NLA solves (approx.)
$$\begin{cases} \texttt{linear system} & : \ \boldsymbol{Ax}=0 \ , \ \boldsymbol{A}\in\mathbb{R}^{n\times n} \\ \texttt{least square} & : \ \min_{x\in\mathbb{R}^n}\|\boldsymbol{Ax}-\boldsymbol{b}\|_2 \ , \ \boldsymbol{A}\in\mathbb{R}^{m\times n}\,(m\geq n) \\ \texttt{eigen-problem} & : \ \boldsymbol{Ax}=\lambda\boldsymbol{x} \ , \ \boldsymbol{A}\in\mathbb{R}^{n\times n},\ \boldsymbol{x}\neq 0,\ \lambda\in\mathbb{C} \\ \texttt{singular value} & : \ \boldsymbol{A}^T\boldsymbol{Ax}=\sigma^2\boldsymbol{x} \ , \ \boldsymbol{A}\in\mathbb{R}^{m\times n},\ \boldsymbol{x}\neq 0,\ \sigma\geq 0 \end{cases}$$

Lin. Space
$$\begin{cases} \mathbb{R}^n,\mathbb{C}^n & : \ \texttt{vec.} \\ \mathbb{R}^{m\times n},\mathbb{C}^{m\times n} & : \ \texttt{mat.} \end{cases}$$

Lin. Trans.
$$\begin{cases} \boldsymbol{A}\in\mathbb{R}^{m\times n} & : \ \mathbb{R}^n\to\mathbb{R}^m \\ \boldsymbol{A}\in\mathbb{C}^{m\times n} & : \ \mathbb{C}^n\to\mathbb{C}^m \end{cases}$$

Lin. Subspace
$$\begin{cases} N(\boldsymbol{A})=\mathrm{Ker}(\boldsymbol{A})\triangleq\{\boldsymbol{x}\in\mathbb{C}^n:\boldsymbol{Ax}=0\} & \subseteq \ \mathbb{C}^n \\ C(\boldsymbol{A})=\mathrm{Ran}(\boldsymbol{A})\triangleq\{\boldsymbol{y}\in\mathbb{C}^m:\boldsymbol{y}=\boldsymbol{Ax},\boldsymbol{x}\in\mathbb{C}^n\} & \subseteq \ \mathbb{C}^m \end{cases}$$

$$\mathrm{span}(\boldsymbol{A})\triangleq\{\alpha_1\boldsymbol{x}_1+\alpha_2\boldsymbol{x}_2+\cdots+\alpha_k\boldsymbol{x}_k\}\begin{cases} \boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n \ \in \ \mathbb{C}^n \\ \alpha_1,\alpha_2,\cdots,\alpha_n \ \in \ \mathbb{C} \\ \boldsymbol{A}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n) \end{cases}$$

Vec. Norms
$$\begin{cases} \|\boldsymbol{x}\|_1=\sum_{i=1}^n|x_i| & \|\boldsymbol{x}\|_2=\left(\sum_{i=1}^n|x_i|^2\right)^{1/2} \\ \|\boldsymbol{x}\|_\infty=\max_{1\leq i\leq n}|x_i| & \|\boldsymbol{x}\|_p=\left(\sum_{i=1}^n|x_i|^p\right)^{1/p} \end{cases}, \quad \begin{matrix} \|\gamma\boldsymbol{x}\|=|\gamma|\cdot\|\boldsymbol{x}\| \\ \|\boldsymbol{x}+\boldsymbol{y}\|\leq\|\boldsymbol{x}\|+\|\boldsymbol{y}\| \end{matrix}$$

† ℓ^p-norms in $\mathbb{R}^n$, $p=1,2,\infty$

Mat. Norms
$$\begin{cases} \|\boldsymbol{A}\|_1=\max_{1\leq j\leq n}\left(\sum_{i=1}^n|a_{ij}|\right) & \|\boldsymbol{A}\|_2=\sqrt{\rho(A^TA)} \\ \|\boldsymbol{A}\|_\infty=\max_{1\leq i\leq n}\left(\sum_{j=1}^n|a_{ij}|\right) & \|\boldsymbol{A}\|_F=\left(\sum_{i=1}^n\sum_{j=1}^n|a_{ij}|^2\right)^{1/2} \end{cases}, \quad \begin{matrix} \|\gamma\boldsymbol{A}\|\leq|\gamma|\cdot\|\boldsymbol{A}\| \\ \|\boldsymbol{A}+\boldsymbol{B}\|\leq\|\boldsymbol{A}\|+\|\boldsymbol{B}\| \\ \|\boldsymbol{AB}\|\leq\|\boldsymbol{A}\|\cdot\|\boldsymbol{B}\| \\ \|\boldsymbol{Ax}\|\leq\|\boldsymbol{A}\|\cdot\|\boldsymbol{x}\| \end{matrix}$$

† ℓ^p-norms in $\mathbb{R}^{n\times n}$, $p=1,2,\infty$ and Frobenius-norm

Special Mat.
$$\begin{cases} \texttt{Hermitian} : \ \boldsymbol{A}^*=\boldsymbol{A} & \texttt{Unitary} : \ \boldsymbol{U}^*\boldsymbol{U}=\boldsymbol{U}\boldsymbol{U}^*=\boldsymbol{I} \\ \texttt{Normal} : \ \boldsymbol{A}^*\boldsymbol{A}=\boldsymbol{A}\boldsymbol{A}^* & \texttt{Orthogonal} : \ \boldsymbol{Q}^T\boldsymbol{T}=\boldsymbol{Q}\boldsymbol{Q}^T=\boldsymbol{I} \end{cases}$$

Eigen-
$$\begin{cases} \texttt{charct. poly.} : \ p(\lambda)=\det(\boldsymbol{A}-\lambda\boldsymbol{I}) & \textbf{eigval.} : \ \{\lambda\in\mathbb{C}:\det(\boldsymbol{A}-\lambda\boldsymbol{I})=0\} \\ \texttt{charct. eq.} : \ \det(\boldsymbol{A}-\lambda\boldsymbol{I})=0 & \textbf{eigvec.} : \ \{\boldsymbol{x}\in\mathbb{C}^n:\boldsymbol{Ax}=\lambda\boldsymbol{x},\lambda\in\mathbb{C}\} \end{cases}$$

† eigval./vec. are only defined for **square** mat.

† real mat. may have **comp.** eigval./vec.

† algebraic multiplicity (AM) of λ_i: $\left\{k\in\mathbb{R}:\det(\boldsymbol{A}-\lambda\boldsymbol{I})=(\lambda-\lambda_i)^k g(\lambda)=0\right\}$

† geometric multiplicity (GM) of λ_i: **dim.** of λ_i's eigenspace

† **similar** mat. ($\boldsymbol{A}=\boldsymbol{XBX}^{-1}$) have **same** eigval.

† mat. **spctm.**: $\sigma(\boldsymbol{A})\triangleq\{\lambda_1,\cdots,\lambda_n\}$

Similarity Trans. $\begin{cases} \boldsymbol{A} = \boldsymbol{X}\boldsymbol{B}\boldsymbol{X}^{-1} \\ \boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{X} = \boldsymbol{B} \end{cases}$

- **Jordan** Canonical From: $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{X} \in \mathbb{C}^{n\times n}$

$$\boldsymbol{X}^{-1}A\boldsymbol{X} = \begin{pmatrix} \boldsymbol{J}_1 & & & \\ & \boldsymbol{J}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{J}_p \end{pmatrix}, \quad \boldsymbol{J}_i = \begin{pmatrix} \boldsymbol{J}_{i1} & & & \\ & \boldsymbol{J}_{i2} & & \\ & & \ddots & \\ & & & \boldsymbol{J}_{i\nu_i} \end{pmatrix}, \quad \boldsymbol{J}_{ik} = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}$$

† ν_i: GM of λ_i

† $\boldsymbol{J}_{ik}$: Jordan block wrt. one eigvec.

† norm-mat. $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T \iff \boldsymbol{A}$ can be **diagonalized** and $\boldsymbol{X}$ is unit-mat.

- **Schur** Decomposition/Triangulation: $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, $\boldsymbol{U} \in \mathbb{C}^{n\times n}$

$$\boldsymbol{U}^*\boldsymbol{A}\boldsymbol{U} = \begin{pmatrix} \lambda_1 & r_{12} & \cdots & r_{1n} \\ & \lambda_2 & \cdots & r_{2n} \\ & & \ddots & \vdots \\ & & & \lambda_n \end{pmatrix} \triangleq \boldsymbol{R} \quad \iff \quad \boldsymbol{A} = \boldsymbol{U}\boldsymbol{R}\boldsymbol{U}^*$$

† simplest form of unitary trans., but $\boldsymbol{U}, \boldsymbol{R}$ are not unique

† norm-mat. $\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^* \iff \boldsymbol{R}$ is diag-mat.

† Hert-mat. $\boldsymbol{A}^* = \boldsymbol{A} \iff \boldsymbol{R}$ is **real** diag-mat.

