12.15

5-3

(13) 令 $\arcsin\sqrt{\frac{x}{1+x}} = t \rightarrow \sqrt{\frac{x}{1+x}} = \sin t$, $x = \frac{\sin^2 t}{1-\sin^2 t} = \tan^2 t$

原式 $= \int_0^{\frac{\pi}{3}} t \cdot d\tan^2 t = t\cdot \tan^2 t \Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan^2 t \, dt$

$= \pi - \int_0^{\frac{\pi}{3}} (\sec^2 - 1)\cdot dt = \pi - \tan t \Big|_0^{\frac{\pi}{3}} + \frac{\pi}{3}$

$= \frac{4}{3}\pi - \sqrt{3}$.

(14) $\int_0^{\pi} (x\sin x)^2 dx = \frac{1}{2}\int_0^{\pi} x^2(1-\cos 2x)\, dx = \frac{1}{6}x^3\Big|_0^{\pi} - \frac{1}{4}\int_0^{\pi} x^2 d\sin 2x$

$= \frac{\pi^3}{6} - \frac{1}{4}x^2\sin 2x\Big|_0^{\pi} + \frac{1}{4}\int_0^{\pi} \sin 2x \cdot 2x \cdot dx$

$= \frac{\pi^3}{6} - \frac{1}{4}\int_0^{\pi} x\, d\cos 2x = \frac{\pi^3}{6} - \frac{1}{4}x\cos 2x\Big|_0^{\pi} + \frac{1}{4}\int_0^{\pi} \cos 2x \cdot dx$

$= \frac{\pi^3}{6} - \frac{\pi}{4} + \frac{1}{8}\sin 2x\Big|_0^{\pi} = \frac{\pi^3}{6} - \frac{\pi}{4}$.

(15) $\int_0^{\pi} e^x\cos^2 x \cdot dx = \frac{1}{2}\int_0^{\pi} e^x(1+\cos 2x)\, dx = \frac{1}{2}e^x\Big|_0^{\pi} + \underline{\frac{1}{2}\int_0^{\pi} \cos 2x \cdot d(e^x)}$. ①

$= \frac{1}{2}(e^{\pi}-1) + \frac{1}{2}\cos 2x\cdot e^x\Big|_0^{\pi} + \cancel{\frac{1}{2}}\int_0^{\pi} \sin 2x \cdot d(e^x)$.

$= \frac{1}{2}(e^{\pi}-1) + \frac{1}{2}(e^{\pi}-1) + \cancel{\int_0^{\pi}} \sin 2x\cdot e^x\Big|_0^{\pi} - 2\int_0^{\pi} e^x \cdot \cos 2x \cdot dx$ ②

由①② 知 $\int_0^{\pi} e^x\cos 2x \cdot dx = \frac{1}{5}(e^{\pi}-1)$

原式 $= \frac{3}{5}(e^{\pi}-1)$.

(17). 原式 $= \int_0^{\pi} \sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}} \cdot dx = \int_0^{\frac{\pi}{2}} \left(\overset{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cancel{\sin\frac{x}{2}-\cos\frac{x}{2}}}\right) \cdot dx + \int_{\frac{\pi}{2}}^{\pi} (\sin x - \cos x)\, dx$

$= 4(\sqrt{2}-1)$.

(20) $\int_0^2 |1-x| \cdot dx = \int_0^1 (1-x) \cdot dx + \int_1^2 (x-1)\, dx = (x-\frac{x^2}{2})\Big|_0^1 + (\frac{x^2}{2}-x)\Big|_1^2$

$= \frac{1}{2} + (0-(-\frac{1}{2})) = 1$.

(21) $\int_0^2 [e^x]\, dx = \int_0^{\ln 2} 1\, dx + \int_{\ln 2}^{\ln 3} 2\, dx + \int_{\ln 3}^{\ln 4} 3\, dx + \cdots + \int_{\ln 7}^{2} 7\, dx$

$= 14 - \ln(7!)$

(26) 原式 $= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\, dx = \begin{cases} \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{n+m}\right]\Big|_{-\pi}^{\pi} = 0, \\ \frac{1}{2}\left[x - \frac{1}{2m}\sin 2mx\right]\Big|_{-\pi}^{\pi} = \pi, m=n. \end{cases}$

3.(1) $\int_0^{\frac{\pi}{2}} = \frac{5}{6} \times \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{32}\pi$ (2) $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \cancel{\times} = \frac{48}{105}$.

(4) $\int_{-\pi}^{\pi} \sin^4\frac{x}{2} \cdot dx \xlongequal[dt=\frac{1}{2}dx]{t=\frac{x}{2}} 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 t\, dt = 4\int_0^{\frac{\pi}{2}} \sin^4 t\, dt = 4 \times \frac{3}{4} \times \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{4}\pi$

6.(1) $I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\tan^2 x + 1)\, dx = \frac{1}{n-1} \tan^{n-1} x \Big|_0^{\frac{\pi}{4}} = \frac{1}{n-1}$.

