1 课本习题

1.1 P114/7

(1)

(i)

$$\begin{aligned} E(Y) = E(2X) &= \int_{-\infty}^{\infty} 2xf(x)\,\mathrm{d}x \\ &= 2\int_{0}^{\infty} xe^{-x}\,\mathrm{d}x = 2; \end{aligned} \tag{1}$$

(ii)

$$E(Y) = E(e^{-2X}) = \int_{0}^{\infty} e^{-2x} \cdot e^{-x}\,\mathrm{d}x = \frac{1}{3}. \tag{2}$$

(2)

X_i 的分布函数为

$$F(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases} \tag{3}$$

(i)

$X_1, X_2, \cdots, X_n$ 独立，所以 $U = \max\{X_1, X_2, \cdots, X_n\}$ 的分布函数为

$$F_U(u) = \begin{cases} 0, & u < 0, \\ u^n, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases} \tag{4}$$

U 的概率密度函数为

$$f_U(u) = \begin{cases} nu^{n-1}, & 0 < u < 1, \\ 0, & \text{其他}. \end{cases} \tag{5}$$

$$E(U) = \int_{0}^{1} u \cdot nu^{n-1}\,\mathrm{d}u = \frac{n}{n+1}. \tag{6}$$

(ii)

$V = \min\{X_1, X_2, \cdots, X_n\}$ 的分布函数为

$$F_V(v) = \begin{cases} 0, & v < 0, \\ 1-(1-v)^n, & 0 \le v < 1, \\ 1, & v \ge 1. \end{cases} \tag{7}$$

V 的概率密度函数为

$$f_V(v) = \begin{cases} n(1-v)^{n-1}, & 0 < v < 1, \\ 0, & \text{其他}. \end{cases} \tag{8}$$

$$E(V) = \int_{0}^{1} v \cdot n(1-v)^{n-1}\,\mathrm{d}v = \frac{1}{n+1}. \tag{9}$$

1.2 P115/10

(1)

由对称性知

$$E(\frac{X^2}{X^2+Y^2}) = E(\frac{Y^2}{X^2+Y^2}). \tag{10}$$

而

$$E(\frac{X^2}{X^2+Y^2}) + E(\frac{Y^2}{X^2+Y^2}) = E(1) = 1. \tag{11}$$

所以

$$E(\frac{X^2}{X^2+Y^2}) = E(\frac{Y^2}{X^2+Y^2}) = \frac{1}{2}. \tag{12}$$

(2)

X 与 Y 独立，因而 (X,Y) 的联合概率密度函数为

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}. \tag{13}$$

原点到 (X,Y) 的距离为 $R = \sqrt{X^2+Y^2}$，期望为

$$E(R) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{X^2+Y^2}\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}\,\mathrm{d}x\mathrm{d}y. \tag{14}$$

换成极坐标系

$$E(R) = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\, r\mathrm{d}r = \sigma\sqrt{\frac{\pi}{2}}. \tag{15}$$

1.3 P115/12

$$\begin{aligned} E(S) = E\left(\frac{\pi d^2}{4}\right) &= \int_a^b \frac{\pi d^2}{4}\frac{1}{b-a}\mathrm{d}d \\ &= \frac{\pi(a^2+ab+b^2)}{12}. \end{aligned} \tag{16}$$

1.4 P115/15

设

$$X_i = \begin{cases} 1, & \text{第 } i \text{ 个球放入了第 } i \text{ 号盒子}, \\ 0, & \text{第 } i \text{ 个球未放入第 } i \text{ 号盒子}, \end{cases} \tag{17}$$

则 X_i 满足 0-1 分布，$P\{X_i=1\} = \frac{1}{n}$，$P\{X_i=0\} = 1-\frac{1}{n}$，因此

$$E(X) = \sum_i E(X_i) = n\cdot\frac{1}{n} = 1. \tag{18}$$

1.5 P116/20

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{+\infty} kp(1-p)^{k-1} \\
&= p\sum_{k=1}^{+\infty} k(1-p)^{k-1} \\
&= p\sum_{k=1}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}(1-p)}(1-p)^{k} \\
&= p\frac{\mathrm{d}}{\mathrm{d}(1-p)}\left(\sum_{k=1}^{+\infty}(1-p)^{k}\right) \\
&= p\cdot\frac{1}{p^2} = \frac{1}{p}, \qquad (19) \\
E(X(X+1)) &= \sum_{k=1}^{+\infty} k(k+1)p(1-p)^{k-1} \\
&= p\sum_{k=1}^{+\infty} \frac{\mathrm{d}^2}{\mathrm{d}(1-p)^2}(1-p)^{k+1} \\
&= p\frac{\mathrm{d}^2}{\mathrm{d}(1-p)^2}\left(\sum_{k=1}^{+\infty}(1-p)^{k+1}\right) \\
&= \frac{2}{p^2}, \qquad (20) \\
D(X) &= E(X^2) - [E(X)]^2 \\
&= E(X(X+1)) - E(X) - [E(X)]^2 \\
&= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}. \qquad (21)
\end{aligned}
$$

