2021 秋高等代数习题课

2021-10-29 第三次习题课

习题 3.2 第 4 题. 计算 n 阶行列式

(1). $\begin{vmatrix} a_1-b_1 & a_1-b_2 & \cdots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \cdots & a_2-b_n \\ \vdots & \vdots & & \vdots \\ a_n-b_1 & a_n-b_2 & \cdots & a_n-b_n \end{vmatrix}$; (3). $\begin{vmatrix} 1 & 3 & 3 & \cdots & 3 \\ 3 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 3 & 3 & \cdots & n-1 & 3 \\ 3 & 3 & \cdots & 3 & n \end{vmatrix}$

解：(1). **令原行列式为** D_n，$\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$，**那么有**

$$\begin{aligned} D_n &= \det(\alpha - b_1\beta, \alpha - b_2\beta, \cdots, \alpha - b_n\beta) \\ &= \det(\alpha - b_1\beta, (b_1-b_2)\beta, \cdots, (b_1-b_n)\beta) \end{aligned}$$

于是，因为后 $n-1$ **列秩至多为** 1，**若** $n \geqslant 3$，**那么** $D_n = 0$；**若** $n \leqslant 2$，**那么**

$$D_n = (b_1-b_2)\det(\alpha - b_1\beta, \beta) = (b_1-b_2)((a_1-b_1)-(a_2-b_1)) = (b_1-b_2)(a_1-a_2).$$

(3). **令原行列式为** D_n，$e_i, i=1,\cdots,n,$ **为** $\mathbb{F}^n$ **的自然基**，$\beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$，**那么有**

$$\begin{aligned} D_n &= \det(-2e_1+3\beta, -e_2+3\beta, 3\beta, e_4+3\beta, \cdots, (n-3)e_n+3\beta) \\ &= \det(-2e_1, -e_2, 3\beta, e_4, \cdots, (n-3)e_n) \end{aligned}$$

于是当 $n \geqslant 3$ **时**，$D_n = 6\cdot(n-3)!$，$D_2 = -4, D_1 = 1.$

习题 3.3 第 1 题. 计算 n 阶行列式

(2). $\begin{vmatrix} x & a & a & \cdots & a \\ -a & x & a & \cdots & a \\ -a & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ -a & -a & -a & \cdots & x \end{vmatrix}$; (4). $\begin{vmatrix} x & 0 & \cdots & 0 & a_n \\ -1 & x & \cdots & 0 & a_{n-1} \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & x & a_2 \\ 0 & 0 & \cdots & -1 & x+a_1 \end{vmatrix}$

解：(2). 令原行列式为 D_n，那么有（$n \geqslant 2$ 时）

$$
\begin{aligned}
D_n &= \begin{vmatrix} x-a & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -a & -a & \cdots & x \end{vmatrix} + \begin{vmatrix} a & a & a & \cdots & a \\ -a & x & a & \cdots & a \\ -a & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ -a & -a & -a & \cdots & x \end{vmatrix} \\
&= (x-a)D_{n-1} + \begin{vmatrix} a & 0 & 0 & \cdots & 0 \\ -a & x+a & 2a & \cdots & 2a \\ -a & 0 & x+a & \cdots & 2a \\ \vdots & \vdots & \vdots & & \vdots \\ -a & 0 & 0 & \cdots & x+a \end{vmatrix} \\
&= (x-a)D_{n-1} + a(x+a)^{n-1} \\
&= (x-a)^2 D_{n-2} + a(x-a)(x+a)^{n-2} + a(x+a)^{n-1} \\
&\cdots \\
&= (x-a)^{n-1}D_1 + a\sum_{i=0}^{n-2}(x-a)^i(x+a)^{n-i-1} \\
&= (x-a)^{n-1}x + a\frac{(x+a)^{n-1}\left(1-\left(\dfrac{x-a}{x+a}\right)^{n-1}\right)}{1-\dfrac{x-a}{x+a}}, \quad a \neq 0 \\
&= (x-a)^{n-1}x + \frac{(x+a)^n - (x+a)(x-a)^{n-1}}{2}, \quad a \neq 0 \\
&= \frac{(x+a)^n + (x-a)^n}{2}, \quad a \neq 0
\end{aligned}
$$