$I_{n-2} + I_{n-4} = \frac{1}{n-3}$

$\cdots$

$I_3 + I_1 = \frac{1}{2}$

(2) $I_n > I_{n+1}$. $2I_n < I_n + I_{n-2} = \frac{1}{n-1}$ $\rightarrow I_n <$

又 $\frac{1}{n+1} = I_n + I_{n+2} < 2I_n \rightarrow I_n > \frac{1}{2(n+1)}$.

9. 原式 $= -\int_0^{\pi} f(x)\, d\cos x + \int_0^{\pi} \sin x\, d(f'(x))$

$= -f(x)\cos x\Big|_0^{\pi} + \int_0^{\pi} f'(x) \cdot \cos x\, dx + f'(x)\sin x\Big|_0^{\pi} - \int_0^{\pi} f'(x)\cos x\, dx$

$= f(\pi) + f(0) = 3 \longrightarrow f(0) = 2$.

10. (1) 令 $x = \frac{\pi}{2} - t$.

原式 $= \int_0^{\frac{\pi}{2}} \frac{f(\cos t)}{f(\cos t)+f(\sin t)} \cdot dt$ 易得 $\int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$ 即原式 $= \frac{\pi}{4}$.

(2) 令 $x = \frac{\pi}{4} - t$.

原式 $= \int_{\frac{\pi}{4}}^{0} \ln[1 + \frac{1-\tan t}{1+\tan t}] \cdot d(-t)$

$= \int_0^{\frac{\pi}{4}} \ln \frac{2}{1+\tan t} \cdot dt = \int_0^{\frac{\pi}{4}} \ln 2 \cdot dt - \int_0^{\frac{\pi}{4}} \ln(1+\tan t)\, dt$

原式 $= \frac{1}{2}\ln 2 \cdot \frac{\pi}{4} = \frac{\ln 2}{8}\pi$.

习题5-5

3. 令 $x = \tan t$ 原式 $= \int_0^{\frac{\pi}{2}} \frac{t \cdot \sec^2 t}{\sec^3 t} \cdot dt = \int_0^{\frac{\pi}{2}} t \cdot \cos t\, dt = \int_0^{\frac{\pi}{2}} t \cdot d\sin t$

$= t\sin t\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin t\, dt = \frac{\pi}{2} - 1$.

4. $\int_0^{+\infty} e^{-ax}\cos bx \cdot dx \ (a>0) = \frac{a}{a^2+b^2}$.

6. $\int_1^{+\infty} \frac{\ln x}{x^2} \cdot dx = -\int \ln x\, d(\frac{1}{x}) = -\ln x \cdot \frac{1}{x}\Big|_1^{+\infty} + \int_1^{+\infty} \frac{1}{x^2} \cdot dx = -\frac{1}{x}\ln x - \frac{1}{x}\Big|_1^{+\infty} =$

7. $\int_1^{+\infty} \frac{dx}{x(x^2+1)} = \int_1^{+\infty} (\frac{1}{x} - \frac{x}{x^2+1}) dx = \ln x \Big|_1^{+\infty} - \frac{1}{2}\ln(x^2+1) \Big|_1^{+\infty} = \ln \frac{x}{\sqrt{x^2+1}} \Big|_1^{+\infty} = \ln$

9. $\int_0^1 \ln x dx = x\ln x \Big|_0^1 - \int_0^1 dx = -\lim_{\varepsilon \to 0^+} \varepsilon \ln \varepsilon - 1 = -1$

11. $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$ 设 $\sqrt{1-x} = t$ 则 $x = 1-t^2$, $dx = -2t \cdot dt$. $2-x = 1+t^2$.

$= -2\int_1^0 \frac{dt}{1+t^2} = -2\arctan t \Big|_1^0 = -2(0-\frac{\pi}{4}) = \frac{\pi}{2}$.

13. $\int_1^{+\infty} \frac{dx}{x\sqrt{x-1}}$ 令 $\sqrt{x-1} = t$, $x = t^2+1$, $dx = 2t \cdot dt$.

$= \int_0^{+\infty} \frac{2t dt}{(t^2+1) \cdot t} = 2\arctan t \Big|_0^{+\infty} = \pi$.

题 5-6.

(1) $x_0 = 0, y_0 = 1$ $h = \frac{1}{8} = 0.125$.

$x_8 = 1, y_8 = \frac{1}{2}$.

$\int_0^1 \frac{dx}{1+x} = h(\frac{y_0+y_8}{2} + \sum_{i=1}^{7} y_i) = 0.69412$.

$|R_n| = \left| \frac{1}{12 \times 8^2} \cdot \frac{2}{(1+\xi)^3} \right| < 0.0027$.

第五章.

1.(1) 原式 $= \int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4}$

(2) 原式 $= \int_0^1 \frac{dx}{\sqrt{1+x}} = 2\sqrt{x+1} \Big|_0^1 = 2(\sqrt{2}-1)$.

2.(2) $\frac{\sin\frac{i\pi}{n}}{n+1} \leq \frac{\sin\frac{i\pi}{n}}{n+\frac{1}{i}} \leq \frac{\sin\frac{i\pi}{n}}{n}$ $i = 1, 2, \cdots, n$.

