Investigating the Path of a Double Pendulum

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PH4C1 (MJW/??)

Abstract

Double pendulum systems are known for their chaotic behaviour - their apparently random states of disorder are governed by underlying patterns with strong sensitivity to initial conditions. Although its motion is known for being chaotic, an interesting question would be whether quasiperiodic and/or perfectly periodic motion could occur, depending on its initial conditions. It is precisely this question that I set out to answer. In this investigation, the starting angles required for these different types of motion were analysed. This was done by using a computer simulation to solve coupled ODEs that were derived using Newtonian mechanics. Here, I show that this simulation mostly agrees with previous work done by university students using a different method. For the cases where it doesn't work, I briefly discuss why that might be the case, and end with an example of a potential application of this model.

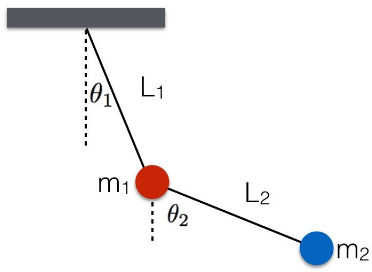
Contents

1 Creating a Simulation

To start off, we shall derive the equations of motion for the pendulum. Newtonian mechanics will be used instead of (the easier option of) Lagrangian mechanics as it is what most readers here will be more familiar with.

1.1 Derivation of Equations of Motion

Consider the diagram below.



1.1.1 Kinematics of the Double Pendulum

Set up a coordinate system with the origin at the top suspension point, with the x-axis defined as the axis in the plane of motion and the y-axis defined upwards. Now, write the equations for x_1, x_2, y_1, and y_2 in terms of θ_1, θ_2, L_1, and L_2.

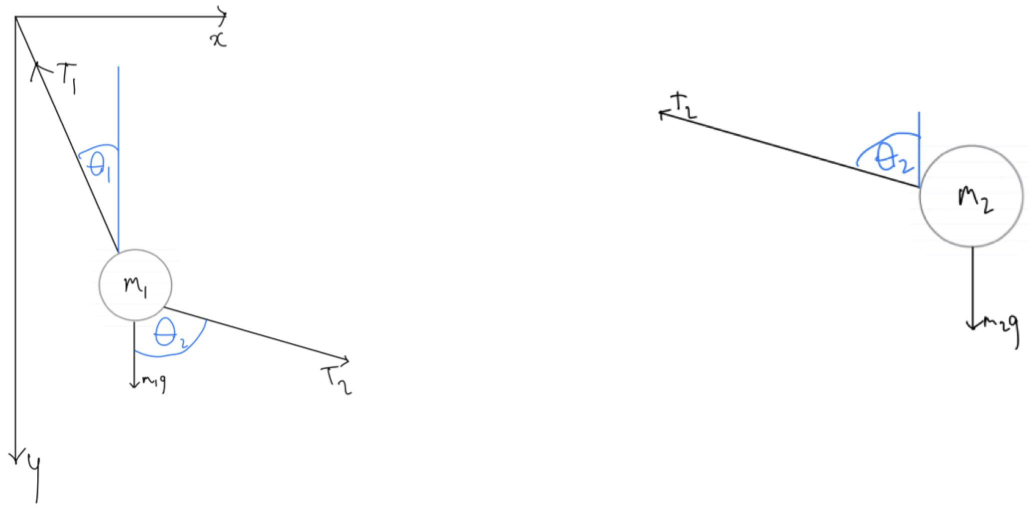
$$\begin{aligned}
x_1 &= L_1 \sin\theta_1 \\
y_1 &= -L_1 \cos\theta_1 \\
x_2 &= x_1 + L_2 \sin\theta_2 \\
y_2 &= y_1 - L_2 \cos\theta_2
\end{aligned}$$

Find the second derivatives of the positions with respect to time (i.e. the acceleration). The angle of pendulum is a function of time, so:

$$\begin{aligned}
x_1'' &= -(\theta_1')^2 L_1 \sin\theta_1 + (\theta_1'') L_1 \cos\theta_1 \\
y_1'' &= (\theta_1')^2 L_1 \cos\theta_1 + (\theta_1'') L_1 \sin\theta_1
\end{aligned}$$

1.1.2 Forces in the Double Pendulum

First, draw a free body diagram for both masses on the pendulum.