- **Schur** Canonical Form: $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$

$$\boldsymbol{Q}^T\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{T}$$

† $\boldsymbol{T}$ is block-tridiag-mat. with $\boldsymbol{T}_{ii}$ of $\left\langle \begin{array}{ll} 1\times 1 & : \text{eigval.} \\ 2\times 2 & : \text{comp. conj. pair of eigvals.} \end{array} \right.$

† $\lambda_i \in \mathbb{R} \iff \boldsymbol{T} = \boldsymbol{R}$ is upper triang-mat. with λ_i on the diag.

Pos. (Semi-)Def. $\begin{cases} \text{Symm. Pos. (semi-)Def.} & : \forall \boldsymbol{x} \neq 0 \;\; \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0\,(\geq 0)\;,\;\; \boldsymbol{A}^T = \boldsymbol{A} \in \mathbb{R}^{n\times n} \\ \text{Herm. Pos. (semi-)Def.} & : \forall \boldsymbol{x} \neq 0 \;\; \boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x} > 0\,(\geq 0)\;,\;\; \boldsymbol{A}^* = \boldsymbol{A} \in \mathbb{C}^{n\times n} \\ \text{Pos. (semi-)Def.} & : \forall \boldsymbol{x} \neq 0 \;\; \mathrm{Re}(\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x}) > 0\,(\geq 0) \end{cases}$

Error $\begin{cases} \text{abs. err.} & : e = x - \tilde{x} \\ \text{rel. err.} & : e_r = \dfrac{x - \tilde{x}}{\tilde{x}} = \dfrac{e}{\tilde{x}} \end{cases}, \quad \begin{array}{ll} x & : \text{approx. val.} \\ \tilde{x} & : \text{true val.} \end{array}$

Sensitivity $\begin{cases} \text{insensi./well-cond.} & : \text{input rel. change} \rightsquigarrow \text{similiar rel. change in sol.} \\ \text{sensi./ill-cond.} & : \text{rel. change in sol. is much larger} \end{cases}$

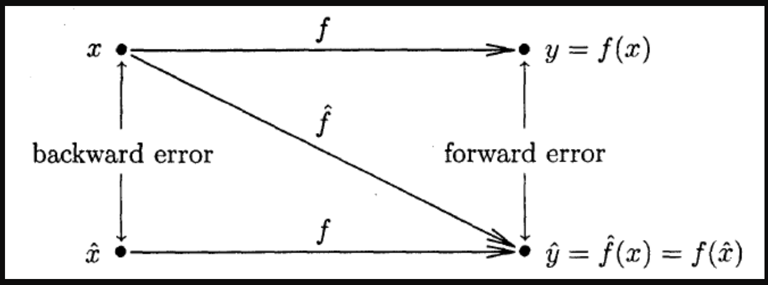
Err. Est. $\begin{cases} \texttt{abs. err. bound } \varepsilon & : \ \exists \varepsilon > 0 : |e| \leq \varepsilon \\ \texttt{rel. err. bound } \varepsilon_r & : \ \exists \varepsilon_r > 0 : |e_r| \leq \varepsilon_r \end{cases}$

- arithmetic $\left\langle \begin{array}{lll} \varepsilon(x_1 + x_2) & \leq & \varepsilon(x_1) + \varepsilon(x_2) \\ \varepsilon(x_1 x_2) & \leq & |x_2|\,\varepsilon(x_1) + |x_1|\,\varepsilon(x_2) \end{array} \right.$, $\varepsilon\left(\dfrac{x_1}{x_2}\right) \leq \dfrac{|x_1|\,\varepsilon(x_1) + |x_1|\,\varepsilon(x_2)}{|x_2|^2}$
- functional $\left\langle \begin{array}{lll} \varepsilon\left(f(x)\right) & \lessapprox & |f'(x)|\,\varepsilon(x) \ [4, 5] \\ \varepsilon_r\left(f(x)\right) & = & \mathrm{cond} \cdot \varepsilon_r(x) \end{array} \right.$, $\mathrm{cond} \triangleq \left| \dfrac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} \right|$

† $\varepsilon\left(f(x)\right) \approx \sum_{k=1}^{n} \left| \dfrac{\partial f(x)}{\partial \tilde{x}_k} \right| \varepsilon(x_k)$, $f(x) = f(x_1, \cdots, x_n)$

† bound the err. rather than compute it (true val. unknown)

Err. Anly. $\begin{cases} \texttt{fore err.} & : \ \Delta y = \hat{y} - y = \hat{f}(x) - f(x) \\ \texttt{back err.} & : \ \Delta x = \hat{x} - x \texttt{ where } f(\hat{x}) = \hat{y} \end{cases}$



‡ x, f: exact input and func.

‡ $\hat{f}$: approx. func.

‡ $\hat{x}$: exact val. for $\hat{f}$

† $\hat{f}(x) = f(\hat{x})$ due to the choice of $\hat{x}$, which defines $\hat{x}$

† back. err. anly is easier, used to measure algo. stability

† comp. err. $\left\langle \begin{array}{ll} \texttt{trunc. err.} & : \texttt{ due to trunc. of inf. series, finite diff., ...} \\ \texttt{round. err.} & : \texttt{ due to finite-precs., rounding arith., ...} \end{array} \right.$

† **total err.** $= \hat{f}(\hat{x}) - f(x) = \underbrace{\left(\hat{f}(\hat{x}) - f(\hat{x})\right)}_{\texttt{comp. err.}} + \underbrace{\left(f(\hat{x}) - f(x)\right)}_{\texttt{data err.}}$

Cond. Num.

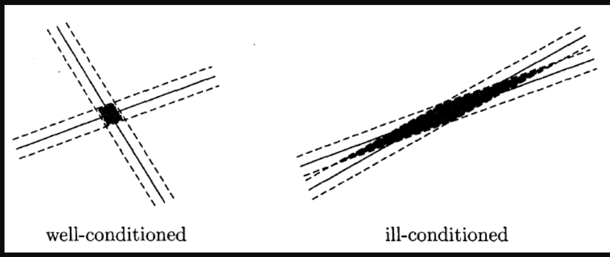
$$\mathrm{cond} \triangleq \frac{|\texttt{rel. change in sol.}|}{|\texttt{rel. change in input}|} = \frac{\left(f(\hat{x}) - f(x)\right)/f(x)}{\left(\hat{x} - x\right)/x} = \frac{|\Delta y/y|}{|\Delta x/x|}$$

† $\mathrm{cond} \geq 1 \ (> 10)$: sensi./ill-cond. prob.

† $|\texttt{rel. fore. err.}| \lessapprox \mathrm{cond} \cdot |\texttt{rel. back. err.}|$

- ∘ cond $\iff$ amplification factor
- ∘ $\lessapprox$: upper bound for the max. cond.

† $\mathrm{cond} \approx \left| \dfrac{x f'(x)}{f(x)} \right|$ [6]



Stability

- math prob. (well-posed)
  - sol. exists, unique, depends continuously
  - small input change $\not\rightarrow$ abrupt change in sol.
  - well-posed $\neq$ sol. is insensi. to pertb.
- algo. desp. (stable)
  - rst. rel. insensi. to pertb. due to approx.
  - rst. is the exact sol. to a nearby prob.
  - weaker: nearly correct rst. for nearly the correct prob.

Accuracy

- closeness of comp. sol. to true sol.
- depends on algo. cond. and stability
- stability $\neq$ accuracy

	alog.	prob.
inaccuracy	stable	ill-cond.
	unstable	well-cond.
accuracy	stable	well-cond.

Mach. Precs.

$$\begin{cases} \varepsilon_m = \arg\min_\varepsilon \mathrm{fl}(1+\varepsilon) > 1 \\ \left|\dfrac{\mathrm{fl}(x)-x}{x}\right| \le \varepsilon_m \end{cases}$$

† $\mathrm{fl}(x)$: round to nearest, $\mathrm{fl}(x \oplus y) = (x \oplus y)(1+\delta)$, $|\delta| \le \varepsilon_m$

† ε_m determines the max. rel. err. in representing a real num. x

Err. Anly. Sum.

	Total Err.		
	comp. err.		data err.
	trunc. err.	round. err.	
howto est.	theoretical anly.	back. err. anly. hard to quantify	cond.ing, cond
howto rdc.	aglo. selction	stable algo. tips†, dbl-precs.	change prob. form. improve cond.ing

† avoid subtractive cancellation in nums. of nearly equal mag.

† avoid adding large and small nums.

† adjust the order of additions

Mat. Cond.

$$\kappa(\boldsymbol{A}) \triangleq \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\|$$

† κ is def. for sqr. non-singlr. mat., $\kappa(\boldsymbol{A}) = \infty$ if $\boldsymbol{A}$ singlr.

