

2021 秋高等代数习题课

2021-10-29 第三次习题课

习题 3.2 第 4 题. 计算 n 阶行列式

$$(1). \begin{vmatrix} a_1 - b_1 & a_1 - b_2 & \cdots & a_1 - b_n \\ a_2 - b_1 & a_2 - b_2 & \cdots & a_2 - b_n \\ \vdots & \vdots & & \vdots \\ a_n - b_1 & a_n - b_2 & \cdots & a_n - b_n \end{vmatrix}; \quad (3). \begin{vmatrix} 1 & 3 & 3 & \cdots & 3 \\ 3 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 3 & 3 & \cdots & n-1 & 3 \\ 3 & 3 & \cdots & 3 & n \end{vmatrix}$$

解: (1). 令原行列式为 D_n , $\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $\beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, 那么有

$$\begin{aligned} D_n &= \det(\alpha - b_1\beta, \alpha - b_2\beta, \cdots, \alpha - b_n\beta) \\ &= \det(\alpha - b_1\beta, (b_1 - b_2)\beta, \cdots, (b_1 - b_n)\beta) \end{aligned}$$

于是, 因为后 $n-1$ 列秩至多为 1, 若 $n \geq 3$, 那么 $D_n = 0$; 若 $n \leq 2$, 那么

$$D_n = (b_1 - b_2) \det(\alpha - b_1\beta, \beta) = (b_1 - b_2)((a_1 - b_1) - (a_2 - b_1)) = (b_1 - b_2)(a_1 - a_2).$$

(3). 令原行列式为 D_n , $e_i, i = 1, \cdots, n$, 为 \mathbb{F}^n 的自然基, $\beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, 那么有

$$\begin{aligned} D_n &= \det(-2e_1 + 3\beta, -e_2 + 3\beta, 3\beta, e_4 + 3\beta, \cdots, (n-3)e_n + 3\beta) \\ &= \det(-2e_1, -e_2, 3\beta, e_4, \cdots, (n-3)e_n) \end{aligned}$$

于是当 $n \geq 3$ 时, $D_n = 6 \cdot (n-3)!$, $D_2 = -4$, $D_1 = 1$.

习题 3.3 第 1 题. 计算 n 阶行列式

$$(2). \begin{vmatrix} x & a & a & \cdots & a \\ -a & x & a & \cdots & a \\ -a & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ -a & -a & -a & \cdots & x \end{vmatrix}; \quad (4). \begin{vmatrix} x & 0 & \cdots & 0 & a_n \\ -1 & x & \cdots & 0 & a_{n-1} \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & x & a_2 \\ 0 & 0 & \cdots & -1 & x + a_1 \end{vmatrix}$$

解: (2). 令原行列式为 D_n , 那么有 ($n \geq 2$ 时)

$$\begin{aligned}
 D_n &= \begin{vmatrix} x-a & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -a & -a & \cdots & x \end{vmatrix} + \begin{vmatrix} a & a & a & \cdots & a \\ -a & x & a & \cdots & a \\ -a & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ -a & -a & -a & \cdots & x \end{vmatrix} \\
 &= (x-a)D_{n-1} + \begin{vmatrix} a & 0 & 0 & \cdots & 0 \\ -a & x+a & 2a & \cdots & 2a \\ -a & 0 & x+a & \cdots & 2a \\ \vdots & \vdots & \vdots & & \vdots \\ -a & 0 & 0 & \cdots & x+a \end{vmatrix} \\
 &= (x-a)D_{n-1} + a(x+a)^{n-1} \\
 &= (x-a)^2 D_{n-2} + a(x-a)(x+a)^{n-2} + a(x+a)^{n-1} \\
 &\dots \\
 &= (x-a)^{n-1} D_1 + a \sum_{i=0}^{n-2} (x-a)^i (x+a)^{n-i-1} \\
 &= (x-a)^{n-1} x + a \frac{(x+a)^{n-1} \left(1 - \left(\frac{x-a}{x+a} \right)^{n-1} \right)}{1 - \frac{x-a}{x+a}}, \quad a \neq 0 \\
 &= (x-a)^{n-1} x + \frac{(x+a)^n - (x+a)(x-a)^{n-1}}{2}, \quad a \neq 0 \\
 &= \frac{(x+a)^n + (x-a)^n}{2}, \quad a \neq 0
 \end{aligned}$$