Using the classic equation $F = m\frac{dv}{dt}$, we can resolve the forces and obtain 4 equations:

$$
\begin{aligned}
m_1 {x_1}'' &= -T_1 \sin\theta_1 + T_2 \sin\theta_2 \\
m_1 {y_1}'' &= T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g \\
m_2 {x_2}'' &= -T_2 \sin\theta_2 \\
m_2 {y_2}'' &= T_2 \cos\theta_2 - m_2 g
\end{aligned}
$$

By rearranging and substituting these equations into one another, either by hand or by using a computer algebra system, we can find differential equations for θ_1'' and θ_2'' in terms of $\theta_1, \theta_2, \theta_1'$ and θ_2'. Doing it by hand takes a (significantly) longer time, so I have skipped the algebraic manipulation here.

$$
\theta_1'' = \frac{-g(2m_1 + m_2)\sin\theta_1 - m_2 g \sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2) m_2 (\theta_2'^2 L_2 + \theta_1'^2 L_1 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2)))}
$$

$$
\theta_2'' = \frac{2\sin(\theta_1 - \theta_2)(\theta_1'^2 L_1(m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + \theta_2'^2 L_2 m_2 \cos(\theta_1 - \theta_2))}{L_2(2m_1 + m_2 - m_2\cos(2(\theta_1 - \theta_2)))}
$$

I checked this by comparing my answer to an MIT derivation in a similar vein [1]. We are then left to solve the coupled ODEs, for which we have various choices of methods. For our purposes, we shall just use the "odeint" command, which uses a solver called LSODA from the FORTRAN odepack, simply for its convenience and suitability.

1.2 Coding the Simulation

I used Python 3.8.1 to code the simulation, due to its ease of use. Any version of Python will suffice, but this was simply the most convenient as I already had it installed. Although this is quite a major part of the project and was what I spent most of my time trying to troubleshoot and figure out (along with the maths that was done earlier), I shall omit the process (as I did with the maths) because I believe that the main focus of this report should be on some of the results. You can access the code by scanning the QR code below:

The code itself is explained via comments there (albeit quite lightly). A basic knowledge of how Python works (or coding in general) is assumed. To make sure that it made sense, I compared my code to someone else's on an online blog [2].

2 | The Actual Investigation

To test that our simulation works, for each type of motion we shall try the initial conditions that were used by previous researchers [3] and compare our results. Then, we shall try to find more initial conditions which result in those types of motion. We shall keep the masses and lengths of the pendulums constant (with arbitrary units of 10 for both) and just change the angles. For each type of motion, I will include a maximum of 3 sets of solutions that I found - to find more, feel free to copy the code from my repository. It is also important to keep in mind that this is being done visually, i.e. I am using no other tools to confirm the type of motion the double pendulum is experiencing. This is quite risky and almost unreliable, but the other tools I can use are very far beyond my scope and ability to cope with new maths so I have decided not to use them.

All angles are in radians, since that is the superior unit (and also calculus/ODEs wouldn't work without radians so it would make sense to use it).

2.1 Periodic Motion

Periodic motion is motion of a system that is continuously and identically repeated.

2.1.1 Testing

We shall use the initial values that are given in the paper for periodic motion: $\theta_1 = 0.157$ and $\theta_2 = 0.342$.

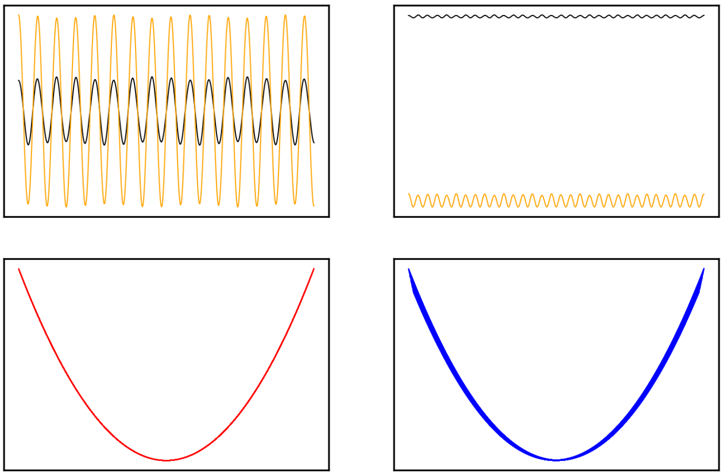


Figure 2.1: For the top graphs, the x and y coordinates of each pendulum are plotted against time. The black line is for the first pendulum (i.e. the one the other pendulum is 'anchored upon'), and the orange is for the second pendulum. The red and blue plots are the paths of the first pendulum and the second pendulum respectively.

