2021 秋高等代数课后习题

第七次作业

习题 3.2 第 1 题. 计算行列式

(1). $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \end{vmatrix}$; (3). $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$;

(5). $\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix}$

解：(1). $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & -2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 8 \end{vmatrix} = -4\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -4\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{vmatrix} = -8$

(3). $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} x & y & 2(x+y) \\ y & x+y & 2(x+y) \\ x+y & x & 2(x+y) \end{vmatrix} = 2(x+y)\begin{vmatrix} 0 & 0 & 1 \\ y-x & x & 1 \\ y & x-y & 1 \end{vmatrix} = 2(x+y)(-(x-y)^2-xy) = -2(x^3+y^3)$

类似的方阵经过行（或列）的调换后可以形成所谓的循环矩阵（Circulant matrix）。关于循环矩阵的行列式，有一般的结论。

(5).

$$\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 2a & 3a+2b & 4a+3b+2c \\ 0 & 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$=a\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

$$=a^2\begin{vmatrix} a & 2a+b \\ 3a & 7a+3b \end{vmatrix} = a^4$$

习题 3.2 第 3 题. 证明

(1). $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$; (2). $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = 0.$

证明：(1).

$$\begin{aligned}\begin{vmatrix}b+c & c+a & a+b\\ q+r & r+p & p+q\\ y+z & z+x & x+y\end{vmatrix} &= \begin{vmatrix}b & c+a & a+b\\ q & r+p & p+q\\ y & z+x & x+y\end{vmatrix} + \begin{vmatrix}c & c+a & a+b\\ r & r+p & p+q\\ z & z+x & x+y\end{vmatrix}\\ &= \begin{vmatrix}b & c+a & a\\ q & r+p & p\\ y & z+x & x\end{vmatrix} + \begin{vmatrix}c & a & a+b\\ r & p & p+q\\ z & x & x+y\end{vmatrix}\\ &= \begin{vmatrix}b & c & a\\ q & r & p\\ y & z & x\end{vmatrix} + \begin{vmatrix}c & a & b\\ r & p & q\\ z & x & y\end{vmatrix}\\ &= 2\begin{vmatrix}a & b & c\\ p & q & r\\ x & y & z\end{vmatrix}\end{aligned}$$

(2).

$$\begin{vmatrix}1 & a & a^2-bc\\ 1 & b & b^2-ac\\ 1 & c & c^2-ab\end{vmatrix} = \begin{vmatrix}1 & a & a^2-bc\\ 0 & b-a & b^2-ac-a^2+bc\\ 0 & c-a & c^2-ab-a^2+bc\end{vmatrix} = \begin{vmatrix}b-a & (b-a)(a+b+c)\\ c-a & (c-a)(a+b+c)\end{vmatrix} = 0$$

习题 3.2 第 4 题. 计算 n 阶行列式

(1). **令原行列式为 D_n，$\alpha = \begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}, \beta = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$，那么有**

$$\begin{aligned}D_n &= \det(\alpha - b_1\beta, \alpha - b_2\beta, \cdots, \alpha - b_n\beta)\\ &= \det(\alpha - b_1\beta, (b_1-b_2)\beta, \cdots, (b_1-b_n)\beta)\end{aligned}$$

于是，因为后 $n-1$ 列秩至多为 1，若 $n \geqslant 3$，那么 $D_n = 0$；若 $n \leqslant 2$，那么

$$D_n = (b_1-b_2)\det(\alpha - b_1\beta, \beta) = (b_1-b_2)((a_1-b_1)-(a_2-b_1)) = (b_1-b_2)(a_1-a_2).$$

(2). $D_n = \begin{vmatrix}a_1 & b_2 & \cdots & b_n\\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n\end{vmatrix}$

按最后一行展开有

$$\begin{aligned}D_n &= a_nD_{n-1} + b_nc_na_2\cdots a_{n-1} = a_n(a_{n-1}D_{n-2} + b_{n-1}c_{n-1}a_2\cdots a_{n-2}) + b_nc_na_2\cdots a_{n-1}\\ &= a_na_{n-1}D_{n-2} + b_nc_na_2\cdots a_{n-1} + b_{n-1}c_{n-1}a_2\cdots a_{n-2}a_n\\ &\cdots\\ &= \left(\prod_{i=2}^{n} a_i\right) D_1 + \sum_{i=2}^{n}\left(b_ic_i \prod_{2\leqslant j\leqslant n, j\neq i} a_j\right)\end{aligned}$$

$$= \prod_{i=1}^{n} a_i + \sum_{i=2}^{n} \left(b_i c_i \prod_{2 \leqslant j \leqslant n, j \neq i} a_j \right)$$

(3). 令原行列式为 D_n，$e_i, i = 1, \cdots, n,$ 为 $\mathbb{F}^n$ 的自然基，$\beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, 那么有

$$\begin{aligned} D_n &= \det(-2e_1 + 3\beta, -e_2 + 3\beta, 3\beta, e_4 + 3\beta, \cdots, (n-3)e_n + 3\beta) \\ &= \det(-2e_1, -e_2, 3\beta, e_4, \cdots, (n-3)e_n) \end{aligned}$$

于是当 $n \geqslant 3$ 时，$D_n = 6 \cdot (n-3)!$，$D_2 = -4, D_1 = 1.$

(4). $D_n = \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x \end{vmatrix}$

有

$$\begin{aligned} D_n &= \begin{vmatrix} x & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & x & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & x & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & a_{n-1} \end{vmatrix} + \begin{vmatrix} x & a_1 & a_2 & \cdots & 0 \\ a_1 & x & a_2 & \cdots & 0 \\ a_1 & a_2 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & x - a_{n-1} \end{vmatrix} \\ &= a_{n-1} \begin{vmatrix} x - a_1 & a_1 - a_2 & a_2 - a_3 & \cdots & 0 \\ 0 & x - a_2 & a_2 - a_3 & \cdots & 0 \\ 0 & 0 & x - a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{vmatrix} + (x - a_{n-1}) D_{n-1} \\ &= a_{n-1} \prod_{i=1}^{n-1} (x - a_i) + (x - a_{n-1}) D_{n-1} \\ &= (x - a_{n-1}) \left((x - a_{n-2}) D_{n-2} + a_{n-2} \prod_{i=1}^{n-2} (x - a_i) \right) + a_{n-1} \prod_{i=1}^{n-1} (x - a_i) \\ &= (x - a_{n-1})(x - a_{n-2}) D_{n-2} + (a_{n-2} + a_{n-1}) \prod_{i=1}^{n-1} (x - a_i) \\ &\cdots \\ &= x \cdot \prod_{i=1}^{n-1} (x - a_i) + \left(\sum_{i=1}^{n-1} a_i \right) \cdot \prod_{i=1}^{n-1} (x - a_i) = \left(x + \sum_{i=1}^{n-1} a_i \right) \cdot \prod_{i=1}^{n-1} (x - a_i) \end{aligned}$$