# 2021 秋高等代数课后习题

## 第八次作业

**习题 3.2 第 1 题.**

(1). $\begin{vmatrix}1&1&1&1\\1&-1&1&-1\\1&2&3&4\\1&3&5&9\end{vmatrix}=\begin{vmatrix}1&0&0&0\\1&-2&0&-2\\1&1&2&3\\1&2&4&8\end{vmatrix}=-4\begin{vmatrix}1&0&1\\1&2&3\\1&2&4\end{vmatrix}=-4\begin{vmatrix}1&0&1\\0&2&2\\0&2&3\end{vmatrix}=-8$

(3). $\begin{vmatrix}x&y&x+y\\y&x+y&x\\x+y&x&y\end{vmatrix}=\begin{vmatrix}x&y&2(x+y)\\y&x+y&2(x+y)\\x+y&x&2(x+y)\end{vmatrix}=2(x+y)\begin{vmatrix}0&0&1\\y-x&x&1\\y&x-y&1\end{vmatrix}=2(x+y)(-(x-y)^2-xy)=-2(x^3+y^3)$

**类似的方阵经过行（或列）的调换后可以形成所谓的循环矩阵（Circulant matrix）。关于循环矩阵的行列式，有一般的结论。**

(5).

$$\begin{vmatrix}a&b&c&d\\a&a+b&a+b+c&a+b+c+d\\a&2a+b&3a+2b+c&4a+3b+2c+d\\a&3a+b&6a+3b+c&10a+6b+3c+d\end{vmatrix}=\begin{vmatrix}a&b&c&d\\0&a&a+b&a+b+c\\0&2a&3a+2b&4a+3b+2c\\0&3a&6a+3b&10a+6b+3c\end{vmatrix}$$

$$=a\begin{vmatrix}a&a+b&a+b+c\\2a&3a+2b&4a+3b+2c\\3a&6a+3b&10a+6b+3c\end{vmatrix}=a\begin{vmatrix}a&a+b&a+b+c\\0&a&2a+b\\0&3a&7a+3b\end{vmatrix}$$

$$=a^2\begin{vmatrix}a&2a+b\\3a&7a+3b\end{vmatrix}=a^4$$

**习题 3.2 第 3 题. 证明：**

(1).

$$\begin{vmatrix}b+c&c+a&a+b\\q+r&r+p&p+q\\y+z&z+x&x+y\end{vmatrix}=\begin{vmatrix}b&c+a&a+b\\q&r+p&p+q\\y&z+x&x+y\end{vmatrix}+\begin{vmatrix}c&c+a&a+b\\r&r+p&p+q\\z&z+x&x+y\end{vmatrix}$$

$$=\begin{vmatrix}b&c+a&a\\q&r+p&p\\y&z+x&x\end{vmatrix}+\begin{vmatrix}c&a&a+b\\r&p&p+q\\z&x&x+y\end{vmatrix}$$

$$=\begin{vmatrix}b&c&a\\q&r&p\\y&z&x\end{vmatrix}+\begin{vmatrix}c&a&b\\r&p&q\\z&x&y\end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

(2).

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-ac-a^2+bc \\ 0 & c-a & c^2-ab-a^2+bc \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(a+b+c) \\ c-a & (c-a)(a+b+c) \end{vmatrix} = 0$$

**习题 3.2 第 4 题.**

(1). **令原行列式为** $D_n$**,** $\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$**, 那么有**

$$\begin{aligned} D_n &= \det(\alpha - b_1\beta, \alpha - b_2\beta, \cdots, \alpha - b_n\beta) \\ &= \det(\alpha - b_1\beta, (b_1-b_2)\beta, \cdots, (b_1-b_n)\beta) \end{aligned}$$

**于是, 因为后** $n-1$ **列秩至多为** $1$**, 若** $n \geqslant 3$**, 那么** $D_n = 0$**; 若** $n \leqslant 2$**, 那么**

$$D_2 = (b_1-b_2)\det(\alpha - b_1\beta, \beta) = (b_1-b_2)((a_1-b_1)-(a_2-b_1)) = (b_1-b_2)(a_1-a_2),$$
$$D_1 = a_1 - b_1$$

