Three Body Problems

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Background

Assume there is a third body that is much smaller in mass than other two and therefore has negligilble influence on their motion. We make efforts to solve for the motion of the third body in the known gravitational field of the first two bodies.

Beginning

We transform the system to a rotating frame of reference in which the first two bodies appear stationary. Then we take the angular velocity of the frame to be 1 and the distance between the two bodies is 1. Then we define a $\mu \in (0, 0.5]$ such that the two masses are in the ratio of $\mu : 1 - \mu$ and are situated repectively at points $(\mu - 1, 0), (\mu, 0)$.

Forbidden Region

We know that the position of the third body is $(\mathrm{x}(\mathrm{t}), \mathrm{y}(\mathrm{t}))$, and

$$\ddot{\mathrm{x}} - 2\dot{\mathrm{y}} = -\frac{\partial \Omega}{\partial \mathrm{x}} \tag{1}$$

$$\ddot{\mathrm{y}} + 2\dot{\mathrm{x}} = -\frac{\partial \Omega}{\partial \mathrm{y}} \tag{2}$$

where

$$\Omega = -\frac{1}{2}(\mathrm{x}^2 + \mathrm{y}^2) - \frac{\mu}{\sqrt{(\mathrm{x} + 1 - \mu)^2 + \mathrm{y}^2}} - \frac{1 - \mu}{\sqrt{(\mathrm{x} - \mu)^2 + \mathrm{y}^2}} \tag{3}$$

i.e. Ω is the potential of the centrifugal and gravitational forces.

Forbidden Region

Show that the quantity

$$J = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Omega(x, y) \tag{4}$$

is constant following the motion.

~~Some Reference~~

~~Chain rule for partial derivative:~~

$$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial x}\frac{dx}{dt} + \frac{\partial \Omega}{\partial y}\frac{dy}{dt}$$

~~Proof~~

$$
\begin{aligned}
\frac{dJ}{dt} &= \frac{1}{2}(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}) + \frac{d\Omega}{dt} \\
&= \dot{x}\ddot{x} + \dot{y}\ddot{y} + \frac{d\Omega}{dt} \\
&= \dot{x}\ddot{x} + \dot{y}\ddot{y} + \frac{\partial\Omega}{\partial x}\cdot\dot{x} + \frac{\partial\Omega}{\partial y}\cdot\dot{y} \\
&= \dot{x}\ddot{x} + \dot{y}\ddot{y} + (-\ddot{x} + 2\dot{y})\cdot\dot{x} + (-\ddot{y} - 2\dot{x})\cdot\dot{y} \\
&= \dot{x}\ddot{x} + \dot{y}\ddot{y} + (-\dot{x}\ddot{x} + 2\dot{x}\dot{y}) + (-\dot{y}\ddot{y} - 2\dot{x}\dot{y}) \\
&= 0
\end{aligned}
$$

~~So it's constant.~~

~~Forbidden Region~~

~~Since J is constant,~~

$$
\begin{aligned}
J &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Omega(x, y) \\
&= \frac{1}{2}(u_0^2 + v_0^2) + \Omega(x_0, y_0) \\
\Omega(x, y) &= \frac{1}{2}(u_0^2 + v_0^2) + \Omega(x_0, y_0) - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \\
&\leq \frac{1}{2}(u_0^2 + v_0^2) + \Omega(x_0, y_0)
\end{aligned}
$$

~~where x_0, y_0, z_0 and v_0 are the initial values of $x, y, \dot{x}$ and $\dot{y}$, respectively.~~

Conclusion

We have come to know that the third body is restricted in a region such that

$$\Omega(x, y) \leq \Omega(x_o, y_o) + \frac{1}{2}(u_0^2 + v_0^2) \tag{5}$$

Space Travel

Assume that the third body is a spacecraft, with the first two bodies being co-orbiting planets of equal mass.

~~Circular motion~~

~~Now we have a new approximation that~~

$$\Omega = -\frac{0.5}{\sqrt{(x-\mu)^2 + y^2}} \tag{6}$$

~~Show that in polar co-ordinates with~~

$$x(t) - \mu = r(t)\cos\theta(t),\ y(t) = r(t)\sin\theta(t) \tag{7}$$

~~the approximate system (1), (2) and (6) is equivalent to~~

$$\dot{\theta} = -1 + kr^{-2},\ \ddot{r} = -V'(r) \tag{8}$$

~~where k is an arbitrary constant and V′(r) is to be found.~~

Solutions

$$\Omega = -\frac{1}{2}((x-\mu)^2+y^2)^{-\frac{1}{2}}$$

$$\frac{\partial\Omega}{\partial x} = \frac{1}{4}((x-\mu)^2+y^2)^{-\frac{3}{2}}2(x-\mu) = \frac{1}{2}r\cos\theta r^{-2} = \frac{\cos\theta}{2r^2} \quad ①$$

$$\frac{\partial\Omega}{\partial y} = \frac{1}{4}((x-\mu)^2+y^2)^{-\frac{3}{2}}2y = \frac{1}{2}r\sin\theta r^{-3} = \frac{\sin\theta}{2r^2} \quad ②$$

~~Solutions~~

$$
\begin{aligned}
& \mathrm{x} = \mathrm{r}\cos\theta + \mu \\
\Rightarrow\ & \dot{\mathrm{x}} = \dot{\mathrm{r}}\cos\theta - \mathrm{r}\sin\theta\dot{\theta} && (3) \\
\Rightarrow\ & \ddot{\mathrm{x}} = \ddot{\mathrm{r}}\cos\theta - \dot{\mathrm{r}}\sin\theta\dot{\theta} - (\dot{\mathrm{r}}\sin\theta\dot{\theta} + \mathrm{r}(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)) \\
\Rightarrow\ & \ddot{\mathrm{x}} = \ddot{\mathrm{r}}\cos\theta - 2\dot{\mathrm{r}}\sin\theta\dot{\theta} - (\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)\mathrm{r} && (4) \\
& \mathrm{y} = \mathrm{r}\sin\theta \\
\Rightarrow\ & \dot{\mathrm{y}} = \dot{\mathrm{r}}\sin\theta + \mathrm{r}\cos\theta\dot{\theta} && (5) \\
\Rightarrow\ & \ddot{\mathrm{y}} = \ddot{\mathrm{r}}\cos\theta + \dot{\mathrm{r}}\sin\theta\dot{\theta} + (\dot{\mathrm{r}}\cos\theta\dot{\theta} + \mathrm{r}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)) \\
\Rightarrow\ & \ddot{\mathrm{y}} = \ddot{\mathrm{r}}\sin\theta + 2\dot{\mathrm{r}}\cos\theta\dot{\theta} + (\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)\mathrm{r} && (6)
\end{aligned}
$$