容易验证, 当 $n=1$ 以及当 $a=0$ 时的特殊情况也符合这个通项公式, 所以

$$D_n = \frac{(x+a)^n + (x-a)^n}{2}.$$

(4). 令原行列式为 D_n , 从第 n 行开始, 逐行乘以 x 加到上一行, 有

$$\begin{aligned}
 D_n &= \begin{vmatrix} 0 & 0 & \cdots & 0 & x^n + a_1 x^{n-1} + \cdots + a_n \\ -1 & 0 & \cdots & 0 & x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1} \\ 0 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & x^2 + a_1 x + a_2 \\ 0 & 0 & \cdots & -1 & x + a_1 \end{vmatrix} \\
 &= (-1)^{1+n} (x^n + a_1 x^{n-1} + \cdots + a_n) \begin{vmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -1 \end{vmatrix} \\
 &= (-1)^{1+n} (x^n + a_1 x^{n-1} + \cdots + a_n) (-1)^{n-1}
 \end{aligned}$$

$$= x^n + a_1 x^{n-1} + \cdots + a_n$$

$$\text{令 } f(x) = x^n + a_1 x^{n-1} + \cdots + a_n, C(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}, \text{那么 } D_n = \det(x - C(f)).$$

方阵 $C(f)$ 被称作多项式 $f(x)$ 的反阵 (Companion Matrix)。

补充题. 设 $f_k(x)$ 是数域 \mathbb{F} 上的 k 次多项式, 其首项系数为 $a_k \neq 0, k = 0, 1, \cdots, n-1$ 。又设 $b_1, b_2, \cdots, b_n \in \mathbb{F}$. 计算下列行列式

$$D = \begin{vmatrix} f_0(b_1) & f_0(b_2) & \cdots & f_0(b_n) \\ f_1(b_1) & f_1(b_2) & \cdots & f_1(b_n) \\ \vdots & \vdots & & \vdots \\ f_{n-1}(b_1) & f_{n-1}(b_2) & \cdots & f_{n-1}(b_n) \end{vmatrix}$$

解: 令 $f_k(x) = \sum_{i=1}^n c_{ik} x^{i-1}, c_{ik} = 0$ 当 $i \geq k+1, c_{kk} = a_k$. 令

$$\beta_i = (b_1^i, b_2^i, \cdots, b_n^i), \quad i = 0, \cdots, n-1,$$

那么有

$$\begin{aligned} D &= \begin{vmatrix} a_0\beta_0 \\ c_{10}\beta_0 + a_1\beta_1 \\ c_{20}\beta_0 + c_{21}\beta_1 + a_2\beta_2 \\ \vdots \\ c_{n0}\beta_0 + c_{n1}\beta_1 + \cdots + a_{n-1}\beta_{n-1} \end{vmatrix} = \begin{vmatrix} a_0\beta_0 \\ a_1\beta_1 \\ a_2\beta_2 \\ \vdots \\ a_{n-1}\beta_{n-1} \end{vmatrix} \\ &= \left(\prod_{k=0}^{n-1} a_k \right) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ b_1^{n-1} & b_2^{n-1} & \cdots & b_n^{n-1} \end{vmatrix} \\ &= \prod_{k=0}^{n-1} a_k \prod_{1 \leq i < j \leq n} (b_j - b_i) \end{aligned}$$

习题 3.2 第 1 题. 计算行列式

$$(1). \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \end{vmatrix}; \quad (3). \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix};$$

$$(5). \begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$$

$$\text{解: (1). } \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & -2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 8 \end{vmatrix} = -4 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -4 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{vmatrix} = -8$$

$$(3). \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} x & y & 2(x+y) \\ y & x+y & 2(x+y) \\ x+y & x & 2(x+y) \end{vmatrix} = 2(x+y) \begin{vmatrix} 0 & 0 & 1 \\ y-x & x & 1 \\ y & x-y & 1 \end{vmatrix} = 2(x+y)(-(x-y)^2 - xy) = -2(x^3 + y^3)$$