† $\kappa(\boldsymbol{A}) \ge 1$, $\kappa(\boldsymbol{I}) = 1$, $\kappa(\gamma\boldsymbol{A}) = \kappa(\boldsymbol{A})$, $\kappa(\boldsymbol{D}) = (\max|d_i|) \Big/ (\min|d_i|)\,, \boldsymbol{D} = \mathrm{diag}(d_i)$

Err. Bounds

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \lessapprox \kappa_1(\boldsymbol{A})\frac{\|\Delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|} \text{ [7]} \quad , \quad \frac{\|\Delta \boldsymbol{x}\|}{\|\hat{\boldsymbol{x}}\|} \lessapprox \kappa_2(\boldsymbol{A})\frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \text{ [8]} \quad , \quad \frac{\|\Delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \lessapprox \kappa_3(\boldsymbol{A})\left(\frac{\|\Delta \boldsymbol{b}\|}{\|\boldsymbol{b}\|} + \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}\right) \text{ [9]}$$

† $\lessapprox$: approx. bounded by

† κ provides a quantitative bound for the err. in comp. sol.

† κ_1 bounds the max. rel. change in sol. due to a given rel. change in vec. $\boldsymbol{b}$

† similar rst. κ_2 holds for rel. change in mat. $\boldsymbol{A}$

† mat. scaling affects κ rather than the singlr.

† $\dfrac{\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \lessapprox \kappa(\boldsymbol{A})\varepsilon_m$ if input data accurate to mach. precs.

Residual $\begin{cases} \boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\hat{\boldsymbol{x}} \text{ where } \hat{\boldsymbol{x}} \text{ is approx. sol.} \\ \text{rel. res.} \triangleq \dfrac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\| \cdot \|\hat{\boldsymbol{x}}\|} \end{cases}$

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\hat{\boldsymbol{x}}\|} \leq \kappa(\boldsymbol{A})\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\| \cdot \|\hat{\boldsymbol{x}}\|} \text{ [10]}$$

† size of res. is considered rel. to the size of prob. and sol.

† small rel. res. $\rightsquigarrow$ small rel. err. iff. $\boldsymbol{A}$ is well-cond.

† large rel. res. $\rightsquigarrow$ large back. err. in mat. $\rightsquigarrow$ unstable algo. [11]

† stable algo. $\iff$ small rel. res. (irresp. to cond.ing)

Tri. Lin. Sys.

- **lower tri. sys.** $\left\langle \begin{array}{l} \boldsymbol{L}\boldsymbol{y} = \boldsymbol{b} \\ y_1 = b_1 / l_{11} \quad , \quad y_i = \left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j\right) / l_{ii} \ , \ i = 2, \cdots, n \end{array} \right.$

Algorithm 1: Fore. Subs. for Unit Lower Tri. Mat.

$y_1 \leftarrow b_1$;
for $i \leftarrow 2$ **to** n **do**
  $s \leftarrow b_i$;
  for $j \leftarrow 1$ **to** $i-1$ **do**
    $s \leftarrow s - l_{ij} \cdot y_j$;
  $y_i \leftarrow s$

- **upper tri. sys.** $\left\langle \begin{array}{l} \boldsymbol{U}\boldsymbol{x} = \boldsymbol{y} \\ x_n = y_n / u_{nn} \quad , \quad x_i = \left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j\right) / u_{ii} \ , \ i = n-1, \cdots, 1 \end{array} \right.$

Algorithm 2: Back. Subs. for Upper Tri. Mat.

$x_n \leftarrow y_n / u_{nn}$;
for $i \leftarrow n-1$ **to** 1 **do**
  $s \leftarrow y_i$;
  for $j \leftarrow i+1$ **to** n **do**
    $s \leftarrow s - u_{ij} \cdot x_j$;
  $x_i \leftarrow s / u_{ii}$

Perm. Mat.

$$\underset{\sim\sim\sim}{P^{-1} = P^T} \quad , \quad \|P\| = 1 \quad , \quad \kappa(P) = 1 \quad , \quad \kappa(PA) = \kappa(A)$$

† P is sqr. mat. with only one 1 in each row/col. and is always non-singlr.

† P is orthogonal mat., prod. of P is also perm. mat.

† PA reorders rows, AP reorders cols.

Elem. Elim. Mat.

$$M_k a = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_n & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \begin{array}{l} m_i = a_i / a_k \\ i = k+1, \cdots, n \end{array}$$

† M_k is unit lower tri., non-singlr., annihilates all entries below **pivot** $a_k \neq 0$

† $\left. \begin{array}{l} M_k = I - m e_k^T \\ M_k^{-1} = I + m e_k^T \triangleq L_k \end{array} \right\rangle$ rever sed signs of multipliers $m = (0, \cdots, 0, m_{k+1}, \cdots, m_n)^T$

† $M_k M_j \ (j > k)$ is their "union", and similar rst. holds for $L_k L_j$ [13]

LU Fact. by Gauss. Elim.

$$\underbrace{M_{n-1} \cdots M_1}_{\text{lower tri. mat.} M} Ax = \overbrace{\underbrace{MA}_{U} x = \underbrace{Mb}_{y}}^{\text{upper tri. sys.}} \quad , \quad LUx = b \left\langle \begin{array}{lll} Ly = b & : & \text{fore. subs.} \\ Ux = y & : & \text{back. subs.} \end{array} \right. \quad [12]$$

† LU Fact. is unique $\Longleftrightarrow$ $\left\langle \begin{array}{l} L \text{ is unit lower tri., } U \text{ is upper tri.} \\ \text{all leading principle submats. are non-singlr.} \end{array} \right.$

† Gauss. Elim. and LU Fact. are two ways of expressing same process

† zero pivot causes breakdown $\rightsquigarrow$ chooes entry with large mag.

Pivoting

- **partial** pvt.

  † M^{-1} is not necessarily lower tri. due to perm.

  † $PA = LU$ where $P = P_{n-1} \cdots P_1$, but can't be determined in advance

  † universally used since N.S. is more than adaquate

- **complete** pvt.

  † entire remaining unreduced submats. is searched for the max. mag.

  † $PAQ = LU$, theoretically superior, but much expensive

$\boldsymbol{LU}$ Fact. Stability

$$\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\|\cdot\|\hat{\boldsymbol{x}}\|} \le \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \le \rho\, n\, \varepsilon_m \quad [11], [Wil61]$$

† $\boldsymbol{E}$: back. err. of mat. $\boldsymbol{A}$

† growth fac. $\rho \triangleq \dfrac{\text{max. ele. of } \boldsymbol{U}}{\text{max. ele. of } \boldsymbol{A}} \left\langle \begin{array}{ll} \text{without pvt.} & : \text{arbitrarily large} \rightsquigarrow \text{unstable} \\ \text{partial pvt.} & : \rho \le 2^{n-1}\ (<10) \rightsquigarrow \text{stable} \\ \text{complete pvt.} & : \rho \text{ much smaller but not worth} \end{array} \right.$

† $\dfrac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \lessapprox n\,\varepsilon_m$ in practice $\overset{\text{always}}{\rightsquigarrow}$ small res. regardless of cond.ing

† small res. $\not\Longleftrightarrow$ accurate sol. unless $\boldsymbol{A}$ is well-cond.

$\boldsymbol{LU}$ Fact. Algo.

Algorithm 3: $\boldsymbol{LU}$ Fact. with Partial PVT.

```
p ← [1 : n] ;                                       // aux. vec.
for i ← 1 to n − 1 do
    a_ki ← max_{i≤j≤n} |a_ji| ;                     // find pvt.
    if k ≠ i then
        for j ← 1 to n do
            t ← a_ij, a_ij ← a_kj, a_kj ← t ;       // row i ⇌ row k
        p(k) ← i, p(i) ← k ;                        // upd. p
    for j ← i + 1 to n do
        a_ji ← a_ji / a_ii ;                        // calc. col. i
    for j ← i + 1 to n do
        for k ← i + 1 to n do
            a_jk ← a_jk − a_ji a_ik ;               // upd. A[i + 1 : n, i + 1 : n]
```

† fact. effectively in-place

† trans. of $\boldsymbol{b}$ can be included or as a sep. step

- storage anly. $\left\{ \begin{array}{l} \boldsymbol{L}, \boldsymbol{U} \text{ overwrite init. mat. } \boldsymbol{A} \left\langle \begin{array}{l} \boldsymbol{U} \text{ on upper tri.} \\ \boldsymbol{L} \text{ on strict lower tri.} \end{array} \right. \\ \text{aux. vec. } \boldsymbol{p} \text{ keeps track of new row order} \end{array} \right.$
- complexity anly.: $\frac{2}{3}n^3 + \mathcal{O}(n^2)$ [14]

Special Lin. Sys. $\left\{ \begin{array}{ll} \text{symm./herm.} & : \boldsymbol{A} = \boldsymbol{A}^T,\ \boldsymbol{A} = \boldsymbol{A}^H \\ \text{pos. def.} & : \forall \boldsymbol{x} \ne 0\ \ \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0,\ \boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x} > 0 \\ \text{banded} & : \forall\, |i-j| > \beta\ \ a_{ij} = 0,\ \text{ where } \beta \triangleq \text{bandwidth} \\ \text{sparse} & : \text{most are zeros} \end{array} \right.$

Symm. Pos. Def. Sys. $\left\{ \begin{array}{l} \text{all eigvals. are pos.} \\ \text{all leading principle mats. are SPD} \\ \text{all elems. on diag. are pos., and } \max\limits_i\{a_{ii}\} > \max\limits_{i\ne j}\{|a_{ij}|\} \\ \text{Cholesky Fact.: } \boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T \end{array} \right.$

Cholesky Fact.

