Independence and Bayesian Networks (Part 1)

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Lecture 3

Outline

Learning Goals

- Given a probabilistic model,
  determine if two variables are unconditionally independent, or conditionally independent given a third variable.
- Give examples of deriving a compact representation of a joint distribution by using independence assumptions.
- Describe components of a Bayesian network.
- Compute a joint probability given a Bayesian network.
- Explain the independence relationships in the three key structures.

(Unconditional) Independence

Definition ((unconditional) independence)

X and Y are (unconditionally) independent iff

$$P(X|Y) = P(X)$$

$$P(Y|X) = P(Y)$$

$$P(X \wedge Y) = P(X)P(Y)$$

Learning Y does NOT influence your belief about X.

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$$P(X \wedge Y) = P(X)P(Y)$$

Learning Y does NOT influence your belief about X.

$\rightarrow$ Convert between the two equations.

To specify joint probability, it is sufficient to specify the individual probabilities.

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$$P(X \wedge Y) = P(X)P(Y)$$

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$$P(X \wedge Y) = P(X)P(Y)$$

Learning Y does NOT influence your belief about X.

$\rightarrow$ To justify that

$$P(X \wedge Y) = P(X)P(Y)$$

we need to make four comparisons.

Conditional Independence

Definition (conditional independence)

X and Y are conditionally independent given Z if

$$P(X|Y \land Z) = P(X|Z).$$

$$P(Y|X \land Z) = P(Y|Z).$$

$$P(Y \land X|Z) = P(Y|Z)P(X|Z).$$

Learning Y does NOT influence your belief about X
if you already know Z.

Conditional Independence

Definition (conditional independence)

X and Y are conditionally independent given Z if

$$P(X|Y \wedge Z) = P(X|Z).$$

$$P(Y|X \wedge Z) = P(Y|Z).$$

$$P(Y \wedge X|Z) = P(Y|Z)P(X|Z).$$

Learning Y does NOT influence your belief about X
if you already know Z.

$\rightarrow$ X is conditionally independent of Y given Z.

Independence does not imply conditional independence, and vice versa.

Conditional Independence

Definition (conditional independence)

X and Y are conditionally independent given Z if

$$P(X|Y \wedge Z) = P(X|Z).$$

$$P(Y|X \wedge Z) = P(Y|Z).$$

$$P(Y \wedge X|Z) = P(Y|Z)P(X|Z).$$

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$$P(Y \wedge X|Z) = P(Y|Z)P(X|Z).$$

Learning Y does NOT influence your belief about X if you already know Z.

$\rightarrow$ To justify that

$$P(X \wedge Y|Z) = P(X|Z)P(Y|Z)$$

we need to make eight comparisons.

Q #1: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

Q #1: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (C) $P(A), P(B|A), P(C|A \wedge B)$. $1 + 2 + 4 = 7$ probabilities

Draw a graph to prove it to yourself.

Q #1: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (C) $P(A, B, C), P(\neg A, B, C), \cdots, P(\neg A, \neg B, \neg C)$. A total of 8 - 1 = 7 probabilities

Q #2: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. Assume that A, B, and C are independent. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

Q #2: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. Assume that A, B, and C are independent. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (A) $P(A), P(B), P(C)$. $1 + 1 + 1 = 3$ probabilities

Draw a graph to prove it to yourself.

Q #3: Deriving a compact representation

Q: Consider a model with three boolean random variables, A, B, C. Assume that A and B are conditionally independent given C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 4

(B) 5

(C) 7

(D) 11

Q #3: Deriving a compact representation

Q: Consider a model with three boolean random variables, A, B, C. Assume that A and B are conditionally independent given C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 4

(B) 5

(C) 7

(D) 11

$\rightarrow$ (B) $P(C), P(A|C), P(B|C)$. $1 + 2 + 2 = 5$ probabilities

Draw a graph to prove it to yourself.

Q #3a: Deriving a compact representation

Q: Given the joint probability distribution about A, B, and C, how many comparisons do we need to make to justify A and B are independent given C?

(A) 1

(B) 4

(C) 8

(D) 6

Q #3a: Deriving a compact representation

Q: Given the joint probability distribution about A, B, and C, how many comparisons do we need to make to justify A and B are independent given C?