类似的方阵经过行（或列）的调换后可以形成所谓的循环矩阵（Circulant matrix）。关于循环矩阵的行列式，有一般的结论。

(5).

$$\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 2a & 3a+2b & 4a+3b+2c \\ 0 & 3a & 6a+3b & 10a+6b+3c \end{vmatrix} \\ = a \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \\ = a^2 \begin{vmatrix} a & 2a+b \\ 3a & 7a+3b \end{vmatrix} = a^4$$

习题 3.2 第 3 题. 证明

$$(1). \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}; \quad (2). \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = 0.$$

证明: (1).

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = \begin{vmatrix} b & c+a & a+b \\ q & r+p & p+q \\ y & z+x & x+y \end{vmatrix} + \begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix} \\ = \begin{vmatrix} b & c+a & a \\ q & r+p & p \\ y & z+x & x \end{vmatrix} + \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix} \\ = \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix} \\ = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

(2).

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-ac-a^2+bc \\ 0 & c-a & c^2-ab-a^2+bc \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(a+b+c) \\ c-a & (c-a)(a+b+c) \end{vmatrix} = 0$$

习题 3.2 第 4 题. 计算 n 阶行列式

$$(2). D_n = \begin{vmatrix} a_1 & b_2 & \cdots & b_n \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix}$$

按最后一行展开有

$$\begin{aligned} D_n &= a_n D_{n-1} + b_n c_n a_2 \cdots a_{n-1} = a_n (a_{n-1} D_{n-2} + b_{n-1} c_{n-1} a_2 \cdots a_{n-2}) + b_n c_n a_2 \cdots a_{n-1} \\ &= a_n a_{n-1} D_{n-2} + b_n c_n a_2 \cdots a_{n-1} + b_{n-1} c_{n-1} a_2 \cdots a_{n-2} a_n \\ &\dots \end{aligned}$$

$$\begin{aligned} &= \left(\prod_{i=2}^n a_i \right) D_1 + \sum_{i=2}^n \left(b_i c_i \prod_{2 \leq j \leq n, j \neq i} a_j \right) \\ &= \prod_{i=1}^n a_i + \sum_{i=2}^n \left(b_i c_i \prod_{2 \leq j \leq n, j \neq i} a_j \right) \end{aligned}$$

$$(4). D_n = \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x \end{vmatrix}$$

有

$$\begin{aligned} D_n &= \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_{n-1} \end{vmatrix} + \begin{vmatrix} x & a_1 & a_2 & \cdots & 0 \\ a_1 & x & a_2 & \cdots & 0 \\ a_1 & a_2 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x - a_{n-1} \end{vmatrix} \\ &= a_{n-1} \begin{vmatrix} x - a_1 & a_1 - a_2 & a_2 - a_3 & \cdots & 0 \\ 0 & x - a_2 & a_2 - a_3 & \cdots & 0 \\ 0 & 0 & x - a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{vmatrix} + (x - a_{n-1}) D_{n-1} \\ &= a_{n-1} \prod_{i=1}^{n-1} (x - a_i) + (x - a_{n-1}) D_{n-1} \\ &= (x - a_{n-1}) \left((x - a_{n-2}) D_{n-2} + a_{n-2} \prod_{i=1}^{n-2} (x - a_i) \right) + a_{n-1} \prod_{i=1}^{n-1} (x - a_i) \\ &= (x - a_{n-1}) (x - a_{n-2}) D_{n-2} + (a_{n-2} + a_{n-1}) \prod_{i=1}^{n-1} (x - a_i) \\ &\dots \\ &= x \cdot \prod_{i=1}^{n-1} (x - a_i) + \left(\sum_{i=1}^{n-1} a_i \right) \cdot \prod_{i=1}^{n-1} (x - a_i) = \left(x + \sum_{i=1}^{n-1} a_i \right) \cdot \prod_{i=1}^{n-1} (x - a_i) \end{aligned}$$