- $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & \cdots & l_{n2} \\ & & \ddots & \vdots \\ & & & l_{nn} \end{pmatrix}$$

$$a_{ij} = \sum_{k=1}^{n} l_{ik} l_{jk} = l_{jj} l_{ij} + \sum_{k=1}^{j-1} l_{ik} l_{jk} \ , \ \ i, j = 1, 2, \cdots, n$$

Algorithm 4: Cholesky $\boldsymbol{L}\boldsymbol{L}^T$ Fact.

```
for j ← 1 to n do
    a_jj ← (a_jj − Σ_{k=1}^{j−1} a_jk^2)^{1/2} ;                    // diag. elem.
    for i ← j + 1 to n do
        a_ij ← (a_ij − Σ_{k=1}^{j−1} a_ik a_jk) / a_jj ;     // elems. below diag. elem.
```

† $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$ is unique when diag. ele. are required to be pos.

† sqr. roots are all of pos. $\rightsquigarrow$ algo is well-def. $\rightsquigarrow$ no pvt. for N.S.: $\rho \le 1$ [15]

† only access, store, overwrite lower tri. mat. of $\boldsymbol{A}$

† $\frac{1}{3} n^3 + \mathcal{O}(n^2)$ [16] (half of $\boldsymbol{L}\boldsymbol{U}$ fact.); $\boxed{\text{mcmd.: } \mathbf{chol}(\boldsymbol{A})}$

- $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ l_{n1} & \cdots & l_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \begin{pmatrix} 1 & l_{21} & \cdots & l_{n1} \\ & 1 & \cdots & l_{n2} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

$$a_{ij} = \sum_{k=1}^{n} l_{ik} d_k l_{jk} = d_j l_{ij} + \sum_{k=1}^{j-1} l_{ik} d_k l_{jk} \ , \ \ i, j = 1, 2, \cdots, n$$

Algorithm 5: Cholesky $\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ Fact.

```
for j ← 1 to n do
    a_jj ← a_jj − Σ_{k=1}^{j−1} a_jk^2 a_kk ;                        // diag. elem.
    for i ← j + 1 to n do
        a_ij ← (a_ij − Σ_{k=1}^{j−1} a_ik a_kk a_jk) / a_jj ;   // elems. below diag. elem.
```

† advantage of not requiring any sqr. roots.

† d_i in $\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ is the sqr. of l_{ii} in $\boldsymbol{L}\boldsymbol{L}^T$

Symm. inDef. Lin. Sys.

† $\rho_{chol} \gg 1 \rightsquigarrow$ breakdown using Cholesky fact. $\rightsquigarrow$ unstable algo.

† pvt. $\rightsquigarrow$ asymm. $\overset{\text{exchange both}}{\underset{\text{rows \& cols.}}{\rightsquigarrow}}$ symm.: $\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$

† stable $\begin{cases} \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T = \boldsymbol{L}\boldsymbol{T}\boldsymbol{L}^T \ , & \boldsymbol{T} \text{ is symm. tri. mat. [Aas71]} \\ \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^T = \boldsymbol{L}\tilde{\boldsymbol{D}}\boldsymbol{L}^T \ , & \tilde{\boldsymbol{D}} \text{ is block tridiag. mat.} \begin{cases} 1 \times 1 \\ 2 \times 2 \end{cases} \text{ [BK77]} \end{cases}$

Tridiag. Lin. Sys.

$$\boldsymbol{A} = \begin{pmatrix} d_1 & u_1 & & \\ l_2 & \ddots & \ddots & \\ & \ddots & \ddots & u_{n-1} \\ & & l_n & d_n \end{pmatrix} \quad , \quad \begin{array}{l} |d_1| > |u_1| > 0 \\ |d_n| > |l_n| > 0 \\ |d_i| \geq |l_i| + |u_i| > 0 \end{array}$$

‡ $>$: irr. strict diag. dom. mat.

‡ $\geq$: irr. weak diag. dom. mat.

$$\boldsymbol{A} = \begin{pmatrix} d_1 & u_1 & & \\ l_2 & \ddots & \ddots & \\ & \ddots & \ddots & u_{n-1} \\ & & l_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & & & \\ l_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & l_n & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \beta_1 & & \\ & 1 & \ddots & \\ & & \ddots & \beta_{n-1} \\ & & & 1 \end{pmatrix} \triangleq \boldsymbol{LU} \quad , \quad \begin{array}{c} \alpha_i = d_i - l_i\beta_{i-1} \\ \beta_i = u_i/\alpha_i \ (\ \beta_0 = 0\ ) \\ i = 1, \cdots, n \end{array}$$

$$\boldsymbol{Ax} = \boldsymbol{b} \Rightarrow \boldsymbol{LUx} = \boldsymbol{b} \left\langle \begin{array}{lll} \boldsymbol{Ly} = \boldsymbol{b} & \Rightarrow & y_i = (b_i - l_i y_{i-1})/\alpha_i \quad , \quad i = 1, \cdots, n \ , \ y_0 = 0 \\ \boldsymbol{Ux} = \boldsymbol{y} & \Rightarrow & x_i = (y_i - \beta_i x_{i+1}) \quad , \quad i = n, \cdots, 1 \ , \ x_{n+1} = 0 \end{array} \right.$$

Algorithm 6: Thomas Algo. for Tridiag. Lin. Sys.

```
β_0 = 0, y_0 = 0, x_{n+1} = 0 ;                        // init. val.
for i ← 1 to n do
    α_i ← d_i − l_i β_{i−1} ;                           // calc. α_i
    β_i ← u_i/α_i ;                                      // calc. β_i
    y_i ← (b_i − l_i y_{i−1})/α_i ;                      // calc. y_i
for i ← n to 1 do
    x_i ← (y_i − β_i x_{i+1}) ;                          // calc. x_i
```

† $0 < |\beta_i| < 1 \rightsquigarrow$ err. in back. subs. is under control

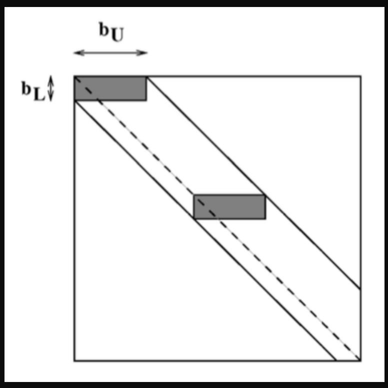
† $0 < |d_i| - |l_i| < |\alpha_i| < |d_i| + |l_i| \ , \ i = 2, 3, \cdots, n \rightsquigarrow \alpha_i \neq 0$

† solve $\boldsymbol{Ly} = \boldsymbol{b}$ and $\boldsymbol{LU}$ fact. simultaneously $\rightsquigarrow$ no need to store α_i but β_i

† complexity: $8n$ [17]

Banded Tri. Lin. Sys.

$$a_{ij} = 0 \quad \text{for} \begin{cases} i > j + b_L \\ i < j - b_U \end{cases}$$



- no pvt. $(\boldsymbol{A} = \boldsymbol{LU})$ $\left\langle \begin{array}{l} \boldsymbol{L} \ : \ \text{lower band} = b_L \\ \boldsymbol{U} \ : \ \text{upper band} = b_U \end{array} \right.$
- partial pvt. $(\boldsymbol{PA} = \boldsymbol{LU})$ $\left\langle \begin{array}{l} \boldsymbol{L} \ : \ \text{lower band} = b_L \text{ with at most } b_L + 1 \text{ nz. in each col.} \\ \boldsymbol{U} \ : \ \text{upper band} \leq b_L + b_U \end{array} \right.$

† complexity: $2nb_Lb_U + 2n(b_L + b_U)$

Pertb. Anly.

$$\left.\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b} \\ (\boldsymbol{A}+\boldsymbol{\delta A})\boldsymbol{x}^* &= \boldsymbol{b}+\boldsymbol{\delta b} \\ \boldsymbol{r} &= \boldsymbol{b}-\boldsymbol{Ax}^* \end{aligned}\right\} \xRightarrow[\text{back. anly.}]{} \begin{aligned} \boldsymbol{\delta x} &= \boldsymbol{x}^* - \boldsymbol{x} \\ &= -\boldsymbol{A}^{-1}\boldsymbol{r} \text{ [18]} \end{aligned} \quad \left\{\begin{aligned} &\boldsymbol{x} &&: \text{ exact sol.} \\ &\boldsymbol{x}^* &&: \text{ approx. sol.} \\ &\boldsymbol{r} &&: \text{ res.} \end{aligned}\right.$$

- $\boldsymbol{\delta x} \leftrightsquigarrow \boldsymbol{x}^*$ ($\boldsymbol{A}$ is non-singlr.)

$$\frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}^*\|} \leq \|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{A}\| \cdot \left(\frac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|} + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{A}\| \cdot \|\boldsymbol{x}^*\|} \right) \text{ [19]}$$

$$\frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}^*\|} \leq \kappa(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|} \quad \text{when} \quad \|\boldsymbol{\delta b}\| = 0$$

- $\boldsymbol{\delta x} \leftrightsquigarrow \boldsymbol{x}$ ($\boldsymbol{A}$ is non-singlr., $\|\boldsymbol{A}^{-1}\| \cdot \|\delta \boldsymbol{A}\| < 1$)

$$\frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}\|} \leq \frac{\kappa(\boldsymbol{A})}{1 - \kappa(\boldsymbol{A}) \cdot \dfrac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|}} \cdot \left(\frac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|} + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{b}\|} \right) \text{ [20]}$$

$$\frac{1}{\kappa(\boldsymbol{A})} \cdot \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{b}\|} \leq \frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{b}\|} \quad \text{when} \quad \|\boldsymbol{\delta A}\| = 0$$

- $\boldsymbol{\delta x} \leftrightsquigarrow \boldsymbol{r}$ ($\boldsymbol{A}$ is non-singlr.)

$$\|\boldsymbol{\delta x}\| \leq \|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{r}\| \text{ [18]}$$

Accuracy Imprv.

