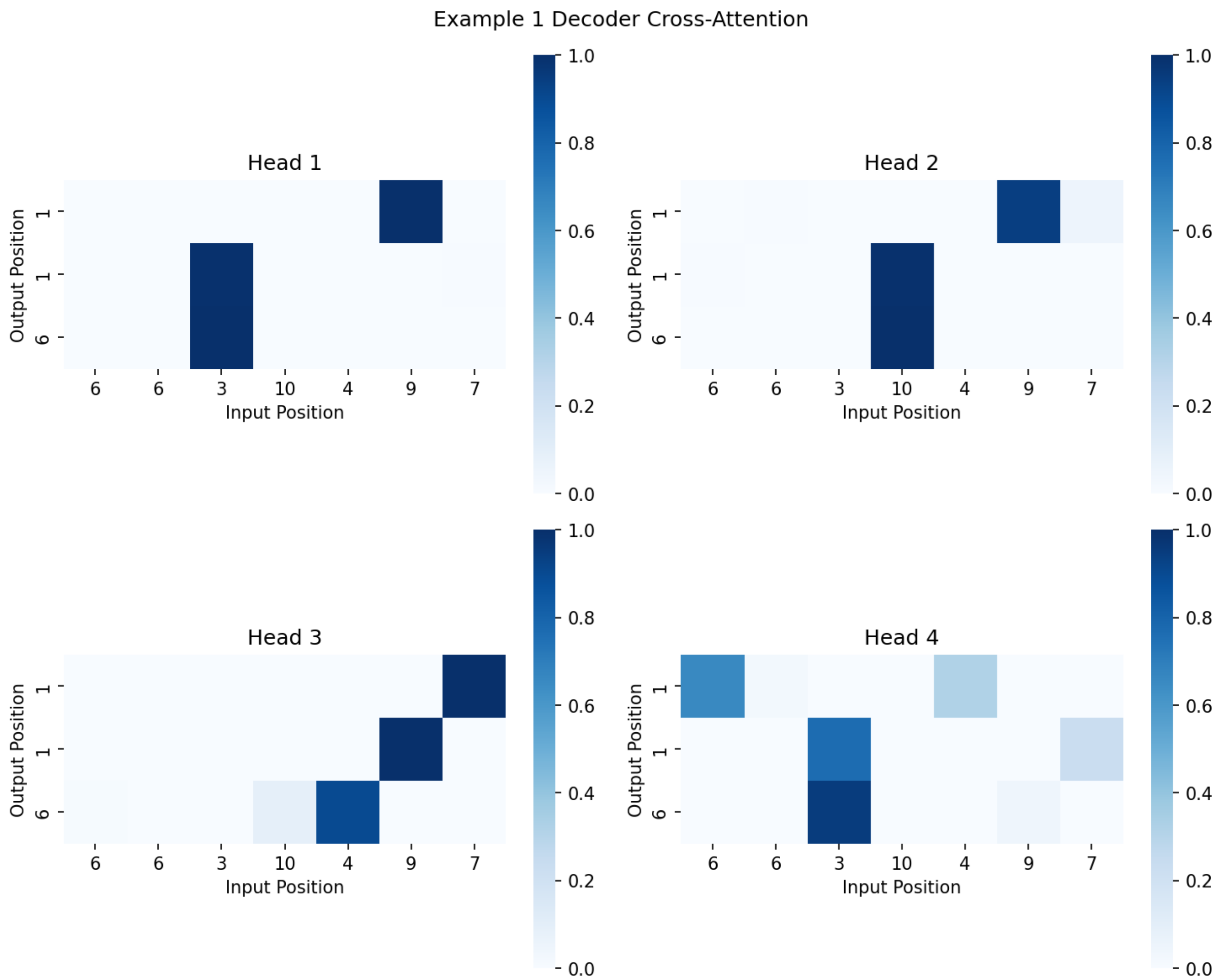
