COMPSCI/SFWRENG 2FA3

Discrete Mathematics with Applications II

Winter 2019

Assignment 3

Dr. William M. Farmer

McMaster University

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Assignment 3 consists of four problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 3 as two files, `Assignment_3_`*`YourMacID`*`.tex` and `Assignment_3_`*`YourMacID`*`.pdf`, to the Assignment 3 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_3_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_3.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_3_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_3_YourMacID
```

This assignment is due **Sunday, February 10, 2019 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_3_`*`YourMacID`*`.tex` and `Assignment_3_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 10.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Background

Let $\Sigma = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be a finite signature of MSFOL, F_Σ be the set of Σ-formulas, and $A \in F_\Sigma$. Recall that the members of F_Σ are certain strings of symbols. A *subformula* of A is a $B \in F_\Sigma$ such that B is a substring of A. For example, let A be the formula $((0 = 2) \wedge (3 \mid 4))$. Then "$(0 = 2)$", "$(3 \mid 4)$", and "$((0 = 2) \wedge (3 \mid 4))$" are the subformulas of A, and "$(0 =$ " and "$\wedge$" are two substrings of A that are not subformulas of A.

Problems

1. **[5 points]** Let $\mathsf{subformulas} : F_\Sigma \to \mathsf{Set}(F_\Sigma)$ be the function that maps a formula $A \in F_\Sigma$ to the set of subformulas of A. Define $\mathsf{subformulas}$ by recursion using pattern matching.

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   a. $\mathsf{subformulas}(t_1 = t_2) = (t_1 = t_2)$
   b. $\mathsf{subformulas}(p(t_1, ..., t_n)) = p(t_1, ..., t_n)$
   c. $\mathsf{subformulas}(\neg A) = \neg A \cup \mathsf{subformulas}(A)$
   d. $\mathsf{subformulas}(Q^* x : \alpha . A) = (Q^* x : \alpha . A) \cup \mathsf{subformulas}(A)$
   e. $\mathsf{subformulas}(A \S^{**} B) = A \S^{**} B \cup \mathsf{subformulas}(A) \cup \mathsf{subformulas}(B)$

   Where

   *Q - any quatifier, like $\forall, \exists, etc.$

   **§ - any logical constant, like $\vee, \wedge, \implies, etc.$

2. **[5 points]** For $A \in F_\Sigma$, let $a(A)$ and $b(A)$ be the number of equalities and predicate applications occurring in A, respectively. Prove that, for all $A \in F_\Sigma$,

$$a(A) + b(A) \leq \frac{|\mathsf{subformulas}(A)| + 1}{2},$$

   where $|S|$ denotes the size of a finite set S.

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3. **[6 points]** Let $R_{\mathsf{sf}} \in F_\Sigma \times F_\Sigma$ be the relation such that

$$A \; R_{\mathsf{sf}} \; B$$

   iff A is a subformula of B. Prove that $(F_\Sigma, R_{\mathsf{sf}})$ is a weak partial order but not a weak total order.

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   Proof. To check this we have to check it for 3 property of week partial order and proof that totality is false

a. **Reflexive:** $\forall A \in F_\Sigma$. A R_{sf} A
For any formula, formula itself is a subformula, so this property holds

b. **Antisymmetric:** $\forall A, B \in F_\Sigma$.(A R_{sf} B $\wedge$ B R_{sf} A $\implies$ A=B)
For any formula, subformula is an subset of symbols of formula, and for subset we have proved that $A \subseteq B \wedge B \subseteq A \iff A = B$ $\implies$ property holds

c. **Transitive:** $\forall A, B, C \in F_\Sigma$.(A R_{sf} B $\wedge$ B R_{sf} C $\implies$ A R_{sf} C)
It is not hard to guess that we will use property of subformulas again and I think that it is pretty obvious that subset of a subset is still a subset

d. $\neg$ **Total:** $\exists A, B \in F_\Sigma$. $\neg(A R_{\mathsf{sf}} B) \wedge \neg(B R_{\mathsf{sf}} A)$
Let A = (0 = 1) and B = (0 < 1), this is good enough example I think

$\square$

4. [**4 points**] Let A be any member of F_Σ and $G = \mathsf{subformulas}(A)$.

   a. What are maximal elements of G in $(F_\Sigma, R_{\mathsf{sf}})$?
   b. What are minimal elements of G in $(F_\Sigma, R_{\mathsf{sf}})$?
   c. Does G have a maximum element in $(F_\Sigma, R_{\mathsf{sf}})$? If so, what is it?
   d. Does G have a minimum element in $(F_\Sigma, R_{\mathsf{sf}})$? If so, what is it?

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a. Maximal: A is an $R_{sf} - maximal$ element of $G \subseteq F_\Sigma$ if $A \in G$ and $\forall B \in F_\Sigma . B \in G \implies \neg(\mathrm{A} R_{sf} \mathrm{B})$

b. Minimal: A is an $R_{sf} - maximal$ element of $G \subseteq F_\Sigma$ if $A \in G$ and $\forall B \in F_\Sigma . B \in G \implies \neg(\mathrm{B} R_{sf} \mathrm{A})$

c. Maximum: A is the maximum element of G as all substrings are eather equal A or less then A.

d. Minimum: G does not have a minimum element as for example A = $(0 = 1) \vee (0 < 1)$ can be breaked down to itself, or $(0 = 1)$ or $(0 < 1)$, last two elements are unbreakeble and also neather of them is subset of other one.