8 Exercise: Stresses

8.1 Stress vectors

Consider the motion of the tetrahedron from the previous exercise:

$$[\underline{\underline{\mathbf{R}}}(t)] = \begin{bmatrix} \cos(\frac{2\pi t}{t_{max}}) & -\sin(\frac{2\pi t}{t_{max}}) & 0 \\ \sin(\frac{2\pi t}{t_{max}}) & \cos(\frac{2\pi t}{t_{max}}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\underline{\mathbf{b}}(t)] = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{20}\frac{t}{t_{max}} \end{bmatrix}$$

with $t_{max} = 1/2$.
Anticipating a global equilibrium, the resulting contact forces and moments acting on the tetrahedron are given by the inertial minus the volume forces and moments calculated in exercise 7:

$$^{con}\underline{\mathbf{f}}(t) = {}^{ine}\underline{\mathbf{f}}(t) - {}^{vol}\underline{\mathbf{f}}(t) \qquad [^{con}\underline{\mathbf{f}}] = \begin{bmatrix} -\frac{\pi^2}{15}(\cos(4\pi t) - \sin(4\pi t)) \\ -\frac{\pi^2}{15}(\cos(4\pi t) + \sin(4\pi t)) \\ \frac{5}{3} \end{bmatrix}$$

$$^{con}\hat{\underline{\mathbf{f}}}_0(t) = {}^{ine}\hat{\underline{\mathbf{f}}}_0(t) - {}^{vol}\hat{\underline{\mathbf{f}}}_0(t) \qquad \left[^{con}\hat{\underline{\mathbf{f}}}_0\right] = \frac{125 + \pi^2(4 + 60t)}{3000} \begin{bmatrix} \sin(4\pi t) + \cos(4\pi t) \\ \sin(4\pi t) - \cos(4\pi t) \\ 0 \end{bmatrix}$$

By analogy with a homogeneous small strain motion of the nodes

$$^{i}\underline{\mathbf{u}}(t) = \underline{\underline{\mathbf{H}}}(t)^{i}\underline{\mathbf{x}} + \underline{\mathbf{b}}(t) = (\underline{\underline{\boldsymbol{\epsilon}}}(t) + \underline{\underline{\mathbf{A}}}(t))^{i}\underline{\mathbf{x}} + \underline{\mathbf{b}}(t)$$

we write a linear form for the force vectors $^{i}\underline{\mathbf{f}}$ acting on the vertices i in the actual configuration

$$^{i}\underline{\mathbf{f}}(t) = -\frac{1}{3}(\underline{\underline{\mathbf{T}}}(t) + \underline{\underline{\mathbf{W}}}(t))\,{}^{i}A_t(t)\,{}^{i}\underline{\mathbf{n}}_t(t) + \frac{1}{4}{}^{con}\underline{\mathbf{f}}(t)$$

where $^{i}A_t\,{}^{i}\underline{\mathbf{n}}$ is the actual face normal weighted by the actual area, $^{con}\underline{\mathbf{f}}(t)$ corresponds to the sum of the contact forces acting on the tetrahedron and $\underline{\underline{\mathbf{T}}}(t)$, $\underline{\underline{\mathbf{W}}}(t)$ are symmetric, respectively antisymmetric second order stress tensors.
Using the usual relation for antisymmetric tensors

$$\underline{\underline{\mathbf{W}}}(t)\,{}^{i}A_t\,{}^{i}\underline{\mathbf{n}}_t(t) = -\frac{1}{2}\underline{\mathbf{w}}(t) \wedge {}^{i}A_t\,{}^{i}\underline{\mathbf{n}}_t(t)$$

and the following definition

$$\underline{\mathbf{w}}(t) = -4((^{cfi}\underline{\mathbf{y}} \cdot {}^{i}A_t{}^{i}\underline{\mathbf{n}}_t)\underline{\underline{\mathbf{I}}} + (^{cfi}\underline{\mathbf{y}} \otimes {}^{i}A_t{}^{i}\underline{\mathbf{n}}_t))^{-1}(^{con}\hat{\underline{\mathbf{f}}}_0 - {}^{c}\underline{\mathbf{y}} \wedge {}^{con}\underline{\mathbf{f}}(t))$$

with summation over i and ${}^{con}\hat{\underline{\mathbf{f}}}_0$ being the sum of the contact moments acting on the tetrahedron with respect to the origin.
Show that

$$\sum_{i=1}^{4} {}^{i}\underline{\mathbf{f}}(t) = {}^{con}\underline{\mathbf{f}}(t) \qquad \forall \underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}^T \tag{1}$$

$$\sum_{i=1}^{4} {}^{i}\underline{\mathbf{y}}(t) \wedge {}^{i}\underline{\mathbf{f}}(t) = {}^{con}\hat{\underline{\mathbf{f}}}_0(t) \qquad \forall \underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}^T \tag{2}$$

