscr{W}$ $(n = 1, 2, \ldots)$. Thus, a strategy for player β has been defined given by $s_\beta = \{\gamma_n\}$. It is also always the case that $\overset{\circ}{W}_{2n-1} \setminus \overline{A_n} \neq \emptyset$ because $\overset{\circ}{W}_{2n-1} \setminus \overline{A_n} = \overset{\circ}{W}_{2n-1} \cap (X \setminus \overline{A_n})$; and since A_n is nowhere dense and that the interior of any set from $\mathscr{W}$ is nonempty, one can see that $X \setminus \overline{A_n}$ is dense (and open) in X and that its intersection with $\overset{\circ}{W}_{2n-1}$ should be a nonempty open set. This means that the strategy s_β does not result in illegal moves.

Now suppose one had a play $\{W_n\}$ consistent with strategy s_β. One then obtains the following chain of relations:

$$\begin{aligned}\bigcap_{n\geq 1} W_n &\subset \bigcap_{n\geq 1}(W_{2n-1} \cap (\overset{\circ}{W}_{2n-1} \setminus \overline{A_n}))\\ &= \bigcap_{n\geq 1}(\overset{\circ}{W}_{2n-1} \setminus \overline{A_n})\\ &\subset \bigcap_{n\geq 1}(X \setminus \overline{A_n})\\ &\subset \bigcap_{n\geq 1}(X \setminus A_n)\\ &= X \setminus \left(\bigcup_{n\geq 1} A_n\right)\\ &= X \setminus A.\end{aligned}$$

This means that player β has a winning strategy given by $s_\beta = \{\gamma_n\}$.

Suppose a winning strategy $s_\beta = \{g_n\}$ for β did exist. Define a *g-chain of order n* to be a sequence of $2n$ sets $W_1, ..., W_{2n}$ from $\mathscr{W}$ such that

$$W_1 \subset W_2 \subset \ldots \subset W_{2n},$$

and for $i = 1, \ldots, n$,

$$W_{2i} = g_i(W_1, \ldots, W_{2i-1}).$$

Then, define the *interior of a g-chain of order n* to be the set $\overset{\circ}{W}_{2n}$.

Lastly, let us define an *g-chain of order $n + k$* to be a continuation of an g-chain of order n such that the first $2n$ elements of both chains are the same. The set of all g-chains can be endowed with a partial ordering induced by the relation of continuation.

First consider the set of all possible g-chains of order 1; then define G_1 to be the maximal family (with respect to the subset relation) of g-chains therein such that the interiors of any two chains in G_1 are disjoint. G_1 exists as guaranteed by Tukey's lemma since the property is of finite character. Define U_1 to be the union of all interiors of members of G_1; then U_1 is a dense open set in X by the maximality of G_1.

Now suppose there exists a family of G_n such that the interiors of any two chains in G_n are disjoint. Now define G_{n+1} to be the largest set of all g-chains of order $n+1$ such that a chain therein is a continuation of G_n and the interiors of any two chains in G_{n+1} are disjoint. Again, such a property is of finite character; so, by Tukey's lemma, such a maximal set must exist. Then the union U_{n+1} of all the interiors of members of G_{n+1} is dense by maximality of G_{n+1}. By way of induction, it has been shown that G_n is defined for all n and that G_{n+1} consists of chains that are continuations chains of G_n and that the union U_n of the interiors all chains in G_n is dense in X.

Now consider $E = \bigcap U_n$ and let $x \in E$. Then there exists a sequence $\{C_n\}$ of g-chains such that $C_n \in G_n$ and x belongs to the interior of C_n. Such a sequence is unique by the property that had been imposed for the interiors of such chains and it is linearly ordered by continuation. Such a sequence yields a play $\{W_n\}$ that is consistent with s_β; since it is a winning strategy by hypothesis, x must be contained in B. Hence, $E = \bigcap U_n \subset B$ and so

$$X \setminus E = \bigcup (X \setminus U_n) \supset A.$$

Since each U_n is a dense open set, its complement in X must be nowhere dense. This, therefore, implies that A should of first category.

This approach could be thought of as a greedy approach as the strategy tries to exclude as much elements of A per move by excluding the countably many sets that "constitute" A one at a time, iteratively following the same move pattern. Another proof of the theorem can also be found in the 2012 book of Kechris [Kec].

