

切比雪夫不等式： 设随机变量 X 的数学期望和方差都存在, 则对任意常数 $\varepsilon > 0$, 有

$$P(|X - EX| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2} \quad \text{或} \quad P(|X - EX| \geq \varepsilon) \geq 1 - \frac{\text{Var}(X)}{\varepsilon^2}$$

存在 $\varepsilon_0 > 0$ 使得等号成立的充要条件为 $P(X = EX - \varepsilon_0) = \frac{1-p}{2}, P(X = EX + \varepsilon_0) = \frac{1-p}{2}$, 其中 $p = P(X = EX)$.

证明：

I、充分性: 如果随机变量满足:

$$P(X = EX - \varepsilon_0) = \frac{1-p}{2}$$

$$P(X = EX) = p$$

$$P(X = EX + \varepsilon_0) = \frac{1-p}{2}$$

则:

$$P(|X - EX| \geq \varepsilon_0) = P(|X - EX| = \varepsilon_0) = P(X = EX + \varepsilon_0) + P(X = EX - \varepsilon_0) = 1 - p$$

$$\text{Var}(X) = E(X - EX)^2 = \varepsilon_0^2 \frac{1-p}{2} + (-\varepsilon_0)^2 \frac{1-p}{2} + 0^2 p = (1-p)\varepsilon_0^2$$

由此可得:

$$P(|X - EX| \geq \varepsilon_0) = \frac{\text{Var}(X)}{\varepsilon_0^2}$$

II、必要性: 设随机变量 X 的分布函数为 $F_X(x)$

由题设可知

$$\varepsilon_0^2 P(|X - EX| \geq \varepsilon_0) = \text{Var}(X)$$

而

$$\text{Var}(X) = \int_{|x-EX| < \varepsilon_0} (x - EX)^2 dF_X(x) + \int_{|x-EX| \geq \varepsilon_0} (x - EX)^2 dF_X(x)$$

假设 $P(0 < |X - EX| < \varepsilon_0) > 0$ 则:

$$\int_{0 < |x-EX| < \varepsilon_0} (x - EX)^2 dF_X(x) > 0$$

于是有:

$$\text{Var}(X) \geq \int_{|x-EX| < \varepsilon_0} (x - EX)^2 dF_X(x) + \varepsilon_0^2 P(|X - EX| \geq \varepsilon_0) > \varepsilon_0^2 P(|X - EX| \geq \varepsilon_0)$$

与题设矛盾, 故 $P(0 < |X - EX| < \varepsilon_0) = 0$, 由前面证明可知

$$\text{Var}(X) = \int_{|x-EX| \geq \varepsilon_0} (x - EX)^2 dF_X(x)$$

假设 $P(|X - EX| > \varepsilon_0) > 0$, 则得:

$$\int_{|x-EX|>\varepsilon_0} (x - EX)^2 dF_x(x) > 0$$

于是有

$$\text{Var}(X) = \varepsilon_0^2 P(|X - EX| = \varepsilon_0) + \int_{|x-EX|>\varepsilon_0} (x - EX)^2 dF_X(x) > \varepsilon_0^2 P(|X - EX| = \varepsilon_0)$$

这与题设矛盾, 故 $P(|X - EX| > \varepsilon_0) = 0$, 于是得到:

$$P(|X - EX| = \varepsilon_0) = 1 - P(|X - EX| = 0) = 1 - p$$

即

$$P(X = EX - \varepsilon_0) = \frac{1 - p}{2}$$

$$P(X = EX + \varepsilon_0) = \frac{1 - p}{2}$$

$$p = P(X = EX)$$