(A) 1

(B) 4

(C) 8

(D) 6

$\rightarrow$ (C) $p(B, A|C) = p(B|C) * p(A|C)$ and
$p(B, \neg A|C) = p(B|C) * p(\neg B|C)$
... A total of 8 equalities!

Is this true?

Q #3a: Deriving a compact representation

Q: Given the joint probability distribution about A, B, and C, how many comparisons do we need to make to justify A and B are independent given C?

(A) 1

(B) 4

(C) 8

(D) 6

Q #3a: Deriving a compact representation

Q: Given the joint probability distribution about A, B, and C, how many comparisons do we need to make to justify A and B are independent given C?

(A) 1

(B) 4

(C) 8

(D) 6

$\rightarrow$ If we already know $p(B, A|T) = p(B|T) * p(A|T); p(\neg B, A|T) = p(\neg B|T) * p(A|T); p(B, \neg A|T) = p(B|T) * p(\neg A|T)$. Do we still need to compare $p(\neg B, \neg A|T)$ and $p(\neg B|T) * p(\neg A|T)$?

Probably not, the answer is (D).

Q #3b: Deriving a compact representation

Q: Read the table to understand whether B and C are independent given A.

A	B	C	Prob
T	T	T	0.16
T	T	F	0.16
T	F	T	0.24
T	F	F	0.24
F	T	T	0.012
F	T	F	0.008
F	F	T	0.108
F	F	F	0.072

(A) B and C are independent given A

(B) B and C are not independent given A

Q #3b: Deriving a compact representation

Read the table to understand whether B and C are independent given A.

A	B	C	Prob
T	T	T	0.16
T	T	F	0.16
T	F	T	0.24
T	F	F	0.24
F	T	T	0.012
F	T	F	0.008
F	F	T	0.108
F	F	F	0.072

- Compute $p(B, C|A)$
- Compute $p(B|A)$ and $p(C|A)$
- Verify $p(B, C|A) = p(B|A) * p(C|A)$

Q #3b: Step-by-Step Derivation $p(B, C|A)$

A	B	C	Prob
T	T	T	0.16
T	T	F	0.16
T	F	T	0.24
T	F	F	0.24
F	T	T	0.012
F	T	F	0.008
F	F	T	0.108
F	F	F	0.072

Table: Merging $p(A, B, C)$.

- $p(B, C|A) = p(B, C, A)/p(A)$
- $p(A) = (0.16 + 0.16 + 0.24 + 0.24, 0.012 + 0.008 + 0.108 + 0.072) = (0.8, 0.2)$

Q #3b: Step-by-Step Derivation $p(B, C|A)$

B	C	(A)	Prob
T	T	T	0.16 / 0.8 = 0.2
T	F	T	0.16 / 0.8 = 0.2
F	T	T	0.24 / 0.8 = 0.3
F	F	T	0.24 / 0.8 = 0.3
T	T	F	0.012 / 0.2 = 0.06
T	F	F	0.008 / 0.2 = 0.04
F	T	F	0.108 / 0.2 = 0.54
F	F	F	0.072 / 0.2 = 0.36

Table: Computing $p(B, C|A)$.

- $p(B, C|A) = p(B, C, A)/p(A)$
- $p(A) = (0.8, 0.2)$
- $p(B, C|A)$ is displayed in the table

Q #3b: Step-by-Step Derivation

A	B	C	Prob
T	T	T	0.16
T	T	F	0.16
T	F	T	0.24
T	F	F	0.24
F	T	T	0.012
F	T	F	0.008
F	F	T	0.108
F	F	F	0.072

Table: Merge $p(A, B, C)$

- Marginalizing over variable C

Q #3b: Step-by-Step Derivation $p(B|A)$

A	B	Prob
T	T	0.32
T	F	0.48
F	T	0.02
F	F	0.18

Table: Computing $p(A, B)$

- Marginalizing over variable C
- Joint $p(A, B)$ is displayed in the table

Q #3b: Step-by-Step Derivation $p(B|A)$

B	(A)	Prob
T	T	$0.32 / 0.8 = 0.4$
F	T	$0.48 / 0.8 = 0.6$
T	F	$0.02 / 0.2 = 0.1$
F	F	$0.18 / 0.2 = 0.9$