容易验证，当 $n=1$ 以及当 $a=0$ 时的特殊情况也符合这个通项公式，所以

$$D_n = \frac{(x+a)^n + (x-a)^n}{2}.$$

(4). 令原行列式为 D_n，从第 n 行开始，逐行乘以 x 加到上一行，有

$$
\begin{aligned}
D_n &= \begin{vmatrix} 0 & 0 & \cdots & 0 & x^n + a_1x^{n-1} + \cdots + a_n \\ -1 & 0 & \cdots & 0 & x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1} \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & x^2 + a_1x + a_2 \\ 0 & 0 & \cdots & -1 & x + a_1 \end{vmatrix} \\
&= (-1)^{1+n}(x^n + a_1x^{n-1} + \cdots + a_n)\begin{vmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -1 \end{vmatrix} \\
&= (-1)^{1+n}(x^n + a_1x^{n-1} + \cdots + a_n)(-1)^{n-1}
\end{aligned}
$$

$$= x^n + a_1 x^{n-1} + \cdots + a_n$$

令 $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$**，** $C(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$**，那么** $D_n = \det(x - C(f))$**。**

方阵 $C(f)$ **被称作多项式** $f(x)$ **的友阵（**Companion Matrix**）。**

补充题. **设** $f_k(x)$ **是数域** $\mathbb{F}$ **上的** K **次多项式，其首项系数为** $a_k \neq 0, k = 0, 1, \cdots, n-1$**。又设** $b_1, b_2, \cdots, b_n \in \mathbb{F}$. **计算下列行列式**

$$D = \begin{vmatrix} f_0(b_1) & f_0(b_2) & \cdots & f_0(b_n) \\ f_1(b_1) & f_1(b_2) & \cdots & f_1(b_n) \\ \vdots & \vdots & & \vdots \\ f_{n-1}(b_1) & f_{n-1}(b_2) & \cdots & f_{n-1}(b_n) \end{vmatrix}$$

解：令 $f_K(x) = \sum\limits_{i=1}^{n} c_{ik} x^{i-1}, c_{ik} = 0$ **当** $i \geqslant k+1, c_{kk} = a_k$. **令**

$$\beta_i = (b_1^i, b_2^i, \cdots, b_n^i), \quad i = 0, \cdots, n-1,$$

那么有

$$D = \begin{vmatrix} a_0\beta_0 \\ c_{10}\beta_0 + a_1\beta_1 \\ c_{20}\beta_0 + c_{21}\beta_1 + a_2\beta_2 \\ \vdots \\ c_{n0}\beta_0 + c_{n1}\beta_1 + \cdots + a_{n-1}\beta_{n-1} \end{vmatrix} = \begin{vmatrix} a_0\beta_0 \\ a_1\beta_1 \\ a_2\beta_2 \\ \vdots \\ a_{n-1}\beta_{n-1} \end{vmatrix}$$

$$= \left(\prod_{k=0}^{n-1} a_k \right) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ b_1^{n-1} & b_2^{n-1} & \cdots & b_n^{n-1} \end{vmatrix}$$

$$= \prod_{k=0}^{n-1} a_k \prod_{1 \leqslant i < j \leqslant n} (b_j - b_i)$$

习题 **3.2** 第 **1** 题. **计算行列式**

(1). $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \end{vmatrix}$;　(3). $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$;

(5). $\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$

解：(1). $\begin{vmatrix}1&1&1&1\\1&-1&1&-1\\1&2&3&4\\1&3&5&9\end{vmatrix}=\begin{vmatrix}1&0&0&0\\1&-2&0&-2\\1&1&2&3\\1&2&4&8\end{vmatrix}=-4\begin{vmatrix}1&0&1\\1&2&3\\1&2&4\end{vmatrix}=-4\begin{vmatrix}1&0&1\\0&2&2\\0&2&3\end{vmatrix}=-8$

(3). $\begin{vmatrix}x&y&x+y\\y&x+y&x\\x+y&x&y\end{vmatrix}=\begin{vmatrix}x&y&2(x+y)\\y&x+y&2(x+y)\\x+y&x&2(x+y)\end{vmatrix}=2(x+y)\begin{vmatrix}0&0&1\\y-x&x&1\\y&x-y&1\end{vmatrix}=2(x+y)(-(x-y)^2-xy)=-2(x^3+y^3)$

类似的方阵经过行（或列）的调换后可以形成所谓的循环矩阵（Circulant matrix）。关于循环矩阵的行列式，有一般的结论。

(5).

$$\begin{vmatrix}a&b&c&d\\a&a+b&a+b+c&a+b+c+d\\a&2a+b&3a+2b+c&4a+3b+2c+d\\a&3a+b&6a+3b+c&10a+6b+3c+d\end{vmatrix}=\begin{vmatrix}a&b&c&d\\0&a&a+b&a+b+c\\0&2a&3a+2b&4a+3b+2c\\0&3a&6a+3b&10a+6b+3c\end{vmatrix}$$

$$=a\begin{vmatrix}a&a+b&a+b+c\\2a&3a+2b&4a+3b+2c\\3a&6a+3b&10a+6b+3c\end{vmatrix}=a\begin{vmatrix}a&a+b&a+b+c\\0&a&2a+b\\0&3a&7a+3b\end{vmatrix}$$

$$=a^2\begin{vmatrix}a&2a+b\\3a&7a+3b\end{vmatrix}=a^4$$

习题 3.2 第 3 题. 证明

(1). $\begin{vmatrix}b+c&c+a&a+b\\q+r&r+p&p+q\\y+z&z+x&x+y\end{vmatrix}=2\begin{vmatrix}a&b&c\\p&q&r\\x&y&z\end{vmatrix}$;　　(2). $\begin{vmatrix}1&a&a^2-bc\\1&b&b^2-ac\\1&c&c^2-ab\end{vmatrix}=0.$