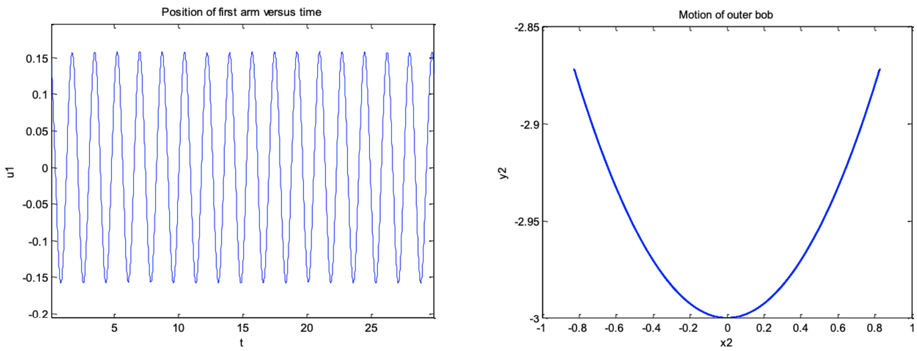


Figure 2.2: The set of results obtained in the paper.

As shown, this gives us the expected results: we get periodic graphs for their coordinates against time, and their paths 'look periodic', i.e. a regular path that we would expect from a periodic system, e.g. a normal pendulum. Hence, we can move on to finding other initial values.

2.1.2 Finding More Periodic Solutions

The strategy for finding interesting solutions is to put in the typical values of angles in radians (e.g. $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$) and see if anything interesting arises. In decimal form, these would be (to 3d.p.) 1.571, 1.047, 0.785, and 0.524 respectively, but we could also try other numbers (e.g. the ones that have already been used), like $\frac{\pi}{20} = 0.157$ to 3 decimal points. **I shall only include two examples for each type of motion**, because if I were to include all possible initial conditions then this would be quite a long report.

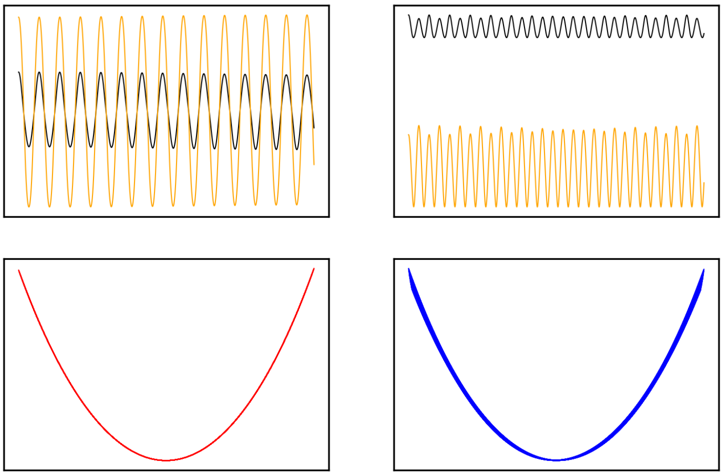


Figure 2.3: $\theta_1 = 0.524 = \frac{\pi}{6}$, $\theta_2 = 0.785 = \frac{\pi}{4}$

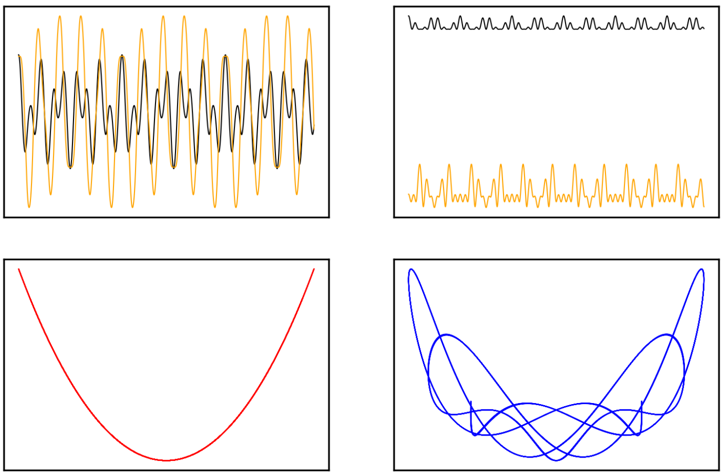


Figure 2.4: $\theta_1 = 0.39$, $\theta_2 = -0.01$. Although it doesn't look like it is periodic, if you look carefully at the graphs of the x and y coordinates against time there are clear repeating paterrns.

