NO
Ex1

Ex1:

① A set of vector $\{v_1 v_2 \cdots v_n\}$ is linear independent if the vector equation
$x_1v_1 + x_2v_2 + \cdots x_kv_k = 0$ has only the trivial solution $x_1v_1 + x_2v_2 + \cdots x_kv_k = 0$. then the vector is linear dependent otherwise

single vector is always linear independence

② A subspace of $\mathbb{R}^d$ is a subset V of $\mathbb{R}^d$ satisfying
① Non-empty: the zero vector is in V.
② if $u.v \in V$, then $u+v \in V$.
③ v is V. c is in $\mathbb{R}$. then $cv \in V$.

③ The rank of a matrix A, rank(A) is the dimension of the column space col(A)

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 + x_4 &= 0 \\ 2x_1 + 4x_2 + 2x_3 + 2x_4 &= 5 \\ -x_1 + 2x_2 + 2x_3 + x_4 &= 7 \\ 4x_3 &= 2 \end{aligned} \qquad \left[\begin{array}{cccc|c} 3 & 2 & 4 & 1 & 0 \\ 2 & 4 & 2 & 2 & 5 \\ -1 & 2 & 2 & 1 & 7 \\ 0 & 0 & 4 & 0 & 2 \end{array}\right]$$

$R_1/3$
$$\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ 2 & 4 & 2 & 2 & 5 \\ -1 & 2 & 2 & 1 & 7 \\ 0 & 0 & 4 & 0 & 2 \end{array}\right]$$

$R_2 - 2R_1$, $R_3 + R_1$
$$\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & \frac{8}{3} & -\frac{2}{3} & \frac{4}{3} & 5 \\ 0 & \frac{8}{3} & \frac{10}{3} & \frac{4}{3} & 7 \\ 0 & 0 & 4 & 0 & 2 \end{array}\right]$$

$R_2 / \frac{8}{3}$
$$\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{15}{8} \\ 0 & \frac{8}{3} & \frac{10}{3} & \frac{4}{3} & 7 \\ 0 & 0 & 4 & 0 & 2 \end{array}\right]$$

$R_1 - \frac{2}{3}R_2$, $R_3 - \frac{8}{3}R_2$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1.5 & 0 & -1.25 \\ 0 & 1 & -0.25 & 0.5 & \frac{15}{8} \\ 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 \end{array}\right]$$

$R_3/4$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1.5 & 0 & -\frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{15}{8} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 4 & 0 & 2 \end{array}\right]$$

$R_1 - 1.5R_3$ $R_2 + \frac{1}{4}R_3$
$R_4 - 4R_3$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -2$
$x_2 + \frac{1}{2}x_4 = 2$
$x_3 = \frac{1}{2}$

① $3x_1 + 2x_2 + 4x_3 + x_4 = 0$
$-6 + 2x_2 + 2 + x_4 = 0$
$x_4 = 2 - 2x_2$

③ $x_4 = 4 - 2x_2$

The rank of the coefficient matrix must be equal to the number of variables in the system

EX2 1. The deteminant is a function
$\det\{A\} \to R$
Satisfying the following properties
① Doing a row replacement on A doesn't change $\det(A)$
② Scaling a row of A by a scalar C multiplies the deteminant by C.
③ Swapping two row of a matrix multiplies the detemanat by -1
④ the deteminant of the identity matrix I_n is 1
$$\det(A) = \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$
invertible: A is an $n \times n$ matrix, we say that A is invertible if there is an $n \times n$ matrix B such that $AB = I_n$ and $BA = I_n$

relation $\det(A^{-1}) = \frac{1}{\det(A)}$

2. 2×2 | 3×3

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \qquad \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$

3. $\cancel{\det A = \sum_{\sigma} sgn(\sigma) a_{1,\sigma}}$

$\det A = \sum \{ sgn(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} a_3 \cdots a_{n,\sigma(n)} \ \sigma \in S_n \}$

where $n = 2$

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$

There are $n!$ permutations in S_n

Ex4. def. A square matrix is called positive definite if it is symmetric and all it eigenvalues λ are positive $\lambda > 0$.

$$A = LL^T$$

Theorem ① if A is positive definite, then $A^{-1} > 0$ $\det(A) > 0$

② A symmetric Matrix A is positive definite if and only if $X^T A X > 0$ for every column $X \neq 0$ in $\mathbb{R}^n$

$\boxed{\text{cholesky Algorithm}}$

Ex3. $\{X \in \mathbb{R}^d \mid c^T x = 0\}$

Q1: is it a linear subspace

Q2: is the dimension $d-1$

$Ax = 0$

$\ker(AX)$

$\{x \in \mathbb{R}^d \mid Ax = 0\}$

$\dim \ker(Ax) + \operatorname{rank}(A) = d$

$\dim(\ker(A)) = d - \operatorname{rank}(A)$

1. 2D plane intersect at one point in $\mathbb{R}^4$

A: Yes.

$$\left\{ x \in \mathbb{R}^4 \,\middle|\, \begin{matrix} a^T x = b \\ c^T x = a \end{matrix} \right\} = P_1$$

$$\left\{ y \in \mathbb{R}^4 \,\middle|\, \begin{matrix} a^T y = b \\ c^T y = a \end{matrix} \right\} = P_2$$

$P_1 = (0, 0, 1, 1)$

$P_2 = \{(1, 1, 0, 0)\}$

$P_1 \cap P_2$

two types of Ex

① simple proofs

② understanding algorithm

Ex 5 ① Let A be an $n \times n$ matrix

1. An eigenvalue of A is a scalar λ such that the equation $Av = \lambda v$ has a nontrivial so

2. An eigenvector of A is a nonzero vector v in $\mathbb{R}$ such that $Av = \lambda v$ for some scalar λ

- describing an characteristic property of A.

②

③ $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 3 \\ 1 & -2 & 2 \end{bmatrix}$ $\det\left(\begin{bmatrix} 1-\lambda & 2 & 0 \\ 3 & -1-\lambda & 3 \\ 1 & -2 & 2-\lambda \end{bmatrix}\right)$

$(1-\lambda)(-1-\lambda)(2-\lambda) + 2\times3\times1 + 0 - (-2\times3)(1-\lambda) - (2-\lambda)\times3\times2 = 0$

$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

$\lambda_1 = 2$

$\lambda_2 = 1$

$\lambda_3 = -1$

~~NO Lab~~ →

NO Ex 2

Sylvester's criterion

positive determinant

upper left 1×1, 2×2, 3×3, ⋮

Ex1: $A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$

$\Delta_1=\det[2]=2$

$\Delta_2=\det\begin{bmatrix}2&-1\\-1&2\end{bmatrix}=2\times2-1=3$

$\Delta_3=\det\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}=2\times2\times2+(-1)(-1)(0)-(-1)(-1)(2)-2(-1)(-1)$
$=8-2-2$
$=4$

Def

正定 $\forall x \quad x^TAx>0$

非正定 $\forall x \quad x^TAx\geq0$

Ex2: $A=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$

$\Delta_1=\det[2]=2 \quad >0$

$\Delta_2=\det\begin{bmatrix}2&-1\\-1&2\end{bmatrix}=3 \quad >0$

Postive semidefinite matrix

$\Delta_3=\det\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}=2\times2\times2+(-1)(-1)(-1)+(-1)(-1)(-1)-(-1)(2)(-1)-(-1)(-1)(2)$
$-2(-1)(-1)$
$=8-1-1-2-2-2$
$=0$

Cholesky 分解

$A=R^TR$

$A=(\Lambda R)^T(\Lambda R)$

R 上三角阵

对角元大于0

$L=\begin{bmatrix}\sqrt{2}&&\\-\frac{1}{\sqrt{2}}&\sqrt{\frac{3}{2}}&\\-\frac{1}{\sqrt{2}}&-\sqrt{\frac{3}{2}}&0\end{bmatrix}$

$A=\begin{bmatrix}A_{11}&A_{12}\\A_{12}^T&A_{22}\end{bmatrix} \quad L=\begin{bmatrix}L_{11}&L_{12}\\0&L_{22}\end{bmatrix} \quad L_1=\sqrt{A_{11}}$

$L^TL=\begin{pmatrix}L_{11}&0\\L_{12}&L_{22}\end{pmatrix}\begin{pmatrix}L_{11}^2&L_{11}L_{12}\\(L_{11}L_{12})^T&(L_{12}L_{12})+(L_{22}L_{22})\end{pmatrix}$

leading priciple minors → upper-left corner 1×1, 2×2, ⋮ ~~sub~~ square submatrix

minor → any ~~submatrix~~ square submatrix

priciple minors → ~~the sum~~ the sum of index rows and columns.