In fact, the nodal forces can be converted to stress vectors ${}^{i}\underline{\mathbf{t}}$ applied on the opposed face:

$$ {}^{i}\underline{\mathbf{t}}(t) = \frac{{}^{con}\underline{\mathbf{f}}(t) - 3\,{}^{i}\underline{\mathbf{f}}(t)}{{}^{i}A} \qquad i = 1,4$$

The stress vectors become a superposition of a homogeneous Cauchy stress and an external contribution that balances forces and moments

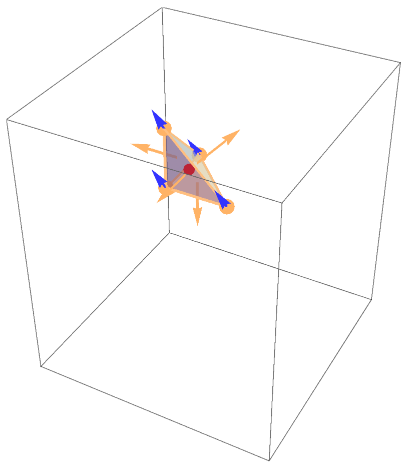
$$ {}^{i}\underline{\mathbf{t}}(t) = \underline{\underline{\mathbf{T}}}(t)^{i}\underline{\mathbf{n}}_t(t) - \frac{1}{2}\underline{\mathbf{w}}(t) \wedge {}^{i}\underline{\mathbf{n}}_t(t) + \frac{1}{4\,{}^{i}A_t(t)}{}^{con}\underline{\mathbf{f}}(t)$$

Show that

$$\sum_{i=1}^{4} {}^{i}A_t\,{}^{i}\underline{\mathbf{t}}(t) = {}^{con}\underline{\mathbf{f}}(t) \qquad \forall \underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}^T \tag{3}$$

$$\sum_{i=1}^{4} {}^{cfi}\underline{\mathbf{y}}(t) \wedge {}^{i}A_t\,{}^{i}\underline{\mathbf{t}}(t) = {}^{con,0}\hat{\underline{\mathbf{f}}}(t) \qquad \forall \underline{\underline{\mathbf{T}}} = \underline{\underline{\mathbf{T}}}^T \tag{4}$$

Plot the nodal forces ${}^{i}\underline{\mathbf{f}}$ at $t = t_{max}$ in the case $\underline{\underline{\mathbf{T}}} = 0$



When inertial and body forces and moments vanish, the Cauchy stress vector is recovered

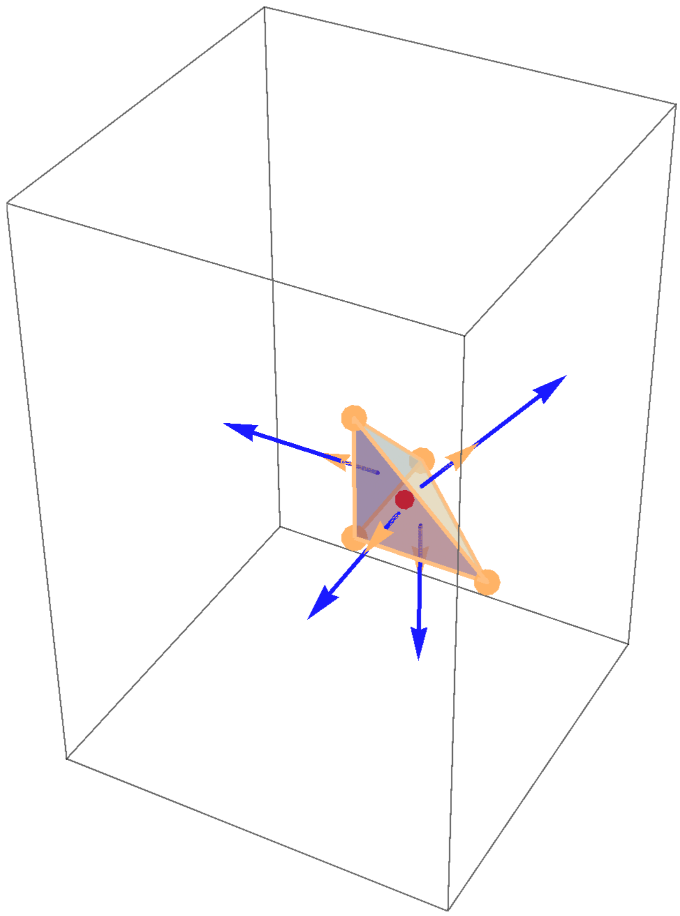
$$^{i}\underline{\mathbf{t}}(t) = \underline{\underline{\mathbf{T}}}(t)\,^{i}\underline{\mathbf{n}}_t(t)$$