2.2 Quasiperiodic Motion

Quasiperiodic motion is motion of a system that is irregular in its periodicity, hence 'quasi-'.

2.2.1 Testing

Again, we use the initial values given in the paper but for quasiperiodic motion: $\theta_1 = 0.78$ and $\theta_2 = -0.156$.

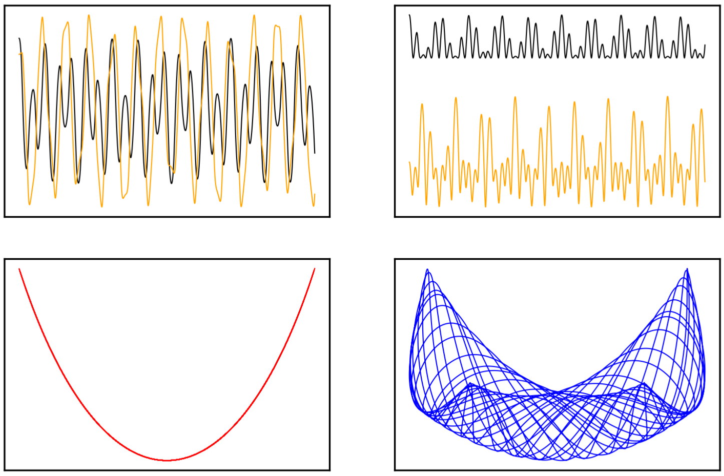


Figure 2.5: $\theta_1 = 0.78$, $\theta_2 = -0.156$. From now on, please just refer back to the first graph for explanations as to what the lines and colours mean, as the styling is consistent throughout.

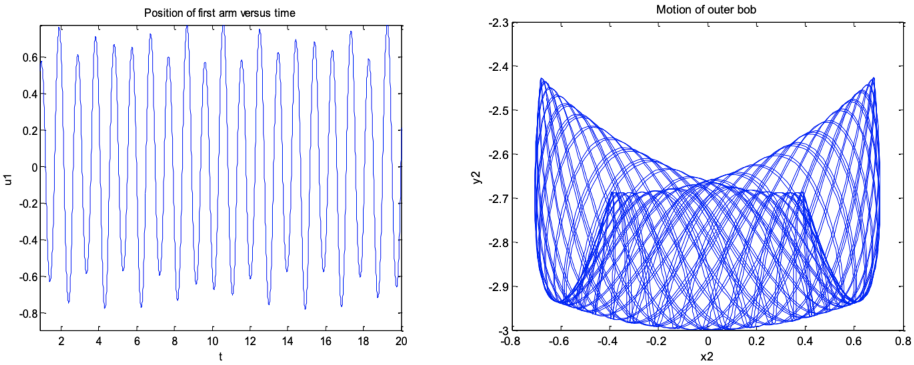


Figure 2.6: The set of results obtained in the paper.

Comparing the above graphs, paired with the tests for the conditions of periodic motion, shows convincing evidence that this simulation works. Thus, we set out to find more values.

2.2.2 Finding More Quasiperiodic Solutions

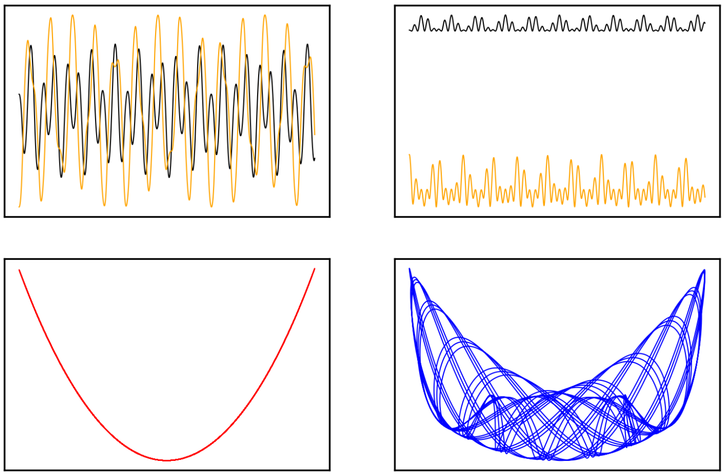


Figure 2.7: $\theta_1 = 0.105 = \frac{\pi}{30}$, $\theta_2 = -0.785 = -\frac{\pi}{4}$