- scaling
  - † scale by multiplying a diag. mat. $\boldsymbol{D}$: $\boldsymbol{DADy} = \boldsymbol{Db}$, $\boldsymbol{x} = \boldsymbol{Dy}$;
  - † scaling affects cond.ing; not practical but worth trying for ill-cond. sys.
  - † accuracy is usually enhanced if all entries have abt. same order of mag.

- iter. refine. $\begin{cases} \text{calc. res. } \boldsymbol{r} \text{ for approx. sol. } \boldsymbol{x}\text{: } \boldsymbol{r} = \boldsymbol{b} - \boldsymbol{Ax}^* \\ \text{sol. } \boldsymbol{Az} = \boldsymbol{r} \text{ [21] with two tri. lin. sys. [22]} : \mathcal{O}(n^2) \end{cases}$

Algorithm 7: Accuracy Enhance. by Iter. Refine.

```
let PA = LU , x* as approx. sol.;
repeat
    calc. r ← b − Ax*;
    sol. Ly = Pr ;                          // y = L^{-1} Pr
    sol. Uz = y ;                           // z = U^{-1} y
    x* ← x* + z ;                           // new approx. sol.
until x* converges;
```

† $\boldsymbol{x}^* + \boldsymbol{z}$ is the exact sol., but $\boldsymbol{z}^*$ is found in practice

† $\|\boldsymbol{r} - \boldsymbol{Az}^*\|$ is smaller than $\|\boldsymbol{r}\|$ $\rightsquigarrow$ $\boldsymbol{x}^* + \boldsymbol{z}^*$ is closer to exact sol. than $\boldsymbol{x}^*$

† $\boldsymbol{A}$ can not be overwritten $\rightsquigarrow$ $\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{U}$ are stored

Lin. LSQ. Prob.

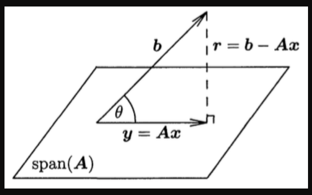
$$\boldsymbol{Ax} \cong \boldsymbol{b} \Leftrightarrow \min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{b}-\boldsymbol{Ax}\|_2 = \min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{r}\|_2 \begin{cases} m=n & : \ \boldsymbol{x}=\boldsymbol{A}^{-1}\boldsymbol{b} \text{ if } \boldsymbol{A} \text{ is non-singlr.} \\ m>n & : \ \text{overdet. (in most cases)} \\ m<n & : \ \text{underdet.} \end{cases}$$

† uniqueness: sol. to $\boldsymbol{Ax} \cong \boldsymbol{b}$ is unique iff. $\boldsymbol{A}$ is full col. rank: $\operatorname{rank}(\boldsymbol{A}) = n$

† existence: sol. always exists but not unique if $\boldsymbol{A}$ is rank-deficient

Cond.ing of Lin. LSQ. Prob.

- pseudoinv. for non-singlr. mat.: $\boldsymbol{A}^+ \triangleq (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$
- cond. num. $\left\langle \begin{array}{ll} \text{full col. rank} & : \ \kappa(\boldsymbol{A}) \triangleq \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^+\| \\ \text{defective rank} & : \ \kappa(A) = \infty \end{array} \right.$



Sensitivity of Lin. LSQ. Prob.

$$\underset{\boldsymbol{b}+\Delta\boldsymbol{b}}{\text{pertb. in } \boldsymbol{b}} \ : \ \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A}) \cdot \frac{1}{\cos\theta} \cdot \frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|} \qquad [23]$$

$$\underset{\boldsymbol{A}+\boldsymbol{E}}{\text{pertb. in } \boldsymbol{A}} \ : \ \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \lessapprox \left(\kappa^2(\boldsymbol{A}) \cdot \tan\theta + \kappa(\boldsymbol{A})\right) \cdot \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \qquad [24]$$

LSQ. Prob. Trans.

$$\underset{\text{overdet.}}{\boldsymbol{Ax} \cong \boldsymbol{b}} \rightsquigarrow \text{sqr. lin. sys.} \rightsquigarrow \text{tri. lin. sys.} \begin{cases} \text{1) normal eq.} & : \ \text{fastest} \\ \text{2) QR Fact.} & : \ \text{most important} \\ \text{3) SVD Fact.} & : \ \text{slowest but most reliable} \end{cases}$$

Normal Eq.

$\boldsymbol{A} \in \mathbb{R}^{m\times n}$ $(m \geq n)$, $\boldsymbol{x}^* \in \mathbb{R}^n$ is lsq. sol. iff. $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{Ax}^* \perp \operatorname{Ran}(\boldsymbol{A})$

$$\Rightarrow \left\langle \begin{array}{ll} \boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b} & : \ \text{lsq. prob. sol.} \iff \text{normal eq. sol.} \\ \boldsymbol{A}^T(\boldsymbol{b}-\boldsymbol{Ax}) = 0 & : \ \text{res. } \boldsymbol{r}=\boldsymbol{b}-\boldsymbol{Ax} \text{ is ortho. to } \operatorname{span}(\boldsymbol{A}) \text{ (geo. view)} \end{array} \right.$$

† rect. mat. $\rightsquigarrow$ sqr. mat. $\rightsquigarrow$ tri. mat.

† $\boldsymbol{A}$ is $\underset{\operatorname{rank}(\boldsymbol{A})=n}{\text{full col. rank}} \Rightarrow \begin{cases} \boldsymbol{A}^T\boldsymbol{A} \text{ is symm. pos. def.} \\ \text{unique lsq. sol.: } \boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} \\ \text{Cholesky Fact.: } \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{LL}^T \Rightarrow \left\langle \begin{array}{l} \boldsymbol{Ly} = \boldsymbol{A}^T\boldsymbol{b} \\ \boldsymbol{L}^T\boldsymbol{x} = \boldsymbol{y} \end{array} \right. \end{cases}$

- ∘ info. can be lost in forming $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}^T\boldsymbol{b}$
- ∘ cond. num. is **sqr.** of $\kappa(\boldsymbol{A})$: $\kappa(\boldsymbol{A}^T\boldsymbol{A}) = \kappa^2(\boldsymbol{A}) \rightsquigarrow$ unstable algo.

† Chol. Fact. complexity of $\boldsymbol{A}^T\boldsymbol{A}$: $mn^2 + \frac{1}{3}n^3 + \mathcal{O}(n^2)$ [26]

$\boldsymbol{QR}$ **Fact.** $\left\langle \begin{array}{ll} \boldsymbol{Q} & : \ \texttt{orthornormal / unitary mat.} \\ \boldsymbol{R} & : \ \texttt{upper tri. mat} \end{array} \right.$ $\rightsquigarrow$ $\begin{array}{c} \boldsymbol{A} = \boldsymbol{QR} \\ (\boldsymbol{A} \in \mathbb{C}^{m \times n}, m \geq n) \end{array}$

† existence: $\exists\, \boldsymbol{Q} \in \mathbb{C}^{m \times n}\ (\boldsymbol{Q}^* \boldsymbol{Q} = \boldsymbol{I})$, $\boldsymbol{R} \in \mathbb{C}^{n \times n}\ (r_{ij} = 0, i > j)$

† uniqueness: $\boldsymbol{QR}$ fact. is unique iff. $\boldsymbol{A}$ is full col. rank, and $r_{ii} > 0$

† app: 1) lin. lsq. prob.; 2) eigval. prob.; 3) $\boldsymbol{Ax} = \boldsymbol{b}$ when $\boldsymbol{A}$ is non-singlr. sqr. mat.