~~Solutions~~

~~From ①, ④, ⑤, (1)~~

$$\cancel{\frac{\cos\theta}{2r^2}} \cancel{= 2\dot{r}\sin\theta + 2r\cos\theta\dot{\theta} - \ddot{r}\cos\theta}$$
$$\cancel{+ 2\dot{r}\sin\theta\dot{\theta} + (\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)r}$$

~~From ②, ③, ⑥, (2)~~

$$\cancel{\frac{\sin\theta}{2r^2}} \cancel{= -2\dot{r}\cos\theta + 2r\sin\theta\dot{\theta} - \ddot{r}\sin\theta}$$
$$\cancel{- 2\dot{r}\cos\theta\dot{\theta} - (\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)r}$$

Solutions

We collect $\sin\theta$'s and $\cos\theta$'s

$$\cos\theta\left(\frac{1}{2r^2} - 2r\dot{\theta} + r - \dot{\theta}^2 r\right) = \sin\theta(2r + 2r\dot{\theta} + \ddot{\theta}r) \qquad (7)$$

$$\sin\theta\left(\frac{1}{2r^2} - 2r\dot{\theta} + r - \dot{\theta}^2 r\right) = \cos\theta(-2r - 2r\dot{\theta} - r\ddot{\theta}) \qquad (8)$$

Then we have 2 possibilities from above equations.

~~Possibility1~~

~~Think of $\frac{7}{8}$, we have~~

$$\frac{1}{\tan\theta} = \frac{\tan\theta}{-1}$$

$$\Rightarrow \tan^2\theta = -1$$

~~which is impossible.~~

Possibility2

The coefficients are zero,

$$\frac{1}{2r^2} - 2r\dot{\theta} + \ddot{r} - \dot{\theta}^2 r = 0 \tag{9}$$

$$2\dot{r} + 2\dot{r}\dot{\theta} + \ddot{\theta}r = 0 \tag{10}$$

$$\Rightarrow \ddot{r} = \frac{2r^3\dot{\theta}^2 + 4r^3\dot{\theta} - 1}{2r^2}$$

$$= r\dot{\theta}^2 + 2r\dot{\theta} - \frac{1}{2r^2} \tag{11}$$

Possibility2

From (10)

$$\begin{aligned}
2\dot{r} + 2\dot{r}\dot{\theta} + r\frac{d\dot{\theta}}{dt} &= 0 \\
r\frac{d\dot{\theta}}{dt} &= -2\frac{dr}{dt}(1+\dot{\theta}) \\
\frac{1}{1+\dot{\theta}}d\dot{\theta} &= -\frac{2}{r}dr \\
\int \frac{1}{1+\dot{\theta}}d\dot{\theta} &= \int -\frac{2}{r}dr \\
\ln|1+\dot{\theta}| &= -2\ln|r| + c \\
\ln|1+\dot{\theta}| &= \ln|\frac{k}{r^2}| \\
\dot{\theta} &= -1 + \frac{k}{r^2} \qquad (12)
\end{aligned}$$

So it's proved.

Solutions

Substitute (12) into (11)

$$\ddot{r} = r(-1+\frac{k}{r^2})^2 + 2r(-1+\frac{k}{r^2}) - \frac{1}{2r^2}$$
$$= r(\frac{k^2}{r^4} + 1 - \frac{2k}{r^2}) - 2r + \frac{2k}{r} - \frac{1}{2r^2}$$
$$= \frac{k^2}{r^3} - r - \frac{1}{2r^2}$$