VI Winning Strategy for the First Player

When one considers a more specific setting of the Banach-Mazur Game, it is actually possible to come up with a necessary and sufficient condition for players α to have a winning strategy. In particular, let us consider the particular scenario where X is a complete metric space.

Theorem. Suppose the underlying topological space X is a complete metric space. The game $BM\langle A, B, X, \mathscr{W}\rangle$ is determined in favor of α if and only if $V \cap B$ is of first category for some set $V \in \mathscr{W}$.

Proof. Suppose $V \cap B$ is of first category so that $V \cap B = \bigcup B_n$ where each B_n is nowhere dense in X. For each n, let us define a mapping

$$\lambda_n : (W_1, ..., W_{2n}) \mapsto W_{2n+1}$$

such that $\overline{W}_{2n+1} \subset G_{2n} \setminus B_n$ and $\operatorname{diam} G_{2n+1} < \frac{1}{n}$. This yields a strategy $s_\alpha = (V, \{\lambda_n\})$ for player α.

Now consider a play $\{W_n\}$ consistent with the strategy s_α. Then, one obtains the following chain of relations:

$$\begin{aligned}V \cap \left(\bigcap_{n\geq 2} W_n\right) &\subset V \cap \left(\bigcap_{n\geq 1}(W_{2n} \cap (W_{2n} \setminus B_n))\right)\\ &\subset \bigcap_{n\geq 1}(W_{2n} \setminus B_n)\\ &\subset \bigcap_{n\geq 1}(X \setminus B_n)\\ &= X \setminus \left(\bigcup_{n\geq 1} B_n\right)\\ &= X \setminus B.\end{aligned}$$

Hence, $\bigcap W_n \subset A$. Then because $\overline{W}_{2n+1} \subset W_{2n}$ for all n,

$$\begin{aligned}\overline{W}_1 \cap \overline{W}_2 \cap \overline{W}_3 \cap \ldots &\subset \overline{W}_1 \cap \overline{W}_2 \cap W_2 \cap \ldots\\ &= \overline{W_1} \cap W_2 \cap W_4 \cap \ldots\\ &= \overline{W_1} \cap (W_1 \cap W_2) \cap (W_3 \cap W_4) \cap ...\\ &= \bigcap W_n.\end{aligned}$$

This implies that $\bigcap W_n = \bigcap \overline{W}_n$ as it was, by the property of the closure operation, already the case that $\bigcap W_n \subset \bigcap \overline{W}_n$. Then noting that $\operatorname{diam} W_{2n+1} < \frac{1}{n}$ for all n and that those sets are subsets of a complete metric space, it must be the case that $\bigcap \overline{W}_n$, the intersection of an infinite number of nested closed sets, should only contain one point. Therefore, $A \cap (\bigcap W_n) = \bigcap \overline{W}_n$ is a nonempty set, i.e., s_α is a winning strategy.

For the proof of the converse, refer to the 1957 paper of Oxtoby [Oxt].

The strategy employed by player α is analogous to the strategy employed by player β in the previous situation, but adapted to take advantage of the fact that, in this case, X is a complete metric space. Thus, one could also think of player α's strategy as a greedy algorithm as well.

VII Analysis and Comparison to Minimax Algorithm

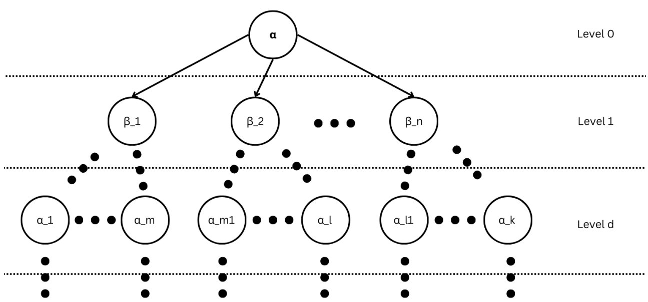

Figure 1: Graphical Representation of a Banach-Mazur Game (a "Game Tree")

The time complexity of the winning strategies for both players α and β is $O(1)$ since the algorithm only depends on the last move of the opponent as stated in the theorem of each winning strategy and that the same number of set-theoretic operations is performed every time. Thus, an algorithm that would attempt to solve a given Banach-Mazur game until a finite depth d (assuming the conditions are in place), would be $O(d)$.