Table: Computing $p(B|A)$

- Marginalizing over variable C
- Conditional $p(B|A)$ is displayed in the table

Q #3b: Step-by-Step Derivation $p(C|A)$

A	B	C	Prob
T	T	T	0.16
T	F	T	0.24
T	T	F	0.16
T	F	F	0.24
F	T	T	0.012
F	F	T	0.108
F	T	F	0.008
F	F	F	0.072

Table: Merging $p(A, B, C)$

- Marginalizing over variable B

Q #3b: Step-by-Step Derivation $p(C|A)$

A	C	Prob
T	T	0.4
T	F	0.4
F	T	0.12
F	F	0.08

Table: Computing $p(A, C)$

- Marginalizing over variable B

Q #3b: Step-by-Step Derivation $p(C|A)$

C	(A)	Prob
T	T	$0.4 / 0.8 = 0.5$
F	T	$0.4 / 0.8 = 0.5$
T	F	$0.12 / 0.2 = 0.6$
F	F	$0.08 / 0.2 = 0.4$

Table: Computing $p(C|A)$

- Marginalizing over variable B
- Computing $p(C|A)$

Q #3b: Step-by-Step Derivation (Verification)

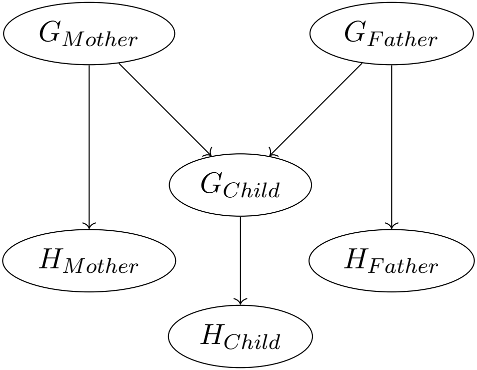
B	(A)	Prob
T	T	0.4
F	T	0.6
T	F	0.1
F	F	0.9

C	(A)	Prob
T	T	0.5
F	T	0.5
T	F	0.6
F	F	0.4

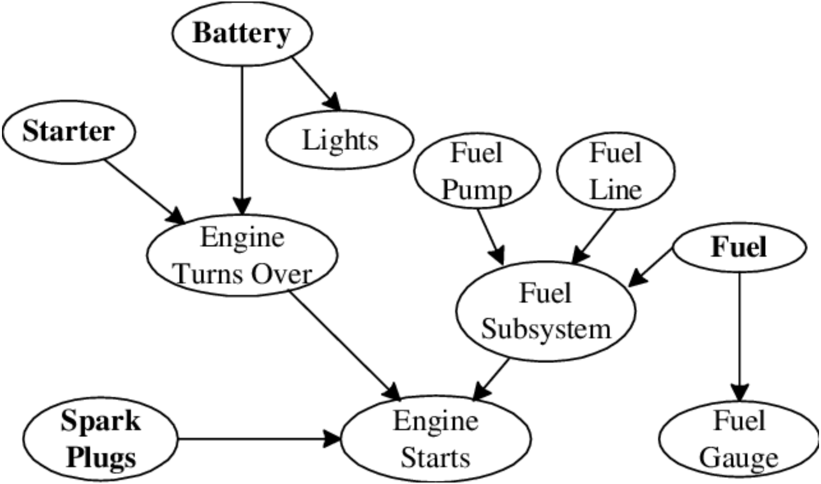
B	C	(A)	Prob
T	T	T	0.5 * 0.4 == 0.2
T	F	T	0.5 * 0.4 == 0.2
F	T	T	0.5 * 0.6 == 0.3
F	F	T	0.5 * 0.6 == 0.3
T	T	F	0.6 * 0.1 == 0.06
T	F	F	0.4 * 0.1 == 0.04
F	T	F	0.6 * 0.9 == 0.54
F	F	F	0.4 * 0.9 == 0.36

All of the probabilities are equal, therefore B and C are independent given A.