则 $\sum_{i=1}^{n} \frac{\sin\frac{i\pi}{n}}{n+1} \leq \sum_{i=1}^{n} \frac{\sin\frac{i\pi}{n}}{n+\frac{1}{i}} \leq \sum_{i=1}^{n} \frac{\sin\frac{i\pi}{n}}{n}$

又 $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin\frac{i\pi}{n}}{n} = \int_0^1 \sin\pi x \cdot dx = \frac{2}{\pi}$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin\frac{i\pi}{n}}{n+1} = \frac{2}{\pi}$. 则 原式 $= \frac{2}{\pi}$.

4.(1) $F(x) = \int_{-a}^{x} (x-t) f(t) \cdot dt + \int_x^a (t-x) f(t) \cdot dt$.

$= x\int_{-a}^{x} f(t) \cdot dt - x\int_x^a f(t) \cdot dt - \int_{-a}^{x} t f(t) dt + \int_x^a t f(t) dt$.

$F'(x) = \int_{-a}^{x} f(t) \cdot dt + x f(x) - \int_x^a f(t) \cdot dt - x f(x) - x f(x) + x f(x)$

$= \int_{-a}^{x} f(t) \cdot dt - \int_x^a f(t) \cdot dt$

$F''(x) = 2f(x) > 0$. 即 $F'(x)$ 单增

(2) $F'(x) = 0 \Rightarrow x = 0$. $F_{min} = 2\int_0^a t f(t) dt$.

~~F_{min}~~

7. $\lim\limits_{x\to0}\int_b^x \frac{\ln(1+t^2)}{t}\,dt = 0$. 得 $b=0$.

原式 $= \lim\limits_{x\to0}\frac{a-\cos x}{\frac{\ln(1+x^3)}{x}} = \lim\limits_{x\to0}\frac{x(a-\cos x)}{\ln(1+x^3)} = \lim\limits_{x\to0}\frac{x(a-\cos x)}{x^3} = \lim\limits_{x\to0}\frac{a-\cos x}{x^2} = c$

得 $a=1 \to c=\frac{1}{2}$.

15、 $I'(x) = \frac{\ln x}{x^2-2x+1} = \frac{\ln x}{(x-1)^2} \geq 0$, $x\in[e,e^2]$

$I_{max} = I(e^2) = \int_e^{e^2}\frac{\ln t}{(t-1)^2}dt = -\int_e^{e^2}\ln t\cdot d\left(\frac{1}{t-1}\right) = -\frac{\ln t}{t-1}\Big|_e^{e^2} + \int_e^{e^2}$

$= -\frac{\ln t}{t-1}\Big|_e^{e^2} + \left(\ln|t-1| - \ln|t|\right)\Big|_e^{e^2}$

$= \ln(e+1) - \frac{e}{e+1}$.

习题5-4. $\begin{cases} y=2x-x^2 \\ x+y=0\end{cases} \to \begin{matrix} x_1=0 \\ x_2=3.\end{matrix}$

1.(3) $S = \int_0^3[2x-x^2-(-x)]dx = \left(\frac{3}{2}x^2-\frac{1}{3}x^3\right)\Big|_0^3 = \frac{9}{2}$.

(4) $S = -\int_{\frac{1}{10}}^1 \ln x\cdot dx + \int_1^{10}\ln x\cdot dx$

$= -x\ln x\Big|_{\frac{1}{10}}^1 + \int_{\frac{1}{10}}^1 1\cdot dx + x\ln x\Big|_1^{10} - \int_1^{10}1\cdot dx$

$= \frac{99}{10}\ln 10 - \frac{81}{10}$

(5) $= 4\int_0^a x\sqrt{a^2-x^2}\cdot dx = -\frac{4}{3}(a^2-x^2)^{\frac{3}{2}}\Big|_0^a = \frac{4}{3}a^3$.

3. (1) $S = 2\cdot\frac{1}{2}\int_0^{\pi}4(1+\cos\theta)^2d\theta = \left(4\theta + 8\sin\theta + \frac{4}{2}\theta + \frac{\sin\theta}{2}\sqrt{1-\sin^2\theta}\right)\Big|_0^{\pi} = 6\pi$.

(2) 设 $x=a\cos^3t$ $y=a\sin^3t$. $(0\leq t\leq\frac{\pi}{2})$.

$S = 4\int_0^a|y|dx = 4\int_{\frac{\pi}{2}}^0|\sin^3t|(-3a\cos^2t\sin t)dt$

$= 12a^2\int_0^{\frac{\pi}{2}}(\sin^4t-\sin^6t)dt = \frac{3\pi a^2}{8}$

4. $r^2=\sin2\varphi \to S = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}\sin2\varphi\,d\varphi = 1$.

5. (1) 令 $y=0$ 得 $x=0$ 或 2.

$V_x = \pi\int_0^2(2x-x^2)^2\cdot dx = \frac{16}{15}\pi$.

$V_y = 2\pi\int_0^2 x\cdot|2x-x^2|\cdot dx = \frac{8}{3}\pi$.