Hence we have

$$V'(r) = -\ddot{r}$$
$$= r + \frac{1}{2r^2} - \frac{k^2}{r^3}$$

Circular-orbit

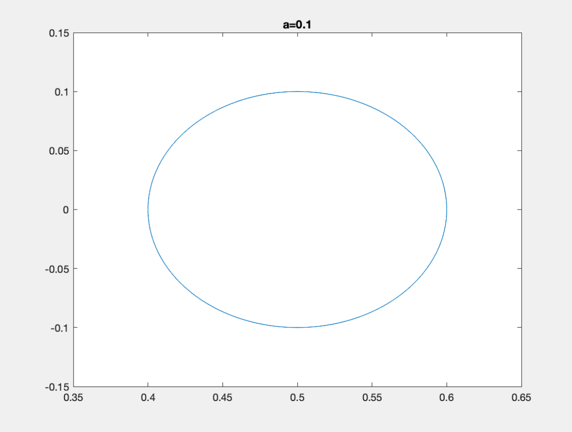


Figure: a=0.1

Circular-orbit

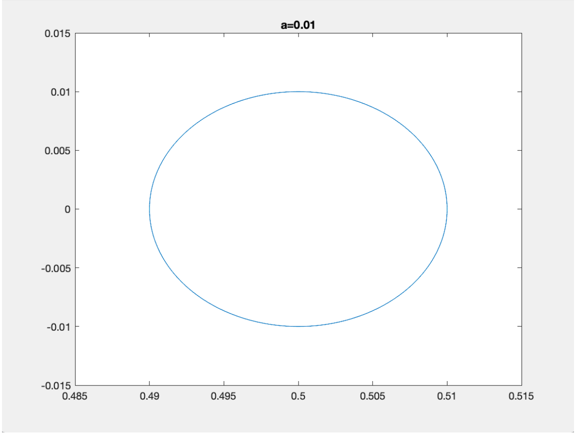


Figure: a=0.01

Circular-orbit

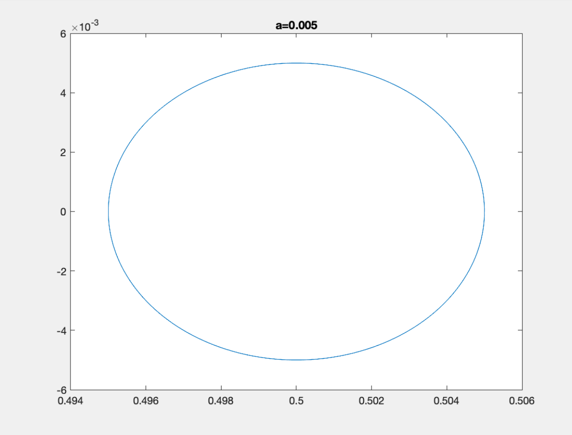


Figure: a=0.005

Codes for Function

```
function dydt = Q2Fun(t,y)
dydt = zeros(4,1);
dydt(1) = y(3);
dydt(2) = y(4);
dydt(3) = -1/2 * (y(1)-1/2) * ((y(1)-1/2)^2+y(2)^2)^(-3/2) + 2*y(4);
dydt(4) = -1/2 * y(2) * ((y(1)-1/2)^2+y(2)^2)^(-3/2) - 2*y(3);
end
```

Codes for Script

```
tspan = [0 20];
a = a;
k = sqrt(a^4+1/2*a);
w = -1 +k/a^2;
v = w*a;
y0 = [0.5+a, 0, 0, v];
opts = odeset('RelTol',1e-10,'AbsTol',1e-10);
[T, Y] = ode45(@Q2Fun, tspan, y0,opts);
plot(Y(:,1),Y(:,2));
title('a=a')
```

Trajectory

Now we return to the original system (1), (2) and (3) with $\mu = 0.5$ and take initial conditions $\mathrm{x} = 0.32, \mathrm{y} = 0, \dot{\mathrm{x}} = 0, \dot{\mathrm{y}} = \mathrm{v}_0$ with v_0 equal to different values.