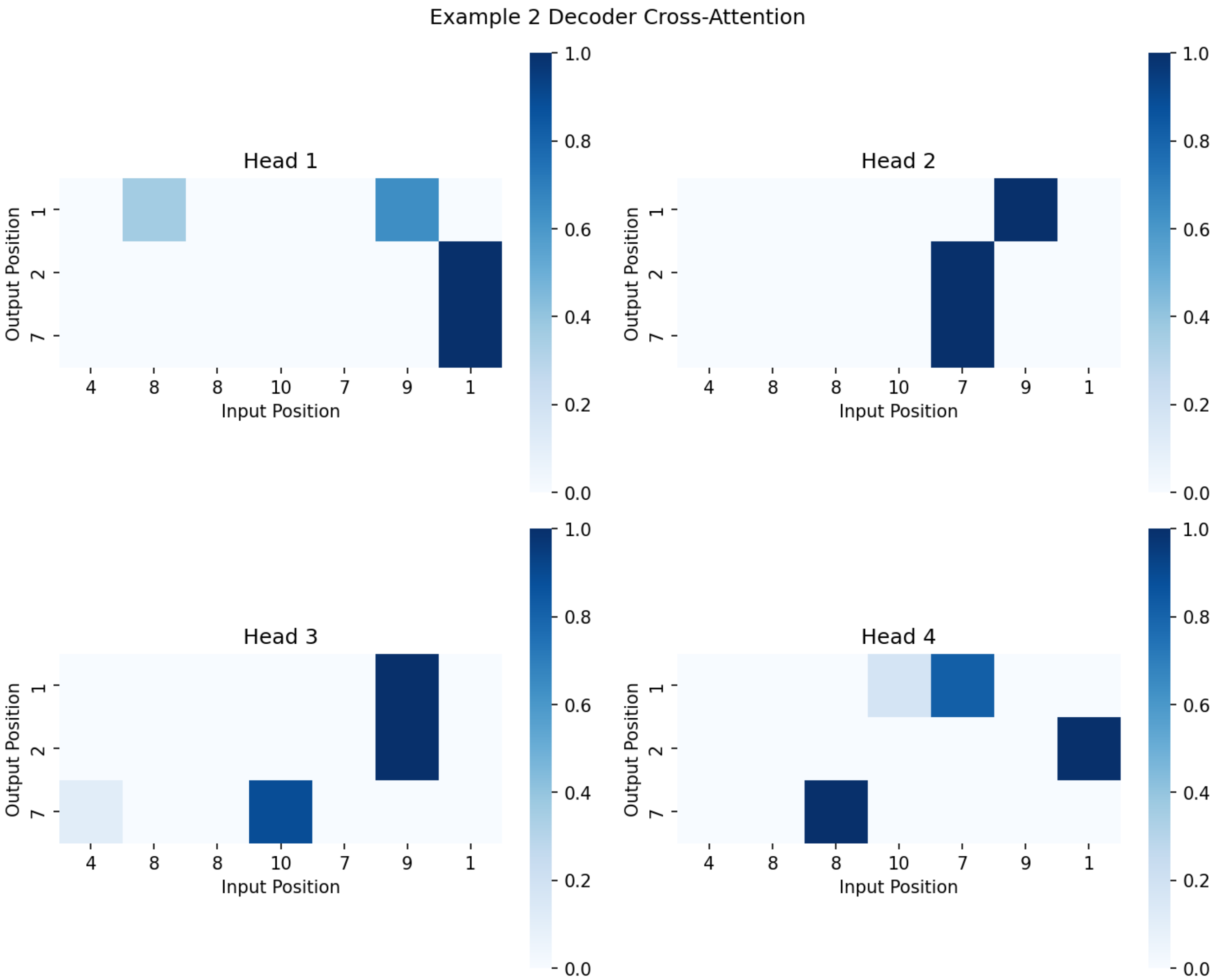
Problem 1