$\boldsymbol{QR}$ Fact. $\leftrightsquigarrow$ Lin. LSQ. Prob.

derv. mmz. of lin. lsq. prob. with $\boldsymbol{QR}$ fact. $\rightsquigarrow$ $\left. \begin{array}{lll} ^{1)}\,\texttt{augment} & \boldsymbol{Q} & : \ [1] \\ ^{2)}\,\texttt{trans.} & \boldsymbol{b} & : \ [2] \\ ^{3)}\,\texttt{normal eq.} & & : \ [3] \end{array} \right\rangle \Rightarrow \boldsymbol{x}^* = \boldsymbol{R}^{-1} \boldsymbol{Q}^T \boldsymbol{b}$

(1)

$$\left. \begin{array}{l} \boldsymbol{A}_{m \times n} = \boldsymbol{Q}_{m \times n} \boldsymbol{R}_{n \times n} \\ (\boldsymbol{Q}, \hat{\boldsymbol{Q}}) \in \mathbb{R}^{m \times m} \end{array} \right\rangle \Rightarrow \left\| \boldsymbol{Ax} - \boldsymbol{b} \right\|_2^2 = \left\| (\boldsymbol{Q}, \hat{\boldsymbol{Q}})^T \cdot (\boldsymbol{Ax} - \boldsymbol{b}) \right\|_2^2 = \left\| \begin{pmatrix} \boldsymbol{Q}^T \\ \hat{\boldsymbol{Q}}^T \end{pmatrix} \cdot (\boldsymbol{QRx} - \boldsymbol{b}) \right\|_2^2$$

$$= \left\| \begin{pmatrix} \boldsymbol{Rx} - \boldsymbol{Q}^T b \\ -\hat{\boldsymbol{Q}}^T \boldsymbol{b} \end{pmatrix} \right\|_2^2 = \left\| \boldsymbol{Rx} - \boldsymbol{Q}^T \boldsymbol{b} \right\|_2^2 + \left\| \hat{\boldsymbol{Q}}^T \boldsymbol{b} \right\|_2^2 \geq \left\| \hat{\boldsymbol{Q}}^T \boldsymbol{b} \right\|_2^2 \Rightarrow \boldsymbol{x}^* = \boldsymbol{R}^{-1} \boldsymbol{Q}^T \boldsymbol{b}$$

(2)

$$\boldsymbol{A}_{m \times n} = \boldsymbol{Q}_{m \times n} \boldsymbol{R}_{n \times n} \ , \ \boldsymbol{b} = (\boldsymbol{QQ}^T + \boldsymbol{I} - \boldsymbol{QQ}^T) \boldsymbol{b}$$

$$\Rightarrow \boldsymbol{Ax} - \boldsymbol{b} = \boldsymbol{Ax} - (\boldsymbol{QQ}^T + \boldsymbol{I} - \boldsymbol{QQ}^T) \boldsymbol{b} = (\boldsymbol{Ax} - \boldsymbol{QQ}^T \boldsymbol{b}) - (\boldsymbol{I} - \boldsymbol{QQ}^T) \boldsymbol{b}$$

$$\dagger \Rightarrow \left\| \boldsymbol{Ax} - \boldsymbol{b} \right\|_2^2 = \left\| \boldsymbol{Ax} - \boldsymbol{QQ}^T \boldsymbol{b} \right\|_2^2 + \left\| (\boldsymbol{I} - \boldsymbol{QQ}^T) \boldsymbol{b} \right\|_2^2 = \left\| \boldsymbol{Q} (\boldsymbol{Rx} - \boldsymbol{Q}^T \boldsymbol{b}) \right\|_2^2 + \left\| (\boldsymbol{I} - \boldsymbol{QQ}^T) \boldsymbol{b} \right\|_2^2$$

$$\geq \left\| (\boldsymbol{I} - \boldsymbol{QQ}^T) \boldsymbol{b} \right\|_2^2 \Rightarrow \boldsymbol{x}^* = \boldsymbol{R}^{-1} \boldsymbol{Q}^T \boldsymbol{b}$$

(3)

$$\boldsymbol{x}^* = (\boldsymbol{A}^T \boldsymbol{A})^{-1} \boldsymbol{A}^T \boldsymbol{b} = (\boldsymbol{R}^T \underbrace{\boldsymbol{Q}^T \boldsymbol{Q}}_{\boldsymbol{I}_{n \times n}} \boldsymbol{R})^{-1} \boldsymbol{R}^T \boldsymbol{Q}^T \boldsymbol{b} = \boldsymbol{R}^{-1} \boldsymbol{R}^{-T} \boldsymbol{R}^T \boldsymbol{Q}^T \boldsymbol{b} = \boldsymbol{R}^{-1} \boldsymbol{Q}^T \boldsymbol{b}$$

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots \tag{4}$$

$$f(x)-f(\tilde{x})\approx f'(\tilde{x})(x-\tilde{x})+\frac{1}{2}f'(\tilde{\xi})(x-\tilde{x})^2 \underset{|e|=|x-\tilde{x}|\le\varepsilon}{\Rightarrow} \begin{aligned}\varepsilon(f(x)) &\le |f'(\tilde{x})|\,\varepsilon(x)+\frac{|f''(\xi)|}{2}\varepsilon^2(x)\\ &\lessapprox |f'(\tilde{x})|\,\varepsilon(x) \underset{f'(x)\approx f'(\tilde{x})}{=} |f'(x)|\,\varepsilon(x)\end{aligned} \tag{5}$$

$$\frac{\Delta y}{y}=\frac{f(x+\Delta x)-f(x)}{f(x)}\approx\frac{f'(x)\Delta x}{f(x)} \Rightarrow \mathrm{cond}\approx\frac{|\Delta y/y|}{|\Delta x/x|}=\frac{|xf'(x)|}{|f(x)|} \tag{6}$$

$$\left.\begin{array}{r}\left.\begin{array}{c}\boldsymbol{Ax}=\boldsymbol{b}\\ \boldsymbol{A\hat{x}}=\boldsymbol{b}+\Delta\boldsymbol{b}\end{array}\right\rangle \underset{\Delta\boldsymbol{x}=\hat{\boldsymbol{x}}-\boldsymbol{x}}{\Rightarrow} \boldsymbol{A}\Delta\boldsymbol{x}=\Delta\boldsymbol{b}\\ \|\boldsymbol{b}\|=\|\boldsymbol{Ax}\|\le\|\boldsymbol{A}\|\cdot\|\boldsymbol{x}\| \\ \|\Delta\boldsymbol{x}\|=\|\boldsymbol{A}^{-1}\boldsymbol{b}\|\le\|\boldsymbol{A}^{-1}\|\cdot\|\Delta\boldsymbol{b}\|\end{array}\right\} \Rightarrow \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\le\|\boldsymbol{A}^{-1}\|\cdot\|\Delta\boldsymbol{b}\|\cdot\frac{\|\boldsymbol{A}\|}{\|\boldsymbol{b}\|}=\kappa(\boldsymbol{A})\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|} \tag{7}$$

$$\begin{aligned}\left.\begin{array}{c}\boldsymbol{Ax}=\boldsymbol{b}\\ (\boldsymbol{A}+\boldsymbol{E})\hat{\boldsymbol{x}}=\boldsymbol{b}\end{array}\right\rangle &\Rightarrow \Delta\boldsymbol{x}=\hat{\boldsymbol{x}}-\boldsymbol{x}=\boldsymbol{A}^{-1}(\boldsymbol{A}\hat{\boldsymbol{x}}-\boldsymbol{b})=\boldsymbol{A}^{-1}(-\boldsymbol{E}\hat{\boldsymbol{x}})=-\boldsymbol{A}^{-1}\boldsymbol{E}\hat{\boldsymbol{x}}\\ &\Rightarrow \|\Delta\boldsymbol{x}\|\le\|\boldsymbol{A}^{-1}\|\cdot\|\boldsymbol{E}\|\cdot\|\hat{\boldsymbol{x}}\| \Rightarrow \frac{\|\Delta\boldsymbol{x}\|}{\|\hat{\boldsymbol{x}}\|}\le\kappa(\boldsymbol{A})\frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}\end{aligned} \tag{8}$$

$$\begin{aligned}\left.\begin{array}{c}\boldsymbol{Ax}=\boldsymbol{b}\\ (\boldsymbol{A}+\boldsymbol{E})\hat{\boldsymbol{x}}=\boldsymbol{b}+\Delta\boldsymbol{b}\end{array}\right\rangle &\Rightarrow \Delta\boldsymbol{x}=\hat{\boldsymbol{x}}-\boldsymbol{x}=\boldsymbol{A}^{-1}(\boldsymbol{A}\hat{\boldsymbol{x}}-\boldsymbol{b})=\boldsymbol{A}^{-1}(\Delta\boldsymbol{b}-\boldsymbol{E}\hat{\boldsymbol{x}})\\ &\Rightarrow \|\Delta\boldsymbol{x}\|\le\|\boldsymbol{A}^{-1}\Delta\boldsymbol{b}\|+\|\boldsymbol{A}^{-1}\boldsymbol{E}\hat{\boldsymbol{x}}\|\le\|\boldsymbol{A}^{-1}\|\cdot\|\Delta\boldsymbol{b}\|+\|\boldsymbol{A}^{-1}\|\cdot\|\boldsymbol{E}\|\cdot\|\boldsymbol{x}\|\\ &\Rightarrow \frac{\Delta\boldsymbol{x}}{\|\boldsymbol{x}\|}\le\frac{\|\boldsymbol{A}\|\cdot\|\boldsymbol{A}^{-1}\|\cdot\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{A}\|\cdot\|\boldsymbol{x}\|}+\frac{\|\boldsymbol{A}\|\cdot\|\boldsymbol{A}^{-1}\|\cdot\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}=\kappa(\boldsymbol{A})\left(\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}+\frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}\right)\end{aligned} \tag{9}$$