(2). $D_n = \begin{vmatrix} a_1 & b_2 & \cdots & b_n \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix}$

**按最后一行展开有**

$$\begin{aligned} D_n &= a_n D_{n-1} + b_n c_n a_2 \cdots a_{n-1} = a_n(a_{n-1}D_{n-2} + b_{n-1}c_{n-1}a_2\cdots a_{n-2}) + b_n c_n a_2 \cdots a_{n-1} \\ &= a_n a_{n-1} D_{n-2} + b_n c_n a_2 \cdots a_{n-1} + b_{n-1}c_{n-1}a_2 \cdots a_{n-2}a_n \\ &\cdots \\ &= \left(\prod_{i=2}^n a_i\right) D_1 + \sum_{i=2}^n \left( b_i c_i \prod_{2\leqslant j \leqslant n, j\neq i} a_j \right) \\ &= \prod_{i=1}^n a_i + \sum_{i=2}^n \left( b_i c_i \prod_{2\leqslant j \leqslant n, j\neq i} a_j \right) \end{aligned}$$

(3). **令原行列式为** $D_n$**,** $e_i, i = 1, \cdots, n,$ **为** $\mathbb{F}^n$ **的自然基,** $\beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$**, 那么有**

$$\begin{aligned} D_n &= \det(-2e_1 + 3\beta, -e_2 + 3\beta, 3\beta, e_4 + 3\beta, \cdots, (n-3)e_n + 3\beta) \\ &= \det(-2e_1, -e_2, 3\beta, e_4, \cdots, (n-3)e_n) \end{aligned}$$

**于是当** $n \geqslant 3$ **时，** $D_n = 6 \cdot (n-3)!$**，** $D_2 = -7, D_1 = 1.$

(4). $D_n = \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x \end{vmatrix}$

**有**

$$
\begin{aligned}
D_n &= \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_{n-1} \end{vmatrix} + \begin{vmatrix} x & a_1 & a_2 & \cdots & 0 \\ a_1 & x & a_2 & \cdots & 0 \\ a_1 & a_2 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x - a_{n-1} \end{vmatrix} \\
&= a_{n-1} \begin{vmatrix} x-a_1 & a_1-a_2 & a_2-a_3 & \cdots & 0 \\ 0 & x-a_2 & a_2-a_3 & \cdots & 0 \\ 0 & 0 & x-a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{vmatrix} + (x-a_{n-1})D_{n-1} \\
&= a_{n-1} \prod_{i=1}^{n-1}(x-a_i) + (x-a_{n-1})D_{n-1} \\
&= (x-a_{n-1}) \left( (x-a_{n-2})D_{n-2} + a_{n-2} \prod_{i=1}^{n-2}(x-a_i) \right) + a_{n-1} \prod_{i=1}^{n-1}(x-a_i) \\
&= (x-a_{n-1})(x-a_{n-2})D_{n-2} + (a_{n-2}+a_{n-1}) \prod_{i=1}^{n-1}(x-a_i) \\
&\cdots \\
&= x \cdot \prod_{i=1}^{n-1}(x-a_i) + \left( \sum_{i=1}^{n-1} a_i \right) \cdot \prod_{i=1}^{n-1}(x-a_i) = \left( x + \sum_{i=1}^{n-1} a_i \right) \cdot \prod_{i=1}^{n-1}(x-a_i)
\end{aligned}
$$

**习题 3.3 第 1 题**.

(1).

$$
D_n = \begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ 0 & x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & \cdots & 0 & x \end{vmatrix} = x \begin{vmatrix} x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & x & y \\ 0 & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{n+1} y \begin{vmatrix} y & 0 & \cdots & 0 & 0 \\ x & y & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & x & y \end{vmatrix}
$$

$$
= x^n + (-1)^{n+1} y^n = x^n - (-y)^n
$$

(3).