M is PSD if and only if all priciple minors ~~both~~ have non negative determinat

Ex3 Max $16x_1 + 14x_2$

st $2x_1 + 3x_2 \le 18$

$2x_1 + 3x_2 \le 24$

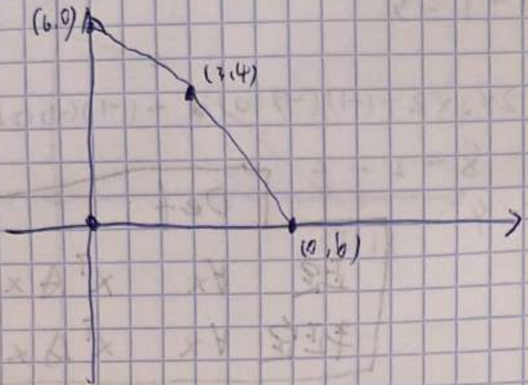
$4x_1 + 3x_2 \le 24$

$x_1, x_2 \ge 0$

$c\binom{4}{2} = 6$

(0,0) (0,6) (9,0) x

(3,4) (6,0) (0,8) x



$z = 0$

$z(0,6) = 84$

$z(3,4) = 104$

$z(6,0) = 96$

Ex4.

n → num item

w_i → weight

p_i → price

H → capacity

Max $z = \sum_{i=1}^{n} p_i x_i$

st $\sum_{i=1}^{n} w_i x_i \le H$

$x_i \in \{0,1\}$ for all $i = 1, 2, 3 \ldots n$

max $\sum_{i=1}^{n} p_i x_i$

st $\sum_{i=1}^{n} x_i w_i \le H$

$x_i \in \{0,1\}$ for all $i = 1, 2, 3, \ldots$

Ex5. min $\sum_{i \in S} \sum_{j \in T} C_{ij} x_{ij}$

st. $\sum_{j \in T} x_{ij} = 1 \quad \forall i \in S$

$\sum_{i \in S} x_{ij} = 1 \quad \forall j \in T$

$x_{ij} \in \{0,1\}$

$\forall_{ij} \ge 0$

max $z = \sum_{e \in E} W_e x_e$

$\sum_{e \in \delta(u)} x_e = 1$ for all $u \in U$

$\sum_{e \in \delta(v)} x_e = 1$ for all $v \in V$

$x_e \in \{0,1\}$ for all $e \in E$

Ex6.

$i = 1\ 2\ 3\ 4$ for $C_1\ C_2\ C_3\ C_4$

$j = 1\ 2\ 3$ for F C Rear

$$\text{Max } 310 \sum_j X_{1j} + 380 \sum_j x_{2j} + 350 \sum_j + 285 \sum_j x_{4j}$$

st

cargo weight
$$\begin{cases} \sum_j x_{1j} \leq 18 \\ \sum_j x_{2j} \leq 15 \\ \sum_j x_{3j} \leq 23 \\ \sum_j x_{4j} \leq 12 \end{cases}$$

comparement weight
$$\begin{cases} \sum_i x_{i1} \leq 10 \\ \sum_i x_{i2} \leq 16 \\ \sum_i x_{i3} \leq 8 \end{cases}$$

space capacity of each comparement
$$\begin{cases} 480 x_{11} + 650 x_{21} + 580 x_{31} + 390 x_{41} \leq 6800 \\ 480 x_{12} + 650 x_{22} + 580 x_{32} + 390 x_{42} \leq 8700 \\ 480 x_{13} + 650 x_{23} + 580 x_{33} + 390 x_{43} \leq 5300 \end{cases}$$

$$\begin{cases} \dfrac{\sum_{i=1}^{4} x_{i1}}{10} = \dfrac{\sum_{i=1}^{4} x_{i2}}{16} \\ \dfrac{\sum_{i=1}^{4} x_{i2}}{16} = \dfrac{\sum_{i=1}^{4} x_{i3}}{8} \end{cases}$$

x.

$i = 1\ 2\ 3\ 4 \quad j = 1\ 2\ 3$

$$\max 310 \sum_j^3 X_{1j} + 380 \sum_j^3 X_{2j} + 350 \sum_j^3 X_{3j} + 285 \sum_j^3 X_{4j}$$

st

cargo
$$\begin{cases} \sum_j X_{1j} \leq 18 \\ \sum X_{2j} \leq 15 \\ \sum X_{3j} \leq 23 \\ \sum X_{4j} \leq 12 \end{cases}$$

comparment
$$\begin{cases} \sum X_{i1} \leq 10 \\ \sum X_{i2} \leq 16 \\ \sum X_{i3} \leq 8 \end{cases}$$

$$\begin{cases} \dfrac{\sum X_{i1}}{10} = \dfrac{\sum X_{i2}}{16} \\ \dfrac{\sum X_{i1}}{10} = \dfrac{\sum X_{i3}}{8} \end{cases}$$

volum
$$\begin{cases} 480 X_{11} + 650 X_{21} + 580 X_{31} + 390 X_{41} \leq 6800 \\ X_{12} \quad X_{22} \quad X_{32} \quad X_{42} \leq 8700 \\ X_{13} \quad X_{23} \quad X_{33} \quad X_{43} \leq 5300 \end{cases}$$

NO
Ex3

~~Lab 3~~ I: Deciding convexity of polynomials of degree ≤ 4 is NP-complete

Ex1: Any $\alpha\beta \geq 0$. $a.b \in C$

$\alpha a \in C$
$2\alpha a \in C$
$\beta b \in C$
$2\beta b \in C$

(cone) property

$\frac{1}{2}(2\alpha a) + \frac{1}{2}(2\beta b) \in C$. (convexity of cone)

Ex3:

Ex4: $H(f) = \begin{bmatrix} \frac{2}{y^4} & -\frac{8x}{y^5} \\ -\frac{8x}{y^5} & 20\frac{x^2}{y^6} \end{bmatrix}$

$$\det = \frac{40x^2}{y^{10}} - \frac{64x^2}{y^{10}} = -\frac{24x^2}{y^{10}} \leq 0$$

Ex5:

Ex3. ex2.