Attention pattern visualizations

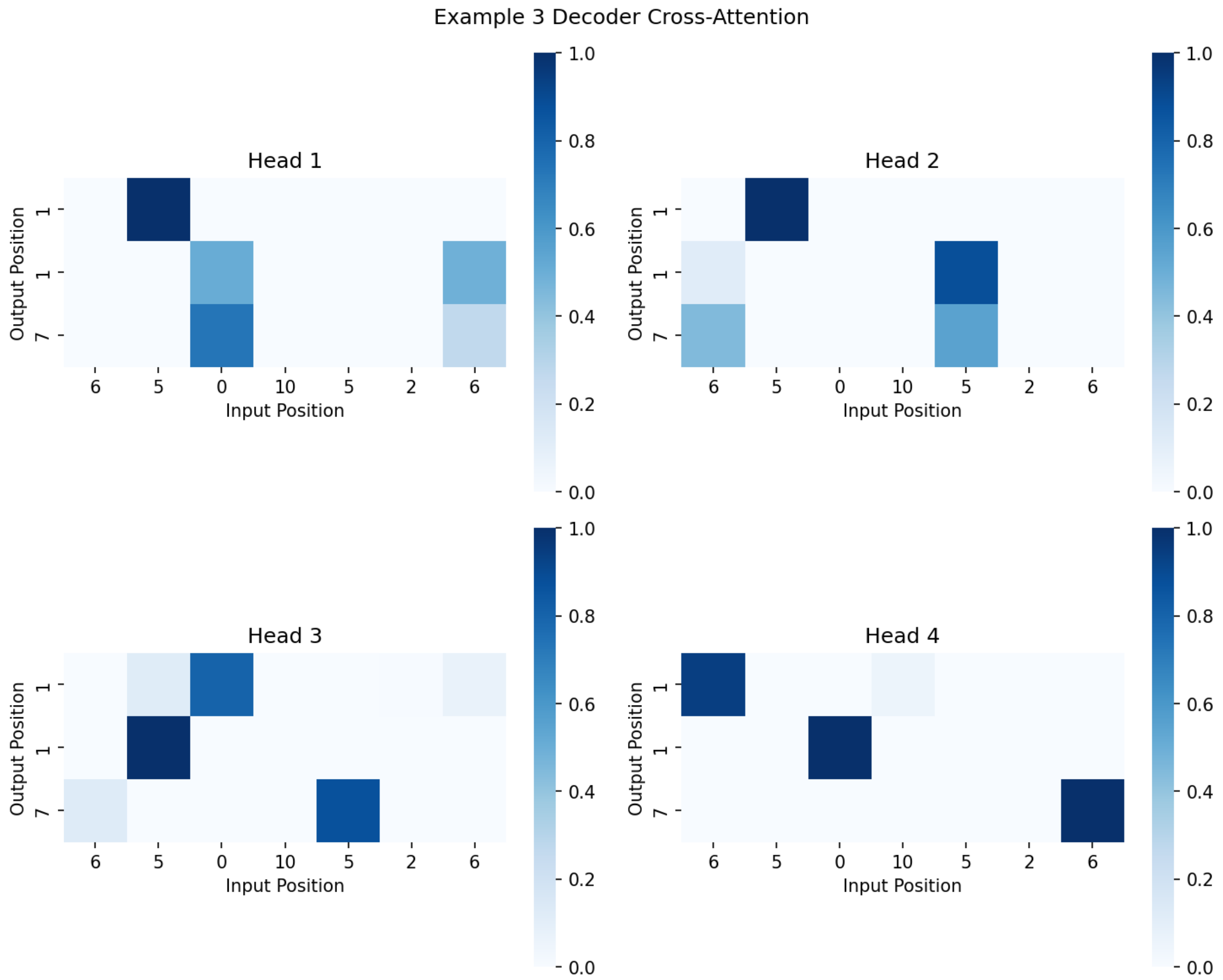
Each figure shows all attention heads; darker cells indicate higher attention weights.