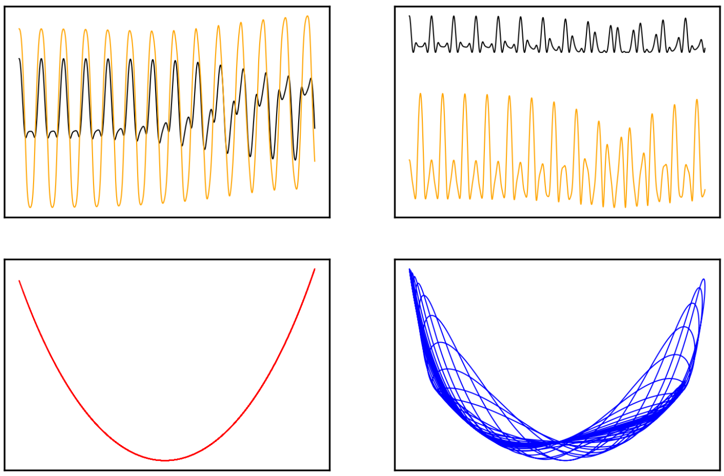


Figure 2.8: $\theta_1 = 0.7$, $\theta_2 = 0.383$

2.3 Chaotic Motion

Chaotic motion, as has already been explained, is motion with random apparent states of disorder. This is, by far, the easiest type of motion to find, since we can most likely just put in any values and will obtain chaotic motion. However, I ran into a problem with the testing, which shows that certain cases don't work on this simulation.

2.3.1 Testing

Using the initial values given ($\theta_1 = 0.157$ and $\theta_2 = 0.157$):

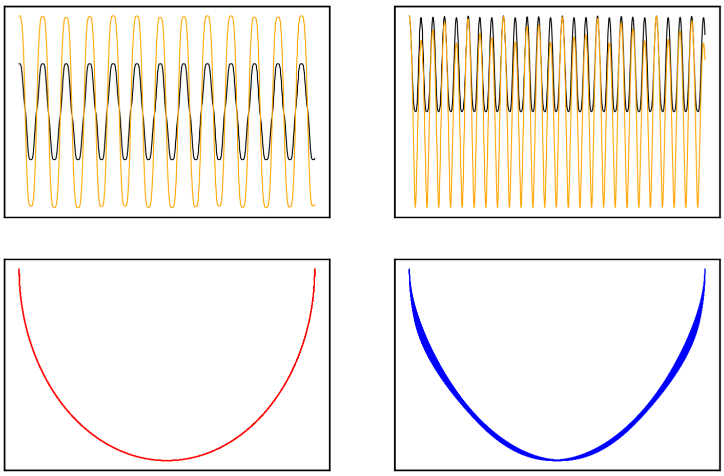


Figure 2.9: $\theta_1 = 0.157$, $\theta_2 = 0.157$

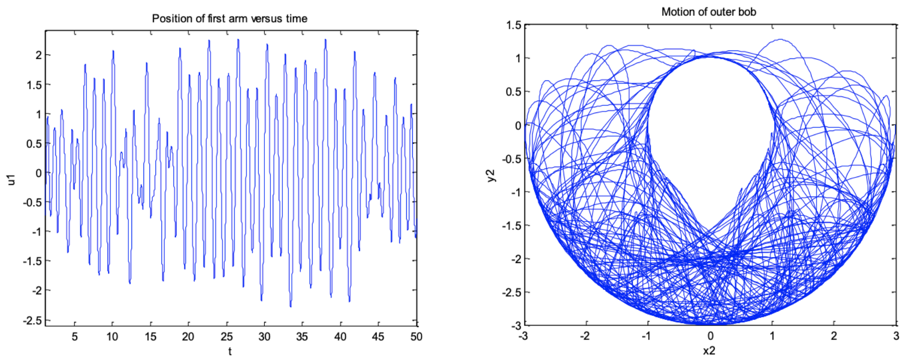


Figure 2.10: The set of results obtained in the paper.