$$f(x,y,z) = 12.5x^2 + 9y^2 + 5.5z^2 + 15xy - 5xz$$

~~the first partial derivative~~

the first order partial derivative

$$\left\{\begin{array}{ll} \dfrac{\partial f}{\partial x} = 25x + 15y - 5z & \left\{\begin{array}{l} \dfrac{\partial}{\partial x}(12.5x^2) = 25x \\ \dfrac{\partial}{\partial x}(15xy) = 15y \\ \dfrac{\partial}{\partial x}(-5xz) = -5z \end{array}\right. \\ \dfrac{\partial f}{\partial y} = 18y + 15x & \left\{\begin{array}{l} \dfrac{\partial}{\partial y}(9y^2) = 18y \\ \dfrac{\partial}{\partial y}(15xy) = 15x \end{array}\right. \\ \dfrac{\partial f}{\partial z} = 11z - 5x & \left\{\begin{array}{l} \dfrac{\partial}{\partial z}(5.5z^2) = 11z \\ \dfrac{\partial}{\partial z}(-5xz) = -5x \end{array}\right. \end{array}\right.$$

$$\left\{\begin{array}{l} \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}(25x + 15y - 5z) = 25 \\ \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial y}(25 + 15 - 5z) = 15 \\ \dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial}{\partial z}(25 - 15y - 5z) = -5 \end{array}\right.$$

$$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}(18y + 15x) = 18$$

$$\dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial}{\partial z}(18y + 15x) = 0$$

$$\dfrac{\partial^2 f}{\partial z^2} = \dfrac{\partial}{\partial z}(11z - 5x) = 11$$

$$H = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

eigenvalues:

$\lambda_1 \approx 37.49$

$\lambda_2 \approx 12.02$

$\lambda_3 \approx 4.5$

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \\ & \dfrac{\partial^2 f}{\partial x_2^2} & \\ & & \cdots \end{bmatrix}$$

$$\det(H - I\vec{x})$$

saddle point

Ex4

Ex 1 & 2

$$\frac{\partial f}{\partial x_i} = s = \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}$$

$$\log \sum_{i=1}^{n} e^{x_i} = f(\bar{x})$$

$$H_f = \left[\frac{* \quad -e_i e^{x_j}}{(\sum_{k=1}^{n} e^{x_i})^2} \quad j_i \right]$$

$$* \frac{e^{x_i}(\sum_{k=1}^{n} e^{x_k}) - e^{x_i}e^{x_i}}{(\sum_k e^{x_k})^2}$$

Ex 1.1 A is symmetric c^2 non negative

$\frac{A}{c^2}$ - positive definite $\iff$ A is positive definite

$$x^T A x \geq 0$$

$$x^T \left(\frac{A}{c^2}\right) x \geq 0 \implies \frac{1}{c^2} x^T A x \geq 0$$

Ex 1.2 $$c^2 = \frac{1}{(\sum_k e^{x_k})^2}$$

Ex 1.3

~~a) $\nabla g = (s_1 \dots s_n)$ $H_g = (s_1(1-s_1) - s_1 s_2$~~

Ex4

Ex 5

Ex 3 $f(x) = \frac{1}{2}(x_1^2 + r x_2^2)$

start point $(r, 1)$

1. $\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_1 \\ r x_2 \end{bmatrix}$

setting gradient to zero $\begin{cases} x_1 = 0 \\ r x_2 = 0 \end{cases}$

Ex 2 x^3

$\|x^3 - y^3\| \leq L \|x - y\|$

$\|x^3 - y^3\| \leq L |x - y|$

$$\frac{|(x-y)(x^2+xy+y^2)|}{|x-y|} \leq L$$

$|x^2 + xy + y^2| \leq L$

$|3x^2 - 3y^2| \leq L |x - y|$

$3|(x+y)(x-y)| \leq L |x-y|$

$3|x+y| \leq L$

$f(x,y) = 7x^2 - x^3y^4 + 5x^4y^3$

$f_x = 7(2x) - 3x^2y^4 + 5(4x^3)y^3$ respect x

$= 14x - 3x^2y^4 + 20x^3y^3$

$f_y = 0 - x^3 4y^3 + 5x^4 3y^2$ respect y.

$= -4x^3y^3 + 15x^4y^2$

Ex2.

① $f(x) = x \log x$ $f'(x) = \log x + 1$. it is not bounded, so it is not Lipschitz common

② $f(x) = x^3$ $f'(x) = 3x^2$ not bounded

$f''(x) = 6x$ not bounded

Ex 6. (ex 1) ~~product~~ demand constrain

$x_i + s_{i-1} \geq d_i$

surplus constrain

$s_i = x_i - d_i$

initial and final surplus

$s_0 = s_{12} = 0$

min $1500 \sum_{i=1}^{11} |x_{i+1} - x_i| + 600 \sum_{i=1}^{12} s_i$

(ex 2) $f(x,y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y$

$$\frac{\partial f}{\partial x} = 4x - 4y = 4x - 4y$$

$$\frac{\partial f}{\partial y} = -4x + 4y^3 + 10y - 10$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(4x - 4y) = 4$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-4x + 4y^3 + 10y - 10)$$

$$= 12y^2 + 10$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}(4x - 4y) = -4$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}(-4x + 4y^3 + 10y - 10) = -4$$

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 12y^2 + 10 \end{bmatrix}$$

1x1 pricipal minor 4

2x2 $4(12y^2+10)-(-4)(-4)$
$= 48y^2 + 56$

non negative so $f(x,y)$ is convex

(ex3)

$$f(x,y) = 2x^2 - 4xy + y^4 + 5y^2 - 10y$$

1. ∇f
2. evaluate ∇f
3. α
4.

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 4x - 4y \\ -4x + 4y^3 + 10y - 10 \end{bmatrix}$$

initial point $(0,0)$, the gradient at this point

$$\nabla f(0,0) = \begin{bmatrix} 4\times 0 - 4\times 0 \\ -4\times 0 - 4(0)^3 + 10(0) - 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

backtracking, start with the initial ~~step~~ step length $\alpha = 1$.
Parameter $c = 0.1$ (sufficient decrease parameter of Armijo)

1. search direction $P = -\nabla f(x,y)$
2. length $\alpha = 1$
3. Armijo condition

Armijo
$f(x_k - \alpha\nabla f(x_k)) \le f(x_k) - c\alpha\|\nabla f(x_k)\|$

$$f(x_k + \alpha P_k) \le f(x_k) + c\alpha \cdot P_k \cdot \nabla f_k$$
$$f(x_k + \alpha P_k) \le f(x_k) + c\alpha \nabla f(x_k)^T P_k$$

$\cancel{f(0 + 1\begin{bmatrix} 0 \\ 10 \end{bmatrix}) = 10400}$

$\cancel{f(0,0) + 0.1 \cdot 1 \cdot \nabla f(0,0) \cdot \begin{bmatrix} 0 \\ 10 \end{bmatrix}}$

$\le f(0,0) + 0.1 \cdot \alpha \cdot \nabla f(0,0) \cdot \begin{bmatrix} 0 \\ -10 \end{bmatrix}$

$f(0 + 1 \begin{bmatrix} 0 \\ -10 \end{bmatrix}$

10600

-10

10

(update rule)

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha\nabla f(x_k, y_k)$$

$1 + 5 - 10 = -4 < -1$ $x_1 =$

ex5

an ellipsoid in $\mathbb{R}^2$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} \leq 1$$

h.k center of ellipsoid

a.b semi-major axis

semi-minor axis

$$x^2+2y^2-8y \leq 0$$

$$x^2+2(y^2-4y) \leq 0$$

$$x^2+2(y^2-4y)+4-4 \leq 0$$

$$x^2+2((y-2)^2-4) \leq 0$$

$$x^2+2(y-2)^2 \leq 8$$

$$\frac{x^2}{8}+\frac{(y-2)^2}{4} \leq 1$$

$$a=\sqrt{8}=2\sqrt{2}$$

$$b=\sqrt{4}=2$$

$$\{x \mid (x-c)^T A^{-1}(x-c) \leq 1\}$$

PD

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ellipsoid in $\mathbb{R}^2$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} \leq 1$$

$$x^2+2y^2-8y \leq 0$$

$$x^2+2(y^2-4y) \leq 0$$

$$x^2+2(y^2-4y+4-4) \leq 0$$

$$x^2+2((y-2)^2-4) \leq 0$$

$$x^2+2(y-2)^2-8 \leq 0$$

$$x^2+2(y-2)^2 \leq 8$$

$$\boxed{Ax^2+By^2 \leq C}$$

$\min f(x,y,z) = x^2 + 13y^2 + 4z^2 + 6xy + 2xz + 10yz + 6x - 2y$

1. first partial derivatives

$$\frac{\partial f}{\partial x} = 2x + 6y + 2z + 6$$

$$\frac{\partial f}{\partial y} = 26y + 6x + 10z - 2$$

$$\frac{\partial f}{\partial z} = 8z + 2x + 10y + 4$$

2. second partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6$$

$$\frac{\partial^2 f}{\partial y^2} = 26 \qquad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 2$$

$$\frac{\partial^2 f}{\partial z^2} = 8 \qquad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 10$$

