2021 秋高等代数课后习题

第九次作业

习题 3.5 第 1 题.

(1).

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ x & 1 & 2 & \cdots & n-1 \\ x & x & 1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x & 1-x & 1 & \cdots & 1 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ x & & & & 1-x \end{vmatrix}$$

$$= \begin{vmatrix} 1-x & 1 & 1 & \cdots & 1 \\ 0 & 1-x & 1 & \cdots & 1 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 \\ x^2 & & & & 1-x \end{vmatrix} = \begin{vmatrix} 1-x & 1 & 0 & \cdots & 0 \\ 0 & 1-x & x & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & \ddots & x \\ x^2 & & & & 1-x \end{vmatrix}$$

$$= (1-x) \begin{vmatrix} 1-x & x & & \\ & \ddots & \ddots & \\ & & \ddots & x \\ & & & 1-x \end{vmatrix} + (-1)^{n+1} x^2 \begin{vmatrix} 1 & & & \\ 1-x & x & & \\ & \ddots & \ddots & \\ & & 1-x & x \end{vmatrix}$$

$$= (1-x)^n + (-1)^{n+1} x^n$$

(2). **令** $\alpha_i = \begin{pmatrix} x_1^i \\ \vdots \\ x_n^i \end{pmatrix}, e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, **那么**

$$\begin{aligned} D_n &= \det(e+\alpha_1, e+\alpha_2, \cdots, e+\alpha_n) \\ &= \det(\alpha_1, \cdots, \alpha_n) + \det(e, \alpha_2, \cdots, \alpha_n) + \cdots + \det(\alpha_1, \cdots, \alpha_{n-1}, e) \\ &= \det(\alpha_1, \cdots, \alpha_n) + \det(e, \alpha_2, \cdots, \alpha_n) + \cdots + (-1)^{n-1} \det(e, \alpha_1, \cdots, \alpha_{n-1}) \\ &= 2\det(\alpha_1, \cdots, \alpha_n) - \left\{ (-1)^2 \det(\alpha_1, \cdots, \alpha_n) + \cdots + (-1)^{n+2} \det(e, \alpha_1, \cdots, \alpha_{n-1}) \right\} \\ &= 2\det(\alpha_1, \cdots, \alpha_n) - \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e & \alpha_1 & \cdots & \alpha_n \end{pmatrix} \\ &= 2x_1 \cdots x_n \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i) - \prod_{0 \leqslant i < j \leqslant n} (x_j - x_i) \quad (\text{其中 } x_0 = 1) \end{aligned}$$

$$= \left(2\prod_{k=1}^{n} x_k - \prod_{k=1}^{n}(x_k - 1)\right) \prod_{1\leqslant i<j\leqslant n}(x_j - x_i)$$

习题 3.5 第 2 题. 令 $\alpha_1 = \begin{pmatrix} -a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \ldots, \alpha_n = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ -a_n \end{pmatrix}, e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$**, 那么类似第一题有**

$$\begin{aligned} D_n &= \det(\alpha_1 + a_1 e, \cdots, \alpha_n + a_n e) \\ &= 2\det(\alpha_1, \cdots, \alpha_n) - \det\begin{pmatrix} 1 & a_1 & \cdots & a_n \\ e & \alpha_1 & \cdots & \alpha_n \end{pmatrix} \end{aligned}$$

这两个都是比较好计算的行列式。