$$D_n = \begin{vmatrix} a+b & a & 0 & \cdots & \cdots & 0 \\ b & a+b & a & \cdots & \cdots & 0 \\ 0 & b & a+b & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & a \\ 0 & 0 & \cdots & \cdots & b & a+b \end{vmatrix} = (a+b)D_{n-1} - b\begin{vmatrix} a & 0 & \cdots & \cdots & 0 \\ b & a+b & a & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a \\ 0 & \cdots & \cdots & b & a+b \end{vmatrix}_{n-1}$$

$$= (a+b)D_{n-1} - baD_{n-2}$$

所以 $D_n = \lambda_1 a^n + \lambda_2 b^n$，代入 $D_1 = a+b, D_2 = a^2+b^2+ab$ 得

$$D_n = \begin{cases} \frac{1}{a-b}(a^{n+1} - b^{n+1}) & a \neq b \\ (n+1)a^b & a = b \end{cases}$$

(5). 与第 (3) 小题类似求得递推公式

$$D_n = (x^2+1)D_{n-1} - x^2 D_{n-2}$$

令 $D_n = \lambda_1 (x^2)^n + \lambda_2 1^n$，代入 $D_1 = x^2+1, D_2 = x^4+x^2+1$ 得

$$D_n = 1 + x^2 + x^4 + \cdots + x^{2n}$$

习题 **3.3** 第 **2** 题. 对于一个奇数阶 $(m \times m)$ 的反对称阵 $M$, 有

$$\det M = \det(-M^T) = (-1)^m \det M^T = -\det M,$$

知 $\det M = 0$.

对于 $A$, 考虑非对角元 $a_{ij}$ 与 $a_{ji}, i \neq j$, 对应的代数余子式 $A_{ij}, A_{ji}$ 相应的方阵 $M_{ij}, M_{ji}$. 由于 $A$ 为反对称阵，容易知 $M_{ij}^T = -M_{ji}$，所以有

$$A_{ij} = (-1)^{i+j} \det M_{ij} = (-1)^{i+j} \det(-M_{ji}^T) = (-1)^{n-1} \cdot (-1)^{i+j} \det(M_{ji}^T) = -A_{ji}$$

对角元 $a_{ii}$ 对应代数余子式 $A_{ii}$ 相应的方阵 $M_{ii}$ 是奇数阶反对称阵，行列式为 $0$.

综上知偶数阶反对称阵所有元的代数余子式之和等于 $0$.

习题 **3.3** 第 **3** 题. 记 $\alpha_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}$ 为 $A$ 的第 $i$ 列，$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, 那么

$$\begin{vmatrix} a_{11}+x_1 & \cdots & a_{1n}+x_n \\ \vdots & & \vdots \\ a_{n1}+x_1 & \cdots & a_{nn}+x_n \end{vmatrix} = \det(\alpha_1 + x_1 e, \alpha_2 + x_2 e, \cdots, \alpha_n + x_n e)$$

$$= \det(\alpha_1, \alpha_2, \cdots, \alpha_n) + \det(x_1 e, \alpha_2, \cdots, \alpha_n) + \det(\alpha_1, x_2 e, \cdots, \alpha_n) + \cdots + \det(\alpha_1, \alpha_2, \cdots, x_n e)$$

$$= \det A + x_1 \det(e, \alpha_2, \cdots, \alpha_n) + x_2 \det(\alpha_1, e, \cdots, \alpha_n) + \cdots + x_n \det(\alpha_1, \alpha_2, \cdots, e)$$

$$
\begin{aligned}
&= \det A + x_1 \sum_{k=1}^{n} A_{k1} + x_2 \sum_{k=1}^{n} A_{k2} + \cdots + + x_n \sum_{k=1}^{n} A_{kn} \\
&= \det A + \sum_{j=1}^{n} x_j \sum_{k=1}^{n} A_{kj}
\end{aligned}
$$