Unlike the other two, our testing failed for the set of initial conditions that they provided. Here, it just looks like the motion is periodic (from looking at our simulation), which is clearly wrong as they had also analysed it with other tools that we shall not use (such as Poincaré sections) and obtained consistent results. Despite this, we strive to test for other conditions which will cause chaotic motion anyways - we shall briefly discuss why some cases might not work later.

2.3.2 Finding More Chaotic Solutions

Interestingly, however, flipping the sign for either of them results in chaotic motion. Whereas for the others the time interval was only set to 100 seconds, for these it was set to 200 seconds because I wanted to see whether I'd coincidentally get anything looking like the graph the researchers obtained. I didn't. I revert to 100 seconds for the next set of graphs.

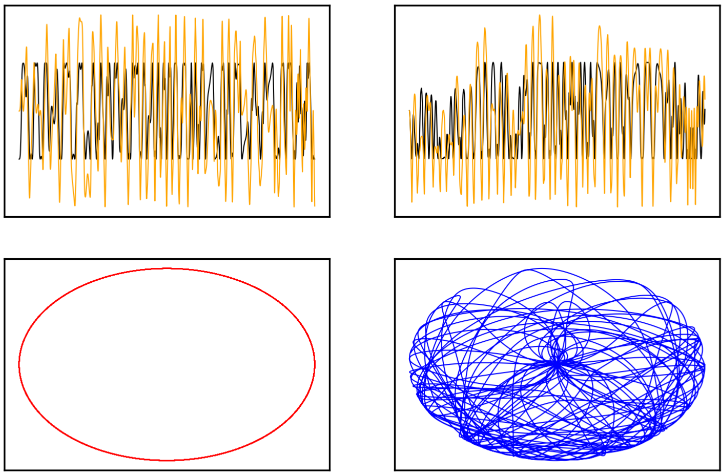


Figure 2.11: $\theta_1 = -0.157$, $\theta_2 = 0.157$

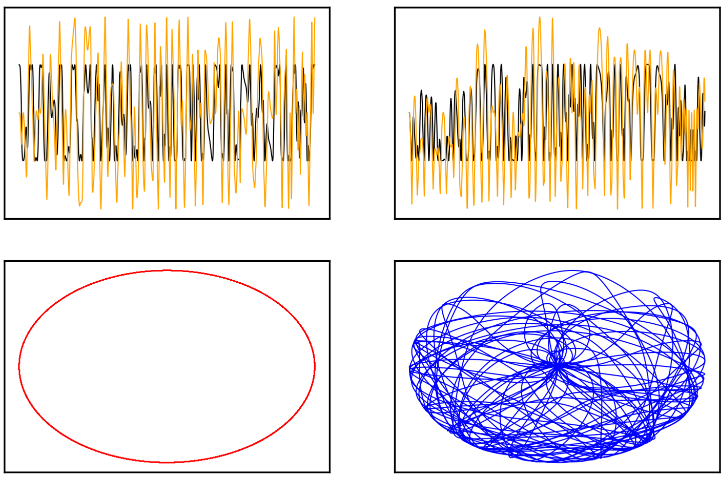


Figure 2.12: $\theta_1 = 0.157$, $\theta_2 = -0.157$

There is a small section in the middle where the paths look identical, but afterwards it just seems like the path is flipped. To show that finding conditions for chaotic motion is a lot easier, I will show another example where the numbers are chosen 'randomly'.

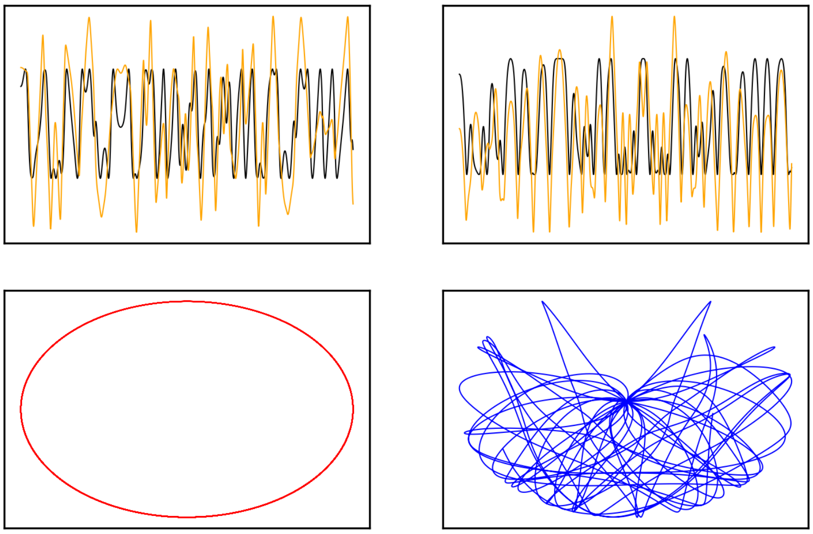


Figure 2.13: $\theta_1 = 2.391$, $\theta_2 = 0.353$ - Two numbers picked by me and a random number generator (respectively).

