COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2019

Assignment 2

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Revised: January 24, 2019

Assignment 2 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 2 as two files, `Assignment_2_`*`YourMacID`*`.tex` and `Assignment_2_`*`YourMacID`*`.pdf`, to the Assignment 2 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_2_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_2.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_2_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_2_YourMacID
```

This assignment is due **Sunday, February 3, 2019 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_2_`*`YourMacID`*`.tex` and `Assignment_2_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 3.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

Let $(S, <)$ be a strict partial order. $(S, <)$ is *dense* if, for all $x, y \in S$ with $x < y$, there is some $z \in S$ such that $x < z < y$. The strict total order $(\mathbb{Q}, <_{\text{rat}})$ of the rationals and the strict total order $(\mathbb{R}, <_{\text{real}})$ of the real numbers are both dense.

Problems

1. **[10 points]** Let $\mathsf{SimpleTree}$ be the inductive set defined by the following constructors:

   a. $\mathsf{Leaf} : \mathbb{N} \to \mathsf{SimpleTree}$.
   b. $\mathsf{Branch1} : \mathsf{SimpleTree} \to \mathsf{SimpleTree}$.
   c. $\mathsf{Branch2} : \mathsf{SimpleTree} \times \mathsf{SimpleTree} \to \mathsf{SimpleTree}$.

   The function $\mathsf{leafs} : \mathsf{SimpleTree} \to \mathbb{N}$ is defined by pattern matching as:

   a. $\mathsf{leafs}(\mathsf{Leaf}(n)) = 1$.
   b. $\mathsf{leafs}(\mathsf{Branch1}(t)) = \mathsf{leafs}(t)$.
   c. $\mathsf{leafs}(\mathsf{Branch2}(t_1, t_2)) = \mathsf{leafs}(t_1) + \mathsf{leafs}(t_2)$.

   The function $\mathsf{branches} : \mathsf{SimpleTree} \to \mathbb{N}$ is defined by pattern matching as:

   a. $\mathsf{branches}(\mathsf{Leaf}(n)) = 0$.
   b. $\mathsf{branches}(\mathsf{Branch1}(t)) = 1 + \mathsf{branches}(t)$.
   c. $\mathsf{branches}(\mathsf{Branch2}(t_1, t_2)) = 1 + \mathsf{branches}(t_1) + \mathsf{branches}(t_2)$.

   Prove that, for all $t \in \mathsf{SimpleTree}$,

   $$\mathsf{leafs}(t) \leq \mathsf{branches}(t) + 1.$$

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   Proof. Let $P(t) \equiv \mathsf{leafs}(t) \leq \mathsf{branches}(t) + 1$. We will proof P(t) for all $t \in \mathsf{SimpleTree}$ using Structural induction.

   Base case: Prove $P(\mathsf{Leaf(n)})$

   $$\begin{aligned} P(\mathsf{Leaf(n)}) &\equiv \mathsf{leafs}(\mathsf{Leaf(n)}) \leq \mathsf{branches}(\mathsf{Leaf(n)}) + 1 && \langle\text{definition of P}\rangle \\ &\equiv 1 \leq 0 + 1 && \langle\text{definition of } \mathsf{leafs} \text{ and } \mathsf{branches}\ \rangle \\ &\equiv 1 \leq 1 && \langle\text{arithmetic}\rangle \end{aligned}$$

Induction step:
Case 1: $t = \mathsf{Branch1(t1)}$. We assume t1 and prove $P(\mathsf{Branch1(t1)})$.

$$
\begin{aligned}
& P(\mathsf{Branch1(t1)}) \\
\equiv\ & \mathsf{leafs(Branch1(t1))} \leq \mathsf{branches(Branch1(t1))} + 1 && \langle \text{definition of P} \rangle \\
\equiv\ & \mathsf{leafs(t1)} \leq 1 + \mathsf{branches(t1)} + 1 && \langle \text{definition of } \mathsf{leafs} \text{ and } \mathsf{branches} \rangle \\
\equiv\ & \mathsf{branches(t1)} + 1 \leq 1 + \mathsf{branches(t1)} + 1 && \langle \text{induction hypothesis} \rangle \\
\equiv\ & \mathsf{branches(t1)} + 1 \leq \mathsf{branches(t1)} + 2 && \langle \text{arithmetic} \rangle
\end{aligned}
$$

Case 2: $t = \mathsf{Branch2(t1,\ t2)}$. We assume t1 and t2, and prove $P(\mathsf{Branch2(t1,\ t2)})$

$$
\begin{aligned}
& P(\mathsf{Branch2(t1,\ t2)}) \\
\equiv\ & \mathsf{leafs(Branch2(t1,\ t2))} \leq \mathsf{branches(Branch2(t1,\ t2))} + 1 && \langle \text{defin} \\
\equiv\ & \mathsf{leafs(t1)} + \mathsf{leafs(t2)} \leq \mathsf{branches(t1)} + \mathsf{branches(t2)} + 2 && \langle \text{definition of } \mathsf{leafs} \text{ and anc} \\
\equiv\ & \mathsf{branches(t1)} + \mathsf{branches(t2)} + 2 \leq 2 + \mathsf{branches(t1)} + \mathsf{branches(t2)} && \langle \text{induction h}
\end{aligned}
$$

□

2. **[10 points]** Let $(S, <)$ be a strict total order such that there exist $a, b \in S$ with $a < b$ (i.e., S has at least two members). Show that, if $(S, <)$ is dense, then $(S, <)$ is not a well-order.

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Proof. $(S, <)$ is dence strict total order with at least two members. To show that it is not well ordered, we just need to find an infinate descending sequence.

First element will be $\mathsf{x}(0) = b$, then, second element will be between a and b, because it is strict dense order, between any two elements must exist one more. According to the task, we have at least a and b, so we can find $\mathsf{x}(1)$ between them. Also it should be obvious that $a < \mathsf{x}(1) < b = \mathsf{x}(0)$.

x(i) will be between a and x(i-1).

x(i) exists because of the way we built it, our first two elements exist, therefore each next will exist ($(S, <)$ is dence). It is also not hard to see that $a < \mathsf{x}$(i) $< \mathsf{x}$(i-1)

We can notice that x(i) is infinate sequance (because it is dense). Also it doent have least element because each next element is less then previous.

Therefore, $(S, <)$ is not well-ordered

$\square$