3. $\Rightarrow \begin{bmatrix} 2 & 6 & 2 \\ 6 & 26 & 10 \\ 2 & 10 & 8 \end{bmatrix}$

4. Leading principal minors

first $\det z_1 = 2 \geq 0$

second $\det \begin{vmatrix} 2 & 6 \\ 6 & 26 \end{vmatrix} = 2 \times 26 - 6 \times 6 = 16 \geq 0$

third $\det(H) = 64 \geq 0$

H is positive semi-definite matrix. Therefore $f(x,y,z)$ is convex.

Ex7

2) Newton method.

$$f(x_1, x_2) = x_1^4 + 2x_1^3 + 2x_1^2 + x_2^2 - 2x_1x_2$$

$\frac{\partial f}{\partial x_1} = 4x_1^3 + 6x_1^2 + 4x_1 - 2x_2$ $\qquad$ $\frac{\partial^2 f}{\partial x_1^2} = 12x_1^2 + 12x_1 + 4$

$\frac{\partial f}{\partial x_2} = 2x_2 - 2x_1$ $\qquad$ $\frac{\partial^2 f}{\partial x_2^2} = 2$

$\frac{\partial^2 f}{\partial x \partial y} = -2$

init $(-1, 0)$

$$\nabla f(-1,0) = \begin{pmatrix} -4+6-4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$Hf(-1,0) = \begin{pmatrix} 12-12+4 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$$

Δx given by $\quad H_f(x,y)\Delta x = -\nabla f(x,y)$

$$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$4\Delta x - 2\Delta y = 2$ $\qquad$ $-\Delta x + \Delta y = -1 \Rightarrow \Delta y = \Delta x - 1$

$-2\Delta x + 2\Delta y = -2$ $\qquad$ $2\Delta x - \Delta y = 1$

$2\Delta x - (\Delta x - 1) = 1$

$\Delta x = 0$

$\Delta y = -1$

$x_{new} = 0 + (-1) = -1$

$y_{new} = 0 + -1 = -1$

Armijo $\quad f(x_{new}) \leq f(x_{old}) + c\alpha \nabla f(x_{old})^T \Delta x$

$$\underbrace{f(-1,-1)}_{0} \leq \underbrace{f(-1,0)}_{1} + 1 \times 0.1 \times (-2,2)\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$1 + (-2) = 0.8$$

satisfied

Accept $(-1, -1)$

$f(x) = -\log(x) + x$

$f'(x) = -\frac{1}{x} + 1$

$f''(x) = \frac{1}{x^2}$ $x^2 > 0$, so it is convex

Ex8

ex1

$$\min \ 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$$
$$\text{st} \quad x_1^2 + x_2^2 \le 5$$
$$3x_1 + x_2 \le 6$$

$$L(x_1, x_2, \lambda_1, \lambda_2) = 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 + \lambda_1(x_1^2 + x_2^2 - 5) + \lambda_2(3x_1 + x_2 - 6)$$

KKT condition

1. stationarity: the partial derivatives of the Lagrangian respect to x_1, x_2 to 0

$$\frac{\partial L}{\partial x_1} = 4x_1 + 2x_2 - 10 + 2\lambda_1 x_1 + 3\lambda_2 = 0$$
$$\frac{\partial L}{\partial x_2} = 2x_1 + 2x_2 - 10 + 2\lambda_1 x_2 + \lambda_2 = 0$$

2. Primal feasibility

$$x_1^2 + x_2^2 - 5 \le 0$$
$$3x_1 + x_2 - 6 \le 0$$

3. Dual feasibility

$$\lambda_1 \ge 0$$
$$\lambda_2 \ge 0$$

4. complementary slackness

$$\lambda_1(x_1^2 + x_2^2 - 5) = 0$$
$$\lambda_2(3x_1 + x_2 - 6) = 0$$

$4x_1 + 2x_2 - 10 = 0$
$2x_1 + 2x_2 - 10 = 0$

$x_1 = 0$
$x_2 = 5$

both inactive

~~first constraints~~

first constraint active ($x_1^2 + x_2^2 \le 5$, $\lambda_2 = 0$),
the second constraint $3x_1 + x_2 \le 6$ is not binding ($\lambda_2 = 0$)

$$\frac{\partial L}{\partial x_1} = 4x_1 + 2x_2 - 10 + 2\lambda_1 x_1$$
$$\frac{\partial L}{\partial x_2} = 2x_1 + 2x_2 - 10 + 2\lambda_1 x_2$$

$$\begin{cases} (4 + 2\lambda_1)x_1 + 2x_2 = 10 \\ 2x_1 + (2 + 2\lambda_1)x_2 = 10 \\ x_1^2 + x_2^2 = 5 \end{cases}$$

$$\begin{cases} (1, 2) \checkmark \\ (2, 1) \times \end{cases}$$

$\lambda_1 = 1$
$\lambda_2 = 0$

second constraint active ($3x_1 + x_2 = 6$, $\lambda_1 = 0$)

$$\begin{cases} \frac{\partial L}{\partial x_1} = 4x_1 + 2x_2 - 10 + 3\lambda_2 = 0 \\ \frac{\partial L}{\partial x_2} = 2x_1 + 2x_2 - 10 + \lambda_2 = 0 \\ 3x_1 + x_2 = 6 \end{cases}$$

$x_1 = \frac{2}{5}$

$x_2 = \frac{24}{5}$

$\lambda_2 = -\frac{5}{2}$

λ_2 has negative value.
not valid under KKT

both active

$\lambda_1 = 1.87$

$\lambda_2 = -0.69$

$x_1 = \left(\frac{9}{5} - \frac{\sqrt{14}}{10}\right)$

$x_2 = \frac{3}{5} + \frac{3\sqrt{14}}{10}$

Linear idependence

LICQ

EX3

$\max\ x_1 x_2$

$\text{st}\ 1 - x_1^2 - x_2^2 \geq 0$

introduce a Lagrange multiplier.

$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(1 - x_1^2 - x_2^2)$

sationarity

1. ~~stationarity~~

$\nabla L = 0$

$\frac{\partial L}{\partial x_1} = x_2 + 2\lambda x_1 = 0$

$\frac{\partial L}{\partial x_2} = x_1 + 2\lambda x_2 = 0$

2. primal feasibility

$g(x_1, x_2) \geq 0$

3. dual feasibility

$\lambda \geq 0$

4. complementary slackness

$\lambda(1 - x_1^2 - x_2^2) = 0$

Solve the KKT conditions

$x_2 + 2\lambda x_1 = 0 \quad \times x_1 \quad x_1 x_2 + 2\lambda x_1^2 = 0$

$x_1 + 2\lambda x_2 = 0 \Rightarrow \times x_2 \Rightarrow x_1 x_2 + 2\lambda x_2^2 = 0 \Rightarrow x_1^2 = x_2^2$

$1 - x_1^2 - x_2^2 \geq 0 \Rightarrow 1 - 2x_1^2 \geq 0 \Rightarrow x_1^2 \leq \frac{1}{2}$

$x_1 = \pm \frac{1}{\sqrt{2}} \quad x_2 = \pm \frac{1}{\sqrt{2}} \quad \max\ 0.5$

EX2

Ex

Ex 4.