Then compare this to the minimax algorithm, which has a worst-case time complexity of $O(b^d)$ where b is the branching factor (which basically encodes the number possible of moves per level of the game tree) and d is the depth of the game tree [GfG].

Considering general topological spaces, there are potentially an infinite number of choices for every level of the game tree as if the underlying topological space X is an infinite set, then one must have that $\wp(X)$ is also an infinite set. Such is true because $|\wp(X)| > |X|$ (Cantor's theorem; see [Jec]). Thus, regardless of the depth one chooses to analyse the Banach-Mazur game, the minimax algorithm has a worst-case time complexity of $O(\infty)$.

The only way for the time complexity to be not infinite is if one instead considers cases where the underlying topological space is a finite set; however, the study of such Banach-Mazur games is outside the scope of the paper.

Therefore the algorithms that have been presented in the proofs given for the theorems in this paper provide a constant-time approach for either player win given certain special conditions. As the theorems were both statements of logical equivalence, the algorithms are thus inadequate for solving other Banach-Mazur games.

VIII Applications

There are many types of Banach-Mazur games (see [Auri]) where most display an infinite number of strategies, and could result to a never-ending game that calls the attention of formulating an effective strategy to reduce the number of options for efficiency. The theorems for each player's strategy is an example of an optimization in deciding which move to pick. Apart from that, the theorems have applications not just as winning strategies.

Characterization of Baire Spaces. In an 2023 article [Rez] written by Reznichenko, it is explained how topological games can be utilized to categorize different Baire spaces. In the case of this paper, the Banach-Mazur game and its other modifications can be used to characterize Baire spaces in which are spaces where the intersection of countably many dense open sets is dense.

If for any family $(U_n)_n$ of open dense subsets of X the intersection G in dense X is nonempty, then A space X is called *nonmeager* or *Baire*.

In mathematics, Baire spaces are important because it has shown strong results in the class of metric spaces (see the article [Rez]). The results can be extended from metric spaces to larger classes of spaces, wherein topological games such as Banach-Mazur games are used. The applications taken from the Baire property is important that modifications of the Banach-Mazur game use.

It can be proven that a topological space X is a Baire space if and only if β has no winning strategy in the Banach-Mazur game $BM\langle A, B, X, W\rangle$ (a nice proof can be found in [Rev]). Thus, in essence, the proven theorems in the algorithm of a winning strategy for β in the paper can be used to characterize Baire spaces.

IX Conclusion

The winning strategies for both players in Banach-Mazur game provide direct and clear strategies that could be utilized by the players given those specialized conditions. The study of the Banach-Mazur game indirectly highlighted the inadequacy of traditional game-solving and decision-making algorithms like the minimax algorithm in analyzing sophisticated and abstract games like the Banach-Mazur game. Through the use of more involved and mathematical methods, the authors were able to show that it was still possible, given certain special conditions, to show whether it is guaranteed that either player would win a Banach-Mazur game.

The issue is that the algorithm and the game involves a high-level of mathematical abstraction; such is the case due to the inherent abstractness and the potential infinitude of topological objects. It is thus unfortunately the case that implementing the game itself in a computer program is not viable and will never be. Those games and particular game states thereof could be sufficiently represented in terms of programming code of reasonable size, owing to their finite and concrete nature; while the general version of the Banach-Mazur game used in this paper is not. With that, determining how to implement a computational logic system to solve this perfect information game is not like making one for chess and tic-tac-toe in terms of difficulty.

Applications of what had been presented in this paper also had been discussed, showing that such research can indeed be used to further research in fields like mathematics. As presented previously, the Baire property is closely related to whether a Banach-Mazur game is (not) determined in favor of the second player, allowing one to classify Baire spaces in accordance to the determinacy of Banach-Mazur games. That being said, the authors suggest that future inquiry be done on aptly "interesting" topological games and the possible theorems that could be derived therefrom in order to potentially make more bridges between the fields of topology and computer science.

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Appendix A: Record of Contribution

The following table is an oversimplified representation on how both authors contributed to the paper in regards to the distribution of work.

Activity	Abino	Yap
Topic Formulation	50	50
Abstract	50	50
Introduction	50	50
Definition of the Algorithm	0	100
Proofs	0	100
Applications	100	0
Analysis and Comparison to Another Algorithm	100	0
Raw total	250	250
Total	50	50