切比雪夫不等式： 设随机变量 X 的数学期望和方差都存在, 则对任意常数 $\varepsilon>0$, 有

$$\mathrm{P}(|X-EX| \geq \varepsilon) \leq \frac{\operatorname{Var}(X)}{\varepsilon^{2}} \quad \text{或} \quad \mathrm{P}(|X-EX| \geq \varepsilon) \geq 1-\frac{\operatorname{Var}(X)}{\varepsilon^{2}}$$

存在 $\varepsilon_0>0$ 使得等号成立的充要条件为 $P\left(X=EX-\varepsilon_{0}\right)=\frac{1-p}{2}, P\left(X=EX+\varepsilon_{0}\right)=\frac{1-p}{2}$, 其中 $p=P(X=EX)$.

证明:

I 、充分性: 如果随机变量满足:

$$P\left(X=EX-\varepsilon_{0}\right)=\frac{1-p}{2}$$
$$P(X=EX)=p$$
$$P\left(X=EX+\varepsilon_{0}\right)=\frac{1-p}{2}$$

则:

$$P\left(|X-EX| \geq \varepsilon_{0}\right)=P\left(|X-EX|=\varepsilon_{0}\right)=P\left(X=EX+\varepsilon_{0}\right)+P\left(X=EX-\varepsilon_{0}\right)=1-p$$

$$\operatorname{Var}(X)=E(X-EX)^{2}=\varepsilon_{0}^{2} \frac{1-p}{2}+\left(-\varepsilon_{0}\right)^{2} \frac{1-p}{2}+0^{2} p=(1-p) \varepsilon_{0}^{2}$$

由此可得:

$$\mathrm{P}\left(|X-EX| \geq \varepsilon_{0}\right)=\frac{\operatorname{Var}(X)}{\varepsilon_{0}^{2}}$$

II 、必要性: 设随机变量 X 的分布函数为 $\mathrm{F_X(x)}$

由题设可知

$$\varepsilon_{0}^{2} P\left(|X-EX| \geq \varepsilon_{0}\right)=\operatorname{Var}(X)$$

而

$$\operatorname{Var}(X)=\int_{|x-EX|<\varepsilon_{0}}(x-EX)^{2} d F_{X}(x)+\int_{|x-EX| \geq \varepsilon_{0}}(x-EX)^{2} d F_{X}(x)$$

假设 $P\left(0<|X-EX|<\varepsilon_{0}\right)>0$ 则:

$$\int_{0<|x-EX|<\varepsilon_{0}}(x-EX)^{2} d F_{X}(x)>0$$

于是有:

$$\operatorname{Var}(X) \geq \int_{|x-EX|<\varepsilon_{0}}(x-EX)^{2} d F_{X}(x)+\varepsilon_{0}^{2} P\left(|X-EX| \geq \varepsilon_{0}\right)>\varepsilon_{0}^{2} P\left(|X-EX| \geq \varepsilon_{0}\right)$$

与题设矛盾, 故 $P\left(0<|X-EX|<\varepsilon_{0}\right)=0$, 由前面证明可知

$$\operatorname{Var}(X)=\int_{|x-EX| \geq \varepsilon_{0}}(x-EX)^{2} d F_{X}(x)$$

假设 $P=(|X-EX|>\varepsilon_0)>0$, 则得:

$$\int_{|x-EX|>\varepsilon_0}(x-EX)^2dF_x(x)>0$$

于是有

$$\operatorname{Var}(X)=\varepsilon_0^2P\left(|X-EX|=\varepsilon_0\right)+\int_{|x-EX|>\varepsilon_0}(x-EX)^2dF_X(x)>\varepsilon_0^2P\left(|X-EX|=\varepsilon_0\right)$$

这与题设矛盾, 故 $P=(|X-EX|>\varepsilon_0)=0$, 于是得到:

$$P=(|X-EX|=\varepsilon_0)=1-P(|X-EX|=0)=1-p$$

即

$$P\left(X=EX-\varepsilon_0\right)=\frac{1-p}{2}$$
$$P\left(X=EX+\varepsilon_0\right)=\frac{1-p}{2}$$
$$p=P(X=EX)$$