v_0=-1.0

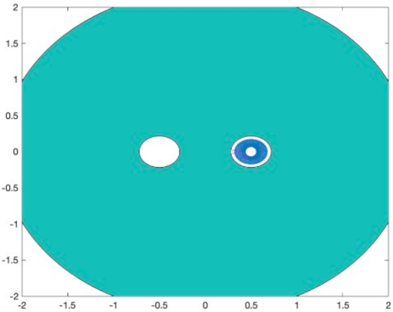


Figure: Forbidden Region

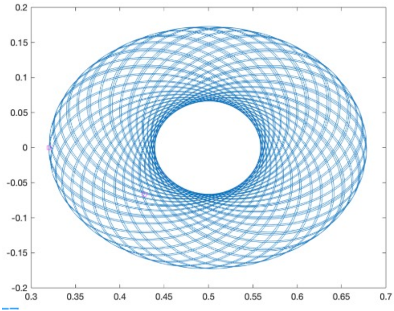


Figure: Trajectory

End point

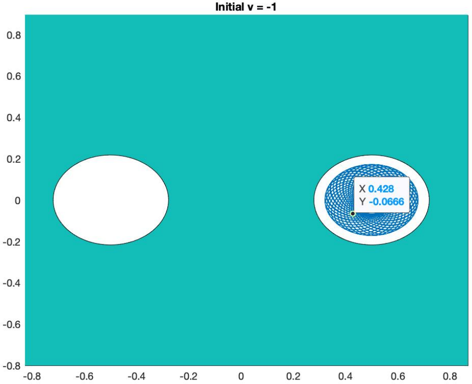


Figure: t=30

v_0=-1.5

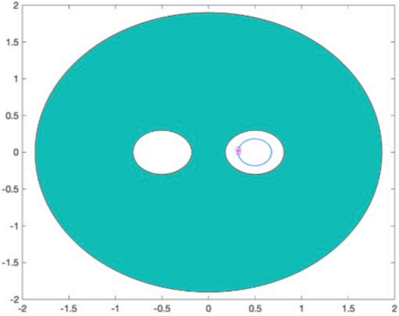


Figure: Forbidden Region

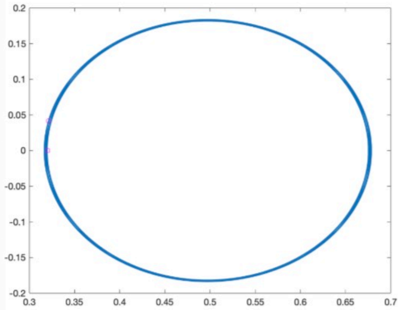


Figure: Trajectory

End point

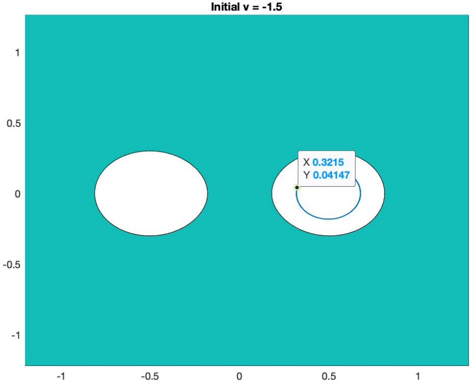


Figure: t=30

v_0=-1.73

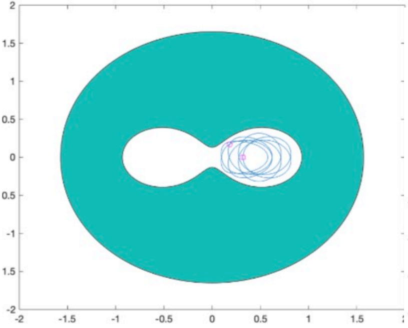


Figure: Forbidden Region

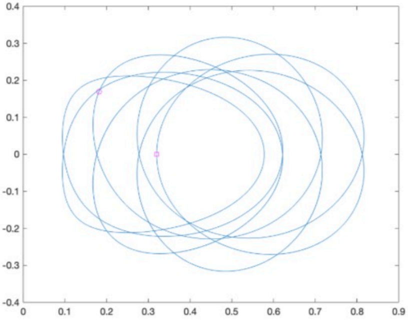


Figure: Trajectory

End point

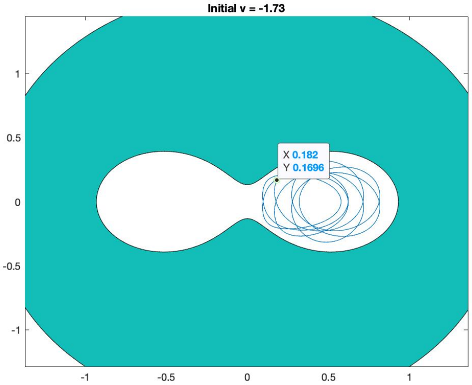


Figure: t=30

v_0=-1.78

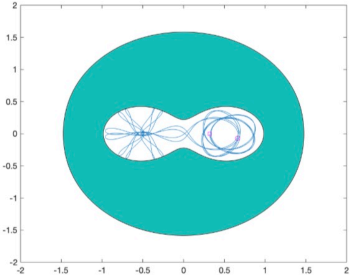


Figure: Forbidden Region

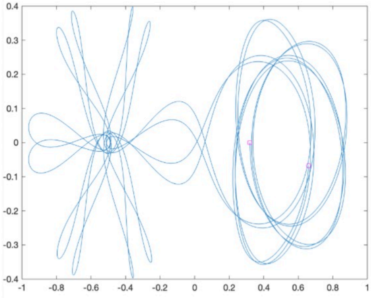


Figure: Trajectory

End point

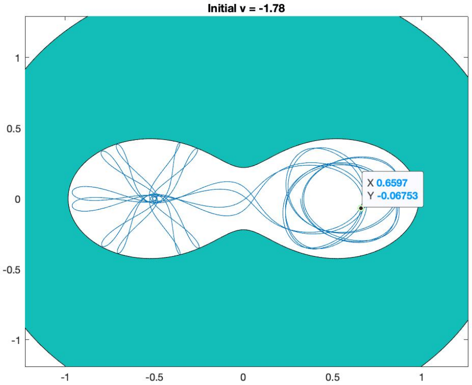


Figure: t=30

v_0=-1.853

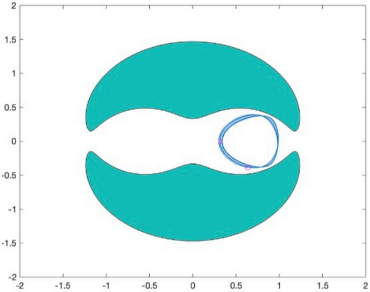


Figure: Forbidden Region

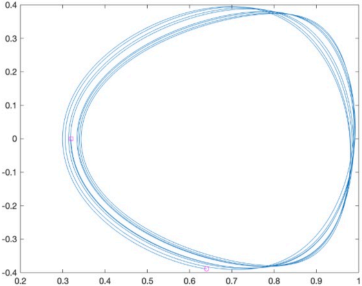


Figure: Trajectory

End point

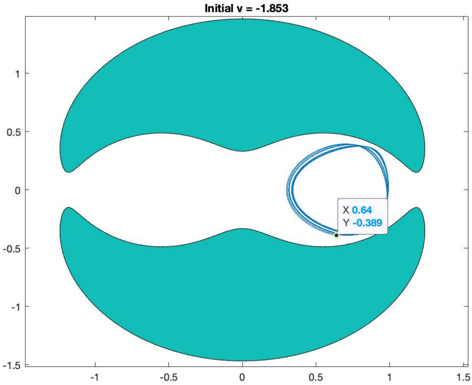


Figure: t=30

v_0=-1.853

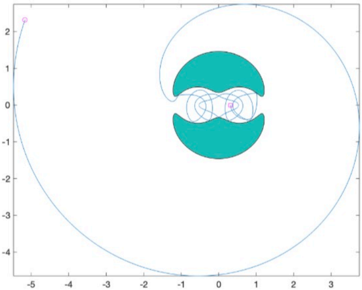


Figure: Forbidden Region

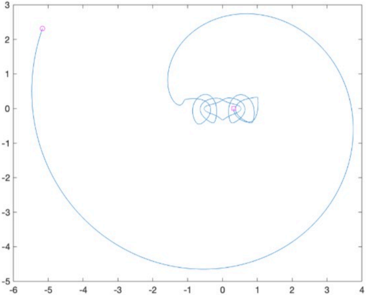


Figure: Trajectory

End point

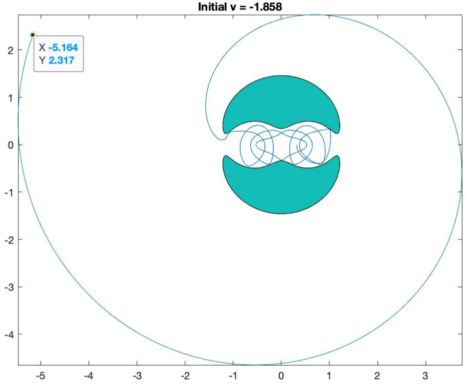


Figure: t=30

v_0=-2.3