General form of a Quadratic function

$g(x) = ax^2 + bx + c$ $\qquad x^2$ positive

1. convex Quadratic function $g(x)$ with solutions $x=-1$ or $x=1$
factorized form

$g(x) = a(x+1)(x-1)$ $\qquad$ ensure convexity
$g(x) = a(x^2-1)$ $\qquad a > 0$.

let $a=1$ $\qquad g(x) = x^2-1$ $\qquad g''(x) = 2$ positive

2. $g_2(x)$ $\qquad x=0 \quad x=1$ $\qquad \boxed{\text{root of a polynomial } 0, 1}$

$g_2(x) = bx(x-1) \iff g_2(x) = bx^2 - bx$

$g_2(x) = b(x-0)(x-1)$

let $b=1$ $\qquad g_2(x) = x^2 - x$

$g_2''(x) = 2$

$x_i = 1$ $\quad S$ $\qquad x_i = x_i$ $\qquad \frac{1-x_ix_j}{2}$

$x_i = -1$ $\quad \bar{S}$

$$\max \sum_{(ij)\in E} w_{ij} \cdot \frac{1-x_ix_j}{2} \Rightarrow \frac{1}{2}\sum_{ij\in E} w_{ij}(1-x_ix_j)$$

s.t.

$$x_i^2 = 1 \quad \forall i \in V$$

$\min \; x_2^2 - x_1$

$g_1 \quad (x_1-1)^2 + (x_2-1)^2 \le 2$

$g_2 \quad (x_1-1)^2 + (x_2+1)^2 \le 2$

$g_3 \quad x_1 \ge 0$

$L(x, \lambda)$

$x_2^2 - x_1 + \lambda_1((x_1-1)^2+(x_2-1)^2-2) + \lambda_2((x_1-1)^2+(x_2-1)^2-2) + \lambda_3(-x_1)$

partial derative

$\nabla L \quad \frac{\partial L}{\partial x_1} = -1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) - \lambda_3 \implies$

$\nabla L = 0 \quad \frac{\partial L}{\partial x_2} = 2x_2 + 2\lambda_1(x_2-1) + 2\lambda_2(x_2+1)$

$-1 + 2(x_1-1)(\lambda_1+\lambda_2) - \lambda_3 = 0 \implies -1 + 2(x_1-1)(\lambda_1+\lambda_2) = \lambda_3$

$2x_2 + 2(x_2-1)\lambda_1 + 2(x_2+1)\lambda_2 = 0$

complementary slackness	feasibility	Dual
$\lambda_1((x_1-1)^2+(x_2-1)^2-2)=0$	g_1	$\lambda_1 \ge 0$
$\lambda_2((x_1-1)^2+(x_2+1)^2-2)=0$	g_2	$\lambda_2 \ge 0$
$\lambda_3 \cdot (-x_1) = 0$	g_3	$\lambda_3 \ge 0$

Solving system

$\lambda_3(-x_1) = 0$

$\lambda_3 = 0$

$x_1 = 0$

$\Downarrow$

$x_1 = 0$

$g_1 \to 0 \le x_2 \le 2$

$g_2 \to -2 \le x_2 \le 0$

$x_2 = 0$

$f(0,0) = 0$

$(x_1-1)^2 + (x_2-1)^2 - ((x_1-1)^2 + (x_2+1)^2) = 0$

$\implies (x_2-1)^2 = (x_2+1)^2 \implies x_2^2 - 2x_2 + 1 = x_2^2 + 2x_2 + 1$

$\implies -2x_2 = 2x_2 \implies x_2 = 0$

$(x_1-1)^2 - (0+1)^2 = 2 \implies x_1 = 2$

$x_1 = 0$

$(0,0) \quad (2,0)$

$f(2,0) = 0^2 - 2) = -2$

$f(0,0) = 0$

$$\min \ -2(x_1-2)^2 - x_2^2$$

$$\text{st. } g_1 \quad x_1^2 + x_2^2 \le 25$$

$$g_2 \quad x_1 \ge 0$$

$$L(x,\lambda) = -2(x_1-2)^2 - x_2^2 + \lambda_1(x_1^2+x_2^2-25) + \lambda_2(-x_1)$$

∇L

$$\frac{\partial L}{\partial x_1} = -4(x_1-2) + \lambda_1 2x_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = -2x_2 + \lambda_1 2x_2 = 0$$

Satinarity

$-2x_2 + \lambda_1 2x_2 \Rightarrow x_2(1+\lambda_1) = 0$

$\lambda_1 = -1$ not valid

$x_2 = 0$

substitude into $\frac{\partial L}{\partial x_1}$

$-4(x_1-2) + 2\lambda_2 x_1 - \lambda_2 = 0$

complementary slackness

$\lambda_1(x_1^2+x_2^2-25) = 0$

$\lambda_2(-x_1) = 0$

Primal feasibility

g_1

g_2

Dual

$\lambda_1 \ge 0$

$\lambda_2 \ge 0$

$x_2 = 0$

$x_1^2 + 0^2 \le 25$

$-5 \le x_1 \le 5 \Rightarrow$

since $x_1 \ge 0$

$0 \le x_1 \le 5$

Ex 12

LICQ

ex.

$$\min \; 2x^2+y^2+xy$$

st

$2x+y \geq 0$ $\qquad g_1(x,y) = 2x+y \geq 0$

$x+2y = 4$ $\qquad h_1(x,y) = x+2y-4 = 0$

$x+3y \geq 5$ $\qquad g_2(x,y) = x+3y-5 \geq 0$

$y \geq 0$ $\qquad g_3(x,y) = y \geq 0$

$$L(x,y,\lambda,\mu_1,\mu_2,\mu_3) = 2x^2+y^2+xy+\mu_1(2x+y)+\lambda(x+2y-4)+\mu_2(x+3y-5) + \mu_3(y)$$

$$L(x,y,\lambda,\mu_1) = 2x^2+y^2+xy+\lambda(x+2y)-4)+\mu_1(2x+y)$$

$$L(x,y,\lambda_1,\mu_1) = 2x^2+y^2+xy+\lambda_1(2x+y)+\mu_1(x+2y-4)$$

λ_1 is the L multiplier active inequality (g_1)

μ_1 is the L multiplier equality constraint (h_1)

KKT

stationarity : $\nabla_x L(x^*,\lambda^*) = 0$

$$\frac{\partial L}{\partial x} = 4x+y+2\lambda_1+\mu_1 = 0$$

$$\frac{\partial L}{\partial y} = 2y+x+\lambda_1+2\mu_1 = 0$$

Primal feasibility

$2x+y \geq 0$

$x+2y = 4$ $\qquad y \geq 0$ $\quad$ (g_2 g_3 are inactive)

Dual feasibility

$\lambda_1 \geq 0$

complementary slackness

$\lambda_1(2x+y) = 0$

$$\begin{cases} 2x+y = 0 \\ x+2y = 4 \end{cases} \Rightarrow \begin{cases} x = -\frac{4}{3} \\ y = \frac{8}{3} \end{cases}$$

Solve the KKT condition

calculate multipliers

$$4\left(-\frac{4}{3}\right)+\frac{8}{3}+2\lambda_1+\mu_1 = 0$$

$$2\left(\frac{8}{3}\right)+\left(-\frac{4}{3}\right)+\lambda_1+2\mu_1 = 0$$

KKT if x local minim and if LICQ holds x. the $\exists \lambda$ a solution to the KKT system