3 | Conclusions

3.1 Analysis of Results

3.1.1 Failures

As we can see from our testing and comparison to results, there are some cases where our simulation doesn't work as it should, and so this is not a perfect simulation and improvements can definitely be made to it. A comparison to similarly designed code [2] reveals that a check to make sure that the total energy of the system 'makes sense' (i.e. the energy doesn't exceed a certain sensible value) may be the cause, but more investigation would have to take place in order to determine the exact cause as to why this has happened as there isn't an apparent reason (and thus far I haven't managed to find anything, as there aren't many people spending time perfecting their double pendulum model for edge cases). If the energy is truly the problem, we might have to resort to some Lagrangian mechanics to solve it as most people online usually do it via energy methods.
There might also be the problem of the time intervals set, or for how long the simulation was run for. For this simulation, I chose the arbitrary number of 100 seconds, but it doesn't change much in terms of the pattern when I increase it, except for the diagram being denser .

3.1.2 Success and Improvements

Despite its shortcomings, I still deem this experiment a success (mostly) due to the fact that it still works for a lot of initial conditions. The only problem is that it's hard to determine which values it does work for, since there isn't much data to compare to.
Something I should definitely improve on for next time would be my Python proficiency, as admittedly I am not very good with coding.
I provide a potential use for the model below for further reading.

3.2 Usefulness In Sports

Interestingly, golfers' and baseballers' swings can theoretically be modelled as the initial part of a double pendulum's action. The first half of the cycle is very predictable, so the double pendulum model could also be used to described the upper or lower limb movements in actions such as throwing, running, or kicking. If you draw a comparison of the double pendulum diagram with the limbs required for each of the aforementioned actions, it is easy to see why that would make for a good model. For example, in golf, the upper length of the pendulum is the person's straight arm, and the lower length is the golf club itself.

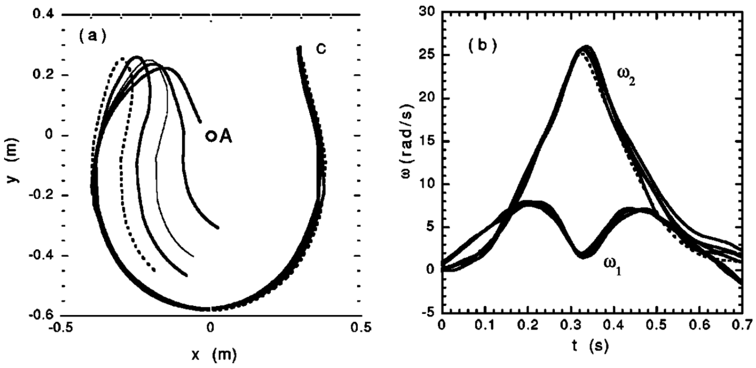


Fig. 4. (a) Trajectories of point C for five swings, all nominally the same, without a stop and without the additional mass on the lower segment. (b) ω_1 and ω_2 vs t for the same five swings.

The figures above come from a famous (in Australian physics circles) University of Sydney researcher's paper [4], physically creating a double pendulum and testing the conditions and proving whether it could actually model these things, which have included in the 'Bibliography' section. It agreed with the theory in that the initial part of the trajectory of the swing is predictable and reproducible, but that afterwards it was highly sensitively dependent on its initial conditions (as you can see in the figures).

Bibliography

[1] https://web.mit.edu/jorloff/www/chaosTalk/double-pendulum/double-pendulum-en.html

[2] https://scipython.com/blog/the-double-pendulum/

[3] https://www.researchgate.net/publication/274634205_THE_DYNAMICS_OF_A_DOUBLE_PENDULUM_CLASSIC_AND_MODERN_APPROACH

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[8] https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/MIT16_07F09_Lec20.pdf