NUMERICAL OPTIMIZATION

Sheet 12: A sample exam

An important disclaimer. These exercises serve as a preview of the type of questions that can appear on the final exam. Please be aware that the exam will contain different questions, including different topics than those mentioned in the tasks below. In fact, any topic covered on the lecture and basic exercises that apply the topics from the lecture can appear in the exam.

Please also note that this sheet is still a homework – in particular, exercises can take you a longer time or a shorter time than the ones in the exam. I recommend not underestimating the exam if some of the following exercises appear easy to you without the time pressure and the classroom environment.

You should also keep in mind that at the exam, there will be no text materials available. I recommend trying to solve the homework with as little Google use as possible, only referencing the syllabus literature or lecture notes if need be.

Finally, the basic rules of any mathematics/theoretical computer science test apply for this test as well. In other words, the final result is important, but it is critical to include all intermediate arguments in your writing, as well as including statements or at least brief explanations of all theorems and definitions that you will be using.

Exercise one

Consider the following optimization problem:

$$\begin{aligned} \text{minimize } & 2x^2 + y^2 + xy \\ \text{subject to } & 2x + y \geq 0 \\ & x + 2y = 4 \\ & x + 3y \geq 5 \\ & y \geq 0. \end{aligned}$$

1. Write the Lagrange function and check the KKT conditions for the case where the first inequality is active but the other two are inactive. Do not forget about any assumptions on the use of KKT. Do the KKT conditions allow a local optimum for this case?
2. Define the concept of a dual problem and strong duality for a constrained optimization problem.
3. Use a theorem from the lecture to conclude that strong duality holds for the problem above.

Exercise two

1. Write the definition of the general *primal-dual path following framework*, which is an interior-point method for solving linear programming. Include as much explanation as possible, including what is the Jacobian, the meaning of the variables $(\Delta x, \Delta\lambda, \Delta s)$, duality measure, and more.
2. Give the definition of the following terms: feasible region, strictly feasible region, central path.
3. The *long-step path following method* uses a neighborhood called $\mathcal{N}_{-\infty}(\gamma)$, parametrized by $\gamma \in (0, 1]$, to maintain its progress. Can you define this set and explain its connection to the central path?

Exercise three

1. Write the general form of duality for any constrained optimization problem (not only for linear programs).
2. Suppose we dualize $\min c^T x$ such that $Ax \leq b$. Write the dual function $h(\lambda)$ in its infimum form. If we wish to find a closed non-infimum form for $h(\lambda)$, what extra constraints do we get

for the dual problem?

EXERCISE FOUR Define the Newton method for solving unconstrained optimization problems. Formally explain what it means that the Newton method is affine invariant. Explain the difference between the pure and damped Newton steps.

EXERCISE FIVE Perform one step of the pure Newton method for minimizing the function

$$f(x_1, x_2) = x_1^4 + 2x_1^3 + 2x_1^2 + x_2^2 - 2x_1x_2,$$

starting in the initial point $(-1, 0)$.