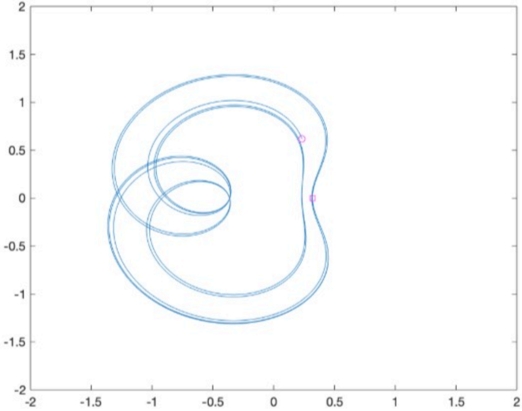


Figure: Trajectory

End point

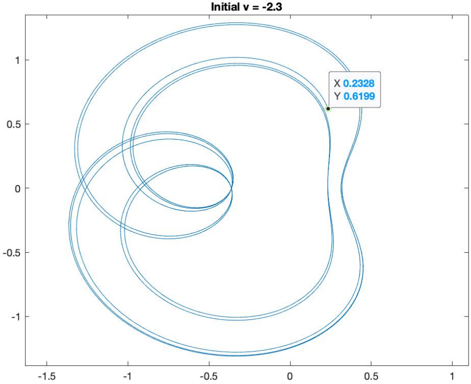


Figure: t=30

v_0=-2.31

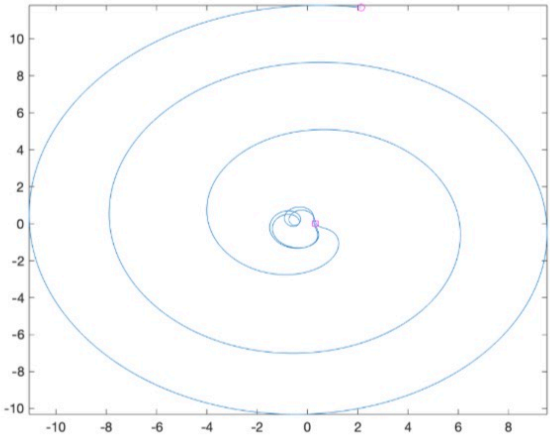


Figure: Trajectory

End point

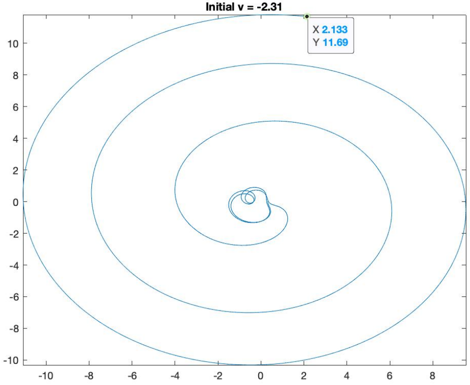


Figure: t=30

Regular

It's easy to get that with initial velocity increasing, the forbidden region becomes smaller. However, the changing route of trajectories is not that obvious.

Critical value

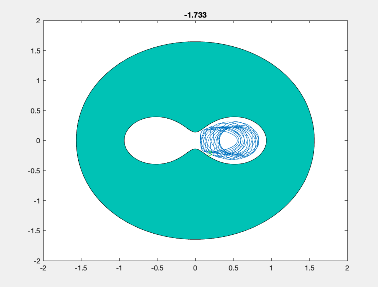


Figure: can't pass through

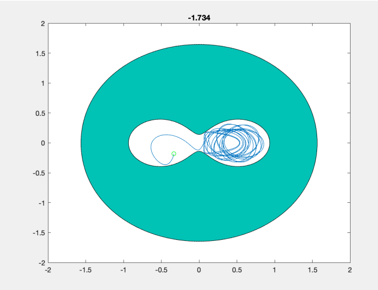


Figure: can pass through

Best-fit v_0

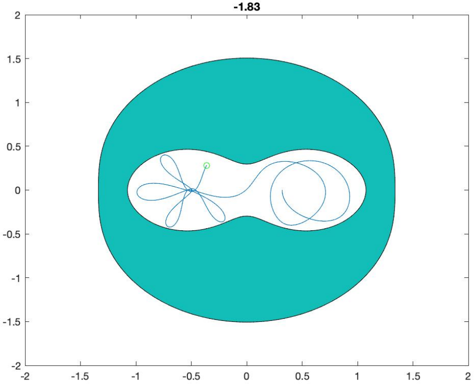


Figure: v=-1.83

Codes for Function

```
%% Define the 2nd order differential equation by transferring into four 1st order's
% the 1st 2nd 3rd 4th entry of y stands for x x' y y' respectively
function dydt = Q3Func(~,y)
u = 0.5; % Set the parameter Greek letter Mu
dydt = zeros(4,1);
dydt(1) = y(2);
dydt(2) = 2*y(4) - (-y(1) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-3/2))*(y(1)+1-u) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*(y(1)-u));
dydt(3) = y(4);
dydt(4) = (-2)*y(2) - (-y(3) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-3/2))*y(3) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*y(3));
end
```