(Ex 5). $f(x_1, x_2) = x_1^4 + 2x_1^3 + 2x_1^2 + x_2^2 - 2x_1 x_2$ $S(-1, 0)$

∇

$\frac{\partial f}{\partial x_1} = 4x_1^3 + 6x_1^2 + 4x_1 - 2x_2$

$\frac{\partial f}{\partial x_2} = 2x_2 - 2x_1$

$\frac{\partial^2 f}{\partial^2 x_1} = 12x_1^2 + 12x_1 + 4$

$\frac{\partial^2 f}{\partial x \partial x_2} = -2$

$H = \begin{bmatrix} 12x_1^2 + 12x_1 + 4 & -2 \\ -2 & 2 \end{bmatrix}$

$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -2$

$\frac{\partial^2 f}{\partial x_2} = 2$

Evaluate Gradient and Hessian at $(-1, 0)$

$\frac{\partial f}{\partial x_2} = 4(-1)^3 + 6(-1)^2 + 4(-1) - 2(0) = -2$

$\frac{\partial f}{\partial x_2} = 2(0) - 2(-1) = 2$

∇f at $(-1, 0) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

H at $(-1, 0) = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$ $\det(H) = 4$

$\boxed{\Delta x_k = x - H^{-1} \nabla f}$

$H^{-} = \frac{1}{\det(H)} \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}$

$\Delta x_k = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Armijo condition

ex3

(1)

$$\min f(x)$$
$$\text{st. } g_i(x) \leq 0 \quad \text{for } i = 1, 2, \dots m \quad (\text{inequality})$$
$$h_j(x) = 0 \quad \text{for } j = 1, 2 \dots p \quad (\text{equality})$$

Lagrangian

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) \qquad \lambda_i \geq 0$$

Dual constraints

$$\lambda_i \geq 0 \quad \text{for all } i \ (\text{non-negativity})$$

Dual objective function

$$g(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$$

Dual

$$\max g(\lambda, \mu)$$
$$\text{st } \lambda_i \geq 0$$

(2)

$$\min c^T x$$
$$\text{st } Ax \leq b$$

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b)$$
$$= x^T (c + A^T \lambda) - b^T \lambda$$

dual $f(\lambda) = \inf_x L(x, \lambda)$

$$-b^T \lambda \quad \text{if } A^T \lambda + c = 0$$

$$f(\lambda) = -\infty$$

$$\max -b^T \lambda$$
$$\text{st } A^T \lambda + c = 0$$
$$\lambda \geq 0$$

EX 8

EX 1

$$A = \begin{bmatrix} 7 & -3 & 1 & -1 \\ -3 & 7 & -1 & 1 \\ 1 & -1 & 7 & -3 \\ -1 & 1 & -3 & 7 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix} \quad x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$k \leftarrow 0$

$R_0 \leftarrow Ax_0 - b \quad P_0 \leftarrow -r_0$

$r = b - Ax = b$

$r = P_0$

$$\alpha_1 = \frac{r^T p}{p^T A p} = \frac{[4\,0\,8\,4]\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix}}{[4\,0\,8\,4]\begin{bmatrix} 7 & & \\ & \ddots & \\ & & 7 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix}} = \frac{3}{16}$$

$$x_1 = x + \alpha_1 p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{16}\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 0 \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$$

$$r_1 = r - \alpha_1 A p = \begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix} - \frac{3}{16}\begin{bmatrix} 7 & \cdots & -1 \\ \vdots & \ddots & \\ -1 & \cdots & 7 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$

$$\beta_1 = \frac{r_1^T r_1}{r_0^T r} = \frac{[-2, 3, -1, 4]\begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix}}{[4\,0\,8\,4]\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix}} = \frac{5}{16}$$

$$P_1 = r_1 + \beta_1 P_0 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix} + \frac{5}{16}\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ 3 \\ \frac{3}{2} \\ 5\frac{1}{4} \end{bmatrix}$$

$$\alpha_2 = \frac{r_1^T r_1}{P_1^T A P_1} = \frac{[-2\;3\;-1\;4]\begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix}}{[-\frac{3}{4}, 3, \frac{3}{2}, \frac{21}{4}]\,[\;\rightarrow\;]^T} = \frac{1}{9}$$

$$x_2 = X_1 + \alpha_2 P_1 = \begin{bmatrix} \frac{3}{4} \\ 0 \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -\frac{3}{4} \\ 3 \\ \frac{3}{2} \\ \frac{21}{4} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{5}{3} \\ \frac{4}{3} \end{bmatrix}$$

$$r_2 = r_1 - \alpha_2 A P_1 = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9} A \begin{bmatrix} -\frac{3}{4} \\ 3 \\ \frac{3}{2} \\ \frac{21}{4} \end{bmatrix} = 0$$

$$\cancel{\begin{bmatrix} \frac{8}{3} \\ -\frac{8}{3} \\ \frac{8}{3} \\ -\frac{8}{3} \end{bmatrix}}$$

$$\cancel{\beta_2 = \frac{r_2^T r_2}{r_0^T r_0} = \frac{8}{27}}$$

$$\cancel{P_2 = r_2 + \beta_2 P_0 = \begin{bmatrix} \frac{8}{3} \\ -\frac{8}{3} \\ \frac{8}{3} \\ -\frac{8}{3} \end{bmatrix} + \frac{8}{27}\begin{bmatrix} 4 \\ 0 \\ 8 \\ 4 \end{bmatrix} =}$$

EX5 $Ax = b$

$$A = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & -2 \end{bmatrix} \quad b = [6, 3, -2, 0]$$

$R_3 - R_1 = R_3$

$R_4 - 2R_1 = R_4$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 1 & 2 & 1 & 0 & -2 \\ 2 & 1 & 3 & -2 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 2 & 0 & 2 & -8 \\ 2 & 1 & 3 & \cancel{0}\,-2 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 2 & 0 & 2 & -8 \\ 0 & 1 & 1 & 2 & -12 \end{array}\right)$$

$R_3 - 2R_2 \Rightarrow R_3$

$R_4 - R_2 \Rightarrow R_4$

$R_3/4 \Rightarrow R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 4 & 2 & -14 \\ 0 & 1 & 1 & 2 & -12 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 4 & 2 & -14 \\ 0 & 0 & 3 & 2 & -15 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0.5 & -3.5 \\ 0 & 0 & 3 & 2 & -15 \end{array}\right)$$

$R_1 - R_3 \Rightarrow R_1$

$R_2 + 2R_3 \Rightarrow R_2$

$R_4 - 3R_3 \Rightarrow R_4$

$R_4/0.5 \Rightarrow R_4$

$R_1 + 2.5R_4 \Rightarrow R_1$

$R_2 - R_4 \Rightarrow R_2$

$R_3 - 0.5R_4 \Rightarrow R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2.5 & 9.5 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0.5 & -3.5 \\ 0 & 0 & 0 & 0.5 & -4.5 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2.5 & 9.5 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0.5 & -3.5 \\ 0 & 0 & 0 & 1 & -9 \end{array}\right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -13 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -9 \end{array}\right)$$

$A_1 = \det(1) = 1$

$A_2 = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$A_3 = \det \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = 1^3 + 0 + 0 - 1 + 4 - 0 = 4$

PSD

$A_4 = \det \begin{vmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & -2 \end{vmatrix} = \cancel{2 + 0 + 0 + 0 - 16 - 0 - 0 + 20}\ 2$

$$\begin{vmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$\boxed{\text{Ex 10}}$

ex1.