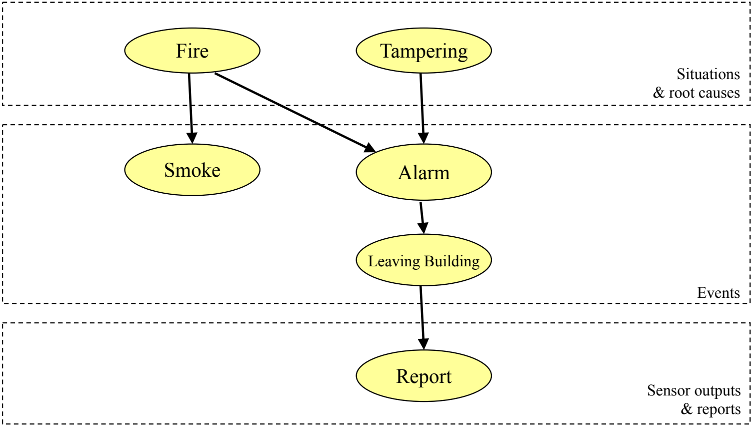
Inheritance of Handedness



Car Diagnostic Network

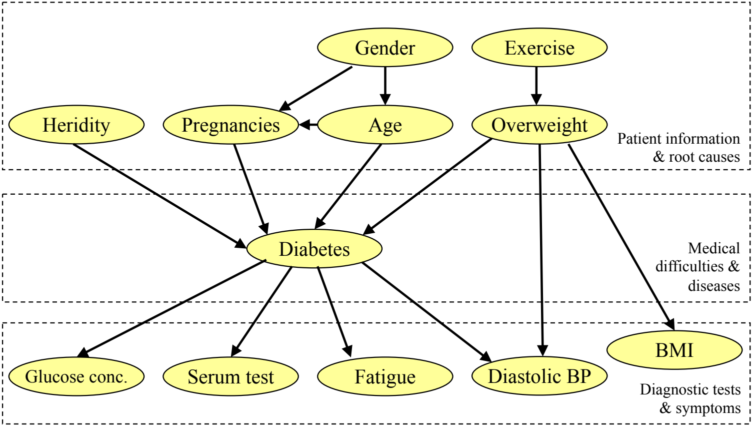


Example: Fire alarms



Report: “report of people leaving building because a fire alarm went off”

Example: Medical diagnosis of diabetes



Why Bayesian Networks?

A probabilistic model of the Holmes scenario:

- The random variables:
  Earthquake, Radio, Burglary, Alarm, Watson, and Gibbon.
- $\#$ of probabilities in the joint distribution: $2^6 = 64$.
- For example,

  $P(E \wedge R \wedge B \wedge A \wedge W \wedge G) = ?$

  $P(E \wedge R \wedge B \wedge A \wedge W \wedge \neg G) = ?$

  ... etc ...

We can compute any probability using the joint distribution, but

- Quickly become intractable as the number of variables grows.
- Unnatural and tedious to specify all the probabilities.

Why Bayesian Networks?

A Bayesian Network

- is a compact version of the joint distribution
- takes advantage of the unconditional and conditional independence among the variables.

Reminder: Modelling the Holmes Scenario

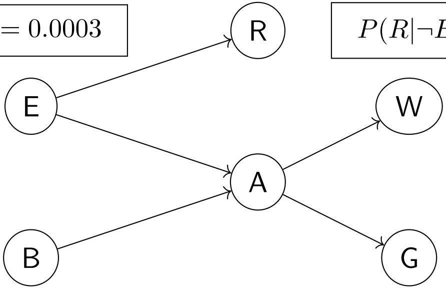
→ The random variables:

- B: A Burglary is happening.
- A: The alarm is going.
- W: Dr. Watson is calling.
- G: Mrs. Gibbon is calling.
- E: Earthquake is happening.
- R: A report of earthquake is on the radio news.

A Bayesian Network for the Holmes Scenario

$P(E) = 0.0003$

$P(R|\neg E) = 0.0002,\ P(R|\ E) = 0.9$



$$P(W|\neg A) = 0.4$$
$$P(W|\ A) = 0.8$$

$$P(G|\neg A) = 0.04$$
$$P(G|\ A) = 0.4$$

$P(B) = 0.0001$

$$P(A|\neg B \wedge \neg E) = 0.01,\ P(A|\neg B \wedge\ E) = 0.2$$
$$P(A|\ B \wedge \neg E) = 0.95,\ P(A|\ B \wedge\ E) = 0.96$$

How many probabilities do we need to encode the Network?

Bayesian Network

A Bayesian Network is a *directed acyclic graph* (DAG).

- Each node corresponds to a random variable.
- X is a parent of Y if there is an arrow from node X to node Y.

  $\rightarrow$ Like a family tree, there are parents, children, ancestors, descendants.

- Each node X_i has a conditional probability distribution $P(X_i|Parents(X_i))$ that quantifies the effect of the parents on the node.