Codes for Script

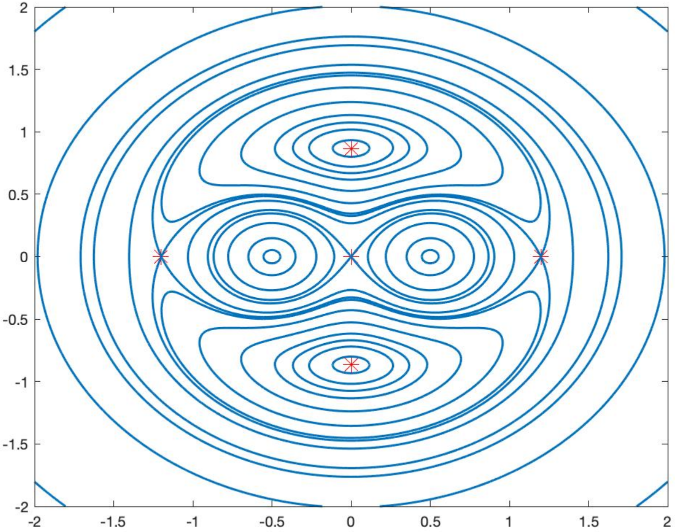
```
%% Plot the trajectory of the third object
tspan = [0 30];% Define the span of time
v = v; % Set the initial speed
y0 = [0.32, 0, 0 , v]; % Define the initial conditions on the motion
opts = odeset('RelTol',1e-10,'AbsTol',1e-10); % Change the precision by controlling Relative &
Absolute Tolerance
[T,Y] = ode45(@Q3Func, tspan, y0, opts); % Call the defined function 'Q3Fun'
plot(Y(:,1),Y(:,3)); % Plot the trace, Y(:,1) means the x-coordinates traversing all the T's
hold on
plot(Y(1,1),Y(1,3),'s','color','m'); % Plot the startpoint of the third object
plot(Y(end,1),Y(end,3),'o','color','m'); % Plot the endpoint of the third object

%% Plot the Forbidden Region
m = linspace(-2,2,1000); % Obtain 1000 points in the interval [-2,2] for x-coordinates
n = linspace(-2,2,1000);
[M, N] = meshgrid(m,n); % Establish grid with the sample in Line14 & 15
w = (-1/2)*(M.^2+N.^2)-1/2*((M+1/2).^2+N.^2).^(-1/2)-(1/2)*((M-1/2).^2+N.^2).^(-1/2);
J = (-1/2)*(y0(1)^2+y0(3)^2)-1/2*((y0(1)+1/2)^2+y0(3)^2)^(-1/2)-(1/2)*((y0(1)-1/2)^2+y0(3)
^2)^(-1/2) + 1/2*(y0(2)^2+y0(4)^2);
contourf(M,N,w,[J J]); % Color the region with Total Potential larger than 'J'
hold off
```

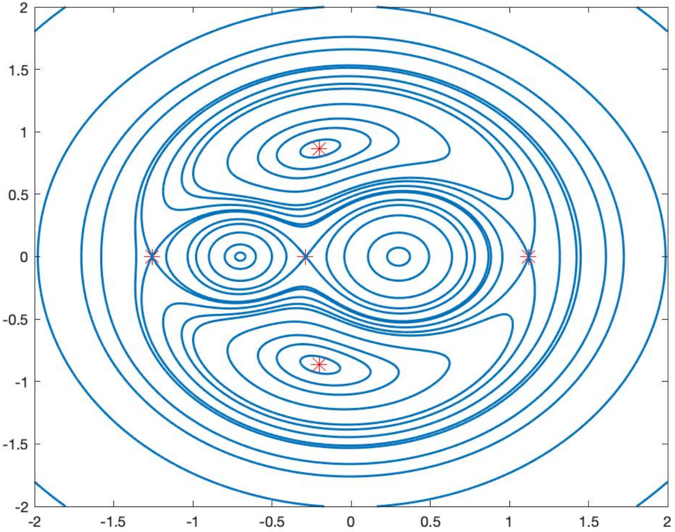
Lagrange points

In the above parts, we only considered the situation as $\mu = 0.5$. In the following parts, we used different values of μ. Then we made efforts to do some contour plots, and we found 5 equilibrium points.