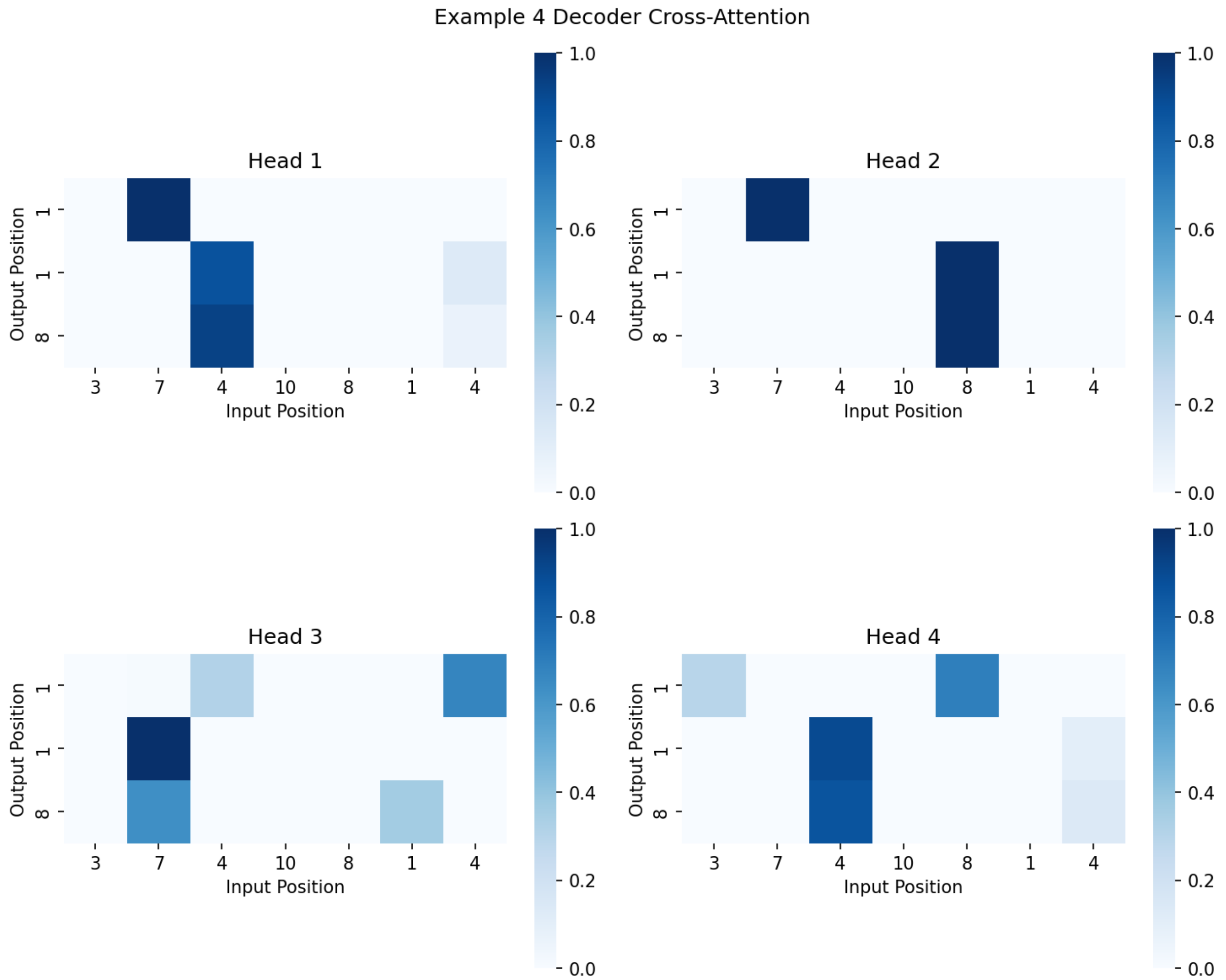
Examples:





Example 2 Decoder Cross-Attention
Head 1
Head 2
Head 3
Head 4
Output Position
Input Position
1
2
7
4
8
8
10
7
9
1
0.0
0.2
0.4
0.6
0.8
1.0


Example 3 Decoder Cross-Attention
Head 1
Head 2
Head 3
Head 4
Output Position
Input Position
1
1
7
6
5
0
10
5
2
6
0.0
0.2
0.4
0.6
0.8
1.0