Primal	dual
min	max
min $c^T x$	max $b^T y$
m constriants n variables	variable constraints
the i-th consteriants $\leq$ the i the constraints $\geq$ the i-th constraints $=$	$y_i \leq 0$ $y_i \geq 0$ $y_i \in \mathbb{R}$
$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	the j-th constraint is $\leq$ the j the constraint is $\geq$ the j-th constraints is $=$

~~min $c^T x$ st $Ax \leq b$, $x \geq 0$ — max $b^T y$, $A^T y \geq c$, $y \geq 0$~~

min $c^T x$ st $Ax \geq b$, $x \geq 0$ — max $b^T y$, $A^T y \leq 0$, $y \geq 0$

max $c^T x$, $Ax \leq b$, $x \geq 0$ — min $b^T y$, $A^T y \geq 0$, $y \geq 0$

P max	Dual min
i th constraint $\leq$	i th variable ≥ 0
$\geq$	≤ 0
$=$	$\in \mathbb{R}$
j the variable ≥ 0	j the constraint $\geq$
≤ 0	$\leq$
$\in \mathbb{R}$	$=$

(P)

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j$$
$$x_j \geq 0$$

(D)

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$$
$$y_i \geq 0$$

$$\min \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$$
$$x_j \geq 0$$

$$\max \sum_{i=1}^{m} b_i y_i$$
$$\sum_{i=1}^{m} a_{ij} y_i \leq 0$$
$$y_i \geq 0$$

ex 2.

$$\min \sum w(v) x_v$$
$$\forall e=(u,v) \in E.\ x_u + x_v \geq 1$$
$$\forall v \in V \quad x_v \geq 0$$

$\Rightarrow$

$$\max \sum_{(u,v)\in E} y_{uv}$$

st ~~$\sum_{u:(u,v)\in E} y_{uv} \leq w(v)$ $\forall v \in V$~~

~~$y_{uv} \geq 0,\ \forall (u,v) \in E$~~

$\forall v \in V.\ \sum y_{(uv)} \leq w(v)$

$$\sum_{u \mid (u,v)\in E} y_{uv} \leq w(v) \quad \forall v \in V$$
$$y_{uv} \geq 0 \quad \forall (u,v) \in E$$

ex3. dual of dual is primal

(P) max $c^T x$
st $Ax \le b$
$x \ge 0$

(D) min $b^T y$
st $A^T y \ge c$
$y \ge 0$

$\times(-1)$ $\Rightarrow$ max $(-b^T)y$
$(-A^T)y \le -c$
$y \ge 0$

(D)(D)

min $-c^T x$
st $-Ax \ge -b$
$x \ge 0$

$\times -1$ $\Rightarrow$ max $c^T x$
st $Ax \le b$
$x \ge 0$

(D)(D)

max $c^T z$
st $Az \le b$
$z \ge 0$

both (P) (D) are infeasible
P is infeasible . D is unbounded
D in infeasible P is unbounded
Both P D feasible $c^T x^* = b^T y^*$

(P) max $\sum_{j=1}^{n} c_j x_j$
st $\sum_{j=1}^{n} a_{ij} x_j \le b_i \; (i=1,\dots,m)$
$x_1 \dots x_n \ge 0$

(D) min $\sum_{i=1}^{m} b_i y_i$
st $\sum_{i=1}^{m} a_{ij} y_i \ge c_j \; (j=1,\dots,n)$
$y_1 \cdots y_m \ge 0$

(D) $-\max \sum_{i=1}^{m} (-b_i) y_i$
$\Rightarrow \sum_{i=1}^{m} (-a_{ij}) y_i \le -c_j$
$y_1 \cdots y_m \ge 0$

(DD) $-\min \sum_{j=1}^{n} (-c_j) x_j$
st $\sum_{j=1}^{n} (a_{ij}) x_j \ge - b_i$
$x_1 \dots x_n \ge 0$

$\Rightarrow$ max $\sum_{j=1}^{n} c_j x_j$
st $\sum_{j=1}^{n} a_{ij} x_j \le b_i$
$x_1 \dots x_n \ge 0$

ex 4.5 min $10x_1 - 4x_2$

$x_1 + 0.6x_3 + 4x_4 \geq 43$

$x_1 - x_2 + 0.6x_3 + 10x_4 \geq 27$

$x_1 - x_2 - 0.4x_3 - x_4 \geq 24$

$x_1 - x_2 - 0.4x_3 - 2x_4 \geq 22$

$x_1 + 3.6x_3 - 3x_4 \geq 56$

$x_1 x_2 x_3 x_4 \geq 0$

$b^T y^*$

$(3.36, 0, 0, 6.48, 0.16)$

$(29.6, 0, 9, 2)$

(D) max $43\lambda_1 + 27\lambda_2 + 24\lambda_3 + 22\lambda_4 + 56\lambda_5$

$43 \times 3.36 + 22 \times 6.48 + 56 \times 0.16 = 296$

$x_1 + 0.6x_3 + 4x_4 = 43$

$x_1 - x_2 - 0.4x_3 - 2x_4 = 22$

$x_1 + 3.6x_3 - 3x_4 = 56$

complementary slackness

$$\left|\begin{array}{ccc|c} 1 & 0.6 & 4 & 43 \\ 1 & -0.4 & -2 & 22 \\ 1 & 3.6 & 3 & 56 \end{array}\right|$$

$x_1 = 29.6$

$x_3 = 9$

$x_4 = 2$

---

ex4

(D)

max $c^T x$

st $Ax \leq b$

$x \geq 0$

(P)

min $b^T y$

st $A^T y \geq c$

$y \geq 0$

Find x, y

st

$Ax \leq b$

$A^T y \geq c$

$x, y \geq 0$

$c^T x = b^T y$

Ex 1

$\min \; x$

$\text{st} \; x \geq 0$

HW2

$L(x, \lambda) = x - \lambda x$

KKT:

stationarity $\frac{\partial L}{\partial x} = 1 - \lambda = 0 \Rightarrow \lambda = 1$

dual feasibility $\lambda \geq 0$

complementary $\lambda x = 0$

from the stationary and complementary, we get $\lambda = 1$ and $x = 0$, then we check all the conditions, which are all satisfied, the we check the LICQ, the gradient of the constraint $\nabla(x) = 1$, it is a non-zero vector. so the LICQ holds, therefore $x = 0$ is the optimal solution.

(P) $\min \; x$ $\quad \text{st} \; x \geq 0$

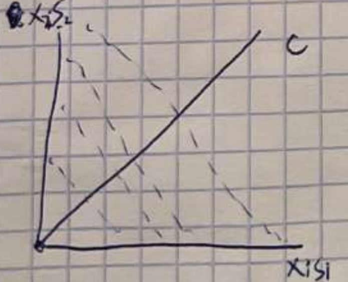
(D) $\max \; \lambda \cdot 0$ $\quad \text{st} \; \lambda + s - 1 = 0$, $\; s \geq 0$

$$\begin{aligned} A^T\lambda + s &= c \\ Ax &= b \\ x_i s_i &= \tau \; (i = 1, 2, \dots, n) \\ (x, s) &> 0 \end{aligned} \Rightarrow \begin{aligned} &\lambda + s = 1 \quad x = 0 \text{ as } b = 0 \text{ ?} \\ &\cancel{x \geq 0 \; x > 0 \; (x > 0 \text{ on the central path})} \\ &x_i s_i = \tau \\ &(x, s) > 0 \\ &\tau > 0 \end{aligned}$$

?

then we define the central path as

$$C = \{(x_\tau, \lambda_\tau, s_\tau) \mid \tau > 0\}$$



$\min \; c^T x$ $\quad \text{st} \; Ax = b, \; x \geq 0$

$\max \; b^T \lambda$ $\quad \text{st} \; A^T\lambda + s = c, \; s \geq 0$

KKT

$$\begin{aligned} A^T\lambda + s &= c \\ Ax &= b \\ x_i s_i &= 0 \quad i = 1 \dots n \\ (x, s) &\geq 0 \end{aligned}$$