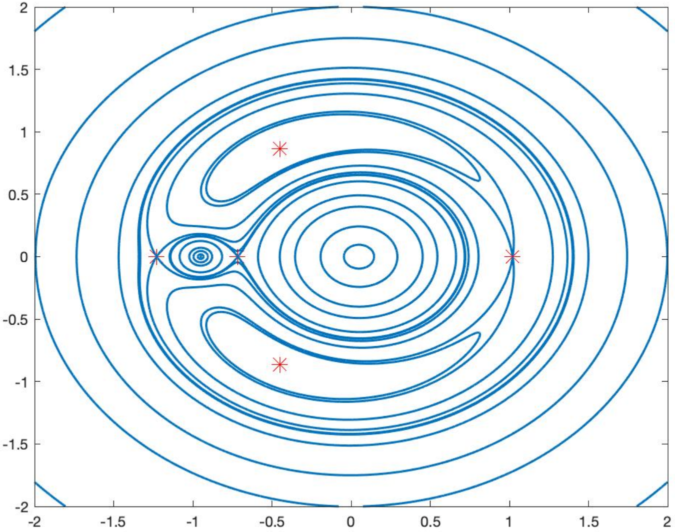
Counter=0.5



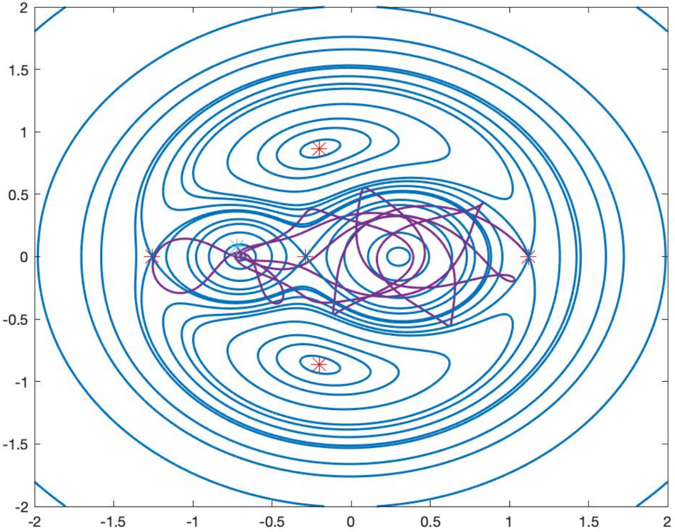
Counter=0.3



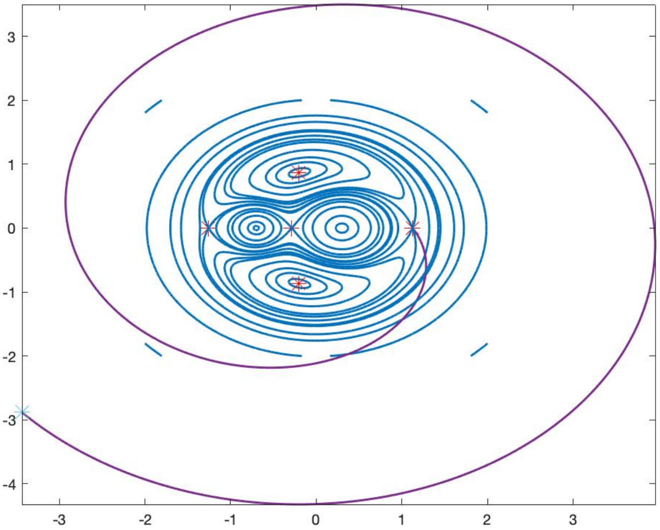
Counter=0.05



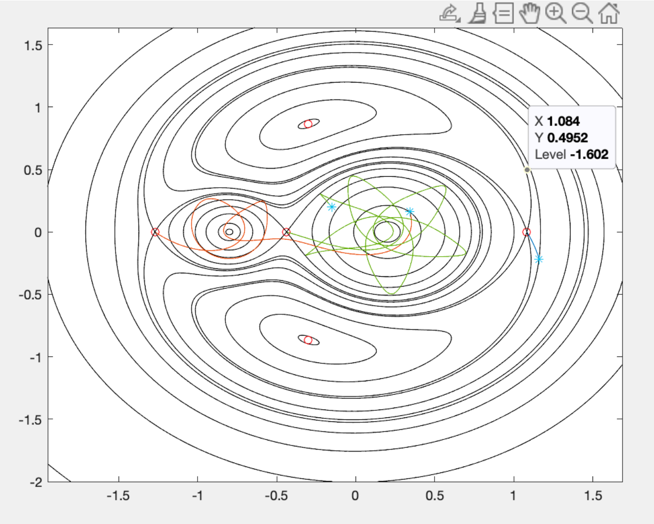
TrajectoryR-L



TrajectoryR-R



Combination of Trace



Codes for Function

```
function Q4i1 = fx1(x)
u = 0.3;
Q4i1 = -x + u*(x+1-u)*((x+1-u).^2).^(-3/2) + (1-u)*(x-u)*((x-u).^2).
^(-3/2);
end
```

Codes for Function

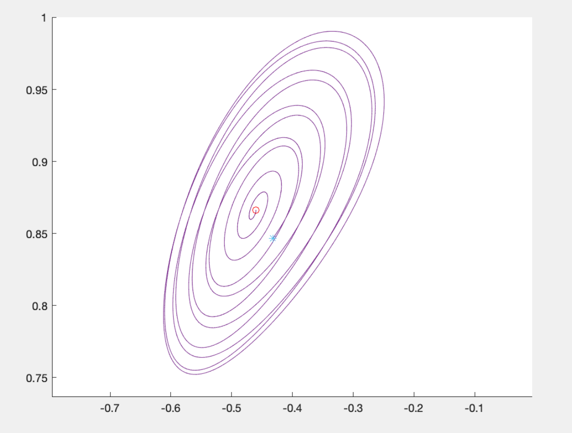
```
% 1/2/3/4-x/x'/y/y'
function dydt = Q3i(t,y)
u = 0.3;
dydt = zeros(4,1);
dydt(1) = y(2);
dydt(2) = 2*y(4) - (-y(1) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-
3/2))*(y(1)+1-u) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*(y(1)-u));
dydt(3) = y(4);
dydt(4) = (-2)*y(2) - (-y(3) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-
3/2))*y(3) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*y(3));
end
```

Codes for Script

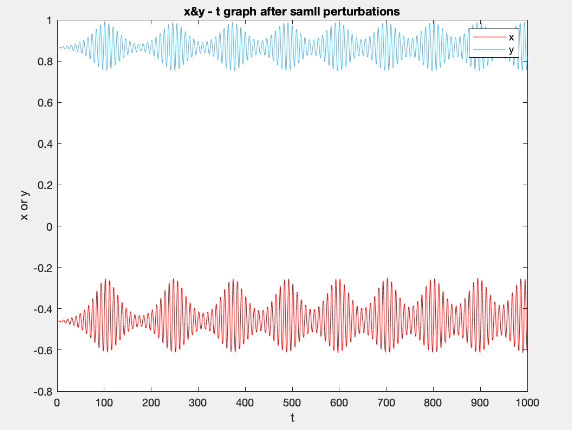
```
%% Initial Setups
% If we change a value of a, we also have to change it in fx1.m and Q3i.m
x = linspace(-2,2,2000);
y = linspace(-2,2,2000);
[X,Y] = meshgrid(x,y);
a = 0.2; % Set Mu
z = (-1/2)*(X.^2+Y.^2)-a*((X+1-a).^2+Y.^2).^(-1/2)-(1-a)*((X-a).^2+Y.^2).^(-1/2);
v = linspace(-1.8,0,20);
u = [-2.1,-2.5,3,-4,-10,-1.45,-1.4,-1.38];
contour(X,Y,z,[v u],'color','k');
hold on
t = zeros(1,1);
%% Plot the trace of points starting at three collinear Lagrange Points
for x0 = [-2 a-0.5 2]
    fun = @fx1;
    i = fzero(fun,x0);
    t = horzcat(t,(i));
    plot(i,0,'O','color','r');
    tspan = [0 9.5];
    v0 = 0;
    y0 = [i+0.0001, 0, 0 , v0];
    opts = odeset('RelTol',1e-10,'AbsTol',1e-10);
    [T,Yn] = ode45(@Q3i, tspan, y0, opts);
    plot(Yn(:,1),Yn(:,3));
    plot(Yn(end,1),Yn(end,3),'*','color',[0.3010 0.7450 0.9330]);
end
plot(a-0.5,sqrt(3)/2,'o','color','r');
plot(a-0.5,-sqrt(3)/2,'o','color','r');
%% Add contours crossing the three collinear and two equilateral Lagrange Points
zi = (-1/2)*(i^2)-a*((i+1-a)^2)^(-1/2)-(1-a)*((i-a)^2)^(-1/2);
contour(X,Y,z,[zi,zi],'color','k');
zi = (-1/2)*(t(1,2)^2)-a*((t(1,2)+1-a)^2)^(-1/2)-(1-a)*((t(1,2)-a)^2)^(-1/2);
contour(X,Y,z,[zi,zi],'color','k');
zi = (-1/2)*(t(1,3)^2)-a*((t(1,3)+1-a)^2)^(-1/2)-(1-a)*((t(1,3)-a)^2)^(-1/2);
contour(X,Y,z,[zi,zi],'color','k');
hold off
```

Stability Properties

We continued with a numerical investigation of the linear stability of the equilateral Lagrange points for different parameter values μ and plot the graph of x and y against t.