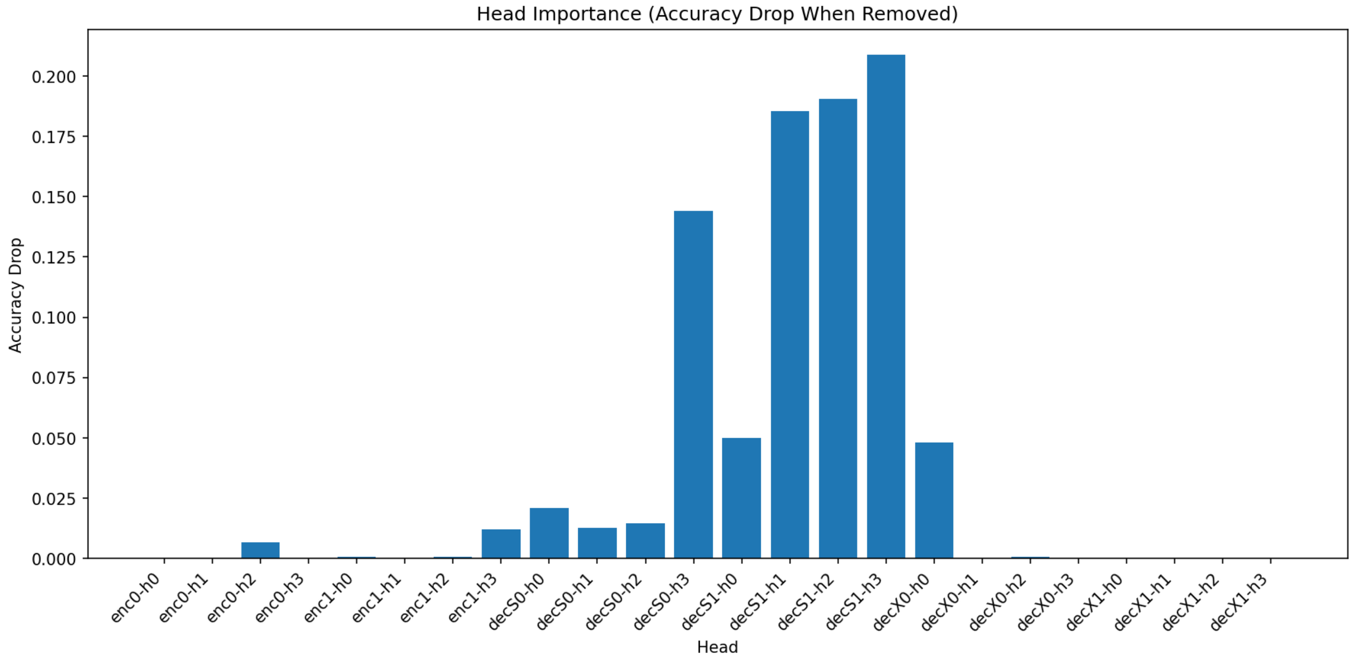
Head ablation study: which heads are critical vs redundant?

- Baseline accuracy: 41.20%.
- Accuracy when ablating individual heads is recorded per layer and module (encoder, decoder self, decoder cross).
- The largest drops are concentrated in decoder self-attention, especially deeper layers.

Top head importance by accuracy drop (higher = more critical):

Rank	Module	Layer	Head	Acc Drop
1	Decoder Self	1	3	0.2090
2	Decoder Self	1	2	0.1905
3	Decoder Self	1	1	0.1855
4	Decoder Self	0	3	0.1440
5	Decoder Self	1	0	0.0500
6	Decoder Cross	0	0	0.0480
7	Encoder	1	3	0.0120

Rank	Module	Layer	Head	Acc Drop
8	Decoder Self	0	2	0.0145



Labels in the plot follow: "Encoder Lx Hy", "Decoder Self Lx Hy", "Decoder Cross Lx Hy".

Discussion: How do attention heads specialize for carry propagation?

- Encoder heads mostly show small impact when ablated (less than 1.2% absolute drop), suggesting they primarily provide general context encoding (e.g., token type/operator awareness) rather than decisive sequence computation.
- Decoder self-attention heads in the deeper layer (layer 1) are critical; ablating them yields large drops (up to ~21%). This pattern is consistent with heads specializing in tracking carry information and aligning generated digits with prior partial sums.
- Cross-attention is mixed: many heads are redundant (no measurable drop), but at least one (layer 0, head 0) contributes moderately (~4.8% drop), likely mediating alignment between source digits and the current decoding position.