The Semantics of Bayesian Networks

Two ways to understand Bayesian Networks:

- A representation of the joint probability distribution
- An encoding of the conditional independence assumptions

Representing the joint distribution

The idea is that, given a random variable X, a small set of variables may exist that directly affect the variable's value in the sense that X is conditionally independent of other variables given values for the directly affecting variables.

- Start with a set of random variables representing all the features of the model.
- Define the **parents** of random variable X_i, written as $parents(X_i)$.
- X_i is independent from other non-descendent variables given the $parents(X_i)$.

Representing the joint distribution

We can compute the full joint probability using the following formula.

$$P(X_n \wedge \cdots \wedge X_1) = \prod_{i=1}^{n} P(X_i | Parents(X_i))$$

Representing the joint distribution

Example: What is the probability that all of the following occur?

- The alarm has sounded
- Neither a burglary nor an earthquake has occurred
- Both Watson and Gibbon call and say they hear the alarm
- There is no radio report of an earthquake

Representing the joint distribution

Example: What is the probability that all of the following occur?

- The alarm has sounded
- Neither a burglary nor an earthquake has occurred
- Both Watson and Gibbon call and say they hear the alarm
- There is no radio report of an earthquake

$\rightarrow$ Formulate as a joint probability:

$$
\begin{aligned}
&P(\neg B \land \neg E \land A \land \neg R \land G \land W) \\
&= P(\neg B)P(\neg E)P(A|\neg B \land \neg E)P(\neg R|\neg E)P(G|A)P(W|A) \\
&= (1 - 0.0001)(1 - 0.0003)(0.01)(1 - 0.0002)(0.4)(0.8) \\
&= 3.2 \times 10^{-3}
\end{aligned}
$$

Q #4: Calculating the joint probability

Q: What is the probability that all of the following occur?

- NEITHER a burglary NOR an earthquake has occurred,
- The alarm has NOT sounded,
- NEITHER of Watson and Gibbon is calling, and
- There is NO radio report of an earthquake?

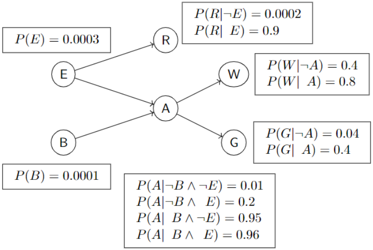
(A) 0.5699

(B) 0.6699

(C) 0.7699

(D) 0.8699

(E) 0.9699



Q #4: Calculating the joint probability

Q: What is the probability that all of the following occur?

- NEITHER a burglary NOR an earthquake has occurred,
- The alarm has NOT sounded,
- NEITHER of Watson and Gibbon is calling, and
- There is NO radio report of an earthquake?

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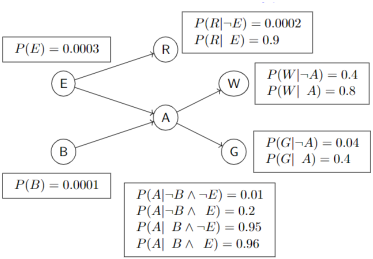
(B) 0.6699

(C) 0.7699

(D) 0.8699

(E) 0.9699



$\rightarrow$ (A)

$$(1 - 0.0001)(1 - 0.0003)(1 - 0.01)(1 - 0.4)(1 - 0.04)(1 - 0.0002) = 0.5699$$

Burglary, Alarm and Watson

Q #5: Unconditional Independence

Q: Are Burglary and Watson independent?

(A) Yes

(B) No

(C) Can't tell.

Q #5: Unconditional Independence

Q: Are Burglary and Watson independent?

$$\text{Burglary} \longrightarrow \text{Alarm} \longrightarrow \text{Watson}$$

(A) Yes

(B) No

(C) Can't tell.

$\rightarrow$ Correct answer is *No*.

If you learned the value of B, would your belief of W change? If B is true, then Alarm is more likely to be true, and W is more likely to be true.

Q #6: Conditional Independence

Q: Are Burglary and Watson conditionally independent given Alarm?

(A) Yes

(B) No

(C) Can't tell

Q #6: Conditional Independence

Q: Are Burglary and Watson conditionally independent given Alarm?