$$\left.\begin{array}{l}\|\Delta\boldsymbol{x}\|=\|\hat{\boldsymbol{x}}-\boldsymbol{x}\|=\|\boldsymbol{A}^{-1}(\boldsymbol{A}\hat{\boldsymbol{x}}-\boldsymbol{b})\|\\ \qquad\underset{\boldsymbol{r}=\boldsymbol{b}-\boldsymbol{A}\hat{\boldsymbol{x}}}{=}\|\boldsymbol{A}^{-1}(-\boldsymbol{r})\|\le\|\boldsymbol{A}^{-1}\|\cdot\|\boldsymbol{r}\|\end{array}\right\rangle \Rightarrow \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\le\frac{\|\boldsymbol{A}\|\cdot\|\boldsymbol{A}^{-1}\|\cdot\|\boldsymbol{r}\|}{\|\boldsymbol{A}\|\cdot\|\boldsymbol{x}\|}=\kappa(\boldsymbol{A})\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\|\cdot\|\boldsymbol{x}\|} \tag{10}$$

$$(\boldsymbol{A}+\boldsymbol{E})\hat{\boldsymbol{x}}=\boldsymbol{b}\Rightarrow\|\boldsymbol{r}\|=\|\boldsymbol{b}-\boldsymbol{A}\hat{\boldsymbol{x}}\|=\|\boldsymbol{E}\hat{\boldsymbol{x}}\|\le\|\boldsymbol{E}\|\cdot\|\hat{\boldsymbol{x}}\| \Rightarrow \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\|\cdot\|\hat{\boldsymbol{x}}\|}\le\frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \tag{11}$$

$$\left.\begin{array}{c}\boldsymbol{MA}=\boldsymbol{U}\\ \boldsymbol{A}=\boldsymbol{LU}\end{array}\right\rangle \Rightarrow \boldsymbol{L}=\boldsymbol{M}^{-1}=(\boldsymbol{M}_{n-1}\cdots\boldsymbol{M}_1)^{-1}=\boldsymbol{M}_1^{-1}\cdots\boldsymbol{M}_{n-1}^{-1}=\boldsymbol{L}_1\cdots\boldsymbol{L}_{n-1} \tag{12}$$

$$\left.\begin{array}{r}\boldsymbol{M}_k\boldsymbol{M}_j=\left(\boldsymbol{I}-\boldsymbol{me}_k^T\right)\left(\boldsymbol{I}-\boldsymbol{te}_j^T\right)=\boldsymbol{I}-\boldsymbol{te}_j^T-\boldsymbol{me}_k^T+\boldsymbol{me}_k^T\boldsymbol{te}_j^T\\ \boldsymbol{L}_k\boldsymbol{L}_j=\left(\boldsymbol{I}+\boldsymbol{me}_k^T\right)\left(\boldsymbol{I}+\boldsymbol{te}_j^T\right)=\boldsymbol{I}+\boldsymbol{te}_j^T+\boldsymbol{me}_k^T+\boldsymbol{me}_k^T\boldsymbol{te}_j^T\end{array}\right\rangle \overset{\boldsymbol{e}_k^T\boldsymbol{t}=0}{\underset{\texttt{if }j>k}{=}} \left\langle\begin{array}{l}\boldsymbol{I}-\boldsymbol{te}_j^T-\boldsymbol{me}_k^T\\ \boldsymbol{I}+\boldsymbol{te}_j^T+\boldsymbol{me}_k^T\end{array}\right. \tag{13}$$

$$\begin{aligned}\boldsymbol{O}_{\mathbf{lu}}&=\sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n}1+\sum_{j=i+1}^{n}\sum_{k=i+1}^{n}2\right)=\sum_{i=1}^{n-1}\left(n-i+2(n-i)^2\right)=\frac{n(n-1)}{2}+2\frac{n(n-1)(2n-1)}{6}\\ &=\frac{2}{3}n^3+\mathcal{O}(n^2)\end{aligned} \tag{14}$$

(15) $$\boldsymbol{A} = \boldsymbol{L}_0 \underbrace{\boldsymbol{D}\boldsymbol{L}_0^T}_{\boldsymbol{U}} \Rightarrow \boldsymbol{U}^T = \boldsymbol{L}_0\boldsymbol{D} \Rightarrow \rho_{\texttt{chol}} = \frac{\max\left\{\left|\boldsymbol{U}^T(i,j)\right|\right\}}{\max\{|a_{ij}|\}} = \frac{\max\{|l_{ij} \cdot l_{jj}|\}}{\max\{|a_{ij}|\}} \leq \frac{\max\left\{l_{ij}^2\right\}}{\max\{a_{ii}\}} \leq 1$$

(16) $$\begin{aligned}\mathcal{O}_{\texttt{chol}} = \sum_{j=1}^{n}\left[\left(1+\sum_{k=1}^{j-1}2+1\right) + \sum_{i=j+1}^{n}\left(1+\sum_{k=1}^{j-1}2+1\right)\right] &= \sum_{j=1}^{n}\left(2j+\sum_{i=j+1}^{n}2j\right) = 2(n+1)\sum_{j=1}^{n}j - 2\sum_{j=1}^{n}j^2 \\ &= n(n+1)^2 - \frac{n(n+1)(2n+1)}{3} = \frac{1}{3}n^3 + \mathcal{O}(n^2)\end{aligned}$$

(17) $$\mathcal{O}_{\texttt{thomas}} = \sum_{i=1}^{n}6 + \sum_{i=n}^{1}2 = 8n$$

(18) $$\boldsymbol{\delta x} = \boldsymbol{x}^* - \boldsymbol{x} = \boldsymbol{x}^* - \boldsymbol{A}^{-1}\boldsymbol{b} = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}) = -\boldsymbol{A}^{-1}\boldsymbol{r} \Rightarrow \|\boldsymbol{\delta x}\| \leq \|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{r}\|$$

(19) $$\left.\begin{aligned}&(\boldsymbol{A}+\boldsymbol{\delta A})\boldsymbol{x}^* = \boldsymbol{b}+\boldsymbol{\delta b} = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{\delta b} \\ &\Rightarrow \boldsymbol{A}(\overbrace{\boldsymbol{x}^*-\boldsymbol{x}}^{\boldsymbol{\delta x}}) = -\boldsymbol{\delta A}\boldsymbol{x}^* + \boldsymbol{\delta b} \\ &\Rightarrow \boldsymbol{\delta x} = \boldsymbol{A}^{-1}(-\boldsymbol{\delta A}\boldsymbol{x}^*+\boldsymbol{\delta b})\end{aligned}\right\} \Rightarrow \begin{aligned}\|\boldsymbol{\delta x}\| &\leq \|\boldsymbol{A}^{-1}\| \cdot (\|\boldsymbol{\delta A}\| \cdot \|\boldsymbol{x}^*\| + \|\boldsymbol{\delta b}\|) \\ \frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}^*\|} &\leq \|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{A}\| \cdot \left(\frac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|} + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{A}\| \cdot \|\boldsymbol{x}^*\|}\right)\end{aligned}$$

(20) $$\begin{aligned}&(\boldsymbol{A}+\boldsymbol{\delta A})\boldsymbol{x}^* = \boldsymbol{b}+\boldsymbol{\delta b} \;\Rightarrow\; \overbrace{\boldsymbol{\delta x}+\boldsymbol{x}}^{\boldsymbol{x}^*} = (\boldsymbol{A}+\boldsymbol{\delta A})^{-1}(\boldsymbol{b}+\boldsymbol{\delta b}) \\ \Rightarrow \quad \boldsymbol{\delta x} \quad &= (\boldsymbol{A}+\boldsymbol{\delta A})^{-1}\Big(\boldsymbol{b}+\boldsymbol{\delta b}-(\boldsymbol{A}+\boldsymbol{\delta A})\boldsymbol{x}\Big) = (\boldsymbol{A}+\boldsymbol{\delta A})^{-1}(\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}-\boldsymbol{\delta A}\boldsymbol{x}+\boldsymbol{\delta b}) \\ &= (\boldsymbol{A}+\boldsymbol{\delta A})^{-1}(-\boldsymbol{\delta A}\boldsymbol{x}+\boldsymbol{\delta b}) \overset{\dagger}{=} (\boldsymbol{I}+\boldsymbol{A}^{-1}\boldsymbol{\delta A})^{-1}\boldsymbol{A}^{-1}(-\boldsymbol{\delta A}\boldsymbol{x}+\boldsymbol{\delta b}) \\ \Rightarrow \quad \frac{\|\boldsymbol{\delta x}\|}{\|\boldsymbol{x}\|} &\leq \|(\boldsymbol{I}+\boldsymbol{A}^{-1}\boldsymbol{\delta A})^{-1}\| \cdot \|\boldsymbol{A}^{-1}\| \cdot \left(\|\boldsymbol{\delta A}\| + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{x}\|}\right) \overset{\ddagger}{\leq} \frac{\|\boldsymbol{A}^{-1}\|}{1-\|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{\delta A}\|} \cdot \left(\|\boldsymbol{\delta A}\| + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{x}\|}\right) \\ &= \frac{\|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{A}\|}{1-\|\boldsymbol{A}^{-1}\| \cdot \|\boldsymbol{A}\| \cdot \dfrac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|}} \cdot \left(\frac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|} + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{A}\| \cdot \|\boldsymbol{x}\|}\right) \\ &= \frac{\kappa(\boldsymbol{A})}{1-\kappa(\boldsymbol{A}) \cdot \dfrac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|}} \cdot \left(\frac{\|\boldsymbol{\delta A}\|}{\|\boldsymbol{A}\|} + \frac{\|\boldsymbol{\delta b}\|}{\|\boldsymbol{b}\|}\right)\end{aligned}$$