8.2 Hydrostatic stress

In the absence of rotation, translation and volume forces, plot the stress vectors on the original face centroids of the tetrahedron for a hydrostatic stress

$$\underline{\underline{\mathbf{T}}} = -\pi\underline{\underline{\mathbf{I}}} \tag{5}$$

with $\pi = 25$ kPa.

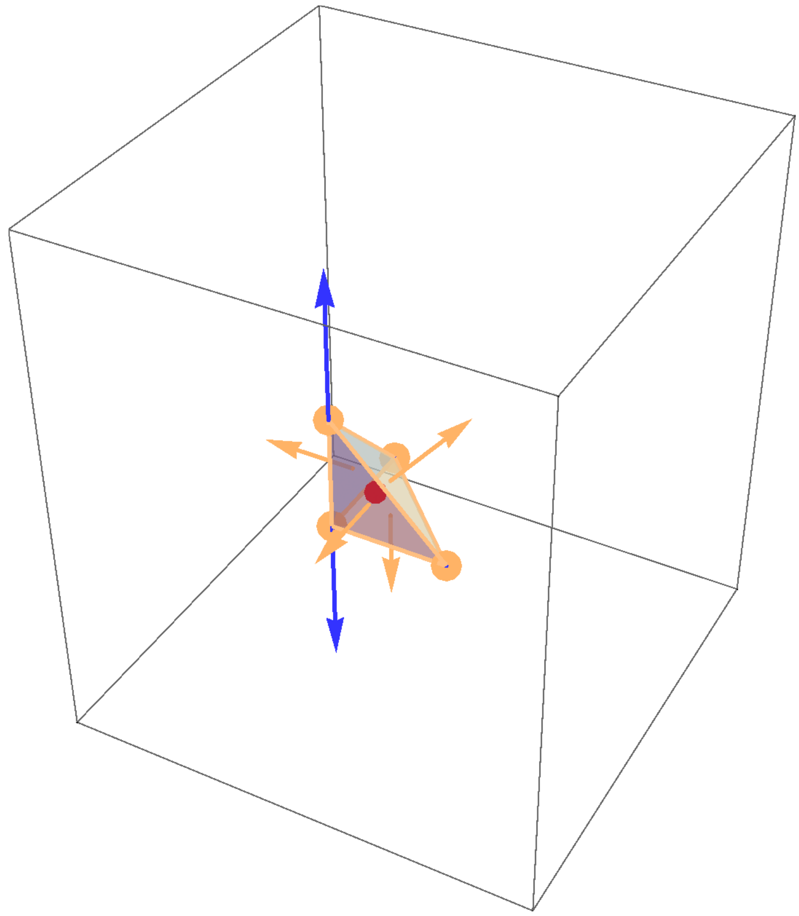


8.3 Uniaxial stress

In the absence of rotation, translation and volume forces, plot the nodal force vectors on the original vertices of the same tetrahedron for a uniaxial stress

$$\underline{\underline{\mathbf{T}}} = \sigma \left({}^3\underline{\mathbf{e}} \otimes {}^3\underline{\mathbf{e}}\right) \tag{6}$$

with $\sigma = 10$ kPa.