$\Downarrow$

$$F(x, \lambda, s) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe \end{bmatrix}$$

$X = \text{diag}(x_1 \dots x_n)$

$S = \text{diag}(s_1 \dots s_n)$

$e = (1, 1, \dots, 1)^T$

$$\mu = \frac{x^T s}{n}$$

$$J(x, \lambda, s) \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = -F(x, \lambda, s)$$

$r_b = Ax - b$

$r_c = A^T\lambda + s - c$

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -XSe \end{bmatrix}$$

$(x, \lambda, s) + \alpha(\Delta x, \Delta \lambda, \Delta s)$

Ex2

$$\min\ x \quad \text{st}\quad x \geq 0 \qquad\qquad \max\ 0 \quad \text{st}\quad \lambda + s - 1 = 0,\ s \geq 0$$

there are no equality constraint ($Ax=0$) in the primal LP. so A and b are non-existent. therefore.

$$F(x,\lambda,s) = \begin{bmatrix} s-1 \\ x\cdot s - \sigma\mu \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial s} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial s} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ s & x \end{bmatrix}$$

Newton's update ~~$\begin{bmatrix}\Delta x \\ \Delta s\end{bmatrix} = -J^{-1}F$~~. ~~give init guess (x_0, s_0) we solve~~

~~$x_0 = 1\ s_0 = 1\ \lambda = 1$~~

$$\begin{bmatrix} 0 & 1 \\ s & x \end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta s\end{bmatrix} = -\begin{bmatrix} s_k - 1 \\ x_k s_k - \sigma\mu_k \end{bmatrix}$$

$$\det(J) = 0\cdot x - 1\cdot s = -s \qquad \mu = \frac{x^T s}{n} = \frac{1\times 1}{1} = 1$$

$$J^{-1} = \frac{1}{-s}\begin{bmatrix} x & -s \\ -1 & 0 \end{bmatrix}$$

give the init $x_0 = 1\ s_0 = 1$ $\sigma = 0.5$ we solve

$$\begin{bmatrix} 0 & 1 \\ s & x \end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta s\end{bmatrix} = \begin{bmatrix} 1-1 \\ 1\times 1 - \sigma \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix}\Delta x \\ \Delta s\end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -1\times 0 + 1\times(-0.5) \\ 1\times 0 + 0\times(-0.5) \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}$$

$$F(x,\lambda,s) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe - \sigma\mu e \end{bmatrix}$$

$$\mu = \frac{x^T s}{n}$$

$$J = \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix}$$

$$J\begin{bmatrix}\Delta x \\ \Delta\lambda \\ \Delta x\end{bmatrix} = -F(x,\lambda,s)$$

$e = [1, 1, \dots, 1]$

σ centering param

$c = 1$

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix}\begin{bmatrix}\Delta x^k \\ \Delta\lambda^k \\ \Delta s^k\end{bmatrix} = \begin{bmatrix} -r^k \\ -r_b^k \\ -x^k s^k e + \sigma_k\mu_k e \end{bmatrix}$$

give $(x^0\ \lambda^0\ s^0)$ $(x^0 s^0) > 0$

for $k = 0, 1, 2 \dots$ $\sigma_k \in [0,1]$

$(x, \lambda, s) + \alpha(\Delta x, \Delta\lambda, \Delta s)$

we define the primal-dual feasible set F and strictly feasible set F^o.

$$F = \{(x,\lambda,s) \mid Ax = b,\ A^T\lambda + s = c,\ (x,s) \geq 0\}$$

$$F^o = \{(x,\lambda,s) \mid Ax = b,\ A^T\lambda + s = c,\ (x,s) > 0\}$$

and central path as an arc of strictly feasible points that for $\tau > 0$:

$$\begin{aligned} A^T\lambda + s &= c, \\ Ax &= b \\ x_i s_i &= \tau \\ (x,s) &> 0, \end{aligned} \Rightarrow C = \{(x_\tau, s_\tau, s_\tau) \mid \tau > 0\}$$

when $\tau \to 0$, the system approach to the KKT condition of primal Problem the C converges to primal-dual solution with $\tau \downarrow 0$. so the path is an arc that strictly reduces $x_i s_i$, $i = 1, 2 \dots n$.

when $(x, \lambda, s) \in \mathcal{F}$, we have $r_b = 0$ $r_c = 0$ such that the search direction satisfy:

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe + \sigma\mu e \end{bmatrix}$$

where $X = \mathrm{diag}(x_1 x_2 \cdots x_n)$ $S = \mathrm{diag}(s_1 s_2 \cdots s_n)$. $\mu = \frac{1}{n}\sum x_i s_i = \frac{x^T s}{n}$

$e = \vec{1}$. $\sigma \in [0, 1]$

the solution of the system gives a $\overbrace{\text{direction}}^{\text{centering}}$, Particularly when $\sigma = 1$, the ~~solution~~ direction point to point on C such that $x_i s_i = \mu$.

When $\sigma = 0$, it becomes a general Newton's step (of the transformation)

$\boxed{Ex3}$

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_c \\ -r_b \\ -XSe + \sigma \mu e \end{bmatrix} \qquad \mu = \frac{x^T s}{n}$$

$-\ (x, \lambda, s) \in F_0$

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe + \sigma \mu_e \end{bmatrix}$$ we can observe that.

from the first row $A^T \Delta \lambda + I \Delta s = 0$

second row $A \Delta x + 0 \Delta \lambda + 0 \Delta s = 0 \Rightarrow A \Delta x = 0$

we muleiply the first equation by Δx^T and second one by $\Delta \lambda^T$

$\Rightarrow A^T \Delta \lambda \Delta x^T + I \Delta s \Delta x^T = 0$

$A \Delta x \Delta \lambda^T = 0$

subtract the ~~first~~ second equation from the first one

$\Rightarrow \Delta s \Delta x^T = 0$ (Lemma 14.2)

$\boxed{Ex4}$

$\min x_1$
st $x_1 + x_2 = 1$
$x_1, x_2 \geq 0$

$-$ (D) $\max \lambda$

$\lambda^+ + s_1 = 1$
$s_1 \geq 0$ λ is unresericted
$\lambda^- + s_2 = 0$
$s_2 \geq 0$

$$F(x, \lambda, s) = \begin{pmatrix} x_1 + x_2 - 1 \\ \lambda + s_1 - 1 \\ \lambda + s_2 \\ x_1 s_1 \\ x_2 s_2 \end{pmatrix} = 0$$

by the complementary, either $x_1 = 0$ or ~~x~~ $s_1 = 0$ and either $x_2 = 0$ or $s_2 = 0$

Let $s_2 = 0$.

$\lambda + 0 = 0 \Rightarrow \lambda = 0$

$\lambda + s_1 - 1 = 0 \Rightarrow 0 + s_1 - 1 = 0 \Rightarrow s_1 = 1$

$x_1 s_1 = 0 \Rightarrow x_1 = 0$

$x_1 + x_2 - 1 \Rightarrow x_2 = 1$

we have a solution

$x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\lambda^* = 0$ $s^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

consider another case $x_2 = 0$

$x_1 + x_2 - 1 = 0 \Rightarrow x_1 = 1$

$x_1 s_1 = 0 \Rightarrow s_1 = 0$

$\lambda + s_1 - 1 = 0 \Rightarrow \lambda = 1$

$\lambda + s_2 = 0 \Rightarrow s_2 = -1$

$x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\lambda^* = (1)$ $s^* = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$s_2 = -1$ violates the constraint