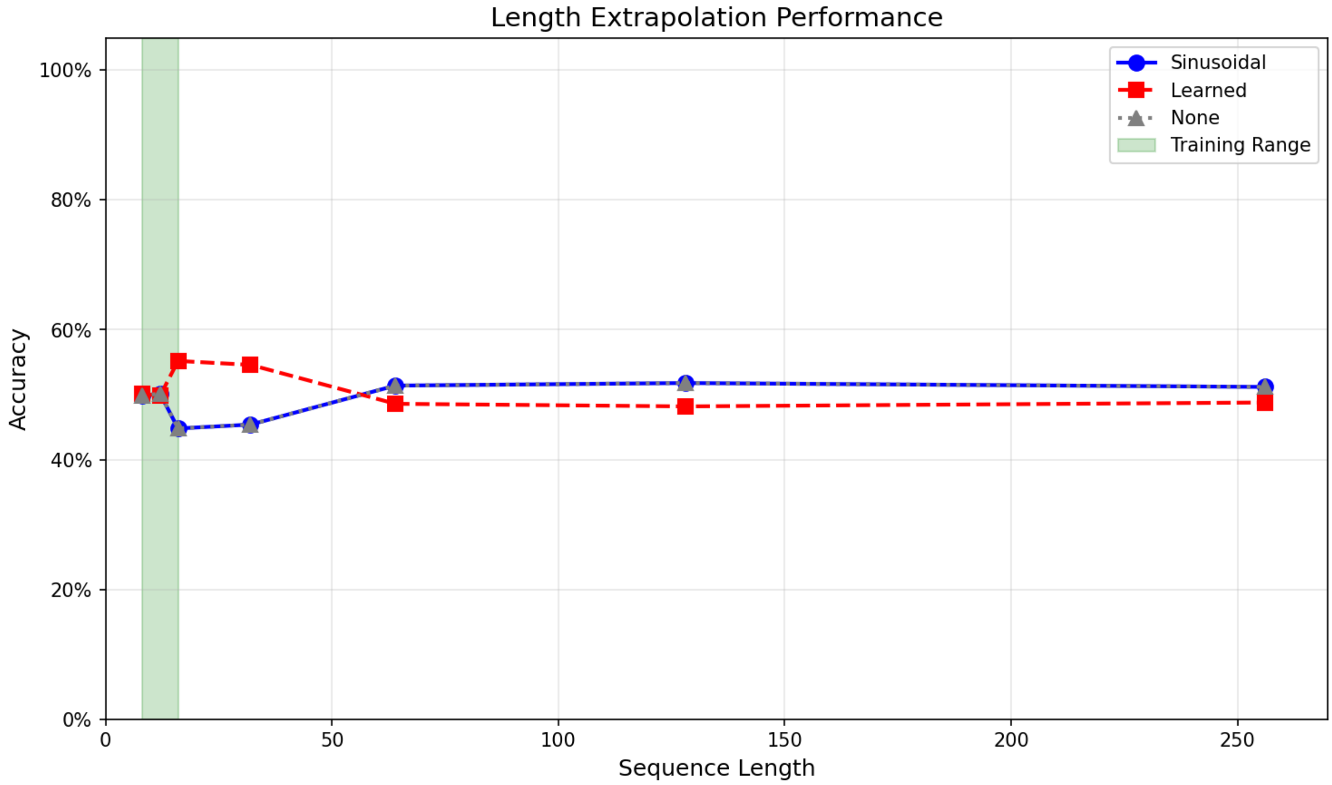
Quantitative pruning result

- Criterion for "minimal accuracy loss": absolute accuracy drop ≤ 0.01.
- Total heads: 24 (2 encoder layers × 4 heads + 2 decoder self × 4 + 2 decoder cross × 4).
- Heads prunable under this criterion: 14 / 24 ≈ 58.3% (most encoder and cross-attention heads; decoder self-attention heads are largely non-prunable).