1.6 P117/28

$$
\begin{aligned}
f_X(x) &= \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} f(x,y)d\,y \\
&= \frac{2}{\pi}\sqrt{1-x^2}, && (22) \\
f_Y(y) &= \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} f(x,y)d\,x \\
&= \frac{2}{\pi}\sqrt{1-y^2}, && (23) \\
E[X] &= \int_{-1}^{1} xf_X(x)d\,x \\
&= 0, && (24) \\
E[Y] &= \int_{-1}^{1} yf_Y(y)d\,y \\
&= 0, && (25) \\
E[XY] &= \int_{-1}^{1} d\,x \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} xyf(x,y)d\,y \\
&= 0, && (26) \\
& && (27)
\end{aligned}
$$

因此 $E[XY] = E[X]E[Y]$，则 $\mathrm{cov}(X,Y) = 0, \rho = 0$，$X$ 与 Y 不相关。但显然 $f_X(x)f_Y(y) \neq f(x,y)$，因此 X 与 Y 不独立。

1.7 P117/32

$$
\begin{aligned}
E[X] &= \int_0^2 d\,x \int_0^2 xf(x,y)d\,y \\
&= \frac{7}{6}, && (28) \\
E[Y] &= \int_0^2 d\,x \int_0^2 yf(x,y)d\,y \\
&= \frac{7}{6}, && (29) \\
E[XY] &= \int_0^2 d\,x \int_0^2 xyf(x,y)d\,y \\
&= \frac{4}{3}, && (30) \\
\text{cov}(X,Y) &= E[XY] - E[X]E[Y] = -\frac{1}{36}, && (31) \\
E[X^2] &= \int_0^2 d\,x \int_0^2 x^2 f(x,y)d\,y \\
&= \frac{5}{3}, && (32) \\
E[Y^2] &= \int_0^2 d\,x \int_0^2 y^2 f(x,y)d\,y \\
&= \frac{5}{3}, && (33) \\
D[X] &= E[X^2] - (E[X])^2 = \frac{11}{36}, && (34) \\
D[Y] &= E[Y^2] - (E[Y])^2 = \frac{11}{36}, && (35) \\
\rho &= \frac{\text{cov}(X,Y)}{\sqrt{D[X]}\sqrt{D[Y]}} = -\frac{1}{11}, && (36) \\
D[X+Y] &= D[X] + D[Y] + 2\text{cov}(X,Y) = \frac{5}{9}. && (37)
\end{aligned}
$$

1.8 P117/33

由于 X，Y 相互独立，$\rho_{XY} = 0$，则

$$
\begin{aligned}
E[Z_1] &= \alpha E[X] + \beta E[Y] = (\alpha+\beta)\mu, && (38) \\
E[Z_2] &= \alpha E[X] - \beta E[Y] = (\alpha-\beta)\mu, && (39) \\
E[Z_1 Z_2] &= E[\alpha^2 X^2 - \beta^2 Y^2] = (\alpha^2-\beta^2)(\mu^2+\sigma^2), && (40) \\
D[Z_1] &= \alpha^2 D[X] + \beta^2 D[Y] = (\alpha^2+\beta^2)\sigma^2, && (41) \\
D[Z_2] &= \alpha^2 D[X] + \beta^2 D[Y] = (\alpha^2+\beta^2)\sigma^2, && (42)
\end{aligned}
$$

因此

$$
\rho_{Z_1 Z_2} = \frac{E[Z_1 Z_2] - E[Z_1]E[Z_2]}{\sqrt{D[Z_1]}\sqrt{D[Z_2]}} = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}. \qquad (43)
$$

1.9 P117/34

(1)

$$E[X^2] = D[X] + (E[X])^2 = 4, \tag{44}$$
$$E[Y^2] = D[Y] + (E[Y])^2 = 16, \tag{45}$$
$$E[XY] = \rho_{XY}\sqrt{D[X]}\sqrt{D[Y]} + E[X]E[Y] = -4, \tag{46}$$

因此

$$\begin{aligned} E[W] &= E[a^2X^2 + 6aXY + 9Y^2] \\ &= a^2E[X^2] + 6aE[XY] + 9E[Y^2] \\ &= 4(a^2 - 6a + 36), \end{aligned} \tag{47}$$