证明：(1).

$$\begin{aligned}\begin{vmatrix}b+c&c+a&a+b\\q+r&r+p&p+q\\y+z&z+x&x+y\end{vmatrix}&=\begin{vmatrix}b&c+a&a+b\\q&r+p&p+q\\y&z+x&x+y\end{vmatrix}+\begin{vmatrix}c&c+a&a+b\\r&r+p&p+q\\z&z+x&x+y\end{vmatrix}\\&=\begin{vmatrix}b&c+a&a\\q&r+p&p\\y&z+x&x\end{vmatrix}+\begin{vmatrix}c&a&a+b\\r&p&p+q\\z&x&x+y\end{vmatrix}\\&=\begin{vmatrix}b&c&a\\q&r&p\\y&z&x\end{vmatrix}+\begin{vmatrix}c&a&b\\r&p&q\\z&x&y\end{vmatrix}\\&=2\begin{vmatrix}a&b&c\\p&q&r\\x&y&z\end{vmatrix}\end{aligned}$$

(2).

$$\begin{vmatrix}1&a&a^2-bc\\1&b&b^2-ac\\1&c&c^2-ab\end{vmatrix}=\begin{vmatrix}1&a&a^2-bc\\0&b-a&b^2-ac-a^2+bc\\0&c-a&c^2-ab-a^2+bc\end{vmatrix}=\begin{vmatrix}b-a&(b-a)(a+b+c)\\c-a&(c-a)(a+b+c)\end{vmatrix}=0$$

习题 3.2 第 4 题. 计算 n 阶行列式

(2). $D_n = \begin{vmatrix} a_1 & b_2 & \cdots & b_n \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix}$

按最后一行展开有

$$\begin{aligned} D_n &= a_n D_{n-1} + b_n c_n a_2 \cdots a_{n-1} = a_n(a_{n-1}D_{n-2} + b_{n-1}c_{n-1}a_2\cdots a_{n-2}) + b_n c_n a_2 \cdots a_{n-1} \\ &= a_n a_{n-1} D_{n-2} + b_n c_n a_2 \cdots a_{n-1} + b_{n-1}c_{n-1}a_2 \cdots a_{n-2}a_n \\ &\cdots \\ &= \left(\prod_{i=2}^{n} a_i\right) D_1 + \sum_{i=2}^{n}\left(b_i c_i \prod_{2\leqslant j\leqslant n, j\neq i} a_j\right) \\ &= \prod_{i=1}^{n} a_i + \sum_{i=2}^{n}\left(b_i c_i \prod_{2\leqslant j\leqslant n, j\neq i} a_j\right) \end{aligned}$$

(4). $D_n = \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x \end{vmatrix}$

有

$$\begin{aligned} D_n &= \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_{n-1} \end{vmatrix} + \begin{vmatrix} x & a_1 & a_2 & \cdots & 0 \\ a_1 & x & a_2 & \cdots & 0 \\ a_1 & a_2 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x-a_{n-1} \end{vmatrix} \\ &= a_{n-1}\begin{vmatrix} x-a_1 & a_1-a_2 & a_2-a_3 & \cdots & 0 \\ 0 & x-a_2 & a_2-a_3 & \cdots & 0 \\ 0 & 0 & x-a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{vmatrix} + (x-a_{n-1})D_{n-1} \\ &= a_{n-1}\prod_{i=1}^{n-1}(x-a_i) + (x-a_{n-1})D_{n-1} \\ &= (x-a_{n-1})\left((x-a_{n-2})D_{n-2} + a_{n-2}\prod_{i=1}^{n-2}(x-a_i)\right) + a_{n-1}\prod_{i=1}^{n-1}(x-a_i) \\ &= (x-a_{n-1})(x-a_{n-2})D_{n-2} + (a_{n-2}+a_{n-1})\prod_{i=1}^{n-1}(x-a_i) \\ &\cdots \\ &= x\cdot\prod_{i=1}^{n-1}(x-a_i) + \left(\sum_{i=1}^{n-1}a_i\right)\cdot\prod_{i=1}^{n-1}(x-a_i) = \left(x+\sum_{i=1}^{n-1}a_i\right)\cdot\prod_{i=1}^{n-1}(x-a_i) \end{aligned}$$