Figure



x&y - t graph after samll perturbations
x or y
t
x
y
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
100
200
300
400
500
600
700
800
900
1000

Figure

Codes for Function

```
function Q4i1 = fx1(x)
u = 0.000954;
Q4i1 = -x + u*(x+1-u)*((x+1-u).^2).^(-3/2) + (1-u)*(x-u)*((x-u).^2).
^(-3/2);
end
```

Codes for Function

```
% 1/2/3/4-x/x'/y/y'
function dydt = Q3i(t,y)
u = 0.039;
dydt = zeros(4,1);
dydt(1) = y(2);
dydt(2) = 2*y(4) - (-y(1) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-
3/2))*(y(1)+1-u) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*(y(1)-u));
dydt(3) = y(4);
dydt(4) = (-2)*y(2) - (-y(3) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-
3/2))*y(3) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*y(3));
end
```

Codes for Script

```
% If we change a value of a, we also have to change it in fx1.m and Q3i.m
x = linspace(-2,2,2000);
y = linspace(-2,2,2000);
[X,Y] = meshgrid(x,y);

a = 0.5; %x

z = (-1/2)*(X.^2+Y.^2)-a*((X+1-a).^2+Y.^2).^(-1/2)-(1-a)*((X-a).^2+Y.^2).^(-1/2);


hold on
t = zeros(1,1);

for x0 = [-2 a-0.5 2]
    fun = @fx1;
    i = fzero(fun,x0);
    t = horzcat(t ,[i]);
    plot(i,0,'O','color','r');
end

tspan = [0 25000];
v0 = 0;
y0 = [a-0.5, 0, sqrt(3)/2+0.0001 , v0];
opts = odeset('RelTol',1e-10,'AbsTol',1e-10);
[T,Yn] = ode45(@Q3i, tspan, y0, opts);

plot(Yn(:,1),Yn(:,3));

plot(Yn(end,1),Yn(end,3),'*','color',[0.3010 0.7450 0.9330]);
```

Codes for Script

```
% If we change a value of a, we also have to change it in fx1.m and Q3i.m
t = zeros(1,1);

a = 0.039; %x

for x0 = [-2 a-0.5 2]
    fun = @fx1;
    i = fzero(fun,x0);
    t = horzcat(t,[i]);
end

tspan = [0 100000000];
v0 = 0;
y0 = [a-0.5, 0, sqrt(3)/2+0.0001 , v0];
[T,Yn] = ode45(@Q3i, tspan, y0);

plot(T,Yn(:,1),'color','r');
hold on
plot(T,Yn(:,3),'color',[0.3010 0.7450 0.9330]);
legend('x','y');

title('x&y - t graph after samll perturbations');
ylabel('x or y');
xlabel('t');
```

Consistence

The(Jupiter) Trojans are asteroids observed near the Sun-Jupiter equilateral Lagrange points, for which $\mu = 9.5 \times 10^{-4}$. We checked whether the persistence of the Trojans near these points consistent with our findings above in the following process.

Codes for Function

```
function Q4i1 = fx1(x)
u = 0.039;
Q4i1 = -x + u*(x+1-u)*((x+1-u).^2).^(-3/2) + (1-u)*(x-u)*((x-u).^2).
^(-3/2);
end
```

Codes for Function

```
% 1/2/3/4-x/x'/y/y'
function dydt = Q3i(t,y)
u = 0.000954;
dydt = zeros(4,1);
dydt(1) = y(2);
dydt(2) = 2*y(4) - (-y(1) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-
3/2))*(y(1)+1-u) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*(y(1)-u));
dydt(3) = y(4);
dydt(4) = (-2)*y(2) - (-y(3) + (u)*(((y(1)+1-u)^2+y(3)^2)^(-
3/2))*y(3) + (1-u)*(((y(1)-u)^2+y(3)^2)^(-3/2))*y(3));
end
```

Codes for Script

```
% If we change a value of a, we also have to change it in fx1.m and Q3i.m
x = linspace(-2,2,2000);
y = linspace(-2,2,2000);
[X, Y] = meshgrid(x,y);
a = 0.000954;
z = (-1/2)*(X.^2+Y.^2)-a*((X+1-a).^2+Y.^2).^(-1/2)-(1-a)*((X-a).^2+Y.^2).^(-1/2);
v = linspace(-1.8,0,10);
u = [-2,-3,-4];
contour(X,Y,z,[v u],'color','k');
hold on
plot(a-0.5,sqrt(3)/2,'o','color','r');
plot(a-0.5,-sqrt(3)/2,'o','color','r');
t = zeros(1,1);
tspan = [0 998];
v0 = 0;
y0 = [a-0.5, 0, sqrt(3)/2+0.000001 , v0];
opts = odeset('RelTol',1e-10,'AbsTol',1e-10);
[T, Yn] = ode45(@Q3i, tspan, y0, opts);
plot(Yn(:,1),Yn(:,3));
plot(Yn(end,1),Yn(end,3),'*','color',[0.3010 0.7450 0.9330]);
t = zeros(1,1);
tspan = [0 998];
v0 = 0;
y0 = [a-0.5, 0, -sqrt(3)/2+0.000001 , v0];
opts = odeset('RelTol',1e-10,'AbsTol',1e-10);
[T, Yn] = ode45(@Q3i, tspan, y0, opts);
plot(Yn(:,1),Yn(:,3));
plot(Yn(end,1),Yn(end,3),'*','color',[0.3010 0.7450 0.9330]);
hold off
```