NUMERICAL OPTIMIZATION

Sheet 1: Linear Algebra refresh

In this sheet, each task will contain a theory refresher and an actual exercise.

Exercise one

Give the definition of:

1. Linear independence of a set of vectors.
2. Vector subspace (sometimes called linear subspace) of $\mathbb{R}^d$.
3. Rank of a set of vectors or of a matrix.

Solve the following system of equalities using Gaussian elimination.

$$\begin{aligned} 3x + 2y + 4z + w &= 0 \\ 2x + 4y + 2z + 2w &= 5 \\ -x + 2y + 2z + w &= 7 \\ 4z &= 2 \end{aligned}$$

What is the geometric shape of the space of all solutions? Also, what is the necessary and sufficient condition on rank (rank of what?) for the system to have a unique solution?

Exercise two

1. Give the definition of a determinant, what an invertible matrix is. What is the connection between a determinant of a matrix and its invertibility?
2. Explain how to compute determinants of 2×2 and 3×3 matrices quickly (the Sarrus rule).
3. Explain the full formula for a determinant that uses permutations of columns and the sgn function.

Exercise three

If you have not heard them in Linear Algebra, familiarize yourselves with the following definitions:

D: A set $A \subseteq \mathbb{R}^d$ is an *affine subspace*, if A is of the form $L + v$ for some linear subspace L and a shift vector $v \in \mathbb{R}^d$. By "A is of the form $L + v$" we mean a bijection between vectors of L and vectors of A given as $b(u) = u + v$. Each affine subspace has a *dimension*, defined as the dimension of its associated linear subspace L.

D: A vector x is an *affine combination* of a finite set of vectors $a_1, a_2, \ldots, a_n$ if $x = \sum_{i=1}^{n} \alpha_i a_i$, where α_i are real numbers satisfying $\sum_{i=1}^{n} \alpha_i = 1$.

A set of vectors $V \subseteq \mathbb{R}^d$ is *affinely independent* if it holds that no vector $v \in V$ is an affine combination of the rest.

D: GIven a set of vectors $V \subseteq \mathbb{R}^d$, we can think of its *affine span*, which is a set of vectors A that are all possible affine combinations of any finite subset of V.

Similar to the linear spaces, affine spaces have a finite basis, so we do not need to consider all finite subsets of V, but we can generate the affine span as affine combinations of the basis.

D: A *hyperplane* is any affine space in $\mathbb{R}^d$ of dimension $d - 1$. Thus, on a 2D plane, any line is a hyperplane. In the 3D space, any plane is a hyperplane, and so on.

A hyperplane splits the space $\mathbb{R}^d$ into two *halfspaces*. We count the hyperplane itself as a part of both halfspaces.

First, solve the following two proving exercises:

1. Prove that every hyperplane can be expressed as the set of solutions of

$$\{x \in \mathbb{R}^d \,|\, c^T x = b\},$$

   where $c \in \mathbb{R}^d$ and $b \in \mathbb{R}$.
2. Prove that each *proper* affine subspace (proper means any subspace which is not the full $\mathbb{R}^d$) can be expressed as an intersection of finitely many affine hyperplanes.

Next, two exercises on your understanding of higher dimensions, which can be solved using the tools above.

1. Can two 2D planes (affine subspaces of dimension 2) intersect in exactly one point, if we are working in $\mathbb{R}^4$?
2. Can two affine subspaces of dimension 3 within $\mathbb{R}^5$ intersect in exactly one point?

Exercise four

1. Give the definition of a *positive definite* matrix. Include at least two more equivalent descriptions of this property (i.e., a matrix is positive definite if and only if ...).
2. One tool to detect if a matrix is positive definite is using the Cholesky decomposition. Give a definition of this decomposition and explain the Cholesky algorithm (or one of its variants).
3. Finally, use the Cholesky algorithm (by hand) to show that this matrix is positive definite:

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$$

Exercise five

1. Give the definition of eigenvalues and eigenvectors.
2. Can you comment on the algorithmic complexity of computing the set of all eigenvalues?
3. One way to compute eigenvalues of a small matrix A uses a formula that ties eigenvalues to roots of a specific polynomial $p_A(t)$ that arises from the determinant. Define this polynomial and use this technique to find the eigenvalues of the following matrix A:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 3 \\ 1 & -2 & 2 \end{pmatrix}$$

Exercise six

One final algorithm that we should learn about is Gram-Schmidt orthogonalization. This is a process to find an orthogonal basis from any given vector set (usually the set is independent, so we are converting from a set of linearly independent vectors to a set of orthogonal vectors, preserving the spanned subspace).

Explain the algorithm and use it (by hand) to convert the following set of vectors to their orthogonal representation:

$V = \{v_1, v_2, v_3\}$, where $v_1 = (1, 0, 1, 1)$, $v_2 = (2, 1, -1, 0)$, and $v_3 = (1, 1, 1, 1)$.