对上式求极值，得到：当 $a = 3$ 时，$E[W]$ 取最小值 108。

(2)

$$\begin{aligned} \mathrm{cov}(W, V) &= E[WV] - E[W]E[V] \\ &= E[X^2 - a^2Y^2] - E[X + aY]E[X - aY] \\ &= E[X^2] - a^2E[Y^2] - (E[X] + aE[Y])(E[X] - aE[Y]) \\ &= \sigma_X^2 + \mu_X^2 - a^2\sigma_Y^2 - a^2\mu_Y^2 - (\mu_X^2 - a^2\mu_Y^2) \\ &= \sigma_X^2 - a^2\sigma_Y^2, \end{aligned} \tag{48}$$

当 $a^2 = \dfrac{\sigma_X^2}{\sigma_Y^2}$ 时，$\mathrm{cov}(W, V) = 0$，即 $\rho_{WV} = 0$。同时由于 W, V 是服从二维正态分布的随机变量的线性组合，因此 W, V 也服从二维正态分布。其相关系数 $\rho = 0$ 说明它们相互独立。

1.10 P117/37

(1) 若 $E(V^2) = 0$，则 $D(V) = 0$，从而 $P\{V = 0\} = 1$，$E(VW) = 0$，不等式取等号。
$E(W^2) = 0$ 时，同理。
(2) 若 $E(V^2) > 0, E(W^2) > 0$，考虑实变量 t 的函数：

$$q(t) = E[(V + tW)^2] = E(V^2) + 2tE(VW) + t^2E(W^2). \tag{49}$$

对于任意 t，$q(t) \geq 0$, $E(W^2) > 0$，所以二次三项式 $q(t)$ 的判别式

$$\Delta = 4[E(VW)]^2 - 4E(V^2)E(W^2) \leq 0, \tag{50}$$

即

$$[E(VW)]^2 \leq E(V^2)E(W^2). \tag{51}$$

取等条件为 $\Delta = 0$, 即 $q(t) = 0$ 有一个根。此时 $t = -E(VW)/E(W^2)$。

$$V = -tW = \frac{E(VW)}{E(W^2)}W. \tag{52}$$

即当 $V = cW$ 时，等号成立 (c 为任一常数)。

2 补充题

2.1

设 X 为三个人中生日在第一季度的人数，$X=0,1,2,3$，则 $X \sim b(3,0.25)$，因此 X 的分布律为

X	0	1	2	3
p_k	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

因此 X 的数学期望为

$$E(X)=\sum_k kp_k=0.75. \tag{53}$$

2.2

(1)

$$\begin{aligned} E\left(\frac{1}{x}\right) &= \int_\alpha^\beta \frac{1}{x}\frac{1}{\beta-\alpha}\mathrm{d}x \\ &= \frac{\ln(\beta/\alpha)}{\beta-\alpha}. \end{aligned} \tag{54}$$

(2)

$$\begin{aligned} 1/E(x) &= 1/\int_\alpha^\beta x\frac{1}{\beta-\alpha}\mathrm{d}x \\ &= \frac{2}{\alpha+\beta}. \end{aligned} \tag{55}$$

2.3

(1)

$$a+0.2+b+0.1=1, \tag{56}$$

$$E(x)=-a+0.2+2b+0.3=1, \tag{57}$$

解得

$$a=0.3,\ b=0.4. \tag{58}$$

(2) 随机变量 $Y=\dfrac{1}{X}$ 的分布律为

$Y=\frac{1}{X}$	-1	$\frac{1}{3}$	$\frac{1}{2}$	1
p_k	0.3	0.1	0.4	0.2

则 $E(Y)=\sum_k kp_k=\dfrac{2}{15}$,

$$\begin{aligned} D(Y) &= \sum_k (k-E(Y))^2 p_k \\ &= 0.593. \end{aligned} \tag{59}$$

2.4

由柯西-施瓦茨不等式

$$E(\frac{1}{X_2})E(X_2) = E((\frac{1}{\sqrt{X_2}})^2)E((\sqrt{X_2})^2) \geq E(1)^2 = 1. \tag{60}$$

等号成立当且仅当存在常数，$\sqrt{X_2} = c/\sqrt{X_2}$，即 $X_2 = c$ 时成立。X_1, X_2 独立，则 $X_1, 1/X_2$ 也独立，所以

$$E(X_1/X_2) = E(X_1)E(1/X_2) \geq E(X_1)/E(X_2) = 1. \tag{61}$$

等号成立当且仅当 X_1, X_2 只取一个常数 c 时成立。