Problem Description - Particles on a Ring

Consider a ring with n equally-spaced points and k particles on them at (clockwise) distances $d_1, \ldots, d_k$ with $d_1 + \cdots + d_k = n$.
At each time step, a particle is selected uniformly at random to move randomly one step to the left or right, with equal probabilities. When two particles meet they are fused into one particle. The process terminates when there is only one particle left.

Let $X(d_1, \ldots, d_k)$ be the number of steps until the process terminates. What is $E[X(d_1, \ldots, d_k)]$, the expected number of steps until the process terminates, as a function of $d_1, \ldots, d_k$? Does this expectation depend on how we select the particle that moves at each step? Give an upper bound for this expectation in terms of n and k.

Analysis:

We can re-frame the problem as a game involving k players. Each player i has d_i units to start with and satisfies :

$d_i > 0, i = 1, \ldots, k$
$d_1 + d_2 + \ldots d_k = n$

At each round, two adjacent players are chosen to play a game (1 is adjacent to k corresponding to the fact that particles form a circle. There are in total k pairs of adjacent players), the winner increases his fortune by 1 unit and loser decreases his fortune by 1 unit. Any player whose fortune drops to 0 is excluded from the game (to count for the fact that particles merge when they meet) and the game ends when a single player has all n units. Each game is equally likely to be won by either of its two players (to count for the fact a chosen particle moves left or right with equal probability). Each pair of players is equally likely to be chosen.

Denote X_i as the number of games player i played. Total number of games played

$$X = \frac{1}{2}\sum_{i=1}^{k} X_i \tag{1}$$

When there are only two players, this is trivially a random walk starting at position 0 and ends when it reaches $-d_1$ or d_2. To see this, we look at the original problem of two particles on a ring. At any round, by randomly choosing one particle and randomly moving it left or right with equal probability leads to the result that either d_1 increases by 1 and d_2 decreases by 1 with probability 1/2, or d_2 increases by 1 and d_1 decreases by 1 with probability 1/2. This is the same as fixing particle 1, always choose particle 2 and move it left or right randomly with equal probability. The position of particle 2 can be seen as position 0 and and position of particle 1 can be seen as position $-d_1$. The process terminates when particle 2 gets to position of particle 1 from the left or from the right, meaning either it reaches $-d_1$ or d_2

The expected time to stop is $d_1 \times d_2$:
Proof
Let $X_i, i = 1, 2, \ldots$ be the amount won in the i^{th} game, $X_i \in \{-1, 1\}$. Then $Z_i = \sum_{j=1}^{i} X_j$ is a martingale with respect to X_i. The first time T satisfies $Z_T \in \{-d_1, d_2\}$ is a stopping time for $X_1, X_2, \ldots$. We can apply the Optional stopping time theorem, because $|Z_i| \le \max(d_1, d_2)$ for every i. We then get

$$E[Z_T] = E[Z_0] = 0$$

On the other hand, if q is the probability of reaching d_2 before reaching $-d_1$, then $E[Z_T] = qd_2 + (1-q)(-d_1)$. Setting this to 0, we get

$$q = \frac{d_1}{d_1 + d_2} \tag{2}$$

Let $Y_t = Z_t^2 - t$ and observe that it is a martingale with respect to $X_1, X_2, \ldots$:

$$\begin{aligned} E[Y_{t+1}|X_1, \ldots, X_t] = E[Z_{t+1}^2|X_1, \ldots, X_t] - (t+1) &= \frac{1}{2}(Z_t+1)^2 + \frac{1}{2}(Z_t-1)^2 - (t+1) \\ &= Z_t^2 - t \\ &= Y_t \end{aligned}$$

We can apply the Optional stopping time theorem because $E[T] < \infty$ and $|Y_{i+1} - Y_i| \le 2\max(d_1, d_2)$. Therefore

$$E[Y_T] = E[Y_0] = 0$$

But $E[Y_T] = E[Z_T^2] - E[T]$. We can compute $E[Z_T^2]$ using the probability q computed previously in (1) :

$$E[Z_T^2] = \frac{d_1}{d_1 + d_2} d_2^2 + \frac{d_2}{d_1 + d_2} d_1^2 = d_1 d_2$$

to find $E[T] = d_1 d_2$. □

We argue that X_i has the same distribution no matter how the pairs of players are chosen in each round. Because from the perspective of player i starting with d_i, he will play independently until his fortune reaches 0 or n, equally likely to win or lose. The number of plays he is involved in is exactly the same as when he has a single opponent with an initial fortune of $n - d_i$:

$$E[X_i] = d_i \cdot (n - d_i) \tag{3}$$

Combining (1) with (3),

$$E[X] = \frac{1}{2}\sum_{i=1}^{k} d_i(n - d_i) = \frac{1}{2}(n^2 - \sum_{i=1}^{k} d_i^2) \tag{4}$$

To find an upper bound for $E[X]$:

Denote $d = [d_1, \ldots, d_k]$. To solve the constrained optimization problem:

$$\text{minimize } f(d) = \sum_{i=1}^{k} d_i^2$$

$$\text{subject to} \sum_{i=1}^{k} d_i = n$$

1. form Lagrangian function with Lagrange multiple α

$$L(d, \alpha) = f(d) - \alpha(\sum_{i=1}^{k} d_i - n)$$

2. to satisfy optimality

$$\frac{\partial L(d, \alpha)}{\partial d_i} = 2d_i - \alpha = 0 \text{ for all } i = 1, \ldots, k$$

Meaning all the d_i's should be the same to yield an optimal solution.

Thus $\sum_{i=1}^{k} d_i^2$ with constraint $\sum_{i=1}^{k} = n$ is minimized when $d_i = \frac{n}{k}, i = 1, \ldots k$. in which case

$$E[X] = \frac{1}{2} n^2 (1 - \frac{1}{k})$$

References

[1] Ross, S. M. (2009).A simple solution to a multiple player gambler's ruin problem.Amer. Math. Monthly 116,77–81.