$$^{\dagger}(\boldsymbol{X}+\boldsymbol{Y})^{-1} = \Big(\boldsymbol{X}(\boldsymbol{I}+\boldsymbol{X}^{-1}\boldsymbol{Y})\Big)^{-1} = (\boldsymbol{I}+\boldsymbol{X}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{X}^{-1} \;, \quad ^{\ddagger}\|(\boldsymbol{I}-\boldsymbol{X})^{-1}\| \leq \frac{1}{1-\|\boldsymbol{X}\|} \;\; \texttt{if} \;\; \|\boldsymbol{X}\| < 1$$

(21) $$\boldsymbol{A}(\boldsymbol{x}^*+\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{A}\boldsymbol{z} = (\boldsymbol{b}-\boldsymbol{r}) + \boldsymbol{r} = \boldsymbol{b}$$

(22) $$\left.\begin{aligned}\boldsymbol{P}\boldsymbol{A} &= \boldsymbol{L}\boldsymbol{U} \\ \boldsymbol{A}\boldsymbol{z} &= \boldsymbol{r}\end{aligned}\right\rangle \Rightarrow \boldsymbol{P}\boldsymbol{A}\boldsymbol{z} = \boldsymbol{L}\boldsymbol{U}\boldsymbol{z} = \boldsymbol{P}\boldsymbol{r} \Rightarrow \left\langle\begin{aligned}\boldsymbol{L}\boldsymbol{y} &= \boldsymbol{P}\boldsymbol{r} \\ \boldsymbol{U}\boldsymbol{z} &= \boldsymbol{y}\end{aligned}\right.$$

(23) $$\left.\begin{aligned}\boldsymbol{A}^T\boldsymbol{A}(\boldsymbol{x}+\Delta\boldsymbol{x}) &= \boldsymbol{A}^T(\boldsymbol{b}+\Delta\boldsymbol{b}) \\ \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{A}^T\boldsymbol{b}\end{aligned}\right\rangle \Rightarrow \boldsymbol{A}^T\boldsymbol{A}\Delta\boldsymbol{x} = \boldsymbol{A}^T\Delta\boldsymbol{b} \Rightarrow \Delta\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\Delta\boldsymbol{b}$$

$$\left.\begin{aligned}\boldsymbol{A}^+ &\triangleq (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T \\ \kappa(\boldsymbol{A}) &\triangleq \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^+\| \\ \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{b}\|} &\triangleq \cos\theta\end{aligned}\right\} \Rightarrow \begin{aligned}\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} &= \|\boldsymbol{A}^+\| \cdot \frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{x}\|} = \kappa(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{b}\|}{\|\boldsymbol{A}\| \cdot \|\boldsymbol{x}\|} \cdot \frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|} \\ &\leq \kappa(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{b}\|}{\|\boldsymbol{A}\boldsymbol{x}\|} \cdot \frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|} = \kappa(\boldsymbol{A}) \cdot \frac{1}{\cos\theta} \cdot \frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}\end{aligned}$$

$$(\boldsymbol{A}+\boldsymbol{E})^T(\boldsymbol{A}+\boldsymbol{E})(\boldsymbol{x}+\Delta\boldsymbol{x}) = (\boldsymbol{A}+\boldsymbol{E})^T\boldsymbol{b}$$

$$\Rightarrow (\boldsymbol{A}^T\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{E}+\boldsymbol{E}^T\boldsymbol{A}+\boldsymbol{E}^T\boldsymbol{E})(\boldsymbol{x}+\Delta\boldsymbol{x}) = (\boldsymbol{A}+\boldsymbol{E})^T\boldsymbol{b}$$

$$\Rightarrow \overbrace{\underline{\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}}}^{\boldsymbol{A}^T\boldsymbol{b}} + \boldsymbol{A}^T\boldsymbol{E}\boldsymbol{x} + \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{E}^T\boldsymbol{E}\boldsymbol{x} + \boldsymbol{A}^T\boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{A}^T\boldsymbol{E}\Delta\boldsymbol{x} + \boldsymbol{E}^T\boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{E}^T\boldsymbol{E}\Delta\boldsymbol{x} = \underline{\boldsymbol{A}^T\boldsymbol{b}} + \boldsymbol{E}^T\boldsymbol{b}$$

$$\begin{aligned} \boldsymbol{A}^T\boldsymbol{A}\Delta\boldsymbol{x} &\approx \boldsymbol{E}^T\boldsymbol{b} - \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}^T\boldsymbol{E}\boldsymbol{x} \quad (\texttt{drop } 2^{\text{nd}} \texttt{ terms of small pertub.}) \\ \Rightarrow \quad &= \boldsymbol{E}^T\underbrace{(\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x})}_{\boldsymbol{r}} - \boldsymbol{A}^T\boldsymbol{E}\boldsymbol{x} \end{aligned}$$

(24) $$\Rightarrow \Delta\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{E}^T\boldsymbol{r} - \underbrace{(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T}_{\boldsymbol{A}^+}\boldsymbol{E}\boldsymbol{x} \lessapprox \|(\boldsymbol{A}^T\boldsymbol{A})^{-1}\| \cdot \|\boldsymbol{E}\| \cdot \|\boldsymbol{r}\| + \|\boldsymbol{A}^+\| \cdot \|\boldsymbol{E}\| \cdot \|\boldsymbol{x}\|$$

$$\Rightarrow \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \lessapprox \|(\boldsymbol{A}^T\boldsymbol{A})^{-1}\| \cdot \|\boldsymbol{E}\| \cdot \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{x}\|} + \|\boldsymbol{A}^+\| \cdot \|\boldsymbol{E}\|$$

$$\Rightarrow \left.\begin{aligned} \kappa^2(\boldsymbol{A}) &\triangleq \|\boldsymbol{A}\|^2 \cdot \|(\boldsymbol{A}^T\boldsymbol{A})^{-1}\| \ [25] \\ \tan\theta &\triangleq \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\boldsymbol{x}\|} \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} &\lessapprox \kappa^2(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \cdot \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\| \cdot \|\boldsymbol{x}\|} + \kappa(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \\ &\le \kappa^2(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \cdot \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\boldsymbol{x}\|} + \kappa(\boldsymbol{A}) \cdot \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \\ &= \Big(\kappa^2(\boldsymbol{A}) \cdot \tan\theta + \kappa(\boldsymbol{A})\Big) \cdot \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|} \end{aligned}$$

(25) $$\kappa^2(\boldsymbol{A}) = ????$$

$$\kappa^2(\boldsymbol{A}) = \kappa(A^TA)$$

(26) $$\left.\begin{aligned} \mathcal{O}(\boldsymbol{A}^T\boldsymbol{A}) &= \cancel{2}m^2n \ (\texttt{symm.}) \\ \mathcal{O}_{\texttt{chol}}(\boldsymbol{A}) &= \frac{1}{3}n^3 + \mathcal{O}(n^2) \ [16] \end{aligned}\right\rangle \Rightarrow \mathcal{O}_{\texttt{chol}}(\boldsymbol{A}^T\boldsymbol{A}) = \underbrace{m^2n}_{\texttt{major}} + \frac{1}{3}n^3 + \mathcal{O}(n^2)$$

approx.	approximate(ly), approximation
vec.	vector
mat.	matrix
lin.	linear(ly)
indp.	independent(ly)
trans.	transformation
eq.	equation
charct.	characteristic
poly.	polynomial
dim.	dimension(al)
spctm.	spectrum

References

[Aas71] Jan Ole Aasen. On the reduction of a symmetric matrix to tridiagonal form. *BIT Numerical Mathematics*, 11(3):233--242, 1971. 8

[BK77] James R Bunch and Linda Kaufman. Some stable methods for calculating inertia and solving symmetric linear systems. *Mathematics of computation*, pages 163--179, 1977. 8

[Wil61] J. H. Wilkinson. Error analysis of direct methods of matrix inversion. *J. ACM*, 8(3):281--330, July 1961. 7

List of Algorithms