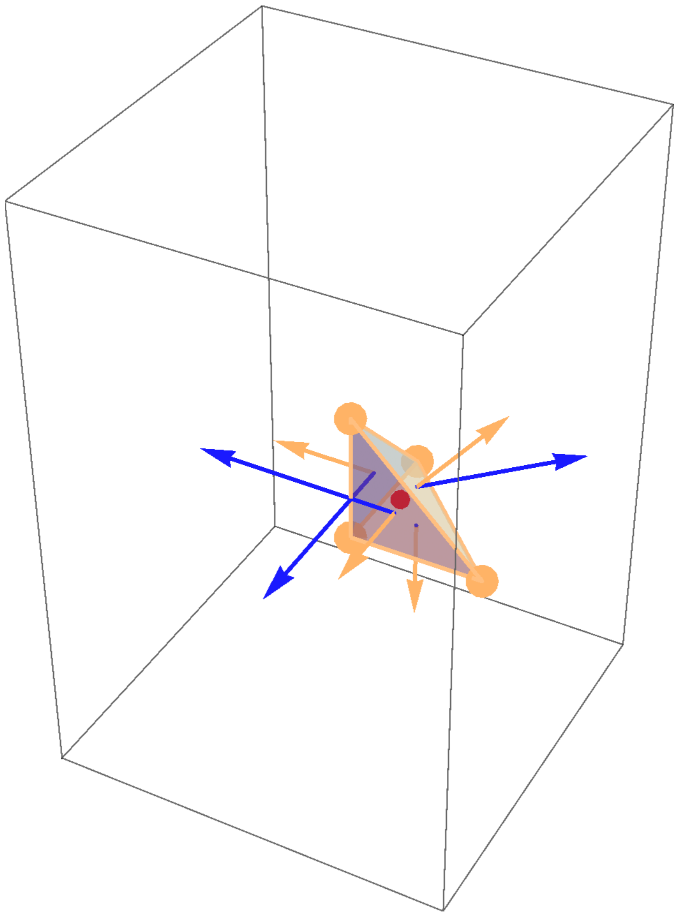


8.4 Shear stress

In the absence of rotation, translation and volume forces, plot the stress vectors on the face centroids of the same tetrahedron for a shear stress

$$\underline{\underline{\mathbf{T}}} = \tau({}^1\underline{\mathbf{e}} \otimes {}^2\underline{\mathbf{e}} + {}^2\underline{\mathbf{e}} \otimes {}^1\underline{\mathbf{e}}) \tag{7}$$

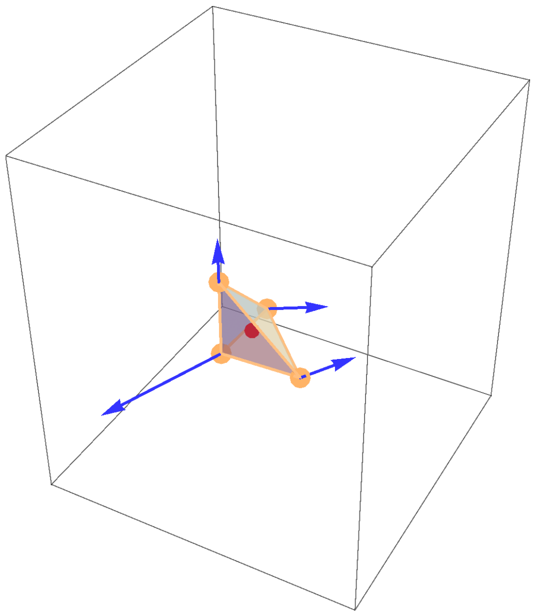
with $\tau = 3.85$ kPa.



8.5 Option: Multiaxial stress

Plot the nodal force vectors on the original vertices for the superposition of the above stresses

$$\underline{\underline{\mathbf{T}}} = -\pi\underline{\underline{\mathbf{I}}} + \sigma({}^3\underline{\mathbf{e}} \otimes {}^3\underline{\mathbf{e}}) + \tau({}^1\underline{\mathbf{e}} \otimes {}^2\underline{\mathbf{e}} + {}^2\underline{\mathbf{e}} \otimes {}^1\underline{\mathbf{e}})$$



Compute hydrostatic pressure and von Mises stress

$$\pi = -\frac{1}{3}\mathrm{tr}\underline{\underline{\mathbf{T}}} = 28333.3\,\mathrm{MPa}$$

$${}^{vM}T = \sqrt{\frac{3}{2}\underline{\underline{\mathbf{T}}}' : \underline{\underline{\mathbf{T}}}'} = 55002.2\,\mathrm{MPa}$$

with

$$\underline{\underline{\mathbf{T}}}' = \underline{\underline{\mathbf{T}}} + \pi\underline{\underline{\mathbf{I}}}$$