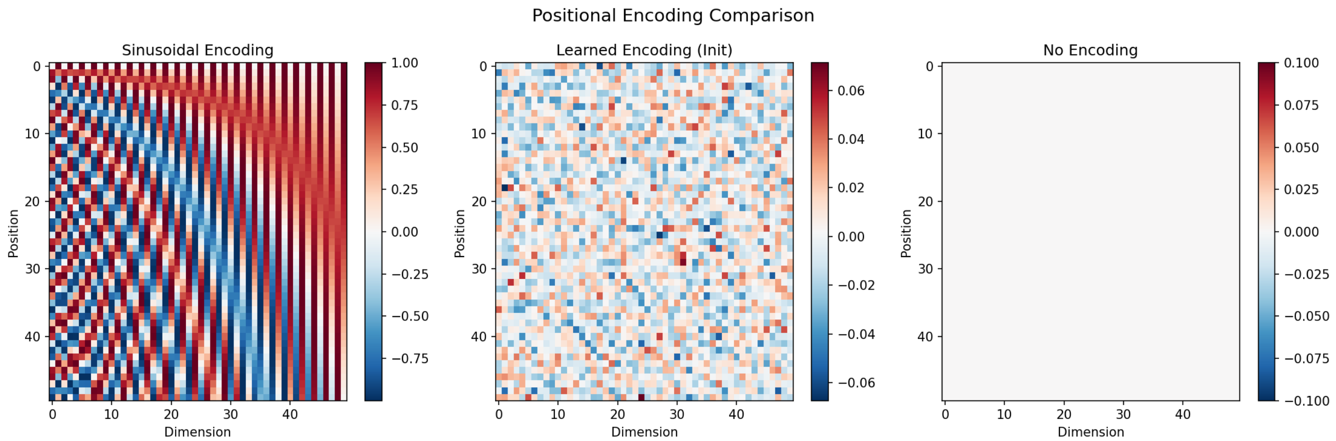
Problem 2

Extrapolation curves

Accuracy vs. sequence length for all three positional encodings.



For reference, a direct visual comparison of encoding patterns:



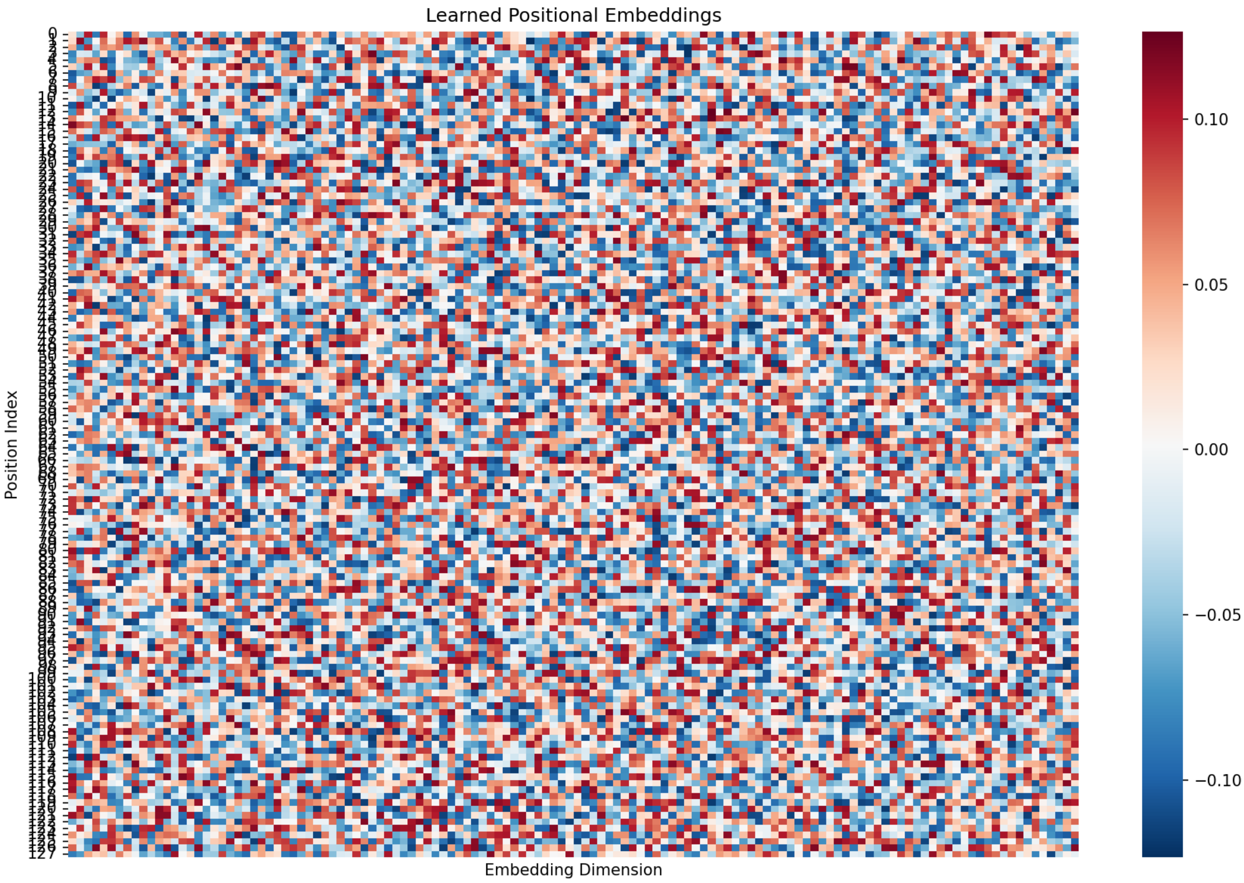
Why does sinusoidal extrapolate but learned fails?