(A) Yes

(B) No

(C) Can't tell

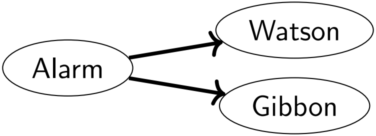
→ Correct answer is *Yes*.

Assume that W does not observe B directly. W only observes A.

B and W could only influence each other through A.

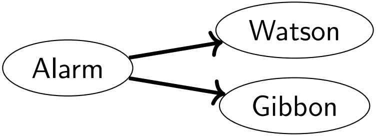
If A is known, then B and W do not affect each other.

Alarm, Watson and Gibbon



Q #7: Unconditional Independence

Q: Are Watson and Gibbon independent?



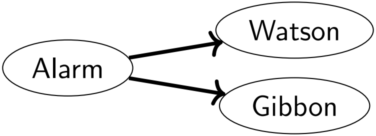
(A) Yes

(B) No

(C) Can't tell

Q #7: Unconditional Independence

Q: Are Watson and Gibbon independent?



(A) Yes

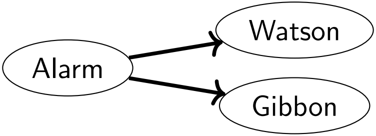
(B) No

(C) Can't tell

$\rightarrow$ Correct answer is No. If Watson is more likely to call, then Alarm is more likely to go off, which means that Gibbon is more likely to call.

Q #8 Conditional Independence

Q: Are Watson and Gibbon conditionally independent given Alarm?



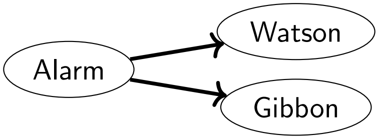
(A) Yes

(B) No

(C) Can't tell

Q #8 Conditional Independence

Q: Are Watson and Gibbon conditionally independent given Alarm?



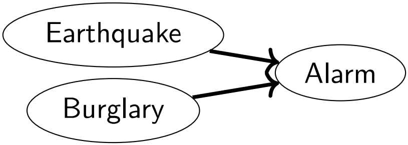
(A) Yes

(B) No

(C) Can't tell

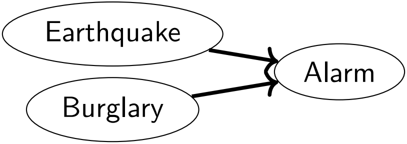
$\rightarrow$ Correct answer is Yes. Watson and Gibbon are both unreliable sensors for Alarm. If Alarm is known, then Watson and Gibbon do not affect each other.

Earthquake, Burglary, and Alarm



Q #9 Unconditional Independence

Q: Are Earthquake and Burglary independent?



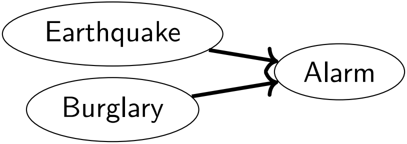
(A) Yes

(B) No

(C) Can't tell

Q #9 Unconditional Independence

Q: Are Earthquake and Burglary independent?



(A) Yes

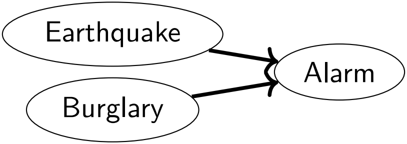
(B) No

(C) Can't tell

$\rightarrow$ Correct answer is Yes.

Q #10: Conditional Independence

Q: Are Earthquake and Burglary conditionally independent given Alarm?



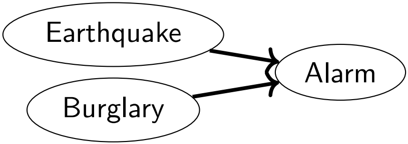
(A) Yes

(B) No

(C) Can't tell

Q #10: Conditional Independence

Q: Are Earthquake and Burglary conditionally independent given Alarm?



(A) Yes

(B) No

(C) Can't tell

→ Correct answer is No. Suppose that the Alarm is going. If there is an Earthquake, then it is less likely that the Alarm is caused by Burglary. If there is a Burglary, it is less likely that the Alarm is caused by Earthquake.

Revisiting Learning Goals

- Given a probabilistic model,
  determine if two variables are unconditionally independent, or conditionally independent given a third variable.
- Give examples of deriving a compact representation of a joint distribution by using independence assumptions.
- Describe components of a Bayesian network.
- Compute a joint probability given a Bayesian network.
- Explain the independence relationships in the three key structures.