- Definition (Fourier features): Let $\omega_i = 10000^{-2i/d}$. The sinusoidal table is $P_{t,2i} = \sin(\omega_i t),; P_{t,2i+1} = \cos(\omega_i t)$, for any integer position $t \ge 0$. This defines a function $P: \mathbb{N} \to \mathbb{R}^d$ with no dependence on training length.
- Relative-position signal: Using the identity $\sin a,\sin b + \cos a,\cos b = \cos(a-b)$, the positional inner product satisfies $P_t^\top P_s = \sum_i \cos\big(\omega_i (t-s)\big)$, which depends only on the displacement $\Delta=t-s$. After linear maps W_Q, W_K, any attention logit term of the form $(P_t W_Q) (P_s W_K)^\top$ becomes a linear combination of $\cos(\omega_i\Delta)$. Hence attention can learn translation-equivariant kernels $g(\Delta)$ that generalize to unseen absolute indices.
- Smooth spectrum and extrapolation: The set ${\cos(\omega_i\Delta)}$ spans low-to-moderate frequencies. Learned weights interpolate these harmonics for new Δ without requiring new parameters, so the same kernel applies at longer lengths.

- Learned absolute embeddings: A table $E\in\mathbb{R}^{L_{\text{max}}\times d}$ defines vectors only for indices seen during training (e.g., $t\in[0,L_{\text{train}}-1]$). For $t\ge L_{\text{train}}$, common strategies are (i) clamp $E_t=E_{L_{\text{train}}-1}$ or (ii) random/untrained vectors. In both cases, dot products $E_t^\top E_s$ are not a function of $t-s$ and introduce distribution shift; with clamping, many long positions collapse to the same vector, destroying relative information.
- Consequence in attention: With sinusoidal PE, the logit between positions t,s has a stable component $g(t-s)$ learned during training, so the model preserves ordering cues beyond the training span. With learned PE, logits depend on absolute IDs; when new IDs appear, that mapping is undefined or degenerate, leading to failures at long lengths.

Learned position embedding visualization

Heatmap of the learned absolute embeddings.



Quantitative comparison (accuracy)

Accuracies at extrapolation lengths 32, 64, 128, 256.

Length	Sinusoidal	Learned	None
32	0.454	0.546	0.454
64	0.514	0.486	0.514
128	0.518	0.482	0.518
256	0.512	0.488	0.512